

STUDY OF A \mathbf{Z} -FORM OF THE COORDINATE RING OF A REDUCTIVE GROUP

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INTRODUCTION

In his famous paper [C1] Chevalley associated to any root datum of adjoint type and to any field k a certain group (now known as a Chevalley group) which in the case where $k = \mathbf{C}$ was the usual adjoint Lie group over \mathbf{C} and which in the case where k is finite led to some new families of finite simple groups. Let \mathbf{O}' be the coordinate ring of a connected semisimple group G over \mathbf{C} attached to a fixed (semisimple) root datum. In a sequel [C2] to [C1], Chevalley defined a \mathbf{Z} -form of \mathbf{O}' . Later, another construction of such a \mathbf{Z} -form was proposed by Kostant [Ko]. Kostant notes that \mathbf{O}' can be viewed as a “restricted” dual of the universal enveloping algebra \mathbf{U} of Lie G ; he defines a \mathbf{Z} -form $\mathbf{U}_{\mathbf{Z}}$ of \mathbf{U} (“the Kostant \mathbf{Z} -form”) and then defines the \mathbf{Z} -form $\mathbf{O}'_{\mathbf{Z}}$ as the set of all elements in \mathbf{O}' which take integral values on $\mathbf{U}_{\mathbf{Z}}$. Then for any commutative ring A with 1 he defines \mathbf{O}'_A as $A \otimes \mathbf{O}'_{\mathbf{Z}}$; this is naturally a Hopf algebra over A . It follows that the set G_A of A -algebra homomorphisms $\mathbf{O}'_A \rightarrow A$ has a natural group structure. Thus the root datum gives rise to a family of groups G_A , one for each A as above.

Unlike Chevalley’s approach which was based on a choice of a faithful representation of G , Kostant’s approach is direct (no choices involved) and generalizes to the quantum case.

In this paper we develop the theory of Chevalley groups following Kostant’s approach. We shall prove that:

(I) *If A is an algebraically closed field, then \mathbf{O}'_A is the coordinate algebra of a connected semisimple algebraic group over A corresponding to the given root datum.* (We treat the reductive case at the same time.) Note that (I) was stated without proof in [Ko].

In this paper we note that Kostant’s definition can be reformulated by replacing \mathbf{U} by a “modified enveloping algebra”. The theory is then developed using extensively the theory of canonical bases of such modified enveloping algebras (presented in [L1]), coming from quantum groups. (See the Notes in [L1] for references to original sources concerning canonical bases.)

We now present the content of this paper in more detail.

Let A be a fixed commutative ring with 1 with a given invertible element $v \in A$.

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In §1 we recall the definition and some properties of the modified (quantized) enveloping algebra $\dot{\mathbf{U}}_A$ over A and its canonical basis $\dot{\mathbf{B}}$. We also define a “completion” $\hat{\mathbf{U}}_A$ of $\dot{\mathbf{U}}_A$ which consists of formal (possibly infinite) A -linear combinations of elements in $\dot{\mathbf{B}}$. We show that the multiplication and “comultiplication” of $\dot{\mathbf{U}}_A$ extend naturally to $\hat{\mathbf{U}}_A$.

In §2 we introduce some invertible elements $s'_{i,e}$ of $\hat{\mathbf{U}}_A$ (where i corresponds to a simple reflection and $e = \pm 1$). We show that conjugation by $s'_{i,e}$ restricted to $\dot{\mathbf{U}}_A$ is essentially the action of a generator in the braid group action on $\dot{\mathbf{U}}_A$ studied in [L1]. Thus $s'_{i,e}$ plays the same role as an element considered in a similar context (with $A = \mathbf{C}(v)$) by Soibelman [So]. But while Soibelman’s element is not explicit and its integrality properties are not clear, our element $s'_{i,e}$ is remarkably simple and has obvious integrality properties.

In §3 we define following [L1, 29.5.2] the Hopf algebra \mathbf{O}_A (a quantum analogue of \mathbf{O}'_A above). We prove that the A -algebra \mathbf{O}_A is finitely generated (see Proposition 3.3) with an explicit set of generators. In 3.7 we show that the A -algebra \mathbf{O}_A can be imbedded into the tensor product of two simpler algebras. (For a closely related result in the case where $A = \mathbf{Q}(v)$, see [Jo, 9.3.13].) In the case where $v = 1$ in A these simpler algebras can be explicitly described in terms of the braid group action; see 3.13. We deduce that, if A is an integral domain and $v = 1$ in A , then \mathbf{O}_A is an integral domain; see Theorem 3.15. (For a similar result in the case where $A = \mathbf{Q}(v)$, see [Jo, 9.1.9].)

In §4 we assume that $v = 1$ in A and we introduce the group G_A in analogy with [Ko]. In Theorem 4.11 we show that \mathbf{O}_A has a property like (I) above.

In §5 we identify (assuming that $v = 1$ in A) our \mathbf{O}_A with Kostant’s \mathbf{O}'_A .

We refer the reader to Borel’s paper in [B2] and to Steinberg’s book [St] for expositions of Chevalley’s approach to his groups which make use of the Kostant \mathbf{Z} -form $\mathbf{U}_{\mathbf{Z}}$. We also refer the reader to Demazure’s paper in [DG, Exposé XXV] where another proof of the existence of a reductive group over \mathbf{Z} with a given root datum is given, assuming that the corresponding group over \mathbf{C} is already known.

Notation. If A' is a commutative ring with 1 and S is a set, we denote by $A'[S]$ the free A' -module with basis indexed by S .

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1. THE ALGEBRAS $\dot{\mathbf{U}}_A, \hat{\mathbf{U}}_A$

1.1. In this section we recall the definition of the modified quantized enveloping algebra $\dot{\mathbf{U}}_A$ (over A) attached to a root datum and we recall the definition and some of the properties of the canonical basis $\dot{\mathbf{B}}$ of $\dot{\mathbf{U}}_A$. We also study a certain completion $\hat{\mathbf{U}}_A$ of $\dot{\mathbf{U}}_A$.

We fix a root datum as in [L1, 2.2]. This consists of two free abelian groups of finite type Y, X with a given perfect pairing $\langle, \rangle : Y \times X \rightarrow \mathbf{Z}$ and a finite set I with given imbeddings $I \rightarrow Y$ ($i \mapsto i$) and $I \rightarrow X$ ($i \mapsto i'$) such that $\langle i, i' \rangle = 2$ for

all $i \in I$ and $\langle i, j' \rangle \in -\mathbf{N}$ for all $i \neq j$ in I ; in addition, we are given a symmetric bilinear form $\mathbf{Z}[I] \times \mathbf{Z}[I] \rightarrow \mathbf{Z}$, $\nu, \nu' \mapsto \nu \cdot \nu'$ such that $i \cdot i \in 2\mathbf{Z}_{>0}$ for all $i \in I$ and $\langle i, j' \rangle = 2i \cdot j / i \cdot i$ for all $i \neq j$ in I . We assume that the matrix $(i \cdot j)_{i,j \in I}$ is positive definite.

Let $X^+ = \{\lambda \in X; \langle i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$. For λ, λ' in X we write $\lambda \geq \lambda'$ or $\lambda' \leq \lambda$ if $\lambda - \lambda' \in \sum_{i \in I} \mathbf{N}i'$. The image of $\nu \in \mathbf{Z}[I]$ under the homomorphism $\mathbf{Z}[I] \rightarrow X$ such that $i \mapsto i'$ for $i \in I$ is denoted again by ν .

Let W be the (finite) subgroup of $\text{Aut}(Y)$ generated by the involutions $s_i : y \mapsto y - \langle y, i' \rangle i$ ($i \in I$) or equivalently the subgroup of $\text{Aut}(X)$ generated by the involutions $s_i : x - \langle i, x \rangle i'$ ($i \in I$); these two subgroups may be identified by taking contragredients. For $i \neq j$ in I let $n_{ij} = n_{ji}$ be the order of $s_i s_j$ in W . Let $l : W \rightarrow \mathbf{N}$ be the standard length function on W with respect to $\{s_i; i \in I\}$. Let $w_0 \in W$ be the unique element such that $l(w_0)$ is maximal.

1.2. Let v be an indeterminate. Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. For $i \in I$ we set $v_i = v^{i \cdot i/2}$.

We fix a commutative ring A with 1 with a given ring homomorphism $\mathcal{A} \rightarrow A$ respecting 1. For $\alpha \in \mathcal{A}$ we shall often denote the image of α under $\mathcal{A} \rightarrow A$ again by α . Whenever we write $A = \mathbf{Q}(v)$ or $A = \mathcal{A}$, we shall understand that A is regarded as an \mathcal{A} -algebra in the obvious way. Whenever we write $A = \mathbf{Q}$ or $A = \mathbf{Z}$, we shall understand that A is regarded as an \mathcal{A} -algebra with $v = 1$ in A .

Let A° be the group of invertible elements of the ring A .

An A -linear map $\phi : H \rightarrow H'$ where H, H' are A -modules is said to be a *split injection* if there exists an A -linear map $\psi : H' \rightarrow H$ such that $\psi\phi = 1$.

1.3. Let \mathbf{f} be the associative $\mathbf{Q}(v)$ -algebra with 1 defined as in [L1, 1.2.5] or equivalently by the generators θ_i ($i \in I$) and the quantum Serre relations

$$\sum_{p,p' \in \mathbf{N}; p+p'=1-\langle i, j' \rangle} (-1)^{p'} (\theta_i^p / [p]_i!) \theta_j (\theta_i^{p'} / [p']_i!)$$

for $i \neq j$ in I , where $[p]_i! = \prod_{s=1}^p (v_i^s - v_i^{-s}) / (v_i - v_i^{-1})$ for $p \in \mathbf{N}$. We have a direct sum decomposition $\mathbf{f} = \bigoplus_{\nu \in \mathbf{Z}[I]} \mathbf{f}_\nu$ as a vector space, where \mathbf{f}_ν is spanned by words in θ_i in which the number of apparitions of θ_i is the coefficient of i in ν , for all $i \in I$. For $i \in I, n \in \mathbf{Z}$ we set $\theta_i^{(n)} = \theta_i^n / [n]_i!$ in \mathbf{f} if $n \geq 0$ and $\theta_i^{(n)} = 0$ if $n < 0$. Let \mathbf{f}_A be the \mathcal{A} -subalgebra with 1 of \mathbf{f} generated by the elements $\theta_i^{(n)}$ with $i \in I, n \in \mathbf{Z}$. We have $\mathbf{f}_A = \bigoplus_{\nu \in \mathbf{Z}[I]} \mathbf{f}_{\nu, A}$ where $\mathbf{f}_{\nu, A} = \mathbf{f}_\nu \cap \mathbf{f}_A$. Let $\mathbf{f}_A = A \otimes_{\mathcal{A}} \mathbf{f}_A$, an A -algebra. We have $\mathbf{f}_A = \bigoplus_{\nu \in \mathbf{Z}[I]} \mathbf{f}_{\nu, A}$ where $\mathbf{f}_{\nu, A} = A \otimes_{\mathcal{A}} \mathbf{f}_{\nu, A}$.

Let \mathbf{B} be the canonical basis of \mathbf{f} (see [L1, 14.4]). Note that \mathbf{B} is also an \mathcal{A} -basis of \mathbf{f}_A . If $b \in \mathbf{B}$, we shall denote the element $1 \otimes_A b \in A \otimes_{\mathcal{A}} \mathbf{f}_A = \mathbf{f}_A$ again by b . Thus \mathbf{B} can be viewed also as an A -basis of \mathbf{f}_A .

1.4. For any $n \in \mathbf{Z}, t \in \mathbf{N}$ and $i \in I$ we define $\begin{bmatrix} n \\ t \end{bmatrix}_i \in \mathcal{A}$ as in [L1, 1.3.1, 1.1.2].

Let $\dot{\mathbf{U}}_A$ be the A -algebra generated by the symbols $x^+ 1_\zeta x'^-, x^- 1_\zeta x'^+$ with $x \in \mathbf{f}_{\nu, A}, x' \in \mathbf{f}_{\nu', A}$ for various ν, ν' and $\zeta \in X$; these symbols are subject to the following relations:

$$\begin{aligned} \mathbf{f}_A &\rightarrow \dot{\mathbf{U}}_A, x \mapsto x^\pm 1_\zeta \text{ is } A\text{-linear for any } \zeta \in X; \\ \theta_i^{(n)+} 1_\zeta \theta_j^{(m)-} &= \theta_j^{(m)-} 1_\zeta \theta_i^{(n)+} \text{ if } m, n \in \mathbf{N}, \zeta \in X, i \neq j; \\ \theta_i^{(n)\pm} 1_{\mp \zeta} \theta_i^{(m)\mp} &= \sum_{t \in \mathbf{N}} [m+n-\langle i, \zeta \rangle t]_i \theta_i^{(m-t)\mp} 1_{\mp \zeta \pm (a+b-t)\nu} \theta_i^{(n-t)\pm} \text{ if } m, n \in \mathbf{N}, \zeta \in X, i \in I; \end{aligned}$$

$$\begin{aligned} x^\pm 1_\zeta &= 1_{\zeta \pm \nu} x^\pm \text{ if } x \in \mathfrak{f}_{\nu, A}, \zeta \in X; \\ (x^\pm 1_\zeta)(1_{\zeta'} x'^{\mp}) &= \delta_{\zeta, \zeta'} x^\pm 1_\zeta x'^{\mp} \text{ if } x, x' \in \mathfrak{f}_A, \zeta \in X; \\ (x^\pm 1_\zeta)(1_{\zeta'} x'^{\pm}) &= \delta_{\zeta, \zeta'} 1_{\zeta \pm \nu} (xx')^\pm \text{ if } x \in \mathfrak{f}_{\nu, A}, x' \in \mathfrak{f}_A, \zeta \in X. \end{aligned}$$

If x or x' in $x^\pm 1_\zeta x'^{\mp}$ or $x^{-1} 1_\zeta x'^{+}$ is 1, we omit writing it.

(See [L1, 31.1.1, 31.1.3].) Note that the two A -linear maps $\mathfrak{f}_A \otimes_A A[X] \otimes_A \mathfrak{f}_A \rightarrow \dot{\mathbf{U}}_A$, $x \otimes \lambda \otimes x' \mapsto x^\pm 1_\lambda x'^{\mp}$ and $x \otimes \lambda \otimes x' \mapsto x^{-1} 1_\lambda x'^{+}$ are isomorphisms of A -modules. Hence we have canonically $\dot{\mathbf{U}}_A = A \otimes_A \dot{\mathbf{U}}_A$ as A -algebras.

Now $\dot{\mathbf{U}}_A$ does not have 1 in general; instead it has a family of elements $1_\lambda (\lambda \in X)$ such that $1_\lambda 1_{\lambda'} = \delta_{\lambda, \lambda'}$ for any λ, λ' in X and such that $\dot{\mathbf{U}}_A = \sum_{\lambda, \lambda' \in X} 1_\lambda \dot{\mathbf{U}}_A 1_{\lambda'}$ (necessarily a direct sum).

In the case where $A = \mathbf{Q}(v)$ we write $\dot{\mathbf{U}}$ instead of $\dot{\mathbf{U}}_A$. Now the obvious map $\dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}$ induced by the inclusion $\mathcal{A} \subset \mathbf{Q}(v)$ is injective and identifies $\dot{\mathbf{U}}_A$ with an \mathcal{A} -subalgebra of $\dot{\mathbf{U}}$. For any $\lambda, \lambda', \lambda_1, \lambda_2, \lambda'_1, \lambda'_2$ in X we define an A -linear map

$$\Delta_{\lambda, \lambda', \lambda_1, \lambda'_1, \lambda_2, \lambda'_2} : 1_\lambda \dot{\mathbf{U}}_A 1_{\lambda'} \rightarrow (1_{\lambda_1} \dot{\mathbf{U}}_A 1_{\lambda'_1}) \otimes_A (1_{\lambda_2} \dot{\mathbf{U}}_A 1_{\lambda'_2})$$

as follows: if $\lambda = \lambda_1 + \lambda_2$, $\lambda' = \lambda'_1 + \lambda'_2$, it is the map obtained by applying $A \otimes_A ?$ to the \mathcal{A} -linear map at the end of [L1, 23.2.3]; otherwise, it is 0. We define an algebra isomorphism S from $\dot{\mathbf{U}}_A$ to $\dot{\mathbf{U}}_A$ with opposed multiplication as follows: if $A = \mathbf{Q}(v)$, S is as in [L1, 23.1.6]; next, $S : \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A$ is the restriction of $S : \dot{\mathbf{U}} \rightarrow \dot{\mathbf{U}}$ to $\dot{\mathbf{U}}_A$; for general A , S is obtained by applying $A \otimes_A ?$ to the \mathcal{A} -linear map $S : \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A$. Let $\omega : \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A$ be the algebra automorphism (with square 1) defined in [L1, 31.1.4].

1.5. Let $\dot{\mathbf{B}}$ be the canonical basis of $\dot{\mathbf{U}}_A$ (see [L1, 25.2]). If $b \in \dot{\mathbf{B}}$, we shall denote the element $1 \otimes_A b \in A \otimes_A \dot{\mathbf{U}}_A = \dot{\mathbf{U}}_A$ again by b . Thus $\dot{\mathbf{B}}$ can be viewed also as an A -basis of $\dot{\mathbf{U}}_A$.

Note that $1_\lambda \in \dot{\mathbf{B}}$ for any $\lambda \in X$. We have $\dot{\mathbf{B}} = \bigsqcup_{\lambda, \lambda' \in X} (\dot{\mathbf{B}} \cap (1_\lambda \dot{\mathbf{U}}_A 1_{\lambda'}))$. For a, b in $\dot{\mathbf{B}}$ we write $ab = \sum_{c \in \dot{\mathbf{B}}} m_{a,b}^c c$ (ab is the product in $\dot{\mathbf{U}}_A$ and $m_{a,b}^c \in A$ is 0 for all but finitely many c). For a, b, c in $\dot{\mathbf{B}}$ we define $\hat{m}_{a,b}^c \in A$ by the following requirement: for any $\lambda, \lambda', \lambda_1, \lambda'_1, \lambda_2, \lambda'_2$ in X and any $c \in \dot{\mathbf{B}} \cap (1_\lambda \dot{\mathbf{U}}_A 1_{\lambda'})$ we have $\Delta_{\lambda, \lambda', \lambda_1, \lambda'_1, \lambda_2, \lambda'_2} c = \sum_{a,b} \hat{m}_{a,b}^c a \otimes b$ where a runs over $\dot{\mathbf{B}} \cap (1_{\lambda_1} \dot{\mathbf{U}}_A 1_{\lambda'_1})$, b runs over $\dot{\mathbf{B}} \cap (1_{\lambda_2} \dot{\mathbf{U}}_A 1_{\lambda'_2})$ (in the last sum $\hat{m}_{a,b}^c$ is 0 for all but finitely many (a, b)).

For any $a \in \dot{\mathbf{B}}$ we have $S(a) = s_a \underline{a}$ where $a \mapsto \underline{a}$ is an involution of $\dot{\mathbf{B}}$ and s_a is \pm a power of v with $s_{\underline{a}} = s_a$ (see [L1, 23.1.7, 26.3.2]). We have $s_{1_\lambda} = 1$, $1_{\underline{\lambda}} = 1_{-\lambda}$ for $\lambda \in X$ (see [L1, 23.1.6]). For any $a \in \dot{\mathbf{B}}$ we have $\omega(a) = e_a a'$ where $a \mapsto a'$ is an involution of $\dot{\mathbf{B}}$ and $e_a = \pm 1$ (see [L1, 26.3.2]). We have $e_{1_\lambda} = 1$, $(1_\lambda)' = 1_{-\lambda}$ for $\lambda \in X$.

Note that the quantities $m_{a,b}^c, \hat{m}_{a,b}^c, s_a, e_a$ (in A) are the images of the corresponding quantities in \mathcal{A} under the given homomorphism $\mathcal{A} \rightarrow A$ and the involutions $a \mapsto \underline{a}, a \mapsto a'$ are independent of A .

As in [L1, 25.4] for any a, b, d, e in $\dot{\mathbf{B}}$ we have

- (i) $\sum_c m_{a,b}^c m_{c,d}^e = \sum_c m_{a,c}^e m_{b,d}^c$;
- (ii) $\sum_c \hat{m}_{a,b}^c \hat{m}_{c,d}^e = \sum_c \hat{m}_{e,c}^a \hat{m}_{b,d}^c$;
- (iii) $\sum_c m_{a,b}^c \hat{m}_{c,d}^e = \sum_{a', b', c', d'} \hat{m}_{a', b'}^{a, b'} \hat{m}_{b', c'}^{c, d'} m_{a', c'}^e m_{b', d'}^d$;
- (iv) $\hat{m}_{1_\lambda}^{a,b} = 1$ if $a = 1_{\lambda'}, b = 1_{\lambda''}, \lambda' + \lambda'' = \lambda$ and $\hat{m}_{1_\lambda}^{a,b} = 0$, otherwise.

From the definitions we have for any a, b, c in $\dot{\mathbf{B}}$:

- (v) $s_c m_{a,b}^c = s_a s_b m_{b,a}^c$;
- (vi) $s_c \hat{m}_c^{b,a} = s_a s_b \hat{m}_c^{a,b}$;
- (vii) $e_c m_{a,b}^c = e_a e_b m_{a',b'}^c$;
- (viii) $e_c \hat{m}_c^{b',a'} = e_a e_b \hat{m}_c^{a,b}$;
- (ix) $\sum_{\lambda \in X} m_{a,1_\lambda}^c = \sum_{\lambda \in X} m_{1_\lambda,a}^c = \delta_{a,c}$;
- (x) $\hat{m}_c^{a,1_0} = \hat{m}_c^{1_0,a} = \delta_{a,c}$;
- (xi) $\sum_{d,e \in \dot{\mathbf{B}}} \hat{m}_a^{d,e} m_{d,\bar{e}}^b s_d$ is 1 if $b = 1_\lambda, a = 1_0$ and is 0 otherwise;
- (xii) $m_{a,b}^{1_0} = \delta_{a,1_0} \delta_{b,1_0}$;
- (xiii) $\hat{m}_c^{a,b} = \hat{m}_c^{b,a}$ in A if $v = 1$ in A .

