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THE CLASSIFICATION OF 2–COMPACT GROUPS

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1. Introduction

In this paper we prove that any connected 2–compact group X is classified, up to isomorphism, by its root datum $\mathbf{D}_X$ over the 2–adic integers $\mathbb{Z}_2$, and in particular the exotic 2–compact group DI(4), constructed by Dwyer–Wilkerson [25], is the only simple 2–compact group not arising as the 2–completion of a compact connected Lie group. This, combined with our previous work with Møller and Viruel [8] for odd primes, finishes the proof of the classification of p–compact groups for all primes p. The classification states that, up to isomorphism, there is a one-to-one correspondence between connected p–compact groups and root data over $\mathbb{Z}_p$. The statement is hence completely parallel to the classification of compact connected Lie groups [9, §4, no. 9], with $\mathbb{Z}$ replaced by $\mathbb{Z}_p$. Our proof is written to work for any prime p and is an induction on the dimension of the group. As a consequence of the classification we prove the maximal torus conjecture, giving a one-to-one correspondence between compact Lie groups and finite loop spaces admitting a maximal torus.

The classification of 2–compact groups has the following consequence, which captures most of the classification statement.

Theorem 1.1 (Classification of 2–compact groups; splitting version)**.** *Let X be a connected 2–compact group. Then $BX \simeq BG\hat{_2} \times B\,\mathrm{DI}(4)^s$, $s \geq 0$, where $BG\hat{_2}$ is the 2–completion of the classifying space of a compact connected Lie group G, and $B\,\mathrm{DI}(4)$ is the classifying space of the exotic 2–compact group* DI(4).

This is the traditional form of the classification conjecture e.g. stated by Dwyer in his 1998 ICM address [20, Conjs. 5.1 and 5.2]. A p–compact group, introduced by Dwyer–Wilkerson [26], can be defined as a pointed, connected, p–complete space BX with $H^*(\Omega BX;\mathbb{F}_p)$ finite over $\mathbb{F}_p$, and X is then the pointed loop space ΩBX. The space BX is hence the *classifying space* of the *loop space* X, justifying the convention of referring to the p–compact group simply by X. A p–compact group is called *connected* if the space X is connected, and two p–compact groups are said to be *isomorphic* if their classifying spaces are homotopy equivalent. For more background on p–compact groups, including details on the history of the classification conjecture, we refer to [8] and the references therein—we also return to it later in this Introduction.

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To make the classification more precise, we now recall the notion of a root datum over $\mathbb{Z}_p$. For $p = 2$ this theory provides a key new input to our proofs and was developed in the paper [29] of Dwyer–Wilkerson and in our paper [7] (see also Section 8); we say more about this later in this Introduction where we give an outline of the proof of Theorem 1.2.

For a principal ideal domain R, an *R–root datum* $\mathbf{D}$ is a triple $(W, L, \{Rb_\sigma\})$, where L is a finitely generated free R–module, $W \subseteq \mathrm{Aut}_R(L)$ is a finite subgroup generated by reflections (i.e., elements σ such that $1-\sigma \in \mathrm{End}_R(L)$ has rank one), and $\{Rb_\sigma\}$ is a collection of rank one submodules of L, indexed by the reflections σ in W, satisfying the two conditions

$$\mathrm{im}(1-\sigma) \subseteq Rb_\sigma \text{ and } w(Rb_\sigma) = Rb_{w\sigma w^{-1}} \text{ for all } w \in W.$$

The element $b_\sigma \in L$, determined up to a unit in R, is called the *coroot* corresponding to σ, and together with σ, it determines a *root* $\beta_\sigma \colon L \to R$ via the formula $\sigma(x) = x + \beta_\sigma(x) b_\sigma$. There is a one-to-one correspondence between $\mathbb{Z}$–root data and classically defined root data by associating $(L, L^*, \{\pm b_\sigma\}, \{\pm\beta_\sigma\})$ to $(W, L, \{\mathbb{Z}b_\sigma\})$; see [29, Prop. 2.16]. For both $R = \mathbb{Z}$ or $\mathbb{Z}_p$, one can, instead of $\{Rb_\sigma\}$, equivalently consider their span, the *coroot lattice*, $L_0 = \sum_\sigma Rb_\sigma \subseteq L$, the definition given in [8, §1] (under the name "R–reflection datum"). For $R = \mathbb{Z}_p$, p odd, the notion of an R–root datum agrees with that of an R–reflection group (W, L); see Section 8. Given two R–root data $\mathbf{D} = (W, L, \{Rb_\sigma\})$ and $\mathbf{D}' = (W', L', \{Rb'_{\sigma'}\})$, an *isomorphism* between $\mathbf{D}$ and $\mathbf{D}'$ is an isomorphism $\varphi \colon L \to L'$ such that $\varphi W \varphi^{-1} = W'$ as subgroups of $\mathrm{Aut}(L')$ and $\varphi(Rb_\sigma) = Rb'_{\varphi\sigma\varphi^{-1}}$ for every reflection $\sigma \in W$. We let $\mathrm{Aut}(\mathbf{D})$ be the automorphism group of $\mathbf{D}$, and we define the *outer automorphism group* as $\mathrm{Out}(\mathbf{D}) = \mathrm{Aut}(\mathbf{D})/W$. A classification of $\mathbb{Z}_p$–root data is given as Theorems 8.1 and 8.13.

We now explain how to associate a $\mathbb{Z}_p$–root datum to a connected p–compact group. By a theorem of Dwyer–Wilkerson [26, Thm. 8.13] any p–compact group X has a maximal torus, which is a map $i \colon BT = (BS^1{}^\wedge_p)^r \to BX$ such that the homotopy fiber has finite $\mathbb{F}_p$–cohomology and non-trivial Euler characteristic. Replacing i by an equivalent fibration, we define the *Weyl space* $\mathcal{W}_X(T)$ as the topological monoid of self-maps $BT \to BT$ over i. The *Weyl group* is defined as $W_X(T) = \pi_0(\mathcal{W}_X(T))$, and the classifying space of the *maximal torus normalizer* is defined as the Borel construction $B\mathcal{N}_X(T) = BT_{h\mathcal{W}_X(T)}$. By definition, W_X acts on $L_X = \pi_2(BT)$, and if X is connected, this gives a faithful representation of W_X on L_X as a finite $\mathbb{Z}_p$–reflection group [26, Thm. 9.7(ii)]. There is also an easy formula for the coroots b_σ in terms of the maximal torus normalizer $\mathcal{N}_X$, for which we refer to Section 8.1 (or [29], [7]). We define $\mathbf{D}_X = (W_X, L_X, \{\mathbb{Z}_p b_\sigma\})$.

We are now ready to state the precise version of our main theorem.

Theorem 1.2 (Classification of p–compact groups)**.** *The assignment, which to a connected p–compact group X associates its $\mathbb{Z}_p$–root datum $\mathbf{D}_X$, gives a one-to-one correspondence between connected p–compact groups, up to isomorphism, and $\mathbb{Z}_p$–root data, up to isomorphism. Furthermore the map $\Phi \colon \mathrm{Out}(BX) \to \mathrm{Out}(\mathbf{D}_X)$ is an isomorphism, and more generally*

$$B\,\mathrm{Aut}(BX) \xrightarrow{\simeq} (B^2\mathcal{Z}(\mathbf{D}_X))_{h\,\mathrm{Out}(\mathbf{D}_X)},$$

where the action of $\mathrm{Out}(\mathbf{D}_X)$ on $B^2\mathcal{Z}(\mathbf{D}_X)$ comes from the canonical action on $\mathcal{Z}(\mathbf{D}_X)$.

Here $\operatorname{Aut}(BX)$ is the space of self-homotopy equivalences of BX, $\operatorname{Out}(BX)$ is its component group, $B^2\mathcal{Z}(\mathbf{D}_X)$ is the double classifying space of the center $\mathcal{Z}(\mathbf{D}_X)$ of the $\mathbb{Z}_p$–root datum $\mathbf{D}_X$ (see Proposition 8.4(1)) and Φ is the standard Adams–Mahmud map [8, Lem. 4.1] given by lifting a self-equivalence BX to BT; see Recollection 8.2. We remark that the existence of the map $B\operatorname{Aut}(BX) \to (B^2\mathcal{Z}(\mathbf{D}_X))_{h\operatorname{Out}(\mathbf{D}_X)}$ in the last part of the theorem requires knowing that the fibration $B\operatorname{Aut}(BX) \to B\operatorname{Out}(BX)$ splits, which was established in [7, Thm. A]. Theorem 1.2 implies that connected p–compact groups are classified by their maximal torus normalizer, the classification conjecture in [20, Conj. 5.3]. For p odd, Theorem 1.2 is [8, Thm. 1.1] (with an improved description of $B\operatorname{Aut}(BX)$ by [7, Thm. A]). Our proof here is written so that it is independent of the prime p; see the outline of proof later in this Introduction for a further discussion.

The main theorem has a number of important corollaries. The "maximal torus conjecture" gives a purely homotopy theoretic characterization of compact Lie groups amongst finite loop spaces:

Theorem 1.3 (Maximal torus conjecture)**.** *The classifying space functor, which to a compact Lie group G associates the finite loop space $(G, BG, e\colon G \xrightarrow{\simeq} \Omega BG)$, gives a one-to-one correspondence between isomorphism classes of compact Lie groups and finite loop spaces with a maximal torus. Furthermore, for G connected, $B\operatorname{Aut}(BG) \simeq (B^2Z(G))_{h\operatorname{Out}(G)}$.*

The automorphism statement above is included for completeness, but follows easily by combining previous work of Jackowski–McClure–Oliver, Dwyer–Wilkerson, and de Siebenthal (cf., [34, Cor. 3.7], [27, Thm. 1.4], and [15, Ch. I, §2, no. 2]). The maximal torus conjecture seems to first have made it into print in 1974, where Wilkerson [68] described it as a "popular conjecture towards which the author is biased".

The "Steenrod problem" from around 1960 (see Steenrod's papers [59, 60]) asks: Which graded polynomial algebras are realized as the cohomology ring of some space? The problem was solved with $\mathbb{F}_p$–coefficients, for p "large enough", by Adams–Wilkerson [2] in 1980, extending the work of Clark–Ewing [13], and for all odd p, by Notbohm [54] in 1999. The case $p = 2$ is different from odd primes, for instance since generators can appear in odd degrees.

Theorem 1.4 (Steenrod's problem for $\mathbb{F}_2$)**.** *Suppose that P^* is a graded polynomial algebra over $\mathbb{F}_2$ in finitely many variables. If $H^*(Y;\mathbb{F}_2) \cong P^*$ for some space Y, then P^* is isomorphic, as a graded algebra, to*

$$H^*(BG;\mathbb{F}_2) \otimes H^*(B\,\mathrm{DI}(4);\mathbb{F}_2)^{\otimes r} \otimes H^*(\mathbb{R}P^\infty;\mathbb{F}_2)^{\otimes s} \otimes H^*(\mathbb{C}P^\infty;\mathbb{F}_2)^{\otimes t}$$

for some $r, s, t \geq 0$, where G is a compact connected Lie group with finite center and $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ denotes infinite-dimensional real and complex projective space, respectively. In particular if P^* has all generators in degree ≥ 3, then P^* is a tensor product of the following graded algebras:

$$\begin{aligned}
H^*(B\,\mathrm{SU}(n);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, \ldots, x_{2n}],\\
H^*(B\,\mathrm{Sp}(n);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_8, \ldots, x_{4n}],\\
H^*(B\,\mathrm{Spin}(7);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8],\\
H^*(B\,\mathrm{Spin}(8);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8, y_8],\\
H^*(B\,\mathrm{Spin}(9);\mathbb{F}_2) &\cong \mathbb{F}_2[x_4, x_6, x_7, x_8, x_{16}],
\end{aligned}$$

$$H^*(BG_2;\mathbb{F}_2) \cong \mathbb{F}_2[x_4, x_6, x_7],$$
$$H^*(BF_4;\mathbb{F}_2) \cong \mathbb{F}_2[x_4, x_6, x_7, x_{16}, x_{24}],$$
$$H^*(B\,\mathrm{DI}(4);\mathbb{F}_2) \cong \mathbb{F}_2[x_8, x_{12}, x_{14}, x_{15}].$$

Since the classification of p–compact groups is a space-level statement, it also gives which graded polynomial algebras over the Steenrod algebra can occur as the cohomology ring of a space; e.g., the decomposition in Theorem 1.4 where P^* is assumed to have generators in degrees ≥ 3 also holds over the Steenrod algebra. It should also be possible to give a more concrete list even without the degree ≥ 3 assumption by finding all polynomial rings which occur as $H^*(BG;\mathbb{F}_2)$ for G a compact connected Lie group with finite center; for G simple a list can be found in [37, Thm. 5.2], cf. Remark 7.1. In a short companion paper [5] we show how the theory of p–compact groups in fact allows for a solution of the Steenrod problem with coefficients in any Noetherian ring of finite Krull dimension.

We can also determine to which extent the realizing space is unique: Recall that two spaces Y and Y' are said to be $\mathbb{F}_p$–equivalent if there exists a space Y'' and a zig-zag $Y \to Y'' \leftarrow Y'$ inducing isomorphisms on $\mathbb{F}_p$–homology. The statement below also holds verbatim when p is odd, where the result is due to Notbohm [54, Cors. 1.7 and 1.8]—complications arise for $p = 2$, e.g., due to the possibility of generators in odd degrees.

Theorem 1.5 (Uniqueness of spaces with polynomial $\mathbb{F}_2$–cohomology)**.** *If A^* is a graded polynomial $\mathbb{F}_2$–algebra over the Steenrod algebra $\mathcal{A}_2$, in finitely many variables, all in degrees ≥ 3, then there exists at most one space Y, up to $\mathbb{F}_2$–equivalence, with $H^*(Y;\mathbb{F}_2) \cong A^*$, as graded $\mathbb{F}_2$–algebras over the Steenrod algebra.*

If P^* is a finitely generated graded polynomial $\mathbb{F}_2$–algebra, then there exists at most finitely many spaces Y up to $\mathbb{F}_2$–equivalence such that $H^*(Y;\mathbb{F}_2) \cong P^*$ as graded $\mathbb{F}_2$–algebras.

The early uniqueness results on p–compact groups starting with [22], which predate root data, or even the formal definition of a p–compact group, were formulated in this language—we give a list of earlier classification results later in this Introduction. The assumption that all generators are in degrees ≥ 3 for the first statement cannot be dropped since for instance $B(S^1 \times \mathrm{SU}(p^2))$ and $B((S^1 \times \mathrm{SU}(p^2))/C_p)$ have isomorphic $\mathbb{F}_p$–cohomology algebras over the Steenrod algebra, but are not $\mathbb{F}_p$–equivalent. Also, the same graded polynomial $\mathbb{F}_p$–algebra can of course often have multiple Steenrod algebra structures, the option left open in the second statement: $B\,\mathrm{SU}(2) \times B\,\mathrm{SU}(4)$ and $B\,\mathrm{Sp}(2) \times B\,\mathrm{SU}(3)$ have isomorphic $\mathbb{F}_p$–cohomology algebras, but with different Steenrod algebra structures at all primes.

Bott's theorem on the cohomology of X/T, the Peter–Weyl Theorem, and Borel's characterization of when centralizers of elements of order p are connected, given for p odd as Theorems 1.5, 1.6, and 1.9 of the paper [8], also hold verbatim for $p = 2$ as a direct consequence of the classification (see Remark 7.3). Likewise [8, Thm. 1.8], stating different conditions for a p–compact group to be p–torsion free, holds verbatim except that condition (3) should be removed; cf. [8, Rem. 10.10]. We also remark that the classification, together with results of Bott for compact Lie groups, gives that $H^*(\Omega X;\mathbb{Z}_p)$ is p–torsion free and concentrated in even degrees for all p–compact groups. This result was first proved by Lin and Kane, in fact in the more general setting of finite mod p H–spaces, in a series of celebrated, but highly technical, papers [39, 40, 41, 36], using completely different arguments.

Theorem 1.2 also implies a classification for non-connected p–compact groups, though, just as for compact Lie groups, the classification is less calculationally explicit: Any disconnected p–compact group X fits into a fibration sequence

$$BX_1 \to BX \to B\pi$$

with X_1 connected, and since our main theorem also includes an identification of the classifying space of such a fibration $B\,\mathrm{Aut}(BX_1)$ with the algebraically defined space $(B^2\mathcal{Z}(\mathbf{D}_{X_1}))_{h\,\mathrm{Out}(\mathbf{D}_{X_1})}$, this allows for a description of the moduli space of p–compact groups with component group π and whose identity component has $\mathbb{Z}_p$–root datum $\mathbf{D}$. More precisely we have the following theorem, which in the case where π is the trivial group recovers our classification theorem in the connected case.

Theorem 1.6 (Classification of non-connected p–compact groups)**.** *Let $\mathbf{D}$ be a $\mathbb{Z}_p$–root datum, π a finite p–group and set $B\,\mathrm{aut}(\mathbf{D}) = (B^2\mathcal{Z}(\mathbf{D}))_{h\,\mathrm{Out}(\mathbf{D})}$. The space*

$$M = (\mathrm{map}(B\pi, B\,\mathrm{aut}(\mathbf{D})))_{h\,\mathrm{Aut}(B\pi)}$$

classifies p–compact groups whose identity component has $\mathbb{Z}_p$–root datum isomorphic to $\mathbf{D}$ and component group isomorphic to π, in the following sense:

(1) *There is a one-to-one correspondence between isomorphism classes of p–compact groups X with $\pi_0(X) \cong \pi$ and $\mathbf{D}_{X_1} \cong \mathbf{D}$, and components of M, given by associating to X the component of M given by the classifying map $B\pi \to B\,\mathrm{Aut}(BX_1) \xrightarrow{\cong} B\,\mathrm{aut}(\mathbf{D}_{X_1})$. In particular the set of isomorphism classes of such p–compact groups identifies with the set of* $\mathrm{Out}(\pi)$*–orbits on $[B\pi, B\,\mathrm{aut}(\mathbf{D})]$, which is finite.*

(2) *For each p–compact group X the corresponding component of M has the homotopy type of $B\,\mathrm{Aut}(BX)$ via the zig-zag*

$$\begin{tikzcd}
 & (\mathrm{map}(B\pi, B\,\mathrm{Aut}(BX_1))_{C(f)})_{h\,\mathrm{Aut}(B\pi)} \arrow[dl, "\simeq"'] \arrow[dr, "\simeq"] & \\
B\,\mathrm{Aut}(BX) & & (\mathrm{map}(B\pi, B\,\mathrm{aut}(\mathbf{D}_{X_1}))_{C(f)})_{h\,\mathrm{Aut}(B\pi)}
\end{tikzcd}$$

where $C(f)$ denotes the $\mathrm{Out}(\pi)$*–orbit on $[B\pi, B\,\mathrm{Aut}(BX_1)]$ of the element classifying $f\colon BX \to B\pi$.*

Finally, we remark that the uniqueness part of the classification Theorem 1.2 can be reformulated as an *isomorphism theorem* stating that the isomorphisms, up to conjugation, between two arbitrary connected p–compact groups are exactly the isomorphisms, up to conjugation, between their root data. For algebraic groups the *isomorphism theorem* can be strengthened to an *isogeny theorem* stating that isogenies of algebraic groups correspond to isogenies of root data; see, e.g., [63]. In another companion paper [4] we deduce from our classification theorem that the same is true for p–compact groups: Homotopy classes of maps $BX \to BX'$ which induce isomorphism in rational cohomology (the notion of an isogeny for p–compact groups) are in one-to-one correspondence with the conjugacy classes of isogenies between the associated $\mathbb{Z}_p$–root data; and this correspondence sends isomorphisms to isomorphisms. Here an isogeny of $\mathbb{Z}_p$–root data $(W, L, \{\mathbb{Z}_p b_\sigma\}) \to (W', L', \{\mathbb{Z}_p b'_\sigma\})$ is a $\mathbb{Z}_p$–linear monomorphism $\varphi\colon L \to L'$ with finite cokernel, such that the induced isomorphism $\varphi\colon \mathrm{Aut}(L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to \mathrm{Aut}(L' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ sends

W isomorphically to W' and such that $\varphi(\mathbb{Z}_p b_\sigma) = \mathbb{Z}_p b'_{\varphi(\sigma)}$, for every reflection $\sigma \in W$ where the corresponding factor of W has order divisible by p. As a special case this theorem also contains the description of rational self-equivalences of p–completed classifying spaces of compact connected Lie groups, the most general of the theorems obtained by Jackowski–McClure–Oliver in [34], illuminating their result.

Structure of the paper and outline of the proof of the classification. Our proof of the classification of 2–compact groups, written to work for any prime, follows the same overall structure as our proof for p odd with Møller and Viruel in [8], but with significant additions and modifications. Most importantly we draw on the theory of root data [29], [7] and have a different way of dealing with the obstruction group problem. We outline our strategy below, and also refer the reader to [8, Sec. 1] where we discuss the proof for p odd.

An inspection of the classification of $\mathbb{Z}_p$–root data, Theorem 8.1, shows that all $\mathbb{Z}_p$–root data have already been realized as root data of p–compact groups by previous work, so only uniqueness is an issue. (For $p = 2$ the root datum of DI(4) [25] is the only irreducible $\mathbb{Z}_2$–root datum not coming from a $\mathbb{Z}$–root datum; for p odd see [8].)

The proof that two connected p–compact groups with isomorphic $\mathbb{Z}_p$–root data are isomorphic, is divided into a prestep and three steps, spanning Sections 2–6. Before describing these steps, we have to recall some necessary results about root data and maximal torus normalizers from [29] and [7]: The first thing to show is that the maximal torus normalizer $\mathcal{N}_X$ can be explicitly *constructed* from $\mathbf{D}_X$. For p odd this follows by a theorem of the first-named author [3] stating that the maximal torus normalizer $\mathcal{N}_X$ is always split (i.e., the fibration $BT \to B\mathcal{N}_X \to B\mathcal{W}_X$ splits). For $p = 2$ this is not necessarily the case, and the situation is more subtle: Recently Dwyer–Wilkerson [29] showed how to extend part of the classical paper of Tits [64] to p–compact groups, in particular reconstructing $\mathcal{N}_X$ from $\mathbf{D}_X$. Since the automorphisms of $\mathcal{N}_X$ differ from those of X, one however for classification purposes has to consider an additional piece of data, namely certain "root subgroups" $\{\mathcal{N}_\sigma\}$, which one can define algebraically for each reflection σ. We constructed these for p–compact groups in [7] (see also Section 8.1) and described a candidate algebraic model for the space $B\operatorname{Aut}(BX)$. (In the setting of algebraic groups, this "root subgroup" $\mathcal{N}_\sigma$ will be the maximal torus normalizer of $\langle U_\alpha, U_{-\alpha}\rangle$, where U_α is the root subgroup in the sense of algebraic groups corresponding to the root α dual to the coroot b_σ; see [7, Rem. 3.1].) For the connoisseur we note that the reliance in [29] on a classification of connected 2–compact groups of rank 2 was eliminated in [7].

In [7], recalled in Recollection 8.2, we furthermore showed that the "Adams–Mahmud" map, which to a homotopy equivalence of BX associates a homotopy equivalence of $B\mathcal{N}$, factors

$$\Phi\colon\ \operatorname{Out}(BX) \to \operatorname{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\}) \xrightarrow{\cong} \operatorname{Out}(\mathbf{D}_X),$$

where $\operatorname{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ is the set of homotopy classes of self-homotopy equivalences of $B\mathcal{N}$ permuting the root subgroups $B\mathcal{N}_\sigma$ (see Recollection 8.2 for the precise definition). As a part of our proof we will show that Φ is an isomorphism by induction.

The main argument proceeds by induction on the cohomological dimension of X, which can be determined from (W_X, L_X) alone. (Almost equivalently one could do induction on the order of W_X.) It is divided into a prestep (Section 2) and three steps (Sections 4–6).

Prestep (Section 2): The first step is to reduce to the case of simple, center-free groups. For this we use rather general arguments with fibrations and their automorphisms, in spirit similar to the arguments in [8]. However the theory of root data and root subgroups is needed both for the statements and results for $p = 2$, and these are incorporated throughout.

With this in hand, we can assume that we have two connected simple, center-free p–compact groups X and X' with isomorphic $\mathbb{Z}_p$–root data $\mathbf{D}_X$ and $\mathbf{D}_{X'}$. As explained in the discussion above, [29] [7] implies that the corresponding maximal torus normalizers and root subgroups are isomorphic, and we can hence assume that they both equal $(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ embedded via maps j and j' in X and X',

$$\begin{tikzcd} & (B\mathcal{N}, \{B\mathcal{N}_\sigma\}) \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ BX \arrow[rr, dotted] & & BX' \end{tikzcd}$$

where the dotted arrow is the one that we want to construct.

Step 1 (Section 4): Using the fact that in a connected p–compact group every element of order p can be conjugated into the maximal torus, uniquely up to conjugation in $\mathcal{N}$, we have, for every element $\nu\colon B\mathbb{Z}/p \to BX$, of order p in X, a diagram of the form:

$$\begin{tikzcd} & (B\mathcal{C}_{\mathcal{N}}(\nu), \{B\mathcal{C}_{\mathcal{N}}(\nu)_\sigma\}) \arrow[dl] \arrow[dr] & \\ B\mathcal{C}_X(\nu) & & B\mathcal{C}_{X'}(\nu) \end{tikzcd}$$

We can furthermore take covers of this diagram with respect to the fundamental group $\pi_1(\mathbf{D})$ of the root datum, which we indicate by adding a tilde $\widetilde{(\cdot)}$. (This uses the formula for the fundamental group of a p–compact group [30], but see also Theorem 8.6.) In Section 4 we prove that one can use the induction hypothesis to construct a homotopy equivalence between $\widetilde{B\mathcal{C}_X(\nu)}$ and $\widetilde{B\mathcal{C}_{X'}(\nu)}$ under $\widetilde{B\mathcal{C}_{\mathcal{N}}(\nu)}$. The tricky point here is that these centralizers need not themselves be connected, so one first has to construct the map on the identity component $B\mathcal{C}_X(\nu)_1$ and then show that it extends; this in turn requires that one has control of the space of self-equivalences of $B\mathcal{C}_X(\nu)_1$.

