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THE FONTAINE-MAZUR CONJECTURE FOR GL_2

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Contents

Introduction

In [FM] Fontaine and Mazur made a remarkable conjecture, predicting that global p-adic Galois representations which are potentially semi-stable at primes dividing p and unramified outside finitely many places ought to come from algebraic geometry. For 2-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, the conjecture asserts that potentially semi-stable representations with odd determinant come from modular forms. The purpose of this paper is to prove that this is so in many cases. Our methods reveal an intimate connection between modularity lifting theorems, the Breuil-Mézard conjecture, and Breuil's p-adic local Langlands correspondence.

To state our main theorem, let $p > 2$, S a set of primes containing $\{p, \infty\}$, $G_{\mathbb{Q},S}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside S, and $G_{\mathbb{Q}_p} \subset G_{\mathbb{Q},S}$ a decomposition group at p. We prove the following

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Theorem. *Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$, having residue field $\mathbb{F}$, and let*

$$\rho : G_{\mathbb{Q},S} \to \mathrm{GL}_2(\mathcal{O})$$

be a continuous representation. Suppose that

(1) *$\rho|_{G_{\mathbb{Q}_p}}$ is potentially semi-stable with distinct Hodge-Tate weights.*
(2) *ρ becomes semi-stable over an abelian extension of $\mathbb{Q}_p$.*
(3) *$\bar\rho : G_{\mathbb{Q},S} \xrightarrow{\rho} \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathbb{F})$ is odd, and $\bar\rho|_{\mathbb{Q}(\zeta_p)}$ is absolutely irreducible.*
(4) *$\bar\rho|_{G_{\mathbb{Q}_p}} \nsim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ for any character $\chi : G_{F_v} \to \mathbb{F}^\times$, where ω denotes the $\bmod p$ cyclotomic character*

Then (up to a twist) ρ is modular.

The condition (2) in the theorem can be removed, assuming a compatibility between the p-adic and classical local Langlands correspondences, which describes the locally algebraic vectors in the p-adic unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to a de Rham representation.[1] (The precise statement is given in §1.2). Assuming (2), this is a result of Colmez and Berger-Breuil [Co 3], [Co 1], [BB 1]. What we prove here is the theorem assuming (1), (3), (4) and this compatibility.

The restriction that $p > 2$ is also likely to be unnecessary, at least in many cases (for example $\bar\rho|_{G_{\mathbb{Q}_p}}$ irreducible) since the p-adic Langlands correspondence is available in this situation, unlike the usual difficulties encountered in integral p-adic Hodge theory when $p = 2$.

In fact we prove the theorem in somewhat greater generality, where $\mathbb{Q}$ is replaced by any totally real field in which p splits completely. Let us also remind the reader that thanks to the results of Khare-Wintenberger [KW 1], [KW 2] and [Ki 5] on Serre's conjecture, the hypothesis that $\bar\rho$ is odd implies that it is modular.

One consequence of the theorem is a conjecture made in [Ki 4, 11.8], which gives a construction of the eigencurve of Coleman-Mazur in purely Galois theoretic terms. This was our original motivation for thinking about modularity lifting theorems. One way to formulate this is the following

Corollary. *Let $\bar\rho$ be as in the Theorem, and denote by $R(\bar\rho)$ its universal deformation ring and by $Z = (\mathrm{Spec}\, R(\bar\rho)[1/p])^{\mathrm{an}}$ the associated analytic space.*

Then the $\bar\rho$-part of the eigencurve is the Zariski-analytic closure of the set of points $(x, \lambda) \in Z \times \mathbb{G}_m$ such that x corresponds to a representation V_x of $G_{\mathbb{Q},S}$ which is potentially semi-stable with Hodge-Tate weights $0, k-1$ with $k \geq 2$, and $D_{\mathrm{cris}}(V_x)^{\varphi=\lambda} \neq 0$.

We now explain how the Breuil-Mézard conjecture and the p-adic local Langlands correspondence enter the proof of the theorem. The first fundamental breakthrough in the direction of the Fontaine-Mazur conjecture was made by Wiles and Taylor-Wiles [Wi], [TW] a little over 10 years ago. They showed how one could deduce the modularity of certain p-adic Galois representations, assuming the mod p reduction was modular. Subsequently a number of authors established modularity lifting theorems for (2-dimensional) potentially Barsotti-Tate representations and more generally representations of small Hodge-Tate weights [Di 2], [CDT], [BCDT],

[1]As this paper went to press, a revised version of [Co 2], which asserts exactly such a compatibility, became available. At present the proof relies on forthcoming work of Emerton on the local-global compatibilty of the p-adic local Landlands correspondence.

[DFG], [Ta 2]. There was also work of Skinner-Wiles establishing the conjecture for ordinary representations [SW 1], [SW 2].

One of the themes in these papers is that in order to prove a modularity lifting theorem, one needs to show a certain local deformation ring is formally smooth (i.e. a power series ring). In [BCDT] the authors considered potentially Barsotti-Tate representations, and they made a conjecture predicting when one could expect this formal smoothness. This conjecture was later generalized by Breuil-Mézard [BM] who predicted that μ_{Gal}, the Hilbert-Samuel multiplicity of the mod p reduction of the local deformation ring, should be given by a certain invariant μ_{Aut} which could be computed representation theoretically.

In [Ki 2] we showed how to modify the Taylor-Wiles argument, so that it applied when the local deformation was not formally smooth. This was used to establish a fairly general modularity lifting theorem for potentially Barsotti-Tate Galois representations. However, another consequence of this modification was that one could use a global argument to show that $\mu_{\mathrm{Gal}} \geq \mu_{\mathrm{Aut}}$ and that establishing a modularity lifting theorem was essentially equivalent to proving the reverse equality. This is explained in §2 of this paper.

The tool which enables us to prove the reverse inequality is the p-adic local Langlands correspondence, whose study was initiated by Breuil [Br 1], [Br 2] and developed by Colmez and Breuil-Berger [Co 1], [BB 1], [BB 2]. A key insight, due to Colmez, is that one can construct instances of this correspondence using Fontaine's theory of φ,Γ-modules. The papers just cited show how to construct unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representations starting with a local Galois representation which Colmez terms *trianguline*. For de Rham representations, this means that the representation becomes semi-stable over an abelian extension of $\mathbb{Q}_p$. In September 2005, at the Montreal conference on p-adic representations, Colmez explained a quite general construction which associated a local Galois representation to a p-adic unitary $\mathrm{GL}_2(\mathbb{Q}_p)$-representation satisfying a mild restriction. This association works integrally, and using it, we show that the local deformation rings we wish to study act faithfully on certain $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. This leads to the required inequality.

We first announced these results at the Montreal conference for ρ which become crystalline over an abelian extension in $\mathbb{Q}_p$ and for $\bar{\rho}$ absolutely irreducible at p. The previous day Colmez had outlined his theory, attaching local Galois representations to certain $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. The arguments we had in mind at that time for proving the inequality $\mu_{\mathrm{Gal}} \leqslant \mu_{\mathrm{Aut}}$ immediately suggested that one should formulate Colmez's correspondence on the level of deformation rings for representations of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_2(\mathbb{Q}_p)$:

$$\Theta : R_{G_{\mathbb{Q}_p}} \to R_{\mathrm{GL}_2(\mathbb{Q}_p)}.$$

(These arguments appear in §§1.5, 1.6 where we consider certain deformations of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations). The advantage of this was that, thanks to the previous work of Colmez and Berger-Breuil, one knew that the image of $\mathrm{Spec}\,\Theta$ contained all trianguline points. A local analogue of an argument of Gouvêa-Mazur [GM] and Böckle [Bö], using the results of [Ki 4], then showed that these points were Zariski dense in $\mathrm{Spec}\,R_{G_{\mathbb{Q}_p}}[1/p]$. This showed that Θ was injective, and its surjectivity was reduced to a calculation involving a map of Ext groups. Colmez has been able to carry out this calculation [Co 2], and the deformation theoretic argument is explained in [Ki 6] (under some mild restrictions).

This allowed the association of a unitary $\mathrm{GL}_2(\mathbb{Q}_p)$ representation to each $G_{\mathbb{Q}_p}$-representation; however this was not yet useful since one could not say much about the locally algebraic vectors in the $\mathrm{GL}_2(\mathbb{Q}_p)$ representation attached to a de Rham representation of $G_{\mathbb{Q}_p}$. On the other hand just the existence of Colmez's functor made possible the application of our method to cases where $\bar{\rho}$ was reducible at p and greatly simplified the arguments.

Finally, let us mention that using Colmez's correspondence, and especially the isomorphism Θ, Emerton has found an alternative approach to the Fontaine-Mazur conjecture (at least in many cases). His method has as a consequence a stronger version of the conjecture made in [Ki 4, 11.8], which we only dared raise as a question [Ki 4, 11.7(2)], namely that a 2-dimensional representation of $G_{\mathbb{Q},S}$ which is trianguline at p arises (up to twist) from an overconvergent modular eigenform.

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1. Breuil-Mézard conjecture and the p-adic local Langlands

(1.0) Notation. Throughout p will denote an odd prime. We denote by $\bar{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$ and we write $G_{\mathbb{Q}_p} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $I_{\mathbb{Q}_p} \subset G_{\mathbb{Q}_p}$ for the inertia subgroup. We will write $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ for the cyclotomic character. We will use class field theory normalized so that the global class field theory isomorphism takes uniformizers to geometric Frobenii.

We denote by $\bar{\mathbb{Z}}_p$ the ring of integers of $\bar{\mathbb{Q}}_p$ and by $\bar{\mathbb{F}}_p$ the residue field of $\bar{\mathbb{Z}}_p$. Let $\mathbb{Q}_p^{\mathrm{ab}} \subset \bar{\mathbb{Q}}_p$ denote the maximal abelian extension of $\mathbb{Q}_p$. Local class field theory gives an inclusion $\mathbb{Q}_p^\times \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$. This allows us to consider characters of $G_{\mathbb{Q}_p}$ as characters of $\mathbb{Q}_p^\times$. With our conventions $\chi_{\mathrm{cyc}}|_{\mathbb{Z}_p^\times}$ is the identity map.

We will denote by $\mathbb{F}$ a finite extension of $\mathbb{F}_p$. We also fix a finite, totally ramified extension $E/W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}$, a uniformizer $\pi \in \mathcal{O}$, and a continuous character $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$. For E'/E a finite extension we will denote by $\mathcal{O}_{E'}$ the ring of integers of E' and by $\pi_{E'}$ a uniformizer of E'.

(1.1) The Breuil-Mézard conjecture. Let V be a finite dimensional E-vector of dimension d, equipped with a continuous action of $G_{\mathbb{Q}_p}$.

Suppose that V is potentially semi-stable in the sense of Fontaine [Fo]. Attached (covariantly) to V is a d-dimensional $\bar{\mathbb{Q}}_p$-representation of the Weil-Deligne group $\mathrm{WD}_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Given a representation $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_d(\bar{\mathbb{Q}}_p)$ with open kernel, we say that V is of *type* τ if the restriction to $I_{\mathbb{Q}_p}$ of the associated Weil-Deligne group representation is equivalent to τ. This is possible only if τ extends to a representation of the Weil group of $\mathbb{Q}_p$. Such τ are said to be of *Galois type.* We will assume in the following that E has been chosen large enough that τ is defined over E.

Now suppose that $d = 2$ and that τ is of Galois type, and fix an integer $k \geq 2$. We will say that V is of *type* (k, τ, ψ) if V is potentially semi-stable of type τ with Hodge-Tate weights $0, k-1$ and determinant $\psi\chi_{\mathrm{cyc}}$. This is possible only if $\psi\chi_{\mathrm{cyc}}^{2-k}|_{I_{\mathbb{Q}_p}} \sim \det \tau$, and we will assume this condition from now on.

Fix a continuous representation

$$\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F}).$$

When $\mathrm{End}_{\mathbb{F}[G_K]}\bar{\rho} = \mathbb{F}$, the representation $\bar{\rho}$ admits a universal deformation to an $\mathcal{O}$-algebra $R(\bar{\rho})$. In [BM] Breuil-Mézard conjectured (for $k < p$) that the deformations of $\bar{\rho}$ to characteristic 0 which are of type (k, τ, ψ) are parameterized by a quotient $R^{\psi}(k, \tau, \bar{\rho})$ of $R(\bar{\rho})$. Moreover, they gave a conjectural formula for the Hilbert-Samuel multiplicity of $R^{\psi}(k, \tau, \bar{\rho})/pR^{\psi}(k, \tau, \bar{\rho})$ in terms of certain representation theoretic data attached to the triple $(k, \tau, \bar{\rho})$.

We will recall this conjecture below. In fact we will define the corresponding invariant in all cases, not just those when $\bar{\rho}$ has trivial endomorphisms. Before giving this definition, we recall a result from [Ki 1], which establishes the existence and basic properties of the ring $R^{\psi}(k, \tau, \bar{\rho})$.

Let $V_{\mathbb{F}}$ denote the underlying $\mathbb{F}$-vector space of $\bar{\rho}$. Recall that the universal framed deformation $\mathcal{O}$-algebra $R^{\square}(\bar{\rho})$ of $\bar{\rho}$ is the $\mathcal{O}$-algebra representing the functor which to a local Artinian $\mathcal{O}$-algebra A with residue field $\mathbb{F}$ attaches the set of isomorphism classes of a deformation V_A of $\bar{\rho}$ to A, together with a lifting to V_A of some fixed choice of basis for $V_{\mathbb{F}}$.

Proposition (1.1.1). *There exists a unique (possibly trivial) quotient $R^{\square,\psi}(k, \tau, \bar{\rho})$ of $R^{\square}(\bar{\rho})$ with the following properties.*

(1) *$R^{\square,\psi}(k, \tau, \bar{\rho})$ is p-torsion free, $R^{\square,\psi}(k, \tau, \bar{\rho})[1/p]$ is reduced and all its irreducible components are smooth and 4-dimensional.*
(2) *If E'/E is a finite extension, then a map $x : R^{\square}(\bar{\rho}) \to E'$ factors through $R^{\square,\psi}(k, \tau, \bar{\rho})$ if and only if the corresponding E'-representation V_x is of type (k, τ, ψ).*

If $\bar{\rho}$ has only scalar endomorphisms, then there exists a quotient $R^{\psi}(k, \tau, \bar{\rho})$ of $R(\bar{\rho})$ with analogous properties, except that the dimension in (1) *is* 1 *rather than* 4.

(1.1.2) Except for the claim about smoothness of the irreducible components, this is a consequence of [Ki 1, 3.3.8]. In fact we will not use the smoothness in this paper; however we give a proof of it in an appendix, which also includes a minor erratum for [Ki 1].

We continue to assume that $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of Galois type, as above. In the appendix to [BM] Henniart shows that there is a unique finite dimensional, irreducible $\bar{\mathbb{Q}}_p$-representation $\sigma(\tau)$ of $\mathrm{GL}_2(\mathbb{Z}_p)$, with open kernel, such that if $\tilde{\tau}$ is any Frobenius semi-simple, continuous representation of $\mathrm{WD}_{\mathbb{Q}_p}$, and $\pi(\tilde{\tau})$ is the smooth representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ associated to $\tilde{\tau}$ by the local Langlands correspondence, then $\pi(\tilde{\tau})|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ contains $\sigma(\tau)$ if and only if $\tilde{\tau}|_{I_{\mathbb{Q}_p}} \sim \tau$. Here the local Langlands correspondence is normalized so that $\pi(\tilde{\tau})$ has central character $\tilde{\tau}|_{\mathbb{Q}_p^{\times}}$, using the convention of (1.0).

We may assume that $\sigma(\tau)$ is defined over E (increasing E if necessary). Following [BM], we set $\sigma(k, \tau) = \sigma(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$. This is a finite dimensional representation of the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$, and hence it contains a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice $L_{k,\tau}$.

Now any irreducible, finite dimensional representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ on an $\mathbb{F}$-vector space is isomorphic to $\sigma_{n,m} = \mathrm{Sym}^n \mathbb{F} \otimes \det^m$ where $n \in \{0, 1, \dots, p-1\}$ and $m \in \{0, 1, \dots, p-2\}$. (Note that such a representation necessarily factors through $\mathrm{GL}_2(\mathbb{F}_p)$, since the normal subgroup $\ker(\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p))$ is a pro-p group and hence has a fixed vector.) Then we have

$$(L_{k,\tau} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \bigoplus_{n,m} \sigma_{n,m}^{a(n,m)}$$

where n and m run over the same ranges explained above.

We set

$$\mu_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}) = \sum_{n,m} a(n,m)\mu_{n,m}(\bar{\rho})$$

where $\mu_{n,m}(\bar{\rho})$ is a non-negative integer which will be defined below.

The following conjecture generalizes the Breuil-Mézard conjecture to the case when $\bar{\rho}$ has non-trivial endomorphisms. It is the crux of our approach to the Fontaine-Mazur conjecture, explained in the introduction, and we will prove many cases of it.