1.6. Let \mathfrak{C}_A be the category whose objects are $\dot{\mathbf{U}}_A$ -modules M which are unital in the following sense: for any $z \in M$ we have $1_\lambda z = 0$ for all but finitely many $\lambda \in X$ and $\sum_{\lambda \in X} 1_\lambda z = z$ and which are finitely generated as an A -module. (A unital $\dot{\mathbf{U}}_A$ -module M is also an A -module by $a : z \mapsto \sum_{\lambda \in X} (a1_\lambda)z$ (all but finitely many terms of the last sum are 0)). A morphism in \mathfrak{C}_A is a $\dot{\mathbf{U}}_A$ -linear map. When $A = \mathbf{Q}(v)$, we write \mathfrak{C} instead of \mathfrak{C}_A .

Let $M, M' \in \mathfrak{C}_A$. The tensor product $M \otimes_A M'$ will be regarded as a $\dot{\mathbf{U}}_A$ -module by the rule $c(z \otimes z') = \sum_{a,b \in \dot{\mathbf{B}}} \hat{m}_c^{a,b} a z \otimes b z'$ where $c \in \dot{\mathbf{B}}, z \in M, z' \in M'$. (All but finitely many terms of the last sum are 0.) The fact that the rule above defines an $\dot{\mathbf{U}}_A$ -module structure follows from 1.5(iii). We have $M \otimes_A M' \in \mathfrak{C}_A$. This makes \mathfrak{C}_A into a monoidal tensor category.

For any M, M' in \mathfrak{C}_A let ${}_f \mathcal{R}_{M,M'} : M' \otimes M \rightarrow M \otimes M'$ be the isomorphism in \mathfrak{C}_A defined in [L1, 32.1.5] in terms of a fixed function $f : X \times X \rightarrow \mathbf{Z}$ as in [L1, 32.1.3]. (In the formula for ${}_f \mathcal{R}_{M,M'}$ in [L1, 32.1.4] we interpret $b^- m, b'^+ m'$ as $\sum_\lambda b^- 1_\lambda m, \sum_\lambda b'^+ 1_\lambda m'$.)

To any object $M \in \mathfrak{C}_A$ we associate a new object ${}^\omega M \in \mathfrak{C}_A$ with the same underlying A -module as M and such that for any $u \in \dot{\mathbf{U}}_A$, the operator u on ${}^\omega M$ coincides with the operator $\omega(u)$ on M .

For any $\lambda \in X^+$ let $\Lambda_\lambda = \mathbf{f}/\mathcal{T}_\lambda$ where $\mathcal{T}_\lambda = \sum_i \mathbf{f} \theta_i^{(i,\lambda)+1}$. Let $\eta_\lambda = 1 + \mathcal{T}_\lambda$. We regard Λ_λ as an object of \mathfrak{C} in which $(\theta_i^+ 1_{\lambda'}) \eta_\lambda = 0$ for $i \in I, \lambda' \in X$ and $(x^- 1_{\lambda'}) \eta_\lambda = \delta_{\lambda,\lambda'} x + \mathcal{T}_\lambda$ for $x \in \mathbf{f}, \lambda' \in X$. We have $\Lambda_\lambda \in \mathfrak{C}$ ([L1, 6.3.4, 23.1.4]).

Let $\Lambda_{\lambda,A} = \dot{\mathbf{U}}_A \eta_\lambda$. This is an \mathcal{A} -lattice in Λ_λ and $\Lambda_{\lambda,A} \in \mathfrak{C}_A$. Note that $\eta_\lambda \in \Lambda_{\lambda,A}$.

Let $\Lambda_{\lambda,A} = A \otimes_A \Lambda_{\lambda,A}$. By extension of scalars $\Lambda_{\lambda,A}$ becomes a $\dot{\mathbf{U}}_A$ -module. We have $\Lambda_{\lambda,A} \in \mathfrak{C}_A$. We write η_λ instead of $1 \otimes_A \eta_\lambda \in A \otimes_A \Lambda_{\lambda,A} = L_{\lambda,A}$. Now $\eta_\lambda \in \Lambda_{\lambda,A}$ regarded as an element of ${}^\omega \Lambda_{\lambda,A}$ will be denoted by $\xi_{-\lambda}$.

For $\lambda \in X^+$ there is a unique subset \mathbf{B}_λ of \mathbf{B} such that $b \mapsto (b^- 1_\lambda) \eta_\lambda$ is a bijection of \mathbf{B}_λ onto a basis $\underline{\mathbf{B}}_\lambda$ of $\Lambda_{\lambda,A}$; for $b \in \mathbf{B} - \mathbf{B}_\lambda$ we have $(b^- 1_\lambda) \eta_\lambda = 0$. There is a unique subset \mathbf{B}'_λ of \mathbf{B} such that $b \mapsto (b^+ 1_{-\lambda}) \xi_{-\lambda}$ is a bijection of \mathbf{B}'_λ onto $\underline{\mathbf{B}}_\lambda$; for $b \in \mathbf{B} - \mathbf{B}'_\lambda$ we have $(b^+ 1_{-\lambda}) \xi_{-\lambda} = 0$.

Note that $\mathbf{B}_\lambda, \mathbf{B}'_\lambda$ are independent of A .

1.7. For $\lambda \in X^+$ let $\dot{\mathbf{U}}^{\geq \lambda}$ be the set of all $u \in \dot{\mathbf{U}}$ such that for any $\lambda' \in X^+$ such that $\lambda' \not\geq \lambda$ we have $u|_{\Lambda_{\lambda'}} = 0$ (a two-sided ideal of $\dot{\mathbf{U}}$). By [L1, 29.1], $\dot{\mathbf{U}}^{\geq \lambda} \cap \dot{\mathbf{B}}$ is a basis of $\dot{\mathbf{U}}^{\geq \lambda}$ and there is a unique partition $\dot{\mathbf{B}} = \bigsqcup_{\lambda \in X^+} \dot{\mathbf{B}}[\lambda]$ such that for any

$\lambda \in X^+$ we have $\dot{U}^{\geq \lambda} \cap \dot{B} = \bigsqcup_{\lambda' \in X^+; \lambda' \geq \lambda} \dot{B}[\lambda']$. By [L1, 29.1.6], $\dot{B}[\lambda]$ is finite for any $\lambda \in X^+$.

Lemma 1.8. *For any $c \in \dot{B}$, the set $\{(a, b) \in \dot{B} \times \dot{B}; m_{a,b}^c \in A - \{0\}\}$ is finite.*

We may assume that $A = \mathcal{A}$. Let $c \in \dot{B}$. We have $c \in \dot{B}[\lambda]$ for a unique $\lambda \in X^+$. Let $(a, b) \in \dot{B} \times \dot{B}$ be such that $m_{a,b}^c \neq 0$. We have $a \in \dot{B}[\lambda']$, $b \in \dot{B}[\lambda'']$ for some λ', λ'' in X^+ . We have $a \in \dot{U}^{\geq \lambda'}$. Since $\dot{U}^{\geq \lambda'} \dot{U} \subset \dot{U}^{\geq \lambda'}$, we have $ab \in \dot{U}^{\geq \lambda'}$. Since c appears with non-zero coefficient in ab and $\dot{U}^{\geq \lambda'} \cap \dot{B}$ is a basis of $\dot{U}^{\geq \lambda'}$, we have $c \in \dot{U}^{\geq \lambda'} \cap \dot{B}$, that is, $c \in \bigcup_{\lambda_1 \in X^+; \lambda_1 \geq \lambda'} \dot{B}[\lambda_1]$. Thus $\lambda \geq \lambda'$. Similarly, $\lambda \geq \lambda''$. Thus $\{(a, b) \in \dot{B} \times \dot{B}; m_{a,b}^c \neq 0\}$ is contained in $\bigsqcup_{(\lambda', \lambda'') \in X^+ \times X^+; \lambda \geq \lambda', \lambda \geq \lambda''} \dot{B}[\lambda'] \times \dot{B}[\lambda'']$; this is a finite set since $\{(\lambda', \lambda'') \in X^+ \times X^+; \lambda \geq \lambda', \lambda \geq \lambda''\}$ is finite and each $\dot{B}[\lambda'] \times \dot{B}[\lambda'']$ is finite. The lemma is proved.

1.9. For any λ, λ' in X^+ there is a unique (finite) subset $\dot{B}_{\lambda, \lambda'}$ of \dot{B} such that $a \mapsto a(\xi_{-\lambda} \otimes \eta_{\lambda'})$ defines a bijection of $\dot{B}_{\lambda, \lambda'}$ onto an A -basis $\underline{B}_{\lambda, \lambda'}$ of ${}^\omega \Lambda_{\lambda, A} \otimes_A \Lambda_{\lambda', A}$; moreover $a(\xi_{-\lambda} \otimes \eta_{\lambda'}) = 0$ for any $a \in \dot{B} - \dot{B}_{\lambda, \lambda'}$. This property appears in [L1, 25.2.1] in the case where $A = \mathbf{Q}(v)$ but it is then automatically true for general A . Note that the subset $\dot{B}_{\lambda, \lambda'}$ of \dot{B} is independent of A . We have $\dot{B} = \bigcup_{\lambda, \lambda' \in X^+} \dot{B}_{\lambda, \lambda'}$.

1.10. We show:

(a) *If $M \in \mathcal{C}_A$, then $\mathfrak{A} := \{a \in \dot{B}; aM \neq 0\}$ is finite.*

Let z_1, z_2, \dots, z_r be a set of generators of the A -module M such that each z_j is contained in $1_{\zeta_j} M$ for some $\zeta_j \in X$. As in the proof of [L1, 31.2.7], for $j \in [1, r]$ we can find $\lambda_j, \lambda'_j \in X^+$ such that $\lambda'_j - \lambda_j = \zeta_j$ and a morphism $\phi_j : {}^\omega \Lambda_{\lambda_j, A} \otimes_A \Lambda_{\lambda'_j, A} \rightarrow M$ in \mathcal{C}_A such that $\phi_j(\xi_{-\lambda_j} \otimes \eta_{\lambda'_j}) = z_j$. If $a \in \mathfrak{A}$, then for some j we have $az_j \neq 0$, hence $a(\xi_{-\lambda_j} \otimes \eta_{\lambda'_j}) \neq 0$ in ${}^\omega \Lambda_{\lambda_j, A} \otimes_A \Lambda_{\lambda'_j, A}$. Thus we have $a \in \dot{B}_{\lambda_j, \lambda'_j}$. We see that $\mathfrak{A} \subset \bigcup_{j \in [1, r]} \dot{B}_{\lambda_j, \lambda'_j}$. Thus \mathfrak{A} is finite.

1.11. Let \hat{U}_A be the A -module consisting of all formal linear combinations

$$\sum_{a \in \dot{B}} n_a a$$

with $n_a \in A$. We define an A -algebra structure on \hat{U}_A by

$$\left(\sum_{a \in \dot{B}} n_a a\right) \left(\sum_{b \in \dot{B}} \tilde{n}_b b\right) = \sum_{c \in \dot{B}} r_c c$$

where $r_c = \sum_{(a,b) \in \dot{B} \times \dot{B}} m_{a,b}^c n_a \tilde{n}_b$. The last sum is well defined since by Lemma 1.8 it has only finitely many non-zero terms. This algebra structure is associative by 1.5(i) and has a unit element $1 = \sum_{\lambda \in X} 1_\lambda$. We have an imbedding of A -algebras $\dot{U}_A \subset \hat{U}_A$ under which $b \in \dot{B} \subset \dot{U}_A$ corresponds to $\sum_{a \in \dot{B}} \delta_{a,b} a \in \hat{U}_A$.

Let $\hat{U}_A^{(2)}$ be the A -module consisting of all formal linear combinations

$$\sum_{(a,a') \in \dot{B} \times \dot{B}} n_{a,a'} a \otimes a'$$

with $n_{a,a'} \in A$. We define an A -algebra structure on $\hat{U}_A^{(2)}$ by

$$\left(\sum_{(a,a') \in \dot{B} \times \dot{B}} n_{a,a'} a \otimes a'\right) \left(\sum_{(b,b') \in \dot{B} \times \dot{B}} \tilde{n}_{b,b'} b \otimes b'\right) = \sum_{(c,c') \in \dot{B} \times \dot{B}} r_{c,c'} c \otimes c'$$

where $r_{c,c'} = \sum_{a,a',b,b'} m_{a,b}^c m_{a',b'}^{c'} n_{a,a'} \tilde{n}_{b,b'}$. (Again this sum is well defined by 1.8.) This algebra is associative by 1.5(i) with unit element $1 = \sum_{\lambda,\lambda' \in X} 1_\lambda \otimes 1_{\lambda'}$. Note that $\hat{\mathbf{U}}_A^{(2)}$ is associated to the direct sum of two copies of our root datum in the same way that $\hat{\mathbf{U}}_A$ is associated to our root datum. For $\xi = \sum_{a \in \dot{\mathbf{B}}} n_a a \in \hat{\mathbf{U}}_A$, $\xi' = \sum_{a' \in \dot{\mathbf{B}}} \tilde{n}_{a'} a' \in \hat{\mathbf{U}}_A$, we set $\xi \otimes \xi' = \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_a \tilde{n}_{a'} a \otimes a' \in \hat{\mathbf{U}}_A^{(2)}$. Note that $\xi, \xi' \mapsto \xi \otimes \xi'$ induces an A -algebra homomorphisms $\hat{\mathbf{U}}_A \otimes_A \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$. We define an A -linear map $\Delta : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$ by

$$c \mapsto \sum_{(a,b) \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} \hat{m}_c^{a,b} a \otimes b$$

for any $c \in \dot{\mathbf{B}}$. By 1.5(iii), Δ is an A -algebra homomorphism.

We define an A -linear map $\mathbf{m} : \hat{\mathbf{U}}_A^{(2)} \rightarrow \hat{\mathbf{U}}_A$ by

$$\sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} a \otimes a' \mapsto \sum_{b \in \dot{\mathbf{B}}} \left(\sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} m_{a,a'}^b \right) b.$$

This is well defined, by 1.8. In particular, if $\xi, \xi' \in \hat{\mathbf{U}}_A$ we have $\mathbf{m}(\xi \otimes \xi') = \xi \xi'$ (product in $\hat{\mathbf{U}}_A$) where $\xi \otimes \xi'$ is regarded as an element of $\hat{\mathbf{U}}_A^{(2)}$ as above.

We define an A -linear map $S : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A$ by $\sum_{a \in \dot{\mathbf{B}}} n_a a \mapsto \sum_{a \in \dot{\mathbf{B}}} s_a n_a \underline{a}$ and A -linear maps $S^{(2)} : \hat{\mathbf{U}}_A^{(2)} \rightarrow \hat{\mathbf{U}}_A^{(2)}$, $S^{(0,1)} : \hat{\mathbf{U}}_A^{(2)} \rightarrow \hat{\mathbf{U}}_A^{(2)}$ by

$$S^{(2)} : \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} a \otimes a' \mapsto \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} s_a s_{a'} n_{a,a'} \underline{a} \otimes \underline{a'},$$

$$S^{(0,1)} : \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} a \otimes a' \mapsto \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} s_{a'} n_{a,a'} a \otimes \underline{a'}.$$

In particular for ξ, ξ' in $\hat{\mathbf{U}}_A$ we have $S^{(2)}(\xi \otimes \xi') = S(\xi) \otimes S(\xi')$, $S^{(0,1)}(\xi \otimes \xi') = \xi \otimes S(\xi')$.

We define an A -linear map $\omega : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A$ by $\sum_{a \in \dot{\mathbf{B}}} n_a a \mapsto \sum_{a \in \dot{\mathbf{B}}} e_a n_a a'$ and an A -linear map $\omega^{(2)} : \hat{\mathbf{U}}_A^{(2)} \rightarrow \hat{\mathbf{U}}_A^{(2)}$ by

$$\sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} a \otimes a' \mapsto \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} e_a e_{a'} n_{a,a'} a' \otimes a''.$$

In particular for ξ, ξ' in $\hat{\mathbf{U}}_A$ we have $\omega^{(2)}(\xi \otimes \xi') = \omega(\xi) \otimes \omega(\xi')$.

For ξ, ξ' in $\hat{\mathbf{U}}_A$ we have $S(\xi \xi') = S(\xi') S(\xi)$, $\omega(\xi \xi') = \omega(\xi) \omega(\xi')$ (see 1.5(v), (vii)).

We define an A -linear map $\epsilon : \hat{\mathbf{U}}_A \rightarrow A$ by $\sum_{a \in \dot{\mathbf{B}}} n_a a \mapsto n_{1_0}$. This is a homomorphism of A -algebras preserving 1, by 1.5(xii).

1.12. Let $M \in \mathfrak{C}_A$. We define an A -linear map $\hat{\mathbf{U}}_A \rightarrow \text{Hom}_A(M, M)$ by $\sum_{a \in \dot{\mathbf{B}}} n_a a : z \mapsto \sum_{a \in \dot{\mathbf{B}}} n_a a z$. The last sum is well defined since, by 1.10(a), all but finitely many of its terms are 0. This defines a structure of left $\hat{\mathbf{U}}_A$ -module on M extending the $\hat{\mathbf{U}}_A$ -module structure. Note that $1 \in \hat{\mathbf{U}}_A$ acts as 1 on M .

Similarly, if $M \in \mathfrak{C}_A$, $M' \in \mathfrak{C}_A$, then the obvious $\hat{\mathbf{U}}_A \otimes \hat{\mathbf{U}}_A$ -module structure on $M \otimes_A M'$ extends to a $\hat{\mathbf{U}}_A^{(2)}$ -module structure on $M \otimes_A M'$ defined by the A -algebra

$$\text{map } \hat{\mathbf{U}}_A^{(2)} \rightarrow \text{Hom}_A(M \otimes_A M', M \otimes_A M'),$$

$$\sum_{(a,a') \in \hat{\mathbf{B}} \times \hat{\mathbf{B}}} n_{a,a'} a \otimes a' : z \otimes z' \mapsto \sum_{(a,a') \in \hat{\mathbf{B}} \times \hat{\mathbf{B}}} n_{a,a'} a z \otimes a' z'.$$

We can now regard $M \otimes_A M'$ as a $\dot{\mathbf{U}}_A$ -module by restricting the $\hat{\mathbf{U}}_A^{(2)}$ -module structure to $\dot{\mathbf{U}}_A$ via the algebra homomorphism $\Delta : \dot{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$. Note that the resulting $\dot{\mathbf{U}}_A$ -module $M \otimes_A M'$ is the one defined in 1.6.

1.13. Let $\lambda, \lambda', \lambda_1, \lambda_2, \lambda'_1, \lambda'_2$ be elements of X^+ such that $\lambda = \lambda_1 + \lambda_2, \lambda' = \lambda'_1 + \lambda'_2$. Let $\mathfrak{T} : \Lambda_{\lambda_1 + \lambda_2, \mathcal{A}} \rightarrow \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}$ be the restriction of the unique morphism $\Lambda_{\lambda_1 + \lambda_2} \rightarrow \Lambda_{\lambda'_1} \otimes \Lambda_{\lambda'_2}$ in \mathfrak{C} such that $\eta_{\lambda_1 + \lambda_2} \mapsto \eta_{\lambda'_1} \otimes \eta_{\lambda'_2}$; see [L1, 25.1.2(a), 25.1.2(b)]. Similarly let $\mathfrak{T}' : {}^\omega \Lambda_{\lambda_1 + \lambda_2, \mathcal{A}} \rightarrow {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}}$ be the restriction of the unique morphism ${}^\omega \Lambda_{\lambda_1 + \lambda_2} \rightarrow {}^\omega \Lambda_{\lambda_1} \otimes {}^\omega \Lambda_{\lambda_2}$ in \mathfrak{C} such that $\xi_{-\lambda_1 - \lambda_2} \mapsto \xi_{-\lambda_1} \otimes \xi_{-\lambda_2}$. Note that $\mathfrak{T}, \mathfrak{T}'$ are morphisms in $\mathfrak{C}_{\mathcal{A}}$. We define

$$\tau : {}^\omega \Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}} \rightarrow {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}$$

as the composition

$$\begin{aligned} & {}^\omega \Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}} \xrightarrow{\mathfrak{T}' \otimes \mathfrak{T}} {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}} \\ & \xrightarrow{1 \otimes v^{f(\lambda_2, \lambda'_1)} f \mathcal{R}_{\omega \Lambda_{\lambda_2, \mathcal{A}}, \Lambda_{\lambda'_1, \mathcal{A}}}^{-1} \otimes 1} {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}, \end{aligned}$$

a morphism in $\mathfrak{C}_{\mathcal{A}}$. Note that τ depends on the choice of $f : X \times X \rightarrow \mathbf{Z}$ which is fixed as in 1.6. Define $\rho : \dot{\mathbf{U}}_{\mathcal{A}} \rightarrow {}^\omega \Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}}$ by $u \mapsto u(\xi_{-\lambda} \otimes \eta_{\lambda'})$. Define

$$\rho' : \hat{\mathbf{U}}_{\mathcal{A}}^{(2)} \rightarrow {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}$$

by $u \mapsto u(\xi_{-\lambda_1} \otimes_{\mathcal{A}} \eta_{\lambda'_1} \otimes_{\mathcal{A}} \xi_{-\lambda_2} \otimes_{\mathcal{A}} \eta_{\lambda'_2})$. Here $M \otimes_A M'$ with $M = {}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}}, M' = {}^\omega \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}$ is regarded as a $\hat{\mathbf{U}}_A^{(2)}$ -module as in 1.12. We show that the diagram

$$\begin{array}{ccc} \dot{\mathbf{U}}_{\mathcal{A}} & \xrightarrow{\Delta} & \hat{\mathbf{U}}_{\mathcal{A}}^{(2)} \\ \rho \downarrow & & \rho' \downarrow \\ {}^\omega \Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}} & \xrightarrow{\tau} & M \otimes_A M' \end{array}$$

is commutative. We regard $\dot{\mathbf{U}}_{\mathcal{A}}$ as a $\dot{\mathbf{U}}_{\mathcal{A}}$ -module via left multiplication. We regard $\hat{\mathbf{U}}_{\mathcal{A}}^{(2)}$ as a $\dot{\mathbf{U}}_{\mathcal{A}}$ -module in which $u \in \dot{\mathbf{U}}_{\mathcal{A}}$ acts by left multiplication by $\Delta(u)$. Then Δ is $\dot{\mathbf{U}}_{\mathcal{A}}$ -linear since Δ is an algebra homomorphism. From the definitions we see that ρ and ρ' are $\dot{\mathbf{U}}_{\mathcal{A}}$ -linear. Thus all maps in our diagram are $\dot{\mathbf{U}}_{\mathcal{A}}$ -linear. Since $\{1_x; x \in X\}$ generate the $\dot{\mathbf{U}}_{\mathcal{A}}$ -module $\dot{\mathbf{U}}_{\mathcal{A}}$, it is enough to show that the two compositions in the diagram coincide when applied to any 1_x with $x \in X$. Thus it is enough to show that

$$\tau(1_x(\xi_{-\lambda} \otimes \eta_{\lambda'})) = \sum_{x', x'' \in X; x' + x'' = x} (1_{x'}(\xi_{-\lambda_1} \otimes \eta_{\lambda'_1})) \otimes (1_{x''}(\xi_{-\lambda_2} \otimes \eta_{\lambda'_2}))$$

or equivalently that

$$\tau(\xi_{-\lambda} \otimes \eta_{\lambda'}) = \xi_{-\lambda_1} \otimes \eta_{\lambda'_1} \otimes \xi_{-\lambda_2} \otimes \eta_{\lambda'_2}.$$

This follows from the definitions using the equality

$$f \mathcal{R}_{\omega \Lambda_{\lambda_2, \mathcal{A}}, \Lambda_{\lambda'_1, \mathcal{A}}}^{-1}(\xi_{-\lambda_2} \otimes \eta_{\lambda'_1}) = v^{-f(\lambda_2, \lambda'_1)} \eta_{\lambda'_1} \otimes \xi_{-\lambda_2}.$$

We now show:

(a) *The \mathcal{A} -linear map τ is a split injection.*

Since τ is the composition of $\mathfrak{T}' \otimes_{\mathcal{A}} \mathfrak{T}$ with an \mathcal{A} -linear isomorphism, it is enough to show that $\mathfrak{T}' \otimes_{\mathcal{A}} \mathfrak{T}$ is a split injection. It is also enough to show that the \mathcal{A} -linear maps \mathfrak{T} and \mathfrak{T}' are split injections. By [L1, 27.1.7], \mathfrak{T} carries $\underline{\mathbf{B}}_{\lambda'}$ (an \mathcal{A} -basis of $\Lambda_{\lambda',\mathcal{A}}$) bijectively onto a subset of $\underline{\mathbf{B}}_{\lambda'_1, \lambda'_2}$ (an \mathcal{A} -basis of $\Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}$). Hence \mathfrak{T} is a split injection.