Now for a general non-trivial elementary abelian p–subgroup $\nu\colon BE \to BX$ of X we can pick an element of order p in E, and restriction provides a map

$$\widetilde{B\mathcal{C}_X(\nu)} \to \widetilde{B\mathcal{C}_X(\mathbb{Z}/p)} \to \widetilde{B\mathcal{C}_{X'}(\mathbb{Z}/p)} \to B\widetilde{X'}.$$

Step 2 (Section 5): To make sure that these maps are chosen in a compatible way, one has to show that this map does not depend on the choice of rank one subgroup of E. In Section 5 we show that this lift does not depend on the choices, relying on techniques developed in [8]. We furthermore show that the maps combine to form an element

$$[\vartheta] \in \lim{}^0_{\nu \in \mathbf{A}(X)} [B\widetilde{\mathcal{C}_X(\nu)}, B\widetilde{X'}]$$

where $\mathbf{A}(X)$ is the Quillen category of X, with objects the non-trivial elementary abelian p–subgroups of X and morphisms induced by conjugation in X.

Step 3 (Section 6): The construction of the element $[\vartheta]$ basically guarantees that X and X' have the same p–fusion, and the last step, which we carry out in Section 6, deals with the rigidification question, where our approach differs significantly from [8]. In particular, since we work on universal covers throughout, we are able to relate our obstruction groups to groups already calculated by Jackowski–McClure–Oliver in [33]. Since the exotic p–compact groups (only DI(4) for $p = 2$) are easily dealt with, we can assume that $BX = BG\hat{_p}$ for some simple, center-free Lie group. It was shown in [33] that $BG\hat{_p}$ can be expressed as a homotopy colimit of certain subgroups P of G, the so-called p–radical (also known as p–stubborn) subgroups. For a p–radical subgroup P of G, our element $[\vartheta]$ above gives maps

$$B\widetilde{P}\hat{_p} \to BC_G(\widetilde{{}_pZ(P)})\hat{_p} \to B\widetilde{X'},$$

where ${}_pZ(P)$ denotes the subgroup of elements of order dividing p in $Z(P)$. These maps combine to form an element in

$$\lim{}^0_{\widetilde{G}/\widetilde{P} \in \mathbf{O}_p^r(\widetilde{G})} [B\widetilde{P}\hat{_p}, B\widetilde{X}'],$$

where $\mathbf{O}_p^r(\widetilde{G})$ is the full subcategory of the p–orbit category of $\widetilde{G}$ with objects $\widetilde{G}/\widetilde{P}$ for $\widetilde{P}$ a p–radical subgroup of $\widetilde{G}$.

The obstructions to rigidifying this to get a map on the homotopy colimit

$$\operatorname{hocolim}_{\widetilde{G}/\widetilde{P} \in \mathbf{O}_p^r(\widetilde{G})} (E\widetilde{G} \times_{\widetilde{G}} \widetilde{G}/\widetilde{P})\hat{_p} \to B\widetilde{X}'$$

lies in obstruction groups which identify with

$$\lim{}^*_{\widetilde{G}/\widetilde{P} \in \mathbf{O}_p^r(\widetilde{G})} \pi_*(Z(\widetilde{P})\hat{_p}).$$

Using extensive case-by-case calculations, Jackowski–McClure–Oliver showed in [33] that these obstructions in fact vanish. Hence we have constructed a map

$$B\widetilde{G}\hat{_p} \xleftarrow{\simeq} (\operatorname{hocolim}_{\widetilde{G}/\widetilde{P} \in \mathbf{O}_p^r(\widetilde{G})} (E\widetilde{G} \times_{\widetilde{G}} \widetilde{G}/\widetilde{P})\hat{_p})\hat{_p} \to B\widetilde{X'}$$

which is easily seen to be an equivalence. Passing to a quotient we get the wanted equivalence $BG\hat{_p} \to BX'$, finishing the proof of uniqueness. The remaining statements of Theorem 1.2 also fall out of this approach.

Section 7 proves the stated consequences of the classification and the Appendix, Section 8, is used to establish a number of general properties of root data over $\mathbb{Z}_p$ used throughout the paper.

Here is the contents in table form:

Contents

We refer to the introduction of our paper [8] with Møller and Viruel for a detailed discussion of the history of the classification for odd primes. The first classification results for 2–compact groups were obtained by Dwyer–Miller–Wilkerson [22] twenty years ago, in the fundamental cases SU(2) and SO(3). Notbohm [53] and Møller–Notbohm [49] covered SU(n), Viruel covered G_2 [67], Viruel–Vavpetič covered F_4 and Sp(n) [65], [66], Morgenroth and Notbohm handled SO($2n$+1) and Spin($2n$+1) [50], [55], and Notbohm proved the result for DI(4) [56], all using arguments specific to the case in question.

Obviously this paper owes a great debt to our earlier work with Møller and Viruel [8] for odd primes. Jesper Møller introduced us to the induction-on-centralizers approach to the classification, and Antonio Viruel gave us the idea of trying to compare the centralizer and p–radical decomposition, a method he had used in the paper [65] in a special case. We are very grateful to them for sharing their insights. We would furthermore like to thank Bill Dwyer and Clarence Wilkerson for helpful correspondence, and for sharing with us an early version of their manuscript [30] on fundamental groups of p–compact groups. We thank Haynes Miller for useful questions and Carles Broto, Natàlia Castellana and Jérôme Scherer for sending us their comments on our paper from the UAB seminar. The results of this paper were announced in Spring 2005 [6]. Independently of our results, Jesper Møller has announced a proof of the classification of connected 2–compact groups (Theorem 1.1) using computer algebra [42, 46, 47]. We benefited from the hospitality of the University of Aarhus, the University of Copenhagen, and the University of Chicago while writing this paper.

2. Reduction to center-free simple p–compact groups

In this section we reduce the classification of connected p–compact groups to the case of simple center-free groups, in the sense that the classification statement, Theorem 1.2, holds for a connected p–compact group if it holds for the simple factors occurring in the corresponding adjoint (center-free) p–compact group (see Propositions 2.1 and 2.4). We do this by extending the proofs given in [8, Sec. 6] for p odd, to all primes by incorporating $\mathbb{Z}_p$–root data and root subgroups. Since this additional data requires a restructuring of most of the proofs, we present this reduction in some detail.

As in [8] we make the following working definition: A connected p–compact group X is said to be *determined by* its $\mathbb{Z}_p$–root datum $\mathbf{D}_X$ if any connected p–compact group X' with $\mathbf{D}_{X'} \cong \mathbf{D}_X$ is isomorphic to X. (Theorem 1.2 will eventually show that this always holds.)

Proposition 2.1 (Product reduction). *Suppose $X = X_1 \times \cdots \times X_k$ is a product of simple p–compact groups.*

(1) *If $\Phi\colon \operatorname{Out}(BX_i) \to \operatorname{Out}(\mathbf{D}_{X_i})$ is injective for each i, then so is $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$.*
(2) *If $\Phi\colon \operatorname{Out}(BX_i) \to \operatorname{Out}(\mathbf{D}_{X_i})$ is surjective and X_i is determined by $\mathbf{D}_{X_i}$ for each i, then $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$ is surjective.*

Proof. The proof of (1) is identical to the proof of [8, Lem. 6.1(2)] (the key fact is knowing that a map out of a connected p–compact group which is trivial when restricted to the maximal torus is in fact trivial, which, e.g., follows from [43, Thm. 6.1]). The statement in (2) is a direct consequence of the description of $\operatorname{Out}(\mathbf{D})$ in Proposition 8.14 together with the assumption that if $\mathbf{D}_{X_i}$ is isomorphic to $\mathbf{D}_{X_j}$, then X_i is isomorphic to X_j. $\square$

The reader might want to note that conversely to Proposition 2.1(2), if $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$ is surjective for all connected p–compact groups X, then all connected p–compact groups are determined by their root datum, as is seen by considering products.

Construction 2.2 (Quotients of p–compact groups). For explicitness we recall the quotient construction for p–compact groups and describe when a self-homotopy equivalence induces a homotopy equivalence on quotients, since this will be used in what follows:

Let X be a p–compact group, A an abelian p–compact group and $i\colon BA \to BX$ a central homomorphism. By assumption BA is homotopy equivalent to $\operatorname{map}(BA, BA)_1$, and $\operatorname{map}(BA, BX)_i \xrightarrow{\text{ev}} BX$ is an equivalence. Recall that the quotient BX/A is defined as the Borel construction of the composition action of $\operatorname{map}(BA, BA)_1$ on $\operatorname{map}(BA, BX)_i$; cf. [26, Pf. of Prop. 8.3]. This action and the resulting quotient space BX/A only depends on the (free) homotopy class of i, even on the point-set level, and we have a canonical quotient map

$$q\colon BX \xleftarrow{\cong} \operatorname{map}(BA, BX)_i \to BX/A$$

well-defined up to homotopy.

Now suppose we have a self-homotopy equivalence $f\colon BX \to BX$ such that there exists a homotopy equivalence $g\colon BA \to BA$ making the diagram

$$\begin{tikzcd} BA \arrow[r, "g"] \arrow[d, "i"'] & BA \arrow[d, "i"] \\ BX \arrow[r, "f"] & BX \end{tikzcd}$$

commute up to homotopy. We claim that f naturally induces a map on quotients:

First, by using the bar construction model for BA, we can without restriction assume that g is induced by a group homomorphism and has a strict inverse g^{-1}. Next note that in general, if $\varphi\colon G \to G'$ is a map of monoids, $h\colon Y \to Y'$ is a map from a G–space Y to a G'–space Y', which is φ–equivariant in the sense that $h(g \cdot y) = \varphi(g) \cdot h(y)$, then there is a canonical induced map on Borel constructions $Y_{hG} \to Y'_{hG'}$ under $h\colon Y \to Y'$ and over $B\varphi\colon BG \to BG'$, e.g., by viewing the Borel construction as a homotopy colimit via the one-sided bar construction.

In the above setup take $\varphi = c_g$, the monoid automorphism of $\mathrm{map}(BA, BA)_1$ given by $c_g(\alpha) = g \circ \alpha \circ g^{-1}$, and h the self-map of $\mathrm{map}(BA, BX)_i$ given by $\beta \mapsto f \circ \beta \circ g^{-1}$. Then f induces a map $\bar{f}\colon BX/A \to BX/A$, which fits into the homotopy commutative diagram:

$$\begin{tikzcd} BX \arrow[r, "f"] \arrow[d, "q"'] & BX \arrow[d, "q"] \\ BX/A \arrow[r, "\overline{f}"] & BX/A \end{tikzcd}$$

The quotient construction furthermore behaves naturally with respect to the maximal torus: If $j\colon BT \to BX$ is a maximal torus of X and $h\colon BT \to BT$ is a lifting of $f\colon BX \to BX$, then $i\colon BA \to BX$ factors through j and $g\colon BA \to BA$ lifts h [27, Lem. 6.5], and the above construction produces a homotopy commutative diagram

$$\begin{tikzcd} BT/A \arrow[r, "\overline{h}"] \arrow[d, "j/A"'] & BT/A \arrow[d, "j/A"] \\ BX/A \arrow[r, "\overline{f}"] & BX/A \end{tikzcd}$$

up to homotopy under the diagram one has before taking quotients.

Lemma 2.3. *Let X be a connected p–compact group with center $i\colon B\mathcal{Z} \to BX$ and let $q\colon BX \to BX/\mathcal{Z}$ denote the quotient map. Then any self-homotopy equivalence $\varphi\colon BX \to BX$ fitting into a homotopy commutative diagram*

$$\begin{tikzcd} & B\mathcal{Z} \arrow[dl, "i"'] \arrow[dr, "i"] & \\ BX \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & BX \arrow[dl, "q"] \\ & BX/\mathcal{Z} & \end{tikzcd}$$

is homotopic to the identity.

Proof. Let $q\colon BX \to BX/\mathcal{Z}$ denote the quotient map, turned into a fibration. By changing φ up to homotopy, we can assume that φ is a map strictly over q. By [17] (see also [21] and [27, Prop. 11.9]) the homotopy class of φ as a map over q corresponds to an element $[\varphi] \in \pi_1(\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_f)$, where $f\colon BX/\mathcal{Z} \to B\,\mathrm{Aut}(B\mathcal{Z})$ is the map classifying the fibration q. Note that the class $[\varphi]$ could a priori depend on how we choose φ, although this turns out not to be the case.

The composite $\bar{f}\colon BX/\mathcal{Z} \to B\,\mathrm{Aut}(B\mathcal{Z}) \to B\,\mathrm{Out}(B\mathcal{Z})$ is null-homotopic since $\pi_1(BX/\mathcal{Z}) = 0$, and obviously $\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Out}(B\mathcal{Z}))_0 \xrightarrow{ev} B\,\mathrm{Out}(B\mathcal{Z})$ is a homotopy equivalence. We thus have a fibration sequence

$$\mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}_1(B\mathcal{Z}))_{[f]} \to \mathrm{map}(BX/\mathcal{Z}, B\,\mathrm{Aut}(B\mathcal{Z}))_f \to B\,\mathrm{Out}(B\mathcal{Z}),$$

where $[f]$ denotes the components which map to the component of f. Since $B\operatorname{Aut}_1(B\mathcal{Z}) \simeq B^2\mathcal{Z}$ is a loop space,

$$\begin{aligned}\pi_1(\operatorname{map}(BX/\mathcal{Z}, B\operatorname{Aut}_1(B\mathcal{Z}))_g) &\cong \pi_1(\operatorname{map}(BX/\mathcal{Z}, B\operatorname{Aut}_1(B\mathcal{Z}))_0)\\ &= [BX/\mathcal{Z}, B\mathcal{Z}] = 0\end{aligned}$$

for any $g\colon BX/\mathcal{Z} \to B\operatorname{Aut}_1(B\mathcal{Z})$, using the fact that $BX/\mathcal{Z}$ is simply connected and $\pi_2(BX/\mathcal{Z})$ is finite (cf. Remark 8.7). Hence $\pi_1(\operatorname{map}(BX/\mathcal{Z}, B\operatorname{Aut}(B\mathcal{Z}))_f) \to \operatorname{Out}(B\mathcal{Z})$ is injective. In terms of self-equivalences of fibrations, this homomorphism sends $[\varphi]$ to the homotopy class of the induced self-equivalence of $B\mathcal{Z}$ over the basepoint in $BX/\mathcal{Z}$. However since $B\mathcal{Z} \to BX$ is the center, our assumption that $\varphi \circ i \simeq i$ shows that $[\varphi]$ maps to the identity in $\operatorname{Out}(B\mathcal{Z})$ (cf. [27, §6]). Since the homomorphism is injective we conclude that $[\varphi]$ is the identity, and in particular φ is homotopic to the identity as wanted. □

Proposition 2.4 (Reduction to center-free case). *Let X be a connected p–compact group with center $\mathcal{Z}$.*

1. *If $X/\mathcal{Z}$ is determined by $\mathbf{D}_{X/\mathcal{Z}}$ and $\Phi\colon \operatorname{Out}(BX/\mathcal{Z}) \to \operatorname{Out}(\mathbf{D}_{X/\mathcal{Z}})$ is surjective, then X is determined by $\mathbf{D}_X$.*
2. *If $\Phi\colon \operatorname{Out}(BX/\mathcal{Z}) \to \operatorname{Out}(\mathbf{D}_{X/\mathcal{Z}})$ is injective, then so is $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$.*
3. *If $\Phi\colon \operatorname{Out}(BX/\mathcal{Z}) \to \operatorname{Out}(\mathbf{D}_{X/\mathcal{Z}})$ is surjective, then so is $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$.*

Proof. The proof of (1) follows the outline of the corresponding statement for odd primes [8, Lem. 6.8(1)], but with the important additional input that we need to keep track of the root subgroups: Suppose that X and X' are connected p–compact groups with the same $\mathbb{Z}_p$–root datum $\mathbf{D}$. By [29, Prop. 1.10] and [3, Thm. 1.2] X and X' have isomorphic maximal torus normalizers $\mathcal{N}_{\mathbf{D}}$; cf. Section 8. By [7, Thm. 3.2(2)] we can choose monomorphisms $j\colon B\mathcal{N}_{\mathbf{D}} \to BX$ and $j'\colon B\mathcal{N}_{\mathbf{D}} \to BX'$ such that the root subgroups in $B\mathcal{N}_{\mathbf{D}}$ with respect to j and j' agree. Furthermore, the centers of X and X' agree, and can be viewed as a subgroup $\mathcal{Z}$ of $\mathcal{N}_{\mathbf{D}}$. Now, $\mathcal{N}_{\mathbf{D}}/\mathcal{Z}$ will be a maximal torus normalizer for both $X/\mathcal{Z}$ and $X'/\mathcal{Z}$ (see, e.g., [45, Thm. 1.2]), and the root subgroups in $B\mathcal{N}_{\mathbf{D}}/\mathcal{Z}$ with respect to $j/\mathcal{Z}$ and $j'/\mathcal{Z}$ also agree by construction. By our assumptions there hence exists a homotopy equivalence $f\colon BX/\mathcal{Z} \to BX'/\mathcal{Z}$ such that

$$\begin{tikzcd} & B\mathcal{N}_{\mathbf{D}}/\mathcal{Z} \arrow[dl, "j/\mathcal{Z}"'] \arrow[dr, "j'/\mathcal{Z}"] & \\ BX/\mathcal{Z} \arrow[rr, "f"] & & BX'/\mathcal{Z} \end{tikzcd}$$

commutes up to homotopy. We need to see that f is a map over $B^2\mathcal{Z}$, since this implies that f induces a homotopy equivalence $BX \to BX'$ as wanted. This is a short argument given as the last part of [8, Pf. of Lem. 6.8(1)].

To see (2) consider $\varphi \in \operatorname{Aut}(BX)$ corresponding to an element in the kernel of $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$. By definition, cf. Recollection 8.2, this means that if

$i\colon BT \to BX$ is a maximal torus, then the diagram

$$\begin{tikzcd} & BT \arrow[dl, "i"'] \arrow[dr, "i"] & \\ BX \arrow[rr, "\varphi"] & & BX \end{tikzcd}$$

commutes up to homotopy. Since the center $B\mathcal{Z} \to BX$ factors through $i\colon BT \to BX$ [27, Thm. 1.2], Construction 2.2 combined with our assumption shows that we get a homotopy commutative diagram

$$\begin{tikzcd} & B\mathcal{Z} \arrow[dl] \arrow[dr] & \\ BX \arrow[rr, "\varphi"] \arrow[dr, "q"'] & & BX \arrow[dl, "q"] \\ & BX/\mathcal{Z} & \end{tikzcd}$$

so Lemma 2.3 gives the desired conclusion.

We now embark on showing (3), i.e., that $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$ is surjective, which requires some preparation. Recall that for any connected p–compact group Y, $\widetilde{Y}$ is the p–compact group whose classifying space $B\widetilde{Y}$ is the fiber of the fibration $BY \xrightarrow{q} P_2BY$, where P_2BY is the second Postnikov section. Let $i\colon BT \to BY$ be a maximal torus, which we can assume to be a fibration, and let $B\widetilde{T}$ denote the fiber of the fibration $q \circ i\colon BT \to P_2BY$. Since $\pi_2(i)\colon \pi_2(BT) \to \pi_2(BY)$ is surjective (see Proposition 8.5), the long exact sequence of homotopy groups shows that $B\widetilde{T}$ is a p–compact torus, and furthermore $B\widetilde{T} \to B\widetilde{Y}$ is a maximal torus by the diagram

$$\begin{tikzcd} \widetilde{Y}/\widetilde{T} \arrow[r] \arrow[d] & B\widetilde{T} \arrow[r] \arrow[d] & B\widetilde{Y} \arrow[d] \\ Y/T \arrow[r] \arrow[d] & BT \arrow[r, "i"] \arrow[d, "q\circ i"] & BY \arrow[d, "q"] \\ * \arrow[r] & P_2BY \arrow[r, equal] & P_2BY \end{tikzcd}$$

Any self-homotopy equivalence $f\colon BY \to BY$ lifts to a self-homotopy equivalence $\widetilde{f}\colon B\widetilde{Y} \to B\widetilde{Y}$ by taking fibers, and it is clear that the assignment $f \mapsto \widetilde{f}$ induces a homomorphism $\operatorname{Out}(BY) \to \operatorname{Out}(B\widetilde{Y})$.

For a Y which satisfies $\pi_1(\mathbf{D}_Y) \xrightarrow{\cong} \pi_1(Y)$, Proposition 8.10 shows that $\mathbf{D}_{\widetilde{Y}} \cong \widetilde{\mathbf{D}_Y}$. Hence, the Adams–Mahmud map Φ, recalled in Recollection 8.2, together with Proposition 8.15, provides the maps in the following diagram:

(2.1)
$$\begin{tikzcd} \operatorname{Out}(BY) \arrow[r, "\Phi"] \arrow[d] & \operatorname{Out}(\mathbf{D}_Y) \arrow[d] \\ \operatorname{Out}(B\widetilde{Y}) \arrow[r, "\Phi"] & \operatorname{Out}(\widetilde{\mathbf{D}_Y}) \end{tikzcd}$$

The diagram commutes, since for a given $f\colon BY \to BY$ both compositions give a map $B\widetilde{T} \to B\widetilde{T}$ over $\widetilde{f}\colon B\widetilde{Y} \to B\widetilde{Y}$, and hence they give the same element in $\mathrm{Out}(\widetilde{\mathbf{D}_Y})$.

By Proposition 8.11(2), $\widetilde{\mathbf{D}_X} \xrightarrow{\cong} \widetilde{\mathbf{D}_{X/\mathcal{Z}}}$, and, chasing through the definitions, $B\widetilde{X} \xrightarrow{\cong} B\widetilde{X/\mathcal{Z}}$. By the fundamental group formula, Theorem 8.6, $\pi_1(\mathbf{D}_{X/\mathcal{Z}}) \xrightarrow{\cong} \pi_1(X/\mathcal{Z})$. (Note that for the proof of Theorem 1.2 we can assume that $X/\mathcal{Z}$ is determined by $\mathbf{D}_{X/\mathcal{Z}}$, making this reference to Theorem 8.6 unnecessary.) Hence applying diagram (2.1) with $Y = X/\mathcal{Z}$ and using the aforementioned identifications produces the diagram

$$\begin{tikzcd}
\mathrm{Out}(BX/\mathcal{Z}) \arrow[r, two heads, "\Phi"] \arrow[d] & \mathrm{Out}(\mathbf{D}_{X/\mathcal{Z}}) \arrow[d, "\cong"] \\
\mathrm{Out}(B\widetilde{X}) \arrow[r, "\Phi"] & \mathrm{Out}(\widetilde{\mathbf{D}_X})
\end{tikzcd}$$

Here the right-hand vertical map is an isomorphism by Proposition 8.11(2) and Corollary 8.17, and the top horizontal map is surjective by assumption. Hence $\Phi\colon\ \mathrm{Out}(B\widetilde{X}) \to \mathrm{Out}(\mathbf{D}_{\widetilde{X}})$ is also surjective.

By [48, Thm. 5.4] there is a short exact sequence $BA \xrightarrow{i} BX' \to BX$, $BX' = B\widetilde{X} \times B\mathcal{Z}(X)_1$, where A is a finite p–group and $i\colon BA \to BX'$ is central. Writing $\mathbf{D}_X = (W, L, \{\mathbb{Z}_p b_\sigma\})$, Proposition 8.4(2) shows that X' has $\mathbb{Z}_p$–root datum $\mathbf{D}_{X'} = (W, L_0, \{\mathbb{Z}_p b_\sigma\}) \times (1, L^W, \emptyset)$, and we have the identification $\mathbf{D}_X \cong \mathbf{D}_{X'}/A$ as in Proposition 8.15.

We are now ready to show that $\mathrm{Out}(BX) \to \mathrm{Out}(\mathbf{D}_X)$ is surjective, by lifting an arbitrary element $\alpha \in \mathrm{Out}(\mathbf{D}_X)$ to $\mathrm{Out}(BX)$. By Proposition 8.15, α identifies with an element $\alpha' \in \mathrm{Out}(\mathbf{D}_{X'})$ with $\alpha'(A) = A$. Since $\Phi\colon\ \mathrm{Out}(B\widetilde{X}) \to \mathrm{Out}(\mathbf{D}_{\widetilde{X}})$ is surjective it follows from Proposition 8.14 that $\Phi\colon\ \mathrm{Out}(BX') \to \mathrm{Out}(\mathbf{D}_{X'})$ is surjective, so we can find a self-homotopy equivalence φ of BX' with $\Phi(\varphi) = \alpha'$. Since $\alpha'(A) = A$ there exists a lift $\varphi'\colon BA \to BA$ of φ fitting into a homotopy commutative diagram

$$\begin{tikzcd}
BA \arrow[r, "\varphi'"] \arrow[d, "i"'] & BA \arrow[d, "i"] \\
BX' \arrow[r, "\varphi"] & BX'
\end{tikzcd}$$

Finally, since $X \cong X'/A$, Construction 2.2 now gives a self-homotopy equivalence $\overline{\varphi}$ of BX with the property that $\Phi(\overline{\varphi}) = \Phi(\varphi)/A = \alpha'/A = \alpha$, as desired. $\square$

3. Preliminaries on self-equivalences of p–compact groups

In this short section we prove a fact about detection of self-equivalences of non-connected p–compact groups on maximal torus normalizers, which we need in the proof of the main theorem, where non-connected groups occur as centralizers of elementary abelian p–groups in connected groups.