Conjecture (1.1.3). *The Hilbert-Samuel multiplicity of $R^{\square,\psi}(k,\tau,\bar{\rho})/(\pi)$ is equal to μ_{Aut}.*

(1.1.4) We have deliberately stated Conjecture (1.1.3) before specifying the values $\mu_{n,m}(\bar{\rho})$. Note that even without specifying these values the equality asserted by Conjecture (1.1.3) amounts to an infinite set of equations (corresponding to the infinitely many possibilities for k and τ) in the finitely many unknowns $\mu_{n,m}(\bar{\rho})$. That these equations *determine* the $\mu_{n,m}(\bar{\rho})$ becomes transparent, if we allow a slight variant of the conjecture.

To explain this, we remark that the results of [Ki 1] show that there is a quotient $R_{\mathrm{cr}}^{\square,\psi}(k,\tau,\bar{\rho})$ of $R^{\square}(\bar{\rho})$ which satisfies Proposition (1.1.1)(1) and such that a map x as in Proposition (1.1.1)(2) factors through $R_{\mathrm{cr}}^{\square,\psi}(k,\tau,\bar{\rho})$ if and only if V_x is potentially *crystalline* and of type (k,τ,ψ). This ring differs from $R^{\square,\psi}(k,\tau,\psi)$ only if τ is scalar.

Correspondingly, we denote by $\sigma_{\mathrm{cr}}(\tau)$ the unique smooth, irreducible representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ such that (using the notation of (1.1.2)) for any $\tilde{\tau}$, $\pi(\tilde{\tau})|_{\mathrm{GL}_2(\mathbb{Z}_p)}$ contains $\sigma_{\mathrm{cr}}(\tau)$ if and only if $\tilde{\tau}|_{I_{\mathbb{Q}_p}} \sim \tau$ and $N=0$ on $\tilde{\tau}$. The existence of $\sigma_{\mathrm{cr}}(\tau)$ again follows from Henniart's results. Concretely, $\sigma_{\mathrm{cr}}(\tau) = \sigma(\tau)$ except when τ is scalar. If $\tau \sim \chi \oplus \chi$ for some character χ, then $\sigma(\tau) \otimes \chi^{-1} \circ \det$ is the $\mathrm{GL}_2(\mathbb{Z}_p)$-representation consisting of E-valued functions on $\mathbb{P}^1(\mathbb{F}_p)$ with average value 0, while $\sigma_{\mathrm{cr}}(\tau) \sim \chi \circ \det$.

We now set $\sigma_{\mathrm{cr}}(k,\tau) = \sigma_{\mathrm{cr}}(\tau) \otimes_E \mathrm{Sym}^{k-2} E^2$. Choosing a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice $L_{k,\tau}^{\mathrm{cr}}$ in $\sigma_{\mathrm{cr}}(k,\tau)$ and taking the reduction modulo π, we obtain

$$(L_{k,\tau}^{\mathrm{cr}} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \xrightarrow{\sim} \bigoplus_{n,m} \sigma(n,m)^{a_{cr}(n,m)}$$

for some non-negative integers $a_{\mathrm{cr}}(n,m)$. Set

$$\mu_{\mathrm{Aut}}^{\mathrm{cr}} = \mu_{\mathrm{Aut}}^{\mathrm{cr}}(k,\tau,\bar{\rho}) = \sum_{n,m} a_{\mathrm{cr}}(n,m)\mu_{n,m}(\bar{\rho})$$

where $\mu_{n,m}(\bar{\rho})$ are the same integers as before. We then have the following variant of the Breuil-Mézard conjecture

Conjecture (1.1.5). *The Hilbert-Samuel multiplicity of $R_{\mathrm{cr}}^{\square,\psi}(k,\tau,\bar{\rho})/(\pi)$ is equal to $\mu_{\mathrm{Aut}}^{\mathrm{cr}}$.*

(1.1.6) The $\mu_{n,m}(\bar\rho)$ are determined by Conjecture (1.1.5), for if we take τ scalar and $k \in [2, p+1]$, then $L_{k,\tau}/\pi L_{k,\tau}$ is an irreducible representation of $\mathrm{GL}_2(\mathbb{F}_p)$ and hence isomorphic to one of the $\sigma_{n,m}$. Moreover each irreducible representation occurs in this way. Thus computing the $\mu_{n,m}(\bar\rho)$ amounts to computing the deformation rings corresponding to crystalline representations of small weight.

We now define the $\mu_{n,m}(\bar\rho)$ explicitly (except in one case).

For i a positive integer, we denote by $\omega_i : I_{\mathbb{Q}_p} \to \bar{\mathbb{F}}_p^\times$ the fundamental character of level i, and we write $\omega = \omega_1$. Recall that if $\mathbb{Q}_{p^i}$ denotes the unramified extension of $\mathbb{Q}_p$, of degree i, and $\mathbb{Z}_{p^i}$ denotes the ring of integers of $\mathbb{Q}_{p^i}$, then ω_i is obtained by composing the maps

$$I_{\mathbb{Q}_p} \xrightarrow{\sim} I_{\mathbb{Q}_{p^i}} \longrightarrow \mathbb{Z}_{p^i}^\times \to \bar{\mathbb{F}}_p^\times$$

where the second map is given by local class field theory normalized as in (1.0). We extend the map $\mathbb{Z}_{p^i}^\times \to \bar{\mathbb{F}}_p^\times$ to $\mathbb{Q}_{p^i}^\times$, by sending p to 1, and view ω_i as a character of $G_{\mathbb{Q}_{p^i}}$ via the class field theory isomorphism. In particular $\omega = \omega_1$ is then the mod p cyclotomic character.

Suppose first that $\bar\rho$ is absolutely irreducible. For $(n,m) \in \{0,1,\dots,p-1\} \times \{0,1,\dots,p-2\}$ we set $\mu_{n,m}(\bar\rho) = 1$ if

$$\bar\rho|_{I_{\mathbb{Q}_p}} \sim \begin{pmatrix} \omega_2^{n+1} & 0 \\ 0 & \omega_2^{p(n+1)} \end{pmatrix} \otimes \omega^m$$

and $\mu_{n,m}(\bar\rho) = 0$ otherwise. Note that, in this case, for a given $\bar\rho$, there are exactly two pairs (n,m) such that $\mu_{n,m}(\bar\rho) \neq 0$.

Suppose now that $\bar\rho$ is reducible. For $\lambda \in \bar{\mathbb{F}}_p^\times$, we denote by $\mu_\lambda : G_{\mathbb{Q}_p} \to \bar{\mathbb{F}}_p^\times$ the unramified character sending the geometric Frobenius to λ. We set $\mu_{n,m}(\bar\rho) = 0$ unless

$$\bar\rho \sim \begin{pmatrix} \omega^{n+1}\mu_\lambda & * \\ 0 & \mu_{\lambda'} \end{pmatrix} \otimes \omega^m$$

for $\lambda, \lambda' \in \bar{\mathbb{F}}_p^\times$, in which case we set $\mu_{n,m}(\bar\rho) = 1$ except in the following cases:

(1) If $n = p-1$, $\lambda = \lambda'$, and $*$ is peu ramifié (including the case $*$ trivial), $\mu_{n,m}(\bar\rho) = 2$.
(2) If $n = 0$, $\lambda = \lambda'$, and $*$ is très ramifié, $\mu_{n,m}(\bar\rho) = 0$.
(3) If $n = p-2$ and $\bar\rho$ is semi-simple, then $\mu_{p-2,m}(\bar\rho) = 2$ if $\lambda \neq \lambda'$, while if $\lambda = \lambda'$, then we do not specify $\mu_{p-2,m}(\bar\rho)$ explicitly, but define it to be the Hilbert-Samuel multiplicity obtained by taking τ trivial and $k = p$ in Conjecture (1.1.5).

It seems quite possible that one could compute the integer $\mu_{p-2,m}(\bar\rho)$ explicitly when $\lambda = \lambda'$ in (3). Global considerations suggest that it is equal to 2 in this case also.

(1.2) Review of Colmez's functor. We review some results of Colmez which allow one to attach a Galois representations to certain representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We begin by recalling the definition of some mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ studied by Barthel-Livne and Breuil.

(1.2.1) Write $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $K = \mathrm{GL}_2(\mathbb{Z}_p)$ and denote by Z the center of $\mathrm{GL}_2(\mathbb{Q}_p)$. If σ is any representation of KZ on a finite dimensional $\mathbb{F}$-vector space V_σ, then we denote by $I(\sigma) = \mathrm{Ind}_{KZ}^G \sigma$ the compact induction of σ.

Recall [BL, Prop. 5] that $I(\sigma)$ has a natural action by the algebra of KZ-bi-invariant functions $\varphi : G \to \mathrm{End}_{\mathbb{F}} V_\sigma$, that is, the functions φ satisfying $\varphi(h_1 g h_2) =$

$\sigma(h_1)\varphi(g_1)\sigma(h_2)$ for all $g \in G$ and $h_1, h_2 \in KZ$. Explicitly, if $f \in I(\sigma)$, then this action is given by

$$\varphi(f)(g) = \sum_{KZy \in KZ\backslash G} \varphi(gy^{-1})f(y) = \sum_{yKZ \in G/KZ} \varphi(y)f(y^{-1}g).$$

Next we regard $\mathbb{F}^2$ as a representation of KZ with $\mathrm{GL}_2(\mathbb{Z}_p)$ acting in the natural way via the map $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ and the element $p \in Z$ acting trivially. Let $r \in [0, p-1]$ be a non-negative integer, and set $\sigma = \mathrm{Sym}^r\mathbb{F}^2$. Denote by T the endomorphism of $I(\sigma)$ corresponding to the KZ-bi-invariant function which is supported on the double coset $KZ\begin{bmatrix} 1 & 0 \\ 0 & p^{-1}\end{bmatrix}KZ$ and which takes $\begin{bmatrix} 1 & 0 \\ 0 & p^{-1}\end{bmatrix}$ to the endomorphism $\mathrm{Sym}^r\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1\end{smallmatrix}\right]$. According to [BL, Prop. 8], $\mathbb{F}[T]$ is the full endomorphism algebra of $I(\sigma)$.

Let $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ be a character, and let $\lambda \in \mathbb{F}$. For $x \in \mathbb{F}$ we denote by $\mu_x : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ the unramified character sending $p \in \mathbb{Q}_p^\times$ to x.

We set $\pi(r,\lambda,\chi) = I(\sigma)/(T-\lambda)I(\sigma) \otimes \chi \circ \det$. The structure of these representations is given by the following result [BL, Thm. 30, Cor. 36], [Br 1, 1, Thm. 1.1, 1.3], where Sp denotes the space of $\mathbb{F}$-valued, locally constant functions on $\mathbb{P}^1(\mathbb{Q}_p)$, modulo the space of constant functions.

Proposition (1.2.2). (1) *$\pi(r,\lambda,\chi)$ is irreducible unless $(r,\lambda) \in \{(0,\pm1),(p-1,\pm1)\}$.*

(2) *If $(r,\lambda) = (0,\pm1)$, then $\pi(r,\lambda,\chi)$ is a non-trivial extension of $\chi\mu_{\pm1} \circ \det$ by $\chi\mu_{\pm1} \circ \det \otimes \mathrm{Sp}$.*

(3) *If $(r,\lambda) = (p-1,\pm1)$, then $\pi(p-1,\lambda,\chi)$ is a non-trivial extension of $\chi\mu_{\pm1} \circ \det \otimes \mathrm{Sp}$ by $\chi\mu_{\pm1} \circ \det$.*

(4) *If (r,λ,χ) and (r',λ',χ') are two such triples, then there exists an isomorphism*

$$\pi(r,\lambda,\chi) \xrightarrow{\sim} \pi(r',\lambda',\chi')$$

exactly in the following cases:

(i) $r = r'$, *and* $\{\chi',\lambda'\}$ *is* $\{\chi,\lambda\}$ *or* $\{\chi\mu_{-1},-\lambda\}$.

(ii) $\lambda = 0$, $r' = p-1-r$ *and* $\chi' \in \{\chi\omega^r, \chi\omega^r\mu_{-1}\}$.

(iii) $\{r,r'\} = \{0,p-1\}$, $\lambda \neq \pm1$, *and* $\{\chi',\lambda'\}$ *is* $\{\chi,\lambda\}$ *or* $\{\chi\mu_{-1},-\lambda\}$.

(1.2.3) Let Π be a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on a $W(\mathbb{F})$-module. If Π has finite length, we say that Π is *admissible* if each of its Jordan-Hölder factors has a central character.

If Π has finite length, then it is a $W_n(\mathbb{F})$-module for some $n \geq 1$, and the admissibility condition implies that there is a finite extension $\mathbb{F}'/\mathbb{F}$ such that the Jordan-Hölder factors of $\Pi \otimes_{W_n(\mathbb{F})} W_n(\mathbb{F}')$ are either 1-dimensional or an infinite dimensional subquotient of some $\pi(r,\lambda,\chi)$ [Br 1, 1,1.2].

Suppose that Π is a representation of $GL_2(\mathbb{Q}_p)$ on an $\mathcal{O}$-module. We will say that Π has a central character if there is a continuous character $\psi : \mathbb{Q}_p^\times \to \mathcal{O}^\times$ such that each $a \in \mathbb{Q}_p^\times \subset \mathrm{GL}_2(\mathbb{Q}_p)$ acts on Π by multiplication by $\psi(a)$.

We have the following result of Colmez [Co 2].[2]

[2]All references to [Co 2] in this paper are to the earlier (much shorter) version of [Co 2] available at `http://people.math.jussieu.fr/colmez/publications.html`.

Theorem (1.2.4). *There exists an exact covariant functor* $\mathbf{V}$ *from the category of finite length, admissible* $\mathrm{GL}_2(\mathbb{Q}_p)$*-representations on an* $\mathcal{O}$*-module* Π*, having a central character, to the category of finite length representations of* $\mathcal{O}[G_{\mathbb{Q}_p}]$*. Moreover, we have*

(1) $\mathbf{V}(\Pi) = 0$ *if* Π *is* 1*-dimensional.*
(2) $\mathbf{V}(\pi(r, \lambda, \chi)) = \omega^{r+1}\mu_\lambda\chi$ *if* $\lambda \neq 0$.
(3) $\mathbf{V}(\pi(r, 0, \chi)) = \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \omega_2^{r+1} \otimes \chi$.

(1.2.5) Suppose now that Π is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an $\mathcal{O}$-module, having an ($\mathcal{O}^\times$-valued) central character, and set $\Pi_n = \Pi \otimes_{\mathbb{Z}} \mathbb{Z}/p^n$. Suppose that Π is p-adically complete and separated, so that $\Pi = \varprojlim \Pi_n$, and that for each n Π_n is of finite length and admissible. We set $\mathbf{V}(\Pi) = \varprojlim \mathbf{V}(\Pi_n)$. Since admissible representations have finite length, inverse limits in this category are exact, so one sees that $\mathbf{V}(\Pi)/p\mathbf{V}(\Pi) = \mathbf{V}(\Pi_1)$ and in particular that $\mathbf{V}(\Pi)$ is a finitely generated $\mathcal{O}$-module, since it is p-adically separated. We call such a representation Π an admissible $\mathcal{O}$-lattice.

We now make the following assumption on our type (k, τ, ψ).[3]

Hypothesis (1.2.6). Let E'/E be a finite extension and V a 2-dimensional E'-vector space equipped with a continuous action of $G_{\mathbb{Q}_p}$. Suppose that V is of type (k, τ, ψ).

Then there exists an admissible $\mathcal{O}_{E'}$-lattice Π with central character ψ such that $\mathbf{V}(\Pi) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} V$. Moreover, there exists a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant inclusion $\sigma(k, \tau) \hookrightarrow \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

(1.2.7) The existence of Π satisfying $\mathbf{V}(\Pi) \otimes \mathbb{Q}_p \xrightarrow{\sim} V$ is proved in [Ki 6, Thm. 0.1] for any 2-dimensional representation V, except when $p = 3$ and the mod p representation attached to V has the form $\left(\begin{smallmatrix} 1 & * \\ 0 & \omega \end{smallmatrix}\right) \otimes \chi$ or $\mathrm{Ind}_{G_{\mathbb{Q}_p^2}}^{G_{\mathbb{Q}_p}} \omega_2^2 \otimes \chi$.

In this paper it is the final property in Hypothesis (1.2.6) which will play the most important role. This can be proved when τ has an abelian extension to $W_{\mathbb{Q}_p}$. We will say that τ is of *abelian type.* Note that this is stronger than asking that τ have abelian image, a condition which always holds when $p > 2$.