We identify the \mathcal{A} -modules ${}^\omega \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} {}^\omega \Lambda_{\lambda_2, \mathcal{A}}$ and $\Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda_1, \mathcal{A}}$ by $x \otimes y \leftrightarrow y \otimes x$. Then \mathfrak{T}' becomes the \mathcal{A} -linear map $\Lambda_{\lambda, \mathcal{A}} \rightarrow \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda_1, \mathcal{A}}$ of the same type as \mathfrak{T} ; hence it is a split injection by the previous argument for \mathfrak{T} . This proves 1.13(a).

1.14. Assume that $\lambda, \lambda', \lambda_1, \lambda_2, \lambda'_1, \lambda'_2$ in X^+ are such that $\lambda = \lambda_1 + \lambda_2, \lambda' = \lambda'_1 + \lambda'_2$. We show:

(a) *If $a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}, c' \in \dot{\mathbf{B}} - \dot{\mathbf{B}}_{\lambda, \lambda'}$ then $\hat{m}_{c'}^{a,b} = 0$ in \mathcal{A} .*

In the commutative diagram in 1.13 we have $\rho(c') = 0$, hence $\rho'(\Delta(c')) = 0$, that is, $\rho'(\sum_{a,b \in \dot{\mathbf{B}}} \hat{m}_{c'}^{a,b} a \otimes b) = 0$ so that

$$\sum_{a,b \in \dot{\mathbf{B}}} \hat{m}_{c'}^{a,b} (a(\xi_{-\lambda_1} \otimes \eta_{\lambda'_1}) \otimes b(\xi_{-\lambda_2} \otimes \eta_{\lambda'_2})) = 0.$$

The term corresponding to (a, b) is 0 unless $(a, b) \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \times \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$; moreover when (a, b) runs through $\dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \times \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$, the elements $a(\xi_{-\lambda_1} \otimes \eta_{\lambda'_1}) \otimes b(\xi_{-\lambda_2} \otimes \eta_{\lambda'_2})$ are linearly independent (they form the set $\dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \otimes \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$). It follows that $\hat{m}_{c'}^{a,b} = 0$ for any $(a, b) \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \times \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$. This proves 1.14(a).

1.15. In the setup of 1.14, we show:

(a) *Let $c \in \dot{\mathbf{B}}_{\lambda, \lambda'}$. There exists a function $h : \dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \times \dot{\mathbf{B}}_{\lambda_2, \lambda'_2} \rightarrow \mathcal{A}$ such that for any $c' \in \dot{\mathbf{B}}$ we have*

$$\sum_{a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a, b) \hat{m}_{c'}^{a,b} = \delta_{c, c'}.$$

For any $\tilde{\lambda}, \tilde{\lambda}'$ in X^+ , we set $b^\dagger = b(\xi_{-\tilde{\lambda}} \otimes \eta_{\tilde{\lambda}'})$ for $b \in \dot{\mathbf{B}}_{\tilde{\lambda}, \tilde{\lambda}'}$ so that $\{b^\dagger; b \in \dot{\mathbf{B}}_{\tilde{\lambda}, \tilde{\lambda}'}\}$ is an \mathcal{A} -basis of ${}^\omega \Lambda_{\tilde{\lambda}, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\tilde{\lambda}', \mathcal{A}}$ and we denote by $\{\hat{b}; b \in \dot{\mathbf{B}}_{\tilde{\lambda}, \tilde{\lambda}'}\}$ the dual \mathcal{A} -basis of $\text{Hom}_{\mathcal{A}}({}^\omega \Lambda_{\tilde{\lambda}, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\tilde{\lambda}', \mathcal{A}}, \mathcal{A})$. In particular the bases $\{\hat{c}'; c' \in \dot{\mathbf{B}}_{\lambda, \lambda'}\}, \{\hat{a}'; a' \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}\}, \{\hat{b}'; b' \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}\}$ of $\text{Hom}_{\mathcal{A}}(\Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}}, \mathcal{A}), \text{Hom}_{\mathcal{A}}(\Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}}, \mathcal{A}), \text{Hom}_{\mathcal{A}}(\Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}}, \mathcal{A})$ are defined. By 1.13(a), we can find an \mathcal{A} -linear map

$$\psi : \Lambda_{\lambda_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_1, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda_2, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda'_2, \mathcal{A}} \rightarrow \Lambda_{\lambda, \mathcal{A}} \otimes_{\mathcal{A}} \Lambda_{\lambda', \mathcal{A}}$$

such that $\psi\tau = 1$ with τ as in 1.13. The transpose maps ${}^t\psi, {}^t\tau$ satisfy ${}^t\tau({}^t\psi) = 1$. We can write uniquely

$${}^t\psi(\hat{c}) = \sum_{a' \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b' \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a', b') \hat{a}' \otimes \hat{b}'$$

with $h(a', b') \in \mathcal{A}$. Hence

$$\hat{c} = \sum_{a' \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b' \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a', b') {}^t\tau(\hat{a}' \otimes \hat{b}').$$

Evaluating both sides at c'^{\dagger} with $c' \in \dot{\mathbf{B}}_{\lambda, \lambda'}$ and using

$$\tau(c'^{\dagger}) = \sum_{a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} \hat{m}_{c'}^{a,b} a^{\dagger} \otimes b^{\dagger}$$

(which follows from the commutative diagram in 1.13), we have

$$\delta_{c,c'} = \sum_{a, a' \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b, b' \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a', b') \hat{m}_{c'}^{a,b} \delta_{a', a} \delta_{b', b},$$

that is,

$$\delta_{c,c'} = \sum_{a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a, b) \hat{m}_{c'}^{a,b}.$$

This also holds when $c' \in \dot{\mathbf{B}} - \dot{\mathbf{B}}_{\lambda, \lambda'}$ (both sides are 0 by 1.14(a)). This proves 1.15(a).

Lemma 1.16. *For any a, b in $\dot{\mathbf{B}}$, the set $\{c \in \dot{\mathbf{B}}; \hat{m}_c^{a,b} \in \mathcal{A} - \{0\}\}$ is finite.*

We have $a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}$, $b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$ for some $\lambda_1, \lambda'_1, \lambda_2, \lambda'_2$ in X^+ . By 1.14(a) we have $\{c \in \dot{\mathbf{B}}; \hat{m}_c^{a,b} \in \mathcal{A} - \{0\}\} \subset \dot{\mathbf{B}}_{\lambda_1 + \lambda_2, \lambda'_1 + \lambda'_2}$. Since $\dot{\mathbf{B}}_{\lambda_1 + \lambda_2, \lambda'_1 + \lambda'_2}$ is finite, the lemma holds.

1.17. We define an A -linear map $\hat{D} : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$ by

$$\sum_{c \in \dot{\mathbf{B}}} n_c c \mapsto \sum_{(a,b) \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} \left(\sum_{c \in \dot{\mathbf{B}}} \hat{m}_c^{a,b} n_c \right) a \otimes b.$$

This make sense: for any $(a, b) \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}$, the sum $\sum_{c \in \dot{\mathbf{B}}} \hat{m}_c^{a,b} n_c$ has only finitely many non-zero terms. (See 1.16.) Using 1.5(iii), we see that \hat{D} is an A -algebra homomorphism. It clearly extends the homomorphism $\Delta : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$. From 1.5(vi), (viii) we see that

$$\hat{D}(S(\xi)) = S^{(2)} \text{tr} \hat{D}(\xi), \quad \hat{D}(\omega(\xi)) = \omega^{(2)} \text{tr} \hat{D}(\xi)$$

for any $\xi \in \hat{\mathbf{U}}_A$ where $S^{(2)}, \omega^{(2)}$ are as in 1.11 and $\text{tr} : \hat{\mathbf{U}}_A^{(2)} \rightarrow \hat{\mathbf{U}}_A^{(2)}$ is the A -linear map given by

$$\sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a,a'} a \otimes a' \mapsto \sum_{(a,a') \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} n_{a',a} a \otimes a'.$$

If $v = 1$ in A , then $\text{tr} \hat{D}(\xi) = \hat{D}(\xi)$ for $\xi \in \hat{\mathbf{U}}_A$ (see 1.5(xiii)). Hence in this case we have $\hat{D}(S(\xi)) = S^{(2)} \hat{D}(\xi)$, $\hat{D}(\omega(\xi)) = \omega^{(2)} \hat{D}(\xi)$ for $\xi \in \hat{\mathbf{U}}_A$.

1.18. For any $i \in I$ and $h \in A$ we set

$$x_i(h) = \sum_{c \in \mathbf{N}, \lambda \in X} h^c \theta_i^{(c)+} 1_{\lambda} \in \hat{\mathbf{U}}_A,$$

$$y_i(h) = \sum_{c \in \mathbf{N}, \lambda \in X} h^c \theta_i^{(c)-} 1_{\lambda} \in \hat{\mathbf{U}}_A.$$

Note that $\theta_i^{(t)\pm} 1_{\lambda} \in \dot{\mathbf{B}}$; see [L1, 25.3.1]. We have

$$y_i(h) = \omega(x_i(h)).$$

We show:

(a) If $v = 1$ in A , then $\hat{D}(x_i(h)) = x_i(h) \otimes x_i(h) \in \hat{U}_A^{(2)}$, $\hat{D}(y_i(h)) = y_i(h) \otimes y_i(h) \in \hat{U}_A^{(2)}$.

Without assumption on $v \in A$ we have (using [L1, 3.1.5]):

$$\begin{aligned} \hat{D}(x_i(h)) &= \sum_{t', t'' \in \mathbf{N}, \lambda', \lambda'' \in X} h^{t'+t''} v_i^{t' t''} v_i^{\langle i, \lambda' \rangle t''} (\theta_i^{(t')} + 1_{\lambda'}) \otimes (\theta_i^{(t'')} + 1_{\lambda''}), \\ \hat{D}(y_i(h)) &= \sum_{t', t'' \in \mathbf{N}, \lambda', \lambda'' \in X} h^{t'+t''} v_i^{-t' t''} v_i^{-\langle i, \lambda'' \rangle t'} (\theta_i^{(t')} - 1_{\lambda'}) \otimes (\theta_i^{(t'')} - 1_{\lambda''}). \end{aligned}$$

The desired formulas follow.

We show:

(b) If $v = 1$ in A and $h, h' \in A$, then $x_i(h+h') = x_i(h)x_i(h') \in \hat{U}_A$, $y_i(h+h') = y_i(h)y_i(h') \in \hat{U}_A$.

Without assumption on $v \in A$ we have

$$\begin{aligned} x_i(h)x_i(h') &= \sum_{t \in \mathbf{N}, \lambda \in X} \left(\sum_{t', t''; t'+t''=t} h^{t'} h^{t''} \begin{bmatrix} t \\ t' \end{bmatrix}_i \right) \theta_i^{(t)+} 1_{\lambda}, \\ y_i(h)y_i(h') &= \sum_{t \in \mathbf{N}, \lambda \in X} \left(\sum_{t', t''; t'+t''=t} h^{t'} h^{t''} \begin{bmatrix} t \\ t' \end{bmatrix}_i \right) \theta_i^{(t)-} 1_{\lambda}. \end{aligned}$$

If $v = 1$, we have $\begin{bmatrix} t \\ t' \end{bmatrix}_i = \binom{t}{t'}$ and $\sum_{t', t''; t'+t''=t} h^{t'} h^{t''} \begin{bmatrix} t \\ t' \end{bmatrix}_i = (h+h')^t$. The desired formulas follow.

1.19. We say that a (possibly infinite) sum $\sum_{f \in F} p_f$ with $p_f \in \hat{U}_A$ is defined in \hat{U}_A if, when setting $p_f = \sum_{a \in \hat{\mathbf{B}}} n_{a,f} a \in \hat{U}_A$ ($n_{a,f} \in A$), the set $\{f \in F; n_{a,f} \neq 0\}$ is finite for any $a \in \hat{\mathbf{B}}$. In this case we assign to the sum the value $\sum_{a \in \hat{\mathbf{B}}} (\sum_{f \in F} n_{a,f}) a \in \hat{U}_A$. Using 1.8 we see that, if the (possibly infinite) sums $\sum_{f \in F} p_f, \sum_{f' \in F'} p'_{f'}$ are defined in \hat{U}_A , then the sum $\sum_{(f, f') \in F \times F'} p_f p'_{f'}$ is defined in \hat{U}_A and we have

$$\left(\sum_{f \in F} p_f \right) \left(\sum_{f' \in F'} p'_{f'} \right) = \sum_{(f, f') \in F \times F'} p_f p'_{f'}, \text{ in } \hat{U}_A.$$

2. THE ELEMENTS $s'_{i,e}, s''_{i,e}$ OF \hat{U}_A

2.1. In this section we introduce and study some elements $s'_{i,e}, s''_{i,e}$ of \hat{U}_A which implement the braid group action [L1, Ch.39] on \hat{U}_A .

Let $i \in I, e = \pm 1$. Let

$$\begin{aligned} s'_{i,e} &= \sum_{y \in \mathbf{Z}, x \in \mathbf{Z}, \lambda \in X; \langle i, \lambda \rangle = x+y} (-1)^y v_i^{ey} \theta_i^{(x)-} 1_{\lambda} \theta_i^{(y)+} \in \hat{U}_A, \\ s''_{i,e} &= \sum_{y \in \mathbf{Z}, x \in \mathbf{Z}, \lambda \in X; \langle i, \lambda \rangle = x+y} (-1)^x v_i^{ex} \theta_i^{(x)-} 1_{\lambda} \theta_i^{(y)+} \in \hat{U}_A. \end{aligned}$$

Note that in the formulas above each $\theta_i^{(x)-} 1_{\lambda} \theta_i^{(y)+}$ belongs to $\hat{\mathbf{B}}$ (see [L1, 25.3.1]). We have $\omega(s'_{i,e}) = s''_{i,e}$. Indeed, using the definition of ω (see 1.4), we have

$$\omega(\theta_i^{(x)-} 1_{\lambda} \theta_i^{(y)+}) = \theta_i^{(x)+} 1_{-\lambda} \theta_i^{(y)-} = \theta_i^{(y)-} 1_{\lambda} \theta_i^{(x)+}$$

for any $y \in \mathbf{Z}, x \in \mathbf{Z}, \lambda \in X$ such that $\langle i, \lambda \rangle = x + y$.

Lemma 2.2. *We preserve the setup of 2.1. For any $r \in \mathbf{Z}$ we consider the sums*

$$(i) \quad \sum_{m,n,p \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = m-n+p-r} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)-} \theta_i^{(n)+} \theta_i^{(p)-} 1_\lambda,$$

$$(ii) \quad \sum_{m,n,p \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)+} \theta_i^{(n)-} \theta_i^{(p)+} 1_\lambda.$$

- (a) *The sums (i), (ii) are defined in $\hat{\mathbf{U}}_A$. Let $\tau'_{i,e,r}, \tau''_{i,e,r}$ be their value in $\hat{\mathbf{U}}_A$.*
- (b) *We have $\tau'_{i,e,0} = s'_{i,e}, \tau''_{i,e,0} = s''_{i,e}$.*
- (c) *If $v = 1$ in A and $r \in \mathbf{Z} - \{0\}$, we have $\tau'_{i,e,r} = 0, \tau''_{i,e,r} = 0$.*
- (d) *If $v = 1$ in A , we have $s''_{i,e} = x_i(1)y_i(-1)x_i(1), s'_{i,e} = y_i(1)x_i(-1)y_i(1)$ in $\hat{\mathbf{U}}_A$.*
- (e) *If $v = 1$ in A , we have $\hat{D}(s'_{i,e}) = s'_{i,e} \otimes s'_{i,e}$ and $\hat{D}(s''_{i,e}) = s''_{i,e} \otimes s''_{i,e}$ in $\hat{\mathbf{U}}_A^{(2)}$.*

We compute formally the sum (ii) for $r \in \mathbf{N}$ (using 1.4):

$$\begin{aligned} & \sum_{m,n,p \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)+} 1_{\lambda+pi'-ni'} \theta_i^{(n)-} \theta_i^{(p)+} \\ &= \sum_{m \in \mathbf{Z}, n \in \mathbf{Z}, p \in \mathbf{N}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \\ & \quad \times \begin{bmatrix} m+n - \langle i, -\lambda - pi' + ni' \rangle \\ t \end{bmatrix}_i \theta_i^{(n-t)-} 1_{\lambda+pi'-ni'+(n+m-t)i'} \theta_i^{(m-t)+} \theta_i^{(p)+} \\ &= \sum_{m \in \mathbf{Z}, n \in \mathbf{Z}, p \in \mathbf{N}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \begin{bmatrix} r+p \\ t \end{bmatrix}_i \\ & \quad \times \begin{bmatrix} m+p-t \\ p \end{bmatrix}_i \theta_i^{(n-t)-} 1_{\lambda+pi'-ni'+(n+m-t)i'} \theta_i^{(m+p-t)+} \\ &= \sum_{m \in \mathbf{Z}, x \in \mathbf{Z}, p \in \mathbf{N}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+x+t-p+r} (-1)^{x+t} v_i^{e(-mp+x+t)} \begin{bmatrix} r+p \\ t \end{bmatrix}_i \\ & \quad \times \begin{bmatrix} m+p-t \\ p \end{bmatrix}_i \theta_i^{(x)-} 1_{\lambda+(m+p-t)i'} \theta_i^{(m+p-t)+} \\ &= \sum_{\substack{y \in \mathbf{N}, x \in \mathbf{N}, \lambda \in X; \\ \langle i, \lambda \rangle = -y+x+r}} (-1)^{x+t} v_i^{e(-(y-p+t)p+x+t)} \begin{bmatrix} r+p \\ t \end{bmatrix}_i \begin{bmatrix} y \\ p \end{bmatrix}_i c_{x,y,\lambda;r} \theta_i^{(x)-} 1_{\lambda+yi'} \theta_i^{(y)+} \end{aligned}$$

where each $\theta_i^{(x)-} 1_{\lambda+yi'} \theta_i^{(y)+}$ belongs to $\dot{\mathbf{B}}$ (see [L1, 25.3.1]) and

$$c_{x,y,\lambda;r} = \sum_{p \in \mathbf{N}, t \in \mathbf{N}} (-1)^{x+t} v_i^{e(-(y-p+t)p+x+t)} \begin{bmatrix} r+p \\ t \end{bmatrix}_i \begin{bmatrix} y \\ p \end{bmatrix}_i.$$

Note that only finitely many terms of this sum can be $\neq 0$ (those for which $p \leq y$ and $t \leq r+p$); here we have used that $r \in \mathbf{N}$. In particular the sum (ii) is defined in $\hat{\mathbf{U}}_A$ and has a value $\tau''_{i,e;r}$ in $\hat{\mathbf{U}}_A$.