Proposition 3.1. *Let X be a (not necessarily connected) p–compact group with maximal torus normalizer $\mathcal{N}$ and identity component X_1. If $\Phi\colon\ \mathrm{Out}(BX_1) \xrightarrow{\cong} \mathrm{Out}(\mathbf{D}_{X_1})$ is an isomorphism, then $\Phi\colon\ \mathrm{Out}(BX) \to \mathrm{Out}(B\mathcal{N})$ is injective.*

Proof. Let $j\colon B\mathcal{N} \to BX$ be a normalizer inclusion map, which we can assume is a fibration. Let $f\colon BX \to B\pi_0(X) = B\pi$ be the canonical fibration and set $q = f \circ j$.

We first argue that we can make the identification

$$\pi_0(\mathrm{Aut}(q)) \xrightarrow{\cong} \{\varphi \in \mathrm{Out}(B\mathcal{N}) \mid \varphi(\ker(\pi_1(q))) = \ker(\pi_1(q))\}.$$

Surjectivity is obvious, so we have to see injectivity, where we first observe that we can pass to a discrete approximation $\breve{q}\colon B\breve{\mathcal{N}} \to B\pi$, where $B\breve{\mathcal{N}}$ and $B\pi$ are the standard bar construction models. The simplicial maps $B\breve{\mathcal{N}} \to B\breve{\mathcal{N}}$ are exactly the group homomorphisms, so any map $\varphi\colon \breve{q} \to \breve{q}$ with $B\breve{\mathcal{N}} \to B\breve{\mathcal{N}}$ homotopic to the identity is induced by conjugation by an element in $\breve{\mathcal{N}}$. Hence φ is homotopic to the identity as a map of fibrations, proving the claim.

Since $B\,\mathrm{Aut}(f) \xrightarrow{\simeq} B\,\mathrm{Aut}(BX)$, we have the following diagram, with horizontal maps fibrations, where $B\mathcal{N}_1$ denotes the fiber of $B\mathcal{N} \to B\pi$:

$$\begin{tikzcd}
\mathrm{map}(B\pi, B\,\mathrm{Aut}(BX_1))_{C(f)} \arrow[r] \arrow[d] & B\,\mathrm{Aut}(BX) \arrow[r] \arrow[d] & B\,\mathrm{Aut}(B\pi) \arrow[d, equal] \\
\mathrm{map}(B\pi, B\,\mathrm{Aut}(B\mathcal{N}_1))_{C(q)} \arrow[r] & B\,\mathrm{Aut}(q) \arrow[r] & B\,\mathrm{Aut}(B\pi)
\end{tikzcd}$$

Here the horizontal fibrations are established in [21] and the subscript $C(f)$ denotes the $\mathrm{Out}(\pi)$–orbit of the element in $[B\pi, B\,\mathrm{Aut}(BX_1)]$ classifying f. The vertical maps are induced by Adams–Mahmud maps, cf. [8, Lem. 4.1], so the diagram is homotopy commutative by the naturality of these maps.

To establish the proposition it is enough to verify that

$$\pi_1(B\,\mathrm{Aut}(BX)) \to \pi_1(B\,\mathrm{Aut}(q))$$

is injective since we already saw that $\pi_1(B\,\mathrm{Aut}(q))$ injects into $\mathrm{Out}(B\mathcal{N})$. By [7, Thm. C] the map $\Phi\colon B\,\mathrm{Aut}(BX_1) \to B\,\mathrm{Aut}(B\mathcal{N}_1)$ factors through the covering space Y of $B\,\mathrm{Aut}(B\mathcal{N}_1)$ with respect to the subgroup $\mathrm{Out}(B\mathcal{N}_1, \{(B\mathcal{N}_1)_\sigma\})$, and $B\,\mathrm{Aut}(BX_1) \to Y$ has left homotopy inverse. Since $Y \to B\,\mathrm{Aut}(B\mathcal{N}_1)$ is a covering,

$$\mathrm{map}(B\pi, Y) \to \mathrm{map}(B\pi, B\,\mathrm{Aut}(B\mathcal{N}_1))$$

is likewise a covering map over each component where it is surjective, and hence induces a monomorphism on π_1 on all components of $\mathrm{map}(B\pi, Y)$. Since $B\,\mathrm{Aut}(BX_1) \to Y$ has a homotopy left inverse,

$$\mathrm{map}(B\pi, B\,\mathrm{Aut}(BX_1)) \to \mathrm{map}(B\pi, B\,\mathrm{Aut}(B\mathcal{N}_1))$$

also induces an injection on π_1 for all choices of base-point. Hence the five-lemma and the long-exact sequence in homotopy groups applied to the pair of fibrations above guarantees that $\mathrm{Out}(BX) = \pi_1(B\,\mathrm{Aut}(BX)) \to \pi_1(B\,\mathrm{Aut}(q))$ is injective, as wanted. ☐

Proposition 3.2. *Suppose that X is a (not necessarily connected) p–compact group such that $\Phi\colon \mathrm{Out}(BX_1) \xrightarrow{\cong} \mathrm{Out}(\mathbf{D}_{X_1})$. Let $i\colon B\mathcal{N}_p \to BX$ be the inclusion of a p–normalizer of a maximal torus, and let $\varphi\colon BX \to BX$ be a self-homotopy equivalence. If $\varphi \circ i$ is homotopic to i, then φ is homotopic to the identity map.*

Proof. Let $j\colon B\mathcal{N} \to BX$ be the inclusion of a maximal torus normalizer, turned into a fibration. By construction of the Adams–Mahmud map, cf. [8, Lem. 4.1], φ

lifts to a map $\varphi'\colon B\mathcal{N} \to B\mathcal{N}$, making the diagram

$$\begin{tikzcd} B\mathcal{N} \arrow[r, "\varphi'"] \arrow[d, "j"] & B\mathcal{N} \arrow[d, "j"] \\ BX \arrow[r, "\varphi"] & BX \end{tikzcd}$$

strictly commute, and the space of such lifts is contractible.

We want to see that φ' is homotopic to the identity, since Proposition 3.1 then implies that φ is homotopic to the identity, as wanted, using the assumption that $\Phi\colon \operatorname{Out}(BX_1) \xrightarrow{\cong} \operatorname{Out}(\mathbf{D}_{X_1})$.

Lift $i\colon B\mathcal{N}_p \to BX$ to a map $k\colon B\mathcal{N}_p \to B\mathcal{N}$. By [8, Pf. of Lem. 4.1] the space of such lifts is contractible, so since $\varphi \circ i$ is homotopic to i, we conclude that $\varphi' \circ k$ is homotopic to k. Replacing k and φ' by discrete approximations, we get the following diagram which commutes up to homotopy:

$$\begin{tikzcd} & B\breve{\mathcal{N}}_p \arrow[dl, "\breve{k}"'] \arrow[dr, "\breve{k}"] & \\ B\breve{\mathcal{N}} \arrow[rr, "\breve{\varphi}'"] & & B\breve{\mathcal{N}} \end{tikzcd}$$

This is a diagram of $K(\pi,1)$'s, so after changing the spaces and maps up to homotopy we can assume that all maps are induced by group homomorphisms and that the diagram commutes *strictly.* But now $\breve{\varphi}'$ is a group homomorphism which is the identity on $\operatorname{im}(\breve{k})$, and in particular it is the identity on $\breve{T}$. Hence $\breve{\varphi}'/\breve{T}\colon W \to W$ is the identity on the Weyl group W_1 of X_1, since W_1 acts faithfully on $\breve{T}$. But W is generated by W_1 and the image of $\operatorname{im}(\breve{k})$, so we conclude that $\breve{\varphi}'/\breve{T}\colon W \to W$ is the identity as well. Hence $\breve{\varphi}'$ is in the image of $\operatorname{Der}(W;\breve{T}) \to \operatorname{Aut}(\breve{\mathcal{N}})$, cf. [8, Pf. of Prop. 5.2], and the homotopy class of $\breve{\varphi}'$ is the image of an element in $H^1(W;\breve{T})$ under the homomorphism $H^1(W;\breve{T}) \to \operatorname{Out}(\breve{\mathcal{N}})$. Since $\breve{\varphi}'$ is the identity on $\operatorname{im}(\breve{k})$, this element restricts trivially to $H^1(W_p;\breve{T})$, where W_p is a Sylow p–subgroup in W. By a transfer argument the element in $H^1(W;\breve{T})$ is therefore also trivial, and $\breve{\varphi}'$ is homotopic to the identity map, as wanted. □

4. First part of the proof of Theorem 1.2: Maps on centralizers

In this section we carry out the first part of the proof of Theorem 1.2 by constructing maps on certain centralizers. We have chosen to be quite explicit about when we replace spaces by homotopy equivalent spaces, since some of these issues become important later on in the proof, where we want to conclude that various constructions really take place in certain over- or undercategories.

Recall that our setup is as follows: Let X and X' be connected center-free simple p–compact groups with isomorphic $\mathbb{Z}_p$–root data $\mathbf{D}_X$ and $\mathbf{D}_{X'}$. We want to prove that BX is homotopy equivalent to BX', by induction on cohomological dimension, where by [28, Lem. 3.8] the cohomological dimension $\operatorname{cd} Y = \max\{n | H^n(Y;\mathbb{F}_p) \neq 0\}$ of a connected p–compact group Y depends only on $\mathbf{D}_Y$. We make the following inductive hypothesis:

($\star$) For all connected p–compact groups Y with $\operatorname{cd} Y < \operatorname{cd} X$, Y is determined by its $\mathbb{Z}_p$–root datum $\mathbf{D}_Y$ and $\Phi\colon \operatorname{Out}(BY) \to \operatorname{Out}(\mathbf{D}_Y)$ is an isomorphism.

Let $\mathbf{D}$ be a fixed $\mathbb{Z}_p$–root datum, isomorphic to both $\mathbf{D}_X$ and $\mathbf{D}_{X'}$, and let $\mathcal{N} = \mathcal{N}_{\mathbf{D}}$ denote the associated normalizer; see Section 8.1. By [7, Thm. 3.2(2)] we can choose maps $j\colon B\mathcal{N} \to BX$ and $j'\colon B\mathcal{N} \to BX'$, making $\mathcal{N}$ a maximal torus normalizer in both X and X' in such a way that the root subgroups $B\mathcal{N}_\sigma$ of $B\mathcal{N}$ with respect to $j\colon B\mathcal{N} \to BX$ and $j'\colon B\mathcal{N} \to BX'$ agree.

By Theorem 8.6, X and X' have canonically isomorphic fundamental groups, which both identify with $\pi_1(\mathbf{D})$ via the inclusions j and j'. Applying the second Postnikov section P_2 to BX and BX' hence gives us a homotopy commutative diagram

$$\begin{tikzcd} & B\mathcal{N} \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ BX \arrow[dr] & & BX' \arrow[dl] \\ & B^2\pi_1(\mathbf{D}) & \end{tikzcd}$$

By changing the diagram up to homotopy, we can assume that all maps are fibrations and that the diagram commutes strictly. After doing this, we now leave these spaces and maps fixed throughout the proof.

Suppose that $\nu\colon BV \to BX$ is a rank one elementary abelian p–subgroup of X, and let $\mu\colon BV \to BT \to B\mathcal{N}$ denote the factorization of ν through the maximal torus T, which exists by [26, Prop. 5.6] and is unique up to conjugacy in $\mathcal{N}$ by [28, Prop. 3.4]. Set $\nu' = j' \circ \mu$ for short.

Taking centralizers, these maps produce the following diagram:

$$\begin{tikzcd} & B\mathcal{C}_{\mathcal{N}}(\mu) \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ B\mathcal{C}_X(\nu) & & B\mathcal{C}_{X'}(\nu') \end{tikzcd} \tag{4.1}$$

where we, by a slight abuse of notation, keep the labeling j and j'. Now the fundamental groups of $B\mathcal{C}_X(\nu)$ and $B\mathcal{C}_{X'}(\nu')$ identify via j and j' with a certain quotient group π of $\pi_1(B\mathcal{C}_{\mathcal{N}}(\mu))$, explicitly described in [27, Rem. 2.11] and Proposition 8.4(3). Passing to the universal covers of $B\mathcal{C}_X(\nu)$ and $B\mathcal{C}_{X'}(\nu')$ and to the cover of $B\mathcal{C}_{\mathcal{N}}(\mu)$ determined by the kernel of $\pi_1(B\mathcal{C}_{\mathcal{N}}(\mu)) \to \pi$ produces a diagram

$$\begin{tikzcd} & B\mathcal{C}_{\mathcal{N}}(\mu)_1 \arrow[dl, "j_\nu"'] \arrow[dr, "j'_{\nu'}"] & \\ B\mathcal{C}_X(\nu)_1 & & B\mathcal{C}_{X'}(\nu')_1 \end{tikzcd} \tag{4.2}$$

where the maps, which are the covers of j and j', are π–equivariant with respect to the natural free action of π on all three spaces in the diagram.

Note that in general if Y is a space with a map $f\colon Y \to BG$, with BG the classifying space of a simplicial group G, a specific model for the homotopy fiber $\widetilde{Y}$ of f is given by the subspace of $Y \times EG$, consisting of pairs whose images in BG agree. In particular it has a canonical free G–action, via the action on the second coordinate, and the projection map $\widetilde{Y} \to Y$ induces a homotopy equivalence

$\widetilde{Y}/G \to Y$. We use this model $\widetilde{(\cdot)}$ for the homotopy fiber in what follows. Note that if Y has a free H–action such that $f\colon Y \to BG$ is H–equivariant (with H acting trivially on BG), then $\widetilde{Y}$ has a free $G \times H$–action.

The spaces in (4.2) all have maps to $B^2\pi_1(\mathbf{D})$ making the obvious diagrams commute, so we can take homotopy fibers of these maps by pulling back along the map $EB\pi_1(\mathbf{D}) \to B^2\pi_1(\mathbf{D})$, as described above. This produces the diagram

(4.3)
$$\begin{tikzcd} & \widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1} \arrow[dl, "\widetilde{j_\nu}"'] \arrow[dr, "\widetilde{j'_{\nu'}}"] & \\ \widetilde{B\mathcal{C}_X(\nu)_1} & & \widetilde{B\mathcal{C}_{X'}(\nu')_1} \end{tikzcd}$$

Note that by construction $K = \pi \times B\pi_1(\mathbf{D})$ acts freely on the spaces in (4.3) and that the maps are K–equivariant. By Propositions 8.4(3) and 8.10, $\widetilde{B\mathcal{C}_X(\nu)_1}$ and $\widetilde{B\mathcal{C}_{X'}(\nu')_1}$ have isomorphic $\mathbb{Z}_p$–root data and strictly smaller cohomological dimension than X, so the inductive assumption $(\star)$ guarantees that they are homotopy equivalent. Furthermore, by construction $\widetilde{j_\nu}$ and $\widetilde{j'_{\nu'}}$ are both maximal torus normalizers, and they define the same root subgroups in $\widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1}$, since this is true for j and j'. Therefore, by the above and the inductive hypothesis $(\star)$ there exists a map $\varphi\colon \widetilde{B\mathcal{C}_X(\nu)_1} \to \widetilde{B\mathcal{C}_{X'}(\nu')_1}$, unique up to homotopy, making the above diagram (4.3) homotopy commute. We now want to argue that this map can be chosen to be K–equivariant so that passing to a quotient of diagram (4.3) with φ inserted produces a left-to-right map making the diagram (4.1) homotopy commute.

Consider the Adams–Mahmud-like zig-zag

(4.4)
$$\Xi\colon\ \operatorname{map}(\widetilde{B\mathcal{C}_X(\nu)_1}, \widetilde{B\mathcal{C}_{X'}(\nu')_1})_\varphi \xleftarrow{\simeq} \operatorname{map}(\widetilde{j_\nu}, \widetilde{j'_{\nu'}})_\varphi \xrightarrow{\simeq} \operatorname{map}(\widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1}, \widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1})_1,$$

where the first map is a homotopy equivalence by [8, Pf. of Lem. 4.1]. That the composite is a homotopy equivalence will follow once we know that the center of $\widetilde{\mathcal{C}_X(\nu)_1}$ agrees with the center of $\widetilde{\mathcal{C}_{\mathcal{N}}(\mu)_1}$, and this follows from Lemma 8.8 applied to the subgroup $\widetilde{\mathcal{C}_X(\nu)_1}$ of $\widetilde{X}$, where the assumptions are satisfied since $\pi_1(\mathbf{D}_{\widetilde{X}}) = 0$ by Proposition 8.10 and Theorem 8.6.

By construction the maps in (4.4) are equivariant with respect to the K–actions. Likewise, since the action on the sources in the mapping spaces is already free, taking homotopy fixed-points agrees up to homotopy with taking actual fixed-points, so the maps in (4.4) induce homotopy equivalences between the fixed-points. This produces homotopy equivalences

$$\begin{aligned} \operatorname{map}_K(\widetilde{B\mathcal{C}_X(\nu)_1}, \widetilde{B\mathcal{C}_{X'}(\nu')_1})_{[\varphi]} &\xleftarrow{\simeq} \operatorname{map}_K(\widetilde{j_\nu}, \widetilde{j'_{\nu'}})_{[\varphi]} \\ &\xrightarrow{\simeq} \operatorname{map}_K(\widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1}, \widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1})_{[1]}, \end{aligned}$$

where the subscript $[\varphi]$ denotes that we are taking all components of maps non-equivariantly homotopy equivalent to φ. We can therefore pick an *equivariant* map $\psi \in \operatorname{map}_K(\widetilde{B\mathcal{C}_X(\nu)_1}, \widetilde{B\mathcal{C}_{X'}(\nu')_1})_{[\varphi]}$ corresponding to

$$1 \in \operatorname{map}_K(\widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1}, \widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)_1})_{[1]}.$$

Define $\widetilde{h_\nu}$ as the composite

$$\widetilde{h_\nu}\colon\ B\widetilde{\mathcal{C}_X(\nu)} \xleftarrow{\simeq} (B\widetilde{\mathcal{C}_X(\nu)}_1)/\pi \xrightarrow[\simeq]{\psi/\pi} (B\widetilde{\mathcal{C}_{X'}(\nu')}_1)/\pi \xrightarrow{\simeq} B\widetilde{\mathcal{C}_{X'}(\nu')},$$

and similarly define

$$h_\nu\colon\ B\mathcal{C}_X(\nu) \xleftarrow{\simeq} (B\widetilde{\mathcal{C}_X(\nu)}_1)/K \xrightarrow[\simeq]{\psi/K} (B\widetilde{\mathcal{C}_{X'}(\nu')}_1)/K \xrightarrow{\simeq} B\mathcal{C}_{X'}(\nu').$$

By construction the maps $\widetilde{h_\nu}$ and h_ν fit into the following homotopy commutative diagram:

$$\begin{tikzcd}
& B\widetilde{\mathcal{C}_{\mathcal{N}}(\mu)} \arrow[dl, "\widetilde{j_\nu}"'] \arrow[d] \arrow[dr, "\widetilde{j'_{\nu'}}"] & \\
B\widetilde{\mathcal{C}_X(\nu)} \arrow[d] \arrow[rr, bend right, "\widetilde{h_\nu}", "\simeq"'] & B\mathcal{C}_{\mathcal{N}}(\mu) \arrow[dl, "j_\nu"'] \arrow[dr, "j'_{\nu'}"] & B\widetilde{\mathcal{C}_{X'}(\nu')} \arrow[d] \\
B\mathcal{C}_X(\nu) \arrow[rr, bend right, "h_\nu", "\simeq"'] & & B\mathcal{C}_{X'}(\nu')
\end{tikzcd}$$

and are in fact uniquely determined up to homotopy by this property by Proposition 3.1 and the inductive assumption $(\star)$.

Let $\widetilde{\varphi_\nu}\colon B\widetilde{\mathcal{C}_X(\nu)} \to B\widetilde{X'}$ be the composite of $\widetilde{h_\nu}$ with the evaluation $B\widetilde{\mathcal{C}_{X'}(\nu')} \to B\widetilde{X'}$, and similarly define $\varphi_\nu\colon B\mathcal{C}_X(\nu) \to BX'$ as the composite of h_ν with the evaluation $B\mathcal{C}_{X'}(\nu') \to BX'$.

We define φ_ν and $\widetilde{\varphi_\nu}$ when $\nu\colon BE \to BX$ is an elementary abelian p–subgroup of rank greater than one, by restricting to a rank one subgroup $V \subseteq E$ and using adjointness as follows: As before $\nu|_V$ factors through T, uniquely as a map to $\mathcal{N}$, and we let $\mu\colon BV \to B\mathcal{N}$ denote the resulting map to $\mathcal{N}$. Restriction produces a map

$$\varphi_{\nu,V}\colon\ B\mathcal{C}_X(\nu) \to B\mathcal{C}_X(\nu|_V) \xrightarrow[\simeq]{h_{\nu|_V}} B\mathcal{C}_{X'}(j'\mu) \to BX'.$$

Similarly, letting $B\widetilde{\mathcal{C}_X(\nu)}$ denote the homotopy fiber of the map $B\mathcal{C}_X(\nu) \to BX \to B^2\pi_1(\mathbf{D})$ as in the rank one case, we define $\widetilde{\varphi_{\nu,V}}$ as

$$\widetilde{\varphi_{\nu,V}}\colon\ B\widetilde{\mathcal{C}_X(\nu)} \to B\widetilde{\mathcal{C}_X(\nu|_V)} \xrightarrow[\simeq]{\widetilde{h_{\nu|_V}}} B\widetilde{\mathcal{C}_{X'}(j'\mu)} \to B\widetilde{X'}.$$

By construction $\varphi_{\nu,V}$ and $\widetilde{\varphi_{\nu,V}}$ fit together, i.e., the diagram

$$\begin{tikzcd}
B\widetilde{\mathcal{C}_X(\nu)} \arrow[r, "\widetilde{\varphi_{\nu,V}}"] \arrow[d] & B\widetilde{X'} \arrow[d] \\
B\mathcal{C}_X(\nu) \arrow[r, "\varphi_{\nu,V}"] & BX'
\end{tikzcd}$$

commutes up to homotopy and is in fact a homotopy pull-back square by construction. This concludes the construction of the maps on centralizers which we will use in the next sections to construct our equivalence $BX \to BX'$. We will in particular prove that $\varphi_{\nu,V}$ and $\widetilde{\varphi_{\nu,V}}$ are independent of the choice of the rank one subgroup $V \subseteq E$, after which we will drop the subscript V.

5. Second part of the proof of Theorem 1.2: The element in $\lim^0$

In this section we prove that the maps $\widetilde{\varphi_{\nu,V}}$ constructed in the previous section are independent of the choice of rank one subgroup $V \subseteq E$ and give coherent maps into $B\widetilde{X'}$. More specifically we prove the following.

Theorem 5.1. *Let X and X' be two connected simple center-free p–compact groups with isomorphic $\mathbb{Z}_p$–root data, and assume the inductive hypothesis $(\star)$. Then the maps*

$$\widetilde{\varphi_{\nu,V}}\colon\ B\widetilde{\mathcal{C}_X(\nu)} \to B\widetilde{X'}$$

constructed in Section 4 are independent of the choice of V and together form an element in $\lim^0_{\nu\in\mathbf{A}(X)}[B\widetilde{\mathcal{C}_X(\nu)}, B\widetilde{X'}]$.

Here $\mathbf{A}(X)$ is the Quillen category of X with objects the non-trivial elementary abelian p–subgroups $\nu\colon BE \to BX$ of X and morphisms given by conjugation (i.e., the morphisms from $(\nu\colon BE \to BX)$ to $(\nu'\colon BE' \to BX)$ are the linear maps $\varphi\colon E \to E'$ such that ν is freely homotopic to $\nu' \circ B\varphi$).

We need the following proposition, whose proof we postpone to after the rest of the proof of Theorem 5.1.

Proposition 5.2. *Let X be a connected simple center-free p–compact group. If $\nu\colon BE \to BX$ is a non-toral elementary abelian p–subgroup of rank two, then $\mathcal{C}_X(\nu)_1$ is non-trivial or $\mathbf{D}_X \cong \mathbf{D}_{\mathrm{PU}(p)\hat{_p}}$.*

Proof of Theorem 5.1. We divide the proof into two steps. Step 1 verifies the independence of the choice of V, and the shorter Step 2 then uses this to construct the element in $\lim^0$.

Step 1: The maps $\widetilde{\varphi_{\nu,V}}$ and $\varphi_{\nu,V}$ are independent of the choice of rank one subgroup V. We divide this step into three substeps a–c. Step 1a assumes ν toral, Step 1b assumes ν rank two non-toral, and finally Step 1c considers the general case.

Step 1a: Assume $\nu\colon BE \to BX$ is a toral elementary abelian p–subgroup. By assumption ν factors through BT to give a map $\mu\colon BE \to B\mathcal{N}$, unique up to conjugation in $\mathcal{N}$, and as in the rank one case we let $\nu' = j'\mu$. We want to say that the map $\widetilde{\varphi_{\nu,V}}$ does not depend on V, basically since it is a map suitably under $B\widetilde{\mathcal{C}_{\mathcal{N}}(\mu)}$, and hence uniquely determined, independently of V. This will follow by adjointness, analogously to [8, Pf. of Thm. 2.2], although a bit of care has to be taken, since we have to verify that this happens over $B^2\pi_1(\mathbf{D})$ in order to be able to pass to the cover $\widetilde{(\cdot)}$, as we now explain.

Recall that by construction the map $h_{\nu|_V}$ is the bottom left-to-right composite in the following diagram:

(5.1)
$$\begin{tikzcd}
 & & B\mathcal{C}_{\mathcal{N}}(\mu|_V) \arrow[ddll, "j_{\nu|_V}"'] \arrow[ddrr, "j'_{\nu'|_V}"] & & \\
 & & (B\widetilde{\mathcal{C}_{\mathcal{N}}(\mu|_V)}_1)/K \arrow[u, "\simeq"'] \arrow[dl] \arrow[dr] & & \\
B\mathcal{C}_X(\nu|_V) & (B\widetilde{\mathcal{C}_X(\nu|_V)}_1)/K \arrow[l, "\simeq"'] \arrow[rr, "\psi/K", "\simeq"'] & & (B\widetilde{\mathcal{C}_{X'}(\nu'|_V)}_1)/K \arrow[r, "\simeq"] & B\mathcal{C}_{X'}(\nu'|_V)
\end{tikzcd}$$

where we notice that all subdiagrams commute up to homotopy *over* BK.

