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# ON THE SUPERRIGIDITY OF MALLEABLE ACTIONS WITH SPECTRAL GAP

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## 1. Introduction

Some of the most interesting aspects of the dynamics of measure preserving actions of countable groups on probability spaces, $\Gamma \curvearrowright (X,\mu)$, are revealed by the study of *group measure space* von Neumann algebras $L^\infty(X) \rtimes \Gamma$ ([MvN1]) and the classification of actions up to *orbit equivalence* (*OE*), i.e. up to isomorphism of probability spaces carrying the orbits of actions onto each other. Although one is in von Neumann algebras and the other in ergodic theory, the two problems are closely related, as an OE of actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ has been shown to implement an algebra isomorphism $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ taking $L^\infty(X)$ onto $L^\infty(Y)$, and vice-versa ([Si], [Dy], [FM]). In particular, the isomorphism class of $L^\infty(X) \rtimes \Gamma$ only depends on the equivalence relation $\mathcal{R}_\Gamma = \{(t, gt) \mid t \in X, g \in \Gamma\}$.

Thus, *von Neumann equivalence* (*vNE*) of group actions, requiring isomorphism of their group measure space algebras, is weaker than OE. Since there are examples of non-OE actions whose associated von Neumann algebras are all isomorphic ([CJ1]), it is in general strictly weaker. On the other hand, OE is manifestly weaker than classical *conjugacy*, which for free actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ requires isomorphism of probability spaces $\Delta : (X,\mu) \simeq (Y,\nu)$ satisfying $\Delta\Gamma\Delta^{-1} = \Lambda$ (so in particular $\Gamma \simeq \Lambda$). How much weaker vNE and OE can be with respect to conjugacy is best seen in the amenable case, where by a celebrated theorem of Connes all free ergodic actions of all (infinite) amenable groups give rise to the same $\mathrm{II}_1$ factor ([C1]) and by ([Dy], [OW], [CFW]) they are undistinguishable under OE as well. Also, any embedding of algebras $L^\infty(X) \rtimes \Gamma \subset L^\infty(Y) \rtimes \Lambda$ with $\Lambda$ amenable forces $\Gamma$ to be amenable.

But the non-amenable case is extremely complex, and for many years progress has been slow ([MvN2], [Dy], [Mc], [C2], [CW], [Sc]), even after the discovery of the first rigidity phenomena by Connes in von Neumann algebras ([C3, C4]) and by Zimmer in OE ergodic theory ([Z1, Z2]). This changed dramatically over the last 7-8 years, with the advent of a variety of striking rigidity results ([Fu1], [G1], [G2], [MoSh], [H], [HK], [P1]–[P8]; see [P9] for a survey; also [Sh] for a survey on OE rigidity).

Our aim in this paper is to investigate the most "extreme" such phenomena, called *strong rigidity*, which show that for certain classes of *source* group actions

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$\Gamma \curvearrowright X$ and *target* actions $\Lambda \curvearrowright Y$ any isomorphism $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ (resp. any OE of $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$) comes from a conjugacy, modulo perturbation by an inner automorphism of $L^\infty(Y) \rtimes \Lambda$ (resp. of $\mathcal{R}_\Lambda$). Ideally, one seeks to prove this under certain conditions on the source group actions $\Gamma \curvearrowright X$ but no condition at all (or very little) on the target $\Lambda \curvearrowright Y$, a type of result labeled *superrigidity.* On the orbit equivalence side, such results appeared first in [Fu1] (for actions of higher rank lattices, such as $SL(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$, $n \geq 3$), then in [MoSh] (for doubly ergodic actions of products of word hyperbolic groups, such as $\mathbb{F}_n \times \mathbb{F}_m$). In the meantime, new developments in von Neumann algebras ([P3, P8]) led to the first vNE strong rigidity result in [P4, P5]. It shows that any isomorphism of group measure space factors $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$, with $\Gamma$ an *infinite conjugacy class* (*ICC*) group having an infinite subgroup satisfying a weak normality condition and with the relative property (T) of Kazhdan-Margulis ($\Gamma$ *w-rigid*) and $\Lambda \curvearrowright Y$ a Bernoulli action of an arbitrary ICC group, comes from a conjugacy. While obtained in a purely von Neumann algebra framework, this result provides new OE rigidity phenomena as well, showing for instance that Bernoulli actions of Kazhdan groups are OE superrigid ([P5]).

The ideas and techniques in [P3, P4, P5] were further exploited in [P1] to obtain a cocycle superrigidity result for Bernoulli actions of w-rigid groups $\Gamma$, from which OE superrigidity is just a consequence. Thus, [P1] shows that any measurable cocycle for $\Gamma \curvearrowright X = [0,1]^\Gamma$ is $\mathcal{U}_{fin}$*-cocycle superrigid* (*CSR*), i.e. any $\mathcal{V}$-valued cocycle for $\Gamma \curvearrowright X$ is cohomologous to a group morphism $\Gamma \rightarrow \mathcal{V}$, whenever $\mathcal{V}$ is a closed subgroup of the unitary group of a separable finite von Neumann algebra, for instance if $\mathcal{V}$ is countable discrete, or separable compact.

The sharp OE and vNE rigidity results in [P1, P5], and in fact in [P1]–[P8], [PS], [IPeP], [PV], [V], [I1, I2] as well, are due to a combination (co-existence) of deformability and rigidity assumptions on the group actions. The deformability condition imposed is often the *malleability* of the action (e.g. in [P1, P5]), a typical example of which are the Bernoulli actions, while the rigidity assumption is always some weak form of property (T) (on the acting group, as in [P1, P5], or on the way it acts, as in [P8]). Thus, the *deformation/rigidity* arguments used in all these papers seemed to depend crucially on the "property (T)-type" assumption.

However, in this paper we succeed to remove this assumption completely. Namely, we prove a new set of rigidity results for malleable actions, in some sense "parallel" to the ones in [P1, P5], but which no longer assume Kazhdan-type conditions on the source group, being surprisingly general in this respect. For instance, we show that if $\mathcal{V} \in \mathcal{U}_{fin}$ and $\Gamma$ is an arbitrary group, then any $\mathcal{V}$-valued cocycle for a Bernoulli $\Gamma$-action can be untwisted on the centralizer (or commutant) of any non-amenable subgroup $H$ of $\Gamma$! More precisely, we prove (compare with 5.2/5.3 in [P1]):

**1.1. Theorem** (CSR: s-malleable actions)**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a m.p. action of a countable group $\Gamma$. Let $H, H' \subset \Gamma$ be infinite commuting subgroups such that:*

($a$) *$H \curvearrowright X$ has stable spectral gap.*
($b$) *$H' \curvearrowright X$ is weak mixing.*
($c$) *$HH' \curvearrowright X$ is s-malleable.*

*Then $\Gamma \curvearrowright X$ is $\mathscr{U}_{fin}$-cocycle superrigid on $HH'$. If in addition $H'$ is w-normal in $\Gamma$, or $H'$ is wq-normal but $\Gamma \curvearrowright X$ is mixing, then $\Gamma \curvearrowright X$ is $\mathscr{U}_{fin}$-cocycle superrigid on all $\Gamma$. Moreover, the same conclusions hold true if we merely assume $HH' \curvearrowright X$*

*to be a relative weak mixing quotient of an m.p. action* $HH' \curvearrowright (X', \mu')$ *satisfying conditions* $(a), (b), (c)$.

The *stable spectral gap* condition $(a)$ in Theorem 1.1 means that the representation implemented by the action $H \curvearrowright X$ on $L^2X\overline{\otimes}L^2X \ominus \mathbb{C}$ has spectral gap, i.e. has no approximately invariant vectors (see Section 3). It automatically implies $H$ is non-amenable. The *s-malleability* condition for an m.p. action $\Gamma_0 \curvearrowright X$ was already considered in [P1]–[P5] and is discussed in Section 2. An action $\Gamma_0 \curvearrowright (X, \mu)$ is a *relative weak mixing* quotient of an m.p. action $\Gamma_0 \curvearrowright (X', \mu')$ if it is a quotient of it and $\Gamma_0 \curvearrowright X'$ is weak mixing relative to $\Gamma_0 \curvearrowright X$ in the sense of [F], [Z3]; see also Definition 2.9 in [P1].

The two "weak normality" conditions considered in Theorem 1.1 are the same as in [P1, P2, P5]: An infinite subgroup $\Gamma_0 \subset \Gamma$ is *w-normal* (resp. *wq-normal*) in $\Gamma$ if there exists a well ordered family of intermediate subgroups $\Gamma_0 \subset \Gamma_1 \subset ... \subset \Gamma_\jmath \subset ... \subset \Gamma_\imath = \Gamma$ such that for each $0 < \jmath \leq \imath$, the group $\Gamma'_\jmath = \bigcup_{n<\jmath} \Gamma_\jmath$ is normal in $\Gamma_\jmath$ (resp. $\Gamma'_\jmath$ is generated by the elements $g \in \Gamma$ with $|g\Gamma'_\jmath g^{-1} \cap \Gamma'_\jmath| = \infty$).

Any generalized Bernoulli action $\Gamma_0 \curvearrowright \mathbb{T}^I$, associated to an action of a countable group $\Gamma_0$ on a countable set $I$, is s-malleable. Given any probability space $(X_0, \mu_0)$ (possibly atomic), the generalized Bernoulli action $\Gamma_0 \curvearrowright (X_0, \mu_0)^I$ is a relative mixing quotient of the s-malleable action $\Gamma_0 \curvearrowright \mathbb{T}^I$. Any Gaussian action $\sigma_\pi : \Gamma_0 \curvearrowright (\mathbb{R}, (2\pi)^{-1/2} \int \cdot e^{-t^2} \mathrm{d}t)^n$ associated to an orthogonal representation $\pi$ of $\Gamma_0$ on the $n$-dimensional real Hilbert space $\mathcal{H}_n = \mathbb{R}^n$, $2 \leq n \leq \infty$, is easily seen to be s-malleable (cf. [Fu2]). The action $\sigma_\pi$ has stable spectral gap on some subgroup $H \subset \Gamma_0$ once the orthogonal representation $\pi_{|H}$ has stable spectral gap. By [P2], a sufficient condition for a generalized Bernoulli action $H \curvearrowright (X_0, \mu_0)^I$ to have stable spectral gap is that $\{g \in H \mid gi = i\}$ be amenable, $\forall i \in I$. Thus, Theorem 1.1 implies:

**1.2. Corollary** (CSR: Bernoulli and Gaussian actions)**.** *Let* $\Gamma$ *be a countable group having infinite commuting subgroups* $H, H'$ *with* $H$ *non-amenable. Let* $\Gamma \curvearrowright X$ *be an m.p. action whose restriction to* $HH'$ *is a relative weak mixing quotient of one of the following:*

1°. *A generalized Bernoulli action* $HH' \curvearrowright (X_0, \mu_0)^I$, *with the actions of* $H, H'$ *on the countable set* $I$ *satisfying* $|H'i| = \infty$ *and* $\{g \in H \mid gi = i\}$ *amenable,* $\forall i \in I$.