Next we compute formally the sum (ii) for $r \in -\mathbf{N}$ (using 1.4):

$$\begin{aligned}
 & \sum_{m,n,p \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)+} \theta_i^{(n)-} 1_{\lambda+pi'} \theta_i^{(p)+} \\
 = & \sum_{m \in \mathbf{N}, n \in \mathbf{Z}, p \in \mathbf{Z}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \begin{bmatrix} n+p - \langle i, \lambda + pi' \rangle \\ t \end{bmatrix}_i \\
 & \times \theta_i^{(m)+} \theta_i^{(p-t)+} 1_{\lambda+pi'-(n+p-t)i'} \theta_i^{(n-t)-} \\
 = & \sum_{m \in \mathbf{N}, n \in \mathbf{Z}, p \in \mathbf{Z}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = -m+n-p+r} (-1)^n v_i^{e(-mp+n)} \begin{bmatrix} m-r \\ t \end{bmatrix}_i \\
 & \times \begin{bmatrix} m+p-t \\ m \end{bmatrix}_i \theta_i^{(m+p-t)+} 1_{\lambda-(n-t)i'} \theta_i^{(n-t)-} \\
 = & \sum_{m \in \mathbf{N}, x \in \mathbf{Z}, y \in \mathbf{Z}, t \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = y-x+r} (-1)^{x+t} v_i^{e(-m(x+t-m)+y+t)} \begin{bmatrix} m-r \\ t \end{bmatrix}_i \\
 & \times \begin{bmatrix} x \\ m \end{bmatrix}_i \theta_i^{(x)+} 1_{\lambda-yi'} \theta_i^{(y)-} \\
 = & \sum_{x \in \mathbf{N}, y \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle = y-x+r} \tilde{c}_{x,y,\lambda;r} \theta_i^{(x)+} 1_{\lambda-yi'} \theta_i^{(y)-},
 \end{aligned}$$

where the last product $\theta_i^{(x)+} 1_{\lambda-yi'} \theta_i^{(y)-}$ belongs to $\hat{\mathbf{B}}$ (see [L1, 25.3.1]) and

$$\tilde{c}_{x,y,\lambda;r} = \sum_{m \in \mathbf{N}, t \in \mathbf{N}} (-1)^{x+t} v_i^{e(-m(x+t-m)+y+t)} \begin{bmatrix} m-r \\ t \end{bmatrix}_i \begin{bmatrix} x \\ m \end{bmatrix}_i.$$

Note that only finitely many terms of this sum can be $\neq 0$ (those for which $m \leq x$ and $t \leq m-r$); here we have used that $r \in -\mathbf{N}$. In particular the sum (ii) is defined in $\hat{\mathbf{U}}_A$ and has a value $\tau''_{i,e;r}$ in $\hat{\mathbf{U}}_A$.

From the computations above we see also that the sum

$$\sum_{m,n,p \in \mathbf{N}, \lambda \in X} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)+} \theta_i^{(n)-} \theta_i^{(p)+} 1_\lambda$$

is defined in $\hat{\mathbf{U}}_A$ and its value $\tau''_{i,e} \in \hat{\mathbf{U}}_A$ is equal to $\sum_{r \in \mathbf{Z}} \tau''_{i,e;r}$.

In the formula for $c_{x,y,\lambda;0}$ we have

$$\sum_{t \in \mathbf{N}} (-1)^{x+t} v_i^{e(-(y-p+t)p+x+t)} \begin{bmatrix} r+p \\ t \end{bmatrix}_i = (-1)^x v_i^{e(-(y-p)p+x)} \delta_{p,0} = (-1)^x v_i^{ex} \delta_{p,0},$$

hence $c_{x,y,\lambda;0} = (-1)^x v_i^{ex}$ and

$$\tau''_{i,e;0} = \sum_{y \in \mathbf{Z}, x \in \mathbf{Z}, \lambda \in X; \langle i, \lambda \rangle = -y+x} (-1)^x v_i^{ex} \theta_i^{(x)-} 1_{\lambda+yi'} \theta_i^{(y)+}.$$

We see that $\tau''_{i,e;0} = s''_{i,e}$. Similarly, we see that the family defining the sum (i) is defined, hence it has a value $\tau'_{i,e;r}$ in $\hat{\mathbf{U}}_A$, that the sum

$$\sum_{m,n,p \in \mathbf{N}, \lambda \in X} (-1)^n v_i^{e(-mp+n)} \theta_i^{(m)-} \theta_i^{(n)+} \theta_i^{(p)-} 1_\lambda$$

is defined in $\hat{\mathbf{U}}_A$ and its value $\tau'_{i,e} \in \hat{\mathbf{U}}_A$ is equal to $\sum_{r \in \mathbf{Z}} \tau'_{i,e,r}$. We see also that $\tau'_{i,e;0} = s'_{i,e}$. (These statements could also be deduced from the earlier part of the proof using that $\tau''_{i,e;r} = \omega(\tau'_{i,e,r})$.)

In the remainder of the proof we assume that $v = 1$ in A . In this case in the formula defining $c_{x,y,\lambda;r}$ with $r > 0$ we have $\sum_{t \in \mathbf{N}} (-1)^{x+t} \begin{bmatrix} r+p \\ t \end{bmatrix}_i = 0$ since $r+p > 0$ and $v = 1$. Hence $c_{x,y,\lambda;r} = 0$ and $\tau''_{i,e;r} = 0$ for $r > 0$. In the formula defining $\tilde{c}_{x,y,\lambda;r}$ with $r < 0$ we have $\sum_{t \in \mathbf{N}} (-1)^{x+t} \begin{bmatrix} m-r \\ t \end{bmatrix}_i = 0$ since $m-r > 0$, hence $\tilde{c}_{x,y,\lambda;r} = 0$ and $\tau''_{i,e;r} = 0$ for $r < 0$. We see that $\tau''_{i,e;r} = 0$ for $r \neq 0$ and $\tau''_{i,e} = \tau''_{i,e;0} = s''_{i,e}$. Applying ω , we see that $\tau'_{i,e;r} = 0$ for $r \neq 0$ and $\tau'_{i,e} = \tau'_{i,e;0} = s'_{i,e}$. It is clear that $\tau''_{i,e} = x_i(1)y_i(-1)x_i(1)$, $\tau'_{i,e} = y_i(1)x_i(-1)y_i(1)$ in $\hat{\mathbf{U}}_A$. Hence (d) follows. Now (e) follows from (d) using 1.18(a) and the fact that $\hat{D} : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A^{(2)}$ is an algebra homomorphism.

2.3. Let $i \in I, e = \pm 1$. Let $T'_{i,e}, T''_{i,e}$ be the algebra automorphisms of $\hat{\mathbf{U}}_A$ defined in [L1, 41.1.8]. For any $M \in \mathfrak{C}_A$ let $T'_{i,e} : M \rightarrow M, T''_{i,e} : M \rightarrow M$ be the A -linear isomorphisms defined in [L1, 41.2.3]. From the definitions we have $T'_{i,e}(z) = \tau'_{i,e;0}z, T''_{i,e}(z) = \tau''_{i,e;0}z$ for $z \in M$ (using the $\hat{\mathbf{U}}_A$ -module structure of M); here we use notation of Lemma 2.2(a). By [L1, 41.2.4], for $u \in \hat{\mathbf{U}}_A, z \in M$ we have

$$T'_{i,e}(uz) = T'_{i,e}(u)(T'_{i,e}(z)), \quad T''_{i,e}(uz) = T''_{i,e}(u)(T''_{i,e}(z)),$$

hence, using Lemma 2.2(b):

$$(a) \quad s'_{i,e}uz = T'_{i,e}(u)(s'_{i,e}z), \quad s''_{i,e}uz = T''_{i,e}(u)(s''_{i,e}z).$$

We show:

$$(b) \quad s'_{i,e}s''_{i,-e} = 1;$$

(c) for any $u \in \hat{\mathbf{U}}_A$ we have $T'_{i,e}(u) = s'_{i,e}us'_{i,e}^{-1}, T''_{i,e}(u) = s''_{i,e}us''_{i,e}^{-1}$.

The proof of 2.3(b) and 2.3(c) is based on the following fact:

(d) Let $u' \in \hat{\mathbf{U}}_A$ be such that $u'z = 0$ for any $M \in \mathfrak{C}_A$ and any $z \in M$. Then $u' = 0$.

We have $u' = \sum_{a \in \hat{\mathbf{B}}} n_a a$ with $n_a \in A$. If $\lambda, \lambda' \in X^+$, we have $u'(\xi_{-\lambda} \otimes \eta_\lambda) = 0$ in ${}^\omega \Lambda_{\lambda,A} \otimes_A \Lambda_{\lambda',A}$. Hence $\sum_{a \in \hat{\mathbf{B}}_{\lambda,\lambda'}} n_a a(\xi_{-\lambda} \otimes \eta_\lambda) = 0$. Since the elements $a(\xi_{-\lambda} \otimes \eta_\lambda)$ with $a \in \hat{\mathbf{B}}_{\lambda,\lambda'}$ form a basis of ${}^\omega \Lambda_{\lambda,A} \otimes_A \Lambda_{\lambda',A}$, it follows that $n_a = 0$ for any $a \in \hat{\mathbf{B}}_{\lambda,\lambda'}$. Since $\hat{\mathbf{B}} = \bigcup_{\lambda,\lambda'} \hat{\mathbf{B}}_{\lambda,\lambda'}$, we see that $n_a = 0$ for any $a \in \hat{\mathbf{B}}$, hence $u' = 0$. Thus, 2.3(d) holds.

We prove 2.3(b). Let $u' = s'_{i,e}s''_{i,-e} - 1 \in \hat{\mathbf{U}}_A$. If $M \in \mathfrak{C}_A$ and $z \in M$, we have $u'z = T'_{i,e}T''_{i,-e}(z) - z = 0$; see [L1, 41.2.4]. Using 2.3(d), we see that $u' = 0$ and 2.3(b) is proved.

We prove 2.3(c). If $M \in \mathfrak{C}_A$ and $z \in M$, we have (using 2.3(b), (a)) $s'_{i,e}us'_{i,e}^{-1}s'_{i,e}z = s'_{i,e}uz = T'_{i,e}(u)(z)$. Thus, setting $u' = s'_{i,e}us'_{i,e}^{-1} - T'_{i,e}(u) \in \hat{\mathbf{U}}_A$, we have $u'z = 0$. Using 2.3(d), we see that $u' = 0$. This proves the first equality in 2.3(c). The second equality is proved similarly.

2.4. Let $i \neq j$ in I . Let $e = \pm 1$. Let $n = n_{ij} = n_{ji}$ be as in 1.1. We show:

(a) $s'_{i,e} s'_{j,e} s'_{i,e} \cdots = s'_{j,e} s'_{i,e} s'_{j,e} \cdots$ in $\hat{\mathbf{U}}_A$;

(b) $s''_{i,e} s''_{j,e} s''_{i,e} \cdots = s''_{j,e} s''_{i,e} s''_{j,e} \cdots$ in $\hat{\mathbf{U}}_A$

(all products in 2.4(a) and 2.4(b) have n factors).

Let $s_1 \in \hat{\mathbf{U}}_A$ be the left hand side minus the right hand side of 2.4(a). If $M \in \mathfrak{C}_A$ and $z \in M$, we have $s_1 z = (T'_{i,e} T'_{j,e} T'_{i,e} \cdots) z - (T'_{j,e} T'_{i,e} T'_{j,e} \cdots) z$ and this is 0 by [L1, 41.2.4(a)]. Thus $s_1 z = 0$. Using 2.3(d), we see that $s_1 = 0$. Thus 2.4(a) holds. The proof of 2.4(b) is similar.

2.5. Let $e = \pm 1$. Let $w \in W$. From 2.4(a), (b) we deduce by a standard argument that there are unique elements $w'_e \in \hat{\mathbf{U}}_A, w''_e \in \hat{\mathbf{U}}_A$ such that $w'_e = s'_{i_1,e} s'_{i_2,e} \cdots s'_{i_r,e}, w''_e = s''_{i_1,e} s''_{i_2,e} \cdots s''_{i_r,e}$ for any sequence i_1, i_2, \dots, i_r in W such that $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ in W with $r = l(w)$. Using 2.3(b), we see that $w'_e (w^{-1})''_e = 1$. From the definitions we have $\omega(w'_e) = w''_e$.

2.6. Let $e = \pm 1$. Assume that $v = 1$ in A . From Lemma 2.2(d) we see that for $i \in I, s'_{i,e}, s''_{i,e}$ are independent of the choice of e ; we denote them by s'_i, s''_i . Using this and the definitions, we see that for $w \in W, w'_e, w''_e$ are independent of the choice of e ; we denote them by w', w'' . Using Lemma 2.2(e) and the fact that \hat{D} is an algebra homomorphism, we see that

(a) $\hat{D}(w') = w' \otimes w', \hat{D}(w'') = w'' \otimes w''$ (in $\hat{\mathbf{U}}_A^{(2)}$) for any $w \in W$.

2.7. Let $\dot{\mathbf{B}}^+ = \{b^+ 1_\lambda; b \in \mathbf{B}, \lambda \in X\} = \{1_{\lambda'} b^+; b \in \mathbf{B}, \lambda' \in X\}$, a subset of $\dot{\mathbf{B}}$; see [L1, 25.2.6]. Let $\dot{\mathbf{B}}^- = \{b^- 1_\lambda; b \in \mathbf{B}, \lambda \in X\} = \{1_{\lambda'} b^-; b \in \mathbf{B}, \lambda' \in X\}$, a subset of $\dot{\mathbf{B}}$. Let $\dot{\mathbf{U}}_A^+$ (resp. $\dot{\mathbf{U}}_A^-$) be the A -submodule of $\dot{\mathbf{U}}_A$ spanned by $\dot{\mathbf{B}}^+$ (resp. by $\dot{\mathbf{B}}^-$). Now $\dot{\mathbf{B}}^+$ (resp. $\dot{\mathbf{B}}^-$) is a basis of the A -module $\dot{\mathbf{U}}_A^+$ (resp. $\dot{\mathbf{U}}_A^-$) and $\dot{\mathbf{U}}_A^+, \dot{\mathbf{U}}_A^-$ are subalgebras of $\dot{\mathbf{U}}_A$. From the definitions we have $\omega(b^+ 1_\lambda) = b^- 1_{-\lambda}$ for $b \in \mathbf{B}, \lambda \in X$. Hence setting $a = b^+ 1_\lambda \in \dot{\mathbf{B}}^+$, we have $e_a = 1$ and $a^! = b^- 1_{-\lambda}$ (see 1.5). Thus, $\omega : \dot{\mathbf{U}}_A \rightarrow \dot{\mathbf{U}}_A$ restricts to a bijection $\dot{\mathbf{B}}^+ \xrightarrow{\sim} \dot{\mathbf{B}}^-$ and to an A -algebra isomorphism $\dot{\mathbf{U}}_A^+ \xrightarrow{\sim} \dot{\mathbf{U}}_A^-$.

For $b, b' \in \mathbf{B}$ we set $b^+ = \sum_{\lambda \in X} b^+ 1_\lambda \in \hat{\mathbf{U}}_A, b'^- = \sum_{\lambda \in X} b'^- 1_\lambda \in \hat{\mathbf{U}}_A$. Then the elements $b^+ 1_\lambda b'^- \in \dot{\mathbf{U}}_A, b'^- 1_\lambda b^+ \in \dot{\mathbf{U}}_A$ (as in 1.4), which are not products in $\dot{\mathbf{U}}_A$, can be interpreted as products $b^+ \cdot 1_\lambda \cdot b'^-, b'^- \cdot 1_\lambda \cdot b^+$ in $\hat{\mathbf{U}}_A$.

2.8. Assume that $v = 1$ in A . Let w_0 be as in 1.1. Let $n = l(w_0)$. We fix a sequence i_1, i_2, \dots, i_n in I such that $s_{i_1} s_{i_2} \cdots s_{i_n} = w_0$. For any $c \in \mathbf{N}$ and $k \in [1, n]$ there is a unique element $x_{c,k} \in \mathfrak{f}_A$ such that in $\hat{\mathbf{U}}_A$ we have

$$T''_{i_1,1} T''_{i_2,1} \cdots T''_{i_{k-1},1} (\theta_{i_k}^{(c)+} 1_{s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1} \lambda}) = x_{c,k}^+ 1_\lambda$$

for any $\lambda \in X^+$. (See [L1, 41.1.3].) For any $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbf{N}^n$ we set

$$x_{\mathbf{c}} = x_{c_1,1} x_{c_2,2} \cdots x_{c_n,n} \in \mathfrak{f}_A.$$

By [L1, 41.1.4], [L1, 41.1.7], the set $\{x_{\mathbf{c}}; \mathbf{c} \in \mathbf{N}^n\}$ is an A -basis of \mathfrak{f}_A . We show:

(a) For any $\mathbf{c} \in \mathbf{N}^n$ we have

$$\hat{D}(x_{\mathbf{c}}^+) = \sum_{\mathbf{c}', \mathbf{c}'' \in \mathbf{N}^n; \mathbf{c}' + \mathbf{c}'' = \mathbf{c}} x_{\mathbf{c}'}^+ \otimes x_{\mathbf{c}''}^+, \text{ in } \hat{\mathbf{U}}^{(2)}.$$

(The last sum is defined since $\mathbf{c}', \mathbf{c}''$ only take finitely many values.) For any $k \in [1, n], c \in \mathbf{N}$ we have

$$\hat{D}(x_{c,k}^+) = \sum_{\substack{\mathbf{c}', \mathbf{c}'' \in \mathbf{N} \\ \mathbf{c}' + \mathbf{c}'' = \mathbf{c}}} x_{\mathbf{c}',k}^+ \otimes x_{\mathbf{c}'',k}^+.$$

(We use the formulas $T''_{i,1}(u) = s''_{i,1} u s''_{i,1}^{-1}$, see 2.3(c), and $\hat{D}(s''_{i,1}) = s''_{i,1} \otimes s''_{i,1}$, see Lemma 2.2(e).) It follows that

$$\begin{aligned} \hat{D}(x_{\mathbf{c}}^+) &= \hat{D}(x_{c_1,1}^+) \hat{D}(x_{c_2,2}^+) \cdots \hat{D}(x_{c_n,n}^+) \\ &= \sum_{\mathbf{c}', \mathbf{c}'' \in \mathbf{N}; \mathbf{c}' + \mathbf{c}'' = \mathbf{c}} (x_{\mathbf{c}'_1,1}^+ x_{\mathbf{c}'_2,2}^+ \cdots x_{\mathbf{c}'_n,n}^+) \otimes (x_{\mathbf{c}''_1,1}^+ x_{\mathbf{c}''_2,2}^+ \cdots x_{\mathbf{c}''_n,n}^+) \end{aligned}$$

which yields 2.8(a).

3. THE HOPF ALGEBRA \mathbf{O}_A

3.1. In this section we define the Hopf algebra \mathbf{O}_A as a submodule of the dual of $\dot{\mathbf{U}}_A$ defined in terms of $\dot{\mathbf{B}}$. We also study some basis properties of \mathbf{O}_A .

For any A -module V we set $V^\diamond = \text{Hom}_A(V, A)$. For any $a \in \dot{\mathbf{B}}$ we define a linear form $a^* : \dot{\mathbf{U}}_A \rightarrow A$ by $a' \mapsto \delta_{a,a'}$ for all $a' \in \dot{\mathbf{B}}$. Let \mathbf{O}_A be the A -submodule of $\dot{\mathbf{U}}_A^\diamond$ spanned by $\{a^*; a \in \dot{\mathbf{B}}\}$. Thus $\{a^*; a \in \dot{\mathbf{B}}\}$ is an A -basis of \mathbf{O}_A . We define an A -algebra structure on \mathbf{O}_A by the rule $a^* b^* = \sum_{c \in \dot{\mathbf{B}}} \hat{m}_c^{a,b} c^*$ for any a, b in $\dot{\mathbf{B}}$. (The sum is well defined by 1.16.) This algebra structure has a unit element, namely 1_0^* . The A -linear map $\delta : \mathbf{O}_A \rightarrow \mathbf{O}_A \otimes_A \mathbf{O}_A$ given by $c^* \mapsto \sum_{(a,b) \in \dot{\mathbf{B}} \times \dot{\mathbf{B}}} m_{a,b}^c a^* \otimes b^*$ is well defined by 1.8; we call it comultiplication. Define an A -linear map (antipode) $S : \mathbf{O}_A \rightarrow \mathbf{O}_A$ by $S(a^*) = s_a a^*$ (see 1.5) for $a \in \dot{\mathbf{B}}$. Using 1.5(i)–(vi), 1.5(ix)–(xii) we see that $(\mathbf{O}_A, \delta, S)$ is a Hopf algebra over A with 1 whose counit $\mathbf{O}_A \rightarrow A$ is given by $a^* \mapsto 1$ if $a = 1_\lambda$ for some $\lambda \in X$ and $a^* \mapsto 0$ if $a \in \dot{\mathbf{B}}$ is not of the form $1_\lambda, \lambda \in X$. From the definitions it is clear that $\mathbf{O}_A = A \otimes_A \mathbf{O}_A$ as a Hopf algebra. Define an A -linear involution $\omega : \mathbf{O}_A \rightarrow \mathbf{O}_A$ by $\omega(a^*) = e_a a^{!*}$ (see 1.5) for $a \in \dot{\mathbf{B}}$. This is an isomorphism of the algebra \mathbf{O}_A onto the algebra \mathbf{O}_A with the opposite multiplication (see 1.5(viii)) preserving 1; moreover it is compatible with the comultiplication of \mathbf{O}_A (see 1.5(vii)).

3.2. We can reformulate 1.15(a) as follows.

(a) Let $c \in \dot{\mathbf{B}}$. Assume that $\lambda, \lambda', \lambda_1, \lambda_2, \lambda'_1, \lambda'_2$ in X^+ are such that $\lambda = \lambda_1 + \lambda_2, \lambda' = \lambda'_1 + \lambda'_2, c \in \dot{\mathbf{B}}_{\lambda, \lambda'}$. There exists a function $h : \dot{\mathbf{B}}_{\lambda_1, \lambda'_1} \times \dot{\mathbf{B}}_{\lambda_2, \lambda'_2} \rightarrow \mathcal{A}$ such that $\sum_{a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a, b) a^* b^* = c^*$ in \mathbf{O}_A . Here $a^* b^*$ is a product in \mathbf{O}_A .