Since BE maps into these spaces via ν and μ, adjointness produces the following diagram:

(5.2)
$$\begin{tikzcd}
 & & B\mathcal{C}_{\mathcal{N}}(\mu) \arrow[ddll, "j_{\nu}"'] \arrow[ddrr, "j'_{\nu'}"] & & \\
 & & B\mathcal{C}_{\mathcal{C}_{\mathcal{N}}(\mu|_V)}(\mu) \arrow[u, "\simeq"'] \arrow[dl] \arrow[dr] & & \\
B\mathcal{C}_X(\nu) & B\mathcal{C}_{\mathcal{C}_X(\nu|_V)}(\nu) \arrow[l, "\simeq"'] & Z \arrow[l, "\simeq"'] \arrow[r, "\mathcal{C}(\psi/K)", "\simeq"'] & B\mathcal{C}_{\mathcal{C}_{X'}(\nu'|_V)}(\nu') \arrow[r, "\simeq"] & B\mathcal{C}_{X'}(\nu')
\end{tikzcd}$$

where $Z = \mathrm{map}(BE, (B\widetilde{\mathcal{C}_X(\nu|_V)}_1)/K)_\nu$ and $\mathcal{C}(\psi/K)$ is the map induced by ψ/K on mapping spaces.

Since the diagram (5.2) homotopy commutes *as a diagram over* $B^2\pi_1(\mathbf{D})$, we get a homotopy commutative diagram by passing to homotopy fibers:

(5.3)
$$\begin{tikzcd}
 & & B\widetilde{\mathcal{C}_{\mathcal{N}}(\mu)} \arrow[ddll, "\widetilde{j_{\nu}}"'] \arrow[ddrr, "\widetilde{j'_{\nu'}}"] & & \\
 & & B\mathcal{C}_{\widetilde{\mathcal{C}_{\mathcal{N}}(\mu|_V)}}(\mu) \arrow[u, "\simeq"'] \arrow[dl] \arrow[dr] & & \\
B\widetilde{\mathcal{C}_X(\nu)} & B\mathcal{C}_{\widetilde{\mathcal{C}_X(\nu|_V)}}(\nu) \arrow[l, "\simeq"'] & \widetilde{Z} \arrow[l, "\simeq"'] \arrow[r, "\widetilde{\mathcal{C}(\psi/K)}", "\simeq"'] & B\mathcal{C}_{\widetilde{\mathcal{C}_{X'}(\nu'|_V)}}(\nu') \arrow[r, "\simeq"] & B\widetilde{\mathcal{C}_{X'}(\nu')}
\end{tikzcd}$$

Denote the bottom left-to-right homotopy equivalence in (5.3) by $\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})$, justified by the fact that by construction the following diagram homotopy commutes:

(5.4)
$$\begin{tikzcd}
B\widetilde{\mathcal{C}_X(\nu)} \arrow[r, "\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})", "\simeq"'] \arrow[d] & B\widetilde{\mathcal{C}_{X'}(\nu')} \arrow[d] \arrow[dr] & \\
B\widetilde{\mathcal{C}_X(\nu|_V)} \arrow[r, "\widetilde{h_{\nu|_V}}", "\simeq"'] & B\widetilde{\mathcal{C}_{X'}(\nu'|_V)} \arrow[r] & B\widetilde{X'}
\end{tikzcd}$$

Diagram (5.3), together with the inductive assumption ($\star$) and Proposition 3.1, shows that the homotopy class of $\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})$ does not depend on V. Hence by diagram (5.4) the same is true for $\widetilde{\varphi_{\nu,V}}$, which is what we wanted. (The key point in the above argument is that we can choose μ once and for all, such that $\mu|_V$ is a factorization of $\nu|_V$ through BT for every $V \subseteq E$.)

Note that the *construction* of $\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})$ in (5.3) does *not* depend on ν being toral, as long as ν' in that case is defined as $BE \xrightarrow{\nu} B\mathcal{C}_X(\nu) \xrightarrow{\varphi_{\nu,V}} BX'$ (instead of $j'\mu$), which makes sense in this more general setting—the top part of diagram (5.3) is only needed to conclude the independence of V. In Step 1b below we will also use the notation $\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})$ for non-toral ν.

Step 1b: Assume $\nu\colon BE \to BX$ is a rank two non-toral elementary abelian p–subgroup. By Proposition 5.2 either $\mathcal{C}_X(\nu)_1$ is non-trivial or $\mathbf{D}_X \cong \mathbf{D}_{\mathrm{PU}(p)\hat{_p}}$.

Assume first that $\mathbf{D}_X \cong \mathbf{D}_{\mathrm{PU}(p)\hat{_p}}$. Since uniqueness for this group is well-known, both for p odd and $p = 2$ by [22], [44], [11] and [8], the statement of course follows for this reason. But one can also argue directly, using a slight modification of the proof of [8, Lem. 3.2] which we quickly sketch: For $\alpha \in W_X(\nu)$ we have the following diagram:

$$\begin{tikzcd}
\widetilde{B\mathcal{C}_X(\nu)} \arrow[r] \arrow[d] & \widetilde{B\mathcal{C}_X(\nu|_V)} \arrow[r, "\widetilde{h_{\nu|_V}}", "\simeq"'] \arrow[d] & \widetilde{B\mathcal{C}_{X'}(j'\mu)} \arrow[r] \arrow[d] & \widetilde{BX'} \arrow[d, equal] \\
\widetilde{B\mathcal{C}_X(\nu)} \arrow[r] & \widetilde{B\mathcal{C}_X(\nu|_{\alpha(V)})} \arrow[r, "\widetilde{h_{\nu|_{\alpha(V)}}}", "\simeq"'] & \widetilde{B\mathcal{C}_{X'}(j'\mu\circ(\alpha^{-1}|_{\alpha(V)}))} \arrow[r] & \widetilde{BX'}
\end{tikzcd}$$

Here all the non-identity vertical maps are given on the level of mapping spaces (i.e., without the tilde) by $f \mapsto f \circ \alpha^{-1}$, which induces a map on the indicated spaces by taking homotopy fibers of the map to $B^2\pi_1(\mathbf{D})$. The left-hand and right-hand squares obviously commute, and the middle square commutes up to homotopy by our inductive assumption ($\star$) and Proposition 3.1. We thus conclude that $\widetilde{\varphi_{\nu,\alpha(V)}} \circ \widetilde{B\mathcal{C}_X(\alpha^{-1})} \simeq \widetilde{\varphi_{\nu,V}}$ for all $\alpha \in W_X(\nu)$. Now, since $\mathbf{D} \cong \mathbf{D}_{\mathrm{PU}(p)\hat{_p}}$ we have $\pi_1(\mathbf{D}) \cong \mathbb{Z}/p$, and because E is non-toral, we see that $B\mathcal{C}_X(\nu) \simeq BE$ and $\widetilde{B\mathcal{C}_X(\nu)} \simeq BP$, where P is the extra-special group p_+^{1+2} of order p^3 and exponent p if p is odd and Q_8 if $p = 2$. Now [8, Prop. 3.1] shows that $W_X(\nu)$ contains $\mathrm{SL}(E)$, and the same is true for X'. The proof of [8, Prop. 3.1] furthermore shows that $\widetilde{\varphi_{\nu,V}} \circ \widetilde{B\mathcal{C}_X(\alpha)} \simeq \widetilde{\varphi_{\nu,V}}$ for $\alpha \in \mathrm{SL}(E)$. (Apply the argument there to the p–group P instead of E.) Since $\mathrm{SL}(E)$ acts transitively on the rank one subgroups of E, combining the above gives that $\widetilde{\varphi_{\nu,V'}} \simeq \widetilde{\varphi_{\nu,V}}$ for any rank one subgroup $V' \subseteq E$, as desired.

We can therefore assume that $\mathcal{C}_X(\nu)_1$ is non-trivial, and the proof in this case is an adaptation of [8, Pf. of Lem. 3.3] to our new setting: Choose a rank one elementary abelian p–subgroup $\eta\colon BU = B\mathbb{Z}/p \to B\mathcal{C}_X(\nu)_1$ in the center of a p–normalizer of a maximal torus in $\mathcal{C}_X(\nu)$. Let $\eta \times \nu\colon BU \times BE \to BX$ be the map defined by adjointness, and for any rank one subgroup V of E, consider the map $\eta \times \nu|_V\colon BU \times BV \to BX$ obtained by restriction. By construction $\eta \times \nu|_V$ is the adjoint of the composite $BU \xrightarrow{\eta} B\mathcal{C}_X(\nu)_1 \xrightarrow{\mathrm{res}} B\mathcal{C}_X(\nu|_V)_1$, so $\eta \times \nu|_V\colon BU \times BV \to BX$ factors through a maximal torus in X by [26, Prop. 5.6].

It is furthermore straightforward to check that $\eta \times \nu$ is a monomorphism (compare [8, Pf. of Lem. 3.3]).

Now consider the following diagram:

$$\begin{tikzcd}
 & B\mathcal{C}_X(\nu|_V) \arrow[dr, "\varphi_{\nu|_V}"] & \\
BU \times BE \arrow[ur] \arrow[r] \arrow[dr] & B\mathcal{C}_X(\eta \times \nu|_V) \arrow[u] \arrow[d] & BX' \\
 & B\mathcal{C}_X(\eta) \arrow[ur, "\varphi_\eta"'] &
\end{tikzcd}$$

Here the left-hand side of the diagram is constructed by taking adjoints of $\eta \times \nu$, and hence it commutes. The right-hand side homotopy commutes by Step 1a (using the inductive assumption ($\star$)), since $\eta \times \nu|_V$ is toral of rank two. We can hence without ambiguity define $(\eta \times \nu)'$ as either the top left-to-right composite (for some rank one subgroup $V \subseteq E$) or the bottom left-to-right composite. Denote by ν' the restriction of $(\eta \times \nu)'$ to BE.

By construction of the map $\mathcal{C}_{\eta\times\nu}(\widetilde{h_{\nu|_V}})$ as the bottom composite in (5.3), the diagram

(5.5)
$$\begin{tikzcd}
B\widetilde{\mathcal{C}_X(\eta \times \nu)} \arrow[r, "\mathcal{C}_{\eta\times\nu}(\widetilde{h_{\nu|_V}})", "\simeq"'] \arrow[d] & B\widetilde{\mathcal{C}_{X'}((\eta \times \nu)')} \arrow[d] \\
B\widetilde{\mathcal{C}_X(\nu)} \arrow[r, "\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})", "\simeq"'] & B\widetilde{\mathcal{C}_{X'}(\nu')}
\end{tikzcd}$$

commutes. Furthermore, since $\eta \times \nu|_V$ is toral, diagram (5.3), applied with μ equal to a factorization of $\eta \times \nu|_V$ through BT, shows that the top horizontal map in (5.5) agrees with $\mathcal{C}_{\eta\times\nu}(\widetilde{h_\eta})$, and in particular it is independent of V, again using Proposition 3.1 and our inductive assumption ($\star$).

We claim that this forces the same to be true for the bottom horizontal map in (5.5): By our choice of η, the centralizer $\mathcal{C}_X(\eta \times \nu)$ contains a p–normalizer of a maximal torus in $\mathcal{C}_X(\nu)$, and hence $\widetilde{\mathcal{C}_X(\eta \times \nu)}$ contains a p–normalizer of a maximal torus in $\widetilde{\mathcal{C}_X(\nu)}$. Proposition 3.2 and our inductive assumption ($\star$) therefore shows that the bottom map in (5.5) is independent of V, so $\widetilde{\varphi_{\nu,V}}\colon B\widetilde{\mathcal{C}_X(\nu)} \xrightarrow{\mathcal{C}_\nu(\widetilde{h_{\nu|_V}})} B\widetilde{\mathcal{C}_{X'}(\nu')} \to B\widetilde{X'}$ is also independent of V, as wanted.

Step 1c: Assume $\nu\colon BE \to BX$ is an elementary abelian p–subgroup of rank ≥ 3. The fact that $\widetilde{\varphi_{\nu,V}}$ is independent of V when E has rank two implies the statement in general: Let $\nu\colon BE \to BX$ be an elementary abelian p–subgroup of rank at least three, and suppose that $V_1 \neq V_2$ are two rank one subgroups of E. Setting

$U = V_1 \oplus V_2$ we get the following diagram:

$$
\begin{tikzcd}
 & \widetilde{B\mathcal{C}_X(\nu|_{V_1})} \arrow[dr, "\widetilde{\varphi_{\nu|_{V_1}}}"] & \\
\widetilde{B\mathcal{C}_X(\nu)} \arrow[ur] \arrow[r] \arrow[dr] & \widetilde{B\mathcal{C}_X(\nu|_U)} \arrow[u] \arrow[d] & B\widetilde{X'} \\
 & \widetilde{B\mathcal{C}_X(\nu|_{V_2})} \arrow[ur, "\widetilde{\varphi_{\nu|_{V_2}}}"'] &
\end{tikzcd}
$$

The left-hand side of this diagram is constructed by adjointness and hence commutes, and the right-hand side of the diagram commutes up to homotopy by Steps 1a and 1b. Thus the top left-to-right composite $\widetilde{\varphi_{\nu,V_1}}$ is homotopic to the bottom left-to-right composite $\widetilde{\varphi_{\nu,V_2}}$, i.e., the map $\widetilde{\varphi_{\nu,V}}$ is independent of the choice of rank one subgroup V, as claimed.

Step 2: An element in $\lim^0$. With the above preparations in place it is easy to see, as in [8, Pf. of Thm. 2.2], that the maps $\widetilde{\varphi_\nu}\colon \widetilde{B\mathcal{C}_X(\nu)} \to B\widetilde{X'}$ fit together to form an element in

$$\lim{}^0_{\nu\in \mathbf{A}(X)}[\widetilde{B\mathcal{C}_X(\nu)}, B\widetilde{X'}].$$

In order not to cheat the reader of the finale, we repeat the short argument from [8, Pf. of Thm. 2.2]: For any morphism $\rho\colon (\nu\colon BE \to BX) \to (\xi\colon BF \to BX)$ in $\mathbf{A}(X)$ we need to verify that

$$
\begin{tikzcd}
\widetilde{B\mathcal{C}_X(\xi)} \arrow[rr, "\widetilde{B\mathcal{C}_X(\rho)}"] \arrow[dr, "\widetilde{\varphi_\xi}"'] & & \widetilde{B\mathcal{C}_X(\nu)} \arrow[dl, "\widetilde{\varphi_\nu}"] \\
 & B\widetilde{X'} &
\end{tikzcd}
$$

commutes up to homotopy. If F has rank one, then ρ is an isomorphism, and we let $\mu\colon BF \to BT \to B\mathcal{N}$ be a factorization of ξ through BT. In this case the claim follows since

(5.6)
$$
\begin{tikzcd}
\widetilde{B\mathcal{C}_X(\xi)} \arrow[r, "\widetilde{h_\xi}", "\simeq"'] \arrow[d, "\widetilde{B\mathcal{C}_X(\rho)}"', "\simeq"] & \widetilde{B\mathcal{C}_{X'}(j'\mu)} \arrow[d, "\widetilde{B\mathcal{C}_{X'}(\rho)}"', "\simeq"] \arrow[dr] & \\
\widetilde{B\mathcal{C}_X(\xi\rho)} \arrow[r, "\widetilde{h_{\xi\rho}}", "\simeq"'] & \widetilde{B\mathcal{C}_{X'}(j'\mu\rho)} \arrow[r] & B\widetilde{X'}
\end{tikzcd}
$$

commutes up to homotopy, since we can view the diagram as taking place under $\widetilde{B\mathcal{C}_{\mathcal{N}}(\mu)} \xrightarrow{\simeq} \widetilde{B\mathcal{C}_{\mathcal{N}}(\mu\rho)}$, up to homotopy, using the inductive assumption $(\star)$ and Proposition 3.1 as in Step 1a. The case where F has arbitrary rank follows from the independence of the choice of rank one subgroup V, established in Step 1, together with the rank one case: For a rank one subgroup V of E set $V' = \rho(V)$

and consider the diagram

$$\begin{tikzcd}
\widetilde{B\mathcal{C}_X(\xi)} \arrow[r] \arrow[d, "\widetilde{B\mathcal{C}_X(\rho)}"'] & \widetilde{B\mathcal{C}_X(\xi|_{V'})} \arrow[d, "\widetilde{B\mathcal{C}_X(\rho|_V)}"'] \arrow[dr] & \\
\widetilde{B\mathcal{C}_X(\nu)} \arrow[r] & \widetilde{B\mathcal{C}_X(\nu|_V)} \arrow[r] & \widetilde{BX'}
\end{tikzcd}$$

The left-hand side commutes by construction, and the right-hand side commutes since the diagram (5.6) commutes, proving the claim.

This constructs an element $[\vartheta] \in \lim^0_{\nu\in\mathbf{A}(X)}[\widetilde{B\mathcal{C}_X(\nu)}, \widetilde{BX'}]$, as wanted. $\square$

We now give the proof of Proposition 5.2, used in the proof of Theorem 5.1, which we postponed. The proof uses case-by-case arguments on the level of $\mathbb{Z}_p$–root data.

Proof of Proposition 5.2. Assume first that (W_X, L_X) is an exotic $\mathbb{Z}_p$–reflection group. We claim that the centralizer of any rank one elementary abelian p–subgroup of X is connected, which in particular implies that there can be no rank two non-toral elementary abelian p–subgroups: For $p > 2$ this follows from [8, Thms. 11.1, 12.2(2) and 7.1] combined with Dwyer–Wilkerson's formula for the Weyl group of a centralizer [27, Thm. 7.6]. For $p = 2$ we have $\mathbf{D}_X \cong \mathbf{D}_{\mathrm{DI}(4)}$ and hence $W_X \cong \mathbb{Z}/2\times\mathrm{GL}_3(\mathbb{F}_2)$, where the central $\mathbb{Z}/2$ factor acts by -1 on L_X and $\mathrm{GL}_3(\mathbb{F}_2)$ acts via the natural representation on $L_X\otimes\mathbb{F}_2$; cf. [8, Pf of Thm. 11.1] or [29, Rem. 7.2]. In particular it follows directly (cf. [29, Pf. of Prop. 9.12, DI(4) case]) that X contains a unique elementary abelian 2–subgroup of rank one up to conjugation and that the centralizer of this subgroup is connected.

By the classification of $\mathbb{Z}_p$–root data, Theorem 8.1(1), we may thus assume that $\mathbf{D}_X$ is of the form $\mathbf{D}_{G\hat{_p}}$ for some simple compact connected Lie group G, and since X is center-free we may assume that so is G. If $\pi_1(G)$ has no p–torsion, [62, Thm. 2.27] implies that the centralizer in G of any element of order p is connected. By the formula for the Weyl group of a centralizer [27, Thm. 7.6] (cf. Proposition 8.4(3)), it then follows that $\mathcal{C}_X(\eta)$ is connected for any rank one elementary abelian p–subgroup $\eta\colon B\mathbb{Z}/p \to BX$ of X, and hence X does not have any rank two non-toral elementary abelian p–subgroups.

We can thus furthermore assume that $\pi_1(G)$ has p–torsion, which implies that $\mathbf{D}_X \cong \mathbf{D}_{G\hat{_p}}$ for one of the following G: $G = \mathrm{PU}(n)$ for $p|n$; $G = \mathrm{SO}(2n+1)$, $n \geq 2$, for $p = 2$; $G = \mathrm{PSp}(n)$, $n \geq 3$, for $p = 2$; $G = \mathrm{PSO}(2n)$, $n \geq 4$, for $p = 2$; $G = PE_6$ for $p = 3$ and $G = PE_7$ for $p = 2$. We want to see that if $\eta\colon B\mathbb{Z}/p \to BX$ has rank one and $\mathbf{D}_X \ncong \mathbf{D}_{\mathrm{PU}(p)\hat{_p}}$, then $\mathcal{C}_X(\eta)$ is not a p–compact toral group, since then [24, Cor. 1.4] implies that for any elementary abelian p–subgroup ν of rank two, $\mathcal{C}_X(\nu)_1$ is non-trivial.

By [27, Thm. 7.6] it is enough to see this for the corresponding Lie group G. So, let $V \subseteq G$ be a rank one elementary abelian p–subgroup of G. If $C_G(V)_1$ is a torus, then $W_{C_G(V)_1} = 1$ and hence $W_{C_G(V)} \cong \pi_0(C_G(V))$. By [9, §5, Ex. 3(b)] or [61, Thm. 9.1(a)] $\pi_0(C_G(V))$ is isomorphic to a subgroup of $\pi_1(G)$, so $|W_{C_G(V)}|$ divides $|\pi_1(G)|$. In particular, if x is a generator of V, then the number of elements in a maximal torus which are conjugate to x in G is at least $|W_G|/|\pi_1(G)|$ (since two elements in a maximal torus are conjugate in G if and only if they are conjugate by a Weyl group element). In particular

$$p^n - 1 \geq \frac{|W_G|}{|\pi_1(G)|},$$

where n is the rank of G. A direct case-by-case check of the above cases shows that this inequality can only hold when $G = \mathrm{PU}(n)$, $p|n$. In this case, a generator of V must have the form $x = [\mathrm{diag}(\lambda_1, \ldots, \lambda_n)]$. For $n > p$, some of the λ's must agree, so $C_G(V)_1$ is not a torus in this case. This proves the claim. □

6. Third and final part of the proof of Theorem 1.2: Rigidification

In this section we finish the proof of Theorem 1.2 by showing that our element in $\lim^0$ from Theorem 5.1 rigidifies to produce a homotopy equivalence $BX \to BX'$. We first need a lemma:

Lemma 6.1. *Suppose that we have a homotopy pull-back square of p–compact groups*

$$\begin{array}{ccc} BX' & \xrightarrow{f'} & BY' \\ \big\downarrow{\scriptstyle g'} & & \big\downarrow{\scriptstyle g} \\ BX & \xrightarrow{f} & BY \end{array}$$

where $f\colon BX \to BY$ is a centric monomorphism (i.e., $\mathrm{map}(BX, BX)_1 \xrightarrow{\simeq} \mathrm{map}(BX, BY)_f$ and Y/X is $\mathbb{F}_p$–finite) and $g\colon BY' \to BY$ is an epimorphism (i.e., Y/Y' is the classifying space of a p–compact group). Then $f'\colon BX' \to BY'$ is a centric monomorphism, and g' is an epimorphism.

Proof. It is clear from the pull-back square that f' is a monomorphism and that g' is an epimorphism. To see that f' is centric, observe that we have a map of fibrations

$$\begin{array}{ccccc} (Y/Y')^{hX'} & \longrightarrow & \mathrm{map}(BX', BX')_{[g']} & \longrightarrow & \mathrm{map}(BX', BX)_{g'} \\ \Big\| & & \big\downarrow & & \big\downarrow \\ (Y/Y')^{hX'} & \longrightarrow & \mathrm{map}(BX', BY')_{[f\circ g']} & \longrightarrow & \mathrm{map}(BX', BY)_{f\circ g'} \end{array}$$

where, e.g., the subscript $[g']$ denotes the components of the mapping space mapping to the component of g'. Since the wanted map

$$\mathrm{map}(BX', BX')_1 \to \mathrm{map}(BX', BY')_{f'}$$

is the restriction of the map of total spaces to a component, it is hence enough to see that the map between the base spaces is an equivalence, which follow since we have equivalences

$$\begin{array}{ccc} \mathrm{map}(BX, BX)_1 & \xrightarrow{\simeq} & \mathrm{map}(BX', BX)_{g'} \\ \big\downarrow{\scriptstyle\simeq} & & \big\downarrow \\ \mathrm{map}(BX, BY)_f & \xrightarrow{\simeq} & \mathrm{map}(BX', BY)_{f\circ g'} \end{array}$$

Here the vertical equivalence follows from the centricity of f and the horizontal equivalences follows from [19, Prop. 3.5] combined with [27, Prop. 10.1] (in [19, Prop. 3.5], taking $E = BX'$, $B = BX$, and $X = BX$ and BY respectively). □

Proof of Theorem 1.2. By the results of Section 2 (Propositions 2.1 and 2.4) we are reduced to the case where we consider two simple, center-free p–compact groups BX and BX' with the same root datum $\mathbf{D}$. Furthermore, in Theorem 5.1 of the previous section we constructed an element $[\vartheta] \in \lim^0_{\nu \in \mathbf{A}(X)}[\widetilde{B\mathcal{C}_X(\nu)}, \widetilde{BX'}]$. We want to use this element to construct the map from BX to BX', and show that it is an equivalence. By the classification of $\mathbb{Z}_p$–root data, Theorem 8.1(1), $\mathbf{D}$ is either exotic or $\mathbf{D} \cong \mathbf{D}_{G^\wedge_p}$ for a compact connected Lie group G, and we handle these two cases separately.