2°. *A Gaussian action associated to an orthogonal representation of* $HH'$ *which has stable spectral gap on* $H$ *and no finite dimensional* $H'$*-invariant subspaces.*

*If* $H'$ *is w-normal in* $\Gamma$, *then* $\Gamma \curvearrowright X$ *is* $\mathscr{U}_{fin}$*-cocycle superrigid. If* $H'$ *is merely wq-normal in* $\Gamma$ *but* $\Gamma \curvearrowright X$ *is a weak mixing quotient of a Bernoulli action, then again* $\Gamma \curvearrowright X$ *is* $\mathscr{U}_{fin}$*-cocycle superrigid.*

Due to Theorems 5.6-5.8 in [P1], the above cocycle superrigidity results imply several superrigidity results in orbit equivalence ergodic theory:

**1.3. Corollary** (OE superrigidity)**.** *Let* $\Gamma$ *be a countable group with no finite normal subgroups and having infinite commuting subgroups* $H, H'$, *with* $H$ *non-amenable. Assume that the free m.p. action* $\Gamma \curvearrowright X$ *is a relative weak mixing quotient of an s-malleable action* $\Gamma \curvearrowright (X', \mu')$ *such that:*

(1.3) $H \curvearrowright X'$ *has stable spectral gap and either* $H'$ *is w-normal in* $\Gamma$ *with* $H' \curvearrowright X'$ *weak mixing, or* $H$ *is merely wq-normal in* $\Gamma$ *but with* $\Gamma \curvearrowright X'$ *mixing.*

*Then* $\Gamma \curvearrowright X$ *satisfies the conclusions in* 5.6, 5.7, 5.8 *of* [P1]. *In particular, any Bernoulli action* $\Gamma \curvearrowright (X,\mu) = (X_0,\mu_0)^\Gamma$ *is OE superrigid, i.e. any OE between* $\Gamma \curvearrowright X$ *and an arbitrary free ergodic m.p. action of a countable group comes from a conjugacy. If in addition* $\Gamma$ *is ICC, then ANY quotient of* $\Gamma \curvearrowright X$ *which is still free on* $\Gamma$ *is OE superrigid.*

By 2.7 in [P2], Corollary 1.2 provides a large class of groups with uncountably many OE inequivalent actions, adding to the numerous examples already found in [Z2], [H], [GP], [P2], [I1], [IPeP]:

**1.4. Corollary.** *Assume* $\Gamma$ *contains a non-amenable subgroup with its centralizer infinite and wq-normal in* $\Gamma$*, e.g.* $\Gamma$ *non-amenable and either a product of two infinite groups, or having infinite center. Then for any countable abelian group* $L$ *there exists a free ergodic action* $\sigma_L$ *of* $\Gamma$ *on a probability space with the first cohomology group* $\mathrm{H}^1(\sigma_L)$ *equal to* $\mathrm{Hom}(\Gamma,\mathbb{T})\times L$*. In particular,* $\Gamma$ *has a continuous family of OE inequivalent actions.*

The cocycle superrigidity Theorem 1.1 is analoguous to 5.2/5.3 in [P1]. The trade-off for only assuming $H \subset \Gamma$ non-amenable in Theorem 1.1, rather than Kazhdan as in [P1], is the spectral gap condition on the action. The proof is still based on a deformation/rigidity argument, but while the malleability is combined in [P1] with property (T) rigidity, here it is combined with spectral gap rigidity. Also, rather than untwisting a given cocycle on $H$, we first untwist it on the group $H'$ commuting with $H$. Due to the weak mixing property (*b*) of Theorem 1.1 and 3.6 in [P1], it then gets untwisted on the w-normalizer of $H'$, thus on $HH'$. Altogether, we rely heavily on technical results from [P1].

We use the same idea of proof, combined this time with technical results from [P4, P5], to obtain a vNE strong rigidity result analogue to 7.1/7.1' in [P5], which derives conjugacy of actions from the isomorphism of their group measure space factors. Note that while the "source" group $\Gamma$ is still required to have a non-amenable subgroup with infinite centralizer, the $\Gamma$-action here is completely arbitrary. In turn, while the "target" group $\Lambda$ is arbitrary, the $\Lambda$-action has to be Bernoulli. Thus, the spectral gap condition, which is automatic for Bernoulli actions, is now on the target side.

**1.5. Theorem** (vNE strong rigidity). *Assume* $\Gamma$ *contains a non-amenable subgroup with centralizer non-virtually abelian and wq-normal in* $\Gamma$*. Let* $\Gamma \curvearrowright (X,\mu)$ *be an arbitrary free ergodic m.p. action. Let* $\Lambda$ *be an arbitrary ICC group and* $\Lambda \curvearrowright (Y,\nu)$ *a free, relative weak mixing quotient of a Bernoulli action* $\Lambda \curvearrowright (Y_0,\nu_0)^\Lambda$*. If* $\theta : L^\infty X \rtimes \Gamma \simeq (L^\infty Y \rtimes \Lambda)^t$ *is an isomorphism of* $\mathrm{II}_1$ *factors, for some* $0 < t \leq 1$*, then* $t = 1$ *and* $\theta$ *is of the form* $\theta = \mathrm{Ad}(u) \circ \theta^\gamma \circ \theta_0$*, where:* $u$ *is a unitary element in* $L^\infty Y \rtimes \Lambda$*;* $\theta^\gamma \in \mathrm{Aut}(L^\infty Y \rtimes \Lambda)$ *is implemented by a character* $\gamma$ *of* $\Lambda$*;* $\theta_0 : L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ *is implemented by a conjugacy of* $\Gamma \curvearrowright X$*,* $\Lambda \curvearrowright Y$*.*

When applied to isomorphisms $\theta$ implemented by OE of actions, Theorem 1.5 above implies an *OE Strong Rigidity* result analogue to 7.6 in [P5]. We in fact derive an even stronger rigidity result, for embeddings of equivalence relations, parallel to 7.8 in [P5]:

**1.6. Theorem** (OE strong rigidity for embeddings). *Let* $\Gamma \curvearrowright (X,\mu)$*,* $\Lambda \curvearrowright (Y,\nu)$ *be as in Theorem* 1.5*. If* $\Delta : (X,\mu) \simeq (Y,\nu)$ *takes each* $\Gamma$*-orbit into a* $\Lambda$*-orbit (a.e.),* *then there exists a subgroup $\Lambda_0 \subset \Lambda$ and $\alpha \in \mathrm{Inn}(\mathcal{R}_\Lambda)$ such that $\alpha \circ \Delta$ conjugates $\Gamma \curvearrowright X$, $\Lambda_0 \curvearrowright Y$.*

Notice that although OE superrigidity results are of a stronger type than OE strong rigidity, Theorem 1.6 cannot be deduced from Corollary 1.3, nor in fact from Theorem 1.1. Likewise, the OE strong rigidity (7.8 in [P5]) cannot be derived from results in [P1].

The idea of combining malleability with spectral gap rigidity in the proofs of Theorems 1.1 and 1.5 is inspired from [P6], where a similar argument was used to prove a unique "McDuff decomposition" result for $\mathrm{II}_1$ factors. We comment on this in Section 6, were we include other remarks as well, showing for instance that if $\Gamma \curvearrowright X$ is a Bernoulli action of a non-amenable group, then $L^\infty X \rtimes \Gamma$ follows prime, due to the same arguments. We also revisit the Connes-Jones counterexample in [CJ1] and point out that, due to results in [P1, P5] and in this paper, it provides cocycle superrigid (in particular OE superrigid) actions $\Gamma \curvearrowright X$ whose equivalence relation $\mathcal{R}_\Gamma$ has trivial fundamental group, $\mathcal{F}(\mathcal{R}_\Gamma) = \{1\}$, while the associated $\mathrm{II}_1$ factor $M = L^\infty(X) \rtimes \Gamma$ has fundamental group equal to $\mathbb{R}_+^*$, so $M$ can be realized by uncountably many OE-inequivalent actions (two of which are free). Section 4 contains the proof of Theorem 1.1 through Corollary 1.4 and Section 5 the proof of Theorems 1.5-1.6. Both sets of proofs rely heavily on technical results from [P1] and respectively [P4, P5]. The present paper should in fact be viewed as a companion to [P1, P4, P5], from which notations and terminology are taken as well.

In Section 2 we comment on s-malleability of actions and transversality; then in Section 3 we define the notion of stable spectral gap for actions and representations of groups, and examine how Bernoulli, Gaussian and Bogoliubov actions (which are the basic examples of s-malleable actions) behave with respect to this property.

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## 2. Transversality of s-malleable actions

In [P1]–[P4] we have considered various degrees of malleability for actions of groups on probability measure spaces $\Gamma \curvearrowright (X, \mu)$ (more generally on von Neumann algebras). The weakest such condition (see 2.1 in [P2], or 4.2 in [P1]) requires the connected component of the identity in the centralizer $\mathrm{Aut}_\Gamma(X \times X, \mu \times \mu)$ of the double action $g(t, t') = (gt, gt')$, $\forall (t, t') \in X \times X$, $g \in \Gamma$, to contain an automorphism $\alpha_1$ satisfying $\alpha_1(L^\infty X \otimes 1) = 1 \otimes L^\infty X$, when viewed as an automorphism of function algebras. More generally, an action on a finite von Neumann algebra $\Gamma \curvearrowright (P, \tau)$ is malleable if it admits an extension to an action on a larger finite von Neumann algebra, $\Gamma \curvearrowright (\tilde{P}, \tilde{\tau})$, such that the connected component of *id* in the centralizer of this action, $\mathrm{Aut}_\Gamma(\tilde{P}, \tilde{\tau})$, contains an automorphism $\alpha_1$ with $P_1 = \alpha_1(P)$ perpendicular to $P$ (with respect to $\tilde{\tau}$) and $\mathrm{sp} P P_1$ dense in $L^2(\tilde{P})$, in other words $L^2(\tilde{P}) = L^2(\mathrm{sp} P P_1) \simeq L^2(P) \overline{\otimes} L^2(P_1)$ (see 1.4, 1.5 in [P4]).

It is this condition that we will generically refer to as (basic) *malleability.* We mention that in all existing examples of malleable actions $\alpha_1$ can in fact be chosen to be the *flip* $(t, t') \mapsto (t', t)$.

A stronger form of malleability in [P1, P3, P4] requires that there actually is a continuous group-like "path" between the identity and $\alpha_1$, i.e. a continuous action $\alpha$ of the reals on $(X \times X, \mu \times \mu)$, commuting with $\Gamma \curvearrowright X \times X$ (resp.

$\alpha(\mathbb{R}) \subset \mathrm{Aut}_\Gamma(\tilde{P}, \tilde{\tau})$) and such that $\alpha_1(L^\infty X \otimes 1) = 1 \otimes L^\infty X$ (resp. $\alpha_1(P) = P_1 \perp P$ as before). We call such a path a *malleable deformation* (or *path*) of $\Gamma \curvearrowright X$. An action having such a deformation was still called *malleable* in [P1]–[P4], but to distinguish it from the above weaker form here we call it *path-malleable*.

