JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 22, Number 3, July 2009, Pages 691–737
S 0894-0347(08)00625-5
Article electronically published on October 2, 2008

THE REDUCED GENUS 1 GROMOV-WITTEN INVARIANTS OF CALABI-YAU HYPERSURFACES

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Contents

0. Introduction

0.1. **Mirror symmetry predictions for a quintic threefold.** Gromov-Witten invariants of a smooth projective variety X are certain counts of curves in X. In many cases, these invariants are known or conjectured to possess rather amazing structure which is often completely unexpected from the classical point of view. For example, a generating function for the genus 0 GW-invariants solves a third-oder PDE in two variables. In the case of the complex projective space $\mathbb{P}^n$, the resulting

Received by the editors July 23, 2007.
2000 *Mathematics Subject Classification.* Primary 14N35, 53D45.
Key words and phrases. Gromov-Witten invariants, genus 1, BCOV prediction.
The author was partially supported by a Sloan Fellowship and DMS Grant 0604874.

PDE condition on the generating function is equivalent to a recursion for counts of rational curves in $\mathbb{P}^n$ and solves the classical problem of enumeration of such curves in $\mathbb{P}^n$; see [RT, Section 10] and [KoM, Section 5].

The above-mentioned PDE property[1] is just one type of structure of GW-invariants motivated by their relation to string theory. The mirror symmetry principle of string theory predicts yet another type of structure whenever X is a Calabi-Yau threefold. It relates GW-invariants of X to an integral on the moduli space of Kahler structures of the "mirror" of X. In the case X is a quintic threefold (degree 5 hypersurface in $\mathbb{P}^4$), this integral was computed in [CaDGP], leading to stunning predictions concerning the Gromov-Witten theory of X. These predictions have been only partially verified.

For each pair of non-negative integers (g,d), let $N_{g,d}$ denote the genus g degree d GW-invariant of a quintic threefold X_5; see equation (0.8) below. For $q=0,1,\ldots$, we define a degree q polynomial $I_q(t)$ in t with coefficients in the power series in e^t by

$$\sum_{q=0}^{\infty} I_q(t)w^q \equiv e^{wt}\sum_{d=0}^{\infty} e^{dt}\frac{\prod_{r=1}^{r=5d}(5w+r)}{\prod_{r=1}^{r=d}((w+r)^5-w^5)}. \tag{0.1}$$

For example,

$$I_0(t) = 1+\sum_{d=1}^{\infty} e^{dt}\frac{(5d)!}{(d!)^5}, \quad I_1(t) = tI_0(t)+\sum_{d=1}^{\infty} e^{dt}\left(\frac{(5d)!}{(d!)^5}\sum_{r=d+1}^{5d}\frac{5}{r}\right). \tag{0.2}$$

Let

$$J_q(t) = I_q(t)/I_0(t) \quad \forall\, q=1,2,\ldots, \qquad T = J_1(t). \tag{0.3}$$

The mirror symmetry prediction of [CaDGP] for the genus 0 GW-invariants of X_5 can be stated as

$$\frac{5}{6}T^3+\sum_{d=1}^{\infty} N_{0,d}e^{dT} = \frac{5}{2}\big(J_1(t)J_2(t)-J_3(t)\big); \tag{0.4}$$

see Appendix B for a comparison of statements of mirror symmetry. A prediction for the genus 1 GW-invariants of X_5 was made in [BCOV], building up on [CaDGP]. Both of these predictions date back to the early days of the Gromov-Witten theory. More recently, predictions for higher-genus GW-invariants of X_5 have been made; the approach of [HKlQ] generates mirror formulas for GW-invariants of X_5 up to genus 51.

While the ODE condition on GW-invariants mentioned above is proved directly, the mathematical approach to the mirror principle has been to compute the relevant GW-invariants in each specific case. However, this is rarely a simple task. The prediction for genus 0 invariants was confirmed mathematically in the mid-1990s. The prediction for genus 1 invariants is verified in this paper.

Theorem 1. *If $N_{1,d}$ denotes the degree d genus 1 Gromov-Witten invariant of a quintic threefold,*

$$2\sum_{d=1}^{\infty} N_{1,d}e^{dT} = \frac{25}{6}\big(J_1(t)-t\big)+\ln\left(I_0(t)^{-62/3}\big(1-5^5e^t\big)^{-1/6}J_1'(t)^{-1}\right). \tag{0.5}$$

[1] It is equivalent to the associativity of the multiplication in quantum cohomology.

This theorem is deduced from Theorem 2 in Subsection 0.3. An outline of this paper is contained in the next subsection.

0.2. **Computing GW-invariants of hypersurfaces.** One approach to computing GW-invariants of a projective hypersurface (and more generally, of a complete intersection) is to relate them to GW-invariants of the ambient projective space as follows. Whenever g, d, and k are non-negative integers and X is a smooth subvariety of $\mathbb{P}^n$, denote by $\overline{\mathfrak{M}}_{g,k}(X,d)$ the moduli space of stable degree d maps into X from genus g curves with k marked points; see [MirSym, Chapter 24]. Let $\mathfrak{U}$ be the universal curve over $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$, with structure map π and evaluation map ev:

$$\begin{array}{ccc} \mathfrak{U} & \xrightarrow{\mathrm{ev}} & \mathbb{P}^n \\ \big\downarrow{\scriptstyle \pi} & & \\ \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d). & & \end{array}$$

In other words, the fiber of π over $[\mathcal{C},f]\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$, where $\mathcal{C}$ is a nodal curve with k marked points and $f\colon\mathcal{C}\longrightarrow\mathbb{P}^n$ is a stable morphism, is $\mathcal{C}/\mathrm{Aut}(\mathcal{C},f)$, while

$$\mathrm{ev}\big([\mathcal{C},f;z]\big)=f(z) \qquad \text{if} \quad z\in\mathcal{C}.$$

A smooth degree a hypersurface X in $\mathbb{P}^n$ is determined by a section s of $\mathcal{O}_{\mathbb{P}^n}(a)$ which is transverse to the zero set:

$$X=s^{-1}(0) \qquad \text{for some } s\in H^0\big(\mathbb{P}^n;\mathcal{O}_{\mathbb{P}^n}(a)\big).^2$$

The section s induces a section $\tilde{s}$ of the sheaf $\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)\longrightarrow\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$ by

$$\tilde{s}\big([\mathcal{C},f]\big)=[s\circ f].$$

It is immediate that

$$\overline{\mathfrak{M}}_{g,k}(X,d)\equiv\big\{[\mathcal{C},f]\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)\colon f(\mathcal{C})\subset X\big\}=\tilde{s}^{-1}(0). \tag{0.6}$$

On the other hand, GW-invariants of X are defined by integration against the virtual fundamental class (VFC) of $\overline{\mathfrak{M}}_{g,k}(X,d)$ constructed in [BeFa], [FuO], and [LiT]:

$$\mathrm{GW}^X_{g,k}(d;\eta)=\big\langle\eta,\big[\overline{\mathfrak{M}}_{g,k}(X,d)\big]^{vir}\big\rangle \qquad \forall\ \eta\in H^*\big(\overline{\mathfrak{M}}_{g,k}(X,d);\mathbb{Q}\big). \tag{0.7}$$

If X is a quintic threefold (or is another Calabi-Yau threefold), then the cycle $[\overline{\mathfrak{M}}_{g,0}(X,d)]^{vir}$ is zero-dimensional and its degree is denoted by $N_{g,d}$:

$$N_{g,d}=\mathrm{GW}^X_{g,0}(d;1)\equiv\big\langle 1,\big[\overline{\mathfrak{M}}_{g,0}(X,d)\big]^{vir}\big\rangle. \tag{0.8}$$

In light of Poincare Duality, equations (0.6) and (0.7) suggest that $\mathrm{GW}^X_{g,k}(d;\eta)$ should be expressible as an integral against $[\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)]^{vir}$ via some sort of euler class of the sheaf $\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)$, whenever η comes from $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$. As can be easily seen from the definition of VFC, this is indeed the case if $g=0$:

$$\mathrm{GW}^X_{0,k}(d;\eta)=\big\langle\eta\cdot e\big(\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)\big),\big[\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)\big]\big\rangle \tag{0.9}$$

for all $\eta\in H^*(\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d);\mathbb{Q})$. The moduli space $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ is a smooth stack (orbifold), and

$$\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)\longrightarrow\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$$

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[2] In other words, s is a holomorphic section of $\gamma_n^{*\otimes a}$, where $\gamma_n\longrightarrow\mathbb{P}^n$ is the tautological line bundle.

is a locally free sheaf (vector bundle). Thus, the right-hand side of (0.9) can be computed via the classical localization theorem of [ABo], though the complexity of this computation increases rapidly with the degree d. Nevertheless, it has been completed in full generality in a number of different of ways, verifying the genus 0 mirror symmetry prediction (0.4). At the end of this subsection we briefly recall Givental's approach [Gi] to dealing with this complexity and describe its interplay with our approach in genus 1; other proofs of (0.4) can be found in [Ber], [Ga], [Le], and [LLY].

The most algebraically natural generalization of (0.9) to positive genera fails. In the genus 1 case, by [LiZ, Theorem 1.1], the most topologically natural analogue of (0.9) does hold for the *reduced* GW-invariants $\mathrm{GW}_{1,k}^{0;X}$ defined in [Z1]:

$$\mathrm{GW}_{1,k}^{0;X}(d;\eta) = \big\langle \eta \cdot e\big(\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)\big), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle \tag{0.10}$$

for all $\eta \in H^*(\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d);\mathbb{Q})$, where $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is the main component of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$, i.e. the closure of the locus in $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$ consisting of maps from smooth domains. While $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is not a smooth stack, it is an equi-dimensional orbi-variety and has a well-defined fundamental class. While the sheaf

$$\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a) \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$$

is not locally free, it is shown in [Z2] that its euler class is well defined. If

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "p_*"] \arrow[d] & \pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a) \arrow[d] \\ \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \arrow[r, "p"] & \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \end{tikzcd} \tag{0.11}$$

is a desingularization of $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ and $\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)$ (i.e. $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is smooth, p is a birational morphism, and $\mathcal{V}_1$ is locally free), then

$$\big\langle \eta \cdot e\big(\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^n}(a)\big), \big[\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle = \big\langle \pi^*\eta \cdot e(\mathcal{V}_1), \big[\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)\big]\big\rangle \tag{0.12}$$

for all $\eta \in H^*(\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d);\mathbb{Q})$. A natural desingularization (0.11) that inherits every torus action from $\mathbb{P}^n$ is constructed in [VaZ]. Thus, the classical localization theorem of [ABo] can be used to compute the right-hand side of (0.10) via (0.12). On the other hand, by [Z1, Section 3], the reduced genus 1 GW-invariant differs from the standard one by a combination of genus 0 GW-invariants. In particular, if X is a quintic threefold, then by [Z1, Theorem 1.1]

$$N_{1,d} \equiv \mathrm{GW}_{1,0}^X(d;1) = N_{1,d}^0 + \frac{1}{12}N_{0,d}, \tag{0.13}$$

where $N_{1,d}^0$ is the reduced genus 1 degree d invariant $\mathrm{GW}_{1,0}^{0;X}(d;1)$ of X.

In this paper we compute the numbers $\mathrm{GW}_{1,0}^{0;X}(d;1)$ for a smooth degree n hypersurface X in $\mathbb{P}^{n-1}$, with $n \geq 3$. By (0.10), (0.12), and the divisor relation (see [MirSym, Section 26.3]),

$$d\,\mathrm{GW}_{1,0}^{0;X}(d;1) = \big\langle e(\mathcal{V}_1)\,\mathrm{ev}_1^*H, \big[\widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)\big]\big\rangle, \tag{0.14}$$

where $H \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class and

$$\mathcal{V}_1 \longrightarrow \widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)$$

is the desingularization of the sheaf

$$\pi_* \mathrm{ev}^* \mathcal{O}_{\mathbb{P}^{n-1}}(n) \longrightarrow \overline{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1}, d)$$

constructed in [VaZ]. Analogously to [Gi], we package all reduced genus 1 GW-invariants (0.14) of a degree n hypersurface X_n in $\mathbb{P}^{n-1}$ into a generating function $\mathcal{F}$; this is a power series with coefficients in the equivariant cohomology of $\mathbb{P}^{n-1}$. In [Gi], an analogous power series provides a convenient way to describe a degree-recursive feature of the genus 0 GW-invariants of X_n. The resulting recursion, [MirSym, Lemma 30.1.1], has a "mystery correction term", which is intrinsically determined by the recursion and another property of the genus 0 generating function called polynomiality (see [MirSym, Section 30.2]). By applying the Atiyah-Bott localization theorem, we relate $\mathcal{F}$ to the genus 0 generating functions (1.22)-(1.24). In the process, we encounter seemingly unwieldy terms which turn out to be similar to the expressions encoded by the genus 0 "correction term" cleverly avoided in [Gi]; the latter are all of the form (2.3). While none of these expressions by itself determines any of our unwieldy terms, the entire series of expressions insures that the relevant genus 0 generating function has the remarkably rigid structure of Definition 2.1, which in turn determines all of our unwieldy terms via (2.4). This leads to Propositions 1.1 and 1.2, which describe the contributions to $\mathcal{F}$ from the two different types of fixed loci in terms of known integrals on $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1}, d)$. In Section 3, we use Lemma 3.3 to extract the non-equivariant part of the expressions in Propositions 1.1 and 1.2, obtaining Theorem 3 on page 721. Theorem 2 in the next subsection follows immediately from Theorem 3 and (1.15).

The approach of this paper to summing over all possible fixed loci involves breaking the graph into trees at a special node. As such trees contribute to certain genus 0 integrals, the desired sum is expressible in terms of these integrals. The same approach directly carries over to computing reduced genus 1 GW-invariants of any complete intersection and should be applicable to localization computations in higher genus.[3] In the latter case, there will be more "special" nodes, but their number will be bounded above by the genus. Once the graphs are broken at the special nodes, there will be a number of distinguished trees and an arbitrary number of "generic" trees. The number of the former will again be bounded by the genus. On the other hand, it should be possible to sum over all possibilities for the latter, using the regularity property of the relevant genus 0 integral described in Subsection 2.2.

0.3. **Mirror symmetry formulas for projective CY-hypersurfaces.** In this subsection we formulate a generalization of Theorem 1 to projective Calabi-Yau hypersurfaces of arbitrary dimensions; see Theorem 2 below. We then take a closer look at its low-dimensional cases, comparing some of them with known results and others with the mirror symmetry predictions of [BCOV] and [KlPa].

[3]This is not to say that higher-genus GW-invariants of projective hypersurfaces are now easily computable. It is far from clear at this point which integrals should be localized in higher genus. No higher-genus analogue of (0.10) has been proved yet, though a conjectural version is stated in [LiZ, Subsection 1.1]. Even with such a higher-genus hyperplane property, one would still need to either figure out how to apply the localization theorem in a singular setting or construct a desingularization of the main component of $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n, d)$.

Let n be a positive integer. For each $q=0,1,\ldots$, define $I_{0,q}(t)$ by

$$\sum_{q=0}^{\infty} I_{0,q}(t)w^q \equiv e^{wt}\sum_{d=0}^{\infty} e^{dt}\frac{\prod_{r=1}^{r=nd}(nw+r)}{\prod_{r=1}^{r=d}((w+r)^n-w^n)} \equiv R(w,t). \tag{0.15}$$

Each $I_{0,q}(t)$ is a degree q polynomial in t with coefficients that are power series in e^t; see (0.2) for explicit formulas for $I_0 \equiv I_{0,0}$ and $I_1 \equiv I_{0,1}$ in the $n=5$ case. For $p,q\in\mathbb{Z}^+$ with $q\geq p$, let

$$I_{p,q}(t) = \frac{d}{dt}\bigg(\frac{I_{p-1,q}(t)}{I_{p-1,p-1}(t)}\bigg). \tag{0.16}$$

By the first statement of Proposition 3.1 below, each of the "diagonal" terms $I_{p,p}(t)$ is a power series in e^t with constant term 1, whenever it is defined. Thus, the division in (0.16) is well-defined for all p. Let

$$T = \frac{I_{0,1}(t)}{I_{0,0}(t)}. \tag{0.17}$$

By (i) of Proposition 3.1, the map $t\longrightarrow T$ is a change of variables; it will be called the mirror map.

Let $\bar{R}(w,t) = R(w,t)/I_{0,0}(t)$. Then, $e^{-wt}\bar{R}(w,t)$ is a power series with e^t-constant term 1 and

$$\mathcal{D}_w^p \ln \bar{R}(w,t) \equiv \frac{1}{p!}\bigg\{\frac{d}{dw}\bigg\}^p\Big(\ln\big(e^{-wt}\bar{R}(w,t)\big)\Big)\bigg|_{w=0} \in \mathbb{Q}[[e^t]]$$

for all $p\geq 2$.

Theorem 2. *For each $n\in\mathbb{Z}^+$, the reduced genus 1 degree d Gromov-Witten invariants of a degree n hypersurface X in $\mathbb{P}^{n-1}$ are given by*

$$\begin{aligned}
\sum_{d=1}^{\infty} e^{dT}\mathrm{GW}_{1,0}^{0;X}(d;1)
&= \bigg(\frac{(n-2)(n+1)}{48} + \frac{1-(1-n)^n}{24n^2}\bigg)(T-t) + \frac{n^2-1+(1-n)^n}{24n}\ln I_{0,0}(t) \\
&\quad - \begin{cases} \frac{n-1}{48}\ln\big(1-n^ne^t\big) + \sum_{p=0}^{(n-3)/2}\frac{(n-1-2p)^2}{8}\ln I_{p,p}(t), & \textit{if } 2\not|n; \\ \frac{n-4}{48}\ln\big(1-n^ne^t\big) + \sum_{p=0}^{(n-4)/2}\frac{(n-2p)(n-2-2p)}{8}\ln I_{p,p}(t), & \textit{if } 2|n; \end{cases} \\
&\quad + \frac{n}{24}\sum_{p=2}^{n-2}\bigg(\mathcal{D}_w^{n-2-p}\frac{(1+w)^n}{(1+nw)}\bigg)\big(\mathcal{D}_w^p \ln\bar{R}(w,t)\big),
\end{aligned}$$

where t and T are related by the mirror map (0.17).

If $n=1$, both sides of the formula in Theorem 2 vanish. If $n=2$, X is a pair of points in $\mathbb{P}^1$. In this case, the right-hand side of the formula in Theorem 2 vanishes by (3.5). This is exactly as one would expect, since there are no positive-degree maps from a curve to a point.

If $n=3$, X is a plane cubic, i.e. a 2-torus embedded as a degree 3 curve in $\mathbb{P}^2$. Thus, its degree d GW-invariant is zero unless d is divisible by 3. Furthermore, its genus 1 degree $3r$ GW-invariant is the number of r-fold (unramified) covers of a torus by a torus divided by r, the order of the automorphism group of each such cover. Since the number σ_r of such covers is given by (B.12), it follows that

$$\sum_{d=1}^{\infty} e^{dT}\mathrm{GW}_{1,0}^{X}(d;1) = -\sum_{d=1}^{\infty}\ln\big(1-e^{3dT}\big). \tag{0.18}$$

On the other hand, Theorem 2 gives

$$\sum_{d=1}^{\infty} e^{dT}\mathrm{GW}_{1,0}^{0;X}(d;1) = \frac{1}{8}(T-t) - \frac{1}{24}\ln\big(1-27e^t\big) - \frac{1}{2}\ln I_0(t), \tag{0.19}$$

where $I_0(t)$ and T are given by

$$I_0(t) = 1+\sum_{d=1}^{\infty} e^{dt}\frac{(3d)!}{(d!)^3}, \qquad T = t+\frac{1}{I_0(t)}\sum_{d=1}^{\infty} e^{dt}\bigg(\frac{(3d)!}{(d!)^3}\sum_{r=d+1}^{3d}\frac{3}{r}\bigg).$$

Since the standard and reduced (genus 1) invariants of a (complex) curve are the same if no descendant classes are involved,

$$\frac{1}{8}(T-t) - \frac{1}{24}\ln\big(1-27e^t\big) - \frac{1}{2}\ln I_0(t) = -\sum_{d=1}^{\infty}\ln\big(1-e^{3dT}\big) \tag{0.20}$$

by (0.18) and (0.19). We do not see a direct proof of (0.20) at this point.

If $n=4$, X is a quartic surface in $\mathbb{P}^3$, i.e. a $K3$. All its GW-invariants are known to be zero. With $n=4$, we find that the two coefficients of $(T-t)$ in Theorem 2 add up to zero; the same is the case for the two coefficients of $\ln I_{0,0}(t)$. Thus, the sum of the terms on the first two lines of the right-hand side in the formula of Theorem 2 is zero. The remaining term is

$$\frac{4}{24}\cdot\frac{1}{2!}\Big\{\frac{d}{dw}\Big\}^2\bigg(\ln\sum_{q=0}^{\infty}J_q(t)w^q\bigg)\bigg|_{w=0} = \frac{1}{6}\bigg(J_2(t)-\frac{1}{2}J_1(t)^2\bigg).$$

Here $J_1(t)$ and $J_2(t)$ are the $n=4$ analogues of the functions in Subsection 0.1:

$$\sum_{q=0}^{\infty} I_q(t)w^q \equiv e^{wt}\sum_{d=0}^{\infty} e^{dt}\frac{\prod_{r=1}^{r=4d}(4w+r)}{\prod_{r=1}^{r=d}((w+r)^4-w^4)}, \qquad J_q(t) = I_q(t)/I_0(t).$$

We note that

$$\begin{aligned}\bigg(J_2(t)-\frac{1}{2}J_1(t)^2\bigg)' &= J_1'(t)\bigg(\frac{J_2'(t)}{J_1'(t)}-J_1(t)\bigg);\\ \bigg(\frac{J_2'(t)}{J_1'(t)}-J_1(t)\bigg)' &= \bigg(\frac{J_2'(t)}{J_1'(t)}\bigg)' - J_1'(t) \equiv I_{2,2}(t)-I_{1,1}(t) = 0.\end{aligned} \tag{0.21}$$

The last equality above holds by the $(n,p)=(4,1)$ case of (3.6). Since $J_2(t)-\frac{1}{2}J_1(t)^2$ is a power series in e^t with no e^t-constant term, (0.21) implies that it is zero as expected.[4]

The $n=5$ case of Theorem 2 implies Theorem 1. In this case, the power series $I_{0,0}(t)$ and $I_{1,1}(t)$ in e^t in the statement of Theorem 2 are $I_0(t)$ and $J_1'(t)$ in the notation of Subsection 0.1. Thus, the sum of the terms on the first two lines of the right-hand side in the formula of Theorem 2 is precisely the right-hand side of (0.5)

[4]For a surface X, the standard and reduced (genus 1) GW-invariants are the same if no descendant classes are involved.

divided by 2. The remaining term in Theorem 2 is a sum of two terms, one of which (the one corresponding to $p=2$) is easily seen to be zero. The other term is

$$\begin{aligned}(0.22)\qquad &\frac{5}{24}\cdot\frac{1}{3!}\cdot\left\{\frac{d}{dw}\right\}^3\left(\ln\sum_{q=0}^{\infty}J_q(t)w^q\right)\bigg|_{w=0}\\ &\qquad=\frac{5}{24}\left(J_3(t)-J_1(t)J_2(t)+\frac{1}{3}J_1(t)^3\right)=-\frac{1}{12}\sum_{d=1}^{\infty}N_{0,d}e^{dT},\end{aligned}$$

with $J_1(t)$, $J_2(t)$, and $J_3(t)$ as in Subsection 0.1. The last equality in (0.22) is immediate from (0.4). Theorem 1 thus follows from Theorem 3 and (0.13).