Suppose that $\mathbf{D}$ is exotic, and notice that in this case $\pi_1(\mathbf{D}) = 0$ by Theorem 8.1(2). If p is odd we are exactly in the situation covered by rather easy arguments in [8] (see [8, Pf. of Thm. 2.2] and [8, Prop. 9.5]). For $p = 2$ uniqueness of $B\,\mathrm{DI}(4)$, as well as the statements about self-maps, are already well-known by the work of Notbohm [56], but we nevertheless quickly remark how a proof also falls out of the current setup, noticing that the arguments from [8] from this point on, in the special case of $\mathrm{DI}(4)$, carry over verbatim: We have $\mathbf{D} \cong \mathbf{D}_{\mathrm{DI}(4)}$, and since $\pi_1(\mathbf{D}) = 0$, the functor $\nu \mapsto \pi_i(\mathrm{map}(\widetilde{B\mathcal{C}_X(\nu)}, \widetilde{BX'})_\nu)$ identifies with $\nu \mapsto \pi_i(B\mathcal{Z}(\mathcal{C}_X(\nu)))$, as explained in detail in [8, Pf. of Thm. 2.2]. Since we are considering $X = \mathrm{DI}(4)$ we know by Dwyer–Wilkerson [25, Prop. 8.1] that $\lim^j_{\mathbf{A}(X)} \pi_i(B\mathcal{Z}(\mathcal{C}_X(\nu))) = 0$ for all $i, j \geq 0$. (The proof of this is a Mackey functor argument, and relies on the regular structure of the Quillen category of $\mathrm{DI}(4)$, due to the fact that its classifying space, like the exotic p–compact groups for p odd, has polynomial $\mathbb{F}_p$–cohomology ring.) The centralizer decomposition theorem [27, Thm. 8.1] now produces a map $BX \to BX'$ which, by standard arguments given in [8, Pf. of Thm. 2.2], is seen to be an equivalence. The statement about self-maps also follows as in [8, Pf. of Thm. 2.2], using $\mathrm{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ instead of $\mathrm{Out}(B\mathcal{N})$.

Now, suppose that $\mathbf{D} \cong \mathbf{D}_{G^\wedge_p}$, for a simple center-free compact Lie group G, with universal cover $\widetilde{G}$. Let $\mathbf{O}^r_p(\widetilde{G})$ be the full subcategory of the orbit category with objects the $\widetilde{G}$–sets $\widetilde{G}/\widetilde{P}$ with $\widetilde{P}$ a p–radical subgroup (i.e, $\widetilde{P}$ is an extension of a torus by a finite p–group, such that $N_{\widetilde{G}}(\widetilde{P})/\widetilde{P}$ is finite and contains no non-trivial normal p–subgroups).

For the p–radical homology decomposition [33, Thm. 4] of the compact Lie group $\widetilde{G}$, one considers the functor $F\colon\ \mathbf{O}^r_p(\widetilde{G}) \to \mathbf{Spaces}$ given by $\widetilde{G}/\widetilde{P} \mapsto E\widetilde{G} \times_{\widetilde{G}} \widetilde{G}/\widetilde{P}$, where **Spaces** denotes the category of topological spaces. Viewed as a functor to the *homotopy category* of spaces, $\mathrm{Ho}(\mathbf{Spaces})$, this functor is isomorphic to the functor $F'\colon\ \mathbf{O}^r_p(\widetilde{G}) \to \mathrm{Ho}(\mathbf{Spaces})$ given on objects by $\widetilde{G}/\widetilde{P} \mapsto B\widetilde{P}$ and on morphisms by sending the $\widetilde{G}$–set map $\widetilde{G}/\widetilde{P} \xrightarrow{f} \widetilde{G}/\widetilde{Q}$ to the map $c_{g^{-1}}\colon\ B\widetilde{P} \to B\widetilde{Q}$, where $g\widetilde{Q} = f(e\widetilde{P})$, via the canonical equivalences $B\widetilde{P} = (E\widetilde{P})/\widetilde{P} \xrightarrow{\simeq} E\widetilde{G} \times_{\widetilde{G}} \widetilde{G}/\widetilde{P}$. (Note that F' is not well-defined as a functor to **Spaces**, since the element g is just an arbitrary coset representative for the coset $g\widetilde{Q}$.) We can hence in what follows replace $E\widetilde{G} \times_{\widetilde{G}} \widetilde{G}/\widetilde{P}$ by $B\widetilde{P}$ in this way, whenever we are working in the homotopy category.

Since $\widetilde{P}$ and $\widetilde{Q}$ are p–radical, the same is true for their images P and Q in G, and $C_G(P) = Z(P)$ and likewise for Q (see [33, Prop. 1.6(i) and Lem. 1.5(ii)]). Hence there is a well-defined induced morphism $c_g\colon\ {}_pZ(Q) \to {}_pZ(P)$ as well as a well-defined (free) homotopy class of maps $c_{g^{-1}}\colon\ B\widetilde{C_G({}_pZ(P))} \to B\widetilde{C_G({}_pZ(Q))}$.

Consider the diagram

$$
\begin{tikzcd}
B\widetilde{P}^\wedge_p \arrow[r] \arrow[d, "c_{g^{-1}}"] & (BC_G(\widetilde{{}_pZ(P)}))^\wedge_p \arrow[r, "\simeq"] \arrow[d, "c_{g^{-1}}"] & B\mathcal{C}_{G^\wedge_p}(\widetilde{i_{({}_pZ(P))}}) \arrow[d, "B\mathcal{C}_{G^\wedge_p}(c_g)"'] \arrow[dr, "\varphi_{\widetilde{i_{({}_pZ(P))}}}"] & \\
B\widetilde{Q}^\wedge_p \arrow[r] & (BC_G(\widetilde{{}_pZ(Q)}))^\wedge_p \arrow[r, "\simeq"] & B\mathcal{C}_{G^\wedge_p}(\widetilde{i_{({}_pZ(Q))}}) \arrow[r, "\varphi_{\widetilde{i_{({}_pZ(Q))}}}"'] & B\widetilde{X'}
\end{tikzcd}
$$

where $i_V\colon BV \to BG^\wedge_p$ denotes the map induced by the inclusion of a subgroup $V \subseteq G$, and where the horizontal maps in the middle square are given by lifting the standard homotopy equivalences given by adjointness to the covers. The first two squares are homotopy commutative by construction, and the right-hand triangle commutes since $[\vartheta] \in \lim^0_{\nu\in\mathbf{A}(X)}[\widetilde{B\mathcal{C}_X(\nu)}, B\widetilde{X'}]$. Hence this diagram produces an element $[\zeta] \in \lim^0_{\widetilde{G}/\widetilde{P}\in\mathbf{O}^r_p(\widetilde{G})}[B\widetilde{P}^\wedge_p, B\widetilde{X'}]$. Denote the composition

$$B\widetilde{P}^\wedge_p \to BC_G(\widetilde{{}_pZ(P)})^\wedge_p \to B\mathcal{C}_{G^\wedge_p}(\widetilde{i_{{}_pZ(P)}}) \xrightarrow{\varphi_{\widetilde{i_{({}_pZ(P))}}}} B\widetilde{X'}$$

by $\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}$ (i.e., the coordinate of $[\zeta]$ corresponding to $\widetilde{G}/\widetilde{P}$), and let $\psi_{G/P}\colon BP^\wedge_p \to BX'$ be the map constructed analogously, using $\varphi_{i_{({}_pZ(P))}}$ instead. We want to lift $[\zeta]$ to a map

$$\operatorname{hocolim}_{\widetilde{G}/\widetilde{P}\in\mathbf{O}^r_p(\widetilde{G})}(E\widetilde{G}\times_{\widetilde{G}}\widetilde{G}/\widetilde{P})^\wedge_p \to B\widetilde{X'}.$$

By [10, Prop. XII.4.1 and XI.7.1] (see also [69, Prop. 3] or [35, Prop. 1.4]) the obstructions to doing this lie in

$$\lim{}^{i+1}_{\widetilde{G}/\widetilde{P}\in\mathbf{O}^r_p(\widetilde{G})}\pi_i(\operatorname{map}(B\widetilde{P}^\wedge_p, B\widetilde{X'})_{\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}}), \quad i \geq 1.$$

By construction, $\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}$ and $\psi_{G/P}$ fit into a homotopy pull-back square:

$$
\begin{tikzcd}
B\widetilde{P}^\wedge_p \arrow[r, "\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}"] \arrow[d] & B\widetilde{X'} \arrow[d] \\
BP^\wedge_p \arrow[r, "\psi_{G/P}"] & BX'
\end{tikzcd}
$$

By [12, Lem. 3.8] the map $\psi_{G/P}$ is centric, so Lemma 6.1 implies that $\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}$ is centric as well. Hence by centricity and naturality, the functor

$$\widetilde{G}/\widetilde{P} \mapsto \pi_i(\operatorname{map}(B\widetilde{P}^\wedge_p, B\widetilde{X'})_{\widetilde{\psi}_{\widetilde{G}/\widetilde{P}}})$$

identifies with the functor $\widetilde{G}/\widetilde{P} \mapsto \pi_{i-1}(Z(\widetilde{P})^\wedge_p)$. Since $\widetilde{G}$ is simple and simply connected, it now follows from the fundamental calculations of Jackowski–McClure–Oliver [33, Thm. 4.1] that

$$\lim{}^{i+1}_{\mathbf{O}^r_p(\widetilde{G})}\pi_{i-1}(Z(\widetilde{P})^\wedge_p) = 0, \text{ for } i \geq 1.$$

Hence by the homology decomposition theorem [33, Thm. 4] we get a map

$$B\widetilde{G}^\wedge_p \xleftarrow{\simeq} (\operatorname{hocolim}_{\widetilde{G}/\widetilde{P}\in\mathbf{O}^r_p(\widetilde{G})}(E\widetilde{G}\times_{\widetilde{G}}\widetilde{G}/\widetilde{P})^\wedge_p)^\wedge_p \to B\widetilde{X'},$$

which by construction is a map under $B\widetilde{\mathcal{N}}_p$, the p–normalizer of a maximal torus in $B\widetilde{G}\hat{_p}$. Dividing out by $\mathcal{Z}(\mathbf{D})$ as explained in Construction 2.2 produces the homotopy commutative diagram

$$\begin{tikzcd} & B\mathcal{N}_p \arrow[dl] \arrow[dr] & \\ BG\hat{_p} \arrow[rr] & & BX' \end{tikzcd}$$

It is now a short argument, given in detail in [8, Pf. of Thm. 2.2], to see that $BX = BG\hat{_p} \to BX'$ is a homotopy equivalence as wanted.

We want to show that $\operatorname{Out}(BX) \to \operatorname{Out}(\mathbf{D}_X)$ is an isomorphism: To see surjectivity, note that if $\alpha \in \operatorname{Out}(\mathbf{D}_X)$, then by [7, Thm. C] and [8, Prop. 5.1], α corresponds to a unique map $\alpha' \in \operatorname{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$. Hence if $j\colon B\mathcal{N} \to BX$ is a maximal torus normalizer, then repeating the above argument with respect to the two maps $j\colon B\mathcal{N} \to BX$ and $j \circ \alpha'\colon B\mathcal{N} \to BX$ gives a map $BX \to BX$ realizing $\alpha \in \operatorname{Out}(\mathbf{D}_X)$. Finally we show injectivity, essentially repeating the argument of Jackowski–McClure–Oliver (cf. [33, Pf. of Thm. 4.2]): We are assuming that $BX \simeq BG\hat{_p}$, for some compact connected center-free Lie group G. As in the proof of Proposition 2.4(3), we have the following commutative diagram:

$$\begin{tikzcd} \operatorname{Out}(BG\hat{_p}) \arrow[r] \arrow[d] & \operatorname{Out}(\mathbf{D}_{G\hat{_p}}) \arrow[d] \\ \operatorname{Out}(B\widetilde{G}\hat{_p}) \arrow[r] & \operatorname{Out}(\mathbf{D}_{\widetilde{G}\hat{_p}}) \end{tikzcd}$$

(Compare diagram (2.1), and note that we use the fact that $\widetilde{\mathbf{D}_{G\hat{_p}}} = \mathbf{D}_{\widetilde{G\hat{_p}}}$ for the right-hand vertical map, which uses Theorem 8.6 for compact Lie groups.) The left-hand vertical map in the above diagram is injective, since factoring out by the center (via the quotient construction recalled in Construction 2.2) provides a left inverse (in fact an actual inverse, though we do not need this here). So we just have to see that $\operatorname{Out}(B\widetilde{G}\hat{_p}) \to \operatorname{Out}(\mathbf{D}_{\widetilde{G}\hat{_p}})$ is injective. By [7, Thm. C], this map factors through $\operatorname{Out}(B\widetilde{\mathcal{N}}, \{B\widetilde{\mathcal{N}}_\sigma\})$ and $\operatorname{Out}(B\widetilde{\mathcal{N}}, \{B\widetilde{\mathcal{N}}_\sigma\}) \to \operatorname{Out}(\mathbf{D}_{\widetilde{G}\hat{_p}})$ is an isomorphism, where $\widetilde{\mathcal{N}}$ denotes the maximal torus normalizer in $\widetilde{G}\hat{_p}$. By the homology decomposition theorem [33, Thm. 4] and obstruction theory [33, Thm. 3.9] (cf. [69, Prop. 3]), injectivity of $\operatorname{Out}(B\widetilde{G}\hat{_p}) \to \operatorname{Out}(B\widetilde{\mathcal{N}})$ follows from Jackowski–McClure–Oliver's calculation of higher limits [33, Thm. 4.1]:

$$\lim{}^i_{\widetilde{G}/\widetilde{P} \in \mathbf{O}_p^r(\widetilde{G})} \pi_{i-1}(Z(\widetilde{P})\hat{_p}) = 0, \quad i \geq 1.$$

We conclude that $\operatorname{Out}(BG\hat{_p}) \to \operatorname{Out}(\mathbf{D}_{G\hat{_p}})$ is also injective, as claimed.

Finally, the last statement in Theorem 1.2 concerning the homotopy type of $B\operatorname{Aut}(BX)$ follows by combining [7] with what we have proved so far: The Adams–Mahmud map factors as $\Phi\colon B\operatorname{Aut}(BX) \to B\operatorname{Aut}(B\mathcal{N}, \{B\mathcal{N}_\sigma\}) \to B\operatorname{Aut}(B\mathcal{N})$, where $B\operatorname{Aut}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ is the covering of $B\operatorname{Aut}(B\mathcal{N})$ with respect to the subgroup $\operatorname{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$ of the fundamental group. Furthermore [7, §5] explains how killing elements in $\pi_2(B\operatorname{Aut}(B\mathcal{N}, \{B\mathcal{N}_\sigma\}))$ constructs a space denoted $B\operatorname{aut}(\mathbf{D}_X)$, whose universal cover is $B^2\mathcal{Z}(\mathbf{D}_X)$, where $B\mathcal{Z}(\mathbf{D}_X) = (B\breve{\mathcal{Z}}(\mathbf{D}_X))\hat{_p}$. It it furthermore shown there that the fibration $B\operatorname{aut}(\mathbf{D}_X) \to B\operatorname{Out}(\mathbf{D}_X)$ is split, i.e., $B\operatorname{aut}(\mathbf{D}_X)$ has the homotopy type of $(B^2\mathcal{Z}(\mathbf{D}_X))_{h\operatorname{Out}(\mathbf{D}_X)}$. Composition gives a map $B\operatorname{Aut}(BX) \to B\operatorname{aut}(\mathbf{D}_X)$, which is an isomorphism on π_i for $i > 1$ by construction and an isomorphism on π_1 by what we have shown in the first part of the theorem.

This concludes the proof of the main Theorem 1.2. □

Remark 6.2 (The fundamental group of a p–compact group). In [8], with Møller and Viruel, we established the fundamental group formula for p–compact groups, Theorem 8.6, for p odd, as a consequence of the classification. In this remark we sketch how one, by modifying the proof of Theorem 1.2, can avoid the reliance on Theorem 8.6, hence proving Theorem 8.6 also for $p = 2$ in this manner. This was the strategy which we had originally envisioned before Dwyer–Wilkerson [30] provided an alternative direct proof of Theorem 8.6 as we were writing this paper.

The fundamental group formula was not used in the reduction to simple center-free groups, except for a reference in the proof of Proposition 2.4(3), where it, for the purpose of inductively proving Theorem 1.2, was only needed in the well-known case of compact Lie groups. Hence we can assume that X and X' are simple center-free p–compact groups with the same $\mathbb{Z}_p$–root datum $\mathbf{D}$ and that we know the fundamental group formula for X and Theorem 1.2 for connected p–compact groups of lower cohomological dimension than X. In the last section we constructed an element $[\vartheta] \in \lim^0_{\nu\in\mathbf{A}(X)}[\widetilde{B\mathcal{C}_X(\nu)}, \widetilde{BX'}]$. By not passing to a universal cover, and hence not using the fundamental group formula, a simplified version of the same argument gives an element $[\bar\vartheta] \in \lim^0_{\nu\in\mathbf{A}(X)}[B\mathcal{C}_X(\nu), BX']$. The only non-obvious change is that since the center formula in Lemma 8.8 does not hold in the presence of direct factors isomorphic to $\mathbf{D}_{\mathrm{SO}(2n+1)_{\hat 2}}$ in the $\mathbb{Z}_2$–root datum, one has to take the target of Ξ in (4.4) to be $B\mathcal{Z}(\mathbf{D}_{\mathcal{C}_X(\nu)_1})$ (obtained as a quotient of $B\mathcal{Z}(\mathcal{C}_{\mathcal{N}}(\mu)_1)$; cf. [7, Lem. 5.1]) to obtain a homotopy equivalence. With this change the rest of Section 4 proceeds as before, but ignoring everything on the level of covers, and one constructs a map as before without any choices. Section 5 has to be modified in the following way: Instead of having maps under the maximal torus normalizer (which we now do not a priori know), we utilize instead that the maps agree in

$$M = (\operatorname{map}(B\pi, B\operatorname{aut}(\mathbf{D})))_{h\operatorname{Aut}(B\pi)},$$

where $B\operatorname{aut}(\mathbf{D}) = (B^2\mathcal{Z}(\mathbf{D}))_{h\operatorname{Out}(\mathbf{D})}$. This means that they are homotopy equivalent, by the description of self-equivalences of non-connected groups (cf. Theorem 1.6), which we know by induction. From the element in $\lim^0$ one can easily get an isomorphism between fundamental groups: Consider the diagram

$$\begin{tikzcd}
& H_*^{\mathbb{Z}_p}(B\mathcal{N}_p) \arrow[dl, two heads, "j"'] \arrow[d] \arrow[dr, two heads, "j'"] & \\
H_*^{\mathbb{Z}_p}(BX) & {}_{\nu\in\mathbf{A}(X)}H_*^{\mathbb{Z}_p}(B\mathcal{C}_X(\nu)) \arrow[l, "\cong"'] \arrow[r, "{[\bar\vartheta]}"] & H_*^{\mathbb{Z}_p}(BX')
\end{tikzcd}$$

where the vertical map is given by choosing a central rank one subgroup ρ of the p–normalizer $\mathcal{N}_p$ and considering the corresponding inclusion $\mathcal{N}_p \to \mathcal{C}_X(j\circ\rho)$. (Here $H_n^{\mathbb{Z}_p}(Y) = \lim H_n(Y;\mathbb{Z}/p^n)$.) Note that the maps j and j' are surjective by a transfer argument, and that the indicated isomorphism follows by the centralizer homology decomposition theorem [27, Thm. 8.1]. This shows that the kernel of j is contained in the kernel of j'. However, since we could reverse the role of X

and X' in all the previous arguments (we have not used any special model for X), we conclude by symmetry that the kernel of j equals the kernel of j', and in particular the kernels of the maps $j\colon \pi_1(\mathbf{D}) \to \pi_1(X)$ and $j'\colon \pi_1(\mathbf{D}) \to \pi_1(X')$ agree, and they are surjective by Proposition 8.5. If $\mathbf{D}$ is exotic, then $\pi_1(\mathbf{D}) = 0$ by Theorem 8.1(2), so there is nothing to prove. If $\mathbf{D}$ is not exotic, then by Theorem 8.1(1), $\mathbf{D} = \mathbf{D}_G \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a compact connected Lie group G, and so we can assume $BX \simeq BG\hat{_p}$. Hence $j\colon \pi_1(\mathbf{D}) \xrightarrow{\cong} \pi_1(X)$ by Lie theory (cf. [9, §4, no. 6, Prop. 11]), so the same holds for X'.

7. Proof of the corollaries of Theorem 1.2

In this section we prove the Theorems 1.3–1.6 from the Introduction. The first theorem is the maximal torus conjecture:

Proof of Theorem 1.3. The proof in the connected case is an extension of [8, Pf. of Thm. 1.10], where a partial result excluding the case $p = 2$ was given: Assume that (X, BX, e) is a *connected* finite loop space with a maximal torus $i\colon BT \to BX$. Let W denote the set of conjugacy classes of self-equivalences φ of BT such that $i\varphi$ is conjugate to i. It is straightforward to see (consult, e.g., [8, Pf. of Thm. 1.10]) that $BT\hat{_p} \to BX\hat{_p}$ is a maximal torus for the p–compact group $X\hat{_p}$ and that $\mathbb{F}_p$–completion allows us to identify $(W_{X\hat{_p}}, L_{X\hat{_p}})$ with $(W, L \otimes \mathbb{Z}_p)$, where $L = \pi_1(T)$. In particular (W, L) is a finite $\mathbb{Z}$–reflection group and all reflections have order 2. Furthermore, for a fixed $\mathbb{Z}$–reflection group (W, L), there is a bijection between $\mathbb{Z}$–root data with underlying $\mathbb{Z}$–reflection group (W, L) and $\mathbb{Z}_2$–root data with underlying $\mathbb{Z}_2$–reflection group $(W, L \otimes \mathbb{Z}_2)$ given by the assignments $\mathbf{D} \mapsto \mathbf{D} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and $(W, L \otimes \mathbb{Z}_2, \{\mathbb{Z}_2 b_\sigma\}) \mapsto (W, L, \{L \cap \mathbb{Z}_2 b_\sigma\})$, as is seen by examining the definitions. Let $\mathbf{D}$ be the $\mathbb{Z}$–root datum with underlying $\mathbb{Z}$–reflection group (W, L) corresponding to $\mathbf{D}_{X\hat{_2}}$.

By the classification of compact connected Lie groups (cf. [9, §4, no. 9, Prop. 16]), there is a (unique) compact connected Lie group G with maximal torus $i'\colon T \to G$ inducing an isomorphism of $\mathbb{Z}$–root data $\mathbf{D}_G \cong \mathbf{D}$. The $\mathbb{Z}_p$–root data of $X\hat{_p}$ and $G\hat{_p}$ are isomorphic at all primes, since the root data at odd primes are determined by $(W, L \otimes \mathbb{Z}_p)$. Theorem 1.2 hence implies that for each p, we have a homotopy equivalence $\varphi_p\colon BX\hat{_p} \to BG\hat{_p}$ such that

$$\begin{tikzcd} & BT\hat{_p} \arrow[dl, "i\hat{_p}"'] \arrow[dr, "i'\hat{_p}"] & \\ BX\hat{_p} \arrow[rr, "\varphi_p"] & & BG\hat{_p} \end{tikzcd}$$

commutes. As in [8, Pf. of Thm. 1.10] we see that $H^*(BX; \mathbb{Q}) \to H^*(BT; \mathbb{Q})^W$ is an isomorphism, and since the same is true for BG we also have an equivalence $BX_{\mathbb{Q}} \to BG_{\mathbb{Q}}$ under BT. We have the following diagram:

$$\begin{tikzcd} \prod_p BX\hat{_p} \arrow[r] \arrow[d, "\simeq"] & (\prod_p BX\hat{_p})_{\mathbb{Q}} \arrow[d, "\simeq"] & BX_{\mathbb{Q}} \arrow[l] \arrow[d, "\simeq"] \\ \prod_p BG\hat{_p} \arrow[r] & (\prod_p BG\hat{_p})_{\mathbb{Q}} & BG_{\mathbb{Q}} \arrow[l] \end{tikzcd} \tag{7.1}$$

The left-hand square in this diagram is homotopy commutative by construction. For the right-hand side note that, since all maps in (7.1) are under BT, the following diagram commutes:

$$\begin{tikzcd}
H_*(\prod_p BX\hat{}_p;\mathbb{Q}) \arrow[dd, "\cong"] & & H_*(BX;\mathbb{Q}) \arrow[ll] \arrow[dd, "\cong"] \\
& H_*(BT;\mathbb{Q})_W \arrow[ul] \arrow[ur, "\cong"] \arrow[dl] \arrow[dr, "\cong"] & \\
H_*(\prod_p BG\hat{}_p;\mathbb{Q}) & & H_*(BG;\mathbb{Q}) \arrow[ll]
\end{tikzcd}$$

This implies commutativity on the level of homotopy groups, and since the involved spaces are all products of rational Eilenberg–Mac Lane spaces (since they have homotopy groups only in even degrees), this implies that the diagram (7.1) homotopy commutes. By changing the maps up to homotopy we can hence arrange that the diagram strictly commutes, and taking homotopy pull-backs produces a homotopy equivalence $BX \to BG$, by the Sullivan arithmetic square [10, VI.8.1], as wanted. This proves that every connected finite loop space BX with a maximal torus is homotopy equivalent to BG for some compact connected Lie group G, and in the course of the analysis, we furthermore saw that G is unique (since we can reconstruct the $\mathbb{Z}$–root datum of G from BX).

We now give the description of $B\operatorname{Aut}(BG)$, also providing a quick description of how the results for the corresponding p–completed spaces are used. By [27, Thm. 1.4] $B^2Z(G) \simeq B\operatorname{Aut}_1(BG)$. (This is a consequence of the equivalence $B^2\mathcal{Z}(G\hat{}_p) \simeq B\operatorname{Aut}_1(BG\hat{}_p)$.) By [34, Cor. 3.7] $\operatorname{Out}(G) \cong \operatorname{Out}(BG)$. (Since $(G/T)^{hT}$ is homotopically discrete with components the Weyl group, as in [8, Lem. 4.1] and [52, Thm. 2.1], any map $f\colon BG \to BG$ gives rise to a map $\varphi\colon BT \to BT$ over f, unique up to Weyl group conjugation and since $\varphi\hat{}_p \in \operatorname{Aut}(\mathbf{D}_{G\hat{}_p})$ for all p one sees that $\varphi \in \operatorname{Aut}(\mathbf{D}_G)$. Now φ determines the collection $\{\varphi\hat{}_p\}$ which determines $\{f\hat{}_p\}$, which again determines f by the arithmetic square.) Finally by a theorem of de Siebenthal [15, Ch. I, §2, no. 2] (see also [9, §4, no. 10]), the short exact sequence $1 \to G/Z(G) \to \operatorname{Aut}(G) \to \operatorname{Out}(G) \to 1$ is split, so the fibration $B\operatorname{Aut}(BG) \to B\operatorname{Out}(BG)$ has a section. (See also [7, §6].) Taken together, these facts establish that $B\operatorname{Aut}(BG) \simeq (B^2Z(G))_{h\operatorname{Out}(G)}$, as claimed.