The strongest condition of this kind considered in [P1]–[P4] requires the $\pm$ directions on the path $\alpha$ to be symmetric with respect to the first coordinate of the double space $X \times X$, a rather natural "geometric" property. In rigorous terms, this means the existence of a period-2 m.p. automorphism $\beta$ of $X \times X$ commuting with the double $\Gamma$-action (resp. $\beta \in \mathrm{Aut}_\Gamma(\tilde{P}, \tilde{\tau})$), which acts as the identity on the first variable (so $\beta(a \otimes 1) = a \otimes 1$, $\forall a \in L^\infty X$; in general $P \subset \tilde{P}^\beta$) and "reverses the direction" of the path $\alpha$, i.e. $\beta \alpha_t \beta = \alpha_{-t}$, $\forall t$. Note that $(\alpha, \beta)$ generate a copy of the group of isometries of $\mathbb{R}$, $\mathrm{Isom}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}$, in the centralizer of the double $\Gamma$-action.

This is called *s-malleability* in [P1, P3, P4], and is a useful strengthening of basic malleability in a "non-commutative environment", e.g. when the probability space is non-commutative (i.e. $\Gamma$ acts on a finite von Neumann algebra with a trace $(P, \tau)$), as in [P3], or when malleability is being used to get information on the von Neumann algebra $L^\infty X \rtimes \Gamma$ and its subalgebras, as in [P4]. Such $(\alpha, \beta)$ plays the role of a "device for patching incremental intertwiners", along the path $\alpha$. We call the pair $(\alpha, \beta)$ a *s-malleable deformation* (or *path*). One should mention that all known examples of malleable deformations of actions (generalized Bernoulli actions [P1]–[P4], Bogoliubov actions [P3] and Gaussian actions [Fu2]) have a natural symmetry $\beta$ and are thus s-malleable.

Let us note that symmetric deformations automatically satisfy a natural "transversality" condition:

**2.1. Lemma.** *Let $\Gamma \curvearrowright (P, \tau)$ be an s-malleable action and $(\alpha, \beta)$ the corresponding s-malleable deformation. Then given any finite von Neumann algebra $(N, \tau)$ the action $\alpha' = \alpha \otimes 1$ of $\mathbb{R}$ on $L^2\tilde{P}\overline{\otimes}L^2N$ satisfies*

$$\|\alpha'_{2s}(x) - x\|_2 \leq 2\|\alpha'_s(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2. \tag{2.1}$$

*Proof.* Let $\beta'$ denote the period 2 automorphism of $L^2\tilde{P}\overline{\otimes}L^2N$ given by $\beta' = \beta \otimes 1$. Since $\beta(x) = x$ for $x \in P$ we have $\beta'(x) = x$ for $x \in L^2P\overline{\otimes}L^2N$. In particular $\beta'(E_{P\overline{\otimes}N}(\alpha'_s(x))) = E_{P\overline{\otimes}N}(\alpha'_s(x))$. Also, $\beta'\alpha'_t\beta' = \alpha'_{-t}$. Thus

$$\begin{aligned}\|\alpha'_s(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2 &= \|\beta'(\alpha'_s(x) - E_{P\overline{\otimes}N}(\alpha'_s(x)))\|_2 \\ &= \|\alpha'_{-s}(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2,\end{aligned}$$

implying that

$$\begin{aligned}\|\alpha'_{2s}(x) - x\|_2 &= \|\alpha'_s(x) - \alpha'_{-s}(x)\|_2 \\ &\leq \|\alpha'_s(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2 + \|\alpha'_{-s}(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2 \\ &= 2\|\alpha'_s(x) - E_{P\overline{\otimes}N}(\alpha'_s(x))\|_2.\end{aligned}$$

□

## 3. Stable spectral gap

**3.1. Definition.** 1°. A unitary representation $\Gamma \curvearrowright \mathcal{H}$ has *spectral gap* (resp. *stable spectral gap*) if $1_\Gamma \not\prec \mathcal{H}$ (resp. $1_\Gamma \not\prec \mathcal{H}\overline{\otimes}\mathcal{H}^*$). An orthogonal representation has spectral gap (resp. stable spectral gap) if its complexification has the property.

2°. An m.p. action $\Gamma \curvearrowright X$ on a probability space $(X, \mu)$ has *spectral gap* (resp. *stable spectral gap*) if the associated representation $\Gamma \curvearrowright L^2X \ominus \mathbb{C}$ has spectral gap (resp. stable spectral gap). More generally, if $(P, \tau)$ is a finite von Neumann algebra, an action $\Gamma \curvearrowright (P, \tau)$ has *spectral gap* (resp. *stable spectral gap*) if the representation $\Gamma \curvearrowright L^2P \ominus \mathbb{C}$ has spectral gap (resp. stable spectral gap).

**Lemma 3.2.** *A representation $\Gamma \curvearrowright^{\pi} \mathcal{H}$ has stable spectral gap if and only if given any representation $\Gamma \curvearrowright^{\rho} \mathcal{K}$, the product representation $(\pi \otimes \rho)(g) = \pi(g) \otimes \rho(g)$ of $\Gamma$ on $\mathcal{H}\overline{\otimes}\mathcal{K}$ has spectral gap.*

*Proof.* Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, we identify the tensor product Hilbert space $\mathcal{K}\overline{\otimes}\mathcal{H}^*$ with the Hilbert space $HS(\mathcal{H}, \mathcal{K})$ of linear bounded operators $T : \mathcal{H} \to \mathcal{K}$ of finite Hilbert-Schmidt norm $\|T\|_{HS} = Tr_{\mathcal{H}}(T^*T)^{1/2} = Tr_{\mathcal{K}}(TT^*)^{1/2} < \infty$, via the map $\xi \otimes \eta^* \mapsto T_{\xi\otimes\eta^*}$, with $T_{\xi\otimes\eta^*}(\zeta) = \langle \zeta, \eta\rangle\xi$, $\zeta \in \mathcal{H}$. In particular, we identify $\mathcal{H}\overline{\otimes}\mathcal{H}^*$ with the Hilbert space $HS(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$.

Note that by the Powers-Størmer inequality ([PoSt]), if for $T, S \in HS(\mathcal{H}, \mathcal{K})$ we denote

$$|T| = (T^*T)^{1/2}, |S| = (S^*S)^{1/2} \in HS(\mathcal{H}),$$

then we have the estimate $\||T| - |S|\|_{HS}^2 \le \|T - S\|_{HS}\|T + S\|_{HS}$.

Now take representations $\Gamma \curvearrowright^{\pi} \mathcal{H}$, $\Gamma \curvearrowright^{\rho} \mathcal{K}$ and still denote by $\pi$ the representation of $\Gamma$ on $\mathcal{H}^*$ given by $\pi(\eta^*) = \pi(\eta)^*$ and by $\rho \otimes \pi$ the representation of $\Gamma$ on $HS(\mathcal{H}, \mathcal{K})$ resulting from the identification of this Hilbert space with $\mathcal{K}\overline{\otimes}\mathcal{H}^*$. Notice that if $T \in HS(\mathcal{H}, \mathcal{K})$, then $(\rho_g \otimes \pi_g(T))^*(\rho_g \otimes \pi_g(T)) = \pi_g(T^*T)$. From the above Powers-Størmer inequality we thus get

$$\begin{aligned} &\|\pi_g(|T|) - |T|\|_{HS}^2 \\ &\le \|T + \pi_g \otimes \rho_g(T)\|_{HS}\|T - \pi_g \otimes \rho_g(T)\|_{HS} \\ &\le 2\|T\|_{HS}\|T - \pi_g \otimes \rho_g(T)\|_{HS}, \end{aligned}$$

showing that if $T$ is almost invariant to $\rho_g \otimes \pi_g$, for $g$ in a finite subset $F \subset \Gamma$, then $|T|$ is almost invariant to $\pi_g, g \in F$. In other words, if $1_\Gamma \prec \mathcal{K}\overline{\otimes}\mathcal{H}^*$, then $1_\Gamma \prec \mathcal{H}\overline{\otimes}\mathcal{H}^*$. ☐

**3.3. Lemma.** 1°. *An orthogonal representation $\Gamma \curvearrowright^{\pi} \mathcal{H}_\infty$ on the infinite dimensional real Hilbert space $\mathcal{H}_\infty$ has stable spectral gap iff the associated Gaussian (resp. Bogoliubov) action $\sigma_\pi$ has stable spectral gap.*

2°. *If $\Gamma$ is non-amenable and $\Gamma \curvearrowright I$ is so that $\{g \in \Gamma \mid gi = i\}$ is amenable $\forall i \in I$, then the generalized Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^I$ has stable spectral gap.*

*Proof.* 1°. For Gaussians, this is clear from the fact that, as a representation on $L^2(\mathcal{H}_\infty, \mu_\infty)$, $\sigma_\pi$ is equivalent to the representation $\bigoplus_{n\ge0} \pi_{\mathbb{C}}^{\odot_s n}$, where $\pi_{\mathbb{C}}$ is the complexification of $\pi$ and for a representation $\rho$ on a (complex) Hilbert space $\mathcal{K}$, $\rho^{\odot_s n}$ denotes its $n$'th symmetric tensor power (see e.g. [CCJJV]). Similarly for Bogoliubov actions.

2°. This is Lemma 1.6.4 in [P2]. ☐

We mention that in the proof of Theorem 1.1 we will in fact need a weaker condition on an action $\Gamma \curvearrowright (P, \tau_P)$ than stable spectral gap, namely a "stable" version of the *strong ergodicity* in [Sc], which we recall requires any asymptotically $\Gamma$-invariant sequence $(x_n) \in (P)_1$ (i.e. $\lim_n \|gx_n - x_n\|_2 = 0$, $\forall g \in \Gamma$) to be asymptotically scalar (i.e. $\lim_n \|x_n - \tau(x_n)1\|_2 = 0$).

**3.4. Definition.** An action $\Gamma \curvearrowright (P, \tau_P)$ is *stably strongly ergodic* if given any action $\Gamma \curvearrowright (Q, \tau_Q)$ on a finite von Neumann algebra, any asymptotically $\Gamma$-invariant sequence of the product action $\Gamma \curvearrowright (P \overline{\otimes} Q, \tau_P \otimes \tau_Q)$ is (asymptotically) contained in $Q$. An m.p. action $\Gamma \curvearrowright (X, \mu)$ is *stably strongly ergodic* if the action it implements on $L^\infty X$ is stably strongly ergodic.