If $n=6$, X is a sextic fourfold in $\mathbb{P}^5$. Theorem 2 in this case gives

$$\begin{aligned}\sum_{d=1}^{\infty}e^{dT}\mathrm{GW}^{0;X}_{1,0}(d;1)&=-\frac{35}{2}\big(T-t\big)+\frac{423}{4}\ln I_0(t)-\ln J_1'(t)-\frac{1}{24}\ln(1-6^6e^t)\\ &\quad+\frac{6}{24}\cdot15\cdot\frac{1}{2!}\left\{\frac{d}{dw}\right\}^2\left(\ln\sum_{q=0}^{\infty}J_q(t)w^q\right)\bigg|_{w=0}\\ &\quad+\frac{6}{24}\cdot\frac{1}{4!}\left\{\frac{d}{dw}\right\}^4\left(\ln\sum_{q=0}^{\infty}J_q(t)w^q\right)\bigg|_{w=0}.\end{aligned}$$

The last two terms above arise from the last term in the formula of Theorem 2. In the $n=3,4,5$ cases, the latter is

$$\mathrm{GW}^{0;X}_{1,k}(d;1)-\mathrm{GW}^{X}_{1,k}(d;1).$$

We show in [Z5] that this is the case for all n. For $n=6$, Theorem 2 would then give

$$\sum_{d=1}^{\infty}e^{dT}\mathrm{GW}^{X}_{1,0}(d;1)=-\frac{35}{2}\big(T-t\big)+\frac{423}{4}\ln I_0(t)-\ln J_1'(t)-\frac{1}{24}\ln(1-6^6e^t),$$

confirming the mirror symmetry prediction of [KlPa, Section 6.1].[5]

1. Equivariant cohomology and stable maps

1.1. **Definitions and notation.** This subsection reviews the notion of equivariant cohomology and sets up related notation that will be used throughout the rest of the paper. For the most part, our notation agrees with [MirSym, Chapters 29,30]; the main difference is that we work with $\mathbb{P}^{n-1}$ instead of $\mathbb{P}^n$.

We denote by $\mathbb{T}$ the n-torus $(\mathbb{C}^*)^n$ (or $(S^1)^n$). It acts freely on $E\mathbb{T}=(\mathbb{C}^\infty)^n-0$ (or $(S^\infty)^n$):

$$\big(e^{\mathrm{i}\theta_1},\ldots,e^{\mathrm{i}\theta_n}\big)\cdot(z_1,\ldots,z_n)=\big(e^{\mathrm{i}\theta_1}z_1,\ldots,e^{\mathrm{i}\theta_n}z_n\big).$$

Thus, the classifying space for $\mathbb{T}$ and its group cohomology are given by

$$B\mathbb{T}\equiv E\mathbb{T}/\mathbb{T}=(\mathbb{P}^\infty)^n\qquad\text{and}\qquad H^*_{\mathbb{T}}\equiv H^*(B\mathbb{T};\mathbb{Q})=\mathbb{Q}[\alpha_1,\ldots,\alpha_n],$$

where $\alpha_i=\pi_i^*c_1(\gamma^*)$ if

$$\pi_i\colon(\mathbb{P}^\infty)^n\longrightarrow\mathbb{P}^\infty\qquad\text{and}\qquad\gamma\longrightarrow\mathbb{P}^\infty$$

[5]The variable t in [KlPa, (46)] is not the same as the variable t in this paper; see Appendix B.

are the projection onto the i-th component and the tautological line bundle, respectively. Denote by $\mathcal{H}_{\mathbb{T}}^*$ the field of fractions of $H_{\mathbb{T}}^*$:

$$\mathcal{H}_{\mathbb{T}}^* = \mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1, \ldots, \alpha_n).$$

A representation ρ of $\mathbb{T}$, i.e. a linear action of $\mathbb{T}$ on $\mathbb{C}^k$, induces a vector bundle over $B\mathbb{T}$:

$$V_\rho \equiv E\mathbb{T} \times_{\mathbb{T}} \mathbb{C}^k.$$

If ρ is one-dimensional, we will call

$$c_1(V_\rho^*) = -c_1(V_\rho) \in H_{\mathbb{T}}^* \subset \mathcal{H}_{\mathbb{T}}^*$$

the weight of ρ. For example, α_i is the weight of representation

$$\pi_i \colon \mathbb{T} \longrightarrow \mathbb{C}^*, \qquad \big(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n}\big) \cdot z = e^{\mathrm{i}\theta_i} z. \tag{1.1}$$

More generally, if a representation ρ of $\mathbb{T}$ on $\mathbb{C}^k$ splits into one-dimensional representations with weights $\beta_1, \ldots, \beta_k$, we will call $\beta_1, \ldots, \beta_k$ the weights of ρ. In such a case,

$$e(V_\rho^*) = \beta_1 \cdot \ldots \cdot \beta_k. \tag{1.2}$$

We will call the representation ρ of $\mathbb{T}$ on $\mathbb{C}^n$ with weights $\alpha_1, \ldots, \alpha_n$ the standard representation of $\mathbb{T}$.

If $\mathbb{T}$ acts on a topological space M, let

$$H_{\mathbb{T}}^*(M) \equiv H^*(BM; \mathbb{Q}), \qquad \text{where} \qquad BM = E\mathbb{T} \times_{\mathbb{T}} M,$$

denote the corresponding equivariant cohomology of M. The projection map $BM \longrightarrow B\mathbb{T}$ induces an action of $H_{\mathbb{T}}^*$ on $H_{\mathbb{T}}^*(M)$. Let

$$\mathcal{H}_{\mathbb{T}}^*(M) = H_{\mathbb{T}}^*(M) \otimes_{H_{\mathbb{T}}^*} \mathcal{H}_{\mathbb{T}}^*.$$

If the $\mathbb{T}$-action on M lifts to an action on a (complex) vector bundle $V \longrightarrow M$, then

$$BV \equiv E\mathbb{T} \times_{\mathbb{T}} V$$

is a vector bundle over BM. Let

$$\mathbf{e}(V) \equiv e(BV) \in H_{\mathbb{T}}^*(M) \subset \mathcal{H}_{\mathbb{T}}^*(M)$$

denote the equivariant euler class of V.

Throughout the paper we work with the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$, i.e. the action induced by the standard action ρ of $\mathbb{T}$ on $\mathbb{C}^n$:

$$\big(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_n}\big) \cdot [z_1, \ldots, z_n] = \big[e^{\mathrm{i}\theta_1} z_1, \ldots, e^{\mathrm{i}\theta_n} z_n\big].$$

Since $B\mathbb{P}^{n-1} = \mathbb{P}V_\rho$,

$$H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) \equiv H^*\big(\mathbb{P}V_\rho; \mathbb{Q}\big) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] \big/ \big(x^n + c_1(V_\rho) x^{n-1} + \ldots + c_n(V_\rho)\big),$$

where $x = c_1(\tilde{\gamma}^*)$ and $\tilde{\gamma} \longrightarrow \mathbb{P}V_\rho$ is the tautological line bundle. Since

$$c(V_\rho) = (1 - \alpha_1) \ldots (1 - \alpha_n),$$

it follows that

$$\begin{aligned} H_{\mathbb{T}}^*(\mathbb{P}^{n-1}) &= \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] \big/ (x - \alpha_1) \ldots (x - \alpha_n), \\ \mathcal{H}_{\mathbb{T}}^*(\mathbb{P}^{n-1}) &= \mathbb{Q}_\alpha[x] \big/ (x - \alpha_1) \ldots (x - \alpha_n). \end{aligned} \tag{1.3}$$

The standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ has n fixed points:

$$P_1 = [1, 0, \ldots, 0], \qquad P_2 = [0, 1, 0, \ldots, 0], \quad \ldots \quad P_n = [0, \ldots, 0, 1].$$

For each $i=1,2,\dots,n$, let

$$\phi_i = \prod_{k\neq i}(x-\alpha_k) \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}). \tag{1.4}$$

By equation (1.10) below, ϕ_i is the equivariant Poincare dual of P_i. We also note that $\tilde{\gamma}|_{BP_i}=V_{\pi_i}$, where π_i is as in (1.1). Thus, the restriction map on the equivariant cohomology induced by the inclusion $P_i \longrightarrow \mathbb{P}^{n-1}$ is given by

$$\begin{aligned} H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) = \mathbb{Q}[x,\alpha_1,\dots,\alpha_n]/\prod_{k=1}^{k=n}(x-\alpha_k) &\longrightarrow H^*_{\mathbb{T}}(P_i) = \mathbb{Q}[\alpha_1,\dots,\alpha_n],\\ x &\longrightarrow \alpha_i. \end{aligned} \tag{1.5}$$

By (1.5),

$$\eta = 0 \in H^*_{\mathbb{T}}(\mathbb{P}^{n-1}) \qquad \Longleftrightarrow \qquad \eta|_{P_i} = 0 \in H^*_{\mathbb{T}} \quad \forall\, i=1,2,\dots,n. \tag{1.6}$$

The tautological line bundle $\gamma_{n-1} \longrightarrow \mathbb{P}^{n-1}$ is a subbundle of $\mathbb{P}^{n-1}\times\mathbb{C}^n$ preserved by the diagonal action of $\mathbb{T}$. Thus, the action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$ naturally lifts to an action on γ_{n-1} and

$$\mathbf{e}\big(\gamma^*_{n-1}\big)\big|_{P_i} = \alpha_i \qquad \forall\, i=1,2,\dots,n. \tag{1.7}$$

The $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ also has a natural lift to the vector bundle $T\mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}$ so that there is a short exact sequence

$$0 \longrightarrow \gamma^*_{n-1}\otimes\gamma_{n-1} \longrightarrow \gamma^*_{n-1}\otimes\big(\mathbb{P}^{n-1}\times\mathbb{C}^n\big) \longrightarrow T\mathbb{P}^{n-1} \longrightarrow 0$$

of $\mathbb{T}$-equivariant vector bundles on $\mathbb{P}^{n-1}$. By (1.2), (1.7), and (1.4),

$$\mathbf{e}\big(T\mathbb{P}^{n-1}\big)\big|_{P_i} = \prod_{k\neq i}(\alpha_i-\alpha_k) = \phi_i|_{P_i} \qquad \forall\, i=1,2,\dots,n. \tag{1.8}$$

If $\mathbb{T}$ acts smoothly on a smooth compact oriented manifold M, there is a well-defined integration-along-the-fiber homomorphism

$$\int_M : H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}$$

for the fiber bundle $BM \longrightarrow B\mathbb{T}$. The classical localization theorem of [ABo] relates it to integration along the fixed locus of the $\mathbb{T}$-action. The latter is a union of smooth compact orientable manifolds F; $\mathbb{T}$ acts on the normal bundle $\mathcal{N}F$ of each F. Once an orientation of F is chosen, there is a well-defined integration-along-the-fiber homomorphism

$$\int_F : H^*_{\mathbb{T}}(F) \longrightarrow H^*_{\mathbb{T}}.$$

The localization theorem states that

$$\int_M \eta = \sum_F \int_F \frac{\eta|_F}{\mathbf{e}(\mathcal{N}F)} \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\, \eta \in H^*_{\mathbb{T}}(M), \tag{1.9}$$

where the sum is taken over all components F of the fixed locus of $\mathbb{T}$. Part of the statement of (1.9) is that $\mathbf{e}(\mathcal{N}F)$ is invertible in $\mathcal{H}^*_{\mathbb{T}}(F)$. In the case of the standard action of $\mathbb{T}$ on $\mathbb{P}^{n-1}$, (1.9) implies that

$$\eta|_{P_i} = \int_{\mathbb{P}^{n-1}} \eta\phi_i \in \mathcal{H}^*_{\mathbb{T}} \qquad \forall\, \eta\in\mathcal{H}^*_{\mathbb{T}}(\mathbb{P}^{n-1}),\ i=1,2,\dots,n; \tag{1.10}$$

see also (1.8).

Finally, if $f: M \longrightarrow M'$ is a $\mathbb{T}$-equivariant map between two compact oriented manifolds, there is a well-defined pushforward homomorphism

$$f_*: H^*_{\mathbb{T}}(M) \longrightarrow H^*_{\mathbb{T}}(M').$$

It is characterized by the property that

$$\int_{M'} (f_*\eta)\,\eta' = \int_M \eta\,(f^*\eta') \qquad \forall\ \eta \in H^*_{\mathbb{T}}(M),\ \eta' \in H^*_{\mathbb{T}}(M'). \tag{1.11}$$

The homomorphism $\int_M$ of the previous paragraph corresponds to M' being a point. It is immediate from (1.11) that

$$f_*\big(\eta\,(f^*\eta')\big) = (f_*\eta)\,\eta' \qquad \forall\ \eta \in H^*_{\mathbb{T}}(M),\ \eta' \in H^*_{\mathbb{T}}(M'). \tag{1.12}$$

1.2. **Setup for localization computation on $\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$.** The standard $\mathbb{T}$-action on $\mathbb{P}^{n-1}$ (as well as any other action) induces $\mathbb{T}$-actions on moduli spaces of stable maps $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d)$ by composition on the right,

$$h\cdot[\mathcal{C},f] = [\mathcal{C},h\circ f] \qquad \forall\ h\in\mathbb{T},\ [\mathcal{C},f]\in\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d),$$

and lifts to an action on $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1},d)$. All the evaluation maps,

$$\mathrm{ev}_i: \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d), \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1},d) \longrightarrow \mathbb{P}^{n-1}, \quad [\mathcal{C},y_1,\ldots,y_k,f] \longrightarrow f(y_i),$$

where $i=1,2,\ldots,k$, are $\mathbb{T}$-equivariant. These actions lift naturally to the universal tangent line bundles

$$L_1,\ldots,L_k \longrightarrow \overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d);$$

see [MirSym, Section 25.2]. Let

$$\psi_i \equiv c_1(L_i^*) \in \mathcal{H}^*_{\mathbb{T}}\big(\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d)\big)$$

denote the equivariant ψ-class.

Via the natural lift of the $\mathbb{T}$-action to $\gamma_{n-1}\longrightarrow\mathbb{P}^{n-1}$ described in Subsection 1.1, the $\mathbb{T}$-actions on $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d)$ and $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1},d)$ lift to $\mathbb{T}$-actions on the sheafs $\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^{n-1}}(a)$ and on the vector bundle

$$\mathcal{V}_1 \longrightarrow \widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$$

introduced in Subsection 0.2. We denote by

$$\mathcal{V}_0 \longrightarrow \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$$

the vector bundle of the locally free sheaf $\pi_*\mathrm{ev}^*\mathcal{O}_{\mathbb{P}^{n-1}}(n)$ over $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$. Let

$$\mathcal{L} = \gamma^{*\otimes n}_{n-1} \longrightarrow \mathbb{P}^{n-1}$$

be the vector bundle corresponding to the locally free sheaf $\mathcal{O}_{\mathbb{P}^{n-1}}(n)\longrightarrow\mathbb{P}^{n-1}$. For $g=0,1$, the equivariant bundle map

$$\widetilde{\mathrm{ev}}_1: \mathcal{V}_g \longrightarrow \mathrm{ev}_1^*\mathcal{L}, \qquad [\mathcal{C},y_1,\ldots,y_k,f,\xi] \longrightarrow \big[\xi(y_1)\big],$$

is surjective. Thus,

$$\mathcal{V}_0' \equiv \ker\widetilde{\mathrm{ev}}_1 \longrightarrow \overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d) \qquad \text{and} \qquad \mathcal{V}_1' \equiv \ker\widetilde{\mathrm{ev}}_1 \longrightarrow \widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$$

are equivariant vector bundles.[6] Furthermore,

$$\mathbf{e}(\mathcal{V}_g) = \mathbf{e}\big(\mathrm{ev}_1^*\mathcal{L}\big)\,\mathbf{e}(\mathcal{V}_g') = n\,\mathrm{ev}_1^*(x)\,\mathbf{e}(\mathcal{V}_g'), \tag{1.13}$$

where $x\in H^2_{\mathbb{T}}(\mathbb{P}^{n-1})$ is the equivariant hyperplane class as in Subsection 1.1.

We denote by α the tuple $(\alpha_1,\ldots,\alpha_n)$. With $\mathrm{ev}_{1,d}$ denoting the evaluation map on $\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$, let

$$\mathcal{F}(\alpha,x,u) = \sum_{d=1}^{\infty} u^d\big(\mathrm{ev}_{1,d*}\mathbf{e}(\mathcal{V}_1)\big) \in \big(H^{n-2}_{\mathbb{T}}(\mathbb{P}^{n-1})\big)[[u]].$$

By (1.3),

$$\mathcal{F}(\alpha,x,u) = \mathcal{F}_0(u)x^{n-2} + \mathcal{F}_1(\alpha,u)x^{n-3} + \ldots + \mathcal{F}_{n-2}(\alpha,u)x^0, \tag{1.14}$$

for some power series $\mathcal{F}_0(u)$ in u and degree p homogeneous α-polynomials

$$\mathcal{F}_p(\alpha,u) \in \mathbb{Q}[[u]]\big[\alpha_1,\ldots,\alpha_n\big].$$

These polynomials must be symmetric in $\alpha_1,\ldots,\alpha_n$. Note that by (0.14) and (1.12),

$$\frac{d}{dT}\sum_{d=1}^{\infty} e^{dT}\mathrm{GW}^{0;X}_{1,0}(d;1) = \mathcal{F}_0(e^T). \tag{1.15}$$

Thus, our aim is to determine the power series $\mathcal{F}_0(u)$ in u defined by (1.14).

By (1.5), (1.10), and (1.11),

$$\begin{aligned}\mathcal{F}(\alpha,\alpha_i,u) = \mathcal{F}(\alpha,x,u)\big|_{P_i} &= \sum_{d=1}^{\infty} u^d\int_{\mathbb{P}^{n-1}}\big(\mathrm{ev}_{1,d*}\mathbf{e}(\mathcal{V}_1)\big)\phi_i\\ &= \sum_{d=1}^{\infty} u^d\int_{\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)}\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i\end{aligned} \tag{1.16}$$

for each $i=1,2,\ldots,n$. By (1.6), the power series $\mathcal{F}_0(u)$ is completely determined by

$$\mathcal{F}(\alpha,\alpha_1,u),\ldots,\mathcal{F}(\alpha,\alpha_n,u) \in \mathbb{Q}_\alpha[[u]],$$

where $\mathbb{Q}_\alpha$ is the ring $\mathbb{Q}(\alpha_1,\ldots,\alpha_n)$ of rational fractions in $\alpha_1,\ldots,\alpha_n$. We will apply the localization formula (1.9) to the last expression in (1.16). In order to do so, we need to describe the fixed loci of the $\mathbb{T}$-action on $\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$ and for each fixed locus F the corresponding triple $(F,\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i|_F,\mathcal{N}F)$ or another triple $(F',\eta,\mathcal{N}')$ such that

$$\int_{F'}\frac{\eta}{\mathbf{e}(\mathcal{N}')} = \int_F\frac{\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i|_F}{\mathbf{e}(\mathcal{N}F)}.$$

In one case in Subsection 1.4, choosing such a replacement turns out to be advantageous.

An element $[\mathcal{C},f]$ of $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},d)$ is the equivalence class of a nodal genus 1 curve $\mathcal{C}$ with one marked point and a stable degree d map $f:\mathcal{C}\longrightarrow\mathbb{P}^{n-1}$. We denote by

$$\mathfrak{M}^{\mathrm{eff}}_{1,1}(\mathbb{P}^{n-1},d) \subset \overline{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1},d)$$

[6] In [MirSym, Chapters 29,30], the analogues of $\mathcal{V}_0$ and $\mathcal{V}_0'$ over $\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^n,d)$ are denoted by $E_{0,d}$ and $E_{0,d}'$, respectively. However, $E_{0,d}'$ is the kernel of the evaluation map at the second marked point.

the open subset of $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},d)$ consisting of the stable maps $[\mathcal{C},f]$ that are not constant on the principal, genus 1 component $\mathcal{C}_P$ of $\mathcal{C}$.[7] By definition, $\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d)$ is dense in $\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$. Let

$$\partial\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)=\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)-\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d).$$

The desingularization $\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ of $\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ is obtained by blowing up along subvarieties contained in $\partial\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$; see [VaZ, Subsection 1.2]. Thus,

$$\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d)\subset\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$$

is a dense open subset. Let

$$\partial\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)=\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)-\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d).$$

Since each of the fixed loci of the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ is contained either in $\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d)$ or in $\partial\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$, each of the fixed loci of the $\mathbb{T}$-action on $\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ is contained in $\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d)$ or in $\partial\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$. Furthermore, the fixed loci contained in $\mathfrak{M}_{1,1}^{\mathrm{eff}}(\mathbb{P}^{n-1},d)$ and the corresponding triples $(F,\eta,\mathcal{N}F)$ as in (1.9) are the same for the $\mathbb{T}$-actions on $\overline{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ and $\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$. These loci and their total contribution to (1.16) are described in Subsection 1.3. The fixed loci contained in $\partial\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ and their total contribution to (1.16) are described in Subsection 1.4 based on [VaZ, Subsection 1.4].[8]

Many expressions throughout the paper involve residues of rational functions in a complex variable $\hbar$. If $f=f(\hbar)$ is a rational function in $\hbar$ and $\hbar_0\in S^2$, we denote by $\mathfrak{R}_{\hbar=\hbar_0}f(\hbar)$ the residue of $f(\hbar)d\hbar$ at $\hbar=\hbar_0$:

$$\mathfrak{R}_{\hbar=\hbar_0}f(\hbar)=\frac{1}{2\pi\mathfrak{i}}\oint f(\hbar)d\hbar,$$

where the integral is taken over a positively oriented loop around $\hbar=\hbar_0$ containing no other singular points of f. With this definition,

$$\mathfrak{R}_{\hbar=\infty}f(\hbar)=-\mathfrak{R}_{w=0}\big\{w^{-2}f(w^{-1})\big\}.$$

If f involves variables other than $\hbar$, $\mathfrak{R}_{\hbar=\hbar_0}f(\hbar)$ will be a function of such variables. If f is a power series in u with coefficients that are rational functions in $\hbar$ and possibly other variables, denote by $\mathfrak{R}_{\hbar=\hbar_0}f(\hbar)$ the power series in u obtained by replacing each of the coefficients by its residue at $\hbar=\hbar_0$. If $\hbar_1,\ldots,\hbar_k$ is a collection of points in S^2, let

$$\mathfrak{R}_{\hbar=\hbar_1,\ldots,\hbar_k}f(\hbar)=\sum_{i=1}^{i=k}\mathfrak{R}_{\hbar=\hbar_i}f.$$

Finally, we will denote by $\bar{\mathbb{Z}}^+$ the set of non-negative integers and by $[n]$, whenever $n\in\bar{\mathbb{Z}}^+$, the set of positive integers not exceeding n:

$$\bar{\mathbb{Z}}^+\equiv\big\{0,1,2,\ldots\big\},\qquad [n]=\big\{1,2,\ldots,n\big\}.$$

[7] The connected curve $\mathcal{C}$ is nodal and has arithmetic genus 1. Thus, either one of the components of $\mathcal{C}$ is a smooth torus or $\mathcal{C}$ contains a circle of one or more spheres (each irreducible component is a $\mathbb{P}^1$ with exactly two nodes). In the first case, $\mathcal{C}_P$ is the smooth torus; in the second, $\mathcal{C}_P$ is the circle of spheres.

[8] We will not describe $\widetilde{\mathfrak{M}}_{1,1}^{0}(\mathbb{P}^{n-1},d)$ in this paper, as this is not necessary.