The non-connected case follows easily from the connected, using our knowledge of self-maps of classifying spaces of compact connected Lie groups: Suppose that X is potentially non-connected, let X_1 be the identity component, and let π be the component group. Note that homotopy classes of spaces Y such that $P_1(Y)$ is homotopy equivalent to $B\pi$ and the fiber $Y\langle 1\rangle$ is homotopy equivalent to BX_1 can alternatively be described as homotopy classes of fibrations $Z \to Y \to K$, with Z homotopic to BX_1 and K homotopic to $B\pi$. (A homotopy equivalence of fibrations means a compatible triple of homotopy equivalences between fibers, total spaces, and base spaces.)

By the first part of the proof we know that $BX_1 \simeq BG_1$ for a unique compact connected Lie group G_1. Homotopy classes of fibrations with base homotopy equivalent to $B\pi$ and fiber homotopy equivalent to BX_1 are classified by $\operatorname{Out}(\pi)$–orbits on the set of free homotopy classes $[B\pi, B\operatorname{Aut}(BG_1)]$. By the results in the connected case the space $B\operatorname{Aut}(BG_1)$ sits in a split fibration

$$B^2Z(G_1) \to B\operatorname{Aut}(BG_1) \to B\operatorname{Out}(G_1).$$

Hence $\operatorname{Out}(\pi)$–orbits on the set

$$[B\pi, B\operatorname{Aut}(BG_1)]$$

correspond to $(\operatorname{Out}(\pi)\times\operatorname{Out}(G_1))$–orbits on the set

$$\coprod_{\alpha\in\operatorname{Hom}(\pi,\operatorname{Out}(G_1))} H^2_\alpha(\pi; Z(G_1)).$$

This agrees with the classification of isomorphism classes of group extensions of the form $1\to H\to ?\to K\to 1$, where H is isomorphic to G_1 and K is isomorphic to π. (Here an isomorphism of group extensions is a compatible triple of isomorphisms.) Since the identity component H is necessarily a characteristic subgroup, isomorphism classes of group extensions as above are in one-to-one correspondence with isomorphism classes of compact Lie groups with identity component isomorphic to G_1 and component group isomorphic to π. These equivalences put together prove that there is a one-to-one correspondence between homotopy classes of spaces Y such that $P_1(Y)\simeq B\pi$ and $Y\langle 1\rangle \simeq BG_1$ and isomorphism classes of compact Lie groups with identity component isomorphic to G_1 and component group isomorphic to π. Hence our BX is homotopy equivalent to BG for a unique compact Lie group G, completing the proof of the theorem. $\square$

Proof of Theorem 1.4. Let Y be a space such that $H^*(Y;\mathbb{F}_2)$ is a graded polynomial algebra of finite type. Let $V = H_1(Y;\mathbb{F}_2)$ (dual to $H^1(Y;\mathbb{F}_2)$) and let Y' denote the fiber of the classifying map $Y\to BV$. Clearly Y' is connected, and since $\pi_1(BV)$ is a finite 2–group it follows from [18] that the Eilenberg–Moore spectral sequence for the fibration $Y'\to Y\to BV$ converges strongly to $H^*(Y';\mathbb{F}_2)$. The map $H^*(BV;\mathbb{F}_2)\to H^*(Y;\mathbb{F}_2)$ is an isomorphism in degree 1 and hence injective since $H^*(Y;\mathbb{F}_2)$ is a polynomial algebra, so $H^*(Y;\mathbb{F}_2)$ is free over $H^*(BV;\mathbb{F}_2)$. Hence the spectral sequence collapses and we get an isomorphism of rings (but not necessarily of algebras over the Steenrod algebra) $H^*(Y;\mathbb{F}_2)\cong H^*(Y';\mathbb{F}_2)\otimes H^*(BV;\mathbb{F}_2)$. In particular $H_1(Y';\mathbb{F}_2)=0$, so by [10, Prop. VII.3.2] Y' is $\mathbb{F}_2$–good and $\pi_1(Y'\hat{{}_2})=0$. So to prove the theorem, we can without restriction assume that Y' is $\mathbb{F}_2$–complete and simply connected.

Write $\pi_2(Y')\cong F\oplus T$, where F is a finitely generated free $\mathbb{Z}_2$–module and T is a finite abelian 2–group, and let Y'' be the fiber of the map $Y'\to B^2F$. The induced homomorphism $H_2(Y';\mathbb{F}_2)\to H_2(B^2F;\mathbb{F}_2)$ is an epimorphism, so $H^*(B^2F;\mathbb{F}_2)\to H^*(Y';\mathbb{F}_2)$ is injective. As above we obtain an isomorphism $H^*(Y';\mathbb{F}_2)\cong H^*(Y'';\mathbb{F}_2)\otimes H^*(B^2F;\mathbb{F}_2)$ as rings.

By construction Y'' is simply connected, $\pi_2(Y'')$ is finite, and by the fiber lemma [10, Lem. II.5.1] Y'' is $\mathbb{F}_2$–complete. Since $H^*(Y'';\mathbb{F}_2)$ is polynomial, the Eilenberg–Moore spectral sequence shows that $H^*(\Omega Y'';\mathbb{F}_2)$ is $\mathbb{F}_2$–finite, so Y'' is the classifying space of a connected 2–compact group. The first part of Theorem 1.4 now follows from the classification Theorem 1.1. The second part follows from this by using the calculation of the mod 2 cohomology of the simple simply connected Lie groups; cf. [37, Thm. 5.2]. $\square$

Remark 7.1. In addition to the list in Theorem 1.4, the only polynomial rings arising as $H^*(BG;\mathbb{F}_2)$ for a simple compact connected Lie group G are $\mathbb{F}_2[x_2,x_3,\ldots,x_n]$

for $G = \mathrm{SO}(n)$, $n \geq 3$, and $\mathbb{F}_2[x_2, x_3, x_8, x_{12}, \ldots, x_{8n+4}]$ for $G = \mathrm{PSp}(2n+1)$, $n \geq 1$; cf. [37, Thm. 5.2]. It is conceivable that any graded polynomial algebra of finite type which is the mod 2 cohomology ring of a space, is a tensor product of these factors and the ones listed in Theorem 1.4.

Proof of Theorem 1.5. The first statement claims that a finitely generated polynomial $\mathbb{F}_2$–algebra A^* with given action of the Steenrod algebra $\mathcal{A}_2$ can be realized by at most one space Y, up to $\mathbb{F}_2$–equivalence, if A^* has all generators in degree ≥ 3. For this notice that, as in the proof of Theorem 1.4, the assumptions assure that $Y\hat{_2} \simeq BX$ for a simply connected 2–compact group X. Using the classification Theorem 1.1, the statement can now easily be checked as done in Proposition 7.2 below.

We now prove the second statement, that for any finitely generated polynomial $\mathbb{F}_2$–algebra P^*, there are only finitely many spaces Y, up to $\mathbb{F}_2$–equivalence, with $H^*(Y;\mathbb{F}_2) \cong P^*$ as rings. By the proof of Theorem 1.4, we can assume that Y is 2–complete, and any such Y fits in a fibration sequence $Y' \to Y \to BV$ where $V = H_1(Y;\mathbb{F}_2)$. It also follows that $H^*(Y';\mathbb{F}_2)$ is a polynomial ring, which is uniquely determined by P^*. In particular $Y' \simeq BX$ for a connected 2–compact group X. By Proposition 8.19, $\mathrm{Out}(\mathbf{D}_X)$ only contains finitely many finite 2–subgroups up to conjugation. Hence the description of $B\,\mathrm{Aut}(BX)$ in Theorem 1.2 implies that $[BV, B\,\mathrm{Aut}(BX)]$ is finite, so it is enough to see that there are only a finite number of possibilities for Y' given $H^*(Y';\mathbb{F}_2)$ as a ring. This again follows easily from the classification of 2–compact groups: The rank of X is bounded above by the Krull dimension of $H^*(Y';\mathbb{F}_2)$, so by the classification of 2–compact groups, Theorem 1.2, it is hence enough to see that there are only a finite number of $\mathbb{Z}_2$–root data with rank less than a fixed rank. This is the result of Proposition 8.18. $\square$

We now give a proof of the auxiliary uniqueness result referred to in the proof of Theorem 1.5.

Proposition 7.2. *Suppose X is a 2–compact group of the form $BX \simeq BG\hat{_2} \times B\,\mathrm{DI}(4)^s$, for a simply connected compact Lie group G and $s \geq 0$, such that $H^*(BX;\mathbb{F}_2)$ is a polynomial algebra. Then X has a unique maximal elementary abelian 2–subgroup $\nu\colon BE \to BX$, and the Weyl group $(W(\nu), E)$ together with the homomorphism $H^6(BX;\mathbb{F}_2) \to H^6(BE;\mathbb{F}_2)$ is an invariant of $H^*(BX;\mathbb{F}_2)$ as an algebra over the Steenrod algebra $\mathcal{A}_2$, which uniquely determines BX up to homotopy equivalence. In particular BX is uniquely determined up to homotopy equivalence by $H^*(BX;\mathbb{F}_2)$ as an $\mathcal{A}_2$–algebra.*

Proof. By Lannes' theory [38, Thm. 0.4] homotopy classes of maps from classifying spaces of elementary abelian 2–groups to BX are determined by $H^*(BX;\mathbb{F}_2)$ as an algebra over the Steenrod algebra. Furthermore, the fact that $H^*(BX;\mathbb{F}_2)$ is assumed to be a polynomial algebra guarantees that there is only one maximal elementary abelian 2–subgroup $\nu\colon BE \to BX$, up to conjugation, by [57, Cor. 10.7] together with the fact that this is true for $B\,\mathrm{DI}(4)$. (This can also be deduced using the unstable algebra techniques of [2] and [32], or simply by inspecting the calculations of Griess [31] referred to below.) Let $(W(\nu), E)$ denote its Weyl group, which also only depends on $H^*(BX;\mathbb{F}_2)$, and we view this as a pair with $W(\nu)$ a subgroup of $\mathrm{GL}(E)$. By [37, Thm. 5.2], G is a direct product of the groups $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$, $\mathrm{Spin}(7)$, $\mathrm{Spin}(8)$, $\mathrm{Spin}(9)$, G_2 and F_4. In these cases, if E is the maximal elementary abelian 2–subgroup of G, then $W_{G\hat{_2}}(\nu) = N_G(E)/C_G(E)$, whose structure is well-known in these cases, e.g., by computations of Griess [31, §5, Thm. 6.1 and Thm. 7.3]. (See also [65, Prop. 3.2] for details on the cases $\mathrm{Spin}(8) \subseteq \mathrm{Spin}(9) \subseteq F_4$.) Since $W_{\mathrm{DI}(4)}(\nu) = \mathrm{GL}_4(\mathbb{F}_2)$ by construction [25], it follows that the group $(W(\nu), E)$ is a direct product of the following (with matrix groups acting on columns):

$$W_{\mathrm{SU}(n)}(\nu) = (\Sigma_n, V'_{n-1}), \qquad W_{\mathrm{Sp}(n)}(\nu) = (\Sigma_n, V_n),$$
$$W_{G_2}(\nu) = \mathrm{GL}_3(\mathbb{F}_2), \qquad W_{\mathrm{DI}(4)}(\nu) = \mathrm{GL}_4(\mathbb{F}_2),$$

$$W_{\mathrm{Spin}(7)}(\nu) = \left[\begin{array}{c|ccc} 1 & * & * & * \\ \hline 0 & & & \\ 0 & & \mathrm{GL}_3(\mathbb{F}_2) & \\ 0 & & & \end{array}\right], \qquad W_{\mathrm{Spin}(8)}(\nu) = \left[\begin{array}{cc|ccc} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ \hline 0 & 0 & & & \\ 0 & 0 & & \mathrm{GL}_3(\mathbb{F}_2) & \\ 0 & 0 & & & \end{array}\right],$$

$$W_{\mathrm{Spin}(9)}(\nu) = \left[\begin{array}{cc|ccc} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ \hline 0 & 0 & & & \\ 0 & 0 & & \mathrm{GL}_3(\mathbb{F}_2) & \\ 0 & 0 & & & \end{array}\right], \qquad W_{F_4}(\nu) = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\mathrm{GL}_2(\mathbb{F}_2)} & * & * & * \\ & & * & * & * \\ \hline 0 & 0 & & & \\ 0 & 0 & & \mathrm{GL}_3(\mathbb{F}_2) & \\ 0 & 0 & & & \end{array}\right].$$

Here V_n is the n–dimensional permutation module for $\mathbb{F}_2[\Sigma_n]$ and V'_{n-1} is the $(n-1)$–dimensional submodule consisting of elements with coordinate sum 0. The Weyl group $W_{\mathrm{Sp}(n)}(\nu) = (\Sigma_n, V_n)$ decomposes as $(\Sigma_n, V'_{n-1}) \times (1, L)$, $L = {V_n}^{\Sigma_n} \cong \mathbb{F}_2$, when n is odd, $n \geq 3$. However, after this decomposition $(W(\nu), E) = (W_1, V_1) \times \ldots \times (W_m, V_m)$, all the listed pairs (W_i, V_i) satisfy that V_i is indecomposable as an $\mathbb{F}_2[W_i]$–module. (Note that this is a priori stronger than saying that (W_i, V_i) does not split as a product; however, it follows from unstable algebra techniques [51, Secs. 5 and 7] that (W_i, V_i) is an $\mathbb{F}_2$–reflection group, and hence the two notions are actually equivalent.) By the Krull–Schmidt theorem [14, Thm. 6.12(ii)] this decomposition $E = V_1 \oplus \ldots \oplus V_m$ as $\mathbb{F}_2[W(\nu)]$–modules is unique up to permutation, and since W_i is characterized as the pointwise stabilizer of $V_1 \oplus \ldots \oplus \widehat{V_i} \oplus \ldots \oplus V_m$, we get that the decomposition of $(W(\nu), E)$ as a product is unique as well, up to permutation. Thus the structure of $(W(\nu), E)$ as a product of finite indecomposable groups is an invariant which almost characterizes BX up to homotopy equivalence, except that for n odd, $n \geq 3$, the group (Σ_n, V'_{n-1}) arises from both $\mathrm{SU}(n)\hat{_2}$ and $\mathrm{Sp}(n)\hat{_2}$. However since $H^6(B\,\mathrm{SU}(n); \mathbb{F}_2) \to H^6(BV'_{n-1}; \mathbb{F}_2)$ is injective and $H^6(B\,\mathrm{Sp}(n); \mathbb{F}_2) \to H^6(BV'_{n-1}; \mathbb{F}_2)$ is trivial, we conclude that $(W(\nu), E)$ together with the homomorphism $H^6(BX; \mathbb{F}_2) \to H^6(BE; \mathbb{F}_2)$ characterizes BX up to homotopy equivalence. This proves the proposition since both data are determined by the $\mathcal{A}_2$–action on $H^*(BX; \mathbb{F}_2)$. □

Proof of Theorem 1.6. It is obvious that there is a one-to-one correspondence between isomorphism classes of p–compact groups with identity component isomorphic to X_1 and component group isomorphic to π, and equivalence classes of fibration sequences $F \to E \xrightarrow{p} B$ with F homotopy equivalent to BX_1 and B homotopy equivalent to $B\pi$. It is likewise obvious that in this case $B\,\mathrm{Aut}(E)$ is homotopy equivalent to $B\,\mathrm{Aut}(p)$, where $\mathrm{Aut}(p)$ is the space of self-homotopy equivalences of the fibration p.

By the classification of fibrations [21], equivalence classes of such fibrations are in one-to-one correspondence with $\mathrm{Out}(B)$–orbits on $[B, B\,\mathrm{Aut}(F)]$, and the space

$B\,\mathrm{Aut}(p)$ equals $(\mathrm{map}(B, B\,\mathrm{Aut}(F))_{C(p)})_{h\,\mathrm{Aut}(B)}$, where $C(p)$ denotes the $\mathrm{Out}(B)$–orbit on $[B, B\,\mathrm{Aut}(F)]$ of the element classifying $p\colon E \to B$.

The above considerations completely reduce the proof of the theorem to our classification theorem for connected p–compact groups, Theorem 1.2, except for the finiteness statement. For this note that Proposition 8.19 implies that $[B\pi, B\,\mathrm{Out}(\mathbf{D})]$ is finite, so the finiteness of $[B\pi, B\,\mathrm{aut}(\mathbf{D})]$ follows. □

Remark 7.3. As stated in the Introduction, our classification also shows that Bott's theorem on the cohomology of X/T, the Peter–Weyl theorem, as well as Borel's characterization of when centralizers of elements of order p are connected, stated as Theorems 1.5, 1.6 and 1.9 in [8], hold verbatim for 2–compact groups. To prove these results it suffices by Theorem 1.1 to check them for DI(4), since they are well-known for compact Lie groups. For DI(4) one argues as follows: Bott's theorem [8, Thm. 1.5] follows from [25, Thm. 1.8(2)], the Peter–Weyl theorem [8, Thm. 1.6] is a result of Ziemiański [70], and it is trivial to check that [8, Thm. 1.9] holds.

8. Appendix: Properties of $\mathbb{Z}_p$–root data

The purpose of this section is to establish some general results about $\mathbb{Z}_p$–root data of p–compact groups, needed in the proof of the main theorem. The analogous results for $\mathbb{Z}$–root data and compact Lie groups are often well-known; see [9, 16]. We build on the paper [29] by Dwyer–Wilkerson and our earlier paper [7].

We briefly recall the definition of root data from the Introduction: For an integral domain R, an *R–reflection group* is a pair (W, L) where L is a finitely generated free R–module and W is a subgroup of $\mathrm{Aut}_R(L)$ generated by reflections (i.e. elements $\sigma \in \mathrm{Aut}_R(L)$ such that $1 - \sigma \in \mathrm{End}_R(L)$ has rank one). If R is a principal ideal domain, we define an *R–root datum* to be a triple $\mathbf{D} = (W, L, \{Rb_\sigma\})$ where (W, L) is a finite R–reflection group, and for each reflection $\sigma \in W$, Rb_σ is a rank one submodule of L with $\mathrm{im}(1-\sigma) \subseteq Rb_\sigma$ and $w(Rb_\sigma) = Rb_{w\sigma w^{-1}}$ for all $w \in W$. If $R \to R'$ is a monomorphism, and $\mathbf{D}$ an R–root datum, we can define an R'–root datum by $\mathbf{D} \otimes_R R' = (W, L \otimes_R R', \{Rb_\sigma \otimes_R R'\})$.

The element b_σ, defined up to a unit in R, is called the *coroot* associated to σ. By definition b_σ determines a unique linear map $\beta_\sigma\colon L \to R$ called the associated *root* such that $\sigma(x) = x + \beta_\sigma(x) b_\sigma$ for $x \in L$. Define the *coroot lattice* $L_0 \subseteq L$ as the sublattice spanned by the coroots b_σ and the *fundamental group* of $\mathbf{D}$ by $\pi_1(\mathbf{D}) = L/L_0$. In general $Rb_\sigma \subseteq \ker(N)$, where $N = 1 + \sigma + \ldots + \sigma^{|\sigma|-1}$ is the norm element (cf. the proof of Lemma 8.9 below), so giving an R–root datum with underlying reflection group (W, L) corresponds to choosing a cyclic R–submodule of $H^1(\langle\sigma\rangle; L)$ for each conjugacy class of reflections σ. In particular for $R = \mathbb{Z}_p$, p odd, the notions of a $\mathbb{Z}_p$–reflection group and a $\mathbb{Z}_p$–root datum agree. If R has characteristic zero, an R–root datum, or an R–reflection group, is called *irreducible* if the representation $W \to \mathrm{GL}(L \otimes_R K)$ is irreducible, where K denotes the quotient field of R, and it is said to be *exotic* if furthermore the values of the character of this representation are not all contained in $\mathbb{Q}$.

We are now ready to state the classification of $\mathbb{Z}_p$–root data, which follows easily from the classification of finite $\mathbb{Z}_p$–reflection groups [8, Thm. 11.1]. This classification is again based on the classification of finite $\mathbb{Q}_p$–reflection groups [13], [23] which states that for a fixed prime p, isomorphism classes of finite irreducible $\mathbb{Q}_p$–reflection groups are in natural one-to-one correspondence with isomorphism classes of finite irreducible $\mathbf{C}$–reflection groups (W,V) [58] for which the values of the character of $W \to \mathrm{GL}(V)$ are embeddable in $\mathbb{Q}_p$; see, e.g., [3, Table 1] for an explicit list of groups and primes.

Theorem 8.1 (The classification of $\mathbb{Z}_p$–root data; splitting version)**.**

(1) *Any $\mathbb{Z}_p$–root datum is isomorphic to a $\mathbb{Z}_p$–root datum of the form $(\mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p) \times \mathbf{D}_2$, where $\mathbf{D}_1$ is a $\mathbb{Z}$–root datum and $\mathbf{D}_2$ is a direct product of exotic $\mathbb{Z}_p$–root data.*

(2) *There is a one-to-one correspondence between isomorphism classes of exotic $\mathbb{Z}_p$–root data and isomorphism classes of exotic $\mathbb{Q}_p$–reflection groups given by the assignment $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\}) \rightsquigarrow (W, L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Moreover $\pi_1(\mathbf{D}) = 0$ for any exotic $\mathbb{Z}_p$–root datum $\mathbf{D}$.*

Proof. For any $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$, [8, Thm. 11.1] gives a splitting $(W,L) \cong (W_1, L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p) \times (W_2, L_2)$ of (W,L), where (W_1, L_1) is a finite $\mathbb{Z}$–reflection group and (W_2, L_2) is a direct product of exotic $\mathbb{Z}_p$–reflection groups. It follows by definition that there are unique $\mathbb{Z}_p$–root data $\mathbf{D}'$ and $\mathbf{D}_2$ with underlying reflection groups $(W_1, L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ and (W_2, L_2) such that $\mathbf{D} \cong \mathbf{D}' \times \mathbf{D}_2$, and by the same argument $\mathbf{D}_2$ splits as a direct product of exotic $\mathbb{Z}_p$–root data. Furthermore, writing $\mathbf{D}' = (W_1, L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p, \{\mathbb{Z}_p b_\sigma\})$ it is clear that $\mathbf{D}' \cong \mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ where $\mathbf{D}_1 = (W_1, L_1, \{L_1 \cap \mathbb{Z}_p b_\sigma\})$. This proves (1).

By [8, Thm. 11.1], the assignment $(W,L) \rightsquigarrow (W, L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ establishes a one-to-one correspondence between exotic $\mathbb{Z}_p$–reflection groups up to isomorphism and exotic $\mathbb{Q}_p$–reflection groups up to isomorphism. To prove the first part of (2) it thus suffices to show that any exotic $\mathbb{Z}_p$–reflection group (W,L) can be given a unique $\mathbb{Z}_p$–root datum structure. For $p > 2$ this holds since $H^1(\langle\sigma\rangle; L) = 0$; cf. the discussion in the beginning of this section. For $p = 2$, $(W,L) \cong (W_{\mathrm{DI}(4)}, L_{\mathrm{DI}(4)})$, where the claim follows by direct inspection (cf. [29, Rem. 7.2]).

For any $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$, the formula $\sigma(x) = x + \beta_\sigma(x) b_\sigma$ shows that the coroot lattice L_0 contains the lattice spanned by the elements $(1 - w)(x)$, $w \in W$, $x \in L$. Hence the final claim follows from the fact that $H_0(W; L) = 0$ for any exotic $\mathbb{Z}_p$–reflection group (W,L) [8, Thm. 11.1]. $\square$

8.1. **The root datum and root subgroups of a p–compact group.** For any connected p–compact group X with maximal torus T, the Weyl group W_X acts naturally on $L_X = \pi_1(T)$ as a finite $\mathbb{Z}_p$–reflection group [26, Thm. 9.7(ii)]. For p odd, $H^1(\langle\sigma\rangle; L) = 0$ for any reflection σ, so the finite $\mathbb{Z}_p$–reflection group (W_X, L_X) gives rise to a unique $\mathbb{Z}_p$–root datum $\mathbf{D}_X$. The construction of root data for connected 2–compact groups, in the present form, is due to Dwyer–Wilkerson [29, §9]: Let $\breve{T}$ be the discrete approximation to T, $\breve{\mathcal{N}}_X$ the discrete approximation to the maximal torus normalizer $\mathcal{N}_X$ and $\sigma \in W_X$ a reflection. Define $\breve{T}^+(\sigma) = \breve{T}^{\langle\sigma\rangle}$ and let $\breve{T}_0^+(\sigma)$ denote its maximal divisible subgroup. Then $X(\sigma) = \mathcal{C}_X(\breve{T}_0^+(\sigma))$ is a connected 2–compact group with Weyl group $\langle\sigma\rangle$ and $\breve{\mathcal{N}}(\sigma) = C_{\breve{\mathcal{N}}_X}(\breve{T}_0^+(\sigma))$ is a discrete approximation to its maximal torus normalizer. Furthermore, let $\breve{T}_0^-(\sigma)$ denote the maximal divisible subgroup of $\breve{T}^-(\sigma) = \ker(\breve{T} \xrightarrow{1+\sigma} \breve{T})$ and define the *root subgroup* $\breve{\mathcal{N}}_{X,\sigma}$ by

$$\breve{\mathcal{N}}_{X,\sigma} = \{x \in \breve{\mathcal{N}}(\sigma) \mid \exists\, y \in \breve{T}_0^-(\sigma) : x \text{ is conjugate to } y \text{ in } X(\sigma)\}.$$

Then there is a short exact sequence

$$1 \to \breve{T}_0^-(\sigma) \to \breve{\mathcal{N}}_{X,\sigma} \to \langle\sigma\rangle \to 1, \tag{8.1}$$

and we define

$$\mathbb{Z}_2 b_\sigma = \begin{cases} \operatorname{im}(L_X \xrightarrow{1-\sigma} L_X) & \text{if (8.1) splits,} \\ \ker(L_X \xrightarrow{1+\sigma} L_X) & \text{otherwise.} \end{cases}$$

The root datum of X is then the $\mathbb{Z}_2$–root datum $\mathbf{D}_X = (W_X, L_X, \{\mathbb{Z}_2 b_\sigma\})$; see [29, §6 and §9] and [7] for a further discussion.