**3.5. Lemma.** *If $\Gamma \curvearrowright (P, \tau_P)$ has (stable) spectral gap, then it is (stably) strongly ergodic.*

*Proof.* This is trivial from the definitions. $\square$

## 4. Proof of cocycle superrigidity

We use in this section the framework and technical results from [P1]. Notations that are not specified, can be found in [P1] as well. We in fact prove a generalized version of Theorem 1.1, for actions of groups on arbitrary finite von Neumann algebras, which is the analogue of 5.5 in [P1]. Recall in this respect that if $\Gamma$ is a discrete group, $\mathcal{N}$ is a finite von Neumann algebra and $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{N})$ an action of $\Gamma$ on $\mathcal{N}$ (i.e. a group morphism of $\Gamma$ into the group of automorphisms $\mathrm{Aut}(\mathcal{N})$ of the von Neumann algebra $\mathcal{N}$), then a (left) *cocycle* for $\sigma$ is a map $w : \Gamma \to \mathcal{U}(\mathcal{N})$ satisfying $w_g \sigma_g(w_h) = w_{gh}$, $\forall g, h \in \Gamma$. Also, two such cocycles $w, w'$ are equivalent if there exists a unitary element $u \in \mathcal{U}(\mathcal{N})$ such that $u^* w_g \sigma_g(u) = w'_g$, $\forall g \in \Gamma$.

**4.1. Theorem** (Cocycle superrigidity: the non-commutative case)**.** *Let $\Gamma \curvearrowright^{\sigma_0} (P, \tau)$ be an action of $\Gamma$ on a finite von Neumann algebra. Let $H, H' \subset \Gamma$ be infinite commuting subgroups such that:*

- $(a)$ *$H \curvearrowright P$ has stable spectral gap.*
- $(b)$ *$H' \curvearrowright P$ is weak mixing.*
- $(c)$ *$HH' \curvearrowright P$ is s-malleable.*

*Let $(N, \tau)$ be an arbitrary finite von Neumann algebra and $\rho$ an action of $\Gamma$ on $(N, \tau)$. Then any cocycle $w$ for the diagonal product action $\sigma_0 \otimes \rho$ of $\Gamma$ on $P \overline{\otimes} N$ is equivalent to a cocycle $w'$ whose restriction to $HH'$ takes values in $N = 1 \otimes N$. If in addition $H'$ is $w$-normal in $\Gamma$, or if $H'$ is $wq$-normal but $\sigma$ is mixing, then $w'$ takes values in $N$ on all $\Gamma$.*

*Moreover, the same result holds true if $\sigma_0$ extends to an s-malleable action $\Gamma \curvearrowright^{\sigma'_0} (P', \tau)$ which satisfies $(a), (b), (c)$ and such that $\sigma'_0$ is a relative weak mixing quotient of $\sigma$, in the sense of* 2.9 *in* [P1].

*Proof of Theorem* 4.1. Denote $\tilde{P} = P \overline{\otimes} P$ and let $\sigma = \sigma_0 \otimes \rho$, $\tilde{\sigma} = \sigma_0 \otimes \sigma_0 \otimes \rho$ be the product actions of $\Gamma$ on $P \overline{\otimes} N$ and resp. $\tilde{P} \overline{\otimes} N$.

Denote $M = P \overline{\otimes} N \rtimes \Gamma$, $\tilde{M} = \tilde{P} \overline{\otimes} N \rtimes \Gamma$ and view $M$ as a subalgebra of $\tilde{M}$ by identifying $P \overline{\otimes} N$ with the subalgebra $(P \otimes 1) \overline{\otimes} N$ of $P \overline{\otimes} P \overline{\otimes} N = \tilde{P} \overline{\otimes} N$ and by identifying the canonical unitaries $\{u_g\}_g$ in $M$, $\tilde{M}$ implementing $\sigma$ on $P \overline{\otimes} N$ and $\tilde{\sigma}$ on $\tilde{P} \overline{\otimes} N$. From now on we denote by $\tau$ the canonical trace on the ambient algebra $\tilde{M}$ and on all its subalgebras.

Since the s-malleable deformation $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{P}, \tau)$ commutes with $\tilde{\sigma}$, it extends to an action of $\mathbb{R}$ on $\tilde{M}$, still denoted $\alpha$, equal to the identity on $N = 1 \otimes N$ and on $\{u_g\}_g$.

Note that if we denote $u'_g = w_g u_g$, then the cocyle relation for $w_g$ is equivalent to the relation $u'_{g_1} u'_{g_2} = u'_{g_1 g_2}, \forall g_1, g_2 \in \Gamma$ in $M \subset \tilde{M}$. Also, denote by $\sigma'$ the action of $\Gamma$ on $P\overline{\otimes}N$ given by $\sigma'_g(x) = \mathrm{Ad}(u'_g)(x) = w_g u_g x u_g^* w_g^* = w_g\sigma_g(x)w_g^*$. (N.B. If $P, N$ are commutative, then this is equal to $\sigma_g(x)$.)

Note then that if we view $L^2\tilde{M}$ as $L^2M\overline{\otimes}L^2P$ via the isomorphism $(x\otimes y)u_h \mapsto (xu_h)\otimes y$, $x \in P\overline{\otimes}N$, $y \in P$, $g \in \Gamma$, then the $\Gamma$-representation $\tilde{\pi}$ on $L^2\tilde{M}$ given by $\tilde{\pi}_g((x\otimes y)u_h) = u'_g((x\otimes y)u_h){u'_g}^*$ corresponds to the $\Gamma$-representation on $L^2M\overline{\otimes}L^2P$ given by $\mathrm{Ad}(u'_g)(xu_h)\otimes\sigma_g(y)$, $\forall x \in P\overline{\otimes}N, y \in P, g, h \in \Gamma$.

Thus, if $\sigma$ has stable spectral gap on $H$, then $\forall\delta > 0$, with $\delta < 2^{-5}$, $\exists F \subset H$ finite and $\delta_0 > 0$ such that: if $u \in \mathcal{U}(\tilde{M})$ satisfies $\|\tilde{\pi}_h(u) - u\|_2 \le \delta_0, \forall h \in F$, then $\|u - E_M(u)\|_2 \le \delta$. Since $\alpha_s(u'_h)$ is continuous in $s$ for all $h \in F$, it follows that for sufficiently small $s > 0$ we have $\|\alpha_{-s/2}(u'_h) - u'_h\|_2 \le \delta_0/2$, $\forall h \in F$. Let $g$ be an arbitrary element in the group $H'$. Since the groups $H, H'$ commute, $u'_g$ commutes with $u'_h, \forall h \in H$, in particular $\forall h \in F$. Thus we get

$$\begin{aligned}\|[\alpha_{s/2}(u'_g), u'_h]\|_2 &= \|[u'_g, \alpha_{-s/2}(u'_h)]\| \\ &\le 2\|\alpha_{-s/2}(u'_h) - u'_h\|_2 \le \delta_0, \forall h \in F, g \in H'.\end{aligned}$$

Since $\|[\alpha_{s/2}(u'_g), u'_h]\|_2 = \|\tilde{\pi}_h(\alpha_{s/2}(u'_g)) - \alpha_{s/2}(u'_g)\|_2$, this implies that the unitaries $u = \alpha_{s/2}(u'_g) \in \tilde{M}, g \in H'$, satisfy the inequality $\|\tilde{\pi}_h(u) - u\|_2 \le \delta_0$. By the above conditions we thus get

$$\|\alpha_{s/2}(u'_g) - E_M(\alpha_{s/2}(u'_g))\|_2 \le \delta\forall g \in H',$$

which by (2.1) implies $\|\alpha_s(u'_g) - u'_g\|_2 \le 2\delta$, $\forall g \in H'$. Let $K = \overline{\mathrm{co}}^w\{u'_g\alpha_s(u'_g)^* \mid g \in H'\}$ and notice that $K$ is a convex weakly compact subset, it is contained in the unit ball of $\tilde{P}\overline{\otimes}N \subset \tilde{M}$ (because $u'_g\alpha_s(u'_g)^* = w_{h'}\alpha_s(w_g)^*$) and for all $\xi \in K$ and $g \in H'$ we have $u'_g\xi\alpha_s(u'_g)^* \in K$. Let $x \in K$ be the unique element of minimal norm $\|\cdot\|_2$. Since $\|u'_g x\alpha_s(u'_g)^*\|_2 = \|x\|_2, \forall g \in H'$, by the uniqueness of $x$ it follows that $u'_g x\alpha_s(u'_g)^* = x, \forall g \in H'$. Thus $x$ intertwines the representations $g \mapsto u'_g$, $g \mapsto \alpha_s(u'_g)$. It follows that the partial isometry $v \in \tilde{P}\overline{\otimes}N$ in the polar decomposition of $x$ is non-zero and still intertwines the representations, i.e. $u'_g v = v\alpha_s(u'_g)$, or equivalently

$$w_g\tilde{\sigma}_g(v) = v\alpha_s(w_g), \forall g \in H'. \tag{4.1}$$

Moreover, since $\|u'_g\alpha_s(u'_g)^* - 1\|_2 = \|u'_g - \alpha_s(u'_g)\|_2 \le 2\delta$ we have $\|\xi - 1\|_2 \le 2\delta, \forall\xi \in K$, thus $\|x - 1\|_2 \le 2\delta$, which by [C1] gives $\|v - 1\|_2 \le 4(2\delta)^{1/2}$.

By using the symmetry $\beta$, viewed as an automorphism of $\tilde{P}\overline{\otimes}N$ (acting as the identity on $N$), the same argument as in the proof of Lemma 4.6 in [P1] shows that starting from (4.1) applied to $s = 2^{-n}$, for some large integer $n$, one can obtain a partial isometry $v_1 \in \tilde{P}\overline{\otimes}N$ such that $w_g\tilde{\sigma}_g(v_1) = v_1\alpha_1(w_g), \forall g \in H'$, and $\|v_1\|_2 = \|v\|_2$. Here we repeat the argument, for completeness.

It is clearly sufficient to show that whenever we have (4.1) for some $s = 2^{-n}$ and a partial isometry $v \in \tilde{P}\overline{\otimes}N$, then there exists a partial isometry $v' \in \tilde{P}\overline{\otimes}N$ satisfying $\|v'\|_2 = \|v\|_2$ and $w_g\tilde{\sigma}_g(v') = v'\alpha_{2s}(w_g)$, $\forall g \in H'$. Indeed, because then the statement follows by repeating the argument $n$ times.