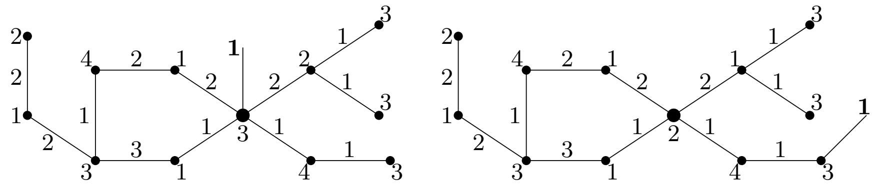

FIGURE 1. A decorated graph of type A_3 and a decorated graph of type $\tilde{A}_{32}$

1.3. **Contributions from fixed loci, I.** As described in detail in [MirSym, Section 27.3], the fixed loci of the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d)$ are indexed by decorated graphs. A graph consists of a set Ver of vertices and a collection Edg of edges, i.e. of two-element subsets of Ver.[9] In Figure 1, the vertices are represented by dots, while each edge $\{v_1,v_2\}$ is shown as the line segment between v_1 and v_2. For the purposes of describing the fixed loci of $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$ and $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^{n-1},d)$, it is sufficient to define a decorated graph as a tuple

$$\Gamma = \big(\mathrm{Ver},\mathrm{Edg};\mu,\mathfrak{d},\eta\big), \tag{1.17}$$

where $(\mathrm{Ver},\mathrm{Edg})$ is a graph and

$$\mu\colon \mathrm{Ver}\longrightarrow [n], \qquad \mathfrak{d}\colon \mathrm{Edg}\longrightarrow \mathbb{Z}^+, \quad \text{and} \quad \eta\colon [k]\longrightarrow \mathrm{Ver}$$

are maps such that

$$\mu(v_1)\neq\mu(v_2) \qquad \text{if} \quad \{v_1,v_2\}\in\mathrm{Edg}. \tag{1.18}$$

In Figure 1, the value of the map μ on each vertex is indicated by the number next to the vertex. Similarly, the value of the map $\mathfrak{d}$ on each edge is indicated by the number next to the edge. The only element of the set $[k]=[1]$ is shown in boldface. It is linked by a line segment to its image under η. By (1.18), no two consecutive vertex labels are the same.

A graph $(\mathrm{Ver},\mathrm{Edg})$ is a tree if it contains no loops, i.e. the set Edg contains no subset of the form

$$\big\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_N,v_1\}\big\}, \qquad v_1,\ldots,v_N\in\mathrm{Ver},\ N\geq 1.$$

For example, the graphs in Figure 2 are trees, while those in Figure 1 contain one loop each. Via the construction of the next paragraph, decorated trees describe the fixed loci of $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$, while decorated graphs with exactly one loop describe the fixed loci of $\mathfrak{M}^{\mathrm{eff}}_{1,k}(\mathbb{P}^{n-1},d)$.

The fixed locus $\mathcal{Z}_\Gamma$ of $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^{n-1},d)$ corresponding to a decorated graph Γ consists of the stable maps f from a genus g nodal curve $\mathcal{C}_f$ with k marked points into $\mathbb{P}^{n-1}$ that satisfy the following conditions. The components of $\mathcal{C}_f$ on which the map f is not constant are rational and correspond to the edges of Γ. Furthermore,

[9] If $g=0$, Edg can be taken to be a subset of the set $\mathrm{Sym}^2(\mathrm{Ver})$ of two-element subsets of Ver. For $g\geq 1$, Edg should be viewed as a map from a finite set Dom(Edg) to $\mathrm{Sym}^2(\mathrm{Ver})$; this map may not be injective (i.e. there can be multiple edges connecting a pair of vertices). In the latter case, $e\in\mathrm{Edg}$ will mean that e is an element of Dom(Edg); if $v\in\mathrm{Ver}$ and $e\in\mathrm{Edg}$, $v\in e$ will mean that v is an element of the image of e in $\mathrm{Sym}^2(\mathrm{Ver})$; a map from Edg will mean a map from Dom(Edg).

if $e=\{v_1,v_2\}$ is an edge, the restriction of f to the component $\mathcal{C}_{f,e}$ corresponding to e is a degree $\mathfrak{d}(e)$ cover of the line

$$\mathbb{P}^1_{\mu(v_1),\mu(v_2)} \subset \mathbb{P}^{n-1}$$

passing through the fixed points $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. The map $f|_{\mathcal{C}_{f,e}}$ is ramified only over $P_{\mu(v_1)}$ and $P_{\mu(v_2)}$. In particular, $f|_{\mathcal{C}_{f,e}}$ is unique up to isomorphism. The remaining, contracted components of $\mathcal{C}_f$ are rational and indexed by the vertices $v\in\mathrm{Ver}$ such that

$$\mathrm{val}(v) \equiv \big|\{e\in\mathrm{Edg}\colon v\in e\}\big| + \big|\{i\in[k]\colon \eta(i)=v\}\big| \geq 3.$$

The map f takes such a component $\mathcal{C}_{f,v}$ to the fixed point $P_{\mu(v)}$. Thus,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \prod_{v\in\mathrm{Ver}} \overline{\mathcal{M}}_{0,\mathrm{val}(v)},$$

where $\overline{\mathcal{M}}_{g',l}$ denotes the moduli space of stable genus g' curves with l marked points. For the purposes of this definition, $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ are one-point spaces. For example, in the case of the first diagram in Figure 1,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_\Gamma \equiv \overline{\mathcal{M}}_{0,5} \times \overline{\mathcal{M}}_{0,3}^2 \times \overline{\mathcal{M}}_{0,2}^5 \times \overline{\mathcal{M}}_{0,1}^4 \approx \overline{\mathcal{M}}_{0,5}$$

is a fixed locus[10] in $\mathfrak{M}^{\mathrm{eff}}_{1,1}(\mathbb{P}^{n-1},19)$, with $n\geq 4$. Since n is fixed throughout the main computation in the paper, each graph Γ completely determines the ambient moduli space containing the fixed locus $\mathcal{Z}_\Gamma$; it will be denoted by $\overline{\mathfrak{M}}_\Gamma$.

Suppose Γ is a decorated graph as in (1.17) and has exactly one loop. By (1.4) and (1.5),

$$\mathrm{ev}_1^*\phi_i\big|_{\mathcal{Z}_\Gamma} = \prod_{k\neq i}\big(\alpha_{\mu(\eta(1))}-\alpha_k\big) = \delta_{i,\mu(\eta(1))}\prod_{k\neq i}(\alpha_i-\alpha_k),$$

where $\delta_{i,\mu(\eta(1))}$ is the Kronecker delta function. Thus, by (1.9), Γ does not contribute to (1.16) unless $\mu(\eta(1))=i$, i.e. the marked point of the map is taken to the point $P_i\in\mathbb{P}^{n-1}$. There are two types of graphs that do (or may) contribute to (1.16); they will be called A_i- and $\tilde{A}_{ij}$-types. In a graph of the A_i-type, the marked point 1 is attached to some vertex $v_0\in\mathrm{Ver}$ that lies inside of the loop and is labeled i. In a graph of the $\tilde{A}_{ij}$-type, the marked point 1 is attached to a vertex that lies outside of the loop and is still labeled i, while the vertex v_0 of the loop which is the closest to the marked point is labeled by some $j\in[n]$. This vertex is thus mapped to the point $P_j\in\mathbb{P}^{n-1}$. Examples of graphs of the two types, with $i=3$ and $j=2$, are depicted in Figure 1.

Whether a graph Γ is of type A_i or $\tilde{A}_{ij}$, it contains a distinguished vertex v_0; it is indicated with a thick dot in Figure 1. If we break Γ at v_0, keeping a copy of v_0 on each of the edges of Γ containing v_0, and cap off each of the "loose" ends with a marked point attached to v_0, we obtain several decorated trees, which will be called the strands of Γ. If $e_-,e_+\in\mathrm{Edg}$ are the two edges in the loop in Γ joined at v_0, the strands are naturally indexed by the set

$$\overline{\mathrm{Edg}}(v_0) \equiv \big\{e\in\mathrm{Edg}\colon v_0\in e\big\}/e_-\sim e_+$$

of edges leaving v_0, with e_- and e_+ identified.[11] The distinguished strand $\Gamma_{e_\pm}$ with two marked points arising from the loop of Γ will be denoted by $\Gamma_\pm$. There are also

[10] After dividing by the appropriate automorphism group; see [MirSym, Section 27.3].

[11] By (1.18), $e_-\neq e_+$, i.e. there is no edge from a vertex to itself.

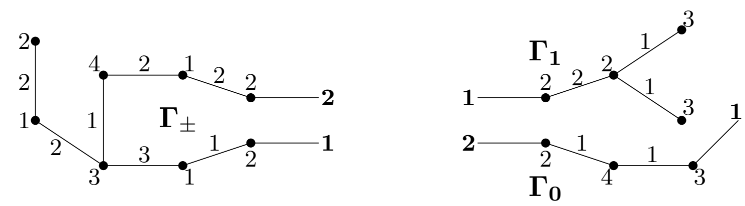

FIGURE 2. The strands of the second graph in Figure 1

$m\ge 0$ strands, $\Gamma_1,\ldots,\Gamma_m$, each of which has exactly one marked point. Finally, if Γ is of type $\tilde{A}_{ij}$, there is also a second distinguished strand with two marked points that contains the marked point 1 of Γ. This strand will be denoted by Γ_0. The strands of the second graph in Figure 1 are shown in Figure 2.

The strands of a one-loop decorated graph Γ correspond to fixed loci of the $\mathbb{T}$-action on $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^{n-1},d)$, with $k=1,2$ and $d\in\mathbb{Z}^+$. Furthermore,

$$\mathcal{Z}_\Gamma \approx \overline{\mathcal{M}}_{0,\mathrm{val}(v_0)} \times \mathcal{Z}_{\Gamma;B} \equiv \overline{\mathcal{M}}_{0,\mathrm{val}(v_0)} \times \prod_{e\in\overline{\mathrm{Edg}}(v_0)} \mathcal{Z}_{\Gamma_e}, \tag{1.19}$$

up to a quotient by a finite group, where B stands for the bubble components. If

$$\pi_P,\pi_B\colon \mathcal{Z}_\Gamma \longrightarrow \overline{\mathcal{M}}_{0,\mathrm{val}(v_0)}, \mathcal{Z}_{\Gamma;B} \qquad \text{and} \qquad \pi_e\colon \mathcal{Z}_{\Gamma;B} \longrightarrow \mathcal{Z}_{\Gamma_e}$$

are the projection maps, then

$$\mathcal{V}_1|_{\mathcal{Z}_\Gamma} \approx \pi_B^*\Big(\bigoplus_{e\in\overline{\mathrm{Edg}}(v_0)} \pi_e^*\mathcal{V}_0'\Big) \quad \Longrightarrow \quad \mathbf{e}(\mathcal{V}_1)\big|_{\mathcal{Z}_\Gamma} = \pi_B^*\Big(\prod_{e\in\overline{\mathrm{Edg}}(v_0)} \pi_e^*\mathbf{e}(\mathcal{V}_0')\Big). \tag{1.20}$$

"Most" of the normal bundle of $\mathcal{Z}_\Gamma$ in $\widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)$ and $\overline{\mathfrak{M}}_{1,1}(\mathbb{P}^{n-1},d)$, as described in [MirSym, Section 27.4], also comes from the components of $\mathcal{Z}_\Gamma$ in the following sense. The marked points on $\overline{\mathcal{M}}_{0,\mathrm{val}(v_0)}$ are naturally indexed by the set

$$\mathrm{Edg}(v_0) \equiv \big\{e\in\mathrm{Edg}\colon v_0\in e\big\}$$

of the edges leaving v_0, along with 1 if Γ is of type A_i. For each $e\in\mathrm{Edg}(v_0)$, let

$$L_e' \longrightarrow \overline{\mathcal{M}}_{1,|\mathrm{val}(v_0)|} \qquad \text{and} \qquad \hbar_e \equiv c_1(L_e') \in H^2\big(\overline{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}\big)$$

be the universal tangent line bundle at the marked point corresponding to e and its first chern class, respectively. Analogously, let

$$L_{e_-}, L_{e_+} \longrightarrow \mathcal{Z}_{\Gamma_\pm} \qquad \text{and} \qquad L_e \longrightarrow \mathcal{Z}_{\Gamma_e},\ e\in \mathrm{Edg}(v_0)-\{e_-,e_+\},$$

be the restrictions to $\mathcal{Z}_{\Gamma_\pm}$ and $\mathcal{Z}_{\Gamma_e}$ of the universal tangent line bundles on $\overline{\mathfrak{M}}_{\Gamma_\pm}$ and $\overline{\mathfrak{M}}_{\Gamma_v}$ at the marked points corresponding to the edges leaving v_0. Let

$$\psi_e = c_1(L_e^*)$$

be the corresponding ψ-classes. The normal bundle of $\widetilde{\mathcal{Z}}_\Gamma$ in $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^{n-1},d)$ is then given by

$$\frac{\mathcal{N}\mathcal{Z}_\Gamma}{T_{\mu(v_0)}\mathbb{P}^{n-1}} = \pi_B^*\Big(\bigoplus_{e\in\overline{\mathrm{Edg}}(v_0)} \pi_e^*\mathcal{N}\mathcal{Z}_{\Gamma_e}\Big) \oplus \bigoplus_{e\in\mathrm{Edg}(v_0)} \frac{\pi_P^*L_e'\otimes\pi_B^*\pi_e^*L_e}{\pi_B^*\pi_e^*T_{\mu(v_0)}\mathbb{P}^{n-1}},$$

where $\mathcal{N}\mathcal{Z}_{\Gamma_e}\longrightarrow\mathcal{Z}_{\Gamma_e}$ is the normal bundle of $\mathcal{Z}_{\Gamma_e}$ in $\overline{\mathfrak{M}}_{\Gamma_e}$. Thus,

$$\frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}=\prod_{e\in\overline{\mathrm{Edg}}(v_0)}\frac{1}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\prod_{e\in\mathrm{Edg}(v_0)}\frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\hbar_e-\psi_e},\tag{1.21}$$

where we omit the pullback maps π_P^*, π_B^*, and π_e^*.

By (1.20) and (1.21),

$$\frac{\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i|_{\mathcal{Z}_\Gamma}}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}\in\mathcal{H}_{\mathbb{T}}^*(\mathcal{Z}_\Gamma)$$

splits into factors coming from the strands of Γ after integrating over the first factor in (1.19). These factors are the contributions of $\mathcal{Z}_{\Gamma_e}$ to integrals over $\overline{\mathfrak{M}}_{\Gamma_e}$ involving $\mathbf{e}(\mathcal{V}_0)$ and ψ_e. We will next describe the total contribution of all graphs of types A_i and $\tilde{A}_{ij}$ to (1.16) in terms of such integrals.

For all $i,j=1,2,\ldots$, let

$$\mathcal{Z}_i^*(\hbar,u)\equiv\sum_{d=1}^{\infty}\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_0')}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i,\tag{1.22}$$

$$\mathcal{Z}_{ij}^*(\hbar,u)\equiv\hbar^{-1}\sum_{d=1}^{\infty}\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_0')}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j,\tag{1.23}$$

$$\widetilde{\mathcal{Z}}_{ij}^*(\hbar_1,\hbar_2,u)\equiv\frac{1}{2\hbar_1\hbar_2}\sum_{d=1}^{\infty}\int_{\overline{\mathfrak{M}}_{0,2}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_0')}{(\hbar_1-\psi_1)(\hbar_2-\psi_2)}\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_j.\tag{1.24}$$

These generating functions have been explicitly computed; see Subsection 3.2. For the moment, we simply note that

$$\mathcal{Z}_i^*,\mathcal{Z}_{ij}^*\in\mathbb{Q}_\alpha(\hbar)[[u]]\qquad\text{and}\qquad\widetilde{\mathcal{Z}}_{ij}^*\in\mathbb{Q}_\alpha(\hbar_1,\hbar_2)[[u]].$$

Thus, the $\hbar$-residues of these power series are well-defined. So is

$$\eta_i(u)=\mathfrak{R}_{\hbar=0}\Big\{\ln\big(1+\mathcal{Z}_i^*(\hbar,u)\big)\Big\}\in\mathbb{Q}_\alpha[[u]],\tag{1.25}$$

since the degree-zero term of the power series $\mathcal{Z}_i^*(\hbar,u)$ is 0. Let

$$\Phi_0(\alpha_i,u)=\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-1}e^{-\eta_i(u)/\hbar}\big(1+\mathcal{Z}_i^*(\hbar,u)\big)\Big\}\in\mathbb{Q}_\alpha[[u]].\tag{1.26}$$

By Lemma 2.3, $e^{-\eta_i(u)/\hbar}\big(1+\mathcal{Z}_i^*(\hbar,u)\big)$ is in fact holomorphic at $\hbar=0$ and thus $\Phi_0(\alpha_i,u)$ is simply the value of this power series at $\hbar=0$. Note that the degree-zero term of $\Phi_0(\alpha_i,u)$ is 1.

Remark. The star in $\mathcal{Z}_i^*$, $\mathcal{Z}_{ij}^*$, and $\widetilde{\mathcal{Z}}_{ij}^*$ indicates that these power series are obtained by removing the u-constant term from certain natural power series $\mathcal{Z}_i$, $\mathcal{Z}_{ij}$, and $\widetilde{\mathcal{Z}}_{ij}$; see [MirSym, Chapter 29] and [Z4, Section 1.1].

Proposition 1.1. *(i) The total contribution $\mathcal{A}_i(u)$ to* (1.16) *from all graphs of type A_i is given by*

$$\mathcal{A}_i(u)=\frac{1}{\Phi_0(\alpha_i,u)}\mathfrak{R}_{\hbar_1=0}\Big\{\mathfrak{R}_{\hbar_2=0}\Big\{e^{-\eta_i(u)/\hbar_1}e^{-\eta_i(u)/\hbar_2}\widetilde{\mathcal{Z}}_{ii}^*(\hbar_1,\hbar_2,u)\Big\}\Big\}.\tag{1.27}$$

(ii) The total contribution $\tilde{\mathcal{A}}_{ij}(u)$ to (1.16) *from all graphs of type $\tilde{A}_{ij}$ is*

$$\tilde{\mathcal{A}}_{ij}(u)=\frac{\mathcal{A}_j(u)}{\prod_{k\neq j}(\alpha_j-\alpha_k)}\mathfrak{R}_{\hbar=0}\Big\{e^{-\eta_j(u)/\hbar}\mathcal{Z}_{ji}^*(\hbar,u)\Big\}.\tag{1.28}$$

It is fairly straightforward to express $\mathcal{A}_i(u)$ and $\tilde{\mathcal{A}}_{ij}(u)$ in terms of sums of products of the residues of $\mathcal{Z}_i^*$, $\mathcal{Z}_{ij}^*$, and $\widetilde{\mathcal{Z}}_{ij}^*$ at every possible $\hbar\in\mathbb{C}^*$. A straightforward application of the Residue Theorem on S^2 then reduces the resulting expressions to sums of products of the residues of $\mathcal{Z}_i^*$, $\mathcal{Z}_{ij}^*$, and $\widetilde{\mathcal{Z}}_{ij}^*$ at $\hbar=0$. However, the products will have either $m+2$ or $m+3$ factors, where m is the number of strands of Γ with one marked point, and must be summed over all possible m. We are able to sum over m because $e^{-\eta_i(u)/\hbar}\big(1+\mathcal{Z}_i^*(\hbar,u)\big)$ turns out to be holomorphic at $\hbar=0$; see Lemmas 2.2 and 2.3. Proposition 1.1 is proved in Subsection 2.3.

1.4. **Contributions from fixed loci, II.** In this subsection we describe the contribution to (1.16) from the fixed loci of the $\mathbb{T}$-action on $\widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)$ that are contained in $\partial\widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)$. We begin by reviewing the description of such loci and their normal bundles given in [VaZ, Subsection 1.4].

A rooted tree is a tree (i.e. a graph with no loops) with a distinguished vertex. A tuple

$$(\mathrm{Ver},\mathrm{Edg},v_0;\mathrm{Ver}_+,\mathrm{Ver}_0)$$

is a refined rooted tree if $(\mathrm{Ver},\mathrm{Edg},v_0)$ is a rooted tree, i.e. v_0 is the distinguished vertex of the tree $(\mathrm{Ver},\mathrm{Edg})$ and $\mathrm{Ver}_+,\mathrm{Ver}_0\subset\mathrm{Ver}-\{v_0\}$ are such that

$$\mathrm{Ver}_+\neq\emptyset,\quad \mathrm{Ver}_+\cap\mathrm{Ver}_0=\emptyset,\quad \{v_0,v\}\in\mathrm{Edg}\;\forall\,v\in\mathrm{Ver}_+\cup\mathrm{Ver}_0.$$

Given such a refined rooted tree, we put

$$\mathrm{Edg}_+=\big\{\{v_0,v\}\colon v\in\mathrm{Ver}_+\big\}\qquad\text{and}\qquad \mathrm{Edg}_0=\big\{\{v_0,v\}\colon v\in\mathrm{Ver}_0\big\}.$$

In the first diagram of Figure 3, the distinguished vertex v_0 is indicated by the thick dot. The elements of Edg_+ and Edg_0 are shown as the thick solid lines and the thin dashed lines, respectively.

A refined decorated rooted tree is a tuple

$$\Gamma=\big(\mathrm{Ver},\mathrm{Edg},v_0;\mathrm{Ver}_+,\mathrm{Ver}_0;\mu,\mathfrak{d},\eta\big), \tag{1.29}$$

where $(\mathrm{Ver},\mathrm{Edg};v_0;\mathrm{Ver}_+,\mathrm{Ver}_0)$ is a refined rooted tree and

$$\mu\colon\mathrm{Ver}-\mathrm{Ver}_0\longrightarrow[n],\qquad \mathfrak{d}\colon\mathrm{Edg}-\mathrm{Edg}_0\longrightarrow\mathbb{Z}^+,\quad\text{and}\quad \eta\colon[k]\longrightarrow\mathrm{Ver}$$

are maps such that

(i) $\mu(v_1)=\mu(v_2)$ and $\mathfrak{d}(\{v_0,v_1\})=\mathfrak{d}(\{v_0,v_2\})$ for all $v_1,v_2\in\mathrm{Ver}_+$;

(ii) if $v_1\in\mathrm{Ver}_+$, $v_2\in\mathrm{Ver}-\mathrm{Ver}_0-\mathrm{Ver}_+$, and $\{v_0,v_2\}\in\mathrm{Edg}$, then

$$\mu(v_1)\neq\mu(v_2)\qquad\text{or}\qquad \mathfrak{d}(\{v_0,v_1\})\neq\mathfrak{d}(\{v_0,v_2\}); \tag{1.30}$$

(iii) if $\{v_1,v_2\}\in\mathrm{Edg}$ and $v_2\notin\mathrm{Ver}_0\cup\{v_0\}$, then

$$\mu(v_2)\neq\mu(v_1)\quad\text{if } v_1\notin\mathrm{Ver}_0\qquad\text{and}\qquad \mu(v_2)\neq\mu(v_0)\quad\text{if } v_1\in\mathrm{Ver}_0;$$

(iv) if $v_1\in\mathrm{Ver}_0$, then $\{v_1,v_2\}\in\mathrm{Edg}$ for some $v_2\in\mathrm{Ver}-\{v_0\}$ and $\mathrm{val}(v_1)\geq3$.

