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# CORRECTION TO "FORMAL DEGREES AND ADJOINT $\gamma$-FACTORS"

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The authors would like to thank Professor Gross for pointing out an error in Lemma 3.3 of [4]. Conjecture 1.4 of [4] does not hold for $d(\pi) = d(\pi, \mu_{G/A,\psi})$ in the non-archimedean case. We need to modify the choice of the Haar measure $\mu_{G/A,\psi}$. Following Gross and Gan [3], we take another Haar measure $\mu'_{G/A,\psi}$ on $G/A$ defined as follows.

Let $F$ be a non-archimedean local field of characteristic zero and let $\psi$ be a non-trivial additive character of $F$. Let $\mathbf{G}$ be a connected reductive algebraic group over $F$ and let $\mathbf{A}$ be the split component of the center of $\mathbf{G}$. We may assume that $\mathbf{A} = \{1\}$. Let $\mathbf{G}_0$ be the split form of $\mathbf{G}$ and choose an isomorphism $\eta_0 : \mathbf{G} \to \mathbf{G}_0$ over $\bar{F}$, which may not be an inner twist. Let $\mathcal{G}_0$ be a Chevalley model of $\mathbf{G}_0$ over $\mathfrak{o}_F$ and choose a differential form $\omega_0$ of top degree on $\mathcal{G}_0$ over $\mathfrak{o}_F$ with non-zero reduction. Put $\omega = \eta_0^*(\omega_0)$. Let $\mu'_{G,\psi}$ denote the Haar measure on $G$ determined by $\omega$ and the self-dual measure on $F$ with respect to $\psi$. The Haar measure $\mu'_{G,\psi}$ does not depend on the choice of $\eta_0$ and $\omega_0$ (cf. [3, §5]).

Conjecture 1.4 of [4] should be modified as follows.

**Conjecture 1.4′.** *Let $\phi : L_F \to {}^L G$ be an elliptic tempered Langlands parameter. Then*

$$d(\pi, \mu'_{G/A,\psi}) = \frac{\langle 1, \pi \rangle}{|\mathcal{S}_\phi^\natural|} \cdot |\gamma(0, \pi, \mathrm{Ad}, \psi)|$$

*for $\pi \in \Pi_\phi(G)$.*

The relation between $\mu_{G/A,\psi}$ and $\mu'_{G/A,\psi}$ has been studied by Gross and Gan [3]. For simplicity, we assume that the connected center of $\mathbf{G}$ is anisotropic. Let $M$ be the motive of $\mathbf{G}$ defined in [1]. Recall that $M = \bigoplus_{d \geq 1} V_d(1-d)$, where $V_d$ is the Artin motive given in [1, §1]. Following [3, §4], we define the Artin conductor $a(M)$ of $M$ by

$$a(M) = \sum_{d \geq 1} (2d-1) a(V_d),$$

where $a(V_d)$ is the Artin conductor of $V_d$. We have

$$\mu'_{G,\psi} = q^{-a(M)/2} \cdot \mu_{G,\psi}$$

(cf. [3, §5]). In particular, $\mu'_{G,\psi} = \mu_{G,\psi}$ if the quasi-split inner form $\mathbf{G}^*$ of $\mathbf{G}$ is unramified.

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Let $\pi_0$ be the Steinberg representation of $G$. By [4, §3.3], we have

$$d(\pi_0, \mu'_{G,\psi}) = |\mathrm{H}^1(F, \mathbf{G})|^{-1} \cdot q^{a(M)/2} \cdot \frac{|L(M^\vee(1))|}{|L(M)|}.$$

Here $\psi$ is a non-trivial additive character of $F$ of order zero. We need to modify Lemma 3.3 of [4] as follows.

**Lemma 3.3′.**

$$|\gamma(0, \pi_0, \mathrm{Ad}, \psi)| = q^{a(M)/2} \cdot \frac{|L(M^\vee(1))|}{|L(M)|}.$$

*Proof.* Set $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$. Let $(\rho, N)$ be the representation of $W'_F$ on $\hat{\mathfrak{g}}$ associated to $\mathrm{Ad} \circ \phi_0$, where $\phi_0 : L_F \to {}^L G$ is the Langlands parameter associated to $\pi_0$. It remains to show that

$$|\epsilon(0, \pi_0, \mathrm{Ad}, \psi)| = q^{a(M)/2} \cdot \prod_{d \geq 1} q^{(d-1) \dim V_d^{I_F}}.$$

By definition, we have

$$|\epsilon(s, \pi_0, \mathrm{Ad}, \psi)| = q^{-a(\hat{\mathfrak{g}})(s-1/2)}.$$

We remark that the equation $a(\hat{\mathfrak{g}}) = \dim \hat{\mathfrak{g}}^{I_F} - \dim \hat{\mathfrak{g}}_N^{I_F}$ in the proof of Lemma 3.3 of [4] is incorrect and should be corrected as

$$a(\hat{\mathfrak{g}}) = a(\rho) + \dim \hat{\mathfrak{g}}^{I_F} - \dim \hat{\mathfrak{g}}_N^{I_F},$$

where $a(\rho)$ is the Artin conductor of $\rho$. By Proposition 5.2 of [2], we have $\hat{\mathfrak{g}} = \bigoplus_{d \geq 1} V_d \otimes \rho_{2d-2}$ as a representation of $\Gamma \times \mathrm{SL}(2, \mathbb{C})$. Hence we have

$$\begin{aligned} a(\hat{\mathfrak{g}}) &= \sum_{d \geq 1} (2d-1) a(V_d) + \sum_{d \geq 1} (2d-1) \dim V_d^{I_F} - \sum_{d \geq 1} \dim V_d^{I_F} \\ &= a(M) + 2 \sum_{d \geq 1} (d-1) \dim V_d^{I_F}. \end{aligned}$$

This completes the proof. □

Thus we obtain

$$d(\pi_0, \mu'_{G,\psi}) = \frac{1}{|\mathcal{S}_{\phi_0}^\natural|} \cdot |\gamma(0, \pi_0, \mathrm{Ad}, \psi)|.$$

In particular, Conjecture 1.4′ holds for $\pi_0$.

Theorem 8.6 of [4] does not hold for $d(\pi_H) = d(\pi_H, \mu_{H,\psi})$ unless $\mathbf{H}$ is unramified. Using Lemma 3.3′, one can show that Proposition 8.5 of [4] holds for $d(\pi_H, \mu'_{H,\psi})$. Thus we obtain the following theorem.

**Theorem 8.6′.** *Let $\mathbf{H} = \mathrm{U}(3)$ be the quasi-split unitary group in three variables. Let $\pi_H$ be a stable discrete series representation of $H$. Then*

$$d(\pi_H, \mu'_{H,\psi}) = \frac{1}{2} \cdot |\gamma(0, \pi_H, \mathrm{Ad}, \psi)|.$$

*In particular, Conjecture 1.4′ holds for $\pi_H$.*

## References

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