Applying $\beta$ to (4.1) and using the fact that $\beta$ commutes with $\tilde{\sigma}$, $\beta(x) = x, \forall x \in P\overline{\otimes}N \subset \tilde{P}\overline{\otimes}N$ and $\beta\alpha_s = \alpha_{-s}\beta$ as automorphisms on $\tilde{P}\overline{\otimes}N$, we get $\beta(w_h) = w_h$ and

$$w_g\tilde{\sigma}_g(\beta(v)) = \beta(v)\alpha_{-s}(w_g), \forall g \in H'. \tag{4.2}$$

Since (3.1) can be read as $v^*w_g = \alpha_t(w_g)\tilde{\sigma}_g(v^*)$, from (4.1) and (4.2) we get the identity

$$v^*\beta(v)\alpha_{-s}(w_g) = v^*w_g\tilde{\sigma}_g(\beta(v))$$
$$= \alpha_s(w_g)\tilde{\sigma}_g(v^*)\tilde{\sigma}_g(\beta(v)) = \alpha_s(w_g)\tilde{\sigma}_g(v^*\beta(v)),$$

for all $g \in H'$. By applying $\alpha_s$ on both sides of this equality, if we denote $v' = \alpha_s(\beta(v)^*v)$, then we further get

$$v'^*w_g = \alpha_{2s}(w_g)\tilde{\sigma}_g(v'^*), \forall g \in H',$$

showing that $w_g\tilde{\sigma}_g(v') = v'\alpha_{2s}(w_g)$, $\forall g \in H'$, as desired. On the other hand, the intertwining relation (4.1) implies that $vv^*$ is in the fixed point algebra $B$ of the action $\mathrm{Ad}w_h \circ \tilde{\sigma}_g = \mathrm{Ad}(u'_g)$ of $H'$ on $\tilde{P}\overline{\otimes}N$. Since $\tilde{\sigma}_{|H'}$ is weak mixing on $(1 \otimes P) \otimes 1 \subset \tilde{P}\overline{\otimes}N$ (because it coincides with $\sigma_0$ on $1_P \otimes P \otimes 1_N \simeq P$) and because $\mathrm{Ad}w_h$ acts as the identity on $(1 \otimes P) \otimes 1$ and leaves $(P \otimes 1)\overline{\otimes}N$ globally invariant, it follows that $B$ is contained in $(P \otimes 1)\overline{\otimes}N$. Thus $\beta$ acts as the identity on it (because it acts as the identity on both $P \otimes 1$ and $1 \otimes N$). In particular $\beta(vv^*) = vv^*$, showing that the right support of $\beta(v^*)$ equals the left support of $v$. Thus, $\beta(v^*)v$ is a partial isometry having the same right support as $v$, implying that $v'$ is a partial isometry with $\|v'\|_2 = \|v\|_2$.

Altogether, this argument shows that $\forall \varepsilon_0 > 0$, $\exists v_1 \in \tilde{P}\overline{\otimes}N$ partial isometry satisfying $w_g\tilde{\sigma}_g(v_1) = v_1\alpha_1(w_g), \forall g \in H'$, and $\|v_1\|_2 \geq 1 - \varepsilon_0/2$. Extending $v_1$ to a unitary $u_1$ in $\tilde{P}\overline{\otimes}N$ it follows that $\|w_g\tilde{\sigma}_g(u_1) - u_1\alpha_1(w_g)\|_2 \leq \varepsilon_0$, $\forall g \in H'$. By 2.12.2° in [P1] it follows that the cocycles $w_g$ and $\alpha_1(w_g), g \in H'$, are equivalent. Since $H' \curvearrowright X$ is assumed weak mixing, we can apply Theorem 3.2 in [P1] to deduce that there exists $u \in \mathcal{U}(P\overline{\otimes}N)$ such that $w'_g = u^*w_g\sigma_g(u)$, $g \in H'$, takes values into $\mathcal{U}(N)$. By the weak mixing of $H' \curvearrowright X$ and Lemma 3.6 in [P1], $w'_g$ takes values into $\mathcal{U}(N)$ for any $g$ in the w-normalizer of $H'$, in particular on all $HH'$. The part of the statement concerning wq-normalizer follows by again applying Lemma 3.6 in [P1], while the part concerning actions $\sigma_0$ that extend to s-malleable actions $\Gamma \curvearrowright^{\sigma'_0} P'$ such that that $\sigma'_0$ is weak mixing relative to $\sigma_0$ follows from Lemma 2.11 in [P1].

This ends the proof of Theorem 4.1. $\square$

*Proof of Theorem* 1.1. Let $(P, \tau) = (L^\infty X, \int \mathrm{d}\mu)$, $(N, \tau_N)$ be a finite von Neumann algebra such that $\mathcal{V}$ is a closed subgroup of $\mathcal{U}(N)$ and $\rho = id$. If $w : X \times \Gamma \to \mathcal{V} \subset \mathcal{U}(N)$ is a measurable (right) cocycle for $\Gamma \curvearrowright X$, as defined for instance in 2.1 of [P1], then we view it as an algebra (left) cocycle $w : \Gamma \to \mathcal{V}^X \subset \mathcal{U}(L^\infty(X, N)) = \mathcal{U}(L^\infty X\overline{\otimes}N) = \mathcal{U}(P\overline{\otimes}N)$ for the action $\Gamma \curvearrowright^\sigma P\overline{\otimes}N$. The result follows then from 4.1 and 3.5 in [P1]. $\square$

*Proof of Corollary* 1.2. This is a trivial consequence of Theorem 1.1 and Lemma 3.3.

*Proof of Theorem* 1.3. This is an immediate consequence of Theorem 1.1, Corollary 1.2 and 5.7-5.9 in [P1].

*Proof of Corollary* 1.4. This follows now readily from Corollary 1.2 and 2.7 in [P2]. $\square$

We end this section by mentioning a non-commutative analogue of Corollary 1.2 which follows from Theorem 4.1:

**4.2. Corollary.** *Let $\Gamma$ be a countable group having infinite commuting subgroups $H, H'$ with $H$ non-amenable. Let $\Gamma \curvearrowright^{\rho} (N, \tau)$ be an arbitrary action. Let $\Gamma \curvearrowright^{\sigma_0} (P, \tau)$ be an action whose restriction to $HH'$ can be extended to an action $HH' \curvearrowright^{\sigma_0'} (P', \tau)$ which is weak mixing relative to $\sigma_{0|HH'}$ and is one of the following:*

$1^\circ$. *A generalized non-commutative Bernoulli action $HH' \curvearrowright (B_0, \tau_0)^I$, with base $(B_0, \tau_0)$ a finite amenable von Neumann algebra and with the actions of $H, H'$ on the countable set $I$ satisfying $|H'i| = \infty$ and $\{g \in H \mid gi = i\}$ amenable, $\forall i \in I$.*

$2^\circ$. *A Bogoliubov action associated to a unitary representation of $HH'$ which has stable spectral gap on $H$ and no finite dimensional $H'$-invariant subspaces.*

*If either $H'$ is w-normal in $\Gamma$, or if $H'$ is merely wq-normal in $\Gamma$ but $\sigma_0$ is mixing, then any cocycle for $\sigma_0 \otimes \rho$ is equivalent to a cocycle with values in $N$.*

## 5. Proof of vNE strong rigidity

To prove Theorem 1.5 we need two technical results about Bernoulli actions, which are of independent interest. In fact, these results hold true for Bernoulli actions with arbitrary finite von Neumann algebras as base, the proof being exactly the same as in the commutative case.

Thus, we will denote by $(B_0, \tau_0)$ an amenable finite von Neumann algebra and by $(B, \tau)$ a von Neumann subalgebra of $(B_0, \tau_0)^\Lambda$ which is invariant to the Bernoulli action $\Lambda \curvearrowright (B_0, \tau_0)^\Lambda$. Let $\tilde{B} = B \overline{\otimes} B$, $M = B \rtimes \Lambda$, $\tilde{M} = \tilde{B} \rtimes \Lambda$, where $\Lambda \curvearrowright \tilde{B}$ is the double of the action $\Lambda \curvearrowright B$. We view $M$ as a subalgebra of $\tilde{M}$ in the obvious way, by identifying $B = B \otimes 1 \subset \tilde{B}$ and by viewing the canonical unitaries $\{v_h \mid h \in \Lambda\} \subset \tilde{M}$ as also implementing $\Lambda \curvearrowright B \otimes 1 = B$.

**5.1. Lemma.** *If $Q \subset M$ is a von Neumann subalgebra with no amenable direct summand, then $Q' \cap \tilde{M}^\omega \subset M^\omega$. Equivalently, $\forall \delta > 0$, $\exists F \subset \mathcal{U}(Q)$ finite and $\delta_0 > 0$ such that if $x \in (\tilde{M})_1$ satisfies $\|[u, x]\|_2 \leq \delta_0$, $\forall u \in F$, then $\|E_M(x) - x\|_2 \leq \delta$.*

*Proof.* By commuting squares of algebras, it is clearly sufficient to prove the case $\Lambda \curvearrowright (B, \tau) = (B_0, \tau_0)^\Lambda$. Let $\zeta_0 = 1, \zeta_1, ...$ be an orthonormal basis of $L^2 B_0$. Denote by $I$ the set of multi-indices $(n_g)_g$ with $n_g \geq 0$, all but finitely many equal to 0. Note that $\Lambda$ acts on $I$ by left translation. For each $i = (n_g)_g$ let $\eta_i = (\zeta_{n_g})_g$. Then $\{\eta_i\}_i$ is an orthonormal basis of $L^2 B$, and $\Lambda \curvearrowright B$ implements a representation $\Lambda \curvearrowright L^2 B$ which on $\xi_i$ is given by $g\xi_i = \xi_{gi}$.

For each $i \in I_0 = I \setminus \{0\}$, let $K_i = \{g \in \Lambda \mid gi = i\}$ be the stabilizer of $i$ in $\Lambda$ and note that $\Lambda i$ with the left translation by $\Lambda$ on it, is the same as $\Lambda / K_i$. Denote $P_i = B \rtimes K_i \subset M$ and note that since $K_i$ is a finite group $P_i$ is amenable. Let us show that $L^2(\text{sp}M(1 \otimes \xi_i)M, \tau) \simeq L^2(\langle M, e_{P_i} \rangle, Tr)$, as Hilbert $M$-bimodules. To this end, we show that $x(1 \otimes \xi_i)y \mapsto x e_{P_i} y$, $x, y \in M$, extends to a well defined isomorphism between the two given Hilbert spaces. It is in fact sufficient to show that

$$\langle x'(1 \otimes \xi_i)y', x(1 \otimes \xi_i)y \rangle_\tau = \langle x' e_{P_i} y', x e_{P_i} y \rangle_{Tr},$$

or equivalently

$$(a) \qquad Tr(y^* e_{P_i} x^* x' e_{P_i} y') = \tau(y^* (1 \otimes \xi_i^*) x^* x' (1 \otimes \xi_i) y'),$$

for all $x, x', y, y' \in M$. Proving this identity for $x = v_g a$, $x' = v_{g'} a'$, $y = v_h$, $y' = v_{h'}$, with $a, a' \in B = B \otimes 1$ and $g, g', h, h' \in \Lambda$ is clearly enough. The left side of $(a)$ is equal to

$$(b) \qquad \delta_{g^{-1}g',K_i}\tau(v_h^* a^* v_{g^{-1}g'} a' v_{h'}),$$

where $\delta_{g^{-1}g',K_i}$ equals 0 if $g^{-1}g' \notin K_i$ and equals 1 if $g^{-1}g' \in K_i$. On the other hand, the right side of $(a)$ equals

$$(c) \qquad \tau(v_h^* a^* v_g^* (1 \otimes \xi_{-gi})(1 \otimes \xi_{g'i}) v_{g'} a' v_{h'}) = \delta_{gi,g'i}\tau(v_h^* a^* v_{g^{-1}g'} a' v_{h'}).$$

Since $gi = g'i$ if and only if $g^{-1}g' \in K_i$, it follows that $(b) = (c)$, showing that $(a)$ is indeed an identity.