In Figure 3, the value of the map μ on each vertex, not in Ver_0, is indicated by the number next to the vertex. Similarly, the value of the map $\mathfrak{d}$ on each edge, not in Edg_0, is indicated by the number next to the edge. The elements of the set $[k]=[1]$ are shown in boldface. Each of them is linked by a line segment to its image under η. The first condition above implies that all of the thick edges have the same labels, and so do their vertices, other than the root v_0. By the second condition, the set of thick edges is a maximal set of edges leaving v_0 which satisfies the first condition. By the third condition, no two consecutive vertex labels are the same. The final condition implies that there are at least two solid lines, at least

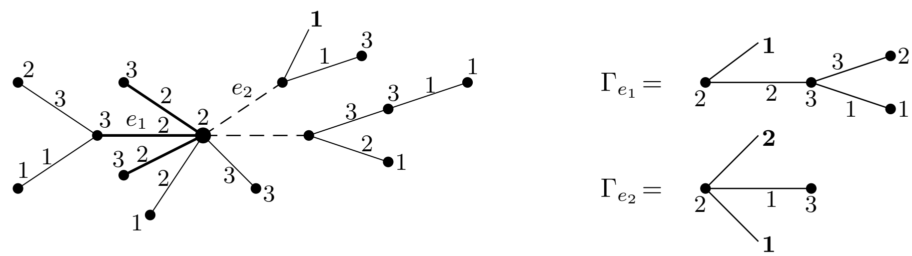

FIGURE 3. A refined decorated rooted tree and some of its strands

one of which is an edge, leaving from every vertex which is connected to the root by a dashed line.

Remark. In [VaZ, Subsection 1.4], refined decorated rooted trees are required to satisfy a fifth condition, $\sum_{e\in\mathrm{Edg}_+}\mathfrak{d}(e)\ge 2$. If this condition is not satisfied (i.e. $\mathfrak{d}(e)\!=\!1$ for the unique element $e\!\in\!\mathrm{Edg}_+$), the locus $\tilde{\mathcal{Z}}_\Gamma$ corresponding to Γ via the construction below will not contribute to (1.16); see also the next remark.

Let Γ be a refined decorated rooted tree as in (1.29). Breaking Γ at v_0, we split Γ into pieces Γ'_e indexed by the set

$$\mathrm{Edg}(v_0)\equiv\big\{e\!\in\!\mathrm{Edg}\colon v_0\!\in\!e\big\}$$

of the edges in Γ leaving v_0. If $e\!\not\in\!\mathrm{Edg}_0$, we will keep the vertex v_0 and the edge e and cap Γ'_e off with a new marked point attached to v_0, just as in Subsection 1.3. If $e=\{v_0,v\}$ with $v\in\mathrm{Ver}_0$, we remove v_0 and e from Γ'_e, cap it off with a new marked point attached to v, and assign the μ-value of v_0 to v. In either case, we denote the resulting decorated tree by Γ_e and call it a strand of Γ; see Figure 3 for two examples. If $v\!\in\!\mathrm{Ver}_+$, let

$$\mu(\Gamma)=\mu(v)\qquad\text{and}\qquad\mathfrak{d}(\Gamma)=\mathfrak{d}\big(\{v_0,v\}\big).$$

By the requirement (i) on Γ, $\mu(\Gamma)$ and $\mathfrak{d}(\Gamma)$ do not depend on the choice of $v\!\in\!\mathrm{Ver}_+$.

Via the construction of Subsection 1.3, each strand Γ_e of Γ determines a $\mathbb{T}$-fixed locus $\mathcal{Z}_{\Gamma_e}$ in a moduli space $\overline{\mathfrak{M}}_{\Gamma_e}$ of genus 0 stable maps. Let

$$\mathcal{Z}_{\Gamma;B}=\prod_{e\in\mathrm{Edg}(v_0)}\mathcal{Z}_{\Gamma_e},$$

where B stands for the "bubble" components. Denote by

$$\pi_e\colon\mathcal{Z}_{\Gamma;B}\longrightarrow\mathcal{Z}_{\Gamma_e}\qquad\text{and}\qquad L_e\longrightarrow\mathcal{Z}_{\Gamma_e}$$

the natural projection map and the restriction to $\mathcal{Z}_{\Gamma_e}$ of the universal tangent line bundle on $\overline{\mathfrak{M}}_{\Gamma_e}$ for the marked point corresponding to the attachment at v_0. Let

$$L_\Gamma=\pi_e^*L_e\longrightarrow\mathcal{Z}_{\Gamma;B}\ \text{if}\ e\!\in\!\mathrm{Edg}_+,\qquad\psi_\Gamma=c_1\big(L_\Gamma^*\big)\in H^2(\mathcal{Z}_{\Gamma;B}),$$
$$F_{\Gamma;B}=\bigoplus_{e\in\mathrm{Edg}_+}\pi_e^*L_e,\quad F^c_{\Gamma;B}=\bigoplus_{e\in\mathrm{Edg}(v_0)-\mathrm{Edg}_+}\pi_e^*L_e,\quad\widetilde{\mathcal{Z}}_{\Gamma;B}=\mathbb{P}F_{\Gamma;B}.$$

By the requirement (i) on Γ, L_Γ is well defined as a $\mathbb{T}$-equivariant line bundle and

$$\widetilde{\mathcal{Z}}_{\Gamma;B}\approx\mathcal{Z}_{\Gamma;B}\times\mathbb{P}^{|\mathrm{Edg}_+|-1}.$$

Let

$$\pi_1, \pi_2 \colon \tilde{\widetilde{\mathcal{Z}}}_{\Gamma;B} \longrightarrow \mathcal{Z}_{\Gamma;B}, \mathbb{P}^{|\mathrm{Edg}_+|-1}$$

be the two projection maps. Up to a quotient by a finite group, the fixed locus of the $\mathbb{T}$-action on $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$ corresponding to Γ (or its equivalent for our purposes; see the next remark) is

$$\widetilde{\mathcal{Z}}_\Gamma \equiv \widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|} \times \tilde{\mathcal{Z}}_{\Gamma;B} \approx \widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|} \times \mathcal{Z}_{\Gamma;B} \times \mathbb{P}^{|\mathrm{Ver}_+|-1}, \tag{1.31}$$

where $\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}$ is a certain blowup of the moduli space of genus 1 curves with $|\mathrm{val}(v_0)|$ marked points constructed in [VaZ, Subsection 2.3].[12] The only property of $\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}$ relevant for the purposes of this paper is (14) below. Denote by

$$\pi_P, \pi_B \colon \widetilde{\mathcal{Z}}_\Gamma \longrightarrow \widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}, \tilde{\mathcal{Z}}_{\Gamma;B}$$

the projection maps.

The normal bundle of $\widetilde{\mathcal{Z}}_\Gamma$ in $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^{n-1}, d)$ (or its equivalent) is given by

$$\mathcal{N}\widetilde{\mathcal{Z}}_\Gamma = \pi_B^*\left(\frac{\pi_1^*\mathcal{N}\mathcal{Z}_{\Gamma;B} \oplus \pi_1^*(L_\Gamma^* \otimes F^c_{\Gamma;B}) \otimes \pi_2^*\gamma^*}{\pi_1^*(L_\Gamma^* \otimes T_{\mu(v_0)}\mathbb{P}^{n-1}) \otimes \pi_2^*\gamma^*}\right) \oplus \pi_P^*\mathbb{L} \otimes \pi_B^*\big(\pi_1^* L_\Gamma \otimes \pi_2^*\gamma\big), \tag{1.32}$$

where $\gamma \longrightarrow \mathbb{P}^{|\mathrm{Ver}_+|-1}$ is the tautological line bundle,

$$\frac{\mathbf{e}\big(\mathcal{N}\mathcal{Z}_{\Gamma;B}\big)}{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})} = \prod_{e \in \mathrm{Edg}(v_0)} \pi_e^*\left(\frac{\mathbf{e}\big(\mathcal{N}\mathcal{Z}_{\Gamma_e}\big)}{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}\right) \in \mathcal{H}^*_{\mathbb{T}}\big(\mathcal{Z}_{\Gamma;B}\big),\text{[13]} \tag{1.33}$$

and $\mathbb{L} \longrightarrow \widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}$ is the universal tangent line bundle constructed in [VaZ, Subsection 2.3]. The only property of this line bundle relevant for our purposes is

$$\int_{\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}} \tilde{\psi}^{\,|\mathrm{val}(v_0)|} = \frac{(|\mathrm{val}(v_0)|-1)!}{24} \qquad \text{if } k=1,\text{[14]} \tag{1.34}$$

where $\tilde{\psi} = c_1(\mathbb{L}^*)$ is the universal ψ-class; see [Z3, Corollary 1.2].

The final piece of localization data we need to recall from [VaZ] is that

$$\mathbf{e}(\mathcal{V}_1')\big|_{\widetilde{\mathcal{Z}}_\Gamma} = \pi_B^*\left(\frac{\mathcal{V}'_{\Gamma;B}}{\pi_1^*(L_\Gamma^* \otimes \mathcal{L}_{\mu(v_0)}) \otimes \pi_2^*\gamma^*}\right), \tag{1.35}$$

where

$$\mathcal{V}'_{\Gamma;B} = \prod_{e \in \mathrm{Edg}(v_0)} \pi_e^*\mathbf{e}(\mathcal{V}_0').\text{[15]} \tag{1.36}$$

---

[12] The blowups $\widetilde{\mathcal{M}}_{1,(I,J)}$ of $\overline{\mathcal{M}}_{1,N}$ constructed in [VaZ] are indexed by ordered partitions (I, J) of $[N]$. The only cases encountered as components of the fixed loci of $\widetilde{\mathfrak{M}}^0_{1,1}(\mathbb{P}^{n-1}, d)$ are $|J| = 0, 1$. In these two cases, the blowups are the same, as are the universal tangent line bundles $\mathbb{L}$ appearing in the following paragraph.

[13] The vector bundle $\mathcal{N}\mathcal{Z}_{\Gamma;B} \longrightarrow \mathcal{Z}_{\Gamma;B}$ is the normal bundle of $\mathcal{Z}_{\Gamma;B}$ in the moduli space $\overline{\mathfrak{M}}_{\Gamma;B}$ of $|\mathrm{Edg}(v_0)|$ tuples of genus-0 stable maps that agree at a distinguished marked point of each element of the tuple.

[14] This assumption on k implies that $|J| = 0, 1$; see footnote 12.

[15] The vector bundle $\mathcal{V}'_{\Gamma;B}$ is the analogue of the vector bundle $\mathcal{V}'_0$ for the moduli space $\overline{\mathfrak{M}}_{\Gamma;B}$; see footnote 13. It is obtained by pulling $\mathcal{L}$ back to the universal curve, then pushing down, and then taking the kernel of a natural evaluation map.

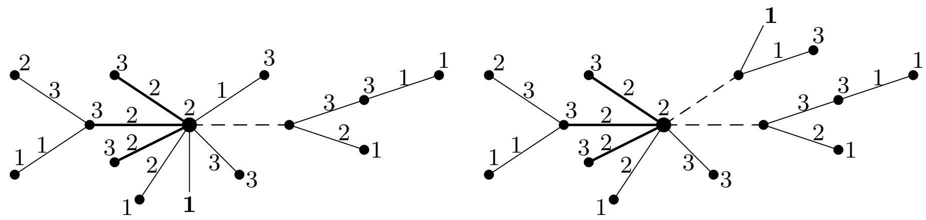

Figure 4. Refined decorated rooted trees of types B_2 and $\tilde{B}_{22}$

Remark. If $\mathfrak{d}(\Gamma)\geq 2$, the pair $(\widetilde{\mathcal{Z}}_\Gamma, \mathcal{N}\widetilde{\mathcal{Z}}_\Gamma)$ described above is precisely the fixed locus corresponding to Γ and its normal bundle as described in [VaZ, Subsection 1.4]. If $\mathfrak{d}(\Gamma)=1$, the actual fixed locus corresponding to Γ has $\mathbb{P}^{|\mathrm{Ver}_+|-2}$ instead of $\mathbb{P}^{|\mathrm{Ver}_+|-1}$ as the last factor in (1.31) and $F^c_{\Gamma;B}$ has an extra component of L_Γ. Thus, by (1.32), the expression

$$\int_{\widetilde{\mathcal{Z}}_\Gamma} \frac{(\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i)|_{\mathcal{Z}_\Gamma}}{\mathbf{e}(\mathcal{N}\widetilde{\mathcal{Z}}_\Gamma)}$$

as described above agrees with the correct one, as the extra dimension in the last factor in (1.31) is canceled by the extra factor of $c_1(\gamma^*)$ in the integrand.[16]

We now consider the refined decorated rooted trees Γ as in (1.29) that contribute to (1.16). As in Subsection 1.3, Γ does not contribute to (1.16) unless $\mu(\eta(1))=i$, i.e. the marked point is mapped to $P_i\in\mathbb{P}^{n-1}$. Similarly to Subsection 1.3, we group all graphs that contribute to (1.16) into two types: B_i and $\tilde{B}_{ij}$. In the graphs of type B_i, $\eta(1)=v_0$, i.e. the marked point 1 lies on the principal contracted component of the domain of the maps (and is mapped to P_i). In the graphs of type B_{ij}, $\eta(1)\neq v_0$, i.e. the marked point 1 lies on one of the strands of Γ, while $\mu(v_0)=j$; see Figure 4 for examples. A graph of type B_i has $m\geq 1$ strands with one marked point. On the other hand, a graph of type $\tilde{B}_{ij}$ has a distinguished strand with two marked points and $m\geq 0$ strands with one marked point. In either case, the first factor in (1.31) is $\widetilde{\mathcal{M}}_{1,m+1}$.

By (1.35), (15), and (1.13),

$$\mathbf{e}(\mathcal{V}_1)|_{\widetilde{\mathcal{Z}}_\Gamma} = n\alpha_{\mu(v_0)}\prod_{e\in\mathrm{Edg}(v_0)}\pi_e^*\mathbf{e}(\mathcal{V}_0')\Big/\big(n\alpha_{\mu(v_0)}+\psi_\Gamma+\lambda\big), \tag{1.37}$$

where $\lambda=c_1(\gamma^*)$ and we omit the pullback maps π_P^* and π_B^*. By (1.32) and (13),

$$\begin{aligned}\frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\mathbf{e}(\mathcal{N}\widetilde{\mathcal{Z}}_\Gamma)} &= -\frac{\mathbf{e}(\gamma^*\otimes L_\Gamma^*\otimes T_{\mu(v_0)}\mathbb{P}^{n-1})}{\psi_\Gamma+\tilde{\psi}+\lambda}\prod_{e\in\mathrm{Edg}(v_0)}\frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\\ &\qquad\times\prod_{e\in\mathrm{Edg}(v_0)-\mathrm{Edg}_+}\frac{1}{\psi_\Gamma-\psi_e+\lambda}.\end{aligned} \tag{1.38}$$

[16] If $\mathfrak{d}(\Gamma)=1$, L_Γ is a direct summand in $T_{\mu(v_0)}\mathbb{P}^{n-1}$. The extra factor of $c_1(\gamma^*)$ in the integrand comes from the direct summand γ^* of the bundle $\pi_1^*(L_\Gamma^*\otimes T_{\mu(v_0)}\mathbb{P}^{n-1})\otimes\pi_2^*\gamma^*$ in (1.32). This summand is canceled by the extra summand of L_Γ in $F^c_{\Gamma;B}$ in [VaZ].

Thus,

$$\frac{\mathbf{e}(\mathcal{V}_1)\mathrm{ev}_1^*\phi_i|_{\widetilde{\mathcal{Z}}_\Gamma}}{\mathbf{e}(\mathcal{N}\widetilde{\mathcal{Z}}_\Gamma)} \in \mathcal{H}^*_{\mathbb{T}}(\widetilde{\mathcal{Z}}_\Gamma)$$

splits into factors coming from the strands of Γ after integrating over the first and last factors in (1.31). These factors are the contributions of $\mathcal{Z}_{\Gamma_e}$ to integrals over $\overline{\mathfrak{M}}_{\Gamma_e}$ involving $\mathbf{e}(\mathcal{V}_0)$ and ψ_e.

Proposition 1.2. *(i) The total contribution $\mathcal{B}_i(u)$ to* (1.16) *from all graphs of type B_i is*

$$\mathcal{B}_i(u) = \frac{n\alpha_i}{24}\mathfrak{R}_{\hbar=0,\infty,-n\alpha_i}\left\{\frac{\prod_{k=1}^{k=n}(\alpha_i-\alpha_k+\hbar)}{(n\alpha_i+\hbar)\hbar^3}\frac{\mathcal{Z}^*_i(\hbar,u)}{1+\mathcal{Z}^*_i(z,u)}\right\}. \tag{1.39}$$

(ii) The total contribution $\tilde{\mathcal{B}}_{ij}(u)$ to (1.16) *from all graphs of type $\tilde{B}_{ij}$ is*

$$\tilde{\mathcal{B}}_{ij}(u) = -\frac{n\alpha_j}{24}\frac{\mathfrak{R}_{\hbar=0,\infty,-n\alpha_j}\left\{\frac{\prod_{k=1}^{k=n}(\alpha_j-\alpha_k+\hbar)}{(n\alpha_j+\hbar)\hbar^2}\frac{\mathcal{Z}^*_{ji}(\hbar,u)}{1+\mathcal{Z}^*_j(z,u)}\right\}}{\prod_{k\neq j}(\alpha_j-\alpha_k)}. \tag{1.40}$$

The proof of this proposition turns out to be quite a bit simpler than that of Proposition 1.1. At an early stage in the computation, Lemma 2.4 reduces a sum of m products of residues to the residue of a product. The resulting products sum over m to the functions of $\hbar$ appearing in the statement of Proposition 1.2. The residues of these functions on $S^2-\{0,\infty,-n\alpha_j\}$ are summed, using the Residue Theorem on S^2, to get $\mathcal{B}_i(u)$ and $\mathcal{B}_{ij}(u)$. Proposition 1.2 is proved in Subsection 2.3.

2. Localization computations

In Subsection 2.1 we give two equivalent characterizations of a property of power series in rational functions that reduces infinite summations involving certain products of residues of such power series to simple expressions. In Subsection 2.2, we dig deeper into Givental's proof of (0.4) to show that a certain generating function for genus 0 GW-invariants satisfies this property. We use these observations to prove Proposition 1.1 in Subsection 2.3, along with Proposition 1.2.

2.1. Regularizable power series in rational functions.

Definition 2.1. A power series $\mathcal{Z}^*=\mathcal{Z}^*(\hbar,u)\in\mathbb{Q}_\alpha(\hbar)[[u]]$ is regularizable at $\hbar=0$ if there exist power series

$$\eta=\eta(u)\in\mathbb{Q}_\alpha[[u]] \qquad \text{and} \qquad \bar{\mathcal{Z}}^*=\bar{\mathcal{Z}}^*(\hbar,u)\in\mathbb{Q}_\alpha(\hbar)[[u]]$$

with no degree-zero term such that $\bar{\mathcal{Z}}^*$ is regular at $\hbar=0$ and

$$1+\mathcal{Z}^*(\hbar,u) = e^{\eta(u)/\hbar}\big(1+\bar{\mathcal{Z}}^*(\hbar,u)\big). \tag{2.1}$$

If $\mathcal{Z}^*$ is regularizable at $\hbar=0$, $\mathcal{Z}^*$ has no degree-zero term and the regularizing pair $(\eta,\bar{\mathcal{Z}}^*)$ is unique. It is determined by

$$\eta(u) = \mathfrak{R}_{\hbar=0}\Big\{\ln\big(1+\mathcal{Z}^*(\hbar,u)\big)\Big\}. \tag{2.2}$$

The logarithm above is a well-defined power series in u, since $\mathcal{Z}^*(\hbar,u)$ has no degree-zero term.

Lemma 2.2. *Suppose $\mathcal{Z}^* = \mathcal{Z}^*(\hbar, u) \in \mathbb{Q}_\alpha(\hbar)[[u]]$ has no degree-zero term.*

(i) The power series $\mathcal{Z}^*$ is regularizable at $\hbar = 0$ if and only if for every $a \geq 0$

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)} \sum_{\substack{\sum_{l=1}^{l=m} a_l = m-2-a \\ a_l \geq 0}} \left(\prod_{l=1}^{l=m} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0} \Big\{ \hbar^{-a_l} \mathcal{Z}^*(\hbar, u) \Big\} \right) = a! \, \mathfrak{R}_{\hbar=0} \Big\{ \hbar^{a+1} \mathcal{Z}^*(\hbar, u) \Big\}. \tag{2.3}$$

(ii) If $(\eta, \bar{\mathcal{Z}}^*)$ is the regularizing pair for $\mathcal{Z}^*$ at $\hbar = 0$, then for every $a \geq 0$

$$\sum_{m=0}^{\infty} \sum_{\substack{\sum_{l=1}^{l=m} a_l = m-a \\ a_l \geq 0}} \left(\prod_{l=1}^{l=m} \frac{(-1)^{a_l}}{a_l!} \mathfrak{R}_{\hbar=0} \Big\{ \hbar^{-a_l} \mathcal{Z}^*(\hbar, u) \Big\} \right) = \frac{\eta(u)^a}{1 + \bar{\mathcal{Z}}^*(0, u)}. \tag{2.4}$$

Note that the sums on the left-hand sides of (2.3) and (2.4) are finite in each u-degree, since $\mathcal{Z}^*$ has no degree-zero term.

Suppose $(\eta, \bar{\mathcal{Z}}^*)$ is the regularizing pair for $\mathcal{Z}^*$ at $\hbar = 0$. Let

$$\bar{\mathcal{Z}}^*(\hbar, u) = \sum_{q=0}^{\infty} C_q(u) \hbar^q \tag{2.5}$$

be the Taylor series expansion for $\bar{\mathcal{Z}}^*(\hbar, u)$ at $\hbar = 0$. If $a \in \mathbb{Z}$, then

$$\mathfrak{R}_{\hbar=0} \big\{ \hbar^a \mathcal{Z}^*(\hbar, u) \big\} = \sum_{\substack{p-q=1+a \\ p,q \geq 0}} \frac{\eta(u)^p}{p!} C_q(u) + \begin{cases} \frac{\eta(u)^{a+1}}{(a+1)!}, & \text{if } a \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

The identities (2.3) and (2.4) follow from (2.6) by a fairly direct computation; see Appendix A.