Proposition 3.3. *The A -algebra \mathbf{O}_A is finitely generated.*

We choose $\lambda^1, \lambda^2, \dots, \lambda^r$ in $X^+ - \{0\}$ such that $X^+ = \mathbf{N}\lambda^1 + \mathbf{N}\lambda^2 + \dots + \mathbf{N}\lambda^r$. Let $\Gamma^+ = \bigcup_{j \in [1, r]} \dot{\mathbf{B}}_{\lambda^j, 0}, \Gamma^- = \bigcup_{j \in [1, r]} \dot{\mathbf{B}}_{0, \lambda^j}, \Gamma = \Gamma^+ \cup \Gamma^-$. We show:

- (a) *The (finite) set $\{b^*; b \in \Gamma\}$ generates the A -algebra \mathbf{O}_A .*
We can assume that $A = \mathcal{A}$. For any $c \in \dot{\mathbf{B}}$ let $N = N_c$ be the smallest integer ≥ 0 such that there exist $(n_1, n_2, \dots, n_r) \in \mathbf{N}^r, (n'_1, n'_2, \dots, n'_r) \in \mathbf{N}^r$ with
- (b) $c \in \dot{\mathbf{B}}_{\lambda, \lambda'}$ where $\lambda = n_1 \lambda^1 + n_2 \lambda^2 + \dots + n_r \lambda^r, \lambda' = n'_1 \lambda^1 + n'_2 \lambda^2 + \dots + n'_r \lambda^r$ and $\sum_j n_j + \sum_j n'_j = N$.

Clearly N_c is well defined. Let \mathfrak{F} be the \mathcal{A} -subalgebra (with 1) of \mathbf{O}_A generated by Γ . It is enough to show that $c^* \in \mathfrak{F}$ for any $c \in \dot{\mathbf{B}}$. We shall prove this by induction on N_c . If $N_c = 0$, then $c \in \dot{\mathbf{B}}_{0,0}$; hence $c = 1_0$ and c^* is the unit element of \mathbf{O}_A . If $N_c = 1$, then $c \in \Gamma$ and $c^* \in \mathfrak{F}$. Assume now that $N_c = N \geq 2$ and that the result is known for any $c' \in \dot{\mathbf{B}}$ with $N_{c'} < N$. We can find λ, λ' in X^+ and $(n_j), (n'_j) \in \mathbf{N}^r$ so that (b) holds. We can find $\lambda_1, \lambda'_1, \lambda_2, \lambda'_2$ in X^+ such that $\lambda_1 + \lambda_2 = \lambda, \lambda'_1 + \lambda'_2 = \lambda'$ and either $(\lambda_1, \lambda'_1) = (\lambda^j, 0)$ for some j or $(\lambda_1, \lambda'_1) = (0, \lambda^j)$ for some j . For any $a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}$ we have $N_a \leq 1$; for any $b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}$ we have $N_b \leq N - 1$. We can write $c^* = \sum_{a \in \dot{\mathbf{B}}_{\lambda_1, \lambda'_1}, b \in \dot{\mathbf{B}}_{\lambda_2, \lambda'_2}} h(a, b) a^* b^*$ as in 3.2(a). By the induction hypothesis, for each (a, b) in the sum we have $a^* \in \mathfrak{F}, b^* \in \mathfrak{F}$; hence $c^* \in \mathfrak{F}$. This completes the inductive proof of Proposition 3.3(a). The proposition is proved.

3.4. Let $\lambda \in X^+$. The sets $\dot{\mathbf{B}}_{\lambda,0}, \dot{\mathbf{B}}_{0,\lambda}$ (see 1.9) can be described as follows:

$$(a) \quad \dot{\mathbf{B}}_{\lambda,0} = \{b^+ 1_{-\lambda}; b \in \mathbf{B}'_\lambda\}, \quad \dot{\mathbf{B}}_{0,\lambda} = \{b^- 1_\lambda; b \in \mathbf{B}_\lambda\}.$$

We only prove the first of these equalities; the proof of the other equality is similar. For $b \in \mathbf{B}'_\lambda$ we have $b^+ 1_{-\lambda}(\xi_{-\lambda} \otimes \eta_0) = (b^+ 1_{-\lambda} \xi_{-\lambda}) \otimes \eta_0 \neq 0$. Since $b^+ 1_{-\lambda} \in \dot{\mathbf{B}}$, it follows that $b^+ 1_{-\lambda} \in \dot{\mathbf{B}}_{\lambda,0}$. Thus $\{b^+ 1_{-\lambda}; b \in \mathbf{B}'_\lambda\} \subset \dot{\mathbf{B}}_{\lambda,0}$. Since this is an inclusion of finite sets with the same cardinal (equal to $\dim \Lambda_\lambda$), it is an equality, as required.

We see that:

(b) *The number of generators of the A-algebra \mathbf{O}_A given by the proof of Proposition 3.3 is at most $2 \sum_{s=1}^r \dim \Lambda_{\lambda^s}$.*

3.5. For any $a \in \dot{\mathbf{B}}^+$ (resp. $a \in \dot{\mathbf{B}}^-$) the restriction of the linear form $a^* : \dot{\mathbf{U}}_A \rightarrow A$ to $\dot{\mathbf{U}}_A^+$ (resp. $\dot{\mathbf{U}}_A^-$) is denoted again by a^* . Let \mathbf{O}_A^+ (resp. \mathbf{O}_A^-) be the A -submodule of $\dot{\mathbf{U}}_A^{+\diamond}$ (resp. $\dot{\mathbf{U}}_A^{-\diamond}$) spanned by $\{a^*; a \in \dot{\mathbf{B}}^+\}$ (resp. $\{a^*; a \in \dot{\mathbf{B}}^-\}$). Note that $\{a^*; a \in \dot{\mathbf{B}}^+\}$ (resp. $\{a^*; a \in \dot{\mathbf{B}}^-\}$) is an A -basis of \mathbf{O}_A^+ (resp. \mathbf{O}_A^-). We define an A -algebra structure on \mathbf{O}_A^\pm by the rule $a^* b^* = \sum_{c \in \dot{\mathbf{B}}^\pm} \hat{m}_c^{a,b} c^*$ for any a, b in $\dot{\mathbf{B}}^\pm$. (The sum is well defined by 1.16.) The (surjective) A -linear map $\pi^+ : \mathbf{O}_A \rightarrow \mathbf{O}_A^+$ given by $a^* \mapsto a^*$ (if $a \in \dot{\mathbf{B}}^+$), $a^* \mapsto 0$ (if $a \in \dot{\mathbf{B}} - \dot{\mathbf{B}}^+$), respects the algebra structures. (If $c \in \dot{\mathbf{B}}^+, a \in \dot{\mathbf{B}}, b \in \dot{\mathbf{B}}, \hat{m}_c^{a,b} \neq 0$, then $a \in \dot{\mathbf{B}}^+, b \in \dot{\mathbf{B}}^+$.) Similarly, the (surjective) A -linear map $\pi^- : \mathbf{O}_A \rightarrow \mathbf{O}_A^-$ given by $a^* \mapsto a^*$ (if $a \in \dot{\mathbf{B}}^-$), $a^* \mapsto 0$ (if $a \in \dot{\mathbf{B}} - \dot{\mathbf{B}}^-$) respects the algebra structures. It follows that the algebras $\mathbf{O}_A^+, \mathbf{O}_A^-$ are associative with 1. Define an A -linear map $\delta^\pm : \mathbf{O}_A^\pm \rightarrow \mathbf{O}_A^\pm \otimes_A \mathbf{O}_A^\pm$ by

$$\delta^\pm(c^*) = \sum_{(a,b) \in \dot{\mathbf{B}}^\pm \times \dot{\mathbf{B}}^\pm} m_{a,b}^c a^* \otimes b^*$$

for any $c \in \dot{\mathbf{B}}^\pm$. The sum is well defined by 1.8. Note that π^\pm is compatible with δ, δ^\pm . Define an A -linear map $S^\pm : \mathbf{O}_A^\pm \rightarrow \mathbf{O}_A^\pm$ by $S^\pm(a^*) = s_a \underline{a}^*$ for $a \in \dot{\mathbf{B}}^\pm$. (We then have $\underline{a} \in \dot{\mathbf{B}}^\pm$.) Note that $(\mathbf{O}_A^\pm, \delta^\pm, S^\pm)$ is a Hopf algebra over A with 1 and with counit $a^* \mapsto 1$ if $a = 1_\lambda$ for some $\lambda \in X$ and $a^* \mapsto 0$ if $a \in \dot{\mathbf{B}}^\pm$ is not of the form $1_\lambda, \lambda \in X$. From the definitions we have $\mathbf{O}_A^\pm = A \otimes_A \mathbf{O}_A^\pm$ as Hopf algebras. Define an A -linear isomorphism $\omega : \mathbf{O}_A^+ \xrightarrow{\sim} \mathbf{O}_A^-$ by $\omega(a^*) = a^{!*}$ (see 1.5) for $a \in \dot{\mathbf{B}}^+$. (Recall that $e_a = 1$.) This is an isomorphism of the algebra \mathbf{O}_A^+ onto the algebra \mathbf{O}_A^- with opposed multiplication.

3.6. Let $\iota : \mathbf{O}_A \rightarrow \mathbf{O}_A^- \otimes_A \mathbf{O}_A^+$ be the A -linear map given by the composition $\mathbf{O}_A \xrightarrow{\delta} \mathbf{O}_A \otimes_A \mathbf{O}_A \xrightarrow{\pi^- \otimes_A \pi^+} \mathbf{O}_A^- \otimes_A \mathbf{O}_A^+$. (Note that ι is an A -algebra homomorphism.)

Lemma 3.7. *The A -linear map ι is a split injection.*

For $\lambda \in X^+$ let Z_λ be the set of pairs $(a, b) \in \dot{\mathbf{B}}^- \times \dot{\mathbf{B}}^+$ such that $a \in \dot{\mathbf{B}}[\lambda], b \in \dot{\mathbf{B}}[\lambda], a1_\lambda = a, 1_\lambda b = b$. By [L2, 4.4] for any $(a, b) \in Z_\lambda$ there is a unique $c \in \dot{\mathbf{B}}[\lambda]$ such that $m_{a,b}^c = \delta_{c,c'}$ for any $c' \in \dot{\mathbf{B}}[\lambda]$; moreover $(a, b) \mapsto c$ is a bijection $\psi_\lambda : Z_\lambda \xrightarrow{\sim} \dot{\mathbf{B}}[\lambda]$. Let $Z = \bigsqcup_{\lambda \in X^+} Z_\lambda$. Then $\psi := \bigsqcup_{\lambda} \psi_\lambda : Z \rightarrow \dot{\mathbf{B}}$ is a bijection.

Let \underline{Z} be the A -submodule of $\mathbf{O}_A^- \otimes_A \mathbf{O}_A^+$ spanned by $\{a^* \otimes b^*; (a, b) \in Z\}$. Define an A -linear map $\rho : \mathbf{O}_A^- \otimes_A \mathbf{O}_A^+ \rightarrow \underline{Z}$ by sending a basis element $a^* \otimes b^*$ to itself if $(a, b) \in Z$ and to 0 otherwise. It is enough to show that $\rho\iota$ is an isomorphism of A -modules.

Let $c \in \dot{\mathbf{B}}[\lambda], \lambda \in X^+$. By definition we have $\rho\iota(c^*) = \sum_{(a,b) \in Z} m_{a,b}^c a^* \otimes b^*$. If $(a, b) \in Z_{\lambda'}, \lambda' \in X^+$ and $m_{a,b}^c \neq 0$, then as in the proof of Lemma 1.8 we have $\lambda' \leq \lambda$. Thus

$$\rho\iota(c^*) = a_0^* \otimes b_0^* + \sum_{\lambda' \in X^+; \lambda' < \lambda} \sum_{(a,b) \in Z_{\lambda'}} m_{a,b}^c a^* \otimes b^*$$

where $(a_0, b_0) = \psi_\lambda^{-1}(c) \in Z_\lambda$. We identify \underline{Z} with \mathbf{O}_A as A -modules via the bijection $a^* \otimes b^* \mapsto c^*$ (with $c = \psi(a, b)$) between the bases of $\underline{Z}, \mathbf{O}_A$. Then we have

$$\rho\iota(c^*) = c^* + \sum_{\lambda' \in X^+; \lambda' < \lambda} \sum_{c' \in \dot{\mathbf{B}}[\lambda']} x_{c',c} c'^*$$

where $x_{c',c} \in A$. We see that $\rho\iota$ is represented by a square upper triangular matrix with entries in A and with 1 on the diagonal. It follows that $\rho\iota$ is an isomorphism. The lemma is proved.

3.8. For $a, b \in \mathbf{B}$ we have $ab = \sum_{c \in \mathbf{B}} \mu_{a,b}^c c$ (in \mathbf{f}_A) where $\mu_{a,b}^c \in A$ are zero for all but finitely many c . In the case where $A = \mathcal{A}$ we define a homomorphism of \mathcal{A} -algebras $\underline{r} : \mathbf{f}_A \rightarrow \mathbf{f}_A \otimes_A \mathbf{f}_A$ as the restriction of the homomorphism $r : \mathbf{f} \rightarrow \mathbf{f} \otimes \mathbf{f}$ given in [L1, 1.2.6]. For general A we define a homomorphism of A -algebras $\underline{r} : \mathbf{f}_A \rightarrow \mathbf{f}_A \otimes_A \mathbf{f}_A$ by applying $A \otimes_A ?$ to the homomorphism $r : \mathbf{f}_A \rightarrow \mathbf{f}_A \otimes_A \mathbf{f}_A$. For any a, b, c in \mathbf{B} we define $\hat{\mu}_c^{a,b} \in A$ by the following requirement: for any $c \in \mathbf{B}$ we have $\underline{r}(c) = \sum_{a,b \in \mathbf{B}} \hat{\mu}_c^{a,b} a \otimes b$ (in the last sum $\hat{\mu}_c^{a,b}$ is 0 for all but finitely many (a, b)).

In the rest of this section we assume that $v = 1$ in A .

We define an A -algebra homomorphism S from \mathbf{f}_A to \mathbf{f}_A with the opposite multiplication by $S(\theta_i^{(m)}) = (-1)^m \theta_i^{(m)}$ for all $i \in I, m \in \mathbf{N}$. For any $a \in \mathbf{B}$ we have $S(a) = q_a \tilde{a}$ where $a \mapsto \tilde{a}$ is an involution of \mathbf{B} and $q_a = \pm 1$. Note that the quantities $\mu_{a,b}^c, \hat{\mu}_c^{a,b}, q_a$ (in A) are the images of the corresponding quantities in \mathcal{A} under the given homomorphism $\mathcal{A} \rightarrow A$. Now $(\mathbf{f}_A, \underline{r}, S)$ is a Hopf algebra over A with 1 whose counit $\mathbf{f}_A \rightarrow A$ is given by $a \mapsto \delta_{a,1}$ for $a \in \dot{\mathbf{B}}$. For any $a \in \mathbf{B}$ let $a^* : \mathbf{f}_A \rightarrow A$ be the A -linear form given by $a^*(b) = \delta_{a,b}$ for $b \in \mathbf{B}$. Let \mathbf{o}_A be the A -submodule of \mathbf{f}_A^\otimes spanned by $\{a^*; a \in \mathbf{B}\}$. Note that $\{a^*; a \in \mathbf{B}^+\}$ is an A -basis of \mathbf{o}_A . We define an A -algebra structure on \mathbf{o}_A by the rule $a^* b^* = \sum_{c \in \mathbf{B}} \hat{\mu}_c^{a,b} c^*$ for any a, b in \mathbf{B}^\pm . (The sum is well defined by homogeneity reasons.) This algebra structure is associative with unit element 1^* . Define an A -linear map (comultiplication)

$\delta_0 : \mathfrak{o}_A \rightarrow \mathfrak{o}_A \otimes_A \mathfrak{o}_A$ by

$$\delta_0(c^*) = \sum_{(a,b) \in \mathbf{B} \times \mathbf{B}} \mu_{a,b}^c a^* \otimes b^*$$

for any $c \in \mathbf{B}$. (The sum is well defined by homogeneity reasons.) Define an A -linear map $S : \mathfrak{o}_A \rightarrow \mathfrak{o}_A$ (antipode) by $S(a^*) = q_a \tilde{a}^*$ for $a \in \mathbf{B}$. Note that $(\mathfrak{o}_A, \delta_0, S)$ is a Hopf algebra over A with 1 and with counit $a^* \mapsto \delta_{a,1}$. From the definitions we have $\mathfrak{o}_A = A \otimes_A \mathfrak{o}_A$ as Hopf algebras. Define a (surjective) A -linear map $\pi^{>0} : \mathfrak{o}_A \rightarrow \mathfrak{o}_A$ by $a^* \mapsto b^*$ if $a = b^+ 1_\lambda$ for some $b \in \mathbf{B}, \lambda \in X$ and $a^* \mapsto 0$ if $a \in \dot{\mathbf{B}}$ is not of the form above. Define a (surjective) A -linear map $\pi^{<0} : \mathfrak{o}_A \rightarrow \mathfrak{o}_A$ by $a^* \mapsto b^*$ if $a = b^- 1_\lambda$ for some $b \in \mathbf{B}, \lambda \in X$ and $a^* \mapsto 0$ if $a \in \dot{\mathbf{B}}$ is not of the form above. Note that $\pi^{>0}$ and $\pi^{<0}$ respect the A -algebra structures and the unit elements. They are also compatible with δ, δ_0 .

3.9. Let $A[X]$ be the group algebra of X with coefficients in A . Define a (surjective) A -linear map $\pi^0 : \mathfrak{o}_A \rightarrow A[X]$ by $a^* \mapsto \lambda$ if $a = 1_\lambda$ for some $\lambda \in X$ and $a^* \mapsto 0$ if $a \in \dot{\mathbf{B}}$ is not of this form. Note that π^0 respects the A -algebra structures and the unit elements.

Let $\iota' : \mathfrak{o}_A \rightarrow \mathfrak{o}_A \otimes_A A[X] \otimes_A \mathfrak{o}_A$ be the A -linear map given by the composition $\mathfrak{o}_A \xrightarrow{(\delta \otimes 1) \delta} \mathfrak{o}_A \otimes_A \mathfrak{o}_A \otimes_A \mathfrak{o}_A \xrightarrow{\pi^{<0} \otimes_A \pi^0 \otimes_A \pi^{>0}} \mathfrak{o}_A \otimes_A A[X] \otimes_A \mathfrak{o}_A$. (Note that ι' is an A -algebra homomorphism.) We have the following variant of Lemma 3.7:

Lemma 3.10. *The A -linear map ι' is a split injection. (Recall that $v = 1$ in A .)*

Let $Z_\lambda, \psi_\lambda, Z, \psi$ be as in 3.7. Let Z' be the A -submodule of $\mathfrak{o}_A \otimes_A A[X] \otimes_A \mathfrak{o}_A$ spanned by $\{a_1^* \otimes \lambda \otimes b_1^*; (a_1^- 1_\lambda, 1_\lambda b_1^+) \in Z\}$. Define an A -linear map

$$\rho' : \mathfrak{o}_A \otimes_A A[X] \otimes_A \mathfrak{o}_A \rightarrow Z'$$

by sending a basis element $a_1^* \otimes \lambda \otimes b_1^*$ to itself if $(a_1^- 1_\lambda, 1_\lambda b_1^+) \in Z$ and to 0 otherwise. It is enough to show that $\rho' \iota'$ is an isomorphism of A -modules. Let $c \in \dot{\mathbf{B}}[\lambda], \lambda \in X^+$. By definition we have

$$\rho' \iota'(c^*) = \sum_{\lambda' \in X} \sum_{(a,b) \in Z_{\lambda'}} m_{a,b}^c a_1^* \otimes \lambda' \otimes b_1^*$$

where $a_1, b_1 \in \mathbf{B}$ are defined in terms of a, b by $a = a_1^- 1_\lambda, b = 1_\lambda b_1^+$. If $(a, b) \in Z_{\lambda'}, \lambda' \in X^+$ and $m_{a,b}^c \neq 0$, then as in the proof of Lemma 1.8 we have $\lambda' \leq \lambda$. Thus

$$\rho' \iota'(c^*) = a_0^* \otimes \lambda \otimes b_0^* + \sum_{\lambda' \in X^+; \lambda' < \lambda} \sum_{(a,b) \in Z_{\lambda'}} m_{a,b}^c a_1^* \otimes \lambda' \otimes b_1^*$$

where $a_0, b_0 \in \mathbf{B}$ are given by $(a_0^- 1_\lambda, 1_\lambda b_0^+) = \psi_\lambda^{-1}(c) \in Z_\lambda$ and a_1, b_1 are as above. We identify Z' with \mathfrak{o}_A as A -modules via the bijection $a_1^* \otimes \lambda \otimes b_1^* \mapsto c^*$ (with $c = \psi(a_1^- 1_\lambda, 1_\lambda b_1^+)$) between the bases of Z', \mathfrak{o}_A . Then we have

$$\rho' \iota'(c^*) = c^* + \sum_{\lambda' \in X^+; \lambda' < \lambda} \sum_{c' \in \dot{\mathbf{B}}[\lambda']} x'_{c',c} c'^*$$

where $x'_{c',c} \in A$. We see that $\rho' \iota'$ is represented by a square upper triangular matrix with entries in A and with 1 on the diagonal. It follows that $\rho' \iota'$ is an isomorphism. The lemma is proved.

3.11. Let $\{x_{\mathbf{c}}; \mathbf{c} \in \mathbf{N}^n\}$ be the A -basis of \mathbf{f}_A defined in 2.8. For any $\mathbf{c} \in \mathbf{N}^n$ we define an A -linear function $\xi_{\mathbf{c}} : \mathbf{f}_A \rightarrow A$ by $\xi_{\mathbf{c}}(x_{\mathbf{c}'}) = \delta_{\mathbf{c},\mathbf{c}'}$ for any $\mathbf{c}' \in \mathbf{N}^n \times X$.

Lemma 3.12. (a) $\{\xi_{\mathbf{c}}; \mathbf{c} \in \mathbf{N}^n\}$ is an A -basis of \mathbf{o}_A .

(b) If $\mathbf{c}', \mathbf{c}'' \in \mathbf{N}^n$, then $\xi_{\mathbf{c}'}\xi_{\mathbf{c}''} = \xi_{\mathbf{c}'+\mathbf{c}''}$, product in \mathbf{o}_A .