Theorem (1.2.8). *Suppose that* τ *is of abelian type. Then a representation* Π *as in Hypothesis* (1.2.6) *exists.*

Proof. Suppose first that V is irreducible. Then the required representation Π is constructed in [BB 1, §4] and [Co 1, Thm. 0.4]. That this Π satisfies $\mathbf{V}(\Pi) \otimes \mathbb{Q}_p \xrightarrow{\sim} V$ follows by comparing [BB 1, Thm. 5.2.7] and [Co 1, Thm. 0.6] with [Co 2, Prop. 4.26]. Note that the assignment $V \mapsto \Pi$ in [BB 1] is *contravariant* so the formulae of [BB 1, Lem. 5.2.4] differ slightly from those of [Co 2, §4.1].

Suppose now that V is reducible. Although this case is much easier than the irreducible one, we could not (at the time of writing) find it explicitly in the literature, so we explain how to deduce it from available results. Let $B \subset \mathrm{GL}_2(\mathbb{Q}_p)$ denote the Borel subgroup of upper triangular matrices. For continuous characters $\chi_1, \chi_2 : B \to \mathcal{O}^\times$, we denote by $\chi_2 \otimes \chi_1 : B \to \mathcal{O}^\times$ the character which

[3]As remarked in the introduction, the latest version of [Co 2] asserts that this hypothesis is always satisfied.

sends $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\chi_2(a)\chi_1(d)$. We denote by $\bar{P}(\chi_1, \chi_2)$ the space of continuous functions $f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathcal{O}$ such that for all $b \in B$, $f(bg) = \chi_2 \otimes \chi_1\chi_{\mathrm{cyc}}^{-1}(b)f(g)$ except when $\chi_1\chi_2^{-1} = \chi_{\mathrm{cyc}}$, in which case $\bar{P}(\chi_1, \chi_2)$ is the quotient of this space by the 1-dimensional subspace of functions which factor through the determinant. By Lemma (1.2.10) below, any such $\bar{P}(\chi_1, \chi_2)$ is an admissible $\mathcal{O}$-lattice and $\mathbf{V}(\bar{P}(\chi_1, \chi_2)) = \mathcal{O}(\chi_1)$.

Suppose now that V is an extension

$$0 \to \mathcal{O}(\chi_1) \to V \to \mathcal{O}(\chi_2) \to 0.$$

Since we are assuming that V is potentially semi-stable with Hodge-Tate weights $0, k-1$ and $k \geq 2$, we may assume that the Hodge-Tate weights of χ_1 and χ_2 are $k-1$ and 0, respectively. Applying $\mathbf{V}$ to any extension

$$0 \to \bar{P}(\chi_1, \chi_2) \to \Pi \to \bar{P}(\chi_2, \chi_1) \to 0 \tag{1.2.9}$$

produces an extension of $\mathcal{O}(\chi_2)$ by $\mathcal{O}(\chi_1)$. Moreover if (1.2.9) is a non-trivial extension, then so is $\mathbf{V}(\Pi)$ by [Co 2, Thm. 5.1]. Hence it suffices to construct a non-trivial extension as in (1.2.9) when $\chi_1\chi_2^{-1} \neq \chi_{\mathrm{cyc}}$ and a 2-dimensional space of such extensions when $\chi_1\chi_2^{-1} = \chi_{\mathrm{cyc}}$. The former has been done by Breuil-Emerton [BE, Thm. 2.2.2] and the latter by Breuil [Br 3]. □

Lemma (1.2.10). *Let $\chi_1, \chi_2 : B \to \mathcal{O}^\times$ be continuous characters and P the space of continuous functions $f : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathcal{O}$ such that for all $b \in B$, $f(bg) = \chi_2 \otimes \chi_1\chi_{\mathrm{cyc}}^{-1}(b)f(g)$. Then P is an admissible $\mathcal{O}$-lattice and $\mathbf{V}(P) \sim \mathcal{O}(\chi_1)$.*

Proof. It is clear that P is p-adically complete and separated. By [BL, Thm. 30], $P/\pi P$ has finite length, so Π is admissible. Moreover this reference together with Theorem (1.2.4) shows that $\mathbf{V}(P/\pi P)$ is 1-dimensional over $\mathbb{F}$. Hence $\mathbf{V}(P)$ is an $\mathcal{O}$-module of rank 1.

For any $\mathcal{O}$-module M we denote by $M^\vee = \mathrm{Hom}(M, E/\mathcal{O})$ its Pontryagin dual. Denote by $J^\vee(P)$ the invariants of $P^\vee$ under the unipotent subgroup of B and by $\tilde{J}^\vee(P)$ the largest finite length submodule of $P^\vee$ stable by $\left(\begin{smallmatrix} \mathbb{Q}_p^\times & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $\tilde{J}^\vee(P)$ is well defined by [Co 2, Prop. 4.34], and there is a canonical isomorphism of $\mathbb{Q}_p^\times$-representations $\tilde{J}^\vee(P)/J^\vee(P) \xrightarrow{\sim} \mathbf{V}(P)^\vee(1)$. In particular $\tilde{J}^\vee(P)/J^\vee(P)$ has corank 1. Now consider the elements $\delta_n \in P^\vee$ given by sending $f \in P$ to the image of $\pi^{-n} f\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in $\pi^{-n}\mathcal{O}/\mathcal{O}$. Then one checks easily that $\delta_n \in \tilde{J}^\vee(P)$ and the δ_n generate a submodule of corank 1 on which $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in B$ acts by $\chi_1^{-1}\chi_{\mathrm{cyc}}(a)$. Hence $G_{\mathbb{Q}_p}$ acts on $\mathbf{V}(P)$ via χ_1, as required. □

(1.2.11) Our arguments can also be made to work assuming only a weaker version of the final statement of Hypothesis (1.2.6). For example, it would be enough to assume that the locally algebraic vectors in $\Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ contain $\mathrm{Sym}^{k-2} \otimes \pi$ for some irreducible, smooth representation π of $\mathrm{GL}_2(\mathbb{Z}_p)$ whose conductor is bounded by that of τ.

(1.3) Hilbert-Samuel multiplicities. Suppose that A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finite A-module. There is a polynomial $P_M^A(X)$ such that $P_M^A(n)$ is equal to the length of $M/\mathfrak{m}^{n+1}M$ for sufficiently large integers n.

If A has dimension d, then P_M^A has degree at most d, and the Hilbert-Samuel multiplicity $e(M, A)$ of M is defined to be $d!$ times the coefficient of X^d in P_M^A.

Suppose now that G is a group and that M is equipped with an action of G. Let α be a collection of irreducible representations of G on finite dimensional $A/\mathfrak{m}$-vector spaces. Then instead of considering the length of $M/\mathfrak{m}^{n+1}M$, one can consider the number of Jordan-Hölder factors of $M/\mathfrak{m}^{n+1}M$ as an $A[G]$-module, which are isomorphic to an element of α. We denote this number by $\chi^A_{M,\alpha}(n)$.

Proposition (1.3.1). *There is a polynomial $P^A_{M,\alpha}$ of degree at most d such that $\chi^A_{M,\alpha}(n) = P^A_{M,\alpha}(n)$ for sufficiently large positive integers n. Moreover the coefficient of X^d in $P^A_{M,\alpha}$ has the form $e_\alpha(M,A)/d!$ where $e_\alpha(M,A)$ is a non-negative integer.*

Proof. The proof is identical to the standard result for G trivial [Ma, §13]. Note that one only has to show that $P^A_{M,\alpha}$, as above, of some degree exists, since the bound on the degree follows from the case when G is trivial. $\square$

Proposition (1.3.2). *If*

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of $A[G]$-modules which are finite over A, then we have

$$e_\alpha(M,A) = e_\alpha(M',A) + e_\alpha(M'',A).$$

Proof. The proof with G trivial goes over unchanged [Ma, Thm. 14.6]. $\square$

(1.3.3) We will sometimes apply Proposition (1.3.2) in the following situation: Suppose that I', I and I'' are ideals of A such that the quotients $A/I', A/I$ and A/I'' have the same dimension and M', M and M'' in Proposition (1.3.2) are modules over $A/I', A/I$ and A/I'', respectively. Then we have

$$e_\alpha(M, A/I) = e_\alpha(M', A'/I') + e_\alpha(M'', A''/I'').$$

This follows from Proposition (1.3.2) applied with $B = A/(I + I' + I'')$ in place of A, since $e_\alpha(M, A/I) = e_\alpha(M, B)$, as $\dim A/I = \dim B$, and similarly for M' and M''.

Proposition (1.3.4). *Let $f : M \to M'$ be a map of A-finite $A[G]$-modules, and $x \in A$ such that M and M' have no x-torsion.*

(1) *If f is an inclusion, then*

$$e_\alpha(M/xM, A/xA) \leqslant e_\alpha(M'/xM', A/xA).$$

(2) *If f is an isomorphism at all the generic points of* $\operatorname{Spec} A$, *then*

$$e_\alpha(M/xM, A/xA) = e_\alpha(M'/xM', A/xA).$$

Proof. Let $P = \ker(f)$. If $\mathfrak{p} \in \operatorname{Spec} A/x \subset \operatorname{Spec} A$ is a minimal prime of A/x, then $P_\mathfrak{p} = 0$. This is clear under the assumptions of (1), since then $P = 0$. In case (2), if $P_\mathfrak{p}$ were non-zero, then $\mathfrak{p}$ would be an associated prime of P [Ma, Thm. 6.5] and $x \in \mathfrak{p}$ would be a zero divisor of M. Hence $e_\alpha(P/xP, A/xA) = 0$, and we may replace M by its image in M' in (2).

Next let $Q \subset M'/M$ be the submodule consisting of elements which are killed by some power of x. Choose $i > 0$ so that x^i kills Q. The sequence

$$0 \to Q[x] \to Q \xrightarrow{x} Q \to Q/xQ \to 0$$

and Proposition (1.3.2) shows that

$$e_\alpha(Q[x], A/xA) = e_\alpha(Q[x], A/x^iA) = e_\alpha(Q/xQ, A/x^iA) = e_\alpha(Q/xQ, A/xA).$$

Hence, if M'' denotes the preimage of Q in M', then using Proposition (1.3.2) we see that $e_\alpha(M/xM, A/xA) = e_\alpha(M''/xM'', A/xA)$. Hence we may replace M by M'' and assume that M'/M is x-torsion free.

Now (1) follows from Proposition (1.3.2), and the same argument as in the first paragraph shows that under the hypothesis of (2), $e_\alpha(M'/(M + xM'), A/xA) = 0$, so (2) also follows. □

Corollary (1.3.5). *Let M, M' be A-finite $A[G]$-modules, and $x \in A$ such that M and M' are x-torsion free. Suppose that for every minimal prime $\mathfrak{p} \subset A$, $M'_\mathfrak{p}$ contains $M_\mathfrak{p}$ as an $A_\mathfrak{p}[G]$-module. Then*

$$e_\alpha(M/xM, A/xA) \leqslant e_\alpha(M'/xM', A/xA).$$

Proof. Let $Q(A)$ denote the localization of A with respect to the set of elements not in any minimal prime of A. Our assumptions imply that there exists an inclusion of $Q(A)[G]$-modules $f : M \otimes_A Q(A) \hookrightarrow M' \otimes_A Q(A)$. Multiplying f by an element not in any minimal prime of A, we may assume that f is induced by a map $f : M \to M'$. Let $M'' = f(M)$. Then using Proposition (1.3.4), we find

$$e_\alpha(M/xM, A/xA) = e_\alpha(M''/xM'', A/xA) \leqslant e_\alpha(M'/xM', A/xA).$$

□

(1.3.6) We now return to the situation without the action of a group. If $\mathfrak{q} \subset A$ is any $\mathfrak{m}$-primary ideal and M is a finite A-module, then there is a polynomial $P_\mathfrak{q}$ of degree at most d such that the length of $M/\mathfrak{q}^{n+1}M$ is given by $P_\mathfrak{q}(n)$. As above, we write $e_\mathfrak{q}(M, A)$ for $d!$ times the leading coefficient of $P_\mathfrak{q}$. If $M = A$, we write simply $e_\mathfrak{q}(A)$ for $e_\mathfrak{q}(A, A)$. If $\mathfrak{q} = \mathfrak{m}$, we sometimes abbreviate $e_\mathfrak{q}(A)$ to $e(A)$.

We will often use the following result which says that $e_\mathfrak{q}(M, A)$ depends only on the behavior of M at minimal primes of maximal dimension [Ma, 14.7].

Proposition (1.3.7). *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ denote the minimal primes $\mathfrak{p}$ of A such that $\dim A/\mathfrak{p} = \dim A$ and $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal. Then*

$$e_\mathfrak{q}(M, A) = \sum_{i=1}^{m} e_{\mathfrak{q}/\mathfrak{p}_i \cap \mathfrak{q}}(A/\mathfrak{p}_i)\ell_{A_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i}).$$

Proposition (1.3.8). *Let κ be a field and let $(A_1, \mathfrak{m}_1)$ and $(A_2, \mathfrak{m}_2)$ be Noetherian, complete local κ-algebras with residue field κ. Let $\mathfrak{n}$ denote the radical of $B = A_1 \widehat{\otimes}_\kappa A_2$. Then*

$$e_\mathfrak{n}(A_1 \widehat{\otimes}_\kappa A_2) = e_{\mathfrak{m}_1}(A_1) e_{\mathfrak{m}_1}(A_2). \tag{1.3.9}$$

Proof. Let κ' denote an algebraic closure of κ. One sees easily that replacing A_1 and A_2 by the completions of $A_1 \otimes_\kappa \kappa'$ and $A_2 \otimes_\kappa \kappa'$ with respect to $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, does not change either side of (1.3.9). Thus, we may assume that κ is infinite.

By [Ma, Thm. 14.14], since κ is infinite, there is an $\mathfrak{m}_1$-primary ideal $\mathfrak{q}_1 \subset A_1$ such that $\mathfrak{m}_1^{r+1} = \mathfrak{q}_1 \mathfrak{m}_1^r$ for some $r > 0$ (an ideal with this property is called a reduction of $\mathfrak{m}_1$) and $\mathfrak{q}_1$ is generated by a system of parameters $x_1, \dots, x_d$ for A_1. Similarly there exists a reduction $\mathfrak{q}_2 \subset A_2$ of $\mathfrak{m}_2$ generated by a system of parameters $y_1, \dots, y_e$. Then $x_1, \dots, x_d, y_1, \dots, y_e$ is a system of parameters for B.

Moreover $\mathfrak{k} = (x_1, \dots, x_d, y_1, \dots, y_e) \subset B$ is a reduction of $\mathfrak{n}$, for if $\mathfrak{m}_1^{r+1} = \mathfrak{q}_1\mathfrak{m}_1^r$ and $\mathfrak{m}_2^{s+1} = \mathfrak{q}_2\mathfrak{m}_2^s$, then

$$\begin{aligned}\mathfrak{n}^{r+s+1} &= (\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s+1} \\ &\subset \mathfrak{q}_1(\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s} + \mathfrak{q}_2(\mathfrak{m}_1 B + \mathfrak{m}_2 B)^{r+s} = (\mathfrak{q}_1 B + \mathfrak{q}_2 B)\mathfrak{n}^{r+s}.\end{aligned}$$

By [Ma, Thm. 14.13] it suffices to show that

$$e_{\mathfrak{k}}(A_1 \widehat{\otimes}_\kappa A_2) = e_{\mathfrak{q}_1}(A_1) e_{\mathfrak{q}_2}(A_2).$$

This follows, for example, from Lech's lemma [Ma, Thm. 14.12] which asserts that

$$e_{\mathfrak{q}_1}(A_1) = \varinjlim \frac{\ell(A_1/(x_1^{\nu_1}, \dots, x_d^{\nu_d}))}{\nu_1 \dots, \nu_d}$$

and similarly for $e_{\mathfrak{q}_2}(A_2)$ and $e_{\mathfrak{k}}(B)$. Here the limit is taken over d-tuples of positive integers $(\nu_1, \dots, \nu_d)$ such that $\min_{i=1}^d \nu_i \to \infty$. □

Proposition (1.3.10). *Let $A \to B$ be a local map of local Noetherian rings with radicals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Let $\mathfrak{p} \subset A$ be a nilpotent prime ideal, and suppose that all the minimal primes of B lie over $\mathfrak{p}$. Then*

$$e_{\mathfrak{n}}(B) \leqslant e_{\mathfrak{n}/\mathfrak{p}B}(B/\mathfrak{p}B)\ell(A_{\mathfrak{p}}).$$

Proof. Let $(0) = \bigcap_{i=1}^m \mathfrak{q}_i$ be a minimal primary decomposition of 0 in A, and suppose that $\mathfrak{q}_1$ is $\mathfrak{p}$-primary. By Proposition (1.3.7), replacing A by $A/\mathfrak{q}_1$ and B by $B/\mathfrak{q}_1$ does not affect either side of the inequality to be proved. Thus we may assume that A has no embedded primes and injects into $A_{\mathfrak{p}}$.