It remains to show that if $\mathcal{Z}^*$ satisfies (2.3) for all $a \geq 0$, then $\mathcal{Z}^*$ admits a regularization. Since $\mathcal{Z}^* \in \mathbb{Q}_\alpha(\hbar)[[u]]$, we can expand $\mathcal{Z}^*$ at $\hbar = 0$ as

$$\mathcal{Z}^*(\hbar, u) = \sum_{d=1}^{\infty} \sum_{q=-N_d}^{\infty} \tilde{C}_{q;d} \hbar^q u^d \equiv \sum_{q \in \mathbb{Z}} \tilde{C}_q(u) \hbar^q,$$

where

$$\tilde{C}_q(u) = \mathfrak{R}_{\hbar=0} \Big\{ \hbar^{-q-1} \mathcal{Z}^*(\hbar, u) \Big\}.$$

Claim. There exists $\eta \in \mathbb{Q}_\alpha[[u]]$ such that

$$\mathfrak{R}_{\hbar=0} \bar{\mathcal{Z}}^*(\hbar, u) = 0, \qquad \text{where} \qquad 1 + \mathcal{Z}^*(\hbar, u) = e^{\eta(u)/\hbar} \big(1 + \bar{\mathcal{Z}}^*(\hbar, u) \big).$$

Since $\bar{\mathcal{Z}}^* \in \mathbb{Q}_\alpha(\hbar)[[u]]$, we can expand $\bar{\mathcal{Z}}^*$ at $\hbar = 0$ as

$$\bar{\mathcal{Z}}^*(\hbar, u) = \sum_{d=1}^{\infty} \sum_{q=-\bar{N}_d}^{\infty} C_{q;d} \hbar^q u^d \equiv \sum_{q \in \mathbb{Z}} C_q(u) \hbar^q,$$

where

$$C_q(u) = \mathfrak{R}_{\hbar=0} \Big\{ \hbar^{-q-1} \bar{\mathcal{Z}}^*(\hbar, u) \Big\}.$$

By assumption on η, $C_{-1}(u)=0$. Let

$$\bar{\mathcal{Y}}^*(\hbar,u)=\sum_{q=0}^{\infty}C_q(u)\hbar^q\in\mathbb{Q}_\alpha(\hbar)[[u]],\qquad 1+\mathcal{Y}^*(\hbar,u)=e^{\eta(u)/\hbar}\big(1+\bar{\mathcal{Y}}^*(\hbar,u)\big).$$

Since $C_{-1}(u)=0$,

$$\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a}\mathcal{Y}^*(\hbar,u)\Big\}=\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a}\mathcal{Z}^*(\hbar,u)\Big\}\qquad\forall\,a\ge 0. \tag{2.7}$$

Since $\bar{\mathcal{Y}}^*$ is holomorphic at $\hbar=0$, $\mathcal{Y}^*$ satisfies (2.3), with $\mathcal{Z}^*$ replaced by $\mathcal{Y}^*$. Thus, for all $a\ge 0$

$$\begin{aligned}
a!\,\mathfrak{R}_{\hbar=0}&\Big\{\hbar^{a+1}\mathcal{Y}^*(\hbar,u)\Big\}\\
&=\sum_{m=2}^{\infty}\frac{1}{m(m-1)}\sum_{\substack{\sum_{l=1}^{l=m}a_l=m-2-a\\ a_l\ge 0}}\left(\prod_{l=1}^{l=m}\frac{(-1)^{a_l}}{a_l!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_l}\mathcal{Y}^*(\hbar,u)\Big\}\right)\\
&=\sum_{m=2}^{\infty}\frac{1}{m(m-1)}\sum_{\substack{\sum_{l=1}^{l=m}a_l=m-2-a\\ a_l\ge 0}}\left(\prod_{l=1}^{l=m}\frac{(-1)^{a_l}}{a_l!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_l}\mathcal{Z}^*(\hbar,u)\Big\}\right)\\
&=a!\,\mathfrak{R}_{\hbar=0}\Big\{\hbar^{a+1}\mathcal{Z}^*(\hbar,u)\Big\}=a!\tilde{C}_{-a-2}.
\end{aligned}$$

Along with (2.7), this implies that $\mathcal{Z}^*=\mathcal{Y}^*$. Thus, $\bar{\mathcal{Z}}^*=\bar{\mathcal{Y}}^*$ is holomorphic at $\hbar=0$.

Proof of the Claim. The required property of η is equivalent to

$$\eta=\sum_{q=0}^{\infty}\frac{(-\eta)^q}{q!}\tilde{C}_{q-1}.$$

Since $\tilde{C}_q\in\mathbb{Q}_\alpha[[u]]$ has no degree-zero term, this equation has a unique power-series solution $\eta\in\mathbb{Q}_\alpha[[u]]$ with no degree-zero term.[17] $\square$

Remark. The identity (2.4) is valid as long as the residue of $\bar{\mathcal{Z}}^*$ at $\hbar$ vanishes, but $\bar{\mathcal{Z}}^*$ is not necessarily holomorphic at $\hbar=0$. In such a case, $\bar{\mathcal{Z}}^*(0,u)$ must be replaced by $\mathfrak{R}_{\hbar=0}\{\hbar^{-1}\bar{\mathcal{Z}}^*(\hbar,u)\}$ on the right-hand side of (2.4). By the proof of the claim, this residue is completely determined by $\mathcal{Z}^*$. The assumption that $\mathcal{Z}^*$ is regularizable at $\hbar=0$ allows us to compute the sum in (2.4) explicitly.

2.2. **Regularizability of GW generating functions.**

Lemma 2.3. *The power series $\mathcal{Z}_i^*=\mathcal{Z}_i^*(\hbar,u)\in\mathbb{Q}_\alpha(\hbar)[[u]]$ defined in (1.22) is regularizable at $\hbar=0$.*

Proof. We will verify that $\mathcal{Z}^*\equiv\mathcal{Z}_i^*$ satisfies (2.3) for all $a\ge 0$. By the string relation (see [MirSym, Section 26.3]),

$$\mathcal{Z}_i'^*(\hbar,u)\equiv\int_{\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^{n-1},d)}\frac{\mathbf{e}(\mathcal{V}_0')}{\hbar-\psi_1}\mathrm{ev}_1^*\phi_i=\hbar\,\mathcal{Z}_i^*(\hbar,u). \tag{2.8}$$

[17] Take $\eta_0=\tilde{C}_{-1}$, $\eta_p=\sum_{q=0}^{p}\frac{(-\eta_{p-1})^q}{q!}\tilde{C}_{q-1}$ for $p\ge 1$. The sequence $\eta_0,\eta_1,\ldots\in\mathbb{Q}_\alpha[[u]]$ converges, since it is constant in degree d after the d-th term.

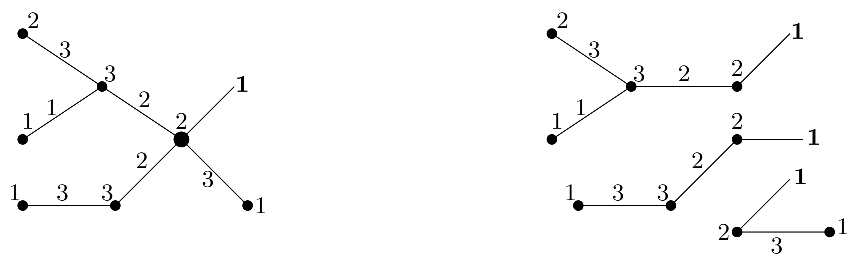

FIGURE 5. A decorated tree contributing to $Q_i(\hbar,u)$, with $i=2$, and its strands

By the same argument as in the proof of [MirSym, Lemma 30.11],

$$\mathcal{Z}_i'^*(\hbar,u) = Q_i(\hbar,u) + \sum_{d=1}^{\infty}\sum_{j\neq i}\frac{1}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\big\{\mathcal{Z}_i'^*(\hbar,u)\big\} \tag{2.9}$$
$$\text{for some}\qquad Q_i\in\mathbb{Q}_\alpha[\hbar,\hbar^{-1}]\big[\big[u\big]\big].$$[18]

The middle term in (2.9) is the sum of contributions to $\mathcal{Z}_i'^*(\hbar,u)$ from the graphs Γ with one marked point such that the marked point is attached to a vertex v_0 of valence at least 3. Similarly to Subsections 1.3 and 1.4, the vertex v_0 of Γ must then be labeled i. An example of such a graph is shown in Figure 5. $\square$

Let Γ be a decorated tree with one marked point as in (1.17) that contributes to $Q_i(\hbar,u)$, i.e.

$$k=1,\quad \mu(v_0)=i,\quad \mathrm{val}(v_0)\geq 3,\qquad \text{where}\quad v_0=\eta(1).$$

As in Subsections 1.3 and 1.4, we break Γ into strands Γ_v indexed by the set

$$\mathrm{Edg}(v_0)=\big\{e\in\mathrm{Edg}\colon v_0\in e\big\}$$

of the edges leaving from v_0. In this case, there are

$$m\equiv\big|\mathrm{Edg}(v_0)\big|\geq 2$$

strands, each with exactly one marked point.

The fixed locus $\mathcal{Z}_\Gamma$ corresponding to Γ, the restriction of $\mathbf{e}(\mathcal{V}_0')$ to $\mathcal{Z}_\Gamma$, and the euler class of the normal bundle of $\mathcal{Z}_\Gamma$ are given by

$$\begin{gathered}\mathcal{Z}_\Gamma=\overline{\mathcal{M}}_{0,|\mathrm{val}(v_0)|}\times\prod_{e\in\mathrm{Edg}(v_0)}\mathcal{Z}_{\Gamma_e},\qquad \mathbf{e}(\mathcal{V}_0')=\prod_{e\in\mathrm{Edg}(v_0)}\pi_e^*\mathbf{e}(\mathcal{V}_0'),\\ \frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}=\prod_{e\in\mathrm{Edg}(v_0)}\bigg(\frac{\mathbf{e}(T_{\mu(v_0)}\mathbb{P}^{n-1})}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})\,(\hbar_e-\pi_e^*\psi_1)}\bigg),\end{gathered} \tag{2.10}$$

where $\hbar_e\equiv c_1(L_e')\in H^*(\overline{\mathcal{M}}_{0,|\mathrm{val}(v_0)|})$ is the first chern class of the universal tangent line bundle for the marked point corresponding to the edge e. By [MirSym, Section 27.2], if $e=\{v_0,v_e\}$, then

$$\psi_1|_{\mathcal{Z}_{\Gamma_e}}=\frac{\alpha_{\mu(v_e)}-\alpha_{\mu(v_0)}}{\mathfrak{d}(e)}=\frac{\alpha_{\mu(v_e)}-\alpha_i}{\mathfrak{d}(e)}. \tag{2.11}$$

[18]The statement of [MirSym, Lemma 30.11] is made for a renormalized version of the power series $\mathcal{Z}_i^*(\hbar,u)$ and is in fact sharper.

Thus, by (2.10) and (1.8),

$$\begin{aligned}
\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_1)\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)} &= \sum_{a=0}^{\infty}\hbar^{-(a+1)}(-1)^m\sum_{\substack{a_e\ge0\\ e\in\mathrm{Edg}(v_0)}}\Bigg\{\int_{\overline{\mathcal{M}}_{0,|\mathrm{val}(v_0)|}}\psi_1^a\prod_{e\in\mathrm{Edg}(v_0)}\hbar_e^{a_e}\\
&\qquad\times\prod_{e\in\mathrm{Edg}(v_0)}\Bigg(\bigg(\frac{\alpha_{\mu(v_e)}-\alpha_i}{\mathfrak{d}(e)}\bigg)^{-(a_e+1)}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\Bigg)\Bigg\}\\
&= \sum_{\substack{a+\sum_{e\in\mathrm{Edg}(v_0)}a_e=m-2\\ a_e\ge0}}\binom{m-2}{a,(a_e)_{e\in\mathrm{Edg}(v_0)}}(-1)^a\hbar^{-(a+1)}\\
&\qquad\times\prod_{e\in\mathrm{Edg}(v_0)}\Bigg(\bigg(\frac{\alpha_{\mu(v_e)}-\alpha_i}{\mathfrak{d}(e)}\bigg)^{-(a_e+1)}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\Bigg).
\end{aligned}\tag{2.12}$$

The first equality holds after dividing the expressions on the right-hand side by the order of the appropriate groups of symmetries; see [MirSym, Section 27.3]. This group is taken into account in the next paragraph.

We now sum up (2.12) over all possibilities for Γ. By (2.9) and its proof, for every $j\in[n]-i$ and $d\in\mathbb{Z}^+$,

$$\begin{aligned}
\sum_{\substack{\Gamma_e\\ \mu(v_e)=j,\mathfrak{d}(e)=d}}\bigg(\frac{\alpha_j-\alpha_i}{d}\bigg)^{-(a_e+1)}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})} &= \mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\Big\{\hbar^{-(a_e+1)}\mathcal{Z}_i'^*(\hbar,u)\Big\}\\
&= \mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\Big\{\hbar^{-a_e}\mathcal{Z}_i^*(\hbar,u)\Big\}.^{19}
\end{aligned}\tag{2.13}$$

Since $\hbar^{-a_e}\mathcal{Z}_i^*(\hbar,u)$ has no residue at $\hbar=\infty$ by Lemma 3.4,

$$\begin{aligned}
\sum_{\Gamma_e}\bigg(\frac{\alpha_j-\alpha_i}{d}\bigg)^{-(a_e+1)}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})} &= \sum_{d=1}^{\infty}\sum_{j\neq i}\mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\Big\{\hbar^{-a_e}\mathcal{Z}_i^*(\hbar,u)\Big\}\\
&= -\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_e}\mathcal{Z}_i^*(\hbar,u)\Big\}
\end{aligned}\tag{2.14}$$

by (2.13), the Residue Theorem on S^2, and Lemma 3.4. By (2.12) and (2.14),

$$\begin{aligned}
\sum_{\substack{\Gamma\\ |\mathrm{Edg}(v_0)|=m}}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)} &= \sum_{a=0}^{\infty}\Bigg\{\hbar^{-(a+1)}(-1)^{m-a}\sum_{\substack{\sum_{e\in\mathrm{Edg}(v_0)}a_e=m-2-a\\ a_e\ge0}}\Bigg\{\\
&\qquad\binom{m-2}{a,(a_e)_{e\in\mathrm{Edg}(v_0)}}\prod_{e\in\mathrm{Edg}(v_0)}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_e}\mathcal{Z}_i^*(\hbar,u)\Big\}\Bigg\}\Bigg\}.
\end{aligned}\tag{2.15}$$

---

[19] The proof is the same as the proof of (3.21) in [MirSym, Chapter 30]. If a graph Γ contributes a factor of $\hbar-(\alpha_j-\alpha_i)/d$ to the denominator of $\mathcal{Z}_i'^*$ (or of $\hbar^{-(a_e+1)}\mathcal{Z}_i'^*$), then the marked point 1 is attached to a vertex of Γ of valence 2. Furthermore, if e is the unique element of $\mathrm{Edg}(v_0)$, then $\mu(v_e)=j$ and $\mathfrak{d}(e)=d$; see [MirSym, Section 30.1] for more details. The second equality in (2.13) is immediate from (2.8).

Taking into account the group of symmetries, i.e. S_m in the case of (2.15), and summing up over all possible values of $|\mathrm{Edg}(v_0)|$, i.e. $m\geq 2$, we obtain

$$\begin{aligned}Q_i(\hbar,u) = \sum_{a=0}^{\infty}\frac{\hbar^{-(a+1)}}{a!}\sum_{m=2}^{\infty}\Bigg\{\frac{1}{m(m-1)}\sum_{\substack{\sum_{l=1}^{l=m} a_l=m-2-a\\ a_l\geq 0}}\Bigg\{\\ \prod_{l=1}^{l=m}\frac{(-1)^{a_l}}{a_l!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_l}\mathcal{Z}_i^*(\hbar,u)\Big\}\Bigg\}\Bigg\}.\end{aligned} \tag{2.16}$$

On the other hand, by the Residue Theorem on S^2, Lemma 3.4, and (2.8),

$$\begin{aligned}\sum_{d=1}^{\infty}\sum_{j\neq i}\frac{1}{\hbar-\frac{\alpha_j-\alpha_i}{d}}\mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\big\{\mathcal{Z}_i'^*(\hbar,u)\big\} &= \sum_{d=1}^{\infty}\sum_{j\neq i}\mathfrak{R}_{z=\frac{\alpha_j-\alpha_i}{d}}\Big\{\frac{1}{\hbar-z}\mathcal{Z}_i'^*(z,u)\Big\}\\ &= -\mathfrak{R}_{z=\hbar,0}\Big\{\frac{1}{\hbar-z}\mathcal{Z}_i'^*(z,u)\Big\}\\ &= \mathcal{Z}_i'^*(\hbar,u)-\sum_{a=0}^{\infty}\hbar^{-(a+1)}\mathfrak{R}_{z=0}\Big\{z^a\mathcal{Z}_i'^*(z,u)\Big\}\\ &= \mathcal{Z}_i'^*(\hbar,u)-\sum_{a=0}^{\infty}\hbar^{-(a+1)}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{a+1}\mathcal{Z}_i^*(\hbar,u)\Big\}.\end{aligned}$$

Comparing with (2.9), we conclude that

$$Q_i(\hbar,u) = \sum_{a=0}^{\infty}\hbar^{-(a+1)}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{a+1}\mathcal{Z}_i^*(\hbar,u)\Big\}. \tag{2.17}$$

By (2.16) and (2.17), $\mathcal{Z}_i^*$ satisfies (2.3) for all $a\geq 0$.

2.3. **Proofs of Propositions 1.1 and 1.2.** In this subsection we prove Propositions 1.1 and 1.2. An argument nearly identical to part of the proof of Lemma 2.2 leads to a long expression like (2.12). In the proof of the first proposition, the Residue Theorem on S^2 reduces it to the form (2.15). We then use the second statement of Lemma 2.2 to deal with the infinite summation. The situation in Proposition 1.2 is a bit different, as the possible strands Γ_e of Γ are not mutually independent due to the requirements (i) and (ii) on Γ in Subsection 1.4. In this case, we will use Lemma 2.4 to reduce a product of residues to the residue of a product; the resulting products are readily summable. The Residue Theorem on S^2 is used at the last step. As the proof of Lemma 2.4 is completely straightforward, we relegate it to Appendix A.

If $f=f(\lambda)$ is holomorphic at $\lambda=0$ and $m\geq 0$, let

$$\mathcal{D}_\lambda^m f = \frac{1}{m!}\frac{d^m}{d\lambda^m}f(\lambda)\Big|_{\lambda=0}.$$

Lemma 2.4. *If* $(f_e = f_e(\lambda))_{e\in E}$ *is a finite collection of functions with at most a simple pole at* $\lambda=0$*, then*

$$\begin{aligned}\mathfrak{R}_{\lambda=0}\Big\{\prod_{e\in E} f_e(\lambda)\Big\} \\ = \sum_{E_+\subset E}\Bigg\{\prod_{e\in E_+}\mathfrak{R}_{\lambda=0}\big\{f_e(\lambda)\big\}\cdot \mathcal{D}_\lambda^{|E_+|-1}\bigg(\prod_{e\notin E_+}\Big(f_e(\lambda)-\lambda^{-1}\mathfrak{R}_{\lambda=0}\big\{f_e(\lambda)\big\}\Big)\bigg)\Bigg\}.\end{aligned}$$

In the case of (i) of Proposition 1.1, equations (1.20) and (1.21) describe the splitting of the integrand corresponding to each fixed locus $\mathcal{Z}_\Gamma$ in the sense of (1.9) between the strands of Γ. Summing over all possible strands as in Subsection 2.2, we find that $\mathcal{A}_i(u)$ is given by

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{\substack{a_-+a_++\sum_{l=1}^{l=m}a_l=m\\ a_\pm,a_l\ge 0}}\Bigg\{\frac{(-1)^{a_-+a_+}}{a_-!a_+!}\mathfrak{R}_{\hbar_1=0}\mathfrak{R}_{\hbar_2=0}\Big\{\hbar_1^{-a_-}\hbar_2^{-a_+}\tilde{\mathcal{Z}}_{ii}^*(\hbar_1,\hbar_2,u)\Big\} \\ \times\prod_{l=1}^{l=m}\frac{(-1)^{a_l}}{a_l!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_l}\mathcal{Z}_i^*(\hbar,u)\Big\}\Bigg\}.\end{aligned}\tag{2.18}$$

This is the analogue of (2.16). In this case, we use Lemma 3.6, in addition to Lemma 3.4 and the Residue Theorem on S^2, to obtain (2.18) from the analogue of (2.12) for $\mathcal{A}_i(u)$. The sum is taken over every possible number m of strands with one marked point. The ends of the distinguished strand $\Gamma_\pm$ are ordered, accounting for the factor of 1/2 in (1.24). By Lemma 2.3, $\mathcal{Z}_i^*$ is regularizable at $\hbar=0$. Therefore, (ii) of Lemma 2.2 reduces the right-hand side of (2.18) to the right-hand side of (1.27).

In the case of (ii) of Proposition 1.1, the analogue of (2.18) is easily seen to be

$$\begin{aligned}\tilde{A}_{ij}(u) = \frac{1}{\prod_{k\neq j}(\alpha_j-\alpha_k)} \\ \times\sum_{m=0}^{\infty}\sum_{\substack{a_-+a_++a_0+\sum_{l=1}^{l=m}a_l=m\\ a_0,a_\pm,a_l\ge 0}}\Bigg\{\frac{(-1)^{a_-+a_+}}{a_-!a_+!}\mathfrak{R}_{\hbar_1=0}\mathfrak{R}_{\hbar_2=0}\Big\{\hbar_1^{-a_-}\hbar_2^{-a_+}\tilde{\mathcal{Z}}_{jj}^*(\hbar_1,\hbar_2,u)\Big\} \\ \times\frac{(-1)^{a_0}}{a_0!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_0}\mathcal{Z}_{ji}^*(\hbar,u)\Big\}\cdot\prod_{l=1}^{l=m}\frac{(-1)^{a_l}}{a_l!}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-a_l}\mathcal{Z}_j^*(\hbar,u)\Big\}\Bigg\}.\end{aligned}$$

In this case, Lemma 3.5 is used in addition to Lemmas 3.4 and 3.6 and the Residue Theorem on S^2. Lemmas 2.2 and 2.3 reduce the right-hand side of the above expression to the right-hand side of (1.28).

In the case of (i) of Proposition 1.2, (1.37) and (1.38) describe the splitting of the integrand for the fixed locus $\widetilde{\mathcal{Z}}_\Gamma$ between the strands of Γ. Let

$$\mathrm{Edg}_- = \mathrm{Edg}(v_0)-\mathrm{Edg}_+, \qquad m_+ = \big|\mathrm{Edg}_+\big|, \qquad \Psi(\hbar,\tilde{\psi}) = -\frac{\prod_{k\neq i}(\alpha_i-\alpha_k+\hbar)}{(n\alpha_i+\hbar)(\hbar+\tilde{\psi})}\,.$$

The analogue of (2.12) is then

$$\int_{\widetilde{\mathcal{Z}}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_1')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\widetilde{\mathcal{Z}}_\Gamma)}=\int_{\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}\times\mathbb{P}^{m_+-1}}\Bigg\{\Psi(\hbar,\tilde{\psi})\prod_{e\in\mathrm{Edg}_+}\bigg(\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\bigg)\times\prod_{e\in\mathrm{Edg}_-}\bigg(\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_e)\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\bigg)\Bigg\}_{\hbar=\psi_\Gamma+\lambda}. \tag{2.19}$$

We now sum (2.19) over all possibilities for Γ_e with $e\in\mathrm{Edg}_-$. In contrast to the three cases encountered above, Γ_e can be any graph with one marked point such that $\mu(1)=i$, as long as the edge leaving the vertex $\mu(1)$ is not labeled $\mathfrak{d}(\Gamma)$ or its other end is not labeled $\mu(\Gamma)$. This restriction is due to (1.30). By (1.9) applied to the function $\mathcal{Z}_i'^*(\hbar,u)$ defined in (2.8), the sum over such graphs Γ_e is

$$\begin{aligned}\sum_{\substack{\Gamma_e\\(\mu(v_e),\mathfrak{d}(e))\neq(\mu(\Gamma),\mathfrak{d}(\Gamma))}}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_e)\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}&=\mathcal{Z}_i'^*(\hbar,u)-\sum_{\substack{\Gamma_e\\(\mu(v_e),\mathfrak{d}(e))=(\mu(\Gamma),\mathfrak{d}(\Gamma))}}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{(\hbar-\psi_e)\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}\\&=\mathcal{Z}_i'^*(\hbar,u)-\frac{1}{\hbar-\psi_\Gamma}\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\},\end{aligned}\tag{2.20}$$

since $\psi_\Gamma=(\alpha_{\mu(\Gamma)}-\alpha_i)/\mathfrak{d}(\Gamma)$. The last equality uses (3.21) and (2.8). On the other hand, summing over all possibilities for Γ_e with $e\in\mathrm{Edg}_+$, without changing $(\mu(\Gamma),\mathfrak{d}(\Gamma))$, we obtain

$$\sum_{\substack{\Gamma_e\\(\mu(v_e),\mathfrak{d}(e))=(\mu(\Gamma),\mathfrak{d}(\Gamma))}}\int_{\mathcal{Z}_{\Gamma_e}}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_{\Gamma_e})}=\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\},\tag{2.21}$$

also by (3.21) and (2.8).