We have thus shown that $L^2\tilde{M} \ominus L^2M \simeq \bigoplus_{i \in I_0} L^2(\langle M, e_{P_i}\rangle, Tr)$. But since $P_i$ are amenable, we have a weak containment of Hilbert $M$-bimodules $L^2(\langle M, e_{P_i}\rangle, Tr) \prec L^2M\overline{\otimes}L^2M$, $\forall i \in I_0$. Thus, we also have such containment as Hilbert $Q$-bimodules.

On the other hand, if $Q' \cap \tilde{M}^\omega \not\subset M^\omega$, then there exists a bounded sequence $(x_n)_n \in \tilde{M}^\omega$ such that $E_M(x_n) = 0$, $\|x_n\|_2 = 1$, $\forall n$, and $\lim_n \|x_n y - y x_n\|_2 = 0$, $\forall y \in Q$. But this implies $L^2Q \prec L^2\tilde{M} \ominus L^2M$ as $Q$-bimodules. From the above, this implies $L^2Q \prec \bigoplus_{i \in I_0} L^2(\langle M, e_{P_i}\rangle, Tr) \prec (L^2M\overline{\otimes}L^2M)^{I_0}$ as Hilbert $Q$-bimodules, which in turn shows that $Q$ has a non-trivial amenable direct summand by Connes' Theorem (see the proof of Lemma 2 in [P7]). $\square$

In the next lemma, the *w-normalizer* of a von Neumann subalgebra $P_0 \subset M$ is the smallest von Neumann subalgebra $P \subset M$ that contains $P_0$ and has the property: if $uPu^* \cap P$ is diffuse for some $u \in \mathcal{U}(M)$, then $u \in P$.

**5.2. Lemma.** *Assume that $\Lambda \curvearrowright (B_0, \tau_0)^\Lambda$ is weak mixing relative to $\Lambda \curvearrowright B$. Let $Q \subset pMp$ be a von Neumann subalgebra with no amenable direct summand and with commutant $Q_0 = Q' \cap pMp$ having no corner embeddable into $B$ inside $M$ (e.g. if $B$ is abelian, one can require $Q_0$ to be $\mathrm{II}_1$; in general one can require $Q_0$ to have no amenable direct summand). Then there exists a non-zero partial isometry $v_0 \in M$ such that $v_0^*v_0 \in Q_0' \cap pMp$ and $v_0Q_0v_0^* \subset L\Lambda$. Moreover, if $\Lambda$ is ICC, then there exists a unitary element $u \in \mathcal{U}(M)$ such that $uQ_0u^* \subset L\Lambda$ and if $P$ denotes the w-normalizer algebra of $Q \vee Q_0$ in $pMp$, then $uPu^* \subset L\Lambda$.*

*Proof.* It is clearly sufficient to prove the statement in case $p = 1$ (by taking appropriate amplifications of $Q \subset pMp$). We may also clearly assume $\Lambda \curvearrowright (B, \tau) = (B_0, \tau_0)^\Lambda$, by the relative weak mixing condition (cf. [P4]). Moreover, we may assume the Bernoulli action $\Lambda \curvearrowright B$ is s-malleable, i.e. $B_0 = L^\infty\mathbb{T}$ in the abelian case and $B_0 = R$ in general. Indeed, because any other abelian (resp. amenable) algebra $B_0$ can be embedded into $L^\infty\mathbb{T}$ (resp. $R$) and $\Lambda \curvearrowright (L^\infty\mathbb{T})^\Lambda = L^\infty(\mathbb{T}^\Lambda)$ (resp. $\Lambda \curvearrowright R^\Lambda$) is weak mixing relative to $\Lambda \curvearrowright B_0^\Lambda$.

Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\tilde{B})$, $\beta \in \mathrm{Aut}(\tilde{B})$, $\beta^2 = id$, give the s-malleable path for the Bernoulli action $\Lambda \curvearrowright B$. Since $\alpha, \beta$ commute with the double action $\Lambda \curvearrowright \tilde{B}$, it follows that $\alpha$ (resp. $\beta$) extends to an action, that we still denote by $\alpha$ (resp. $\beta$), of $\mathbb{R}$ (resp. $\mathbb{Z}/2\mathbb{Z}$) on $\tilde{M}$.

We first prove that there exists a non-zero partial isometry $w \in \tilde{M}$ such that $w^*w \in Q_0' \cap M$, $ww^* \in \alpha_1(Q_0' \cap M)$, $wy = \alpha_1(y)w, \forall y \in Q_0$.

Fix $\varepsilon > 0$. By Lemma 5.1, there exists a finite set $F \subset \mathcal{U}(Q)$ and $\delta_0 > 0$ such that if $x \in (\tilde{M})_1$ satisfies $\|[u, x]\|_2 \leq \delta_0, \forall u \in F$, then $\|E_M(x) - x\|_2 \leq \varepsilon/2$.

Since $\alpha_s(Q)$ commutes with $\alpha_s(Q_0)$ and $\alpha_s(u)$ is a continuous path, $\forall u \in F$, it follows that there exists $n$ such that $s = 2^{-n}$ satisfies

$$\|[u, \alpha_{s/2}(x)]\|_2 = \|[\alpha_{-s/2}(u), x]\|_2$$
$$\leq 2\|\alpha_{-s/2}(u) - u\|_2 \leq \delta_0, \forall x \in (Q_0)_1, \forall u \in F.$$

Thus $\|E_M(\alpha_{s/2}(x)) - \alpha_{s/2}(x)\|_2 \leq \varepsilon/2$, $\forall x \in (Q_0)_1$, in particular for all $x = v \in \mathcal{U}(Q_0)$. By (2.1) and the choice of $\delta$, it follows that $\|\alpha_s(v) - v\|_2 \leq \varepsilon$, $\forall v \in \mathcal{U}(Q_0)$. As in the proof of Theorem 1.1 in Section 4, this implies there exists a partial isometry $V \in \tilde{M}$ such that $Vv = \alpha_s(v)V$, $\forall v \in \mathcal{U}(Q_0)$ and $\|v - 1\|_2 \leq 4\varepsilon^{1/2}$. In particular, $V^*V \in Q_0' \cap \tilde{M}$, $VV^* \in \alpha_s(Q_0' \cap \tilde{M})$ and $V \neq 0$ if $\varepsilon < 1/16$. Since $Q_0$ has no corner that can be embedded into $B$ inside $M$, by Theorem 3.2 in [P4] we have $Q_0' \cap \tilde{M} = Q_0' \cap M$. But then exactly the same argument as in the proof of Theorem 1.1 in Section 4 gives a partial isometry $V_1 \in \tilde{M}$ such that $\|V_1\|_2 = \|V\|_2 \neq 0$ and $V_1 v = \alpha_1(v)V_1$, $\forall v \in \mathcal{U}(Q_0)$.

By Steps 4 and 5 on page 395 in [P4], it then follows that there exists a non-zero partial isometry $v_0 \in M$ such that $v_0^* v_0 \in Q_0' \cap M$ and $v_0 Q_0 v_0^* \subset L\Lambda$.

Assume now that $\Lambda$ is ICC, equivalently $L\Lambda$ is a factor. As in the proof of 4.4 in [P4], to show that we can actually get a unitary element $v_0$ satisfying $v_0 Q_0 v_0^* \subset L\Lambda$, we use a maximality argument. Thus, we consider the set $\mathcal{W}$ of all families $(\{p_i\}_i, u)$ where $\{p_i\}_i$ are partitions of 1 with projections in $Q_0' \cap M$, $u \in M$ is a partial isometry with $u^*u = \Sigma_i p_i$ and $u(\Sigma_i Q_0 p_i)u^* \subset L\Lambda$. We endow $\mathcal{W}$ with the order given by $(\{p_i\}_i, u) \leq (\{p_j'\}_j, u')$ if $\{p_i\}_i \subset \{p_j'\}_j$, $u = u'(\Sigma_i p_i)$. $(\mathcal{W}, \leq)$ is clearly inductively ordered.

Let $(\{p_i\}_i, u)$ be a maximal element. If $u$ is a unitary element, then we are done. If not, then denote $q' = 1 - \Sigma_i p_i \in Q_0' \cap M$ and take $q \in Q_0$ such that $\tau(qq') = 1/n$ for some integer $n \geq 1$. Denote $Q_1 = M_{n\times n}(qQ_0qq')$ regarded as a von Neumann subalgebra of $M$, with the same unit as $M$. Then the relative commutant of $Q_1$ in $M$ has no amenable direct summand, so by the first part there exists a non-zero partial isometry $w \in M$ such that $w^*w \in Q_1' \cap M$ and $wQ_1w^* \subset L\Lambda$. Since $qq' \in Q_1$ has scalar central trace in $Q_1$, it follows that there exists a non-zero projection in $w^*wQ_1w^*w$ majorized by $qq'$ in $Q_1$.

It follows that there exists a non-zero projection $q_0 \in qq'Q_1qq' = qQ_0qq'$ and a partial isometry $w_0 \in M$ such that $w_0^*w_0 = q_0$ and $w_0(qQ_0qq')w_0^* \subset L\Lambda$. Moreover, by using the fact that $Q_0$ is diffuse, we may shrink $q_0$ if necessary so that it is of the form $q_0 = q_1 q' \neq 0$ with $q_1 \in \mathcal{P}(Q_0)$ of central trace equal to $m^{-1}z$ for some $z \in \mathcal{Z}(Q_0)$ and $m$ an integer. But then $w_0$ trivially extends to a partial isometry $w_1 \in M$ with $w_1^*w_1 = q'z \in Q_0' \cap M$ and $w_1 Q_0 w_1^* \subset L\Lambda$. Moreover, since $L\Lambda$ is a factor, we can multiply $w_1$ from the left with a unitary element in $L\Lambda$ so that $w_1w_1^*$ is perpendicular to $uu^*$. But then $(\{p_i\}_i \cup \{q'z\}, u_1)$, where $u_1 = u + w_1$, is clearly in $\mathcal{W}$ and is (strictly) larger than the maximal element $(\{p_i\}_i, u)$, a contradiction.

We have thus shown that there exists a unitary element $u \in \mathcal{U}(M)$ such that $uQ_0u^* \subset L\Lambda$. But then by 3.1 in [P4] it follows that $uQu^* \subset L\Lambda$ as well, and in fact all the w-normalizer of $Q \vee Q_0$ is conjugated by $u$ into $L\Lambda$. Thus, $uPu^* \subset L\Lambda$. $\square$

*Proof of Theorem* 1.5. Let $H \subset \Gamma$ be a non-amenable group with centralizer $H' = \{g \in \Gamma \mid gh = hg, \forall h \in H\}$ non-virtually abelian and wq-normal in $\Gamma$. With the above notations, we can take $B_0 = L^\infty\mathbb{T}$. Let $Q = \theta(LH) \subset pMp$ and $Q_0 = \theta(LH)' \cap pMp$. By hypothesis, $Q$ has no amenable direct summand and $Q_0$ is type $\mathrm{II}_1$. Thus, by Lemma 5.2 it follows that there exists $u \in \mathcal{U}(M)$ such that

$uQ_0u^* \subset L\Lambda$. Moreover, since $\theta(L\Gamma)$ is contained in the w-normalizer algebra $P$ of $Q_0$, it follows that $u\theta(L\Gamma)u^* \subset L\Lambda$. From this point on, the results in [P5] apply to conclude the proof. $\square$

*Proof of Theorem* 1.6. We may assume $L^\infty X \rtimes \Gamma \subset L^\infty Y \rtimes \Lambda = M$, $L^\infty X = L^\infty Y = A$ and for each $g \in \Gamma$ there exists a partition of 1 with projections $\{p_h^g\}_{h\in\Lambda}$ such that $u_g = \Sigma_h p_h^g v_h$ give the canonical unitaries implementing $\Gamma \curvearrowright A$. Thus, $Q = LH$ has no amenable direct summand, $Q_0 = L(H')$ is type $\mathrm{II}_1$ and $L\Gamma$ is contained in the w-normalizer algebra of $Q_0$.