If $\mathfrak{p}$ is the zero ideal, there is nothing to prove. Choose an ideal $J \subset \mathfrak{p}$, such that $J_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module of length 1. In particular we have $\mathfrak{p}J = 0$, since this is true after localizing at $\mathfrak{p}$. Using induction on the length of $A_{\mathfrak{p}}$, we find that

$$e_{\mathfrak{n}}(B/JB, B) = e_{\mathfrak{n}/JB}(B/JB) \leqslant e_{\mathfrak{n}/\mathfrak{p}B}(B/\mathfrak{p}B)\ell_{A_{\mathfrak{p}}}((A/J)_{\mathfrak{p}}).$$

On the other hand

$$e_{\mathfrak{n}}(JB, B) \leqslant e_{\mathfrak{n}}(J \otimes_{A/\mathfrak{p}} B/\mathfrak{p}, B) = e_{\mathfrak{n}}(B/\mathfrak{p}, B) = e_{\mathfrak{n}/\mathfrak{p}B}(B/\mathfrak{p}).$$

Here the second equality follows from Proposition (1.3.7) as $J_{\mathfrak{p}}$ has length 1. Hence we find

$$\begin{aligned}e_{\mathfrak{n}}(B) &= e_{\mathfrak{n}}(JB, B) + e_{\mathfrak{n}}(B/JB, B) \\ &\leqslant e_{\mathfrak{n}/\mathfrak{p}B}(B/\mathfrak{p}B)\ell_{A_{\mathfrak{p}}}((A/J)_{\mathfrak{p}} + 1) = e_{\mathfrak{n}/\mathfrak{p}B}(B/\mathfrak{p}B)\ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}).\end{aligned}$$

□

(1.4) Representations and pseudo-representations. In this subsection we compare deformation rings of Galois representations with those of the corresponding pseudo-representations.

(1.4.1) Let G be a group and R a commutative ring with 1. Recall [Ta 1, §1] that a pseudo-representation of G over R of dimension d is a function $T : G \to R$ such that T has the following properties of the trace of a representation of G on a finite free R-module:

(1) $T(1) = d$.
(2) $T(g_1 g_2) = T(g_2 g_1)$ for $g_1, g_2 \in G$.

(3) $\sum_{\sigma\in S_{d+1}} \varepsilon(\sigma)T_\sigma(g_1,\dots,g_{d+1}) = 0$ for $g_1,\dots,g_{d+1}\in G$, where S_{d+1} is the symmetric group on $d+1$ letters, $\varepsilon(\sigma)$ denotes the sign of σ, and if σ has the cycle decomposition

$$(i_1^1, i_1^2,\dots,i_1^{k_1})(i_2^1,\dots,i_2^{k_2})\dots(i_{m_\sigma}^1,\dots,i_{m_\sigma}^{k_{m_\sigma}}),$$

then $T_\sigma : G^{d+1}\to R$ is the function

$$(g_1,\dots,g_{d+1})\mapsto T(g_{i_1^1},\dots,g_{i_1^{k_1}})T(g_{i_2^2},\dots,g_{i_2^{k_2}})\dots T(g_{i^1_{m_\sigma}},\dots,g_{i^{k_{m_\sigma}}_{m_\sigma}}).$$

If $A\to A'$ is a surjection of rings and $T_{A'}: G\to A'$ is a pseudo-representation, then by a deformation of $T_{A'}$ to A we mean a lifting of $T_{A'}$ to an A-valued pseudo-representation. If T is a pseudo-representation of G over R, then we may regard T as a map $R[G]\to R$ by linearity.

In the following we shall work with a profinite, finitely topologically generated group G. Let κ be a topological field in which 2 is invertible. If κ is discrete and has characteristic $p>0$, then we set W equal either to κ or to a characteristic 0, discrete valuation ring with residue field κ. In all other cases, we set $W=\kappa$. (In this paper we will apply this with $\kappa=\mathbb{F}$ and $W=\mathcal{O}$; however the results below can be used to study deformations of p-adic Galois representations by taking κ a finite extension of $\mathbb{Q}_p$.)

Suppose that $T_\kappa : G\to\kappa$ is a continuous pseudo-representation of dimension d. For a local Artinian W-algebra A with residue field κ, denote by $D^{\mathrm{ps}}_{T_\kappa}(A)$ the set of continuous deformations of T_κ to A.

Lemma (1.4.2). *Suppose that $d!$ is invertible in κ. Then $D^{\mathrm{ps}}_{T_\kappa}$ is (pro-)represented by a Noetherian, complete local W-algebra $R^{\mathrm{ps}}_{T_\kappa}$.*

Proof. By [Ta 1, Thm. 1] there is a finite subset $S\subset G$ such that a continuous pseudo-representation of G is determined by its values on S. This implies that the tangent space $D^{\mathrm{ps}}_{T_\kappa}(\kappa[\epsilon])$ is finite dimensional over κ. The lemma now follows directly from Grothendieck's representability criterion [Maz, §18]. □

Lemma (1.4.3). *Let $\omega_1,\omega_2 : G\to\kappa^\times$ be distinct characters, and V_κ a non-trivial extension of ω_1 by ω_2. Suppose that $\mathrm{Ext}^1_{\kappa[G]}(\omega_1,\omega_2)$ is 1-dimensional over κ, and let $V_{\kappa[\epsilon]}$ be a deformation of V_κ to the dual numbers $\kappa[\epsilon]$. If $V_{\kappa[\epsilon]}$ induces the trivial deformation on pseudo-representations, then $V_{\kappa[\epsilon]}$ is the trivial deformation.*

Proof. We shall adapt an argument of Carayol which applies when V_κ is absolutely irreducible [Ca, Thm. 1]. Fix a basis of V_κ such that the resulting representation $\bar\rho:\kappa[G]\to M_2(\kappa)$ is upper triangular. Let $A=\kappa[\epsilon]/\epsilon^2$ denote the dual numbers over κ, and suppose that $\rho : A[G]\to M_2(A)$ is a deformation of $\bar\rho$, which satisfies $\mathrm{tr}\rho(\sigma)=\mathrm{tr}\bar\rho(\sigma)$ for $\sigma\in A[G]$. Write $\rho(\sigma)=\bar\rho(\sigma)+\Delta(\sigma)$, where $\Delta(\sigma)\in M_2(\epsilon\kappa)$, and we again denote by $\bar\rho : A[G]\to M_2(A)$ the A-linear extension of $\bar\rho$. Since ρ is a ring map, one sees that

$$\mathrm{tr}(\bar\rho(\sigma_1)\Delta(\sigma_2)+\Delta(\sigma_1)\bar\rho(\sigma_2)) = \mathrm{tr}(\Delta(\sigma_1\sigma_2)) = 0$$

for $\sigma_1,\sigma_2\in A[G]$. Taking $\sigma_1\in\ker\bar\rho$, we see that $\mathrm{tr}(\Delta(\sigma_1)\bar\rho(\sigma_2))=0$ for all $\sigma_2\in\kappa[G]$. Our hypotheses imply that $\bar\rho(\kappa[G])$ consists of all upper triangular matrices in $M_2(\kappa)$, so $\Delta(\sigma_1)$ has the form $\left(\begin{smallmatrix}0 & *\\ 0 & 0\end{smallmatrix}\right)$.

Now for $\sigma\in A[G]$ write $\Delta(\sigma)=\left(\begin{smallmatrix}\alpha(\sigma) & \beta(\sigma)\\ \gamma(\sigma) & -\alpha(\sigma)\end{smallmatrix}\right)\cdot\epsilon$ and $\bar\rho(\sigma)=\left(\begin{smallmatrix}a(\sigma) & b(\sigma)\\ 0 & d(\sigma)\end{smallmatrix}\right)$. The above calculation shows that $\alpha(\sigma)=0$ for $\sigma\in\ker\bar\rho$. If $\tau_1,\tau_2,\tau_4\in\kappa[G]$ have image under $\bar\rho$ equal to $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$, respectively, then computing the traces of $\Delta(\tau_1^2)$ and $\Delta(\tau_4^2)$ shows that $\alpha(\tau_1) = \alpha(\tau_4) = 0$. Let $\sigma \in A[G]$ and set $\sigma' = a(\sigma)\tau_1 + b(\sigma)\tau_2 + d(\sigma)\tau_4$. Then $\sigma - \sigma' \in \ker(\bar\rho)$, so that $\alpha(\sigma - \sigma') = 0$, and

$$\alpha(\sigma) = \alpha(\sigma') = \alpha(\tau_1)a(\sigma) + \alpha(\tau_2)b(\sigma) + \alpha(\tau_4)d(\sigma) = \alpha(\tau_2)b(\sigma).$$

Hence after replacing ρ by $U\rho U^{-1}$, where $U = \left(\begin{smallmatrix}1&0\\ \alpha(\tau_2)\cdot\epsilon&1\end{smallmatrix}\right)$, we may assume that $\alpha(\sigma) = 0$ for all $\sigma \in A[G]$.

But now $\sigma \mapsto b(\sigma) + \beta(\sigma)$ gives a $\mathbb{F}[\epsilon]$-valued cocycle corresponding to an extension of ω_1 by ω_2. Since $\mathrm{Ext}^1_{\mathbb{F}[G]}(\omega_1, \omega_2)$ is 1-dimensional, this cocycle vanishes on $\ker\bar\rho$, and so Δ vanishes on the kernel of $\bar\rho$. In particular we can view Δ as a derivation on $\bar\rho(\kappa[G])$.

The restriction of Δ to the separable subalgebra $\kappa^2 \subset \bar\rho(\kappa[G])$ consisting of elements of the form $\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)$ is necessarily an inner derivation, so after conjugating ρ by an element of $1 + M_2(\epsilon\kappa)$, we may assume that Δ vanishes on this subalgebra. If τ_2 is as above, then

$$0 = \Delta(\tau_2^2) = \bar\rho(\tau_2)\Delta(\tau_2) + \Delta(\tau_2)\bar\rho(\tau_2) = \begin{pmatrix}\gamma(\tau_2)&0\\0&\gamma(\tau_2)\end{pmatrix}$$

implies that $\gamma(\tau_2) = 0$. Similarly,

$$0 = \Delta(\tau_2\tau_1) = \begin{pmatrix}\alpha(\tau_2)&0\\ \gamma(\tau_2)&0\end{pmatrix}$$

so $\alpha(\tau_2) = 0$. Hence Δ is the inner derivation corresponding to $\left(\begin{smallmatrix}\beta(\tau_2)&0\\0&0\end{smallmatrix}\right) \in \bar\rho(\kappa[G])$, so conjugating ρ by $1 + \left(\begin{smallmatrix}\beta(\tau_2)\cdot\epsilon&0\\0&0\end{smallmatrix}\right)$ shows that $\rho \sim \bar\rho$. $\square$

Corollary (1.4.4). *Let V_κ be a finite dimensional κ-vector space equipped with a continuous action of G such that $\mathrm{End}_{\kappa[G]}V_\kappa = \kappa$. Let R_{V_κ} denote the universal deformation W-algebra of V_κ and T_κ the pseudo-representation corresponding to V_κ. Let*

$$\theta : R^{\mathrm{ps}}_{T_\kappa} \to R_{V_\kappa}$$

denote the map induced by sending a G-representation to its trace.

(1) *If V_κ is absolutely irreducible, then θ is an isomorphism.*
(2) *If V_κ is a non-trivial extension of ω_1 by ω_2 for two distinct characters ω_1 and ω_2 of G and $\mathrm{Ext}^1_{\kappa[G]}(\omega_1, \omega_2)$ is 1-dimensional over κ, then θ is a surjection.*

Proof. (1) follows from a result of Nyssen [Ny]. To prove (2), it suffices to show that θ induces a surjection on tangent spaces, and this follows from Lemma (1.4.3). $\square$

Corollary (1.4.5). *Suppose that $G = G_{\mathbb{Q}_p}$ and $\kappa \subset \bar{\mathbb{F}}_p$ and that V_κ is as in Corollary* (1.4.4) *and satisfies one of the conditions* (1) *or* (2)*. Then the map*

$$(R^{\mathrm{ps}}_{T_\kappa}[1/p])^{\mathrm{red}} \to (R_{V_\kappa}[1/p])^{\mathrm{red}} \qquad (1.4.6)$$

induced by θ is an isomorphism.

Proof. When V_κ satisfies Corollary (1.4.4)(1), there is nothing to show. Suppose that it satisfies Corollary (1.4.4)(2). Note that (1.4.6) is a surjection between reduced Jacobson rings [GD, IV, 10.5.7], and so it suffices to check that it induces a surjection on closed points. If $E/W[1/p]$ is a finite extension and x is an E-valued point of $(R^{\mathrm{ps}}_{T_\kappa}[1/p])^{\mathrm{red}}$, then after replacing E by a finite extension, we may assume that x corresponds to a semi-simple G-representation on a finite dimensional E-representation V_x.

Let $\mathcal{O}_E$ denote the ring of integers of E and let π_E denote a uniformizer. If V_x is absolutely irreducible, then it contains a lattice whose reduction mod π_E is a non-trivial extension of ω_1 by ω_2, and so x corresponds to a point of R_{V_κ}. If V_x is reducible, then it is a sum of two characters $\tilde{\omega}_1$ and $\tilde{\omega}_2$ lifting ω_1 and ω_2, respectively. Now any extension of $\tilde{\omega}_1$ by $\tilde{\omega}_2$ gives rise to the pseudo-representation corresponding to x. Thinking of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ as $\mathcal{O}_E^\times$-valued characters, consider the map

$$\mathrm{Ext}^1_{\mathcal{O}_E[G_{\mathbb{Q}_p}]}(\tilde{\omega}_1, \tilde{\omega}_2) \to \mathrm{Ext}^1_{\kappa[G_{\mathbb{Q}_p}]}(\omega_1, \omega_2).$$

Since the left hand side is a finitely generated $\mathcal{O}_E$-module, the image of this map is non-zero, and hence it is surjective. It follows that there is an extension of $\tilde{\omega}_1$ by $\tilde{\omega}_2$ which gives rise to V_κ, so x is again induced by a point of R_{V_κ}. □

Corollary (1.4.7). *Let $T_{\mathbb{F}}$ be a 2-dimensional pseudo-representation of $G_{\mathbb{Q}_p}$ over $\mathbb{F}$ and assume that $T_{\mathbb{F}}$ is either irreducible or a sum of two distinct pseudo-representations of dimension 1, given by $\mathbb{F}^\times$-valued characters ω_1 and ω_2 of $G_{\mathbb{Q}_p}$. If $p = 3$, assume also that $\omega_1\omega_2^{-1} \neq \omega$.*

Denote by $R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$ the image of $R^{\mathrm{ps}}_{T_{\mathbb{F}}}$ in $(R^{\mathrm{ps}}_{T_{\mathbb{F}}}[1/p])^{\mathrm{red}}$. Then there is a finite free $R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$-module M of rank 2, equipped with a continuous action of $G_{\mathbb{Q}_p}$, such that for $\sigma \in G_{\mathbb{Q}_p}$ the trace of σ on M is given by $T(\sigma) \in R^{\mathrm{ps},\circ}_{T_{\mathbb{F}}}$.

Proof. This follows from Corollary (1.4.5) once we remark that, if ω_1 and ω_2 are distinct, then $\mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\omega_1, \omega_2)$ is 1-dimensional, provided that $\omega_2\omega_1^{-1}$ is not the mod p cyclotomic character. Since we can exchange the roles of ω_1 and ω_2, the only case in which Corollary (1.4.5) does not apply is when $\omega_2\omega_1^{-1} = \omega = \omega^{-1}$, which can happen only if $p = 3$. □

(1.4.8) We return for a moment to the situation of an arbitrary topological κ and V_κ a finite dimensional κ-representation of G. We denote by $R^\square_{V_\kappa}$ the universal framed deformation ring of V_κ (cf. (1.1)) and by R_{V_κ} its universal deformation ring when V_κ has trivial endomorphisms.

We will need an extension of the universal mapping property for $R^\square_{V_\kappa}$ and R_{V_κ}. To state it, let R be one of these rings, and write V_R for the universal G-representation over R. Let κ' be a field, $\eta : R \to \kappa'$ a map of rings, and set $V_{\kappa'} = V_R \otimes_R \kappa'$. Suppose that A is a local Artin ring with residue field κ'. Let A^+ denote the preimage of $\eta(R)$ in A. Then A^+ is a union of subrings A_λ which surject onto $\eta(R)$, with finitely generated kernel. Each of these A_λ is a complete local ring with residue field κ. If V_A is a deformation of $V_{\kappa'}$ to A, we say that V_A is *continuous* if it is induced by a deformation of $V_R \otimes_R \eta(R)$ to some A_λ.

Lemma (1.4.9). *Suppose $R = R^\square_{V_\kappa}$. Given κ' and A as above, there is a bijection between maps $R^\square_{V_\kappa} \to A$ lifting η and the set of isomorphism classes of continuous deformations of $V_{\kappa'}$ to V_A together with a lifting of the chosen basis on $V_{\kappa'}$.*

An analogous statement holds for $R = R_{V_\kappa}$ and unframed deformations.