Note that each summand $\mathbf{f}_{\nu,A}$ of \mathbf{f}_A is a free A -module of finite rank with basis given by its intersection with \mathbf{B} ; this A -module also has as a basis its intersection with the A -basis $\{x_{\mathbf{c}}; \mathbf{c} \in \mathbf{N}^n\}$ of \mathbf{f}_A . It follows that $\{\xi_{\mathbf{c}}; \mathbf{c} \in \mathbf{N}^n\}$ and $\{b^*; b \in \mathbf{B}\}$ span the same A -submodule of \mathbf{f}_A° , namely the set of A -linear functions $\mathbf{f}_A \rightarrow A$ which vanish on $\mathbf{f}_{\nu,A}$ for all but finitely many ν . This proves Lemma 3.12(a).

We prove Lemma 3.12(b). For any $\mathbf{c} \in \mathbf{N}^n$ we can write uniquely $x_{\mathbf{c}} = \sum_{b \in \mathbf{B}} h_{\mathbf{c},b} b$ where $h_{\mathbf{c},b} \in A$ are zero for all but finitely many b ; for any $b \in \mathbf{B}$ we can write uniquely $b = \sum_{\mathbf{c} \in \mathbf{N}^n} \tilde{h}_{b,\mathbf{c}} x_{\mathbf{c}}$ where $\tilde{h}_{b,\mathbf{c}} \in A$ are zero for all but finitely many \mathbf{c} . We then have $\xi_{\mathbf{c}} = \sum_{b \in \mathbf{B}} \tilde{h}_{b,\mathbf{c}} b^*$ and $b^* = \sum_{\mathbf{c} \in \mathbf{N}^n} h_{\mathbf{c},b} \xi_{\mathbf{c}}$. We have

$$\xi_{\mathbf{c}'}\xi_{\mathbf{c}''} = \left(\sum_{b' \in \mathbf{B}} \tilde{h}_{b',\mathbf{c}'} b'^*\right) \left(\sum_{b'' \in \mathbf{B}} \tilde{h}_{b'',\mathbf{c}''} b''^*\right) = \sum_{a,b',b'' \in \mathbf{B}} \tilde{h}_{b',\mathbf{c}'} \tilde{h}_{b'',\mathbf{c}''} \hat{\mu}_a^{b',b''} a^*;$$

hence Lemma 3.12(b) is equivalent to the set of equalities

$$(c) \quad \sum_{b',b'' \in \mathbf{B}} \tilde{h}_{b',\mathbf{c}'} \tilde{h}_{b'',\mathbf{c}''} \hat{\mu}_a^{b',b''} = \tilde{h}_{a,\mathbf{c}'+\mathbf{c}''}$$

for any $\mathbf{c}', \mathbf{c}'', a$. The equality in 2.8(a) can be rewritten as

$$\underline{r}(x_{\mathbf{c}}) = \sum_{\mathbf{c}',\mathbf{c}'' \in \mathbf{N}^n; \mathbf{c}'+\mathbf{c}''=\mathbf{c}} x_{\mathbf{c}'} \otimes x_{\mathbf{c}''},$$

hence as

$$\sum_{a',a'',b \in \mathbf{B}} h_{\mathbf{c},b} \hat{\mu}_b^{a',a''} a' \otimes a'' = \sum_{a',a'' \in \mathbf{B}, \mathbf{c}',\mathbf{c}'' \in \mathbf{N}^n; \mathbf{c}'+\mathbf{c}''=\mathbf{c}} h_{\mathbf{c}',a'} h_{\mathbf{c}'',a''} a' \otimes a''.$$

Thus

$$\sum_{b \in \mathbf{B}} h_{\mathbf{c},b} \hat{\mu}_b^{a',a''} = \sum_{\mathbf{c}',\mathbf{c}'' \in \mathbf{N}^n; \mathbf{c}'+\mathbf{c}''=\mathbf{c}} h_{\mathbf{c}',a'} h_{\mathbf{c}'',a''}$$

for any $a', a'' \in \mathbf{B}, \mathbf{c} \in \mathbf{N}^n$. We multiply both sides by $\tilde{h}_{\mathbf{c},\mathbf{c}} \tilde{h}_{a',\mathbf{c}'} \tilde{h}_{a'',\mathbf{c}''}$ and sum over all \mathbf{c}, a', a'' . We obtain the equalities Lemma 3.12(c). This proves Lemma 3.12(b). The lemma is proved.

Lemma 3.13. Assume that $v = 1$ in A . Recall that $n = l(w_0)$. Let $A[t_1, t_2, \dots, t_n]$ be the algebra of polynomials with coefficients in A in the indeterminates t_1, t_2, \dots, t_n . Define an A -linear map $\kappa : \mathbf{o}_A \rightarrow A[t_1, t_2, \dots, t_n]$ by $\xi_{\mathbf{c}} \mapsto t_1^{c_1} t_2^{c_2} \dots t_n^{c_n}$ for any $\mathbf{c} = (c_1, c_2, \dots, c_n) \in \mathbf{N}^n$. Then κ is an A -algebra isomorphism.

By Lemma 3.12(a), κ is an A -linear isomorphism. It is compatible with the algebra structures by Lemma 3.12(b). The lemma is proved.

3.14. Define $\tilde{\iota}' : \mathbf{O}_A \rightarrow A[t_1, t_2, \dots, t_n] \otimes_A A[X] \otimes_A A[t_1, t_2, \dots, t_n]$ as the composition $(\kappa \otimes 1 \otimes \kappa)\iota'$ where $\iota' : \mathbf{O}_A \rightarrow \mathbf{o}_A \otimes_A A[X] \otimes_A \mathbf{o}_A$ is as in 3.9.

Theorem 3.15. Recall the assumption that $v = 1$ in A . The A -linear map $\tilde{\iota}'$ is a split injection. It is also an imbedding of A -algebras with 1. Thus, the A -algebra \mathbf{O}_A is isomorphic to a finitely generated A -subalgebra (with 1) of $A[t_1, t_2, \dots, t_n] \otimes_A A[X] \otimes_A A[t_1, t_2, \dots, t_n]$. In particular, if A is an integral domain, then \mathbf{O}_A is an integral domain.

The fact that $\tilde{\iota}'$ is a split injection follows from 3.10 and the fact that $\kappa \otimes 1 \otimes \kappa$ is an isomorphism of A -modules. The fact that $\tilde{\iota}'$ is an algebra homomorphism follows from the fact that ι' is an algebra homomorphism and that κ is an algebra isomorphism. The fact that the algebra \mathbf{O}_A is finitely generated follows from Proposition 3.3. If A is an integral domain, then $A[t_1, t_2, \dots, t_n] \otimes_A A[X] \otimes_A A[t_1, t_2, \dots, t_n]$ is an integral domain and hence so is \mathbf{O}_A . The theorem is proved.

4. THE GROUP G_A

4.1. In this section we assume that $v = 1$ in A .

In analogy with [Ko], we define G_A as the set of A -algebra homomorphisms $\mathbf{O}_A \rightarrow A$ which take 1 to 1. Since \mathbf{O}_A is a Hopf algebra over A with 1 and with counit, G_A has a natural group structure. We now describe the group structure of G_A in more detail. Note that G_A is the set of all A -linear functions $\phi : \mathbf{O}_A \rightarrow A$ such that $\phi(a^*b^*) = \phi(a^*)\phi(b^*)$ for all $a, b \in \mathbf{B}$ and such that $\phi(1_0^*) = 1$. By the correspondence $\phi \mapsto \sum_{a \in \mathbf{B}} \phi(a^*)a$ we identify G_A with the subset of $\hat{\mathbf{U}}_A$ defined as

$$\left\{ \sum_{a \in \mathbf{B}} n_a a \in \hat{\mathbf{U}}_A; n_{1_0} = 1, \sum_{c \in \mathbf{B}} \hat{m}_c^{a,b} n_c = n_a n_b \text{ for all } a, b \in \mathbf{B} \right\}$$

or equivalently as $\{\xi \in \hat{\mathbf{U}}_A; \hat{D}(\xi) = \xi \otimes \xi, \epsilon(\xi) = 1\}$. Note that G_A is closed under the multiplication in the algebra $\hat{\mathbf{U}}_A$. Thus the multiplication in $\hat{\mathbf{U}}_A$ induces an associative monoid structure on G_A . Since the unit element 1 of $\hat{\mathbf{U}}_A$ satisfies $\hat{D}(1) = 1 \otimes 1, \epsilon(1) = 1$, it belongs to G_A and plays the role of a unit element for the monoid structure of G_A . If $\xi \in G_A$, we have $\hat{D}(S(\xi)) = S^{(2)}(\hat{D}(\xi)) = S^{(2)}(\xi \otimes \xi) = S(\xi) \otimes S(\xi)$ and $\epsilon(S(\xi)) = \epsilon(\xi) = 1$. Thus $S(\xi) \in G_A$. From 1.5(xi) we have $\mathbf{m}(S^{(0,1)}(\hat{D}(\xi))) = \xi$. In our case this can be written as $\mathbf{m}(S^{(0,1)}(\xi \otimes \xi)) = \xi$, that is, $\mathbf{m}(\xi \otimes S(\xi)) = \xi$ and $\xi S(\xi) = \xi$. We see that any element of G_A has a right inverse in the monoid G_A . Hence any element of G_A has a left inverse in the monoid G_A . Thus G_A is a group (a subgroup of the group of invertible elements of the algebra $\hat{\mathbf{U}}_A$). From the definitions we see that the algebra involution $\omega : \hat{\mathbf{U}}_A \rightarrow \hat{\mathbf{U}}_A$ restricts to an involution $\omega : G_A \rightarrow G_A$ which is a group isomorphism.

4.2. Let $\hat{\mathbf{U}}_A^{>0}$ (resp. $\hat{\mathbf{U}}_A^{<0}$) be the A -submodule of $\hat{\mathbf{U}}_A$ consisting of all elements of the form $\sum_{b \in \mathbf{B}, \lambda \in X} n_b(1_\lambda b^+)$ (resp. $\sum_{b \in \mathbf{B}, \lambda \in X} n_b(b^- 1_\lambda)$) where $n_b \in A$. Let \mathfrak{G}_A be the set of A -algebra homomorphisms $\mathbf{o}_A \rightarrow A$ which take 1 to 1. Since \mathbf{o}_A is a Hopf algebra over A with 1 and with counit, \mathfrak{G}_A has a natural group structure. Note that \mathfrak{G}_A is the set of A -linear maps $\phi : \mathbf{o}_A \rightarrow A$ such that $\phi(a^*b^*) = \phi(a^*)\phi(b^*)$ for all $a, b \in \mathbf{B}$ and such that $\phi(1) = 1$. (Here a^*b^* is the product in \mathbf{o}_A .) By the correspondence $\phi \mapsto \sum_{a \in \mathbf{B}, \lambda \in X} \phi(a^*)a^- 1_\lambda$ we identify \mathfrak{G}_A with the set of all $\xi \in \hat{\mathbf{U}}_A^{<0}$ such that $\hat{D}(\xi) = \xi \otimes \xi$ (in $\hat{\mathbf{U}}_A^{(2)}$) and $\epsilon(\xi) = 1$ or equivalently with $G_A^{<0} := G_A \cap \hat{\mathbf{U}}_A^{<0}$. Similarly, by the correspondence $\phi \mapsto \sum_{a \in \mathbf{B}, \lambda \in X} \phi(a^*)1_\lambda a^+$ we identify \mathfrak{G}_A with the set of all $\xi \in \hat{\mathbf{U}}_A^{>0}$ such that $\hat{D}(\xi) = \xi \otimes \xi$ (in $\hat{\mathbf{U}}_A^{(2)}$) and $\epsilon(\xi) = 1$ or equivalently with $G_A^{>0} := G_A \cap \hat{\mathbf{U}}_A^{>0}$. The group structure on \mathfrak{G}_A coincides with that induced from the group structure of G_A via either one of the isomorphisms $\mathfrak{G}_A \xrightarrow{\sim} G_A^{<0}, \mathfrak{G}_A \xrightarrow{\sim} G_A^{>0}$.

Let $\hat{\mathbf{U}}_A^0$ be the set of elements of $\hat{\mathbf{U}}_A$ of the form $\sum_{\lambda \in X} n_\lambda 1_\lambda$ with $n_\lambda \in A$. Let $T_A = \hat{\mathbf{U}}_A^0 \cap G_A$. Thus T_A is the set of elements of $\hat{\mathbf{U}}_A$ of the form $\sum_{\lambda \in X} n_\lambda 1_\lambda$ with $n_\lambda \in A^\circ$ such that $\lambda \mapsto n_\lambda$ is a group homomorphism $X \rightarrow A^\circ$. Clearly, T_A

is an abelian subgroup of G_A . Note that T_A can be identified with $\text{Hom}(X, A^\circ) = A^\circ \otimes_{\mathbf{Z}} Y$. The isomorphism between $A^\circ \otimes Y$ and T_A associates to $d \otimes y$ (where $d \in A^\circ$, $y \in Y$) the element $\sum_{\lambda \in X} d^{(y, \lambda)} 1_\lambda$ of T_A . (Since $d \in A^\circ$, it can be raised to any integer power.) Note that T_A can also be identified with the set of homomorphisms of A -algebras $A[X] \rightarrow A$ preserving 1, by $[g = \sum_{\lambda \in X} n_\lambda 1_\lambda \in T_A] \mapsto [\lambda \mapsto n_\lambda]$. We show:

(a) *Multiplication in G_A defines an injective map $G_A^{<0} \times T_A \times G_A^{>0} \rightarrow G_A$.*

Let \hat{U}_A^- be the set of elements of \hat{U}_A of the form $\sum_{a \in \mathbf{B}^-} n_a a$. Let $\xi_1, \xi'_1 \in G_A^{<0}$, $\xi_2, \xi'_2 \in T_A$, $\xi_3, \xi'_3 \in G_A^{>0}$ be such that $\xi_1 \xi_2 \xi_3 = \xi'_1 \xi'_2 \xi'_3$ in G_A . Then $\xi'_3 \xi_3^{-1} = \xi'_2^{-1} \xi'_1^{-1} \xi_1 \xi_2$. The right hand side is contained in \hat{U}_A^- since $G_A^{<0}, T_A$ are contained in \hat{U}_A^- and \hat{U}_A^- is closed under multiplication in \hat{U}_A . Thus we have $\xi'_3 \xi_3^{-1} \in \hat{U}_A^-$. Since $G_A^{>0} \in \hat{U}_A^{>0}$ and $\hat{U}_A^{>0}$ is closed under multiplication in \hat{U}_A , we also have $\xi'_3 \xi_3^{-1} \in \hat{U}_A^{>0}$. Thus $\xi'_3 \xi_3^{-1} \in \hat{U}_A^- \cap \hat{U}_A^{>0}$. From the definitions we have $\hat{U}_A^- \cap \hat{U}_A^{>0} = \{1\}$. Thus $\xi'_3 \xi_3^{-1} = 1$ so that $\xi'_3 = \xi_3$. It follows that $\xi_1 \xi_2 = \xi'_1 \xi'_2$. Then $x_1^{-1} x_1 = x_2 x_2^{-1}$ belongs to both $\hat{U}_A^{<0}$ and to $\hat{U}_A^{>0}$ which have intersection $\{1\}$. Thus $x_1^{-1} x_1 = x_2 x_2^{-1} = 1$ and 4.2(a) follows.

From the definitions we see that the involution $\omega : G_A \rightarrow G_A$ interchanges $G_A^{<0}$ with $G_A^{>0}$ and its restriction to T_A is given by $g \mapsto g^{-1}$.

4.3. For any $i \in I$, $h \in A$ we have $x_i(h) \in G_A^{>0}$, $y_i(h) \in G_A^{<0}$. (We use 1.18(a).)

For any $w \in W$, the elements w', w'' of \hat{U}_A (see 2.5, 2.6) belong to G_A . (We use 2.6(a).) We have $w'(w^{-1})'' = 1$; see 2.5.

Now W acts on T_A by $w : \sum_{\lambda \in X} n_\lambda 1_\lambda \mapsto \sum_{\lambda \in X} n_{w^{-1}(\lambda)} 1_\lambda$. This corresponds to the W -action on $A^\circ \otimes Y$ given by $w : d \otimes y \mapsto d \otimes w(y)$. This W action on T_A is denoted by $w : g \mapsto w(g)$. We show:

(a) *If $w \in W$, $g \in T_A$, then $w(g) = w' g w'^{-1} = w'' g w''^{-1}$.*

Using the definition of w' , we see that to prove the first equality in 4.3(a) we may assume that $w = s_i$ for some $i \in I$. In this case it is enough to show that $1_{s_i(\lambda)} = s'_i 1_\lambda (s'_i)^{-1}$ or that $1_{s_i(\lambda)} = T'_{i,1}(1_\lambda)$ for any $\lambda \in X$. This follows from [L1, 41.1.2]. The proof of the second equality in 4.3(a) is similar.

4.4. Let $i \in I$, $u \in A^\circ$. Let $t_i(u) = \sum_{\lambda \in X} u^{(i, \lambda)} 1_\lambda \in T_A$. We show:

(a) $s'_i t_i(-1) = s'_i t_i(-1)$ in G_A .

We have

$$s'_i t_i(-1) = \sum_{y \in \mathbf{Z}, x \in \mathbf{Z}, \lambda, \lambda' \in X; \lambda' = \lambda - y i', \langle i, \lambda \rangle = x + y} (-1)^y (-1)^{\langle i, \lambda' \rangle} \theta_i^{(x)-} 1_\lambda \theta_i^{(y)+}.$$

It is enough to show that $(-1)^{y + \langle i, \lambda - y i' \rangle} = (-1)^x$ in the sum above. This follows from $\langle i, i' \rangle = 2$ and $y + \langle i, \lambda \rangle = x \pmod 2$.

4.5. Let $i \in I$. Let $e, f, g \in A$ be such that $d := eg + f^2 \in A^\circ, f \in A^\circ$. Define $e', f', g' \in A$ by $e' = ed^{-1}, f' = fd^{-1}, g' = gd^{-1}$. Note that $d' := e'g' + f'^2 = d^{-1} \in A^\circ, f' \in A^\circ$ and $e = e'd'^{-1}, f = f'd'^{-1}, g = g'd'^{-1}$. We show:

(a) $y_i(e) t_i(f^{-1}) x_i(g) = x_i(g') t_i(f') y_i(e')$.

We compute the left hand side in \hat{U}_A :

$$\begin{aligned} & \sum_{\substack{c,c' \in \mathbf{N}, \\ \lambda \in X}} e^c f^{-\langle i, \lambda \rangle} g^{c'} \theta_i^{(c)-} 1_\lambda \theta_i^{(c') +} = \sum_{\substack{c,c' \in \mathbf{N}, \lambda \in X; \\ \langle i, \lambda \rangle \geq c+c'}} e^c f^{-\langle i, \lambda \rangle} g^{c'} \theta_i^{(c)-} 1_\lambda \theta_i^{(c') +} \\ & + \sum_{\substack{c,c',t \in \mathbf{N}, \lambda \in X; \\ \langle i, \lambda \rangle < c+c'}} e^c f^{-\langle i, \lambda \rangle} g^{c'} \binom{c+c' - \langle i, \lambda \rangle}{t} \theta_i^{(c'-t)+} 1_{\lambda - (c+c'-t)i'} \theta_i^{(c'-t)-} \\ & = \sum_{c,c' \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle \geq c+c'} e^c f^{-\langle i, \lambda \rangle} g^{c'} \theta_i^{(c)-} 1_\lambda \theta_i^{(c') +} \\ & + \sum_{\substack{r,r' \in \mathbf{N}, \lambda' \in X \\ \langle i, -\lambda' \rangle > r+r'}} \sum_{t \in \mathbf{N}} e^{r'+t} f^{-\langle i, \lambda' + (r+r'+t)i' \rangle} g^{r+t} \binom{-r-r' - \langle i, \lambda' \rangle}{t} \theta_i^{(r)+} 1_{\lambda'} \theta_i^{(r')-} \\ & = \sum_{c,c' \in \mathbf{N}, \lambda \in X; \langle i, \lambda \rangle \geq c+c'} e^c f^{-\langle i, \lambda \rangle} g^{c'} \theta_i^{(c)-} 1_\lambda \theta_i^{(c') +} \\ & + \sum_{\substack{r,r' \in \mathbf{N}, \lambda' \in X; \\ \langle i, -\lambda' \rangle > r+r'}} e^{r'} f^{-\langle i, \lambda' + (r+r')i' \rangle} g^r (1 + e f^{-2} g)^{-r-r' - \langle i, \lambda' \rangle} \theta_i^{(r)+} 1_{\lambda'} \theta_i^{(r')-}. \end{aligned}$$

We compute the right hand side of 4.5(a) in \hat{U}_A :

$$\begin{aligned} & \sum_{\substack{c,c' \in \mathbf{N}, \\ \lambda \in X}} g^{c'} f^{\langle i, \lambda \rangle} e^{c'} \theta_i^{(c)+} 1_\lambda \theta_i^{(c') -} = \sum_{\substack{c,c' \in \mathbf{N}, \lambda \in X; \\ -\langle i, \lambda \rangle > c+c'}} g^{c'} f^{\langle i, \lambda \rangle} e^{c'} \theta_i^{(c)+} 1_\lambda \theta_i^{(c') -} \\ & + \sum_{\substack{c,c',t \in \mathbf{N}, \lambda \in X; \\ -\langle i, \lambda \rangle \leq c+c'}} g^{c'} f^{\langle i, \lambda \rangle} e^{c'} \binom{c+c' + \langle i, \lambda \rangle}{t} \theta_i^{(c'-t)-} 1_{\lambda + (c+c'-t)i'} \theta_i^{(c'-t)+} \\ & = \sum_{c,c' \in \mathbf{N}, \lambda \in X; -\langle i, \lambda \rangle > c+c'} g^{c'} f^{\langle i, \lambda \rangle} e^{c'} \theta_i^{(c)+} 1_\lambda \theta_i^{(c') -} \\ & + \sum_{\substack{s,s' \in \mathbf{N}, \lambda' \in X; \\ \langle i, \lambda' \rangle \geq s+s'}} \sum_{t \in \mathbf{N}} g^{s'+t} f^{\langle i, \lambda' - (s+s+t)i' \rangle} e^{s+t} \binom{-s-s' + \langle i, \lambda' \rangle}{t} \theta_i^{(s)-} 1_{\lambda'} \theta_i^{(s') +} \\ & = \sum_{c,c' \in \mathbf{N}, \lambda \in X; -\langle i, \lambda \rangle > c+c'} g^{c'} f^{\langle i, \lambda \rangle} e^{c'} \theta_i^{(c)+} 1_\lambda \theta_i^{(c') -} \\ & + \sum_{\substack{s,s' \in \mathbf{N}, \lambda' \in X; \\ \langle i, \lambda' \rangle \geq s+s'}} g^{s'} f^{\langle i, \lambda' - (s+s)i' \rangle} e^{s'} (1 + g' f'^{-2} e')^{-s-s' + \langle i, \lambda' \rangle} \theta_i^{(s)-} 1_{\lambda'} \theta_i^{(s') +}. \end{aligned}$$

To complete the proof, it is enough to show that

$$\begin{aligned} e^c f^{-\langle i, \lambda \rangle} g^{c'} &= g^{c'} f^{\langle i, \lambda - (c+c')i' \rangle} e^{c'} (1 + g' f'^{-2} e')^{-c-c' + \langle i, \lambda \rangle}, \\ g^{c'} f^{\langle i, \lambda \rangle} e^{c'} &= e^{c'} f^{-\langle i, \lambda + (c+c')i' \rangle} g^c (1 + e f^{-2} g)^{-c-c' - \langle i, \lambda \rangle} \end{aligned}$$

for any $c, c' \in \mathbf{N}, \lambda \in X$. This is immediate.