Conversely, the maximal torus normalizer and the root subgroups of a connected p–compact group can be reconstructed from its root datum: For a $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ there is [29, Def. 6.15], [7, §3] an algebraically defined extension

$$1 \to \breve{T} \to \breve{\mathcal{N}}_{\mathbf{D}} \to W \to 1$$

called the *normalizer extension* with a subextension $1 \to \breve{T}_0^-(\sigma) \to \breve{\mathcal{N}}_{\mathbf{D},\sigma} \to \langle\sigma\rangle \to 1$ for each reflection $\sigma \in W$. For a connected p–compact group X with $\mathbb{Z}_p$–root datum $\mathbf{D}_X$ there is an isomorphism of extensions [29, Prop. 1.10], [7, Thm. 3.2(2)]

$$\begin{tikzcd} 1 \arrow[r] & \breve{T} \arrow[r] \arrow[d, equal] & \breve{\mathcal{N}}_{\mathbf{D}} \arrow[r] \arrow[d, "\cong"] & W \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \breve{T} \arrow[r] & \breve{\mathcal{N}}_X \arrow[r] & W \arrow[r] & 1 \end{tikzcd}$$

sending the *root subgroups* $\breve{\mathcal{N}}_{\mathbf{D},\sigma}$ to $\breve{\mathcal{N}}_{X,\sigma}$ for all reflections σ, and any such isomorphism is unique up to conjugation by an element in $\breve{T}$.

We define $B\mathcal{N}_{\mathbf{D}}$ as the fiber-wise $\mathbb{F}_p$–completion [10, Ch. I, §8] of $B\breve{\mathcal{N}}_{\mathbf{D}}$ and likewise introduce the (non-discrete) root subgroups $B\mathcal{N}_{X,\sigma}$ and $B\mathcal{N}_{\mathbf{D},\sigma}$ by fiber-wise $\mathbb{F}_p$–completion of the corresponding discrete versions $B\breve{\mathcal{N}}_{X,\sigma}$ and $B\breve{\mathcal{N}}_{\mathbf{D},\sigma}$.

Recollection 8.2 (The Adams–Mahmud map)**.** By [8, Lem. 4.1] we have an "Adams–Mahmud" homomorphism $\Phi\colon \operatorname{Out}(BX) \to \operatorname{Out}(\breve{\mathcal{N}})$, given by associating to $f\colon BX \to BX$ the homomorphism $\Phi(f)\colon \breve{\mathcal{N}} \to \breve{\mathcal{N}}$, unique up to conjugation, such that the diagram

$$\begin{tikzcd} B\breve{\mathcal{N}} \arrow[r, "B\Phi(f)"] \arrow[d] & B\breve{\mathcal{N}} \arrow[d] \\ BX \arrow[r, "f"] & BX \end{tikzcd}$$

commutes up to homotopy.

By [7, Thm. C], Φ factors through $\operatorname{Out}(\breve{\mathcal{N}}, \{\breve{\mathcal{N}}_\sigma\}) = \{[\varphi] \in \operatorname{Out}(\breve{\mathcal{N}}) \mid \varphi(\breve{\mathcal{N}}_\sigma) = \breve{\mathcal{N}}_{\varphi(\sigma)}\}$, which is isomorphic to $\operatorname{Out}(\mathbf{D}_X)$ via restriction to $\breve{T}$. We again denote this map by Φ. Likewise, as e.g. explained in [8, Prop. 5.1], fiber-wise $\mathbb{F}_p$–completion [10, Ch. I, §8] induces a natural isomorphism $\operatorname{Out}(\breve{\mathcal{N}}) \xrightarrow{\cong} \operatorname{Out}(B\mathcal{N})$, and we can hence equivalently view $\Phi(f)$ as an element in $\operatorname{Out}(B\mathcal{N})$, and we will not notationally distinguish between the two cases. We denote the subgroup of $\operatorname{Out}(B\mathcal{N})$ corresponding to $\operatorname{Out}(\breve{\mathcal{N}}, \{\breve{\mathcal{N}}_\sigma\})$ by $\operatorname{Out}(B\mathcal{N}, \{B\mathcal{N}_\sigma\})$.

8.2. **Centers and fundamental groups.** If $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ is a $\mathbb{Z}_p$–root datum, we define a *subdatum* of $\mathbf{D}$ to be a $\mathbb{Z}_p$–root datum of the form $(W', L, \{\mathbb{Z}_p b_\sigma\}_{\sigma\in\Sigma'})$, where (W', L) is a reflection subgroup of (W, L) and Σ' is the set of reflections in W'. For the next result, recall [27, Def. 4.1] that a homomorphism $f\colon BX \to BY$ is called a *monomorphism of maximal rank* if the homotopy fiber Y/X is $\mathbb{F}_p$–finite and X and Y have the same rank.

Proposition 8.3. *Let X and Y be connected p–compact groups. If $f\colon BX \to BY$ is a monomorphism of maximal rank, then $\mathbf{D}_X$ naturally identifies with a subdatum of $\mathbf{D}_Y$.*

Proof. By definition there is a maximal torus $i\colon BT \to BX$ such that $f\circ i\colon BT \to BY$ is a maximal torus for Y. Thus we can identify $L_X = L_Y = \pi_1(T)$, and by [27, Lem. 4.4] we have an induced monomorphism $W_X \to W_Y$. This proves the result for p odd, since in that case the $\mathbb{Z}_p$–root data $\mathbf{D}_X$ and $\mathbf{D}_Y$ are uniquely determined by their underlying reflection groups (W_X, L_X) and (W_Y, L_Y). In the case $p = 2$ the result follows from the construction of the $\mathbb{Z}_2$–root data of X and Y; cf. [29, Pf. of Lem. 9.16]. □

For a $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$, we let $\breve{T} = L \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty$ be the associated discrete torus, and for a reflection $\sigma \in W$, we define $h_\sigma = b_\sigma/2 \in \breve{T}$. Clearly h_σ is independent of the choice of b_σ, and conversely h_σ determines $\mathbb{Z}_p b_\sigma$; cf. [29, §2 and §6]. So instead of $(W, L, \{\mathbb{Z}_p b_\sigma\})$ we might as well work with $(W, \breve{T}, \{h_\sigma\})$; we will use these two viewpoints interchangeably without further mention. Also note that $h_\sigma = 1$ for p odd. When $\sigma \in W$ is a reflection, we define the *singular set* $S(\sigma)$ by

$$S(\sigma) = \langle \breve{T}_0^+(\sigma), h_\sigma\rangle = \ker(\beta_\sigma \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty\colon\ \breve{T} \to \mathbb{Z}/p^\infty);$$

cf. [27, Def. 7.3] and [7, (3.2)]. Define the *discrete center* $\breve{\mathcal{Z}}(\mathbf{D})$ of $\mathbf{D}$ as $\bigcap_\sigma S(\sigma)$, where the intersection is taken over all reflections $\sigma \in W$. In other words, letting M_0 denote the *root lattice*, i.e., the $\mathbb{Z}_p$–sublattice of L^* spanned by the roots β_σ, we have the identification

$$\breve{\mathcal{Z}}(\mathbf{D}) = \ker\left(\breve{T} = \operatorname{Hom}_{\mathbb{Z}_p}(L^*, \mathbb{Z}/p^\infty) \twoheadrightarrow \operatorname{Hom}_{\mathbb{Z}_p}(M_0, \mathbb{Z}/p^\infty)\right). \tag{8.2}$$

The following proposition translates into current language the results of Dwyer–Wilkerson [27, §7] on how to compute the center of X and centralizers of toral subgroups of X.

Proposition 8.4. *Let X be a connected p–compact group with $\mathbb{Z}_p$–root datum $\mathbf{D}_X = (W, L, \{\mathbb{Z}_p b_\sigma\})$.*

1. *The center $B\mathcal{Z}(X)$ of X is canonically homotopy equivalent to the* center $B\mathcal{Z}(\mathbf{D}) = (B\breve{\mathcal{Z}}(\mathbf{D}))^\wedge_p$ *of* $\mathbf{D}$.
2. *The identity component $\mathcal{Z}(X)_1$ of the center has $\mathbb{Z}_p$–root datum $(1, L^W, \emptyset)$.*
3. *For $A \subseteq \breve{T}$, let $W(A)$ be the pointwise stabilizer of A in W, Σ_A the set of reflections σ with $A \subseteq S(\sigma)$, and $W(A)_1$ the subgroup of W generated by Σ_A. Then the centralizer $\mathcal{C}_X(A)$ has Weyl group $W(A)$ and its identity component $\mathcal{C}_X(A)_1$ has $\mathbb{Z}_p$–root datum equal to the subdatum $\mathbf{D}_A = (W(A)_1, L, \{\mathbb{Z}_p b_\sigma\}_{\sigma\in\Sigma_A})$ of $\mathbf{D}$.*

Proof. The first part is [27, Thm. 7.5]. The second part follows easily from this since the maximal divisible subgroup of $\breve{\mathcal{Z}}(\mathbf{D})$ equals

$$\bigcap_{\sigma} \breve{T}_0^+(\sigma) = \bigcap_{\sigma} \left(L^+(\sigma) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty\right) = L^W \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty.$$

Part (3) follows by combining [27, Thm. 7.6] and Proposition 8.3 since $B\mathcal{C}_X(A)_1 \to BX$ is a monomorphism of maximal rank [27, Prop. 4.3]. $\square$

Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a $\mathbb{Z}_p$–root datum. Recall that the coroot lattice $L_0 \subseteq L$ is the $\mathbb{Z}_p$–lattice spanned by the coroots b_σ and that the fundamental group $\pi_1(\mathbf{D})$ is the quotient L/L_0. Define $H_n^{\mathbb{Z}_p}(X) = \lim_k H_n(X; \mathbb{Z}/p^k)$. The following proposition, which refines [8, Prop. 10.2], constructs a canonical epimorphism $\pi_1(\mathbf{D}_X) \to \pi_1(X)$ which will be shown to be an isomorphism in Theorem 8.6 below.

Proposition 8.5. *Let X be a connected p–compact group with maximal torus T and $\mathbb{Z}_p$–root datum $\mathbf{D}_X = (W, L, \{\mathbb{Z}_p b_\sigma\})$. Then the homomorphism $L = H_2^{\mathbb{Z}_p}(BT) \to H_2^{\mathbb{Z}_p}(BX) \cong \pi_1(X)$ factors through $\pi_1(\mathbf{D}_X)$ and the induced homomorphism $\pi_1(\mathbf{D}_X) \to \pi_1(X)$ is surjective with finite kernel.*

Proof. For p odd we have $\mathrm{im}(1-\sigma) = \mathbb{Z}_p b_\sigma$ for all σ, so $L/L_0 \cong H_0(W; L)$, and the result follows from [8, Prop. 10.2].

Assume now that $p = 2$ and let $\sigma \in W$ be a reflection. To see the first part we have to show that the homomorphism $L = H_2^{\mathbb{Z}_2}(BT) \to H_2^{\mathbb{Z}_2}(BX) \cong \pi_1(X)$ vanishes on the coroots b_σ. This follows from the construction of the root datum of X: $X(\sigma) = \mathcal{C}_X(\breve{T}_0^+(\sigma))$ is a connected 2–compact group, and Proposition 8.4(3) shows that $\mathbf{D}_{X(\sigma)}$ equals the subdatum $(\langle\sigma\rangle, L, \{\mathbb{Z}_2 b_\sigma\})$ of $\mathbf{D}_X$. The commutative diagram

$$\begin{tikzcd}
L = H_2^{\mathbb{Z}_2}(BT) \arrow[r] \arrow[d, equal] & H_2^{\mathbb{Z}_2}(BX(\sigma)) \cong \pi_1(X(\sigma)) \arrow[d] \\
L = H_2^{\mathbb{Z}_2}(BT) \arrow[r] & H_2^{\mathbb{Z}_2}(BX) \cong \pi_1(X)
\end{tikzcd}$$

shows that it suffices to prove the claim for $X(\sigma)$. However since $X(\sigma)$ is a connected 2–compact group of semi-simple rank 1, it follows (cf. [29, pp. 1369–1370]) that $X(\sigma) \cong G_{\hat{2}}$ for $G = \mathrm{SU}(2) \times (S^1)^{r-1}$, $\mathrm{SO}(3) \times (S^1)^{r-1}$ or $\mathrm{U}(2) \times (S^1)^{r-2}$, where r is the rank of X. The well-known formula for the fundamental group of a compact connected Lie group in terms of its root datum (cf. [9, §4, no. 6, Prop. 11] or [1, Thm. 5.47]) now establishes the first part of the proposition.

Since $\mathrm{im}(1-\sigma) \subseteq \mathbb{Z}_2 b_\sigma$ by definition, $L \to L/L_0 = \pi_1(\mathbf{D}_X)$ factors through $H_0(W; L)$, so the final claim now follows from [8, Prop. 10.2]. $\square$

We will also be using the following formula for the fundamental group of a p–compact group, proved by Dwyer–Wilkerson [30] by a transfer argument as this paper was being written (see also [8, Rem. 10.3]). The formula was previously known for p odd, by our classification [8], and we sketch in Remark 6.2 how one can bypass the use of this formula also in the classification for $p = 2$ by a more cumbersome argument which we had originally envisioned using in this paper; in particular providing an independent proof.

Theorem 8.6 (Dwyer–Wilkerson [30] and Remark 6.2)**.** *Let X be a connected p–compact group. Then $\pi_1(\mathbf{D}_X) \xrightarrow{\cong} \pi_1(X)$ induced by the maximal torus $T \to X$.*

Proof. By Theorem 8.1(1) we may write $\mathbf{D}_X = \mathbf{D}_1 \times \mathbf{D}_2$, where $\mathbf{D}_1$ is of the form $\mathbf{D}' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a $\mathbb{Z}$–root datum $\mathbf{D}'$, and $\mathbf{D}_2$ is a direct product of exotic $\mathbb{Z}_p$–root data. By [28, Thm. 1.4] this induces a splitting $BX \simeq BX_1 \times BX_2$ with $\mathbf{D}_{X_i} \cong \mathbf{D}_i$. We have to show that the kernel of $L = \pi_1(T) \to \pi_1(X)$ equals the coroot lattice L_0; by the above it suffices to treat the case where $\mathbf{D}_X$ is exotic and the case where $\mathbf{D}_X$ is of the form $\mathbf{D}' \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a $\mathbb{Z}$–root datum $\mathbf{D}'$.

In the first case, Theorem 8.1(2) shows that $\pi_1(\mathbf{D}_X) = 0$, so the result follows from Proposition 8.5. In the second case we have $\mathbf{D}_X \cong \mathbf{D}_{G\hat{_p}}$ for some compact connected Lie group G. By the result of Dwyer–Wilkerson [30, Thm. 1.1] the kernel of $L = \pi_1(T) \to \pi_1(X)$ equals the kernel of $H_2^{\mathbb{Z}_p}(BT_X) \to H_2^{\mathbb{Z}_p}(B\mathcal{N}_X)$. Since the maximal torus normalizer may be reconstructed from the root datum by [29, Prop. 1.10] for $p = 2$ and [3, Thm. 1.2] for p odd, we may identify the homomorphism $H_2^{\mathbb{Z}_p}(BT_X) \to H_2^{\mathbb{Z}_p}(B\mathcal{N}_X)$ with the homomorphism $H_2(BT_G;\mathbb{Z}) \to H_2(BN_G;\mathbb{Z})$ tensored by $\mathbb{Z}_p$. The result now follows from the corresponding result for compact Lie groups; cf. [9, §4, no. 6, Prop. 11] or [1, Thm. 5.47]. □

Remark 8.7. Note that the composition $L^W \hookrightarrow L \to L/L_0 = \pi_1(\mathbf{D})$ is injective with finite cokernel by Lemma 8.9 below. Hence, for a connected p–compact group X, the canonical homomorphism $\pi_1(\mathcal{Z}(X)_1) \to \pi_1(X)$ is injective with finite cokernel by Propositions 8.4(2) and 8.5. In particular the center of X is finite if and only if the fundamental group of X is finite.

8.3. **Covers and quotients.** We now start to address how the root datum behaves upon taking covers and quotients of a p–compact group.

Lemma 8.8. *Let $f\colon BX \to BY$ be a monomorphism of maximal rank between connected p–compact groups X and Y. If $\pi_1(\mathbf{D}_Y) = 0$, then $B\mathcal{Z}(X) \simeq B\mathcal{Z}(\mathcal{N}_X)$.*

Proof. For p odd, the conclusion holds for any connected p–compact group X [27, Rem. 7.7], so we may suppose $p = 2$. Since $\pi_1(\mathbf{D}_Y) = 0$, $\mathbf{D}_Y$ does not have any direct factors isomorphic to $\mathbf{D}_{\mathrm{SO}(2n+1)\hat{_2}}$, so [7, Lem. 5.1] implies that the singular set $S_Y(\sigma)$ with respect to Y equals $\breve{T}^+(\sigma)$ for any reflection $\sigma \in W_Y$. By Proposition 8.3, $\mathbf{D}_X$ identifies with a subdatum of $\mathbf{D}_Y$, and hence $S_X(\sigma) = \breve{T}^+(\sigma)$ for all reflections $\sigma \in W_X$. The claim now follows from Proposition 8.4(1). □

Lemma 8.9. *Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a $\mathbb{Z}_p$–root datum with coroot lattice L_0. Then $L_0 \cap L^W = 0$ and $L_0 \oplus L^W$ has finite index in L. In particular W acts faithfully on L_0.*

Proof. The $\mathbb{Q}_p[W]$–module $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ decomposes as $V = V^W \oplus U$, where $U^W = 0$. Writing $b_\sigma = x + y$ with $x \in V^W$ and $y \in U$ we have $\sigma(b_\sigma) = x + \sigma(y)$, and hence $\sigma(b_\sigma) - b_\sigma \in U$. Also $\sigma(b_\sigma) \neq b_\sigma$: If $N = 1 + \sigma + \ldots + \sigma^{|\sigma|-1}$ is the norm element, then $\beta_\sigma(x)N(b_\sigma) = N(\sigma - 1)(x) = 0$ for all $x \in L$. Since $\sigma \neq 1$ we have $\beta_\sigma \neq 0$ so $Nb_\sigma = 0$. Thus $\sigma(b_\sigma) \neq b_\sigma$, since otherwise $Nb_\sigma = |\sigma| b_\sigma \neq 0$. This proves that $(\sigma - 1)(b_\sigma) = rb_\sigma$ with $r \neq 0$ so $b_\sigma \in U$. Thus $L_0 \subseteq U$ and $L_0 \cap L^W = 0$, as desired.

Since $|W|x = \left(\sum_{w \in W} wx\right) + \sum_{w \in W}(x - wx) \in L^W + L_0$, for any $x \in L$ we see that $L_0 \oplus L^W$ has finite index in L, and in particular W acts faithfully on L_0. □

Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a $\mathbb{Z}_p$–root datum and let L_0 be the coroot lattice. If L' is a $\mathbb{Z}_p$–lattice with $L_0 \subseteq L' \subseteq L$, the formula $\sigma(x) = x + \beta_\sigma(x) b_\sigma$ shows that L' is W–invariant. By Lemma 8.9, W acts faithfully on L_0 and hence also on L', so $(W, L', \{\mathbb{Z}_p b_\sigma\})$ is a $\mathbb{Z}_p$–root datum. We define a *cover* of $\mathbf{D}$ to be any $\mathbb{Z}_p$–root datum of this form. In particular the *universal cover* $\widetilde{\mathbf{D}}$ of $\mathbf{D}$ is defined by $\widetilde{\mathbf{D}} = (W, L_0, \{\mathbb{Z}_p b_\sigma\})$. Note that by definition, $\pi_1(\widetilde{\mathbf{D}}) = 0$. For the reduction in Section 2 we need the following result which does not rely on the fundamental group formula Theorem 8.6.

Proposition 8.10. *Let X be a connected p–compact group with $\mathbb{Z}_p$–root datum $\mathbf{D}_X$ and let H be a subgroup of $\pi_1(X)$. Let $Y \to X$ be the cover of X corresponding to H. Then $\mathbf{D}_Y$ is the cover of $\mathbf{D}_X$ corresponding to the kernel of the composition $L_X \to \pi_1(\mathbf{D}_X) \to \pi_1(X) \to \pi_1(X)/H$.*

Proof. By construction BY is the fiber of $BX \to B^2(\pi_1(X)/H)$. Let $BT \to BX$ be a maximal torus of X and let $B\mathcal{N}_X \to BX$ be the maximal torus normalizer. Now consider the following diagram obtained by pulling the fibration $BY \to BX \to B^2(\pi_1(X)/H)$ back along $BT \to B\mathcal{N}_X \to BX$:

$$\begin{tikzcd} BT' \arrow[r] \arrow[d] & B\mathcal{N}' \arrow[r] \arrow[d] & BY \arrow[d] \\ BT \arrow[r] \arrow[d] & B\mathcal{N}_X \arrow[r] \arrow[d] & BX \arrow[d] \\ B^2(\pi_1(X)/H) \arrow[r, equal] & B^2(\pi_1(X)/H) \arrow[r, equal] & B^2(\pi_1(X)/H) \end{tikzcd} \tag{8.3}$$

Thus $BT' \to BY$ is a maximal torus and $B\mathcal{N}' \to BY$ is a maximal torus normalizer by [45, Thm. 1.2]. The above diagram shows that the Weyl group of Y identifies with the Weyl group W of X. For a reflection $\sigma \in W$ we have the diagram

$$\begin{tikzcd} B\breve{T}' \arrow[r] \arrow[d] & B\breve{\mathcal{N}}'(\sigma) \arrow[r] \arrow[d] & BY(\sigma) \arrow[d] \\ B\breve{T} \arrow[r] & B\breve{\mathcal{N}}_X(\sigma) \arrow[r] & BX(\sigma) \end{tikzcd}$$

where $X(\sigma) = \mathcal{C}_X(\breve{T}_0^+(\sigma))$, $\breve{\mathcal{N}}_X(\sigma) = C_{\breve{\mathcal{N}}_X}(\breve{T}_0^+(\sigma))$ and similarly for $Y(\sigma)$ and $\breve{\mathcal{N}}'(\sigma)$. It now follows by definition (cf. [29, §9]) that the image of $h'_\sigma \in \breve{T}'$ equals $h_\sigma \in \breve{T}$. Diagram (8.3) produces the short exact sequence $0 \to L_Y \to L_X \to \pi_1(X)/H \to 0$ so $\mathbb{Z}_p b'_\sigma \subseteq L_Y$ maps to $\mathbb{Z}_p b_\sigma \subseteq L_X$. This shows the claim. $\square$

We next introduce quotients of root data. Let $\mathbf{D} = (W, \breve{T}, \{h_\sigma\})$ be a $\mathbb{Z}_p$–root datum, and let $A \subseteq \breve{\mathcal{Z}}(\mathbf{D})$ be a subgroup of the discrete center. We define a *quotient* of $\mathbf{D}$ to be a $\mathbb{Z}_p$–root datum of the form $\mathbf{D}/A = (W, \breve{T}/A, \{\overline{h_\sigma}\})$, where $\overline{h_\sigma}$ denotes the image of h_σ in $\breve{T}/A$; the fact that this is a $\mathbb{Z}_p$–root datum is part of the following result.

Proposition 8.11.