Proof. Since G is topologically finitely generated, R is a Noetherian ring. Hence η has finitely generated kernel and any map $R \to A$ lifting factors through one of the A_λ. The lemma is now a simple consequence of the definition of a continuous deformation and the universal property of $R^\square_{V_\kappa}$ and R_{V_κ}. □

(1.5) $\mathrm{GL}_2(\mathbb{Q}_p)$**-representations mod** p**.** In this subsection we study certain (pro-)finite length, admissible $\mathrm{GL}_2(\mathbb{Q}_p)$-representations built out of irreducible mod p $\mathrm{GL}_2(\mathbb{Z}_p)$-representations and the Galois representations obtained from them by applying the functor $\mathbf{V}$ introduced in (1.2).

(1.5.1) As in (1.2), we fix an integer $r \in [0, p-1]$, and we consider the representations $\sigma = \mathrm{Sym}^r\mathbb{F}^2$ of KZ obtained by letting $p \in K$ act trivially. We also fix a character $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ and an element $\lambda \in \mathbb{F}$.

The operator T introduced in (1.2.1) acts on $I(\sigma) = \mathrm{Ind}_{KZ}^G \mathrm{Sym}^r\mathbb{F}^2$ and hence on $I_\chi(\sigma) = I(\sigma) \otimes \chi \circ \det$. We set

$$\Pi(r, \lambda, \chi) = \varprojlim I_\chi(\sigma)/(T-\lambda)^n I_\chi(\sigma).$$

The $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\Pi(r, \lambda, \chi)$ is naturally a module over $\mathbb{F}[\![S]\!]$, where S acts on $I_\chi(\sigma)/(T-\lambda)^n I_\chi(\sigma)$ by $T-\lambda$. We will sometimes write $T-\lambda$ for S. As mentioned in (1.2.5), inverse limits on the category of admissible representations are exact. In particular, one sees that $\Pi(r, \lambda, \chi)/(T-\lambda)^n \xrightarrow{\sim} I_\chi(\sigma)/(T-\lambda)^n$. The list of possibilities for $\pi(r, \lambda\, \chi) = \Pi(r, \lambda, \chi)/(T-\lambda)$ is given in Proposition (1.2.2).

We assume that the central character of $I_\chi(\sigma)$ (and hence also of $\Pi(r, \lambda, \chi)$) is equal to the reduction of ψ modulo π.

Lemma (1.5.2). *Let*

$$\mathbf{V}(\Pi(r, \lambda, \chi)) = \varprojlim \mathbf{V}(I_\chi(\sigma)/(T-\lambda)^n I_\chi(\sigma)).$$

Then $\mathbf{V}(\Pi(r, \lambda, \chi))$ *is a finite free* $\mathbb{F}[\![S]\!]$*-module which has rank* 1 *if* $\lambda \neq 0$ *and has rank* 2 *if* $\lambda = 0$.

Proof. Let $i = 1$ if $\lambda \neq 0$ and 2 if $\lambda = 0$. The exactness of $\mathbf{V}$ and Theorem (1.2.4) imply that

$$\mathbf{V}(\Pi(r, \lambda, \chi))/S\mathbf{V}(\Pi(r, \lambda, \chi)) \xrightarrow{\sim} \mathbf{V}(\pi(r, \lambda, \chi))$$

has $\mathbb{F}$-dimension i, and hence one sees that there is a surjection

$$\mathbb{F}[\![S]\!]^i/S^n \to \mathbf{V}(\Pi(r, \lambda, \chi)/(\pi-\lambda)^n\Pi(r, \lambda, \chi)).$$

Since $I_\chi(\sigma)$ has no $T-\lambda$ torsion (this is easily seen using the fact that the functions in $I_\chi(\sigma)$ are compactly supported), $\Pi(r, \lambda, \chi)/(T-\lambda)^n\Pi(r, \lambda, \chi)$ has a filtration of length n where the associated graded pieces are isomorphic to $\pi(r, \lambda, \chi)$. Hence $\mathbf{V}(\Pi(r, \lambda, \chi)/(T-\lambda)^n\Pi(r, \lambda, \chi))$ has length ni by Theorem (1.2.4), and this surjection is an isomorphism. The lemma follows by passing to the limit over n. □

Lemma (1.5.3). $\mathbf{V}(\Pi(r, 0, \chi))$ *is a deformation to* $\mathbb{F}[\![T]\!]$ *of the absolutely irreducible* 2*-dimensional* $\mathbb{F}$*-representation* $\mathbf{V}(\pi(r, 0, \chi))$ *of* $G_{\mathbb{Q}_p}$*. If* R *denotes the universal deformation ring of this representation, then the map* $R \to \mathbb{F}[\![T]\!]$ *is surjective.*

Proof. It suffices to consider the case when χ is trivial. The first claim follows from Lemma (1.5.2). To prove the second, we first consider the case when $r \in [0, p-2]$. Let $E/W(\mathbb{F})[1/p]$ be a finite totally ramified extension, as in (1.0), and write $E(T) \in W(\mathbb{F})[T]$ for the Eisenstein polynomial of π and $e = [E : W(\mathbb{F})[1/p]]$.

Consider $\mathrm{Sym}^r W(\mathbb{F})^2$ viewed as a KZ-module, by letting $p \in KZ$ act trivially. The compact induction $\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2$ is a $W(\mathbb{F})[T]$-module, where T acts via the KZ-bivariant function on $\mathrm{Sym}^r W(\mathbb{F})^2$ which is supported on KZ and takes $\begin{bmatrix} 1 & 0 \\ 0 & p^{-1} \end{bmatrix}$ to $\mathrm{Sym}^r \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$ (cf. (1.2.1)). Then [Br 2, Prop. 3.3.3] asserts that

$$(\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2)/(E(T)) \xrightarrow{\sim} (\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathcal{O}^2)/(T-\pi) \tag{1.5.4}$$

is p-torsion free with central character χ_{cyc}^r and that its reduction modulo π is isomorphic to $\pi(r,0,1)$. By [BB 1, Thm. 4.3.1][4] and the discussion in Theorem (1.2.8), applying $\mathbf{V}$ to the p-adic completion of $\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r W(\mathbb{F})^2/(E(T))$ yields a lattice in a 2-dimensional, crystalline E-representation V_π of $G_{\mathbb{Q}_p}$, having Hodge-Tate weights $0, r+1$. Moreover, if $D_{\mathrm{cris}}^*(V_\pi)$ denotes the weakly admissible module contravariantly associated to V_π, then the trace of the Frobenius φ on $D_{\mathrm{cris}}^*(V_\pi)$ is equal to π.

Let $R^{0,r+1}$ denote the quotient of R corresponding to crystalline deformations having Hodge-Tate weights $0, r+1$. Suppose that A is any finite local $W(\mathbb{F})$-algebra, and consider a map of $W(\mathbb{F})$-algebras $\theta : R^{0,r+1} \to A$. Denote by V_A the corresponding A-representation of $G_{\mathbb{Q}_p}$. The theory of Fontaine-Laffaille [FL] implies that there is an element $a_p \in R^{0,r+1}$ such that for any A and θ as above, the trace of φ on $D_{\mathrm{cris}}(V_A^*)$ is equal to $\theta(a_p)$. Here V_A^* denotes the A-dual of V_A.

Now the reduction of (1.5.4) modulo p is

$$\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathbb{F}^2/T^e \xrightarrow{\sim} \Pi(r,0,1)/T^e\Pi(r,0,1).$$

It follows that $R \to \mathbb{F}[\![T]\!]/T^e$ factors through the quotient $R^{0,r+1}$ and sends a_p to T. Since we can make this argument for any totally ramified extension E of $W(\mathbb{F})$, this holds for any e, and the lemma follows when $r \in [0, p-2]$. When $r = p-1$, it follows from the case $r = 0$ and Lemma (1.5.5) below. $\square$

Lemma (1.5.5). *There is a morphism of* $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$*-modules*

$$\mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1}\mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1}$$

which induces a continuous isomorphism of $\mathbb{F}[\![T]\!][\mathrm{GL}_2(\mathbb{Q}_p)]$*-modules*

$$\Pi(0,\lambda,\chi) \xrightarrow{\sim} \Pi(p-1,\lambda,\chi)$$

for $\lambda \in \mathbb{F}\backslash\{\pm 1\}$.

Proof. It suffices to consider the case $\chi = 1$.

We recall the notation of [Br 1, 2.3]. Suppose that σ, V_σ and $I(\sigma)$ are as in (1.2). If $g \in G$ and $v \in V_\sigma$, we denote by $[g,v] \in I(\sigma)$ the function which is supported on KZg^{-1} and given by $[g,v](g') = \sigma(g'g)v$ for $g' \in KZg^{-1}$. If $\varphi : G \to \mathrm{End}_{\mathbb{F}} V_\sigma$ is a KZ-bivariant function, then the corresponding operator T_φ on $I(\sigma)$ is given by [Br 1, 2.4]

$$T_\varphi([g,v]) = \sum_{g'KZ \in G/KZ} [gg', \varphi(g'^{-1})(v)]. \tag{1.5.6}$$

We identify $\mathrm{Sym}^{p-1}\mathbb{F}^2$ with the space of polynomials in $\mathbb{F}[x,y]$ which are homogeneous of degree $p-1$, with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acting by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x^{p-1-j}y^j = (ax+cy)^{p-1-j}(bx+dy)^j.$$

Let $I \subset \mathrm{GL}_2(\mathbb{Z}_p)$ denote the Iwahori subgroup consisting of matrices whose reduction modulo p is upper triangular. Then we identify $I\backslash K$ with $\mathbb{P}^1(\mathbb{F}_p)$ via $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c,d)$, and we may think of x, y as projective co-ordinates on $\mathbb{P}^1(\mathbb{F}_p)$, so that $\mathrm{Sym}^{p-1}\mathbb{F}^2$ becomes a subspace of $\mathrm{Ind}_I^K \mathbf{1}$, consisting of the functions with average value 0.

[4] Note also the alternate description of $(\mathrm{Ind}_{KZ}^G \mathrm{Sym}^r \mathcal{O}^2)/(T-\pi)$ given by [Br 2, 3.2.1(i)].

Set $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$, and denote by T the operator introduced in (1.2), which corresponds to the KZ-bi-invariant function φ_α supported on $KZ\alpha^{-1}KZ$ and sending α^{-1} to $\mathrm{Sym}^r \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. A simple calculation (cf. [Br 1, Prop. 4.1.2]) shows that the elements $[\alpha, 1]$ and $[1, 1]$ of $\mathrm{Ind}_{KZ}^G \mathbf{1}$ are I-invariant, and hence so is $b = [\alpha, 1] - T[1, 1]$. By Frobenius reciprocity we obtain a map

$$\mathrm{Ind}_I^K \mathbf{1} \to \mathrm{Ind}_{KZ}^G \mathbf{1} \tag{1.5.7}$$

which sends the characteristic function of the coset I to $-b$.

We claim that this map annihilates the space of constant functions. To see this, it suffices to show that $\sum_{k\in K/I} k \cdot b = 0$. Now

$$\sum_{k\in K/I} k \cdot T[1, 1] = T(\sum_{k\in K/I} [k, 1]) = (p + 1)T[1, 1] = T[1, 1],$$

while

$$\sum_{k\in K/I} k \cdot [\alpha, 1] = \sum_{k\in K/I} [k\alpha, 1] = T[1, 1],$$

where the final equality follows from (1.5.6) and the fact that the map $K/I \overset{k\mapsto k\cdot\alpha}{\to} KZ\alpha KZ/KZ$ is an isomorphism. Hence (1.5.7) kills the constant functions as claimed and induces a map

$$\mathrm{Sym}^{p-1}\mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1}$$

taking x^{p-1} to b. We denote by

$$h_b : \mathrm{Ind}_{KZ}^G \mathrm{Sym}^{p-1}\mathbb{F}^2 \to \mathrm{Ind}_{KZ}^G \mathbf{1} = I(\mathbf{1})$$

the map obtained by Frobenius reciprocity. h_b is characterized by the property that $h_b([1, x^{p-1}]) = b$.

We now check that h_b is compatible with the action of T. Let $C \subset K$ denote the set of matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ i & 1 \end{smallmatrix}\right)$ with $i = 0, 1, \ldots, p - 1$ together with the matrix $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $C\alpha$ consists of a set of representatives for $KZ\alpha KZ/KZ$, and we compute

$$\begin{aligned} T([1, x^{p-1}]) &= \sum_{gKZ\in G/KZ} [g, \varphi_\alpha(g^{-1})(x^{p-1})] \\ &= \sum_{k\in C} (k\alpha) \cdot [1, \varphi_\alpha(\alpha^{-1}k^{-1})(x^{p-1})] = \sum_{k\in C} (k\alpha) \cdot [1, (\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} k^{-1})(x^{p-1})] \\ &= \sum_{k\in C\setminus\{1\}} (k\alpha) \cdot [1, y^{p-1}] = \sum_{k\in C\setminus\{1\}} (k\alpha w) \cdot [1, x^{p-1}]. \end{aligned}$$

Hence we have

$$h_b(T[1, x^{p-1}]) = \sum_{k\in C\setminus\{1\}} (k\alpha w) \cdot b.$$

Note that

$$k\alpha w[\alpha, 1] = [k\alpha w\alpha, 1] = [kpw, 1] = [1, 1],$$

so

$$h_b(T[1,x^{p-1}]) = \sum_{k\in C\backslash\{1\}} ([1,1]-T[k\alpha w,1])$$
$$= \sum_{k\in C\backslash\{1\}} -T[k\alpha,1] = T([\alpha,1]-T[1,1]) = Th_b([1,x^{p-1}]).$$

Since any map of $\mathrm{GL}_2(\mathbb{Q}_p)$ modules with source $\mathrm{Ind}_{KZ}^G\mathrm{Sym}^{p-1}\mathbb{F}^2$ is characterized by its value on $[1,x^{p-1}]$, this shows that h_b is a map of $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$-modules.

Now let $\lambda\in\mathbb{F}$. Then h_b is non-zero modulo $T-\lambda$, for if $(T-\lambda)c = b$ for some $c\in\mathrm{Ind}_{KZ}^G\mathbf{1}$, then by comparing supports one finds that c must be in $\mathbb{F}\cdot[1,1]$ (see [BL, Lem. 20]). But then $[\alpha,1]$ would be in the span of $[1,1]$ and $T[1,1]$, which is not the case.

Suppose that $\lambda\neq\pm1$. Taking the reduction of h_b modulo $(T-\lambda)^n$ gives a map of $\mathbb{F}[T][\mathrm{GL}_2(\mathbb{Q}_p)]$-modules

$$I(\mathrm{Sym}^{p-1}\mathbb{F}^2)/(T-\lambda)^n \to I(\mathbf{1})/(T-\lambda)^n. \tag{1.5.8}$$

Since $I(\mathbf{1})/(T-\lambda)$ is irreducible and (1.5.8) is non-zero modulo $T-\lambda$, it is surjective by Nakayama's lemma. Since both sides have the same length, (1.5.8) is an isomorphism. Passing to the limit over n yields the isomorphism of the lemma. □

Lemma (1.5.9). *If $\lambda\in\mathbb{F}^\times$, then the action of $G_{\mathbb{Q}_p}$ on $\mathbf{V}(\Pi(r,\lambda,\chi))$ is given by the $\mathbb{F}[\![S]\!]^\times$-valued character $\chi^{-1}\psi\chi_{\mathrm{cyc}}\mu_T$, where μ_T is the unramified character of $G_{\mathbb{Q}_p}$ sending the geometric Frobenius corresponding to p, Frob_p^{-1}, to $T=S+\lambda$.*

Proof. We use the notation of the proof of Lemma (1.5.3). Again it suffices to consider the case when χ is trivial. Let $[\lambda]\in W(\mathbb{F})$ be the Teichmüller representative of λ and consider the quotient

$$(\mathrm{Ind}_{KZ}^G\mathrm{Sym}^r W(\mathbb{F})^2)/(E(T-[\lambda])) \xrightarrow{\sim} (\mathrm{Ind}_{KZ}^G\mathrm{Sym}^r\mathcal{O}^2)/(T-([\lambda]+\pi)). \tag{1.5.10}$$

By [BB 2, Thm. 7.2.2] and Lemma (1.2.10), p-adically completing (1.5.10) and applying $\mathbf{V}$ produces the character $\mu_{\lambda_1}\chi_{\mathrm{cyc}}^{r+1}$, where $\mu_{\lambda_1}: G_{\mathbb{Q}_p}\to\mathcal{O}^\times$ is the unramified character sending Frob_p^{-1} to the unit root of the quadratic equation $X^2-([\lambda]+\pi)X+p^{r+1}$. Hence applying $\mathbf{V}$ to $(\mathrm{Ind}_{KZ}^G\mathrm{Sym}^r\mathbb{F}^2)/(T-\lambda)^e$ produces the character

$$\mu_T\chi_{\mathrm{cyc}}^{r+1}: G_{\mathbb{Q}_p}\to(\mathcal{O}/\pi^e)^\times \xrightarrow{\sim} (\mathbb{F}[\![S]\!]/S^e)^\times.$$

Since we are assuming that χ is trivial, $\psi=\chi_{\mathrm{cyc}}^r$, and the lemma follows as in the proof of Lemma (1.5.3). □

Lemma (1.5.11). *Let $V_{\mathbb{F}}$ be a continuous representation of $G_{\mathbb{Q}_p}$ on a 2-dimensional $\mathbb{F}$-vector space, with determinant $\psi\chi_{\mathrm{cyc}}$. Denote by $\bar{\mathfrak{r}}$ the associated pseudo-representation and by $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ the universal deformation $\mathcal{O}$-algebra of $\bar{\mathfrak{r}}$. Suppose that $\mathbf{V}(\pi(r,\lambda,\chi))$ is a Jordan-Hölder factor of $V_{\mathbb{F}}$.*

Then there is a map $\theta: R^{\mathrm{ps}}(\bar{\mathfrak{r}})\to\mathbb{F}[\![S]\!]$ such that for $\sigma\in G_{\mathbb{Q}_p}$, the element $\theta(T(\sigma))\in\mathbb{F}[\![S]\!]$ acts on $\mathbf{V}(\Pi(r,\lambda,\chi))$ by $\sigma+\psi\chi_{\mathrm{cyc}}(\sigma)\sigma^{-1}$.