By Lemma 5.2 it follows that there exists a unitary element $u$ in $M$ such that $uL\Gamma u^* \subset L\Lambda$. Since $\Lambda \curvearrowright A$ is Bernoulli, by Lemma 4.5 in [P5] it follows that $\Gamma \curvearrowright A$ is mixing, thus Theorem 5.2 in [P5] applies to conclude that there exists $u \in \mathcal{U}(M)$ such that $u(\{u_h\}_h)u^* \subset \mathbb{T}\{v_h\}_h$ and $uAu^* = A$. $\square$

## 6. Final remarks

**6.1. vNE versus OE: the Connes-Jones example.** Formal definitions show that OE $\Rightarrow$ vNE, but are these notions of equivalence really different, and if they are, then how much different? In other words: If $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are free ergodic m.p. actions, does $L^\infty X \rtimes \Gamma \simeq L^\infty Y \rtimes \Lambda$ imply $\mathcal{R}_\Gamma \simeq \mathcal{R}_\Lambda$? If $\theta$ denotes the isomorphism between the $\mathrm{II}_1$ factors, this is same as asking whether there always exists $\rho \in \mathrm{Aut}(L^\infty Y \rtimes \Lambda)$ such that $\rho(\theta(L^\infty X)) = L^\infty Y$.

Two sets of results give a positive answer to this question for certain classes of group actions: On the one hand, if $\Gamma, \Lambda$ are amenable, then by [OW] there does exist an automorphism $\rho$ of $L^\infty Y \rtimes \Lambda \simeq R$ taking $\theta(L^\infty X)$ onto $L^\infty Y$; in fact by [CFW] any two *Cartan subalgebras* of $R$ are conjugated by an automorphism of $R$. On the other hand, all vNE rigidity results in [P1, P4, P5, P8], [IPeP], [PV] are about showing that for any isomorphisms $\theta$ between certain group measure space factors $L^\infty X \rtimes \Gamma, L^\infty Y \rtimes \Lambda$ (or even amplifications of such) $\exists u \in \mathcal{U}(L^\infty Y \rtimes \Lambda)$ such that $\mathrm{Ad}u \circ \theta(L^\infty X) = L^\infty Y$. This is unlike the amenable case though, where one can decompose $R$ in uncountably many ways, $R = L^\infty X_i \rtimes_{\sigma_i} \mathbb{Z}$, with $\mathbb{Z} \curvearrowright^{\sigma_i} X_i$ free ergodic actions (which can even be taken cojugate to the same given $\mathbb{Z}$-action), such that no inner automorphism of $R$ can take the subalgebras $L^\infty X_i \subset R$ onto each other, for different $i$'s ([FM]).

Nevertheless, the answer to "vNE $\Rightarrow$ OE?" is negative in general, as shown by Connes and Jones in [CJ1] through the following example: Let $\Gamma_0$ be any non-amenable group and $\Gamma_1 = \Sigma_n H_n$ an infinite direct sum of non-abelian groups. Let $H_n \curvearrowright [0,1]$ be any free $H_n$-action preserving the Lebesgue measure (e.g. a Bernoulli $H_n$-action) and let $\Gamma_1 = \Sigma_n H_n \curvearrowright [0,1]^{\mathbb{N}}$ be the product of these actions. Finally, denote $X = ([0,1]^{\mathbb{N}})^{\Gamma_0}$ and let $\Gamma_0$ act on $X$ by (left) Bernoulli shifts and $H_0$ act diagonally, identically on each copy of $[0,1]^{\mathbb{N}}$. Since the $\Gamma_0, \Gamma_1$ actions commute they implement an action of $\Gamma = \Gamma_0 \times \Gamma_1$ on $(X, \mu)$, which is easily seen to be free.

Since $\Gamma_0$ is non-amenable and $\Gamma_0 \curvearrowright X$ is Bernoulli, $\Gamma \curvearrowright X$ is strongly ergodic (it even has spectral gap), thus $\mathcal{R}_\Gamma$ is strongly ergodic as well. However, since any sequence of canonical unitaries $v_{h_n}$ with $h_n \in H_n$ is central for $M = L^\infty X \rtimes \Gamma$, by the non-commutativity of the $H_n$'s it follows that $M' \cap M^\omega$ is non-commutative, so by McDuff's theorem $M \simeq M \overline{\otimes} R$. Thus $M$ can also be decomposed as $M = (L^\infty X \rtimes \Gamma)\overline{\otimes}(L^\infty([0,1]) \rtimes H) = L^\infty(X \times [0,1]) \rtimes (\Gamma \times H)$, where $H \curvearrowright [0,1]$ is any free ergodic m.p. action of an amenable group $H$. Such $H \curvearrowright L^\infty([0,1])$ always has non-trivial approximately invariant sequences, i.e. it is not strongly ergodic.

Thus $\Gamma \times H \curvearrowright L^\infty(X \times [0,1])$ is not strongly ergodic either, so it cannot be OE to $\Gamma \curvearrowright X$ although both actions give the same $\mathrm{II}_1$ factor, i.e. are vNE. Thus vNE $\not\Rightarrow$ OE.

Note that by taking $H_n = H$ , $\forall n$, one gets the same group $\Gamma \simeq \Gamma \times H$ having two actions, one strongly ergodic the other not, both giving rise to the same $\mathrm{II}_1$ factor. Moreover, if $\Gamma_0$ is taken Kazhdan, or merely w-rigid, then the action $\Gamma_0 \curvearrowright X$ satisfies the hypothesis of 5.2/5.3 in [P1], so it is cocycle superrigid. Since it is weakly mixing, its extension to $\Gamma \curvearrowright X$ is also cocycle superrigid. Similarly, if $\Gamma_0$ is taken as a product between a non-amenable group and an infinite group, then $\Gamma \curvearrowright X$ follows cocycle superrigid by Theorem 1.1. In particular, in both cases $\Gamma \curvearrowright X$ is OE superrigid so, by 5.7 in [P1] and Corollary 1.3, $\mathscr{F}(\mathcal{R}_\Gamma)$ is countable. If in addition $\Gamma_0$ and $H_n$ have no finite normal subgroups, $\forall n$, then $\mathscr{F}(\mathcal{R}_\Gamma) = 1$.

In other words, there exists a free ergodic cocycle superrigid action $\Gamma \curvearrowright X$ which is strongly ergodic, satisfies $\mathscr{F}(\mathcal{R}_\Gamma) = 1$, but the associated $\mathrm{II}_1$ factor $M = L^\infty X \rtimes \Gamma$ can also be realized as $M = L^\infty Y \rtimes \Gamma'$ with $\Gamma' \curvearrowright Y$ a free ergodic but not a strongly ergodic action with $\mathcal{R}_{\Gamma'} \simeq \mathcal{R}_{\Gamma'} \times \mathcal{R}_{hyp}$, $M \simeq M \overline{\otimes} R$. In particular $\mathscr{F}(M) = \mathscr{F}(\mathcal{R}_{\Gamma'}) = \mathbb{R}_+^*$. Moreover, one can take $\Gamma \simeq \Gamma'$.

**6.2. On the transversality of mallebale actions.** Although all existing examples of malleable actions are in fact s-malleable, it would be interesting to give a proof of Theorem 1.1 that would only use (basic) malleability, even if this means sacrificing some of the generality on the side of the target groups. For instance, to prove Theorem 1.1 for cocycles of malleable actions with abelian, compact or discrete groups as targets. But it seems to us that any alternative argument would still need some sort of "transversality" property for an appropriate family $\{\alpha_s\}_s$ of automorphisms commuting with the double action $\Gamma \curvearrowright X \times X$ and relating $id$ to the flip, requiring that if $\alpha_s(x)$ close to $L^\infty X \otimes 1$ for some $x \in L^\infty X \otimes 1$, then $\alpha(x)$ is close to $x$. Besides s-malleability, another sufficient condition for this to happen is the following:

(6.2) There exists a Hilbert space $\mathcal{K}$ containing $L^2(X \times X, \mu \times \mu)$, an orthonormal system $\{\xi_n\}_n \subset \mathcal{K}$ satisfying $L^2 X \otimes 1 \subset \overline{\Sigma_n \mathbb{C}\xi_n}$, and an extension of $\alpha_s$ to a unitary element $\alpha'_s$ on $\mathcal{K}$, such that $\langle \xi_n, \alpha'_s(\xi_m)\rangle = \delta_{nm} c_n$, with $c_n \in \mathbb{R}$, $\forall n, m$.

Indeed, it is easy to see that if an automorphism $\alpha_s$ satisfies (6.2), then

$$\|\alpha_s^2(x) - x\|_2 \leq 2\sqrt{2}\|\alpha_s(x) - E_{L^\infty X}(\alpha_s(x))\|_2, \ \forall x \in L^\infty X \otimes 1.$$

In fact, in an initial version of this paper we used property (6.2) to derive the transversality (2.1), and proved that Bernoulli, Gaussian and Bogoliubov actions satisfy (2.1) by showing they satisfy (6.2). It was Stefaan Vaes and the referee who pointed out to us that in fact s-malleability trivially implies the transversality condition (2.1) (i.e. Lemma 2.1).

Nevertheless, condition (6.2) seems interesting in its own right. Related to it, note that if $\Gamma \curvearrowright X$ is so that $\mathrm{Aut}_\Gamma(X \times X)$ contains a finite group $K$ that has the flip in it and for which there exists an extension of $K \curvearrowright L^2 X \otimes L^2 X$ to a representation $K \curvearrowright \mathcal{K}$, with an orthonormal system $\{\xi_n\}_n \subset \mathcal{K}$ spanning $L^2 X \otimes 1$, such that the Hilbert spaces $\mathcal{K}_n = \mathrm{sp}\{k\xi_n \mid k \in K\}$ are mutually orthogonal and have dimensions majorized by some constant $c = c(|K|)$ with the property that $\forall n$, $\exists k \in K \setminus \{e\}$ with $\|k\xi_n - \xi_n\|_2 < 1$, then $\Gamma \curvearrowright X$ would automatically satisfy a cocycle superrigidity result, with no additional requirements on the group $\Gamma$, or on the way it acts on $X$.