By (2.20) and (2.21), the sum of the terms in (2.19) with $\mathrm{Edg}(v_0)$, Edg_+, and $(\mu(\Gamma),\mathfrak{d}(\Gamma))$ fixed is

$$\begin{aligned}&\int_{\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}\times\mathbb{P}^{m_+-1}}\Bigg\{\prod_{e\in\mathrm{Edg}_+}\Big(\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\}\Big)\Psi(\hbar,\tilde{\psi})\\&\qquad\times\prod_{e\in\mathrm{Edg}_-}\Big(\mathcal{Z}_i'^*(\hbar,u)-\frac{1}{\hbar-\psi_\Gamma}\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\}\Big)\Bigg\}_{\hbar=\psi_\Gamma+\lambda}\\&=\int_{\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}}\Bigg\{\prod_{e\in\mathrm{Edg}_+}\Big(\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\}\Big)\\&\qquad\times\mathcal{D}_\lambda^{m_+-1}\bigg(\Psi(\psi_\Gamma+\lambda,\tilde{\psi})\prod_{e\in\mathrm{Edg}_-}\Big(\mathcal{Z}_i'^*(\psi_\Gamma+\lambda,u)-\lambda^{-1}\mathfrak{R}_{z=\psi_\Gamma}\big\{\mathcal{Z}_i'^*(z,u)\big\}\Big)\bigg)\Bigg\}.\end{aligned}\tag{2.22}$$

By the last expression in (2.22) and Lemmas 2.4 and 3.4, the sum of the terms in (2.19) with only $m\equiv|\mathrm{Edg}(v_0)|$ and $(\mu(\Gamma),\mathfrak{d}(\Gamma))$ fixed is

$$\begin{aligned}
&\int_{\widetilde{\mathcal{M}}_{1,|\mathrm{val}(v_0)|}} \mathfrak{R}_{z=\psi_\Gamma}\Bigg(\Psi(z,\tilde{\psi})\prod_{e\in\mathrm{Edg}(v_0)}\mathcal{Z}_i'^{*}(z,u)\Bigg)\\
(2.23)\qquad &= \frac{(-1)^m m!}{24}\mathfrak{R}_{z=\frac{\alpha_{\mu(\Gamma)}-\alpha_i}{\mathfrak{d}(\Gamma)}}\left\{z^{-(m+2)}\frac{\prod_{k\neq i}(\alpha_i-\alpha_k+z)}{(n\alpha_i+z)}\mathcal{Z}_i'^{*}(z,u)^m\right\}\\
&= \frac{(-1)^m m!}{24}\mathfrak{R}_{z=\frac{\alpha_{\mu(\Gamma)}-\alpha_i}{\mathfrak{d}(\Gamma)}}\left\{\frac{\prod_{k\neq i}(\alpha_i-\alpha_k+z)}{(n\alpha_i+z)z^2}\mathcal{Z}_i^{*}(z,u)^m\right\}.
\end{aligned}$$

The first equality above follows from (14), while the second one follows from (2.8).

Finally, taking into account (1.13) and the group of symmetries, i.e. S_m, and summing (2.23) over all possible numbers of one-pointed strands, i.e. $m\geq 1$, and all possible values of $(\mu(\Gamma),\mathfrak{d}(\Gamma))$, we obtain

$$\begin{aligned}
\mathcal{B}_i(u) &= \frac{n\alpha_i}{24}\sum_{j\neq i}\sum_{d=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\left\{\frac{\prod_{k\neq i}(\alpha_i-\alpha_k+\hbar)}{(n\alpha_i+\hbar)\hbar^2}\mathcal{Z}_i^{*}(\hbar,u)^m\right\}\\
(2.24)\qquad &= -\frac{n\alpha_i}{24}\sum_{j\neq i}\sum_{d=1}^{\infty}\mathfrak{R}_{\hbar=\frac{\alpha_j-\alpha_i}{d}}\left\{\frac{\prod_{k=1}^{k=n}(\alpha_i-\alpha_k+\hbar)}{(n\alpha_i+\hbar)\hbar^3}\frac{\mathcal{Z}_i^{*}(\hbar,u)}{1+\mathcal{Z}_i^{*}(\hbar,u)}\right\}.
\end{aligned}$$

The first claim of Proposition 1.2 now follows from the Residue Theorem on S^2 and Lemma 3.4.

The proof of (ii) of Proposition 1.2 is nearly identical. In the starting equation (2.19), i is replaced by j and the entire expression is divided by $\prod_{k\neq j}(\alpha_j-\alpha_k)$. One of the strands now has two marked points, and there are $m\geq 0$ other strands. The two-pointed strand can appear as an element of Edg_+ as well as of Edg_- and contributes to $\mathcal{Z}_{ji}^{*}(\hbar,u)$ instead of $\mathcal{Z}_j^{*}(\hbar,u)$. The number $|\mathrm{val}(v_0)|$ is still $m+1$. Therefore, (2.24) becomes

$$\begin{aligned}
\tilde{\mathcal{B}}_{ij}(u) &= \frac{1}{\prod\limits_{k\neq j}(\alpha_j-\alpha_k)}\frac{n\alpha_j}{24}\\
&\quad\times\sum_{l\neq j}\sum_{d=1}^{\infty}\sum_{m=0}^{\infty}\left\{(-1)^m\mathfrak{R}_{\hbar=\frac{\alpha_l-\alpha_j}{d}}\left\{\frac{\prod_{k\neq j}(\alpha_j-\alpha_k+\hbar)}{(n\alpha_j+\hbar)\hbar}\mathcal{Z}_j^{*}(\hbar,u)^m\mathcal{Z}_{ji}^{*}(\hbar,u)\right\}\right\}\\
&= \frac{1}{\prod\limits_{k\neq j}(\alpha_j-\alpha_k)}\frac{n\alpha_j}{24}\sum_{l\neq j}\sum_{d=1}^{\infty}\mathfrak{R}_{\hbar=\frac{\alpha_l-\alpha_j}{d}}\left\{\frac{\prod_{k=1}^{k=n}(\alpha_j-\alpha_k+\hbar)}{(n\alpha_j+\hbar)\hbar^2}\frac{\mathcal{Z}_{ji}^{*}(\hbar,u)}{1+\mathcal{Z}_j^{*}(\hbar,u)}\right\}.
\end{aligned}$$

The second claim of Proposition 1.2 now follows from the Residue Theorem, along with Lemmas 3.4 and 3.5.

Remark. In (i) and (ii) of Proposition 1.1, $|\mathrm{val}(v_0)|=m+3$, where m is the number of one-pointed strands. Of the extra 3, 2 comes from the distinguished strand $\Gamma_\pm$. The remaining 1 comes from the marked point 1 that lies on the contracted component $\mathcal{C}_{f,v_0}$ corresponding to the vertex v_0 in (i) and from the second distinguished strand Γ_0 in (ii). In (i) and (ii) of Proposition 1.2, $|\mathrm{val}(v_0)|=m+1$, where m is again the number of one-pointed strands. The extra 1 comes from the marked point 1 that lies on $\mathcal{C}_{f,v_0}$ in (i) and from the two-pointed strand in (ii).

3. Algebraic computations

In this section we use Lemmas 3.3-3.6 to deduce the main theorem of this paper, Theorem 3 below, from Propositions 1.1, 1.2, 3.1, and 3.2. Theorem 3 expresses the contribution from each of the two types of $\mathbb{T}$-fixed loci in $\widetilde{\mathfrak{M}}_{1,1}^0(\mathbb{P}^{n-1},d)$ to the generating function $\mathcal{F}_0(u)$ for the reduced genus 1 GW-invariants of a degree n hypersurface in $\mathbb{P}^{n-1}$ in terms of hypergeometric series. Along with (1.15), it immediately implies Theorem 2. Propositions 3.1 and 3.2 describe the structure of the function $R=R(w,t)$ defined in (2.6) at $w=0$ and $w=\infty$, respectively; they are proved in [ZaZ]. Lemma 3.3 serves as a tool for extracting the non-equivariant part of an equivariant cohomology class, while Lemmas 3.4-3.6 recall the relevant information about genus 0 generating functions. With R as in (0.15), let

$$\mu(e^t) \equiv \mathfrak{R}_{\hbar=0}\big\{ \ln R(\hbar^{-1},t)\big\} - t. \tag{3.1}$$

Theorem 3. *The generating function $\mathcal{F}_0(u)$ defined in (1.14) is given by*

$$\mathcal{F}_0(e^T) = \frac{d}{dT}\big(\tilde{A}(e^t)+\tilde{B}(e^t)\big),$$

where T and t are related by the mirror map (0.17), and

$$\begin{aligned}
\tilde{A}(e^t) &= \frac{1}{2}\Bigg(\frac{(n-2)(n+1)}{24}\mu(e^t) - \frac{(n-2)(3n-5)}{24}\ln\big(1-n^ne^t\big) \\
&\qquad\qquad - \sum_{p=0}^{n-3}\binom{n-1-p}{2}\ln I_{p,p}(t)\Bigg) \\
&= \frac{(n-2)(n+1)}{48}\mu(e^t) \\
&\quad - \begin{cases} \frac{n+1}{48}\ln\big(1-n^ne^t\big) + \sum_{p=0}^{(n-3)/2}\frac{(n-1-2p)^2}{8}\ln I_{p,p}(t), & \text{if } 2\nmid n, \\ \frac{n-2}{48}\ln\big(1-n^ne^t\big) + \sum_{p=0}^{(n-4)/2}\frac{(n-2p)(n-2-2p)}{8}\ln I_{p,p}(t), & \text{if } 2|n, \end{cases}
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\tilde{B}(e^t) &= \bigg(\frac{(n-2)(n+1)}{48} + \frac{1-(1-n)^n}{24n^2}\bigg)(T-t) - \frac{(n-2)(n+1)}{48}\mu(e^t) \\
&\quad + \frac{1}{24}\ln\big(1-n^ne^t\big) + \frac{n^2-1+(1-n)^n}{24n}\ln I_{0,0}(t) \\
&\quad + \frac{n}{24}\sum_{p=2}^{n-2}\bigg(\mathcal{D}_w^{n-2-p}\frac{(1+w)^n}{(1+nw)}\bigg)\big(\mathcal{D}_w^p\ln\bar{R}(w,t)\big)
\end{aligned} \tag{3.3}$$

are T-integrals of the contributions of the effective fixed loci of Subsection 1.3 and of the boundary fixed loci of Subsection 1.4, respectively.

The next two propositions, which are proved in [ZaZ], are used in the proof of Theorem 3 in Subsection 3.3.

Proposition 3.1. *(i) There exist $\tilde{I}_{p,r}\in\mathbb{Q}[[e^t]]$ for $p,r\in\bar{\mathbb{Z}}^+$ with $r\geq p$ such that*

$$I_{p,q}(t) = \sum_{r=p}^{r=q}\frac{t^{q-r}}{(q-r)!}\tilde{I}_{p,r}(t) \qquad \forall\ p,q\in\bar{\mathbb{Z}}^+ \text{ with } q\geq p$$

and the constant term of $\tilde{I}_{p,r}$ is 1 for $r=p$ and is 0 for $r>p$.

(ii) The power series $I_{p,p}$ in e^t with $p=0,1,\ldots,n-1$ satisfy

$$I_{0,0}(t)I_{1,1}(t)\ldots I_{n-1,n-1}(t)\big(1-n^ne^t\big)=1, \tag{3.4}$$

$$I_{0,0}(t)^{n-1}I_{1,1}(t)^{n-2}\ldots I_{n-1,n-1}(t)^0\big(1-n^ne^t\big)^{(n-1)/2}=1, \tag{3.5}$$

$$I_{p,p}(t)=I_{n-1-p,n-1-p}(t) \qquad \forall\ p=0,1,\ldots,n-1. \tag{3.6}$$

Proposition 3.2. *For all $n\in\mathbb{Z}^+$,*

$$\mu(e^t)=\int_0^{e^t}\frac{(1-n^nu)^{-1/n}-1}{u}du\in e^t\cdot\mathbb{Q}[[e^t]]. \tag{3.7}$$

The coefficients of the power series

$$Q(\hbar,e^t)\equiv e^{-(t+\mu(e^t))/\hbar}R(\hbar^{-1},t)\in\mathbb{Q}(\hbar)[[e^t]] \tag{3.8}$$

are holomorphic at $\hbar=0$. Furthermore,

$$\Phi_0(e^t)\equiv Q(0,e^t)=\big(1-n^ne^t\big)^{-1/n}; \tag{3.9}$$

$$\Phi_1(e^t)\equiv\frac{d}{d\hbar}Q(\hbar,e^t)\Big|_{\hbar=0}=\frac{(n-2)(n+1)}{24n}\Big(\big(1-n^ne^t\big)^{-1/n}-\big(1-n^ne^t\big)^{-1}\Big). \tag{3.10}$$

Any one of the identities in (3.4)-(3.6) is implied by the others. We state them all for convenience. A simple algorithm for determining all coefficients of the expansion of Q at $\hbar=0$ is provided by [ZaZ, Theorem 1.5]; these may be needed for computing higher-genus GW-invariants of projective CY-hypersurfaces.

3.1. **Linear independence in symmetric rational functions.** In this subsection we prove a lemma showing that most terms appearing in our computation of $\mathcal{F}(\alpha,x,u)$ can be ignored if our only aim is to determine $\mathcal{F}_0(u)$.

For each $p\in[n]$, let σ_p be the p-th elementary symmetric polynomial in $\alpha_1,\ldots,\alpha_n$. Denote by

$$\mathbb{Q}[\alpha]^{S_n}\equiv\mathbb{Q}[\alpha_1,\ldots,\alpha_n]^{S_n}\subset\mathbb{Q}[\alpha_1,\ldots,\alpha_n]$$

the subspace of symmetric polynomials, by

$$\mathcal{I}\subset\mathbb{Q}[\alpha]^{S_n}$$

the ideal generated by $\sigma_1,\ldots,\sigma_{n-1}$, and by

$$\tilde{\mathbb{Q}}[\alpha]^{S_n}\equiv\mathbb{Q}[\alpha_1,\ldots,\alpha_n]^{S_n}_{\langle\alpha_j,(\alpha_j-\alpha_k)|j\neq k\rangle}\subset\mathbb{Q}_\alpha$$

the subalgebra of symmetric rational functions in $\alpha_1,\ldots,\alpha_n$ whose denominators are products of α_j and $(\alpha_j-\alpha_k)$ with $j\neq k$. For each $i=1,\ldots,n$, let

$$\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\equiv\mathbb{Q}[\alpha_1,\ldots,\alpha_n]^{S_{n-1}}_{\langle\alpha_i,(\alpha_i-\alpha_k)|k\neq i\rangle}\subset\mathbb{Q}_\alpha$$

be the subalgebra consisting of rational functions symmetric in $\{\alpha_k:k\neq i\}$ and with denominators that are products of α_i and $(\alpha_i-\alpha_k)$ with $k\neq i$. Let

$$\mathcal{K}_i\equiv\mathrm{Span}\big\{\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}},\{1,\alpha_i,\ldots,\alpha_i^{n-3},\alpha_i^{n-1}\}\cdot\tilde{\mathbb{Q}}[\alpha]^{S_n}\big\} \tag{3.11}$$

be the linear span (over $\mathbb{Q}$) of $\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}$ and $\alpha_i^p\cdot\tilde{\mathbb{Q}}[\alpha]^{S_n}$ with $p=0,1,\ldots,n$–3, n–1. If $f,g\in\mathbb{Q}_\alpha$, we will write

$$f\cong_i g \qquad \text{if} \qquad f-g\in\mathcal{K}_i. \tag{3.12}$$

Lemma 3.3. *(i) The ideal $\mathcal{I}$ does not contain the product of any powers of σ_n and*

$$D \equiv \prod_{j\neq k}(\alpha_j - \alpha_k).$$

(ii) If $n\geq 2$, the linear span of α_i^{n-2} is disjoint from $\mathcal{K}_i$:

$$\mathrm{Span}\{\alpha_i^{n-2}\} \cap \mathcal{K}_i = \{0\} \subset \mathbb{Q}_\alpha.$$

Proof. (i) Suppose $m\in\bar{\mathbb{Z}}^+$. If $\alpha_1,\ldots,\alpha_n$ are the n distinct roots of the polynomial x^n-1, then

$$\sigma_1(\alpha_1,\ldots,\alpha_n),\ldots,\sigma_{n-1}(\alpha_1,\ldots,\alpha_n)=0, \quad \sigma_n(\alpha_1,\ldots,\alpha_n)^m D(\alpha_1,\ldots,\alpha_n)^m \neq 0.$$

Thus, $\sigma_n^m D^m \notin \mathcal{I}$.

(ii) Every polynomial in $\alpha_1,\ldots,\alpha_n$ which is symmetric in $\{\alpha_k : k\neq i\}$ can be written as a polynomial in α_i with coefficients in $\mathbb{Q}[\alpha]^{S_n}$. Thus, suppose

$$\alpha_i^{n-1} = \frac{\sum_{r=0}^{r=N}\alpha_i^r f_r}{\alpha_i^m\prod_{k\neq i}(\alpha_i-\alpha_k)^m} + \frac{\sum_{p=0}^{n-2}\alpha_i^p g_p + \alpha_i^n g_n}{\sigma_n^m\prod_{j\neq k}(\alpha_j-\alpha_k)^m}, \tag{3.13}$$

$$\text{where} \qquad m\in\bar{\mathbb{Z}}^+,\ N=n(m+1)-2,\ f_r\in\mathcal{I},\ g_p\in\mathbb{Q}[\alpha]^{S_n}.$$

Multiplying out the denominators, we obtain

$$\alpha_i^{n-1}\sigma_n^m D^m = \sum_{r=0}^{r=\tilde{N}}\alpha_i^r F_r + \sum_{p=0}^{n-2}\alpha_i^p g_p + \alpha_i^n g_n, \quad \text{where} \quad \tilde{N}=n(mn+1)-2,\ F_r\in\mathcal{I}.$$

It follows that

$$\begin{split}\sum_{i=1}^{i=n}\mathfrak{R}_{x=\alpha_i}\left\{\frac{x^{n-1}}{\prod_{k=1}^{k=n}(x-\alpha_k)}\sigma_n^m D^m\right\} \\ = \sum_{i=1}^{i=n}\mathfrak{R}_{x=\alpha_i}\left\{\frac{1}{\prod_{k=1}^{k=n}(x-\alpha_k)}\left(\sum_{r=0}^{r=\tilde{N}}x^p F_r + \sum_{p=0}^{n-2}x^p g_p + x^n g_n\right)\right\}.\end{split} \tag{3.14}$$

By the Residue Theorem on S^2, both sides of (3.14) are equal to the negative of the residues of the corresponding one-forms at $x=\infty$. Thus,

$$1\cdot\sigma_n^m D^m = \sum_{r=n-1}^{r=\tilde{N}}\tilde{F}_r + \sigma_1 g_n, \qquad \text{where} \qquad \tilde{F}_r\in\mathcal{I}. \tag{3.15}$$

Note that $\tilde{F}_r\in\mathcal{I}$ because $F_r\in\mathcal{I}$. Thus, (3.15) contradicts the first statement of the lemma, and (3.13) cannot hold. □

Remark. The proof of (ii) of Lemma 3.3 shows that its statement remains valid if n–2 is replaced by any $p=0,1,\ldots,n$–1, in the definition of $\mathcal{K}_i$ and in the statement of (ii).

3.2. **The genus 0 generating functions.** By Subsections 1.2-1.4, the generating function $\mathcal{F}(\alpha,\alpha_i,u)$ for the reduced genus 1 GW-invariants of a degree n hypersurface in $\mathbb{P}^{n-1}$ is given by

$$\mathcal{F}(\alpha,\alpha_i,u) = \mathcal{A}_i(u) + \sum_{j=1}^{j=n}\tilde{\mathcal{A}}_{ij}(u) + \mathcal{B}_i(u) + \sum_{j=1}^{j=n}\tilde{\mathcal{B}}_{ij}(u); \tag{3.16}$$

see (1.16) for the definition of $\mathcal{F}(\alpha,\alpha_i,u)$. Propositions 1.1 and 1.2 express the four terms on the right-hand side of (3.16) in terms of the generating functions for genus 0 GW-invariants defined in (1.22)-(1.24). These functions have been previously computed in terms of hypergeometric series. We describe them in this subsection. For the rest of Section 3, we assume that $n\geq 2$.

Let

$$\mathcal{Y}(\hbar,x,e^t)=\frac{1}{I_{0,0}(t)}\sum_{d=0}^{\infty}e^{dt}\frac{\prod_{r=1}^{r=nd}(nx+r\hbar)}{\prod_{r=1}^{r=d}\big(\prod_{k=1}^{k=n}(x-\alpha_k+r\hbar)-\prod_{k=1}^{k=n}(x-\alpha_k)\big)}. \tag{3.17}$$

If $p\in\bar{\mathbb{Z}}^+$, let

$$\mathcal{Y}_p(\hbar,x,e^t)=e^{-xt/\hbar}\left\{\frac{\hbar}{I_{p,p}(t)}\frac{d}{dt}\right\}\dots\left\{\frac{\hbar}{I_{1,1}(t)}\frac{d}{dt}\right\}\big(e^{xt/\hbar}\mathcal{Y}(\hbar,x,e^t)\big). \tag{3.18}$$

In particular,

$$\mathcal{Y}_0(\hbar,x,e^t)=\mathcal{Y}(\hbar,x,e^t),\qquad \mathcal{Y}_1(\hbar,x,e^t)=\left\{x\frac{dt}{dT}+\hbar\frac{d}{dT}\right\}\mathcal{Y}(\hbar,x,e^t), \tag{3.19}$$

if T and t are related by the mirror transformation (0.17). Let

$$\begin{aligned}\widetilde{\mathcal{Y}}(\hbar_1,\hbar_2,x,e^t)=x\sum_{\substack{p+q=n-2\\ p,q\geq 0}}&\mathcal{Y}_p(\hbar_1,x,e^t)\mathcal{Y}_q(\hbar_2,x,e^t)\\ &+x^{-(n-1)}\mathcal{Y}_{n-1}(\hbar_1,x,e^t)\mathcal{Y}_{n-1}(\hbar_2,x,e^t).\end{aligned} \tag{3.20}$$

Lemma 3.4. *The power series $\mathcal{Z}_i^*(\hbar,u)$ is rational in $\hbar\in S^2$ and vanishes to second order at $\hbar=\infty$. It has simple poles at $\hbar=(\alpha_j-\alpha_i)/d$, with $j\neq i$ and $d\in\mathbb{Z}^+$, and another pole at $\hbar=0$. Furthermore,*

$$\mathfrak{R}_{\hbar=(\alpha_j-\alpha_i)/d}\big\{\mathcal{Z}_i^*(\hbar,u)\big\}=\sum_{\Gamma}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}, \tag{3.21}$$

where the sum is taken over the two-pointed trees Γ as in (1.17) such that the marked point 1 is attached to a vertex $v_0=\eta(1)$ of valence 2, $\mu(v_0)=i$, $\mu(v)=j$ for the unique vertex v adjacent to v_0, and $\mathfrak{d}(\{v_0,v\})=d$. Finally, for all $z=0,\infty,-n\alpha_i$ and $a\in\mathbb{Z}$,

$$\begin{aligned}\mathfrak{R}_{\hbar=z}\left\{\frac{\hbar^a}{n\alpha_i+\hbar}\big(1+\mathcal{Z}_i^*(\hbar,e^T)-e^{(t-T)\alpha_i/\hbar}\mathcal{Y}(\hbar,\alpha_i,e^t)\big)\right\}&\\ \in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big]&.\end{aligned} \tag{3.22}$$