Moreover, the map θ is surjective unless $V_{\mathbb{F}}$ has scalar semi-simplification (so $r=p-2$ and $\lambda=\pm1$), in which case the image of θ has the form $\mathbb{F}[\![S']\!]$, where $S'\in\mathbb{F}[\![S]\!]$ is an element of S-adic valuation 2.

If $V_{\mathbb{F}}$ is reducible, then θ depends only on $V_{\mathbb{F}}^{\mathrm{ss}}$ and not on (r,λ,χ).

Proof. If $V_{\mathbb{F}}$ is irreducible, this follows from Lemma (1.5.3), since $\mathbf{V}(\Pi(r,\lambda,\chi))$ is a deformation of $V_{\mathbb{F}}$ to $\mathbb{F}[\![S]\!]$.

If $V_{\mathbb{F}}$ is reducible, then $\mathbf{V}(\Pi(r,\lambda,\chi))$ is a direct summand of the deformation $\chi\mu_{T^{-1}} \oplus \chi^{-1}\psi\chi_{\mathrm{cyc}}\mu_T$ of $V_{\mathbb{F}}^{\mathrm{ss}}$ by Lemma (1.5.9), and this gives a map $\theta : R^{\mathrm{ps}}(\bar{\mathfrak{t}}) \to \mathbb{F}[\![S]\!]$, with $T(\sigma)$ acting as claimed. Since this deformation depends only on $V_{\mathbb{F}}^{\mathrm{ss}}$, so does θ.

Now if $\sigma \in G_{\mathbb{Q}_p}$ acts via the geometric Frobenius on the residue field of $\bar{\mathbb{Q}}_p$, then

$$\theta(T(\sigma)) = \chi(\sigma)(S+\lambda)^{-1} + \chi^{-1}\psi\chi_{\mathrm{cyc}}(\sigma)(S+\lambda). \tag{1.5.12}$$

The coefficient of S in the above expression is $-\chi(\sigma)\lambda^{-2} + \chi^{-1}\psi\chi_{\mathrm{cyc}}(\sigma)$, which is 0 for all such σ if and only if $(\chi\mu_{\lambda^{-1}})^2 = \psi\chi_{\mathrm{cyc}}$. Since $\psi\chi_{\mathrm{cyc}} = \omega^{r+1}\chi^2$, this condition is equivalent to asking that $\mu_{\lambda^{-1}}^2 = \omega^{r+1}$, which holds exactly if $r = p-2$ and $\lambda = \pm 1$. This is equivalent to asking that $V_{\mathbb{F}}$ have scalar semi-simplification.

For $i \geq 2$, the coefficient of S^i in (1.5.12) is $(-1)^i\chi(\sigma)\lambda^{-i-1} \neq 0$, so if the coefficient of S in (1.5.12) is 0, then we may take $S' = \sum_{i=2}^{\infty}(-1)^i S^i \lambda^{-i-1}$. □

(1.6) Local patching and multiplicities. In this subsection we give a construction of certain finite modules over deformation rings for 2-dimensional pseudo-representations of $G_{\mathbb{Q}_p}$. We assume from now on that the triple (k,τ,ψ) satisfies the Hypothesis (1.2.6).

(1.6.1) We now return to the notation of (1.1). In particular $k \geq 2$ and $\psi : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ are as in Proposition (1.1.1), $\tau : I_{\mathbb{Q}_p} \to \mathrm{GL}_2(E)$ is of Galois type, and

$$L_{k,\tau} \subset \sigma(k,\tau) = \sigma(\tau) \otimes_E \mathrm{Sym}^{k-2}E^2$$

is a $\mathrm{GL}_2(\mathbb{Z}_p)$-stable $\mathcal{O}$-lattice. We will regard ψ as a character of $\mathbb{Q}_p^\times$, as before, and we remind the reader that we have

$$\psi|_{\mathbb{Z}_p^\times} = \chi_{\mathrm{cyc}}^{k-2} \det \tau|_{\mathbb{Z}_p^\times} = \sigma(k,\tau)|_{\mathbb{Z}_p^\times}$$

where the final term means the central character of $\sigma(k,\tau)$.

We again denote by $V_{\mathbb{F}}$ the underlying $\mathbb{F}$-vector space of $\bar{\rho}$, and we denote by $\bar{\mathfrak{t}} : G_{\mathbb{Q}_p} \to \mathbb{F}$ the pseudo-representation given by the trace of $\bar{\rho}$, and by $R^{\mathrm{ps}}(\bar{\mathfrak{t}})$ the universal deformation $\mathcal{O}$-algebra of $\bar{\mathfrak{t}}$.

Suppose that E' is a finite extension of E and that $\mathfrak{t}$ is a deformation of $\bar{\mathfrak{t}}$ to $\mathcal{O}_{E'}$. Enlarging E', if necessary, and regarding $\mathfrak{t}$ as an E'-valued pseudo-representation, there is a representation $V_{\mathfrak{t}}$ of $G_{\mathbb{Q}_p}$ on a 2-dimensional E'-vector space, so that $\mathfrak{t}$ is given by the trace of $V_{\mathfrak{t}}$. Moreover the semi-simplification of $V_{\mathfrak{t}}$ is uniquely determined.

Suppose that $V_{\mathfrak{t}}$ is of type (k,τ,ψ). By Hypothesis (1.2.6) there is an admissible $\mathcal{O}_{E'}$-lattice $\Pi_{\mathfrak{t}}$ such that $V_{\mathfrak{t}} \xrightarrow{\sim} \mathbf{V}(\Pi_{\mathfrak{t}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, $\Pi_{\mathfrak{t}}$ has central character ψ and there is a $K = \mathrm{GL}_2(\mathbb{Z}_p)$-equivariant embedding $\sigma(k,\tau) \hookrightarrow \Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. If $V_{\mathfrak{t}}$ is absolutely irreducible, then we may and do assume that $\Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ has no non-zero, proper, closed, $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant subspaces, while if $V_{\mathfrak{t}}$ is reducible, we take $\Pi_{\mathfrak{t}}$ to be one of the representations constructed in the proof of Theorem (1.2.8).

Since $\Pi_{\mathfrak{t}}$ has central character ψ, the embedding $\sigma(k,\tau) \hookrightarrow \Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ becomes KZ-equivariant if we let Z act on $\sigma(k,\tau)$ via ψ, and hence we obtain a map from the compact induction

$$\mathrm{Ind}_{KZ}^{G}\sigma(k,\tau) \to \Pi_{\mathfrak{t}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Multiplying this map by a power of p, if necessary, we may assume that it induces a map

$$\text{Ind}_{KZ}^{G} L_{k,\tau} \to \Pi_{\mathfrak{r}}. \tag{1.6.2}$$

Denote by $\Pi(\mathfrak{r})$ the closure of the image of (1.6.2). It is an admissible $\mathcal{O}$-lattice, whose E'-span is $\Pi_{\mathfrak{r}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ if $V_{\mathfrak{r}}$ is absolutely irreducible and is a proper closed submodule otherwise [BE, 2.2.1] or [Co 1, Thm. 0.4]. More precisely, in the latter case $\Pi(\mathfrak{r}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $(\Pi_{\mathfrak{r}}/\Pi(\mathfrak{r})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ are infinite dimensional, admissible $\text{GL}_2(\mathbb{Q}_p)$-representations which depend only on the semi-simplification of $V_{\mathfrak{r}}$. In particular applying $\mathbf{V}$ to any admissible $\mathcal{O}$-lattice in either of these representations gives a non-zero representation of $G_{\mathbb{Q}_p}$ on a finite free $\mathcal{O}$-module of rank 1. Moreover, since $\mathbf{V}(\Pi(\mathfrak{r}))$ depends only on the semi-simplification of $V_{\mathfrak{r}}$, the action of $G_{\mathbb{Q}_p}$ on $\mathbf{V}(\Pi(\mathfrak{r}))$ is via χ_{cyc}^{k-1} times a finitely ramified character.

Let $V(\mathfrak{r}) = \mathbf{V}(\Pi(\mathfrak{r}))$. The remarks of the previous paragraph show that the E'-span of the image of the composite

$$V(\mathfrak{r}) \to \mathbf{V}(\Pi_{\mathfrak{r}}) \to V_{\mathfrak{r}}$$

is $V_{\mathfrak{r}}$ if $V_{\mathfrak{r}}$ is absolutely irreducible and is a 1-dimensional E'-subspace of $V_{\mathfrak{r}}$ otherwise.

Next suppose that we are given a finite collection of distinct deformations $U = \{\mathfrak{r}_1, \dots, \mathfrak{r}_n\}$ of $\bar{\mathfrak{r}}$ (possibly defined over different fields E') and for each $\mathfrak{r}_i$ a potentially semi-stable representation $V_{\mathfrak{r}_i}$ of type (k, τ, ψ) giving rise to $\mathfrak{r}_i$. Then we obtain a map as in (1.6.2) for each $\mathfrak{r}_i$, and we denote by $\Pi(U)$ the closure of the image of

$$\text{Ind}_{KZ}^{G} L_{k,\tau} \to \bigoplus_{i=1}^{n} \Pi_{\mathfrak{r}_i}.$$

This is again an admissible $\mathcal{O}$-lattice.

Finally if we are given a countable collection $U = \{\mathfrak{r}_i\}_{i \geq 1}$ of deformations and a potentially semi-stable representation $V_{\mathfrak{r}_i}$ of type (k, τ, ψ) giving rise to $\mathfrak{r}_i$, then we set

$$\Pi(U) = \varprojlim \Pi(U')$$

where U' runs over finite subsets of U. We set $V(U) = \varprojlim \mathbf{V}(\Pi(U'))$.

Lemma (1.6.3). *Suppose $U = \{\mathfrak{r}_i\}_{i \geq 1}$ is as above. Then*

(1) *$V(U)$ is naturally an $R^{\text{ps}}(\bar{\mathfrak{r}})$-module.*
(2) *If $U' \subset U$ is any subset, then the natural map $V(U) \to V(U')$ is a map of $R^{\text{ps}}(\bar{\mathfrak{r}})$-modules.*
(3) *If $\mathfrak{r} \in U$, then $R^{\text{ps}}(\bar{\mathfrak{r}})$ acts on $V(\mathfrak{r})$ via the image of the corresponding map $x_{\mathfrak{r}} : R^{\text{ps}}(\bar{\mathfrak{r}}) \to \mathcal{O}_{E'}$. In particular $V(\mathfrak{r})$ is an $x_{\mathfrak{r}}(R^{\text{ps}}(\bar{\mathfrak{r}}))$-module.*

Proof. It suffices to prove the lemma when $U = \{\mathfrak{r}_1, \dots, \mathfrak{r}_n\}$ is finite, where the $\mathfrak{r}_i$ are distinct pseudo-representations.

Note that we have an inclusion $V(U) \hookrightarrow \bigoplus_{i=1}^n V_{\mathfrak{r}_i}$, and $R^{\text{ps}}(\bar{\mathfrak{r}})$ acts on each $V_{\mathfrak{r}_i}$ via the corresponding character $x_{\mathfrak{r}_i} : R^{\text{ps}} \to \mathcal{O}_{E'}$. We saw in (1.4) that R^{ps} is topologically generated by the elements $T(\sigma)$ with $\sigma \in G_{\mathbb{Q}_p}$. Hence it suffices to check that the map $T(\sigma) : V(U) \to \bigoplus_{r=1}^n V_{\mathfrak{r}_i}$ induced by $T(\sigma)$ has image in $V(U)$.

The operator $\sigma^2 - T(\sigma)\sigma + \psi(\sigma)\chi_{\text{cyc}}(\sigma)$ acts on each $V_{\mathfrak{r}_i}$ by 0, so that $T(\sigma)$ on $\bigoplus_{i=1}^n V_{\mathfrak{r}_i}$ is given by $\sigma + \psi\chi_{\text{cyc}}(\sigma)\sigma^{-1}$, which preserves $V(U)$ since $V(U) \subset \bigoplus_{i=1}^n V_{\mathfrak{r}_i}$ is a $G_{\mathbb{Q}_p}$-stable subspace.

This shows (1). Since the map in (2) respects $G_{\mathbb{Q}_p}$ actions, (2) also follows, and (3) is clear. □

(1.6.4) Suppose that Q is a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ on an $\mathbb{F}$-vector space and that we are given a finite collection P of representations of the form $\pi(r, \lambda, \chi)$, all with some fixed central character ψ. We set $Q_{\widehat{P}} = \varprojlim Q'$ where Q' runs over finite length quotients of Q all of whose Jordan-Hölder factors are isomorphic to a subquotient of a representation $\pi(r, \lambda, \chi) \in P$.

It is clear that the functor $Q \mapsto Q_{\widehat{P}}$ is right exact. We write $\mathbf{V}(Q_{\widehat{P}}) = \varprojlim \mathbf{V}(Q')$.

Lemma (1.6.5). *Let $Q = \mathrm{Ind}_{KZ}^G L$ where $L = \mathrm{Sym}^r \mathbb{F}^2 \otimes \chi \circ \det$ is an irreducible representation of KZ on a finite dimensional $\mathbb{F}$-vector space (so $r \in [0, p-1]$). Suppose that P is a finite collection of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations of the form $\pi(r', \lambda', \chi')$ with central character $\psi = \omega^r \chi^2$, and let Λ be the set of $\lambda \in \mathbb{F}$ such that $\pi(r, \lambda, \chi)$ has the same semi-simplification as an element of P.*

Then $Q_{\widehat{P}}$ is a successive extension of the $|\Lambda|$ representations $\Pi(r, \lambda, \chi)$, $\lambda \in \Lambda$, introduced in (1.5)*.*

Proof. Write $P(T) = \prod_{\lambda \in \Lambda}(T - \lambda)$, and let Q' be any finite length quotient of Q all of whose Jordan-Hölder factors are isomorphic to a subquotient of a representation $\pi(r', \lambda', \chi') \in P$. For any polynomial $R(T) \in \mathbb{F}[T]$ we will write $R(T)Q'$ for the image of $R(T)Q$ in Q'.

For $m \geq 0$, the submodules $P(T)^m Q' \subset Q'$ form a descending sequences of subrepresentations of Q', and since Q' has finite length, they stabilize for $m \geq m_0$ for some integer m_0. By [BL, Prop. 32] any irreducible quotient of $\mathrm{Ind}_{KZ}^G L$ is a quotient of $\mathrm{Ind}_{KZ}^G L/(T - \lambda)\mathrm{Ind}_{KZ}^G L$ for some λ. Since $P(T)^{m_0} Q'$ is a quotient of Q via the composite

$$Q \overset{P(T)^{m_0}}{\to} P(T)^{m_0} Q \to P(T)^{m_0} Q',$$

if this module is non-zero, it admits a non-zero quotient which is also a quotient of $\pi(r, \lambda, \chi) = Q/(T - \lambda)Q$ for some $\lambda \in \mathbb{F}$. But then $\pi(r, \lambda, \chi)$ has a Jordan-Hölder factor in common with an element of P, and Proposition (1.2.2) shows that this implies that the factors of $\pi(r, \lambda, \chi)$ are the same as the Jordan-Hölder factors of a single element of P. That is, $\lambda \in \Lambda$, so

$$P(T)^{m_0+1} Q' \subset (T - \lambda) P(T)^{m_0} Q' \subsetneq P(T)^{m_0} Q',$$

contradicting our assumption on m_0.