**6.3. $\mathcal{CS}$ and $\mathcal{OES}$ groups.** Related to Remark 6.7 in [P1], we re-iterate here the following question: What is the class $\mathcal{CS}$ of groups $\Gamma$ for which the Bernoulli action $\Gamma \curvearrowright \mathbb{T}^\Gamma$ is $\mathcal{U}_{fin}$-cocycle superrigid? (N.B. Any relative weak mixing quotient of $\Gamma \curvearrowright \mathbb{T}^\Gamma$, for $\Gamma \in \mathcal{CS}$, is then automatically $\mathcal{U}_{fin}$-CSR as well, by results in [P1].) The class $\mathcal{CS}$ cannot contain free products with amalgamation $\Gamma = \Gamma_1 *_H \Gamma_2$, with $H$ a finite subgroup of $\Gamma_i$, $H \neq \Gamma_i$, $i = 1, 2$ (see e.g. [P2]). The class covered by Theorem 1.1 does not contain word hyperbolic groups. Hyperbolic groups with Haagerup property are not covered by 5.2/5.3 in [P1] either, because they cannot have infinite subgroups with the relative property (T).

The following question is equally interesting: What is the class of groups $\Gamma$ for which any OE between a Bernoulli $\Gamma$-action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ and an arbitrary Bernoulli action $\Lambda \curvearrowright (Y_0, \nu_0)^\Lambda$ comes from a conjugacy? It is very possible that this class consists of all non-amenable groups. It would be very interesting to decide this question for the free groups. A related question is to characterize the sub-class $\mathcal{OES}$ of groups $\Gamma$ for which the Bernoulli action $\Gamma \curvearrowright \mathbb{T}^\Gamma$ is OE Superrigid. $\mathcal{OES}$ doesn't contain any free product of infinite amenable groups, by [OW], [CFW].

**6.4. Examples of prime factors.** Lemma 5.2 allows deriving new examples of prime $\mathrm{II}_1$ factors, i.e. factors $M$ that cannot be decomposed as tensor products $M = Q\overline{\otimes}Q_0$ with $Q, Q_0$ $\mathrm{II}_1$ factors (see [O1], [O2], [Pe] for other examples of such factors):

**6.4.1. Theorem.** *Let $\Lambda$ be an arbitrary non-amenable group and $\Lambda \curvearrowright Y$ a free relative weak mixing quotient of a Bernoulli action. Then $L^\infty Y \rtimes \Lambda$ is prime. More generally, if $B \subset R^\Lambda$ is a von Neumann algebra invariant to the action $\Lambda \curvearrowright R^\Lambda$, such that $\Lambda \curvearrowright B$ is free and $\Lambda \curvearrowright R^\Lambda$ is weak mixing relative to $\Lambda \curvearrowright B$, then $B \rtimes \Lambda$ is prime. In particular $L^\infty \mathbb{T}^\Lambda \rtimes \Lambda$ and $R^\Lambda \rtimes \Lambda$ are prime.*

*Proof.* Denote $M = L^\infty Y \rtimes \Lambda$. Assume $M = Q\overline{\otimes}Q_0$. Since $M$ is non($\Gamma$) (see e.g. [I1]), it follows that both $Q, Q_0$ are non($\Gamma$), thus non-amenable. By the first part of Lemma 5.2, there exists a non-zero $p \in Q_0' \cap M = Q$ and a unitary element $u \in M$ such that $u(Q_0 p)u^* \subset L\Lambda$. By 3.1 in [P4] it follows that $up(Q \vee Q_0)pu^* \subset L\Lambda$. But the left hand side is equal to $p'Mp'$, where $p' = upu^*$. This means $p'L\Lambda p' = p'Mp'$, a contradiction. □

We mention that a more careful handling of the proof of Lemmas 5.1, 5.2 allows us to prove that factors $B \rtimes \Lambda$ associated to Bernoulli actions $\Lambda \curvearrowright (B, \tau) = (B_0, \tau_0)^\Lambda$, with an arbitrary finite von Neumann algebra $B_0 \neq \mathbb{C}$ as base, are prime for any non-amenable $\Lambda$ (see [I2] for related rigidity results on such factors).

Note that Lemmas 5.1, 5.2 show that if $\Lambda$ is an ICC group such that $M = L\Lambda$ has the property:

(6.4.1) *If $Q \subset M$ has type* $\mathrm{II}_1$ *relative commutant $Q' \cap M$, then $Q$ is amenable*,

then given any free, relative weak mixing quotient $\Lambda \curvearrowright Y$ of the Bernoulli action $\Lambda \curvearrowright \mathbb{T}^\Lambda$, the $\mathrm{II}_1$ factor $M = L^\infty Y \rtimes \Lambda$ has property (6.4.1) as well. Indeed, because if $Q \subset M$ has no amenable direct summand and $Q_0 = Q' \cap M$ is of type $\mathrm{II}_1$, then by the last part of Lemma 5.2 there exists a unitary element $u \in M$ such that $u(Q \vee Q_0)u^* \subset L\Lambda$, contradicting the property for $L\Lambda$. This result should be compared with a result in [O2], showing that if $\Lambda$ satisfies property AO and $H$ is an abelian group, then the wreath product $H \wr \Lambda$ has the property AO as well. By [O1] this implies $L(\Lambda \ltimes H) = L^\infty \hat{H} \rtimes \Lambda$ is solid, thus prime.

**6.5. On spectral gap rigidity.** The results of Theorem 1.1 through Theorem 1.6 add to the plethora of rigidity phenomena involving product groups that have been discovered in recent years in group theory, OE ergodic theory, Borel equivalence relations and von Neumann algebras/$\mathrm{II}_1$ factors ([MoSh], [HK], [OP], [Mo], [BSh], etc). It would of course be interesting to find some common ground (explanation) to these results. The idea behind our approach is very much in the spirit of $\mathrm{II}_1$ factor theory, but is otherwise rather elementary. It grew out from an observation in [P6], where for the first time spectral gap rigidity was used to prove a structural rigidity result for $\mathrm{II}_1$ factors. The starting point of all deformation/spectral gap rigidity arguments we have used in this paper and in [P6], [P7] is the following observation, which can be viewed as a general "spectral gap rigidity principle":

**6.5.1. Lemma.** *Let $\mathcal{U}$ be a group of unitaries in a $\mathrm{II}_1$ factor $\tilde{M}$ and $M, \tilde{P} \subset \tilde{M}$ von Neumann subalgebras such that $\mathcal{U}$ normalizes $\tilde{P}$ and the commutant of $\mathcal{U}$ in $\tilde{P}$, $Q_0 = \mathcal{U}' \cap \tilde{P}$, is contained in $M$. Assume:*

(6.5.1) *The action $\mathrm{Ad}\mathcal{U}$ on $\tilde{P}$ has spectral gap relative to $M$, i.e. for any $\varepsilon > 0$, there exist $F(\varepsilon) \subset \mathcal{U}$ finite and $\delta(\varepsilon) > 0$ such that if $x \in (\tilde{P})_1$, $\|uxu^* - x\|_2 \leq \delta(\epsilon)$, $\forall u \in F(\epsilon)$, then $\|E_M(x) - x\|_2 \leq \varepsilon$. (Note that this is equivalent to the condition $\mathcal{U}' \cap \tilde{P}^\omega \subset M^\omega$.)*

*Then any deformation of $id_{\tilde{M}}$ by automorphisms $\theta_n \in \mathrm{Aut}(\tilde{M})$ satisfies:*

$$\lim_n(\sup\{\|\theta_n(y) - E_M(\theta_n(y))\|_2 \mid y \in (Q_0)_1\}) = 0. \tag{6.5.2}$$

*In other words, the unit ball of $\theta_n(Q_0)$ tends to be contained into the unit ball of $M$, as $n \to \infty$.*

*Proof.* Fix $\varepsilon > 0$ and let $F(\varepsilon) \subset \mathcal{U}$, $\delta(\varepsilon) > 0$, as given by (6.5.1). Let $n$ be large enough so that $\|\theta_n(u) - u\|_2 \leq \delta/2$, $\forall u \in F$. If $x \in (\theta_n(Q_0))_1$, then $x$ commutes with $\theta_n(F)$ and thus $\|uxu^* - x\|_2 \leq 2\|u - \theta_n(u)\|_2 \leq \delta$. By (6.5.1), this implies $\|x - E_M(x)\|_2 \leq \varepsilon$. ☐

In the proof of Theorem 1.1, Lemma 6.5.1 is used for $\tilde{P} = L^\infty X \overline{\otimes} L^\infty X \overline{\otimes} N$, $\tilde{M} = \tilde{P} \rtimes \Gamma$, $M = (L^\infty X \otimes 1 \overline{\otimes} N) \rtimes \Gamma$ and $\mathcal{U} = \{u_h \mid h \in H\}$.

In the proof of Theorems 1.5 and 1.6, Lemma 6.5.1 is used for $\tilde{M} = \tilde{P} = L^\infty Y \overline{\otimes} L^\infty Y \rtimes \Lambda$, $M = L^\infty Y \rtimes \Lambda$ and $\mathcal{U} = \theta(\{u_h \mid h \in H\})$.

In the proof of Theorem 1 in [P7] it is used for $\tilde{M} = \tilde{P} = L\mathbb{F}_n * L\mathbb{F}_n$, $M = L\mathbb{F}_n * \mathbb{C}$, $\mathcal{U} = \mathcal{U}(Q)$.

In all these cases the deformation of $id_{\tilde{M}}$ is by automorphisms of a malleable path $\alpha_s, s \in \mathbb{R}$.

The initial result in [P7], where a "baby version" of spectral gap rigidity was used, states that if $\tilde{M} = Q \overline{\otimes} R$ is a McDuff $\mathrm{II}_1$ factor, with $Q$ non$(\Gamma)$, then any other tensor product decomposition $\tilde{M} = N \overline{\otimes} P$ with $N$ non$(\Gamma)$ and $P \simeq R$ is unitary conjugate to it, after re-scaling. In this case one applies Lemma 6.5.1 for $\tilde{P} = \tilde{M}$, $\mathcal{U} = \mathcal{U}(Q)$, $M = Q_0 = R$. The trick then is to take a deformation by inner automorphisms $\theta_n = \mathrm{Ad}(v_n)$ with $v_n \in \mathcal{U}(R_n)$ where $R_n \subset R$ is a decreasing sequence of subfactors splitting off the $2^n$ by $2^n$ matrices in $R$, i.e. $R = R_n \otimes M_{2^n \times 2^n}(\mathbb{C})$, and satisfying $\bigcap_n R_n = \mathbb{C}1$. By Lemma 6.5.1 one then gets $vQ_0v^* \approx Q_0$ (unit balls) uniformly in $v \in \mathcal{U}(R_n)$, for $n$ large, implying that $\mathcal{U}(R_n) \underset{\sim}{\subset} Q_0$, thus $R_n \underset{\sim}{\subset} Q_0$ (unit balls), so by [OP] there exists $u \in \mathcal{U}(\tilde{M})$ with the required properties. Note that there is an alternative way to carry out this argument, using the deformation by conditional expectations $E_{R_n' \cap \tilde{M}}$, as explained in §5 of [P7].

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