(1) *If $\mathbf{D} = (W, \breve{T}, \{h_\sigma\})$ is a $\mathbb{Z}_p$–root datum and $A \subseteq \breve{\mathcal{Z}}(\mathbf{D})$, then $\mathbf{D}/A = (W, \breve{T}/A, \{\overline{h_\sigma}\})$ is a $\mathbb{Z}_p$–root datum. Moreover $\widetilde{\mathbf{D}/A} \cong \widetilde{\mathbf{D}}$ and $\breve{\mathcal{Z}}(\mathbf{D}/A) \cong \breve{\mathcal{Z}}(\mathbf{D})/A$.*

(2) *Let X be a connected p–compact group with $\mathbb{Z}_p$–root datum $\mathbf{D}_X$ and $A \to X$ a central monomorphism. If $\breve{A}$ denotes the discrete approximation to A, then the $\mathbb{Z}_p$–root datum $\mathbf{D}_{X/A}$ of the p–compact group X/A identifies with the quotient datum $\mathbf{D}_X/\breve{A}$. In particular $\widetilde{\mathbf{D}_{X/A}} \cong \widetilde{\mathbf{D}_X}$.*

Proof. Write $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ where $L = \operatorname{Hom}_{\mathbb{Z}_p}(\mathbb{Z}/p^\infty, \breve{T})$ is the associated $\mathbb{Z}_p$–lattice and the $b_\sigma \in L$ are the associated coroots. By (8.2) the sequence of discrete tori $\breve{T} \to \breve{T}/A \to \breve{T}/\breve{\mathcal{Z}}(\mathbf{D})$ corresponds to the sequence $L \to L_{\breve{T}/A} \to {M_0}^*$ of $\mathbb{Z}_p$–lattices, where M_0 is the root lattice spanned by the β_σ. Note that (W, L^*) is a reflection group via the action $\sigma(\alpha) = \alpha + \alpha(b_{\sigma^{-1}})\beta_{\sigma^{-1}}$, so $(W, L^*, \{\mathbb{Z}_p\beta_{\sigma^{-1}}\})$ is a $\mathbb{Z}_p$–root datum (the *dual* of $\mathbf{D}$). Applying Lemma 8.9 to this $\mathbb{Z}_p$–root datum shows that $M_0 \oplus (L^*)^W$ has finite index in L^*, so the dual homomorphism $L \to {M_0}^* \oplus \Big((L^*)^W\Big)^*$ is injective. The last summand is isomorphic to $\operatorname{Hom}_{\mathbb{Z}_p}(H_0(W;L), \mathbb{Z}_p)^*$ which again identifies with $H_0(W;L)$ modulo its torsion subgroup. The formula $\sigma(x) - x = \beta_\sigma(x)b_\sigma$ shows that the image of the composition $L_0 \to L \to H_0(W;L)$ is torsion, so by the above the composition $L_0 \to L \to L_{\breve{T}/A} \to {M_0}^*$ is injective. In particular W acts faithfully on $L_{\breve{T}/A}$ by Lemma 8.9. The homomorphism $\beta_\sigma \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty \colon \breve{T} \to \mathbb{Z}/p^\infty$ factors through $\breve{T}/A$, and we get a corresponding homomorphism $\beta'_\sigma \colon L_{\breve{T}/A} \to \mathbb{Z}_p$ such that the composition $L \to L_{\breve{T}/A} \xrightarrow{\beta'_\sigma} \mathbb{Z}_p$ agrees with β_σ. We now claim that $\sigma(x) = x + \beta'_\sigma(x)b_\sigma$ for $x \in L_{\breve{T}/A}$. To see this note that the image of L in $L_{\breve{T}/A}$ has finite index since we have the exact sequence $L \to L_{\breve{T}/A} \to \operatorname{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}/p^\infty, A)$ with $\operatorname{Ext}^1_{\mathbb{Z}_p}(\mathbb{Z}/p^\infty, A)$ finite. Thus the claim follows from the above by using the corresponding formula $\sigma(x) = x + \beta_\sigma(x)b_\sigma$ for $x \in L$. This proves that $\mathbf{D}/A = (W, L_{\breve{T}/A}, \{\mathbb{Z}_p b_\sigma\})$ is a $\mathbb{Z}_p$–root datum. In particular we obtain $\widetilde{\mathbf{D}/A} = (W, L_0, \{\mathbb{Z}_p b_\sigma\}) = \widetilde{\mathbf{D}}$ by definition.

To see the claim about $\breve{\mathcal{Z}}(\mathbf{D}/A)$, note that by the above $\beta_\sigma \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty \colon \breve{T} \to \mathbb{Z}/p^\infty$ identifies with the composition $\breve{T} \to \breve{T}/A \xrightarrow{\beta'_\sigma \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty} \mathbb{Z}/p^\infty$. Hence the singular sets for $\mathbf{D}$ and $\mathbf{D}/A$ satisfy $S_{\mathbf{D}}(\sigma)/A = S_{\mathbf{D}/A}(\sigma)$, and the claim follows.

To see part (2), let $i\colon BT \to BX$ be a maximal torus and let $f\colon BA \to BX$ be the central monomorphism. Then f factors through BT by [26, Prop. 8.11] to give a central monomorphism $g\colon BA \to BT$. Moreover i factors through the maximal torus normalizer $\mathcal{N}$ of X, and we obtain the diagram

$$\begin{tikzcd}
BT \arrow[r] \arrow[d] & B\mathcal{N} \arrow[r] \arrow[d] & BX \arrow[d] \\
BT/A \arrow[r] & B\mathcal{N}/A \arrow[r] & BX/A
\end{tikzcd}$$

cf. Construction 2.2. It follows that T/A is a maximal torus in X/A and $\mathcal{N}/A$ is the maximal torus normalizer. The Weyl groups of X and X/A naturally identify; cf. [48, Thm. 4.6]. By construction the elements $h'_\sigma \in \breve{T}/\breve{A}$ corresponding to

$\mathbf{D}_{X/A}$ are the images of the elements $h_\sigma \in \breve{T}$ corresponding to X (cf. the proof of Proposition 8.10). This shows that $\mathbf{D}_{X/A} \cong \mathbf{D}_X/\breve{A}$, as desired. □

As a special case of the quotient construction, we define the *adjoint* $\mathbf{D}_{\mathrm{ad}}$ of a $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ by $\mathbf{D}_{\mathrm{ad}} = \mathbf{D}/\breve{\mathcal{Z}}(\mathbf{D})$. Note that by Proposition 8.11(1) we have $\breve{\mathcal{Z}}(\mathbf{D}_{\mathrm{ad}}) = 0$ and that it follows from the proof that $\mathbf{D}_{\mathrm{ad}} = (W, {M_0}^*, \{\mathbb{Z}_p b_\sigma\})$, where M_0 is the *root lattice*, i.e., the sublattice of L^* spanned by the roots β_σ.

Proposition 8.12. *Any $\mathbb{Z}_p$–root datum with $\breve{\mathcal{Z}}(\mathbf{D}) = 0$ or $\pi_1(\mathbf{D}) = 0$ splits as a direct product $\mathbf{D} \cong \mathbf{D}_1 \times \ldots \times \mathbf{D}_n$ of irreducible $\mathbb{Z}_p$–root data $\mathbf{D}_i$.*

Proof. The case $\breve{\mathcal{Z}}(\mathbf{D}) = 0$ is essentially proved by Dwyer–Wilkerson [28, Pf. of Thm. 1.5]; for completeness we briefly sketch the argument: Let (W, L) be the $\mathbb{Z}_p$–reflection group associated to $\mathbf{D}$ and let $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_1 \oplus \ldots \oplus V_n$ be the decomposition of $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ into irreducible $\mathbb{Q}_p[W]$–modules. Define $L_i = L \cap V_i$. Since $\breve{\mathcal{Z}}(\mathbf{D}) = 0$ we have $\bigcap_\sigma \breve{T}_0^+(\sigma) = 0$ as well, and it follows [28, Pf. of Thm. 1.5] that the homomorphism $L_1 \times \ldots \times L_n \to L$ is an isomorphism. Letting W_i denote the pointwise stabilizer of $L_1 \oplus \ldots \oplus \widehat{L_i} \oplus \ldots \oplus L_n$ we hence (cf. [28, Prop. 7.1]) get a product decomposition $(W, L) \cong (W_1, L_1) \times \ldots \times (W_n, L_n)$. It is now clear that there is a unique $\mathbb{Z}_p$–root datum structure $\mathbf{D}_i$ on (W_i, L_i) such that we get a product decomposition $\mathbf{D} \cong \mathbf{D}_1 \times \ldots \times \mathbf{D}_n$ into irreducible $\mathbb{Z}_p$–root data.

The case where $\pi_1(\mathbf{D}) = 0$ is easily reduced to the first case using the previous results: By Proposition 8.11(1), $\breve{\mathcal{Z}}(\mathbf{D}_{\mathrm{ad}}) = 0$, so we can write $\mathbf{D}_{\mathrm{ad}} \cong \mathbf{D}_1 \times \ldots \times \mathbf{D}_n$, where the $\mathbf{D}_i$ are irreducible. Proposition 8.11(1) now shows that $\mathbf{D} = \widetilde{\mathbf{D}} \cong \widetilde{\mathbf{D}_{\mathrm{ad}}} \cong \widetilde{\mathbf{D}_1} \times \ldots \times \widetilde{\mathbf{D}_n}$, as claimed. □

Theorem 8.13 (The classification of $\mathbb{Z}_p$–root data; structure version)**.**

(1) *Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a $\mathbb{Z}_p$–root datum with coroot lattice L_0, and let $\mathbf{D}' = (W, L_0 \oplus L^W, \{\mathbb{Z}_p b_\sigma\}) = \widetilde{\mathbf{D}} \times \mathbf{D}_{\mathrm{triv}}$, where $\mathbf{D}_{\mathrm{triv}} = (1, L^W, \emptyset)$ is a trivial $\mathbb{Z}_p$–root datum. Then $\mathbf{D} \cong \mathbf{D}'/A$ for a finite central subgroup $A \subseteq \breve{\mathcal{Z}}(\mathbf{D}')$, and there is a splitting $\widetilde{\mathbf{D}} \cong \mathbf{D}_1 \times \ldots \times \mathbf{D}_n$ of $\widetilde{\mathbf{D}}$ into irreducible $\mathbb{Z}_p$–root data $\mathbf{D}_i$ with $\pi_1(\mathbf{D}_i) = 0$.*

(2) *For $p > 2$, the assignment $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\}) \rightsquigarrow (W, L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is a one-to-one correspondence between isomorphism classes of irreducible $\mathbb{Z}_p$–root data $\mathbf{D}$ with $\pi_1(\mathbf{D}) = 0$ and isomorphism classes of non-trivial irreducible $\mathbb{Q}_p$–reflection groups. For $p = 2$, the assignment is surjective and the preimage of every element consists of a single element, except for*

$$(W_{\mathrm{Sp}(n)}, L_{\mathrm{Sp}(n)} \otimes \mathbb{Q}_2) \cong (W_{\mathrm{Spin}(2n+1)}, L_{\mathrm{Spin}(2n+1)} \otimes \mathbb{Q}_2),$$

whose preimage consists of $\mathbf{D}_{\mathrm{Sp}(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and $\mathbf{D}_{\mathrm{Spin}(2n+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_2$ which are non-isomorphic for $n \geq 3$.

Proof. The fact that $\mathbf{D}'$ is a $\mathbb{Z}_p$–root datum follows from Lemma 8.9. The short exact sequence $0 \to L_0 \oplus L^W \to L \to F \to 0$, where F is finite with trivial W–action, produces a short exact sequence $1 \to A \to \breve{T}' \to \breve{T} \to 1$ between the associated discrete tori. Since the roots β'_σ for $\mathbf{D}'$ are given by $L_0 \oplus L^W \to L \xrightarrow{\beta_\sigma} \mathbb{Z}_p$, it follows that $A \subseteq S_{\mathbf{D}'}(\sigma) = \ker((L_0 \oplus L^W) \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty \to \mathbb{Z}/p^\infty)$ for any reflection σ. Hence $A \subseteq \breve{\mathcal{Z}}(\mathbf{D}')$ is central and $\mathbf{D} \cong \mathbf{D}'/A$. The last part of (1) follows from Proposition 8.12.

We now prove (2). For any prime p, Theorem 8.1(2) shows that the assignment in (2) gives a one-to-one correspondence between isomorphism classes of exotic $\mathbb{Z}_p$–root data and isomorphism classes of exotic $\mathbb{Q}_p$–reflection groups, and that $\pi_1(\mathbf{D}) = 0$ for all exotic $\mathbb{Z}_p$–root data $\mathbf{D}$. Hence it suffices by Theorem 8.1(1) to show the claim for $\mathbb{Z}_p$–root data of the form $\mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a $\mathbb{Z}$–root datum $\mathbf{D}_1$.

In this case it is clear that if $\pi_1(\mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p) = 0$, then we can find a $\mathbb{Z}$–root datum $\mathbf{D}_1'$ with $\pi_1(\mathbf{D}_1') = 0$ and $\mathbf{D}_1' \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ (simply choose $\mathbf{D}_1'$ to be the universal cover of $\mathbf{D}_1$; this is defined for $\mathbb{Z}$–root data in the same way as for $\mathbb{Z}_p$–root data). Hence it suffices to study the assignment $\mathbf{D} = (W, L, \{\mathbb{Z}b_\sigma\}) \rightsquigarrow (W, L \otimes_{\mathbb{Z}} \mathbb{Q})$ from irreducible $\mathbb{Z}$–root data with $\pi_1(\mathbf{D}) = 0$ to non-trivial irreducible $\mathbb{Q}$–reflection groups. It is well-known (cf. [9, §4]) that this assignment is surjective and that it only fails to be injective in that the $\mathbb{Z}$–root data $\mathbf{D}_{\mathrm{Sp}(n)}$ and $\mathbf{D}_{\mathrm{Spin}(2n+1)}$ which are non-isomorphic for $n \geq 3$ maps to the same $\mathbb{Q}$–reflection group. This proves part (2) since for $n \geq 3$, the $\mathbb{Z}_p$–root data $\mathbf{D}_{\mathrm{Sp}(n)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $\mathbf{D}_{\mathrm{Spin}(2n+1)} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ are non-isomorphic for $p = 2$ and isomorphic for $p \neq 2$. □

8.4. **Automorphisms.** Recall that an *isomorphism* between two $\mathbb{Z}_p$–root data $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ and $\mathbf{D}' = (W', L', \{\mathbb{Z}_p b'_{\sigma'}\})$ is an isomorphism $\varphi\colon L \to L'$ with the property that $\varphi W \varphi^{-1} = W'$ as subgroups of $\mathrm{Aut}(L')$ and $\varphi(\mathbb{Z}_p b_\sigma) = \mathbb{Z}_p b'_{\varphi\sigma\varphi^{-1}}$ for every reflection $\sigma \in W$. We denote the automorphism group of $\mathbf{D}$ by $\mathrm{Aut}(\mathbf{D})$; clearly W is a normal subgroup of $\mathrm{Aut}(\mathbf{D})$, and we define the *outer automorphism group* $\mathrm{Out}(\mathbf{D})$ by $\mathrm{Out}(\mathbf{D}) = \mathrm{Aut}(\mathbf{D})/W$.

Recall that a $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ is called irreducible if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an irreducible $\mathbb{Q}_p[W]$–module. The following proposition is a restatement of [8, Prop. 5.4].

Proposition 8.14. *Suppose $\mathbf{D}_i = (W_i, L_i, \{\mathbb{Z}_p b_\sigma\}_{\sigma \in \Sigma_i})$, $i = 0, \ldots, k$, is a collection of pairwise non-isomorphic irreducible $\mathbb{Z}_p$–root data. Assume that $W_0 = 1$ but that W_i is non-trivial for $i \geq 1$. Let $\mathbf{D} = \prod_{i=0}^k \mathbf{D}_i^{m_i}$ denote a product of these $\mathbb{Z}_p$–root data with $m_i \geq 1$ for $i \geq 1$. Then*

$$\mathrm{GL}_{m_0}(\mathbb{Z}_p) \times \left(\prod_{i=1}^k \mathrm{Out}(\mathbf{D}_i) \wr \Sigma_{m_i} \right) \xrightarrow{\cong} \mathrm{Out}(\mathbf{D}).$$

Proof. This follows directly by combining [8, Prop. 5.4] with [7, Rem. 4.5]. □

The following two results are needed for the reduction in Section 2.

Proposition 8.15. *Let $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ be a $\mathbb{Z}_p$–root datum with coroot lattice L_0. Let $\mathbf{D}' = (W, L_0 \oplus L^W, \{\mathbb{Z}_p b_\sigma\}) = \widetilde{\mathbf{D}} \times \mathbf{D}_{\mathrm{triv}}$, where $\mathbf{D}_{\mathrm{triv}} = (1, L^W, \emptyset)$ is a trivial $\mathbb{Z}_p$–root datum. Then $\mathbf{D} \cong \mathbf{D}'/A$ for a finite subgroup $A \subseteq \check{\mathcal{Z}}(\mathbf{D}')$, and the restriction*

$$\mathrm{Aut}(\mathbf{D}) \longrightarrow \mathrm{Aut}(\mathbf{D}') = \mathrm{Aut}(\widetilde{\mathbf{D}}) \times \mathrm{Aut}(\mathbf{D}_{\mathrm{triv}})$$

is an isomorphism onto the subgroup $\{\varphi \in \mathrm{Aut}(\mathbf{D}') \mid \varphi(A) = A\}$. In particular $\mathrm{Out}(\mathbf{D})$ identifies with a subgroup of finite index in $\mathrm{Out}(\mathbf{D}')$.

Remark 8.16. For a $\mathbb{Z}_p$–root datum $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$ with $\pi_1(\mathbf{D}) = 0$ we have $\mathbb{Z}_p b_\sigma = \ker(L \xrightarrow{N} L)$, where $N = 1 + \sigma + \ldots + \sigma^{|\sigma|-1}$ is the norm element: By Theorem 8.1(1) either $\mathbf{D} \cong \mathbf{D}_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ for a $\mathbb{Z}$–root datum $\mathbf{D}_1$ with $\pi_1(\mathbf{D}_1) = 0$ (cf. the proof of Theorem 8.13(2)) or $\mathbf{D}$ is exotic. In the first case the result is well-known [9, §4], and in the second case the claim follows since $H^1(\langle\sigma\rangle; L) = 0$ for all reflections σ (cf. the proof of Theorem 8.1(2)). In particular we obtain $\mathrm{Out}(\mathbf{D}) = N_{\mathrm{GL}(L)}(W)/W$, and this group is explicitly computed for all finite irreducible $\mathbb{Z}$–reflection groups (W, L) in [8, Thm. 13.1].

Proof of Proposition 8.15. The fact that $\mathbf{D} \cong \mathbf{D}'/A$ for a finite subgroup $A \subseteq \breve{\mathcal{Z}}(\mathbf{D}')$ is part of Theorem 8.13(1), and the identification $\mathrm{Aut}(\mathbf{D}') = \mathrm{Aut}(\widetilde{\mathbf{D}}) \times \mathrm{Aut}(\mathbf{D}_{\mathrm{triv}})$ comes from Proposition 8.14. Clearly both L_0 and L^W are invariant under $\mathrm{Aut}(\mathbf{D})$, so we get a restriction homomorphism $\mathrm{Aut}(\mathbf{D}) \longrightarrow \mathrm{Aut}(\mathbf{D}')$ which is injective since $L_0 \oplus L^W$ has finite index in L by Lemma 8.9. If $\varphi \in \mathrm{Aut}(\mathbf{D})$, then φ corresponds to an automorphism $\varphi\colon \breve{T} \to \breve{T}$ and the restriction of φ to $L_0 \oplus L^W$ corresponds to a lift $\widetilde{\varphi}\colon \breve{T}' \to \breve{T}'$ which sends A to itself. Conversely, if $\varphi \in \mathrm{Aut}(\mathbf{D}')$ with $\varphi(A) = A$, then φ clearly induces an automorphism of $\mathbf{D}$. The last claim follows from the fact that the orbit of A under $\mathrm{Out}(\mathbf{D}')$ is finite since $\breve{\mathcal{Z}}(\mathbf{D}')$ has only finitely many subgroups isomorphic to A. □

Corollary 8.17. *For any $\mathbb{Z}_p$–root datum $\mathbf{D}$ there is a canonical isomorphism*

$$\mathrm{Aut}(\widetilde{\mathbf{D}}) \xleftrightarrow{\cong} \mathrm{Aut}(\mathbf{D}_{\mathrm{ad}}).$$

Proof. Write $\mathbf{D} = (W, L, \{\mathbb{Z}_p b_\sigma\})$, and let $M_0 \subseteq L^*$ denote the root lattice, i.e., the lattice spanned by the roots β_σ. Then $\widetilde{\mathbf{D}} = (W, L_0, \{\mathbb{Z}_p b_\sigma\})$, and the roots for $\widetilde{\mathbf{D}}$ are given by the composition $L_0 \to L \xrightarrow{\beta_\sigma} \mathbb{Z}_p$. Hence we can identify the root lattice for $\widetilde{\mathbf{D}}$ with M_0, and hence $\widetilde{\mathbf{D}}/\breve{\mathcal{Z}}(\widetilde{\mathbf{D}}) = (\widetilde{\mathbf{D}})_{\mathrm{ad}} \cong (W, {M_0}^*, \{\mathbb{Z}_p b_\sigma\}) = \mathbf{D}_{\mathrm{ad}}$. The result now follows from Proposition 8.15. □

8.5. **Finiteness properties.** In this final subsection we prove that there are only finitely many $\mathbb{Z}_p$–root data of a given rank and that for a fixed $\mathbb{Z}_p$–root datum $\mathbf{D}$, $\mathrm{Out}(\mathbf{D})$ only contains finitely many finite subgroups up to conjugation. These results are used for the proofs of the finiteness statements in Theorems 1.5 and 1.6 given in Section 7.

Proposition 8.18. *For any prime p there is, up to isomorphism, only finitely many $\mathbb{Z}_p$–root data of a fixed rank.*

Proof. Note first that the order of a finite subgroup $G \subseteq \mathrm{GL}_n(\mathbb{Z}_p)$ is bounded above: It is easily seen that the composition $G \to \mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{F}_p)$ is injective for $p > 2$ and has kernel of order at most 2^{n^2} for $p = 2$ (cf. [8, Lem. 11.3]). Hence G has order at most $|\mathrm{GL}_n(\mathbb{F}_p)|$ for $p > 2$ and $2^{n^2} \cdot |\mathrm{GL}_n(\mathbb{F}_2)|$ for $p = 2$. Since there are only finitely many isomorphism classes of groups of a given order, the local version of the Jordan–Zassenhaus theorem [14, Thm. 24.7] now implies that, up to isomorphism, there are only finitely many finite $\mathbb{Z}_p$–reflection groups of fixed rank. Finally, choosing a $\mathbb{Z}_p$–root datum for a finite $\mathbb{Z}_p$–reflection group (W, L) corresponds to choosing a cyclic subgroup of the finite group $H^1(\langle\sigma\rangle; L)$ for each conjugacy class of reflections σ. Hence any finite $\mathbb{Z}_p$–reflection group gives rise to only finitely many $\mathbb{Z}_p$–root data. This proves the result. □

Proposition 8.19. *Let $\mathbf{D}$ be a $\mathbb{Z}_p$–root datum. Then* $\mathrm{Out}(\mathbf{D})$ *contains only finitely many conjugacy classes of finite subgroups.*

Proof. Let (W, L) be the finite $\mathbb{Z}_p$–reflection group associated to $\mathbf{D}$. From the first part of the proof of Proposition 8.18 it follows that there is an upper bound on the order of a finite subgroup of $N_{\mathrm{GL}(L)}(W) \subseteq \mathrm{GL}(L)$. Since $\mathrm{Out}(\mathbf{D})$ is contained in

$N_{\mathrm{GL}(L)}(W)/W$ and W is finite, it follows that there is also an upper bound on the order of a finite subgroup of $\mathrm{Out}(\mathbf{D})$. It thus suffices to show that $\mathrm{Rep}(G, \mathrm{Out}(\mathbf{D}))$ is finite for any finite group G (where $\mathrm{Rep}(G, H)$ as usual denotes the set of homomorphisms $G \to H$ modulo conjugation in H). By Theorem 8.13(1), we can write $\mathbf{D} \cong \mathbf{D}'/A$ where $\mathbf{D}' = \widetilde{\mathbf{D}} \times \mathbf{D}_{\mathrm{triv}}$ and $\mathbf{D}_{\mathrm{triv}}$ is a trivial $\mathbb{Z}_p$–root datum. This identifies $\mathrm{Out}(\mathbf{D})$ with a subgroup of finite index in $\mathrm{Out}(\mathbf{D}')$, cf. Proposition 8.15, so it is enough to prove that $\mathrm{Rep}(G, \mathrm{Out}(\mathbf{D}'))$ is finite.

By Theorem 8.13(1) and Proposition 8.14, $\mathrm{Out}(\mathbf{D}')$ is isomorphic to a direct product of $\mathrm{GL}_{m_0}(\mathbb{Z}_p)$ and groups of the form $\mathrm{Out}(\mathbf{D}_i) \wr \Sigma_{m_i}$, where the $\mathbf{D}_i$ are irreducible $\mathbb{Z}_p$–root data, so we need to show that $\mathrm{Rep}(G, \mathrm{GL}_m(\mathbb{Z}_p))$ and $\mathrm{Rep}(G, \mathrm{Out}(\mathbf{D}) \wr \Sigma_m)$ are finite for any m and any irreducible $\mathbb{Z}_p$–root datum $\mathbf{D}$. The first claim follows directly from the local version of the Jordan–Zassenhaus theorem [14, Thm. 24.7].

To see the second claim, let W denote the Weyl group of $\mathbf{D}$ and note that since $\mathbf{D}$ is irreducible, Schur's lemma implies that the image of the central homomorphism $\mathbb{Z}_p^\times \to \mathrm{Out}(\mathbf{D})$ equals the kernel of the canonical homomorphism $\mathrm{Out}(\mathbf{D}) \to \mathrm{Out}(W)$. Since $\mathbb{Z}_p^\times \cong \mathbb{Z}_p \times C$, where C is finite ($C = \mathbb{Z}/2$ for $p = 2$ and $C = \mathbb{Z}/(p-1)$ for $p > 2$) and $\mathrm{Out}(W)$ is finite, it follows that $\mathrm{Out}(\mathbf{D})$ is a finite central extension of $\mathbb{Z}_p$. In particular $E = \mathrm{Out}(\mathbf{D}) \wr \Sigma_m$ fits into an extension of the form

$$1 \to (\mathbb{Z}_p)^m \to E \xrightarrow{\pi} Q \to 1, \tag{8.4}$$

where Q is finite. Any homomorphism $\varphi\colon G \to E$ gives a homomorphism $\pi \circ \varphi\colon G \to Q$ by composition, and for a fixed homomorphism $\alpha\colon G \to Q$, the set of homomorphisms $\varphi\colon G \to E$ with $\pi \circ \varphi = \alpha$ equals the set of splittings of the pull-back $1 \to (\mathbb{Z}_p)^m \to E' \to G \to 1$ of (8.4) along $\alpha\colon G \to Q$. The set of such splittings, modulo conjugation by elements in $(\mathbb{Z}_p)^m$, is in one-to-one correspondence with $H^1(G; (\mathbb{Z}_p)^m)$, which is finite for all actions of G on $(\mathbb{Z}_p)^m$. Hence the set of homomorphisms $\varphi\colon G \to E$ with $\pi \circ \varphi = \alpha$ is finite modulo conjugation in $(\mathbb{Z}_p)^m \subseteq E$. Since Q is finite, there are only finitely many homomorphisms $\alpha\colon G \to Q$, so we conclude that $\mathrm{Rep}(G, E)$ is finite, as claimed. $\square$

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