Lemma 3.5. *The power series $\hbar\mathcal{Z}_{ji}^*(\hbar,u)$ is rational in $\hbar\in S^2$ and vanishes at $\hbar=\infty$. It has simple poles at $\hbar=(\alpha_l-\alpha_j)/d$, with $l\neq j$ and $d\in\mathbb{Z}^+$, and another pole at $\hbar=0$. Furthermore,*

$$\mathfrak{R}_{\hbar=(\alpha_l-\alpha_j)/d}\big\{\hbar\mathcal{Z}_{jl}^*(\hbar,u)\big\}=\sum_{\Gamma}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_j\mathrm{ev}_2^*\phi_i}{\mathbf{e}(\mathcal{N}\mathcal{Z}_\Gamma)}, \tag{3.23}$$

where the sum is taken over the two-pointed trees Γ as in (1.17) such that the marked point 1 is attached to a vertex $v_0=\eta(1)$ of valence 2, $\mu(v_0)=j$, $\mu(v)=l$ for the *unique vertex v adjacent to v_0, and $\mathfrak{d}(\{v_0,v\})=d$. Finally, for all $z=0,\infty,-n\alpha_j$ and $a\in\mathbb{Z}$,*

$$\begin{aligned}(3.24)\quad &\mathfrak{R}_{\hbar=z}\Big\{\frac{\hbar^a}{n\alpha_j+\hbar}\big(\alpha_i^{n-2}\alpha_j+\hbar\mathcal{Z}^*_{ji}(\hbar,e^T)-\alpha_i^{n-2}e^{(t-T)\alpha_j/\hbar}\mathcal{Y}_1(\hbar,\alpha_j,e^t)\big)\Big\}\\ &\qquad\in\big(\{1,\alpha_i,\ldots,\alpha_i^{n-3},\alpha_i^{n-1}\}\cdot\tilde{\mathbb{Q}}_j[\alpha]^{S_{n-1}}\oplus\alpha_i^{n-2}\mathcal{I}\cdot\tilde{\mathbb{Q}}_j[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big].\end{aligned}$$

Lemma 3.6. *The power series $\hbar_1\hbar_2\widetilde{\mathcal{Z}}^*_{ii}(\hbar_1,\hbar_2,u)$ is rational in $\hbar_1\in S^2$ and vanishes at $\hbar_1=\infty$. It has simple poles at $\hbar_1=(\alpha_j-\alpha_i)/d$, with $j\neq i$ and $d\in\mathbb{Z}^+$, and another pole at $\hbar_1=0$. Furthermore,*

$$(3.25)\qquad \mathfrak{R}_{\hbar_1=(\alpha_j-\alpha_i)/d}\big\{2\hbar_1\hbar_2\widetilde{\mathcal{Z}}^*_{ii}(\hbar_1,\hbar_2,u)\big\}=\sum_{\Gamma}\int_{\mathcal{Z}_\Gamma}\frac{\mathbf{e}(\mathcal{V}_0')\mathrm{ev}_1^*\phi_i\mathrm{ev}_2^*\phi_i}{(\hbar_2-\psi_2)\mathbf{e}\big(\mathcal{N}\mathcal{Z}_\Gamma\big)},$$

where the sum is taken over the two-pointed trees Γ as in (1.17) *such that the marked point* 1 *is attached to a vertex $v_0=\eta(1)$ of valence* 2, *$\mu(v_0)=i$, $\mu(v)=j$ for the unique vertex v adjacent to v_0, and $\mathfrak{d}(\{v_0,v\})=d$. The analogous statements hold for $\hbar_2$. Finally, for all $a_1,a_2\in\mathbb{Z}^-$,*

$$\begin{aligned}(3.26)\quad &\mathfrak{R}_{\hbar_1=0}\mathfrak{R}_{\hbar_2=0}\Bigg\{\hbar_1^{a_1}\hbar_2^{a_2}\Bigg(2\hbar_1\hbar_2\widetilde{\mathcal{Z}}^*_{ii}(\hbar,e^T)\\ &\qquad-\frac{e^{(t-T)\alpha_i(\hbar_1^{-1}+\hbar_2^{-1})}}{\hbar_1+\hbar_2}\widetilde{\mathcal{Y}}(\hbar_1,\hbar_2,\alpha_i,e^t)\Bigg)\Bigg\}\in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big].\end{aligned}$$

All statements concerning rationality of $\mathcal{Z}^*_i$, $\mathcal{Z}^*_{ji}$, and $\mathcal{Z}^*_{ii}$ in these three lemmas refer to rationality of the coefficients of the powers of e^t. Lemma 3.4 is proved in [MirSym, Chapter 30]; Lemmas 3.5 and 3.6 are proved in [Z4].[20] For example, the conclusion of [MirSym, Section 30.4] is that

$$1+\mathcal{Z}^*_i(\hbar,e^T)=e^{C(e^t)\sigma_1/\hbar}e^{(t-T)\alpha_i/\hbar}\mathcal{Y}(\hbar,\alpha_i,e^t),$$

where

$$C(u)=-\sum_{d=1}^{\infty}u^d\bigg(\frac{(nd)!}{(d!)^n}\sum_{r=1}^{r=d}\frac{1}{r}\bigg).$$

This statement clearly implies (3.22), provided $n\geq 2$ so that $\sigma_1\in\mathcal{I}$.

The differences in (3.22), (3.24), and (3.26) are of course symmetric in the α's in every appropriate sense. For example, any of the differences in (3.22) is the evaluation at $x=\alpha_i$ of a power series in e^t with coefficients in the rational functions in $x,\sigma_1,\ldots,\sigma_{n-1}$. This is immediate from the explicit formulas for $\mathcal{Z}^*_i$, $\mathcal{Z}^*_{ji}$, and $\widetilde{\mathcal{Z}}_{ij}$ in [MirSym, Chapter 30] and in [Z4]. This symmetry is used in the next subsection in the computation of the contributions of $\tilde{\mathcal{A}}_{ij}$ and $\tilde{\mathcal{B}}_{ij}$.

3.3. **Proof of Theorem 3.** We will use Lemma 3.3, along with Lemmas 3.4-3.6, to extract the coefficients of α_i^{n-2} from the expressions of Propositions 1.1 and 1.2 modulo $\mathcal{K}_i[[u]]$. In the notation of Theorem 3, the two coefficients are $\frac{d}{dT}\tilde{A}(e^t)$ and $\frac{d}{dT}\tilde{B}(e^t)$. Let $\cong_i$ be as in (3.12), with its meaning extended to power series in e^t in the natural way.

[20] In fact, Lemma 3.4 is essentially the main result of [MirSym, Chapter 30], while Lemmas 3.5 and 3.6 are essentially the main results of [Z4].

We begin by defining the analogues of the power series $\mathcal{Y}(\hbar,x,u)$ and $\mathcal{Y}_p(\hbar,x,u)$ without the α's. Let

$$Y(\hbar,x,e^t)=\frac{1}{I_{0,0}(t)}\sum_{d=0}^{\infty}e^{dt}\frac{\prod_{r=1}^{r=nd}(nx+r\hbar)}{\prod_{r=1}^{r=d}\big((x+r\hbar)^n-x^n\big)}. \tag{3.27}$$

If $p\in\bar{\mathbb{Z}}^+$, let

$$Y_p(\hbar,x,e^t)=e^{-xt/\hbar}\bigg\{\frac{\hbar}{I_{p,p}(t)}\frac{d}{dt}\bigg\}\dots\bigg\{\frac{\hbar}{I_{1,1}(t)}\frac{d}{dt}\bigg\}\big(e^{xt/\hbar}Y(\hbar,x,e^t)\big). \tag{3.28}$$

Similarly to (3.19),

$$Y_0(\hbar,x,e^t)=Y(\hbar,x,e^t),\qquad Y_1(\hbar,x,e^t)=\bigg\{x\frac{dt}{dT}+\hbar\frac{d}{dT}\bigg\}Y(\hbar,x,e^t), \tag{3.29}$$

if T and t are related by the mirror transformation (0.17). Let

$$\begin{aligned}\widetilde{Y}(\hbar_1,\hbar_2,x,e^t)=x\sum_{\substack{p+q=n-2\\ p,q\ge 0}}&Y_p(\hbar_1,x,e^t)Y_q(\hbar_2,x,e^t)\\ &+x^{-(n-1)}Y_{n-1}(\hbar_1,x,e^t)Y_{n-1}(\hbar_2,x,e^t).\end{aligned} \tag{3.30}$$

We note that

$$\bigg(\prod_{k=1}^{k=n}(x-\alpha_k+r\hbar)-\prod_{k=1}^{k=n}(x-\alpha_k)\bigg)-\big((x+r\hbar)^n-x^n\big)\in\mathcal{I}[\hbar,x]. \tag{3.31}$$

It follows that

$$\begin{aligned}&\mathfrak{R}_{\hbar=z}\bigg\{\frac{\hbar^a}{n\alpha_i+\hbar}\mathcal{Y}(\hbar,\alpha_i,u)\bigg\}-\bigg(\mathfrak{R}_{\hbar=z'}\bigg\{\frac{\hbar^a}{nx+\hbar}Y(\hbar,x,u)\bigg\}\bigg)\bigg|_{x=\alpha_i}\\ &=\mathfrak{R}_{\hbar=z}\bigg\{\frac{\hbar^a}{n\alpha_i+\hbar}\mathcal{Y}(\hbar,\alpha_i,u)\bigg\}-\mathfrak{R}_{\hbar=z}\bigg\{\frac{\hbar^a}{n\alpha_i+\hbar}Y(\hbar,\alpha_i,u)\bigg\}\\ &\in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[u\big]\big]\end{aligned} \tag{3.32}$$

for all $a\in\mathbb{Z}$, $z=0,\infty,-n\alpha_i$ and the corresponding $z'=0,\infty,-nx$. The equality in (3.32) holds because the evaluation at $x=\alpha_i$ of the residue of $\hbar^aY(\hbar,x,u)/(nx+\hbar)$ at $\hbar=z'$ is defined; i.e. the evaluation of $Y(\hbar,x,u)$ at $x=\alpha_i$ does not change the order of the pole of (the coefficient of each u^d in) $Y(\hbar,x,u)$. This is the reason we modify the denominators of [MirSym, Chapter 30] and of [Z4] by subtracting off $\prod_{k=1}^{k=n}(x-\alpha_k)$. This modification has no effect on the evaluation maps $x\longrightarrow\alpha_i$.

We now use Lemmas 3.4-3.6 and (3.32) to extract the "relevant" part from the expressions of Proposition 1.1 and 1.2. If $\eta_i(u)$ and $\mu(u)$ are as in (1.25) and (3.1), respectively, then

$$\eta_i(e^T)-\big(t-T+\mu(e^t)\big)\alpha_i\in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big] \tag{3.33}$$

by (3.22) and (3.32). Similarly, if $\Phi_0(\alpha_i,u)$ and $\Phi_0(u)$ are as in (1.26) and (3.9), respectively, then

$$\Phi_0(\alpha_i,e^T)-\frac{\Phi_0(e^t)}{I_{0,0}(e^t)}\in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big] \tag{3.34}$$

by (3.22), (3.32), and (3.33). By (1.27), (3.26), (3.33), and (3.34),

$$\mathcal{A}_i(e^T)-A(e^t)\alpha_i^{n-2}\in\big(\mathcal{I}\cdot\tilde{\mathbb{Q}}_i[\alpha]^{S_{n-1}}\big)\big[\big[e^t\big]\big], \tag{3.35}$$

where

$$A(e^t)=\frac{I_{0,0}(e^t)}{2\Phi_0(e^t)}\mathfrak{R}_{\hbar_1=0}\mathfrak{R}_{\hbar_2=0}\bigg\{\hbar_1^{-1}\hbar_2^{-1}\frac{e^{-\mu(e^t)(\hbar_1^{-1}+\hbar_2^{-1})}}{\hbar_1+\hbar_2}\widetilde{Y}(\hbar_1,\hbar_2,1,e^t)\bigg\}. \tag{3.36}$$

On the other hand, by (3.24), (3.32), and (3.33),

$$\mathfrak{R}_{\hbar=0}\big\{e^{-\eta_j(e^T)/\hbar}\mathcal{Z}^*_{ji}(\hbar,e^T)\big\}\cong_i\big(-1+\Delta(e^t)\big)\alpha_i^{n-2}\alpha_j, \tag{3.37}$$

where

$$\begin{aligned}\Delta(e^t)&\equiv\mathfrak{R}_{\hbar=0}\bigg\{\hbar^{-1}e^{-\mu(e^t)/\hbar}Y_1(\hbar,1,e^t)\bigg\}\\&=\mathfrak{R}_{\hbar=0}\bigg\{\hbar^{-1}\Big\{\frac{d(t+\mu(e^t))}{dT}+\hbar\frac{d}{dT}\Big\}\big(e^{-\mu(e^t)/\hbar}Y(\hbar,1,e^t)\big)\bigg\}\\&=\frac{d(t+\mu(t))}{dT}\mathfrak{R}_{\hbar=0}\Big\{\hbar^{-1}e^{-\mu(e^t)/\hbar}Y(\hbar,1,e^t)\Big\}=\frac{d(t+\mu(e^t))}{dT}\frac{\Phi_0(e^t)}{I_{0,0}(t)}.\end{aligned} \tag{3.38}$$

The first equality in (3.38) follows from (3.29), the second from the holomorphicity statement of Proposition 3.2, and the last from (3.27) and (3.9). By (1.28), (3.35), and (3.37),

$$\begin{aligned}\sum_{j=1}^{j=n}\tilde{\mathcal{A}}_{ij}(e^T)&\cong_i\alpha_i^{n-2}\big(-1+\Delta(e^t)\big)\sum_{j=1}^{j=n}\frac{\alpha_j\mathcal{A}_j(e^T)}{\prod_{k\neq j}(\alpha_j-\alpha_k)}\\&\cong_i\alpha_i^{n-2}\big(-1+\Delta(e^t)\big)A(e^t)\sum_{j=1}^{j=n}\mathfrak{R}_{z=\alpha_j}\bigg\{\frac{z^{n-1}}{\prod_{k=1}^{k=n}(z-\alpha_k)}\bigg\}\\&=-\alpha_i^{n-2}\big(-1+\Delta(e^t)\big)A(e^t)\mathfrak{R}_{z=\infty}\bigg\{\frac{z^{n-1}}{\prod_{k=1}^{k=n}(z-\alpha_k)}\bigg\}\\&=\alpha_i^{n-2}\big(-1+\Delta(e^t)\big)A(e^t).\end{aligned} \tag{3.39}$$

The first equality above follows from the Residue Theorem on S^2. Thus, by (3.35) and (3.38),

$$\mathcal{A}_i(e^T)+\sum_{j=1}^{j=n}\tilde{\mathcal{A}}_{ij}(e^T)\cong_i\alpha_i^{n-2}\frac{d(t+\mu(t))}{dT}\frac{\Phi_0(e^t)}{I_{0,0}(t)}A(e^t), \tag{3.40}$$

with $A(e^t)$ defined by (3.36).

We next reduce the right-hand side of (3.36) to the explicit form of Theorem 3. Let

$$L(e^t)=\big(1-n^ne^t\big)^{1/n},\qquad f_p(e^t)=\frac{1}{L(e^t)I_{p,p}(t)}\ \forall\,p=0,1,\ldots. \tag{3.41}$$

By (3.8), (3.28), and (3.7),

$$\begin{aligned}&e^{-\mu(e^t)/\hbar}Y_p(\hbar,1,e^t)\\&\quad=\frac{1}{I_{p,p}(t)}\bigg\{\frac{d(t+\mu(t))}{dt}+\hbar\frac{d}{dt}\bigg\}\ldots\frac{1}{I_{1,1}(t)}\bigg\{\frac{d(t+\mu(t))}{dt}+\hbar\frac{d}{dt}\bigg\}\bigg(\frac{Q(\hbar,e^t)}{I_{0,0}(t)}\bigg)\\&\quad=f_p(e^t)\bigg\{1+\hbar L(e^t)\frac{d}{dt}\bigg\}\ldots f_1(e^t)\bigg\{1+\hbar L(e^t)\frac{d}{dt}\bigg\}\bigg(\frac{Q(\hbar,e^t)}{I_{0,0}(t)}\bigg)\end{aligned}$$

for all $p\geq 0$. Thus, by (3.9), (3.10), and the regularity statement of Proposition 3.2,

$$\Theta_p^{(0)}(e^t)\equiv \mathfrak{R}_{h=0}\Big\{\hbar^{-1}e^{-\mu(e^t)/\hbar}Y_p(\hbar,1,e^t)\Big\}=\prod_{r=0}^{r=p}f_r(e^t), \tag{3.42}$$

$$\begin{aligned}\Theta_p^{(1)}(e^t)&\equiv \mathfrak{R}_{h=0}\Big\{\hbar^{-2}e^{-\mu(e^t)/\hbar}Y_p(\hbar,1,e^t)\Big\}\\ &=L(e^t)\bigg(\prod_{r=0}^{r=p}f_r(e^t)\bigg)\bigg(\Phi_1(e^t)+\sum_{r=0}^{p-1}(p-r)\frac{f_r'(e^t)}{f_r(e^t)}\bigg),\end{aligned} \tag{3.43}$$

where $'$ denotes the derivative with respect to t. Note that by (3.4) and (3.5),

$$\Theta_{n-1}^{(0)}(e^t)=1 \qquad\text{and}\qquad \Theta_{n-1}^{(1)}(e^t)=L(e^t)\Phi_1(e^t).$$

Thus, by (3.30), (3.42), and (3.43),

$$\begin{aligned}&\mathfrak{R}_{h_1=0}\mathfrak{R}_{h_2=0}\bigg\{\frac{e^{-\mu(e^t)(\hbar_1^{-1}+\hbar_2^{-1})}}{\hbar_1\hbar_2(\hbar_1+\hbar_2)}\widetilde{Y}(\hbar_1,\hbar_2,1,e^t)\bigg\}\\ &\qquad=\sum_{\substack{p+q=n-2\\ p,q\geq 0}}\Theta_p^{(1)}(e^t)\Theta_q^{(0)}(e^t)+\Theta_{n-1}^{(1)}(e^t)\Theta_{n-1}^{(0)}(e^t)\\ &\qquad=L(e^t)\bigg(n\Phi_1(e^t)+\sum_{p=0}^{n-2}\sum_{r=0}^{p-1}(p-r)\frac{f_r'(e^t)}{f_r(e^t)}\bigg).\end{aligned} \tag{3.44}$$

The last equality uses (3.6), followed by (3.4).

By (3.40), (3.36), (3.44), (0.17), (3.7), and (3.10), the contribution of the fixed loci of Proposition 1.1 is

$$\begin{aligned}&\frac{1}{2I_{1,1}(t)}\bigg(\frac{(n-2)(n+1)}{24}\Big(L(e^t)^{-1}-L(e^t)^{-n}\Big)+\sum_{p=0}^{n-2}\binom{n-1-p}{2}\frac{f_p'(e^t)}{f_p(e^t)}\bigg)\\ &=\frac{1}{2I_{1,1}(t)}\frac{d}{dt}\bigg(\frac{(n-2)(n+1)}{24}\Big(\big(t+\mu(e^t)\big)-\big(t-n\ln L(e^t)\big)\Big)\\ &\qquad\qquad+\sum_{p=0}^{n-3}\binom{n-1-p}{2}\ln f_p(e^t)\bigg)\\ &=\frac{1}{2}\frac{d}{dT}\bigg(\frac{(n-2)(n+1)}{24}\mu(e^t)-\frac{(n-2)(3n-5)}{24}\ln\big(1-n^ne^t\big)\\ &\qquad\qquad-\sum_{p=0}^{n-3}\binom{n-1-p}{2}\ln I_{p,p}(t)\bigg).\end{aligned}$$

The first equality above uses (3.41) and (3.7); the second one also uses (0.16) and (0.17). We have now proved the first statement of Theorem 3; the second form of the expression in (3.2) is easily obtainable from the first using (3.4)-(3.6).

We compute the contributions of the terms $\mathcal{B}_i(u)$ and $\tilde{\mathcal{B}}_{ij}(u)$ of Proposition 1.2 to $\mathcal{F}_0(u)$ similarly. However, before proceeding, we observe that the α-free analogue of the term $\prod_{k=1}^{k=n}(\alpha_i-\alpha_k+\hbar)$ in the numerators in (1.39) and (1.40) is $(x+\hbar)^n-x^n$. The reason is the $r=1$ case of (3.31) and the fact that subtracting off $\prod_{k=1}^{k=n}(x-\alpha_k)$ from the numerators has no effect on the evaluation maps $x\longrightarrow\alpha_i$.

By (1.39), (3.22), and (3.32),

$$\mathcal{B}_i(e^T) \cong_i \alpha_i^{n-2} B(e^t), \qquad \text{where}$$
$$B(e^t) = \frac{n}{24}\mathfrak{R}_{\hbar=0,\infty,-n}\left\{\frac{(1+\hbar)^n-1}{(n+\hbar)\hbar^3}\,\frac{e^{(t-T)/\hbar}Y(\hbar,1,e^t)-1}{e^{(t-T)/\hbar}Y(\hbar,1,e^t)}\right\}.$$

On the other hand, by (1.40), (3.22), (3.24), (3.32), and (3.29),

$$\begin{aligned}
\tilde{\mathcal{B}}_{ij}(e^T) &\cong_i -\frac{n\alpha_j\alpha_i^{n-2}}{24\prod_{k\neq j}(\alpha_j-\alpha_k)} \\
&\qquad\times \mathfrak{R}_{\hbar=0,\infty,-n\alpha_j}\left\{\frac{(\alpha_j+\hbar)^n-\alpha_j^n}{(n\alpha_j+\hbar)\hbar^3}\,\frac{e^{(t-T)\alpha_j/\hbar}Y_1(\hbar,\alpha_j,e^t)-\alpha_j}{e^{(t-T)\alpha_j/\hbar}Y(\hbar,\alpha_j,e^t)}\right\} \\
&= -\frac{n\alpha_i^{n-2}\alpha_j^{n-1}}{24\prod_{k\neq j}(\alpha_j-\alpha_k)} \\
&\qquad\times \mathfrak{R}_{\hbar=0,\infty,-n}\left\{\frac{(1+\hbar)^n-1}{(n+\hbar)\hbar^3}\,\frac{\left\{1+\hbar\frac{d}{dT}\right\}\left(e^{(t-T)/\hbar}Y(\hbar,1,e^t)\right)-1}{e^{(t-T)/\hbar}Y(\hbar,1,e^t)}\right\} \\
&= \frac{\alpha_i^{n-2}\alpha_j^{n-1}}{\prod_{k\neq j}(\alpha_j-\alpha_k)}\Big(-B(e^t)+\frac{d}{dT}\tilde{B}(e^t)\Big),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{B}(e^t) \equiv \tilde{B}_0(e^t)+\tilde{B}_\infty(e^t)+\tilde{B}_{-n}(e^t),\\
&\tilde{B}_z(e^t) = -\frac{n}{24}\mathfrak{R}_{\hbar=z}\left\{\frac{(1+\hbar)^n-1}{(n+\hbar)\hbar^2}\,\ln\Big(e^{(t-T)/\hbar}Y(\hbar,1,e^t)\Big)\right\}.
\end{aligned} \tag{3.45}$$

Thus, applying the Residue Theorem on S^2 as in (3.39), we obtain

$$\mathcal{B}_i(e^T)+\sum_{j=1}^{j=n}\tilde{\mathcal{B}}_{ij}(e^T) \cong_i \alpha_i^{n-2}\frac{d}{dT}\tilde{B}(e^t). \tag{3.46}$$

It remains to compute the three residues $\tilde{B}_z(e^t)$. For $z=-n$, the pole is simple. Since

$$e^{t/\hbar}Y(\hbar,1,e^t) = \bar{R}(\hbar^{-1},t) \qquad \text{and} \qquad \bar{R}(-n^{-1},t) = e^{-t/n}/I_{0,0}(t)$$

with $\bar{R}$ as in Subsection 0.3, we obtain

$$\tilde{B}_{-n}(e^t) = -\frac{n}{24}\cdot\frac{(1-n)^n-1}{n^2}\left(\frac{t-T}{-n}-\ln I_{0,0}(t)\right). \tag{3.47}$$

The residue at $z=0$ is computed using Proposition 3.2:

$$\tilde{B}_0(e^t) = -\frac{n}{24}\left(\frac{(n-2)(n+1)}{2n}\big(t-T+\mu(e^t)\big)+\ln\Phi_0(e^t)-\ln I_{0,0}(e^t)\right). \tag{3.48}$$

Finally, note that

$$e^{wt}Y(w,1,e^t) = \bar{R}(w,t) = 1+Tw+\sum_{q=2}^{\infty}\frac{I_{0,q}(t)}{I_{0,0}(t)}w^q.$$

Thus,

$$
\begin{aligned}
\tilde{B}_\infty(e^t) &= \frac{n}{24}\mathfrak{R}_{w=0}\left\{\frac{(1+w)^n}{(1+nw)w^{n-1}}\big(-Tw+\ln\bar{R}(w,t)\big)\right\}\\
&= \frac{n}{24}\left\{-T\,\mathcal{D}_w^{n-3}\left(\frac{(1+w)^n}{(1+nw)}\right)+\mathcal{D}_w^{n-2}\left(\frac{(1+w)^n}{(1+nw)}\ln\bar{R}(w,t)\right)\right\}\\
&= \frac{n}{24}\sum_{p=2}^{n-2}\left(\mathcal{D}_w^{n-2-p}\frac{(1+w)^n}{(1+nw)}\right)\big(\mathcal{D}_w^p\ln\bar{R}(w,t)\big).
\end{aligned}
\tag{3.49}
$$

The remaining statement of Theorem 3 is obtained by adding (3.47)-(3.49).