4.6. Let $i \in I$ and let $u \in A^\circ$. We have

(a)
$$x_i(u-1)y_i(1)x_i(u^{-1}-1)y_i(-u) = t_i(u) \in T_A.$$

Taking $e = 1, f = 1, g = u^{-1} - 1$ in 4.5(a), we obtain

$$y_i(1)x_i(u^{-1} - 1) = x_i(-u + 1)t_u y_i(u)$$

and 4.6(a) follows.

4.7. Let $w \in W, i \in I$ be such that $l(ws_i) = l(w) + 1$. Let $h \in A$. We show:

(a) $w''x_i(h)w''^{-1} \in G_A^{>0}$.

We can find a sequence i_1, i_2, \dots, i_n in I as in 2.8 such that $w = s_{i_1}s_{i_2}\dots s_{i_{k-1}}, i = i_k, k - 1 = l(w)$. The equality

$$T''_{i_1,1}T''_{i_2,1}\dots T''_{i_{k-1},1}(\theta_{i_k}^{(c)+}1_{s_{i_{k-1}}s_{i_{k-2}}\dots s_{i_1}\lambda}) = x_{c,k}^+1_\lambda$$

in 2.8 (with $c \in \mathbf{N}, \lambda \in X$, and with $x_{c,k} \in \mathbf{f}_A$ independent of λ) can be rewritten as

$$w''(\theta_{i_k}^{(c)+}1_{s_{i_{k-1}}s_{i_{k-2}}\dots s_{i_1}\lambda})w''^{-1} = x_{c,k}^+1_\lambda.$$

(See 2.3(c).) We multiply this by h^c and sum over all $\lambda \in X$ and $c \in \mathbf{N}$; we obtain

$$w''x_i(h)w''^{-1} = \sum_{c \in \mathbf{N}} h^c x_{c,k}^+.$$

The sum in the right hand side is well defined since for some $\nu \in \mathbf{N}[I] - \{0\}$ we have $x_{c,k}^+ \in \mathbf{f}_{c\nu,A}$. Thus we have $w''x_i(h)w''^{-1} \in \hat{\mathbf{U}}_A^{>0}$. Since $w''x_i(h)w''^{-1} \in G_A$ and $G_A \cap \hat{\mathbf{U}}_A^{>0} = G_A^{>0}$, the result follows.

4.8. In the setup of 2.8, for any $\mathbf{h} = (h_1, h_2, \dots, h_n) \in A^n$ we set

$$\mathbf{x}_\mathbf{h} = x_{i_1}(h_1)(s''_{i_1}x_{i_2}(h_2)s''_{i_1}{}^{-1})\dots(s''_{i_1}s''_{i_2}\dots s''_{i_{n-1}}x_{i_n}(h_n)s''_{i_{n-1}}{}^{-1}\dots s''_{i_2}{}^{-1}s''_{i_1}{}^{-1}).$$

From 4.7(a) we see that $\mathbf{x}_\mathbf{h} \in G_A^{>0}$. We show:

(a) *The map $A^n \rightarrow G_A^{>0}, \mathbf{h} \mapsto \mathbf{x}_\mathbf{h}$ is a bijection.*

For \mathbf{h} as above we have (using 2.3(c)):

$$\begin{aligned} \mathbf{x}_\mathbf{h} &= \sum_{c_1, c_2, \dots, c_n \in \mathbf{N}, \lambda_1, \lambda_2, \dots, \lambda_n \in X} h_1^{c_1} h_2^{c_2} \dots h_n^{c_n} \\ &\times \theta_{i_1}^{(c_1)+}1_{\lambda_1} T''_{i_1,1}(\theta_{i_2}^{(c_2)+}1_{\lambda_2}) \dots T''_{i_1,1} T''_{i_2,1} \dots T''_{i_{n-1},1}(\theta_{i_n}^{(c_n)+}1_{\lambda_n}) \\ &= \sum_{\mathbf{c} \in \mathbf{N}^n, \lambda \in X} h_1^{c_1} h_2^{c_2} \dots h_n^{c_n} x_\mathbf{c} 1_\lambda. \end{aligned}$$

Hence for $\mathbf{c} \in \mathbf{N}^n$ we have

(b)
$$\mathbf{x}_\mathbf{h}(\xi_\mathbf{c}) = h_1^{c_1} h_2^{c_2} \dots h_n^{c_n}.$$

Now the A -algebra homomorphisms $A[t_1, t_2, \dots, t_n] \rightarrow A$ (preserving 1) are clearly in bijection with the points of A^n ; to $(h_1, h_2, \dots, h_n) \in A^n$ corresponds the algebra homomorphism which takes t_r to h_r for $r \in [1, n]$. Composing with κ in 3.13, we obtain a bijection between the set of A -algebra homomorphisms $\mathbf{o}_A \rightarrow A$ (preserving 1) and the set A^n . From 4.8(b) we see that under this bijection $\mathbf{x}_\mathbf{h} \in G_A^{>0}$ corresponds to $\mathbf{h} \in A^n$. This proves 4.8(a).

4.9. For any $\lambda \in X$ we define a homomorphism $\chi_\lambda : T_A \rightarrow A^\circ$ by $d \otimes y \mapsto d^{(y,\lambda)}$ for $d \in A^\circ$, $y \in Y$ or equivalently by $\sum_{\lambda \in X} n_\lambda \lambda \mapsto n_\lambda$. For $t \in T_A$, $h \in A$, $i \in I$, we show:

$$(a) \quad tx_i(h)t^{-1} = x_i(\chi_{i'}(t)h).$$

Writing $t = \sum_{\lambda \in X} n_\lambda \lambda$, we see that the left hand side of 4.9(a) is

$$\begin{aligned} & \sum_{c \in \mathbf{N}, \lambda, \lambda' \in X} n_\lambda^{-1} n_{\lambda'} h^c 1_{\lambda'} \theta_i^{(c)+} 1_\lambda = \sum_{c \in \mathbf{N}, \lambda, \lambda' \in X} n_\lambda^{-1} n_{\lambda'} h^c \theta_i^{(c)+} 1_{\lambda' - ci'} 1_\lambda \\ & = \sum_{c \in \mathbf{N}, \lambda \in X} n_\lambda^{-1} n_{\lambda + ci'} h^c \theta_i^{(c)+} 1_\lambda = \sum_{c \in \mathbf{N}, \lambda \in X} n_{\lambda'}^c h^c \theta_i^{(c)+} 1_\lambda = x_i(n_{i'} h) = x_i(\chi_{i'}(t)h). \end{aligned}$$

This proves 4.9(a).

In the setup of 2.8, let $k \in [1, n]$. Define a homomorphism $f_k : A \rightarrow G_A^{>0}$ by $h \mapsto s''_{i_1} s''_{i_2} \dots s''_{i_{k-1}} x_{i_k}(h) s''_{i_{k-1}}^{-1} \dots s''_{i_2}^{-1} s''_{i_1}^{-1}$. From 3.13 we see that f_k is an isomorphism of A onto its image $f_k(A)$. We set $\lambda_k = s_{i_1} s_{i_2} \dots s_{i_{k-1}}(i'_k) \in X$. For $t \in T_A$, $h \in A$ we show:

$$(b) \quad tf_k(h)t^{-1} = f_k(\chi_{\lambda_k}(t)h).$$

Let $t' = s_{i_{k-1}} \dots s_{i_2} s_{i_1}(t) \in T_A$. Using 4.3(a), we see that 4.9(b) is equivalent to $t' x_{i_k}(h) t'^{-1} = x_{i_k}(\chi_{\lambda_k}(t)h)$, that is, using 4.9(a), to $x_{i_k}(\chi_{i'_k}(t')h) = x_{i_k}(\chi_{\lambda_k}(t)h)$. It is enough to show that $\chi_{i'_k}(t') = \chi_{\lambda_k}(t)$. This is immediate from the definitions.

We show:

(c) Let $w \in W$, $i \in I$. There exists $u \in A^\circ$ such that for any $h \in A$ we have $\omega(w'' x_i(h) w''^{-1}) = w'' y_i(uh) w''^{-1}$.

Writing $w = s_{j_1} s_{j_2} \dots s_{j_r}$ with j_1, j_2, \dots, j_r in I and $r = l(w)$, we have

$$w'' = s'_{j_1} t_{j_1} (-1) s'_{j_2} t_{j_2} (-1) \dots s'_{j_r} t_{j_r} (-1) = w' t$$

for some $t \in T_A$. Hence

$$w'' x_i(h) w''^{-1} = w' t x_i(h) t^{-1} w'^{-1} = w' x_i(uh) w'^{-1}$$

where $u = \chi_{i'}(t) \in A^\circ$; see 4.9(a). Using $\omega(w'') = w'$, $\omega(y_i(uh)) = x_i(uh)$, we obtain $w'' x_i(h) w''^{-1} = \omega(w'' y_i(uh) w''^{-1})$ and 4.9(c) is proved.

4.10. We now assume that A is an algebraically closed field. Since, by 3.15, \mathbf{O}_A is a finitely generated A -algebra which is an integral domain, we see that G_A is a connected algebraic group over A with coordinate ring \mathbf{O}_A . Since, by 3.13, \mathfrak{o}_A is a finitely generated A -algebra which is an integral domain, we see that \mathfrak{G}_A (and hence $G_A^{>0}$ and $G_A^{<0}$) are connected algebraic groups over A with coordinate ring \mathfrak{o}_A . Since the inclusions of $G_A^{>0}$ and $G_A^{<0}$ into G_A correspond to surjective algebra homomorphisms $\mathbf{O}_A \rightarrow \mathfrak{o}_A$, we see that $G_A^{>0}$ and $G_A^{<0}$ are closed subgroups of G_A . Also T_A is clearly a subtorus of G_A .

Theorem 4.11. *In the setup of 4.10, G_A is a connected reductive group over A with associated root datum the same as the one given in 1.1.*

From 3.13 we see that as an affine variety $G_A^{>0}$ is isomorphic to the affine space A^n . It follows that $G_A^{>0}$ is a connected unipotent group. Similarly $G_A^{<0}$ is a connected unipotent group.

The T_A -action (conjugation) on the unipotent group $G_A^{>0}$ satisfies the hypotheses of [B1, 14.4, IV]: the induced T_A -action on $\text{Lie } G^{>0}$ is through one dimensional weight spaces corresponding to weights given by the distinct non-zero elements

λ_k ($k \in [1, n]$) of X (see 4.9) and $\lambda_k, \lambda_{k'}$ are linearly independent in X for $k \neq k'$. (This is a well known property of the W -action on X .) Using *loc. cit.*, we see that any non-trivial closed subgroup of $G_A^{>0}$ which is stable under conjugation by T contains the subgroup $f_k(A)$ (see 4.9) for some $k \in [1, n]$; moreover, the centralizer of T_A in $G_A^{>0}$ is $\{1\}$.

From 3.10 we see that the morphism $G_A^{<0} \times T_A \times G_A^{>0} \rightarrow G_A$ given by multiplication in G_A has dense image. Since T_A normalizes $G_A^{<0}$, we see that $G_A^{>0}T_A$ is a closed subgroup of G_A . Since $G_A^{<0}T_AG_A^{>0}$ is the orbit of 1 for an action of $(G_A^{<0}T) \times G_A^{>0}$ on G_A , we see that this orbit is locally closed in G_A . Since it is dense in G_A , we see that $G_A^{<0}T_AG_A^{>0}$ is open in G_A .

Since $G_A^{<0}T_A$ is solvable and connected, it is contained in some Borel subgroup B of G_A . Assume that $G_A^{<0}T_A$ is contained properly in B . Now $(G_A^{<0}T_AG_A^{>0}) \cap B$ is open in B . If ξ_1, ξ_2, ξ_3 are elements of $G_A^{>0}, T_A, G_A^{>0}$ such that $\xi_1\xi_2\xi_3 \in B$, then using $\xi_1\xi_2 \in B$, we see that $\xi_3 \in B$. Thus $(G_A^{<0}T_AG_A^{>0}) \cap B = (G_A^{<0}T_A)(G_A^{>0} \cap B)$. If $G_A^{>0} \cap B = \{1\}$, it would follow that $(G_A^{>0}T_AG_A^{>0}) \cap B = G_A^{<0}T_A$ which is a proper closed subset of B ; this contradicts the fact that $(G_A^{>0}T_AG_A^{>0}) \cap B$ is open (non-empty) in B . Thus we have $G_A^{>0} \cap B \neq \{1\}$. Since $G_A^{>0} \cap B$ is a unipotent subgroup of B , it is equal to $G_A^{>0} \cap B_u$ where B_u is the unipotent radical of B . Since $G_A^{>0} \cap B_u$ is a non-trivial closed subgroup of $G_A^{>0}$, stable under conjugation by T_A , we see that we have $f_k(A) \subset G_A^{>0} \cap B_u$ for some $k \in [1, n]$. Since $f_k(A) \subset G_A^{>0}$, we have $\omega(f_k(A)) \subset G_A^{<0}$. Since $G_A^{<0}$ is a unipotent subgroup of B , we have $G_A^{<0} \subset B_u$, hence $\omega(f_k(A)) \subset B_u$. Thus there exists $w \in W$ and $i \in I$ such that $w''x_i(A)w'' \subset B_u$ and $\omega(w''x_i(A)w'') \subset B_u$. Hence, using 4.9(c), we have $w''y_i(A)w''^{-1}$. Thus $x_i(A)$ and $y_i(A)$ are contained in the unipotent group $w''^{-1}B_uw''$. Using 4.6, we deduce that for any $u \in A^\circ$ we have $t_i(u) \in w''^{-1}B_uw''$. Since $t_i(u) \in T_A$ is semisimple, it follows that $t_i(u) = 1$ for all $u \in A - \{0\}$. From the definition of $t_i(u)$ we see that if we choose $u \in A - \{0, 1\}$ (recall that A is algebraically closed), then $t_i(u) \neq 1$. We have a contradiction. We see that $B = G_A^{<0}T_A$ so that $G_A^{<0}T_A$ is a Borel subgroup of G_A . Applying ω , we see that $\omega(G_A^{<0}T_A) = G_A^{>0}T_A$ is a Borel subgroup of G_A . From 4.2(a) we see that $(G_A^{<0}T_A) \cap (G_A^{>0}T_A) = T_A$. Since the unipotent radical of G_A is contained in both Borel subgroups $G_A^{<0}T_A, G_A^{>0}T_A$, it must also be contained in their intersection, the torus T_A . Thus, the unipotent radical of G_A is $\{1\}$. We see that G_A is reductive. Since T_A is the intersection of two Borel subgroups of G_A , it must be a maximal torus of G_A . From the definitions we see that the weights of the adjoint action of T_A on the Lie algebra of G_A are exactly λ_k ($k \in [1, n]$), their negatives (each with multiplicity 1) and the 0 weight with multiplicity $\dim T_A$. It follows that the root datum associated to G_A is exactly the root datum given in 1.1. The theorem is proved.

5. FROM ENVELOPING ALGEBRAS TO MODIFIED ENVELOPING ALGEBRAS

5.1. In this subsection we recall some definitions from [Ko].

Let $\mathbf{U}_{\mathbf{Q}}$ be the \mathbf{Q} -algebra with 1 generated by the symbols x^+, x^- with $x \in \mathbf{f}_{\mathbf{Q}}$ and \underline{y} with $y \in Y$; these symbols are subject to the following relations:

$$\begin{aligned} \mathbf{f}_{\mathbf{Q}} &\rightarrow \mathbf{U}_{\mathbf{Q}}, x \mapsto x^{\pm} \text{ is a } \mathbf{Q}\text{-algebra homomorphism preserving 1;} \\ Y &\rightarrow \mathbf{U}_{\mathbf{Q}}, y \mapsto \underline{y} \text{ is } \mathbf{Z}\text{-linear;} \\ \underline{yy'} &= \underline{y'y} \text{ for } y, y' \in Y; \end{aligned}$$

$$\begin{aligned} \underline{y}\theta_i^\pm &= \theta_i^\pm(\underline{y} \pm \langle y, i \rangle) \text{ for } y \in Y, i \in I; \\ \theta_i^+\theta_j^- - \theta_j^-\theta_i^+ &= \delta_{i,j}\underline{i} \text{ for } i, j \in I. \end{aligned}$$

Thus $\mathbf{U}_{\mathbf{Q}}$ is the universal enveloping algebra attached to our root datum. There is a unique algebra homomorphism $\Delta : \mathbf{U}_{\mathbf{Q}} \rightarrow \mathbf{U}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{U}_{\mathbf{Q}}$ such that $\Delta(\underline{y}) = \underline{y} \otimes 1 + 1 \otimes \underline{y}$ for $y \in Y$, $\Delta(\theta_i^\pm) = \theta_i^\pm \otimes 1 + 1 \otimes \theta_i^\pm$ for $i \in I$. There is a unique algebra homomorphism S from $\mathbf{U}_{\mathbf{Q}}$ to $\mathbf{U}_{\mathbf{Q}}$ with the opposed multiplication such that $S(\underline{y}) = -\underline{y}$ for $y \in Y$, $S(\theta_i^\pm) = -\theta_i^\pm$ for $i \in I$. There is a unique algebra homomorphism $\epsilon : \mathbf{U}_{\mathbf{Q}} \rightarrow \mathbf{Q}$ preserving 1 such that $\epsilon(\underline{y}) = 0$ for $y \in Y$, $\epsilon(\theta_i^\pm) = 0$ for $i \in I$. Then $(\mathbf{U}_{\mathbf{Q}}, \Delta, S)$ is a Hopf algebra over \mathbf{Q} with 1 and with counit ϵ .

Let $\mathcal{C}'_{\mathbf{Q}}$ be the category whose objects are unital $\mathbf{U}_{\mathbf{Q}}$ -modules M such that $M = \bigoplus_{\lambda \in X} M^\lambda$ (as a vector space) where for any $\lambda \in X$ we have $M^\lambda = \{z \in M; yz = \langle y, \lambda \rangle z \text{ for any } y \in Y\}$.

For any $M \in \mathcal{C}'_{\mathbf{Q}}$ let C'_M be the \mathbf{Q} -subspace of $\mathbf{U}_{\mathbf{Q}}^\circ$ spanned by the functions $u \mapsto z'(uz)$ for various $z \in M, z' \in M^\circ$. Let $\mathbf{O}'_{\mathbf{Q}} = \sum_{M \in \mathcal{C}'_{\mathbf{Q}}} C'_M$, a \mathbf{Q} -subspace of $\mathbf{U}_{\mathbf{Q}}^\circ$. Now $\mathbf{O}'_{\mathbf{Q}}$ is a Hopf algebra over \mathbf{Q} in which the multiplication is the “transpose” of $\Delta : \mathbf{U}_{\mathbf{Q}} \rightarrow \mathbf{U}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{U}_{\mathbf{Q}}$, the comultiplication is the “transpose” of the multiplication $\mathbf{U}_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{U}_{\mathbf{Q}} \rightarrow \mathbf{U}_{\mathbf{Q}}$, the antipode is the “transpose” of $S : \mathbf{U}_{\mathbf{Q}} \rightarrow \mathbf{U}_{\mathbf{Q}}$, the unit is the counit of $\mathbf{U}_{\mathbf{Q}}$ and the counit is given by $f \mapsto f(1)$.

Let $\mathbf{U}_{\mathbf{Z}}$ be the subring of $\mathbf{U}_{\mathbf{Q}}$ generated by the elements

$$\binom{\underline{y}}{k} = \underline{y}(\underline{y} - 1) \dots (\underline{y} - k + 1)/k!$$

for $y \in Y, k \in \mathbf{N}$ and $\theta_i^{(m)+}, \theta_i^{(m)-}$ for $i \in I, m \in \mathbf{N}$. This is a \mathbf{Z} -lattice in $\mathbf{U}_{\mathbf{Q}}$, called the Kostant \mathbf{Z} -form of $\mathbf{U}_{\mathbf{Q}}$. Let $\mathbf{O}'_{\mathbf{Z}} = \{f \in \mathbf{O}'_{\mathbf{Q}}; f(\mathbf{U}_{\mathbf{Z}}) \subset \mathbf{Z}\}$. This is a \mathbf{Z} -subalgebra with 1 of $\mathbf{O}'_{\mathbf{Q}}$; it inherits from $\mathbf{O}'_{\mathbf{Q}}$ a comultiplication, an antipode and a counit, which make it into a Hopf algebra over \mathbf{Z} (here we use the fact that $\mathbf{O}'_{\mathbf{Z}}$ is a torsion free \mathbf{Z} -module).

If $v = 1$ in A , we set $\mathbf{O}'_A = A \otimes \mathbf{O}'_{\mathbf{Z}}$; this is naturally a Hopf algebra over A . The following results show that our results about \mathbf{O}'_A apply also to \mathbf{O}'_A .

Proposition 5.2. *If $v = 1$ in A , we have canonically $\mathbf{O}'_A = \mathbf{O}_A$ as Hopf algebras over A .*

The proof is given in 5.9.