Hence $P(T)^{m_0} Q' = 0$, and Q' is a quotient $Q/P(T)^{m_0} Q$. Conversely, for any $m \geq 0$ each Jordan-Hölder factor of $Q/P(T)^m Q$ is a subquotient of an element of P. It follows that $Q_{\widehat{P}} = \varprojlim_m Q/P(T)^m Q$.

Now order $\Lambda = \{\lambda_1, \dots, \lambda_{|\Lambda|}\}$, and set $P_i(T) = \prod_{j \leqslant i}(T - \lambda_j)$ for $i = 0, \dots, |\Lambda|$. Define a decreasing filtration $(Q/P(T)^m Q)^i = P_i(T)^m Q/P(T)^m Q$ on $Q/P(T)^m Q$. For $0 \leqslant i \leqslant |\Lambda| - 1$, the i^{th} graded piece of this filtration is isomorphic to $Q/(T - \lambda_{i+1})^m Q$. Moreover the natural projection $Q/P(T)^{m+1} Q \to Q/P(T)^m Q$ maps $(Q/P(T)^{m+1} Q)^i$ onto $(Q/P(T)^m Q)^i$ since multiplication by $P_i(T)$ induces an automorphism of $P_i(T)^m Q/P(T)^m Q$ as $\lambda_i \neq \lambda_j$ for $j > i$. It follows that the filtration on $Q/P(T)^m Q$ for $m \geq 1$ induces a filtration on $Q_{\widehat{P}}$, and the i^{th} graded piece of the latter filtration is isomorphic to $\Pi(r, \lambda_{i+1}, \chi)$. □

Lemma (1.6.6). *$V(U)$ is a finite $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module of dimension $\leqslant 2$. In particular, if $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ denotes the image of $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ in $\operatorname{End} V(U)$, then $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ is a flat $\mathcal{O}$-algebra of relative dimension at most 1.*

If $\dim R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ has relative dimension 1 over $\mathcal{O}$ and $V_{\mathbb{F}}$ is reducible, then the reduced ring $(R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})/\pi)^{\mathrm{red}}$ may be identified with the image of the map θ in Lemma (1.5.11). In particular it is 1-dimensional and formally smooth.

Proof. From the construction, one sees that $V(U)$ is p-adically separated (and even $\mathfrak{m}_{R^{\mathrm{ps}}(\bar{\mathfrak{r}})}$-adically separated, where $\mathfrak{m}_{R^{\mathrm{ps}}(\bar{\mathfrak{r}})}$ is the maximal ideal of R^{ps}). Hence to prove the first claim in the lemma, it suffices to show that $V(U)/\pi V(U)$ is a finitely generated $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module of dimension at most 1. The claim regarding the dimension of $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ follows from this.

Let P be the set of $\pi(r,\lambda,\chi)$ with central character ψ, such that $\mathbf{V}(\pi(r,\lambda,\chi))$ is a Jordan-Hölder factor in $V_{\mathbb{F}}$. Then P is a finite set, and $V(U)/\pi V(U)$ is a quotient of $\mathbf{V}((\operatorname{Ind}^G_{KZ}\bar{L}_{k,\tau})_{\widehat{P}})$ where $\bar{L}_{k,\tau} = L_{k,\tau}/\pi L_{k,\tau}$.

Let $\{0\} = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_u = \bar{L}_{k,\tau}$ be a filtration by KZ-stable subspaces, such that L_{i+1}/L_i is an irreducible KZ-module for $i = 0,\dots,u-1$. For $i = 1,\dots,u$, let $V(U)_i$ denote the image of the composite

$$\mathbf{V}((\operatorname{Ind}^G_{KZ}L_i)_{\widehat{P}}) \to \mathbf{V}((\operatorname{Ind}^G_{KZ}\bar{L}_{k,\tau})_{\widehat{P}}) \to V(U)/\pi V(U).$$

Then $V(U)_i$ is a $G_{\mathbb{Q}_p}$-stable subspace of $V(U)$ and hence is $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-stable, since the elements $T(\sigma)$ of $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ act on $V(U)/\pi V(U)$ via $\sigma + \psi\chi_{\mathrm{cyc}}\sigma^{-1}$. Hence it suffices to show that $V(U)_{i+1}/V(U)_i$ is a finitely generated $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module of dimension at most 1.

Now for $i = 0,\dots,u-1$, $(\operatorname{Ind}^G_{KZ}(L_{i+1}/L_i))_{\widehat{P}}$ is isomorphic to a successive extension of representations of the form $\Pi(r,\lambda,\chi)$ by Lemma (1.6.5). Hence $V(U)_{i+1}/V(U)_i$ has a $G_{\mathbb{Q}_p}$-stable filtration whose associated graded pieces are quotients of the $G_{\mathbb{Q}_p}$-modules $\mathbf{V}(\Pi(r,\lambda,\chi))$. As above, since the filtration is $G_{\mathbb{Q}_p}$-stable, it is a filtration by $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-submodules, and the $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module structure on $\mathbf{V}(\Pi(r,\lambda,\chi))$ given by Lemma (1.5.11) is compatible with that on the graded pieces. Finally the first claim in the lemma follows because $\mathbf{V}(\Pi(r,\lambda,\chi))$ is a 1-dimensional $R^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module by Lemma (1.5.11).

To prove the final claim, suppose that $V_{\mathbb{F}}$ is reducible, so that θ depends only on $V_{\mathbb{F}}^{\mathrm{ss}}$. Then it suffices to show that every point of $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{r}})$ in the support of $V(U)/\pi V(U)$ contains the kernel of θ. However, we have just seen that $V(U)/\pi V(U)$ has a finite filtration by $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$-submodules which are $\theta(R^{\mathrm{ps}}(\bar{\mathfrak{r}}))$-modules. □

(1.6.7) Let $I_{\mathfrak{r}}$ be the kernel of the map $x_{\mathfrak{r}}$ of Lemma (1.6.3)(3), corresponding to a deformation $\mathfrak{r}$ of $\bar{\mathfrak{r}}$ which arises from a representation of type (k,τ,ψ). Let I be the intersection of all the ideals $I_{\mathfrak{r}}$ with $\mathfrak{r}$ of this kind. The set of such $\mathfrak{r}$ has a countable subset U_0 such that $I = \bigcap_{\mathfrak{r}\in U_0} I_{\mathfrak{r}}$.[5] In particular, any $R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})$ is a quotient of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$. Note that $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is flat of relative dimension 1 over $\mathcal{O}$. Indeed, by Lemma (1.6.6) the only other possibility is that $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is a finite $\mathcal{O}$-algebra, and this is not the case, for example by Proposition (1.1.1).

[5] This holds for any collection of ideals of a complete local ring: For $n \geq 1$ let $I_{\mathfrak{r},n} = I_{\mathfrak{r}} + \mathfrak{m}^n_{R^{\mathrm{ps}}(\bar{\mathfrak{r}})}$. By the Artin-Rees lemma $I_{\mathfrak{r}} = \bigcap_n I_{\mathfrak{r},n}$, so $I = \bigcap_{\mathfrak{r},n} I_{\mathfrak{r},n}$. However the collection $\{I_{\mathfrak{r},n}\}_{\mathfrak{r},n}$ contains only countably many distinct ideals.

Suppose now that $\bar\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ is indecomposable with trace given by $\bar{\mathfrak{t}}$. Set

$$\mu'_{\mathrm{Aut}} = \mu'_{\mathrm{Aut}}(k,\tau,\bar\rho) = \sum_{n,m} a(n,m)\mu'_{n,m}(\bar\rho)$$

where $a(n,m)$ is as in (1.1.2), $\mu'_{n,m}(\bar\rho) = 0$ if $\mu_{n,m}(\bar\rho) = 0$ and $\mu'_{n,m}(\bar\rho) = 1$ otherwise.

We will use the notion of Proposition (1.3.2)

Lemma (1.6.8). *Let $\alpha = \{\bar\rho\}$ if $\bar\rho$ is absolutely irreducible and $\alpha = \{\omega^{n+1+m}\mu_{\lambda\lambda'}\}$ if $\bar\rho \sim \begin{pmatrix} \omega^{n+1}\mu_\lambda & * \\ 0 & \mu_{\lambda^{-1}} \end{pmatrix} \otimes \omega^m\mu_{\lambda'}$ with $n,m \in [0,p-2]$ and $\lambda,\lambda' \in \mathbb{F}^\times$. If $\bar\rho$ is reducible, $n=0$, and $\lambda = \pm 1$, then we assume that $\bar\rho$ is a peu ramifié extension. Then*

$$e_\alpha(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})) \leqslant \mu'_{\mathrm{Aut}}$$

unless $\bar\rho$ has scalar semi-simplification, in which case

$$e_\alpha(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})) \leqslant 2\mu'_{\mathrm{Aut}}.$$

Proof. We use the notation of the proof of Lemma (1.6.6). Let $i \in [0, u-1]$ and suppose that $L_{i+1}/L_i = \mathrm{Sym}^r\mathbb{F}^2 \otimes \chi \circ \det$ for some $\chi : \mathbb{Q}_p^\times \to \mathbb{F}^\times$ with $\chi|_{\mathbb{Z}_p^\times} = \omega^s$. Denote by R_i the image of $R^{\mathrm{ps}}(\bar{\mathfrak{t}})$ in the endomorphisms of $\mathbf{V}((\mathrm{Ind}_{KZ}^G(L_{i+1}/L_i))_{\widehat{P}})$. We claim that

$$e_\alpha = e_\alpha(\mathbf{V}((\mathrm{Ind}_{KZ}^G(L_{i+1}/L_i))_{\widehat{P}}), R_i) = \mu'_{r,s}(\bar\rho) \tag{1.6.9}$$

unless $\bar\rho$ has scalar semi-simplification, in which case it is equal to $2\mu'_{r,s}(\bar\rho)$.

Since $\det\bar\rho = \psi\chi_{\mathrm{cyc}}$ and L_{i+1}/L_i has central character ψ, comparing the definition of $\mu'_{r,s}(\bar\rho)$ with the formulas of Theorem (1.2.4), one sees that $\mu'_{r,s}(\bar\rho) \neq 0$ if and only if there exists a $\lambda \in \mathbb{F}$ such that $\mathbf{V}(\pi(r,\lambda,\chi))$ is the unique element of α. (This is where we use the hypothesis that if $n = 0$ and $\lambda = \pm 1$, then $\bar\rho$ is peu ramifié, since in general whether $\mu'_{0,s}(\bar\rho) \neq 0$ also depends on the extension class $*$.) Thus if $\mu'_{r,s}(\bar\rho) = 0$, then $\mathbf{V}((\mathrm{Ind}_{KZ}^G(L_{i+1}/L_i))_{\widehat{P}})$ has no subquotients isomorphic to α by Lemma (1.6.5), and $e_\alpha = 0$.

If $\mu'_{r,s} \neq 0$, then there is a unique $\lambda \in \mathbb{F}$ such that $\mathbf{V}(\pi(r,\lambda,\chi))$ is the unique element of α. By Lemma (1.6.5), we have

$$e_\alpha(\mathbf{V}((\mathrm{Ind}_{KZ}^G(L_{i+1}/L_i))_{\widehat{P}}), R_i) = e_\alpha(\mathbf{V}(\Pi(r,\lambda,\chi)), R_i),$$

the $R^{\mathrm{ps}}(\bar{\mathfrak{t}})$-module structure on $\mathbf{V}(\Pi(r,\lambda,\chi))$ being given by Lemmas (1.5.3) and (1.5.11). These lemmas also show that $\dim R_i = 1$ and that $e_\alpha = 1$ unless $\bar\rho$ has scalar semi-simplification, in which case $e_\alpha = 2$.

This proves the claim, and the lemma now follows by applying Proposition (1.3.2) and the remark (1.3.3), keeping in mind that we have already seen that if the two sides of (1.6.9) are non-zero, then

$$\dim R_i = 1 = \dim R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}}).$$

□

Proposition (1.6.10). *Suppose that $\bar\rho$ is absolutely irreducible. Then*

$$e(R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar\rho).$$

Proof. By (1.4.6), $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is a quotient of the universal deformation $\mathcal{O}$-algebra of $V_{\mathbb{F}}$ and hence carries a finite free $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$-module of rank 2 equipped with a continuous action of $G_{\mathbb{Q}_p}$. Denote this module by $M(U_0)$.

Let $\sigma \in R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$. By definition

$$P_\sigma(X) = X^2 - T(\sigma)X + \frac{1}{2}(T(\sigma)^2 - T(\sigma^2))$$

is the characteristic polynomial of σ acting on $M(U_0)$, and by construction $P_\sigma(\sigma)$ annihilates $V(U_0)$. Hence, by Lemma (1.6.11) below (and because $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is reduced by construction) $M(U_0)$ is isomorphic to an $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$-submodule of $V(U_0)$ at each generic point of $\operatorname{Spec} R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$.

Set $\alpha = \{V_{\mathbb{F}}\}$. Since $M(U_0)$ is a finite free $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$-module of rank 2 and $V_{\mathbb{F}}$ is absolutely irreducible, we see that the Hilbert-Samuel multiplicity of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ is equal to $e_\alpha(M(U_0)/\pi M(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}}))$. Using Lemma (1.6.8) and Corollary (1.3.5), we find that

$$\begin{aligned} &e_\alpha(M(U_0)/\pi M(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \\ &\qquad\leqslant e_\alpha(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \leqslant \mu'_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}. \end{aligned}$$

□

Lemma (1.6.11). *Let κ be a field and V and W representations of a group G on finite dimensional κ-vector spaces. Suppose that V is absolutely irreducible, and for $\sigma \in \kappa[G]$ let $P_\sigma(X) = \det(X - \sigma|V)$. If $P_\sigma(\sigma)|_W = 0$ for all $\sigma \in \kappa[G]$, then W is V-isotypic.*

Proof. It suffices to consider the case when W is absolutely irreducible and nonzero. Let $I \subset \kappa[G]$ be the two-sided ideal generated by the elements $P_\sigma(\sigma)$ for $\sigma \in \kappa[G]$, and let J (resp. J') be the kernel of $\kappa[G]$ acting on V (resp. W). By Burnside's theorem $\kappa[G]$ surjects onto $\operatorname{End}_\kappa W$ and $\operatorname{End}_\kappa V$, so in particular $\kappa[G]/J$ and $\kappa[G]/J'$ are simple κ-algebras and $(J+J')/J'$ is either 0 or $\kappa[G]/J'$. If $\sigma \in J$, then $P_\sigma(X) = X^d$ where $d = \dim V$, and so $\sigma^d \in J'$. Hence J is contained in the radical of J'. It follows that $(J+J')/J' \neq \kappa[G]$ and $J = J'$.

It follows that V and W both have dimension d and that if we consider $\kappa[G]$ as a $\kappa[G]$ module via multiplication on the left, then we find that

$$V^d \sim \kappa[G]/J = \kappa[G]/J' \sim W^d;$$

hence $V \sim W$ as required. □

(1.6.12) Suppose that $Z \subset \operatorname{Spec} R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})[1/p]$ is an irreducible component. We say that Z is of irreducible type if the pseudo-representation of $G_{\mathbb{Q}_p}$ at the generic point of Z corresponds to an absolutely irreducible representation. Otherwise we say that Z is of reducible type. Note that although the representation at the generic point of Z is a priori defined over some finite extension of the residue field at that point, Corollary (1.4.7) guarantees that it is actually defined over the residue field itself in almost all cases. Of course all components are of irreducible type if $V_{\mathbb{F}}$ is irreducible. In fact one can show that a component of irreducible type cannot meet a component of reducible type, but we shall not need this here. The following lemma gives an explicit description of the components of reducible type.

Lemma (1.6.13). *The set of components of reducible type is empty unless τ extends to an abelian representation of $G_{\mathbb{Q}_p}$ and (after possibly increasing $\mathcal{O}$) there exist finite order characters $\varepsilon_1, \varepsilon_2 : I_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that*

1. $\tau \sim \varepsilon_1 \oplus \varepsilon_2$.
2. $\varepsilon_1\varepsilon_2 = \chi_{\mathrm{cyc}}^{2-k}\psi|_{I_{\mathbb{Q}_p}}$.
3. $\bar{\mathfrak{r}}|_{I_{\mathbb{Q}_p}} = \bar{\varepsilon}_1 + \bar{\varepsilon}_2\chi_{\mathrm{cyc}}^{k-1}$, *where $\bar{\varepsilon}_1, \bar{\varepsilon}_2 : I_{\mathbb{Q}_p} \to \mathbb{F}^\times$ denote the reductions of ε_1 and $\varepsilon_2 \bmod \pi$.*

If these conditions hold, then $\bar{\mathfrak{r}} = \omega_1 + \omega_2$ is a sum of two characters, and the reducible components are parameterized by distinct pairs of characters of the form $\{\omega_i, \varepsilon_j\}$ with $i, j = 1, 2$ such that $\omega_i|_{I_{\mathbb{Q}_p}} = \bar{\varepsilon}_j$. If we set $i' = 3-i$ and $j' = 3-j$, then the $\bar{\mathbb{Z}}_p$-points of such a component correspond to liftings $\mathfrak{r}$ of $\bar{\mathfrak{r}}$ such that $\mathfrak{r} = \tilde{\omega}_i + \tilde{\omega}_{i'}$ with $\tilde{\omega}_i$ and $\tilde{\omega}_{i'}$ lifting ω_i and $\omega_{i'}$, respectively, having restriction to inertia equal to ε_j and $\varepsilon_{j'}\chi_{\mathrm{cyc}}^{k-1}$, respectively, and with $\tilde{\omega}_i\tilde{\omega}_{i'} = \chi_{\mathrm{cyc}}\psi$.