Appendix A. Some combinatorics

This appendix contains the computational steps omitted in the proof of Lemma 2.2, as well as the proof of Lemma 2.4.

Lemma A.1. *For all $b\geq 0$, $N\geq 0$, and $q_1,\ldots,q_N\geq 0$,*

$$
\sum_{\substack{\sum_{l=1}^{l=N}\beta_l=b\\ \beta_l\geq 0}}\prod_{l=1}^{l=N}\binom{q_l}{\beta_l}=\binom{q_1+\ldots+q_N}{b}.
\tag{A.1}
$$

For all $q\geq 0$ and $a\geq 1$,

$$
\sum_{b=0}^{\infty}(-1)^b\binom{q}{b}\frac{1}{a+b}=\frac{(a-1)!q!}{(a+q)!}.
\tag{A.2}
$$

For all $q\geq 0$ and $a,s\geq 0$,

$$
\sum_{b=0}^{\infty}(-1)^b\binom{q}{b}\prod_{r=a-s+1}^{r=a}(r+b)=(-1)^q s!\binom{a}{s-q}.
\tag{A.3}
$$

Proof. (1) Each summand on the left side of (A.1) is the total number of ways to choose β_l elements from a q_l-element set for $l=1,\ldots,N$. Thus, the number on the left side of (A.1) is the number of ways to choose $b=\sum_{l=1}^{l=N}\beta_l$ elements from a set with $\sum_{l=1}^{l=N}q_l$ elements.

(2) The identity (A.2) is satisfied for $q=0$. Suppose (A.2) holds for all $a\geq 1$ and some $q\geq 0$. Then,

$$
\begin{aligned}
\sum_{b=0}^{\infty}\frac{(-1)^b}{a+b}\binom{q+1}{b} &= \sum_{b=0}^{\infty}\frac{(-1)^b}{a+b}\left(\binom{q}{b}+\binom{q}{b-1}\right)\\
&= \sum_{b=0}^{\infty}\frac{(-1)^b}{a+b}\binom{q}{b}-\sum_{b=0}^{\infty}\frac{(-1)^b}{a+1+b}\binom{q}{b}\\
&= \frac{(a-1)!\,q!}{(a+q)!}-\frac{a!\,q!}{(a+1+q)!}=\frac{(a-1)!\,(q+1)!}{(a+q+1)!},
\end{aligned}
$$

as needed.

(3) With q, a, and s as in (A.3),

$$\begin{aligned}\sum_{b=0}^{\infty}(-1)^b\binom{q}{b}\prod_{r=a-s+1}^{r=a}(r+b)
&=\left\{\frac{d}{dx}\right\}^s\left(\sum_{b=0}^{\infty}(-1)^b\binom{q}{b}x^{a+b}\right)\bigg|_{x=1}\\
&=\left\{\frac{d}{dx}\right\}^s\big((1-x)^q x^a\big)\Big|_{x=1}=\binom{s}{q}(-1)^q q!\cdot\frac{a!}{(a-s+q)!},\end{aligned}$$

as claimed. □

For each $k\in\mathbb{Z}$, let

$$[k]=\big\{l\in\mathbb{Z}^+: 1\le l\le k\big\}$$

as before. If $\alpha=(\alpha_1,\ldots,\alpha_k)$ is a tuple of non-negative integers, define

$$|\alpha|=\sum_{i=1}^{i=k}\alpha_i,\qquad \alpha!=\prod_{i=1}^{i=k}\alpha_i!\,,\qquad S(\alpha)=\big\{(i,j)\colon\ j=1,\ldots,\alpha_i;\ i=1,\ldots,k\big\}.$$

If b is an integer, possibly negative, let

$$\binom{|\alpha|+b}{\alpha,b}=\binom{|\alpha|+b}{\alpha_1,\ldots,\alpha_k,b}.$$

Denote by $(\bar{\mathbb{Z}}^+)^{\emptyset}$ the 0-dimensional lattice.

Proof of (2.3). Suppose $k\in\bar{\mathbb{Z}}^+$, $\alpha\in(\mathbb{Z}^+)^k$, and $q\in(\bar{\mathbb{Z}}^+)^k$ is a tuple of distinct non-negative integers. With C_q as in (2.5), we will compare the coefficients of

$$C_q^{\alpha}\equiv\prod_{i=1}^{i=k}C_{q_i}^{\alpha_i}$$

on the two sides of (2.3). By (2.6), for each $\beta\in(\bar{\mathbb{Z}}^+)^{S(\alpha)}$ and every choice of k distinguished disjoint subsets of $[a+2+|\beta|]$ of cardinalities $\alpha_1,\ldots,\alpha_k$, the term C_q^{α} appears in the $m=a+2+|\beta|$ summand on the left side of (2.3) with the coefficient

$$\begin{aligned}\eta^{m-|\alpha|}&\prod_{(i,j)\in S(\alpha)}\left(\frac{(-1)^{\beta_{i,j}}}{\beta_{i,j}!}\cdot\frac{\eta^{q_i+1-\beta_{i,j}}}{(q_i+1-\beta_{i,j})!}\right)\\
&=\frac{\eta^{a+2+\alpha\cdot q}}{(q+1)!^{\alpha}}\prod_{(i,j)\in S(\alpha)}(-1)^{\beta_{i,j}}\binom{q_i+1}{\beta_{i,j}},^{21}\end{aligned}\tag{A.4}$$

where

$$\alpha\cdot q=\sum_{i=1}^{i=k}\alpha_i q_i\qquad\text{and}\qquad (q+1)!^{\alpha}=\prod_{i=1}^{i=k}\big((q_i+1)!\big)^{\alpha_i}.$$

Since the number of above choices is

$$\binom{a+2+|\beta|}{\alpha,\,a+2+|\beta|-|\alpha|},$$

[21] In the first product, the (i,j)-factor is defined to be zero if $\beta_{i,j}>(q_i+1)$.

it follows that the coefficient of C_q^α on the left-hand side of (2.3) is

$$\frac{\eta^{a+2+\alpha\cdot q}}{(q+1)!^\alpha}\sum_{\beta\in(\bar{\mathbb{Z}}^+)^{S(\alpha)}}\Bigg\{\frac{1}{(a+2+|\beta|)(a+1+|\beta|)}\binom{a+2+|\beta|}{\alpha,\,a+2+|\beta|-|\alpha|} \times\prod_{(i,j)\in S(\alpha)}(-1)^{\beta_{i,j}}\binom{q_i+1}{\beta_{i,j}}\Bigg\}. \tag{A.5}$$

If $k=0$ and thus $(\bar{\mathbb{Z}}^+)^{S(\alpha)}=\{0\}$, this expression reduces to $a!\eta^{a+2}/(a+2)!$. By (2.6), this is the term on the right-hand side of (2.3) that does not involve any C_q. If $k\ge 1$, (A.5) becomes

$$\begin{aligned}\frac{\eta^{a+2+\alpha\cdot q}}{(q+1)!^\alpha\alpha!}\sum_{b=0}^{\infty}(-1)^b\binom{\alpha\cdot q+|\alpha|}{b}\frac{1}{(a+2+b)(a+1+b)}\frac{(a+2+b)!}{(a+2+b-|\alpha|)!}\\ = a!\frac{\eta^{a+2+|q|}}{(a+|q|+2)!}\times\begin{cases}1, & \text{if } |\alpha|=1,\\ 0, & \text{if } |\alpha|\ge 2,\end{cases}\end{aligned}$$

by Lemma A.1. By (2.6), this is also the coefficient of C_q^α on right-hand side of (2.3). □

Proof of (2.4). The coefficient of C_q^α in the $m=a+|\beta|$ summand on the left side of (2.4) is again given by the first expression in (A.4). Thus, the coefficient of C_q^α on the left-hand side of (2.4) is

$$\frac{\eta^{a+\alpha\cdot q}}{(q+1)!^\alpha}\sum_{\beta\in(\bar{\mathbb{Z}}^+)^{S(\alpha)}}\binom{a+|\beta|}{\alpha,\,a+|\beta|-|\alpha|}\prod_{(i,j)\in S(\alpha)}(-1)^{\beta_{i,j}}\binom{q_i+1}{\beta_{i,j}}. \tag{A.6}$$

If $k=0$, this expression reduces to η^a. If $k\ge 1$, (A.6) becomes

$$\frac{\eta^{a+\alpha\cdot q}}{(q+1)!^\alpha\alpha!}\sum_{b=0}^{\infty}(-1)^b\binom{\alpha\cdot q+|\alpha|}{b}\frac{(a+b)!}{(a+b-|\alpha|)!}=(-1)^{|\alpha|}\eta^a\times\begin{cases}1, & \text{if } |q|=0,\\ 0, & \text{if } |q|\ge 1,\end{cases}$$

by Lemma A.1, as required. □

Proof of Lemma 2.4. For each $e\in E$, let

$$r_e=\mathfrak{R}_{\lambda=0}\{f_e(\lambda)\},\quad g_e(\lambda)=f_e(\lambda)-r_e\lambda^{-1},\quad h_e(\lambda)=\lambda f_e(\lambda)=r_e+\lambda g_e(\lambda).$$

Since f_e has at most a simple pole at $\lambda=0$, g_e and h_e are holomorphic at $\lambda=0$ and

$$\mathcal{D}_\lambda^{j_e+1}h_e=\mathcal{D}_\lambda^{j_e}g_e\qquad\forall\ j_e\ge 0.$$

Since $\prod_{e\in E} f_e(\lambda)$ has a pole of order at most $|E|$ at $\lambda=0$,

$$\begin{aligned}
\mathfrak{R}_{\lambda=0}\Big\{\prod_{e\in E} f_e(\lambda)\Big\} &= \mathcal{D}_\lambda^{|E|-1}\Big\{\lambda^{|E|}\prod_{e\in E} f_e\Big\} = \mathcal{D}_\lambda^{|E|-1}\Big\{\prod_{e\in E} h_e\Big\}\\
&= \sum_{\sum_{e\in E} j_e=|E|-1}\prod_{e\in E}\mathcal{D}_\lambda^{j_e} h_e\\
&= \sum_{E_+\subset E}\bigg\{\Big(\prod_{e\in E_+} r_e\Big)\sum_{\sum_{e\notin E_+} j_e=|E_+|-1}\Big(\prod_{e\notin E_+}\mathcal{D}_\lambda^{j_e+1} h_e\Big)\bigg\}\\
&= \sum_{E_+\subset E}\bigg\{\Big(\prod_{e\in E_+} r_e\Big)\sum_{\sum_{e\notin E_+} j_e=|E_+|-1}\Big(\prod_{e\notin E_+}\mathcal{D}_\lambda^{j_e} g_e\Big)\bigg\}\\
&= \sum_{E_+\subset E}\bigg\{\Big(\prod_{e\in E_+} r_e\Big)\mathcal{D}_\lambda^{|E_+|-1}\Big(\prod_{e\notin E_+} g_e\Big)\bigg\},
\end{aligned}$$

as claimed. □

Appendix B. Comparison of mirror symmetry formulations

In this appendix we compare a number of mirror symmetry formulations for genus 0 and genus 1 curves in a quintic threefold. In all cases, the predictions are of the form

$$F_g^{top}(T) = F_g(t),$$

where $g=0,1$, $F_g^{top}(T)$ is a generating function for the genus g GW-invariants of a quintic threefold (related to an A-model correlation function), $F_g(t)$ is an explicit function of t (related to a B-model correlation function), and $T=\mathcal{J}(t)$ for some function $\mathcal{J}$ (called a mirror transformation).

In [CaDGP], the variables on the B-side and A-side are ψ and t, respectively. Let

$$\varpi_0(\psi) = 1+\sum_{d=1}^{\infty}\frac{(5d)!}{(d!)^5}(5\psi)^{-5d};$$

this is equation (3.8) in [CaDGP] with n replaced by d. The mirror transformation is defined by equation (5.9):

$$t=\mathcal{J}(\psi)\equiv -\frac{5}{2\pi\mathfrak{i}}\bigg\{\ln(5\psi)-\frac{1}{\varpi_0(\psi)}\sum_{d=1}^{\infty}\frac{(5d)!}{(d!)^5}(5\psi)^{-5d}\Big(\sum_{r=d+1}^{5d}\frac{1}{r}\Big)\bigg\}.$$

The mirror symmetry prediction for genus 0 curves is given in [CaDGP] in equation (5.13):

$$5+\sum_{d=1}^{\infty} n_{0,d}d^3\frac{e^{2\pi\mathfrak{i}dt}}{1-e^{2\pi\mathfrak{i}dt}} = \frac{5\psi^2}{(1-\psi^5)\varpi_0^2(\psi)}\bigg(\frac{5}{2\pi\mathfrak{i}}\Big(\frac{d\psi}{dt}\Big)\bigg)^3, \tag{B.1}$$

where $n_{0,d}$ is the genus 0 degree d instanton number of a quintic. These numbers are related to the genus 0 GW-invariants by the formula

$$N_{0,d}=\sum_{d_1d_2=d}\frac{n_{0,d_1}}{d_2^3} \qquad\Longleftrightarrow\qquad \sum_{d=1}^{\infty}N_{0,d}d^3q^d=\sum_{d=1}^{\infty}n_{0,d}d^3\frac{q^d}{1-q^d}. \tag{B.2}$$

The right-hand side of (B.1) replaces the function κ_{ttt} appearing in [CaDGP, (5.13)], using [CaDGP, (5.11)] and the following two lines.

In [CoKa, Chapter 2], the B-side variable is x and the variables on A-side are s and $q=e^s$. Let

$$y_0(x) = 1 + \sum_{d=1}^{\infty} \frac{(5d)!}{(d!)^5}(-x)^d,$$

$$y_1(x) = y_0(x)\ln(-x) + 5\sum_{d=1}^{\infty} \frac{(5d)!}{(d!)^5}(-x)^d \Big(\sum_{r=d+1}^{5d} \frac{1}{r}\Big).$$

These equations are (2.23) and (2.24) in [CoKa]. The mirror transformation is given by

$$s = \mathcal{J}(x) \equiv y_1(x)/y_0(x), \qquad q = e^s = e^{y_1(x)/y_0(x)}. \tag{B.3}$$

The mirror symmetry prediction for genus 0 curves is given in [CoKa] in equation (2.26):

$$5 + \sum_{d=1}^{\infty} n_{0,d} d^3 \frac{q^d}{1-q^d} = \frac{5}{(1+5^5x)y_0^2(x)} \Big(\frac{q}{x}\frac{dx}{dq}\Big)^3. \tag{B.4}$$

The relation with variables in [CaDGP] is

$$x = -(5\psi)^{-5}, \qquad s = 2\pi\mathfrak{i}t, \qquad q = e^{2\pi\mathfrak{i}t}. \tag{B.5}$$

With these identifications, $y_0(x) = \varpi_0(\psi)$, $\mathcal{J}(x)$ of [CoKa] is $\mathcal{J}(\psi)$ of [CaDGP] times $2\pi\mathfrak{i}$, and the right-hand sides of (B.1) and (B.4) are the same.

In [MirSym, Chapters 29,30], the variables on the B and A sides are t and T. The mirror transformation is given by

$$T = \mathcal{J}(t) \equiv J_1(t) = I_1(t)/I_0(t),$$

where I_q and J_q are as in (0.1) and (0.3). The mirror symmetry prediction for genus 0 curves is formulated in [MirSym, (29.2)] as

$$e^{HT} + \frac{H^2}{5}\sum_{d=1}^{\infty} n_{0,d} d^3 \sum_{k=1}^{\infty} \frac{e^{(H+kd)T}}{(H+kd)^2} = \sum_{i=0}^{i=3} J_i(t)H^i \qquad \mathrm{mod}\ H^4. \tag{B.6}$$

Using (B.2), (B.6) can be re-written as

$$\sum_{i=0}^{i=3} \frac{H^iT^i}{i!} + \frac{H^2}{5}\sum_{d=1}^{\infty} N_{0,d}(d-2H)e^{(H+d)T} = \sum_{i=0}^{i=3} J_i(t)H^i \qquad \mathrm{mod}\ H^4. \tag{B.7}$$

The relation (0.4) is obtained by extracting the H^2 and H^3-terms from (B.7); it is the statement of [MirSym, Exercise 29.2.2], minus a typo. The relation with variables in [CoKa] is

$$x = -e^t, \qquad s = T, \quad \text{and} \quad q = e^T. \tag{B.8}$$

With these identifications, $I_i(t) = y_i(x)$ and $\mathcal{J}(t)$ of [MirSym] is precisely $\mathcal{J}(x)$ of [CoKa]. It is shown at the very end of [CoKa, Chapter 2] that the third derivative of the right-hand side of (0.4) with respect to $s=T$ is the right-hand side of (B.4). We note that throughout Section 2.6 of [CoKa] (in contrast to Section 2.4),

$$Y(q) = \frac{5}{(1+5^5x)y_0^2(x)} \Big(\frac{q}{x}\Big(\frac{dx}{dq}\Big)\Big)^3.$$

The relation between x and q is described in the previous paragraph. In Section 2.5, this function Y is called the normalized Yukawa coupling regarded as a function of q.

As in [CaDGP], the B-side and A-side variables in [BCOV] are ψ and t. However, they are now related by the mirror transformation

$$t=\mathcal{J}(\psi)\equiv 5\bigg\{\ln(5\psi)-\frac{1}{\varpi_0(\psi)}\sum_{d=1}^{\infty}\frac{(5d)!}{(d!)^5}(5\psi)^{-5d}\Big(\sum_{r=d+1}^{5d}\frac{1}{r}\Big)\bigg\}=-T,$$

with T as in [MirSym, Chapters 29,30]. This is contrary to the suggestion in the paper that ψ and t are related in the same way as in [CaDGP]. Let

$$F_1(\psi)=\ln\bigg(\Big(\frac{\psi}{\varpi_0(\psi)}\Big)^{62/3}(1-\psi^5)^{-1/6}\Big(\frac{d\psi}{dt}\Big)\bigg); \tag{B.9}$$

see equation (23) in [BCOV]. The mirror symmetry prediction for genus 1 curves is given in [BCOV] in equation (24):

$$\frac{25}{6}-2\sum_{d_1,d_2=1}^{\infty}n_{1,d_1}d_1d_2\frac{q^{d_1d_2}}{1-q^{d_1d_2}}-\frac{1}{6}\sum_{d=1}^{\infty}n_{0,d}d\frac{q^d}{1-q^d}=\partial_tF_1(\psi), \tag{B.10}$$

where $q=e^{-t}$ and $n_{1,d}$ is the genus 1 degree d instanton number of the quintic. These numbers are related to the genus 1 GW-invariants by the formula

$$\begin{aligned}N_{1,d}&=\sum_{d_1d_2=d}n_{1,d_1}\frac{\sigma_{d_2}}{d_2}+\frac{1}{12}\sum_{d_1d_2=d}n_{0,d_1}\frac{1}{d_2}\\ \Longleftrightarrow\sum_{d=1}^{\infty}N_{1,d}dq^d&=\sum_{d_1,d_2=1}^{\infty}n_{1,d_1}d_1d_2\frac{q^{d_1d_2}}{1-q^{d_1d_2}}+\frac{1}{12}\sum_{d=1}^{\infty}n_{0,d}d\frac{q^d}{1-q^d},\end{aligned} \tag{B.11}$$

where σ_r is the number of degree r unramified connected covers of a smooth genus 1 surface or equivalently of subgroups of $\mathbb{Z}^2$ of index r. Since this number is the same as the sum of positive integer divisors of r,

$$\sum_{r=1}^{\infty}\sigma_rq^r=\sum_{r=1}^{\infty}r\frac{q^r}{1-q^r}. \tag{B.12}$$

This identity implies equivalence of the two equalities in (B.11).

Integrating both sides of (B.10) with respect to t and using (B.11), we find that (B.10) is equivalent to

$$C+\frac{25}{6}t+2\sum_{d=1}^{\infty}N_{1,d}q^d=\ln\bigg(\Big(\frac{\psi}{\varpi_0(\psi)}\Big)^{62/3}(1-\psi^5)^{-1/6}\Big(\frac{d\psi}{dt}\Big)\bigg) \tag{B.13}$$

for some constant C. This equality should be interpreted by moving $25t/6$ to the right-hand side and expanding as a power series in q at $q=0$. The relation between the variables in [BCOV] and [CoKa] is

$$x=-(5\psi)^{-5},\qquad s=-t,\quad\text{and}\quad q=q.$$

Thus, (B.13) is equivalent to

$$C'-\frac{25}{6}s+2\sum_{d=1}^{\infty}N_{1,d}q^d=\ln\bigg(x^{-25/6}y_0(x)^{-62/3}\big(1+5^5x\big)^{-1/6}\Big(\frac{q}{x}\frac{dx}{dq}\Big)\bigg), \tag{B.14}$$

with $q = e^s$ and s and x related by the mirror transformation (B.3). In the notation of [MirSym], i.e. with identifications (B.8), (B.14) becomes

$$C'' - \frac{25}{6}T + 2\sum_{d=1}^{\infty} N_{1,d}e^{dT} = -\frac{25}{6}t + \ln\Big(I_0(t)^{-62/3}\big(1-5^5e^t\big)^{-1/6}J_1'(t)^{-1}\Big). \tag{B.15}$$

It is straightforward to see that $C''=0$. Thus, (B.10) and (B.15) are equivalent to (0.5).

Remark. The conventions used in [KlPa] to formulate a mirror symmetry prediction for the genus 1 GW-invariants of a sextic fourfold are the same as in [BCOV], except 5 above is replaced by 6.

Acknowledgments

The author is very grateful to D. Zagier for taking interest in the statements of and collaborating on the proof of Propositions 3.1 and 3.2, to Jun Li for bringing [BCOV] to the author's attention and collaborating on related work concerning genus 1 GW-invariants, to R. Pandharipande for generously sharing his expertise in the Gromov-Witten theory, and the referee for suggestions on improving the exposition. The author would also like to thank S. Katz, Y.-P. Lee, D. Maulik, A. Mustata, A. Postnikov, and R. Vakil for many enlightening discussions.

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