5.3. Let \mathbf{P} be the \mathbf{Q} -algebra of functions $X \rightarrow \mathbf{Q}$ generated by the functions $\underline{y} : \lambda \mapsto \langle y, \lambda \rangle$ for various $y \in Y$. We regard \mathbf{P} as a \mathbf{Q} -subalgebra with 1 of $\hat{\mathbf{U}}_{\mathbf{Q}}^0$ by $\phi \mapsto \sum_{\lambda \in X} \phi(\lambda)1_\lambda$. For $y \in Y, h \in \mathbf{Z}$ and $k \in \mathbf{N}$ we set

$$\binom{\underline{y} + h}{k} = (\underline{y} + h)(\underline{y} + h - 1) \dots (\underline{y} + h - k + 1)/k! \in \mathbf{P}.$$

Let $\mathbf{P}_{\mathbf{Z}}$ be the subring of \mathbf{P} generated by the elements $\binom{\underline{y}}{k}$ for various $y \in Y, k \in \mathbf{N}$. Note that

- (a) $\mathbf{P}_{\mathbf{Z}} \subset \hat{\mathbf{U}}_{\mathbf{Z}}^0$,
- (b) $\binom{\underline{y} + h}{k} \in \mathbf{P}_{\mathbf{Z}}$ for any $y \in Y, h \in \mathbf{Z}, k \in \mathbf{N}$.

Let $\{y_1, y_2, \dots, y_r\}$ be a basis of the \mathbf{Z} -module Y . The following result is well known.

(c) The elements $\phi_{\mathbf{k}} = \binom{y_1}{k_1} \binom{y_2}{k_2} \dots \binom{y_r}{k_r}$ (with $\mathbf{k} = (k_1, k_2, \dots, k_r) \in \mathbf{N}^r$) form a \mathbf{Z} -basis of the \mathbf{Z} -module $\mathbf{P}_{\mathbf{Z}}$ and a \mathbf{Q} -basis of the \mathbf{Q} -vector space \mathbf{P} .

We show:

(d) Let $N \geq 1$. Let $X_N = \{\lambda \in X; -N \leq \langle y_j, \lambda \rangle \leq N \text{ for } j = 1, 2, \dots, r\}$. Let \mathcal{M}_N be the set of all elements $\sum_{\lambda' \in X} n_{\lambda'} 1_{\lambda'} \in \hat{\mathbf{U}}_{\mathbf{Z}}^0$ such that $n_{\lambda'} = 0$ for $\lambda' \in X_N$. Let $\lambda \in X$. Then $1_{\lambda} \in \mathbf{P}_{\mathbf{Z}} + \mathcal{M}_N$.

If $\lambda \notin X_N$, then $1_{\lambda} \in \mathcal{M}_N$ and the result holds. Now assume that $\lambda \in X_N$. We have

$$(e) \quad \prod_{j \in [1, r]} \binom{y_j + 2N - \langle y_j, \lambda \rangle}{2N} = \sum_{\substack{\lambda' \in X; \langle y_j, \lambda' \rangle \in (\infty, -\langle y_j, \lambda \rangle - 2N - 1] \cup [\langle y_j, \lambda \rangle, \infty) \\ \text{for } j \in [1, r]}} n_{\lambda'} 1_{\lambda'}$$

where $n_{\lambda'} \in \mathbf{Z}$ are such that $n_{\lambda} = 1$. For λ' in the last sum such that $\langle y_j, \lambda' \rangle \in (\infty, -\langle y_j, \lambda \rangle - 2N - 1]$ for some j we have $\langle y_j, \lambda' \rangle \leq -N - 1$ (since $-N \leq \langle y_j, \lambda \rangle$), hence $\lambda' \notin X_N$; the sum over all such λ' is in \mathcal{M}_N . On the other hand, the left hand side of 5.3(e) is in $\mathbf{P}_{\mathbf{Z}}$. Thus 5.3(e) implies

$$\sum_{\lambda' \in X_N; \langle y_j, \lambda' \rangle \geq \langle y_j, \lambda \rangle \text{ for } j \in [1, r]} n_{\lambda'} 1_{\lambda'} \in \mathbf{P}_{\mathbf{Z}} + \mathcal{M}_N$$

where $n_{\lambda'} \in \mathbf{Z}$ are such that $n_{\lambda} = 1$. For any $\lambda' \in X_N$ we set $q_{\lambda'} = \sum_{j \in [1, r]} \langle y_j, \lambda' \rangle$ so that $-Nr \leq q_{\lambda'} \leq Nr$. We have

$$1_{\lambda} + \sum_{\lambda' \in X_N; \langle y_j, \lambda' \rangle \geq \langle y_j, \lambda \rangle \text{ for } j \in [1, r], q_{\lambda'} > q_{\lambda}} n_{\lambda'} 1_{\lambda'} \in \mathbf{P}_{\mathbf{Z}} + \mathcal{M}_N.$$

This shows by descending induction on q_{λ} (starting with $q_{\lambda} = Nr$) that $1_{\lambda} \in \mathbf{P}_{\mathbf{Z}} + \mathcal{M}_N$ for any $\lambda \in X_N$. This proves 5.3(d).

5.4. There is a unique algebra homomorphism $\mathbf{U}_{\mathbf{Q}} \rightarrow \hat{\mathbf{U}}_{\mathbf{Q}}$ preserving 1 such that $\underline{y} \mapsto \sum_{\lambda \in X} \langle y, \lambda \rangle 1_{\lambda}$ for any $y \in Y$, $\theta_i^{\pm} \mapsto \theta_i^{\pm}$ (as in 2.7) for any $i \in I$. This homomorphism is injective and we identify $\mathbf{U}_{\mathbf{Q}}$ with a subalgebra of $\hat{\mathbf{U}}_{\mathbf{Q}}$ via this homomorphism. We identify \mathbf{P} with a subalgebra of $\mathbf{U}_{\mathbf{Q}}$ by $\underline{y} \mapsto y$. This makes \mathbf{P} a subalgebra of $\hat{\mathbf{U}}_{\mathbf{Q}}^0$ (in the same way as in 5.3). Note that:

(a) The elements $b^+ \phi_{\mathbf{k}} b'^-$ with $b, b' \in \mathbf{B}$ and $\mathbf{k} \in \mathbf{N}^r$ form a \mathbf{Z} -basis of $\mathbf{U}_{\mathbf{Z}}$ and a \mathbf{Q} -basis of $\mathbf{U}_{\mathbf{Q}}$.

We show that

(b) $\mathbf{U}_{\mathbf{Z}} \subset \hat{\mathbf{U}}_{\mathbf{Z}}$.

By 5.4(a) it is enough to show that for $b \in \mathbf{B}$ we have $b^+ \in \hat{\mathbf{U}}_{\mathbf{Z}}$ (this is clear), $b^- \in \hat{\mathbf{U}}_{\mathbf{Z}}$ (this is clear) and that for $\mathbf{k} \in \mathbf{N}$ we have $\phi_{\mathbf{k}} \in \hat{\mathbf{U}}_{\mathbf{Z}}^0$ (this follows from 5.3(a)).

The category $\mathcal{C}_{\mathbf{Q}}$ (see 1.6) is equivalent to the category $\mathcal{C}'_{\mathbf{Q}}$: if $M \in \mathcal{C}_{\mathbf{Q}}$, then M can be regarded as a $\hat{\mathbf{U}}_{\mathbf{Q}}$ -module as in 1.12 and then as a $\mathbf{U}_{\mathbf{Q}}$ -module via the imbedding $\mathbf{U}_{\mathbf{Q}} \subset \hat{\mathbf{U}}_{\mathbf{Q}}$. This $\mathbf{U}_{\mathbf{Q}}$ -module structure makes M into an object of $\mathcal{C}'_{\mathbf{Q}}$. This gives a functor $\mathcal{C}_{\mathbf{Q}} \rightarrow \mathcal{C}'_{\mathbf{Q}}$ which is easily seen to be an equivalence of categories. It is well known that

(c) Any object in $\mathcal{C}_{\mathbf{Q}}$ is semisimple and the simple objects of $\mathcal{C}_{\mathbf{Q}}$ are exactly the objects $\Lambda_{\lambda, \mathbf{Q}}$ for various $\lambda \in X^+$.

For any $M \in \mathcal{C}_{\mathbf{Q}}$ let C_M be the \mathbf{Q} -subspace of $\hat{\mathbf{U}}_{\mathbf{Q}}^{\circ}$ spanned by the functions $u \mapsto z'(uz)$ for various $z \in M, z' \in M^{\circ}$.

Lemma 5.5. *Let $f \in \dot{\mathbf{U}}_{\mathbf{Q}}^{\circ}$. The following three properties are equivalent:*

- (i) $f \in \mathbf{O}_{\mathbf{Q}}$;
- (ii) $f \in \sum_{M \in \mathfrak{C}_{\mathbf{Q}}} C_M$.
- (iii) $f \in \sum_{\lambda \in X^+} C_{\Lambda_{\lambda, \mathbf{Q}}}$.

Assume that (i) holds. To show (ii), we can assume that $f = a^*$ where $a \in \dot{\mathbf{B}}_{\lambda, \lambda'}$ for some λ, λ' in X^+ . Let $M = {}^{\omega}\Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda', \mathbf{Q}}$. Let $z = \xi_{\lambda} \otimes \eta_{\lambda'} \in M$. Define $z' \in M^{\circ}$ by $z'(a'z) = \delta_{a, a'}$ for any $a' \in \dot{\mathbf{B}}_{\lambda, \lambda'}$. For $a'' \in \dot{\mathbf{B}}$ we have $z'(a''z) = a^*(a'')$ (if $a'' \notin \dot{\mathbf{B}}_{\lambda, \lambda'}$, both sides are 0 since $a''z = 0$; if $a'' \in \dot{\mathbf{B}}_{\lambda, \lambda'}$, both sides are $\delta_{a, a''}$). Hence $z'(uz) = a^*(u)$ for all $u \in \dot{\mathbf{U}}_{\mathbf{Q}}$. Hence $a^* \in C_M$, as required.

Assume that (iii) holds. To show (i), we may assume that there exist $z \in M$, $z' \in M^{\circ}$ with $M = \Lambda_{\lambda, \mathbf{Q}}$, $\lambda \in X^+$ such that $f(u) = z'(uz)$ for all $u \in \dot{\mathbf{U}}_{\mathbf{Q}}$. Recall that, if $b \in \dot{\mathbf{B}} - \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']$, then b acts as 0 on M and hence $f(b) = 0$. Let $f_1 = f - \sum_{b \in \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']} f(b)b^* \in \dot{\mathbf{U}}_{\mathbf{Q}}^{\circ}$. Note that $f_1(b) = 0$ if $b \in \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']$. If $b' \notin \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']$, then both f and $\sum_{b \in \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']} f(b)b^*$ vanish on b' . Hence $f_1(b') = 0$. We see that $f_1 = 0$ and $f = \sum_{b \in \bigcup_{\lambda'; \lambda \geq \lambda'} \dot{\mathbf{B}}[\lambda']} f(b)b^*$. Thus $f \in \mathbf{O}_{\mathbf{Q}}$, as required.

Assume that (ii) holds. Then (iii) holds since $C_{M' \oplus M''} = C_{M'} + C_{M''}$ for any M', M'' in $\mathfrak{C}_{\mathbf{Q}}$ and any $M \in \mathfrak{C}_{\mathbf{Q}}$ is a direct sum of finitely many objects of the form $\Lambda_{\lambda, \mathbf{Q}}$, $\lambda \in X^+$. The lemma is proved.

5.6. As in the proof of Lemma 5.5 we see that for $f \in \mathbf{U}_{\mathbf{Q}}^{\circ}$ the following two properties are equivalent:

- (i) $f \in \sum_{M \in \mathfrak{C}'_{\mathbf{Q}}} C'_M$.
 - (ii) $f \in \sum_{\lambda \in X^+} C'_{\Lambda_{\lambda, \mathbf{Q}}}$.
- (Recall that $\mathfrak{C}_{\mathbf{Q}} = \mathfrak{C}'_{\mathbf{Q}}$.)

5.7. Let $\mathcal{O} = \bigoplus_{\lambda \in X^+} \Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda, \mathbf{Q}}^{\circ}$, a \mathbf{Q} -vector space. We show:

(a) *The linear map $\alpha : \mathcal{O} \rightarrow \dot{\mathbf{U}}_{\mathbf{Q}}^{\circ}$ whose restriction to $\Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda, \mathbf{Q}}^{\circ}$ is $z \otimes z' \mapsto [u \mapsto z'(uz)]$ is injective.*

For any finite subset F of X^+ let $\mathcal{O}_F = \bigoplus_{\lambda \in F} \Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda, \mathbf{Q}}^{\circ}$. It is enough to verify that for any F as above, the linear map $\mathcal{O}_F \rightarrow \dot{\mathbf{U}}^{\circ}$ (restriction of $\mathcal{O} \rightarrow \dot{\mathbf{U}}^{\circ}$ in 5.7(a)) is injective.

Let $(\xi_{\lambda})_{\lambda \in F} \in \ker(\mathcal{O}_F \rightarrow \dot{\mathbf{U}}^{\circ})$. For $\lambda \in F$ we have $\xi_{\lambda} = \sum_j z_j^{\lambda} \otimes z_j'^{\lambda}$ with $z_j^{\lambda} \in \Lambda_{\lambda, \mathbf{Q}}$, $z_j'^{\lambda} \in \Lambda_{\lambda, \mathbf{Q}}^{\circ}$. For any $u \in \dot{\mathbf{U}}_{\mathbf{Q}}$ we have $\sum_{\lambda \in F} \sum_j z_j'^{\lambda} (uz_j^{\lambda}) = 0$. Using 5.4(c), we see that for any collection $(e_{\lambda})_{\lambda \in F}$, $e_{\lambda} \in \text{End}(\Lambda_{\lambda, \mathbf{Q}})$, we can find $u \in \dot{\mathbf{U}}_{\mathbf{Q}}$ such that u acts on $\Lambda_{\lambda, \mathbf{Q}}$ as e_{λ} . Hence we have $\sum_{\lambda \in F} \sum_j z_j'^{\lambda} (e_{\lambda} z_j^{\lambda}) = 0$ for any $(e_{\lambda})_{\lambda \in F}$ as above. Hence for any $\lambda \in F$ we have $\sum_j z_j'^{\lambda} (e z_j^{\lambda}) = 0$ for any $e \in \text{End}(\Lambda_{\lambda, \mathbf{Q}})$. Hence $\sum_j z_j^{\lambda} \otimes z_j'^{\lambda} = 0$ for any $\lambda \in F$. Thus $\xi_{\lambda} = 0$ for any $\lambda \in F$. This proves 5.7(a).

An entirely similar proof gives the following result.

(b) *The linear map $\alpha' : \mathcal{O} \rightarrow \mathbf{U}_{\mathbf{Q}}^{\circ}$ whose restriction to $\Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda, \mathbf{Q}}^{\circ}$ is $z \otimes z' \mapsto [u \mapsto z'(uz)]$ is injective.*

From 5.5, 5.6, we see that the image of α is $\mathbf{O}_{\mathbf{Q}}$ and the image of α' is $\mathbf{O}'_{\mathbf{Q}}$. Using 5.7(a), (b), we see that α, α' define isomorphisms $\mathcal{O} \xrightarrow{\sim} \mathbf{O}_{\mathbf{Q}}$, $\mathcal{O} \xrightarrow{\sim} \mathbf{O}'_{\mathbf{Q}}$.

Now let $\hat{\alpha} : \mathcal{O} \rightarrow \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ}$ be the linear map whose restriction to $\Lambda_{\lambda, \mathbf{Q}} \otimes_{\mathbf{Q}} \Lambda_{\lambda, \mathbf{Q}}^{\circ}$ is $z \otimes z' \mapsto [u \mapsto z'(uz)]$. Let $\gamma : \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ} \rightarrow \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ}$, $\gamma' : \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ} \rightarrow \mathbf{U}_{\mathbf{Q}}^{\circ}$ be the linear maps transpose to the imbeddings $\hat{\mathbf{U}}_{\mathbf{Q}} \rightarrow \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ}$, $\mathbf{U}_{\mathbf{Q}} \rightarrow \hat{\mathbf{U}}_{\mathbf{Q}}^{\circ}$. From the definitions we have $\gamma\hat{\alpha} = \alpha$, $\gamma'\hat{\alpha} = \alpha'$. Since α is injective, we see that $\hat{\alpha}$ is injective. Hence $\hat{\alpha}$ defines an isomorphism $\mathcal{O} \rightarrow \hat{\mathbf{O}}_{\mathbf{Q}}$ where $\hat{\mathbf{O}}_{\mathbf{Q}}$ is the image of $\hat{\alpha}$. We see that the image of $\mathbf{O}_{\mathbf{Q}}$ under γ is equal to the image of $\mathbf{O}'_{\mathbf{Q}}$ under γ' and both these images are equal to $\hat{\mathbf{O}}_{\mathbf{Q}}$. Now γ, γ' restrict to isomorphisms $\mathbf{O}_{\mathbf{Q}} \xrightarrow{\sim} \hat{\mathbf{O}}_{\mathbf{Q}}$, $\mathbf{O}'_{\mathbf{Q}} \xrightarrow{\sim} \hat{\mathbf{O}}_{\mathbf{Q}}$. Thus both $\mathbf{O}_{\mathbf{Q}}, \mathbf{O}'_{\mathbf{Q}}$ can be canonically identified with $\hat{\mathbf{O}}_{\mathbf{Q}}$ and hence also with each other. From the definitions we see that this identification of $\mathbf{O}_{\mathbf{Q}}, \mathbf{O}'_{\mathbf{Q}}$ is compatible with the Hopf algebra structures.

Lemma 5.8. *Let F be a finite subset of X^+ .*

(i) *Let $u \in \dot{\mathbf{U}}_{\mathbf{Z}}$. There exists $u' \in \mathbf{U}_{\mathbf{Z}}$ such that u, u' act in the same way on $\Lambda_{\lambda, \mathbf{Q}}$ for any $\lambda \in F$.*

(ii) *Let $u' \in \mathbf{U}_{\mathbf{Z}}$. There exists $u \in \dot{\mathbf{U}}_{\mathbf{Z}}$ such that u, u' act in the same way on $\Lambda_{\lambda, \mathbf{Q}}$ for any $\lambda \in F$.*

We can find $N \geq 1$ such that $\{\lambda \in X; 1_{\lambda}\Lambda_{\lambda, \mathbf{Q}} \neq 0\} \subset X_N$ (notation of 5.3(d)).

To prove Lemma 5.8(i), we may assume that $u = b^+ 1_{\lambda} b'^- \in \dot{\mathbf{U}}_{\mathbf{Z}}$ where $b, b' \in \mathbf{B}$, $\lambda \in X$. By 5.3(d) we can find $u_1 \in \mathbf{P}_{\mathbf{Z}}$, $u_2 \in \hat{\mathbf{U}}_{\mathbf{Z}}^0$ such that $1_{\lambda} = u_1 + u_2$ and u_2 acts as 0 on $\Lambda_{\lambda, \mathbf{Q}}$ for any $\lambda \in F$. Then $b^+ u_2 b'^- \in \hat{\mathbf{U}}_{\mathbf{Z}}$ acts as 0 on $\Lambda_{\lambda, \mathbf{Q}}$ for any $\lambda \in F$. Hence $u = b^+ u_1 b'^- + b^+ u_2 b'^-$ acts in the same way as $b^+ u_1 b'^-$ on $\Lambda_{\lambda, \mathbf{Q}}$ for any $\lambda \in F$. Hence $u' = b^+ u_1 b'^- \in \mathbf{U}_{\mathbf{Z}}$ is as required in Lemma 5.8(i).

We prove Lemma 5.8(ii). We set $u = u' \sum_{\lambda \in X_N} 1_{\lambda} \in \dot{\mathbf{U}}_{\mathbf{Q}}$. Since $u' \in \hat{\mathbf{U}}_{\mathbf{Z}}$ (see 5.4(b)), we see that $u \in \hat{\mathbf{U}}_{\mathbf{Z}}$. Thus, $u \in \dot{\mathbf{U}}_{\mathbf{Q}} \cap \hat{\mathbf{U}}_{\mathbf{Z}} = \dot{\mathbf{U}}_{\mathbf{Z}}$. Clearly, u is as required by Lemma 5.8(ii). The lemma is proved.

5.9. We prove Proposition 5.2. We can assume that $A = \mathbf{Z}$. Let Z (resp. Z') be the set of all $(\xi_{\lambda})_{\lambda \in X^+} \in \mathcal{O}$, $\xi_{\lambda} = \sum_j z_j^{\lambda} \otimes z_j'^{\lambda}$ with $z_j^{\lambda} \in \Lambda_{\lambda, \mathbf{Q}}$, $z_j'^{\lambda} \in \Lambda_{\lambda, \mathbf{Q}}^{\circ}$, such that for any $u \in \dot{\mathbf{U}}_{\mathbf{Z}}$ (resp. any $u \in \mathbf{U}_{\mathbf{Z}}$) we have $\sum_{\lambda \in X^+} \sum_j z_j'^{\lambda} (uz_j^{\lambda}) \in \mathbf{Z}$.

Under the identification $\mathcal{O} = \mathbf{O}_{\mathbf{Q}}$ (resp. $\mathcal{O} = \mathbf{O}'_{\mathbf{Q}}$) induced by α (resp. α') in 5.7, the subset $\mathbf{O}_{\mathbf{Z}}$ (resp. $\mathbf{O}'_{\mathbf{Z}}$) corresponds to the subset Z (resp. Z') of \mathcal{O} .

Since any element $(\xi_{\lambda})_{\lambda \in X^+} \in \mathcal{O}$ has only finitely many non-zero components, we see, using Lemma 5.8(i), that $Z' \subset Z$ and, using Lemma 5.8(ii), that $Z \subset Z'$. Thus $Z = Z'$. It follows that under the identification $\mathbf{O}_{\mathbf{Q}} = \mathbf{O}'_{\mathbf{Q}}$, $\mathbf{O}_{\mathbf{Z}}$ corresponds to $\mathbf{O}'_{\mathbf{Z}}$. Proposition 5.2 follows.

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