Proof. It is clear that $V_{\mathbb{F}}^{\mathrm{ss}}$ has reducible liftings of type (k, τ, ψ) if and only if the conditions (1)–(3) are satisfied. Since ε_1 and ε_2 extend to characters of $G_{\mathbb{Q}_p}$, (3) implies that $\bar{\mathfrak{r}}$ is not irreducible.

Suppose that (1)–(3) are satisfied and let $\{\omega_i, \varepsilon_j\}$ be a pair as in the lemma. We first show that $\operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ has a component whose $\bar{\mathbb{Z}}_p$-points correspond to lifts $\mathfrak{r}$ as in the lemma. Without loss of generality we may assume that $i = j = 1$. Fix liftings $\tilde{\omega}_1, \tilde{\omega}_2 : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ of ω_1 and ω_2, respectively, such that $\tilde{\omega}_1|_{I_{\mathbb{Q}_p}} = \varepsilon_1$, $\tilde{\omega}_2|_{I_{\mathbb{Q}_p}} = \varepsilon_2\chi_{\mathrm{cyc}}^{k-1}$ and $\tilde{\omega}_1\tilde{\omega}_2 = \psi\chi_{\mathrm{cyc}}$. If $\{\tilde{\omega}_1', \tilde{\omega}_2'\}$ is another such pair of liftings, then $\tilde{\omega}_1' = \tilde{\omega}_1\mu$ and $\tilde{\omega}_2' = \tilde{\omega}\mu^{-1}$ for some unramified character μ having trivial reduction.

Now consider the deformation of the pseudo-representation $\bar{\mathfrak{r}}$ to $\mathcal{O}[\![S]\!]$ given by

$$\sigma \mapsto \tilde{\omega}_1\mu_T + \tilde{\omega}_2\mu_T^{-1}$$

where $T = 1 + S$ and, as in Lemma (1.5.9), μ_T denotes the unramified character of $G_{\mathbb{Q}_p}$ sending the geometric Frobenius to T. This gives a map $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to \mathcal{O}[\![S]\!]$. The composite

$$R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to \mathcal{O}[\![S]\!] \to \mathbb{F}[\![S]\!]$$

is surjective if $\omega_1 \neq \omega_2$ and contains an element of S-adic valuation 2 otherwise (cf. the proof of Lemma (1.5.11)). Hence $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to \mathcal{O}[\![S]\!]$ is a finite map and induces a surjection of $\operatorname{Spec} \mathcal{O}[\![S]\!]$ onto a component of $\operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$, since $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ is pure of dimension 2. The $\bar{\mathbb{Z}}_p$-points of this component are of the required kind.

It remains to show that every reducible component of $R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ is one of those just described. By definition, any component Z of $\operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})[1/p]$ has a Zariski dense set of closed points such that the corresponding pseudo-representation $\mathfrak{r}$ is the trace of a representation of type (k, τ, ψ). If Z is of reducible type, then $\mathfrak{r}$ must be reducible and of the form $\tilde{\omega}_1 + \tilde{\omega}_2$, with $\tilde{\omega}_i$ a character lifting ω_i for $i = 1, 2$. Since $\mathfrak{r}$ is the trace of a representation of type (k, τ, ψ), one of $\tilde{\omega}_1, \tilde{\omega}_2$ — say $\tilde{\omega}_1$ — is finitely ramified, with restriction to inertia equal to one of ε_1 or ε_2 — say ε_1. This shows that $\mathfrak{r}$ lies on the component corresponding to $\{\omega_1, \varepsilon_1\}$. It follows that Z has a Zariski dense set of points in common with one of the components already constructed, and hence it must be equal to one of these. □

(1.6.14) Suppose that $V_{\mathbb{F}}^{\mathrm{ss}} \sim \omega_1 \oplus \omega_2$ is reducible. Let Z be a component of reducible type, and let $x \in Z$ be a closed point, which corresponds to an absolutely reducible representation of $G_{\mathbb{Q}_p}$, V_x. Since V_x has distinct Hodge-Tate weights, V_x

is in fact reducible, and its semi-simplification V_x^{ss} is uniquely determined by x. Suppose $V_x^{\mathrm{ss}} \sim \tilde{\omega}_1 \oplus \tilde{\omega}_2$ with $\tilde{\omega}_i$ reducing to ω_i, for $i = 1, 2$. If we insist that V_x be potentially semi-stable and indecomposable, then this determines which of $\tilde{\omega}_1$ and $\tilde{\omega}_2$ appears as a subspace of V_x. We say that the point x is of type ω_i if $\tilde{\omega}_i$ appears as a subspace. Explicitly this means that the image of inertia in $\tilde{\omega}_i(1-k)$ is finite. By Lemma (1.6.13), either all points on Z are of type ω_1 or all are of type ω_2, and we say that Z is of type ω_1 or ω_2, respectively.

Proposition (1.6.15). *Suppose that $\bar{\rho}$ is a non-trivial extension of ω_2 by ω_1, with $\omega_1 \neq \omega_2, \omega\omega_2$. Choose $U = U_{\omega_1}$ so that* $\operatorname{Spec} R_U^{\mathrm{ps}}(\bar{\mathfrak{r}}) \subset \operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ *is the closure of the union of the components of irreducible type and of type ω_1. Then*

$$e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})) \leqslant \mu_{\mathrm{Aut}}(k, \tau, \bar{\rho}).$$

Proof. Let $I^{\mathrm{irr}} \subset R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$ be the ideal corresponding to the components of irreducible type, and let $I^{\omega_1} \subset R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$ be the ideal corresponding to components of type ω_1. Write $V(U)^{\mathrm{irr}}$ and $V(U)^{\omega_1}$ for $V(U)/I^{\mathrm{irr}}$ and $V(U)/I^{\omega_1}$, respectively. We denote by $U_{\omega_1}^{\mathrm{red}} \subset U_{\omega_1}$ the points which lie on a component of type ω_1.

Let M be a finite $R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$-module. In order to lighten notation, we will write $e(M)$ for $e(M, R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}}))$. Similarly, if M carries an action of $G_{\mathbb{Q}_p}$, we will write $e_{\{\omega_1\}}(M)$ for $e_{\{\omega_1\}}(M, R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}}))$.

Since $\operatorname{Ext}^1_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\omega_2, \omega_1)$ is 1-dimensional, $R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$ carries a finite free module of rank 2, $M(U)$ equipped with a continuous action of $G_{\mathbb{Q}_p}$, by Corollary (1.4.7), and $M(U)/I^{\omega_1}M(U)$ has a finite free rank $R^{\mathrm{ps}}_{U_{\omega_1}^{\mathrm{red}}}(\bar{\mathfrak{r}})$-submodule L^{ω_1}, of rank 1, on which $G_{\mathbb{Q}_p}$ acts via a character $\tilde{\omega}_1 : G_{\mathbb{Q}_p} \to (R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/I^{\omega_1})^\times = R^{\mathrm{ps}}_{U_{\omega_1}^{\mathrm{red}}}(\bar{\mathfrak{r}})^\times$ which lifts ω_1.

The same argument as in Proposition (1.6.10), using Lemma (1.6.11), shows that

$$e_{\{\omega_1\}}(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/(I^{\mathrm{irr}}, \pi)) \leqslant e_{\{\omega_1\}}(V(U)^{\mathrm{irr}}/\pi V(U)^{\mathrm{irr}}). \tag{1.6.16}$$

Similarly, if we extend $\tilde{\omega}_1$ to a linear map $R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}] \to R^{\mathrm{ps}}_{U_{\omega_1}^{\mathrm{red}}}(\bar{\mathfrak{r}})$, then, as remarked in (1.6.1), for $\sigma \in R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})[G_{\mathbb{Q}_p}]$, $\sigma - \tilde{\omega}_1(\sigma)$ annihilates $V(U_{\omega_1}^{\mathrm{red}})$. Hence we have

$$\begin{aligned} e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/(I^{\omega_1}, \pi)) &= e_{\{\omega_1\}}(L^{\omega_1}/\pi L^{\omega_1}) \\ &\leqslant e_{\{\omega_1\}}(V(U_{\omega_1}^{\mathrm{red}})/\pi V(U_{\omega_1}^{\mathrm{red}})) \leqslant e_{\{\omega_1\}}(V(U)^{\omega_1}/\pi V(U)^{\omega_1}), \end{aligned} \tag{1.6.17}$$

where the first inequality follows from Corollary (1.3.5) and Lemma (1.6.11).

Now the map

$$V(U) \to V(U)^{\mathrm{irr}} \oplus V(U)^{\omega_1}$$

is an isomorphism at all the generic points of $R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$. Hence combining (1.6.16) and (1.6.17) and using Proposition (1.3.4) and Lemma (1.6.8), one finds that

$$\begin{aligned} e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})) &= e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/(I^{\mathrm{irr}}, \pi)) + e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/(I^{\omega_1}, \pi)) \\ &\leqslant e_{\{\omega_1\}}(V(U)^{\mathrm{irr}}/\pi V(U)^{\mathrm{irr}}) + e_{\{\omega_1\}}(V(U)^{\omega_1}/\pi V(U)^{\omega_1}) \\ &= e_{\{\omega_1\}}(V(U)/\pi V(U)) \leqslant \mu'_{\mathrm{Aut}} = \mu_{\mathrm{Aut}}. \end{aligned}$$

□

Proposition (1.6.18). *Suppose that $\bar{\rho}$ has scalar semi-simplification. Let $U_{\mathrm{irr}} \subset U_0$ be a dense set of points on the components of irreducible type in* $\operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})[1/p]$, *and denote by* $\mathcal{C}_{\mathrm{red}}$ *the set of components of reducible type. Then*

$$e(R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})) + |\mathcal{C}_{\mathrm{red}}| \leqslant \mu'_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. Using the notation of Lemma (1.6.8), we see that for every point of U_0 the corresponding representation $V(\mathfrak{r})$ has the property that all the Jordan-Hölder factors of $V(\mathfrak{r})/\pi V(\mathfrak{r})$ are equal and 1-dimensional, isomorphic to the unique element of α. Hence

$$(1.6.19)\quad e(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) = e_\alpha(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})) \leqslant 2\mu'_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Since $V(U_0)$ has a quotient of rank 2 at a dense set of points on any component of irreducible type, it has rank ≥ 2 on any such component, and

$$(1.6.20)\quad e(R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})) \leqslant e(V(U_{\mathrm{irr}})/\pi V(U_{\mathrm{irr}}), R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}}))/2 = e(V(U_{\mathrm{irr}})/\pi V(U_{\mathrm{irr}}), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}}))/2.$$

Let Z be a component of reducible type, and denote by $U_Z \subset U_0$ a Zariski dense set of points of Z. Consider the map $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}}) \to \mathcal{O}[\![S]\!]$ introduced in the proof of Lemma (1.6.13), corresponding to a reducible component Z, and denote by $\tilde{\omega}_1, \tilde{\omega}_2$ the characters introduced in that lemma. We claim that the $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$-module structure on $V(U_Z)$ extends to a structure of $\mathcal{O}[\![S]\!]$-module. To see this, consider a pseudo-representation $\mathfrak{r}$ corresponding to a point of U_Z. We remarked in (1.6.1) that the action of $G_{\mathbb{Q}_p}$ on $V(\mathfrak{r})$ is via the character $\tilde{\omega}_2\mu_T^{-1}$. In particular, $V(\mathfrak{r})$ has a natural action of $S = T-1$, which is compatible with the action of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$. If $\mathfrak{r}_1, \dots, \mathfrak{r}_n \in U_Z$, then we see as in the proof of Lemma (1.6.3) that the image of $V(U_Z) \to \bigoplus_{i=1}^n V_{\mathfrak{r}_i}$ is $G_{\mathbb{Q}_p}$-stable and hence stable by the action of S. This proves our claim.

Now we saw in the proof of Lemma (1.5.11) that $R_Z = \mathrm{Im}(R^{\mathrm{ps}}_{U_0} \to \mathbb{F}[\![S]\!])$ is a discrete valuation ring and that $\mathbb{F}[\![S]\!]$ is free of rank 2 over R_Z. Since $V(U_Z)/\pi V(U_Z)$ is a finite $\mathbb{F}[\![S]\!]$-module of dimension 1, it is a faithful $\mathbb{F}[\![S]\!]$-module, and we have

$$(1.6.21)\quad 1 \leqslant e(V(U_Z)/\pi V(U_Z), \mathbb{F}[\![S]\!]) = e(V(U_Z)/\pi V(U_Z), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}}))/2.$$

Summing (1.6.20) and (1.6.21) for all reducible components Z and using (1.6.19) and Proposition (1.3.4) gives

$$e(R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_{\mathrm{irr}}}(\bar{\mathfrak{r}})) + |\mathcal{C}_{\mathrm{red}}| \leqslant e(V(U_0)/\pi V(U_0), R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}}))/2 \leqslant \mu'_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

□

(1.7) From pseudo-representations to representations. In this subsection we deduce bounds on the Hilbert-Samuel multiplicities of deformation rings for Galois representations from the corresponding bounds on the deformation rings for pseudo-representations.

Lemma (1.7.1). *The natural map* $R^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to R^{\psi,\square}(k,\tau,\bar{\rho})$ *factors through* $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$. *Similarly, if* $\bar{\rho}$ *has only scalar endomorphisms, then* $R^{\mathrm{ps}}(\bar{\mathfrak{r}}) \to R^{\psi}(k,\tau,\bar{\rho})$ *factors through* $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$.

If $\bar{\rho}$ *is a non-trivial extension of* ω_2 *by* ω_1 *for some characters* $\omega_1, \omega_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$, *then these maps factors through* $R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{r}})$, *where* U_{ω_1} *is as in Proposition* (1.6.15).

Proof. The claim regarding the map $R^{\mathrm{ps}}(\bar{\mathfrak{t}}) \to R^{\psi}(k,\tau,\bar{\rho})$ when $\bar{\rho}$ has scalar endomorphisms follows from that for $R^{\mathrm{ps}}(\bar{\mathfrak{t}}) \to R^{\psi,\square}(k,\tau,\bar{\rho})$ since $R^{\psi,\square}(k,\tau,\bar{\rho})$ is formally smooth over $R^{\psi}(k,\tau,\bar{\rho})$.

Let $U = U_{\omega_1}$ if $\bar{\rho}$ is a non-trivial extension of ω_2 by ω_1 and $U = U_0$ otherwise. Since $R^{\psi,\square}(k,\tau,\bar{\rho})$ is p-torsion free and reduced by Proposition (1.1.1), it suffices to check that for any finite extension E'/E, an E'-valued point x of $R^{\psi,\square}(k,\tau,\bar{\rho})$ gives rise to an E'-valued point of $R^{\mathrm{ps}}_U(\bar{\mathfrak{t}})$. Now x corresponds to a 2-dimensional E'-representation V_x which is of type (k,τ,ψ) and admits an $\mathcal{O}_{E'}$-lattice which is a deformation of $\bar{\rho}$ to $\mathcal{O}_{E'}$. Hence the trace of V_x is a deformation $\mathfrak{t}$ of $\bar{\mathfrak{t}}$.

Let $I_{\mathfrak{t}}$ be is as in (1.6.7). Since $\bigcap_{\mathfrak{t}\in U_0} I_{\mathfrak{t}} = \bigcap_{\mathfrak{t}} I_{\mathfrak{t}}$, where in the second intersection $\mathfrak{t}$ runs over all deformations of $\bar{\mathfrak{t}}$ arising from a deformation of $V_{\mathbb{F}}$ of type (k,τ,ψ), x induces an E'-valued point of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})$, which completes the proof if $\bar{\rho}$ is semi-simple. If $\bar{\rho}$ is a non-trivial extension of ω_2 by ω_1, then x lies either on a component of irreducible type or of type ω_1 and hence factors through $R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{t}})$. □

Corollary (1.7.2). *Suppose that $\bar{\rho}$ is either absolutely irreducible or a non-trivial extension of ω_2 by ω_1, where $\omega_1,\omega_2 : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$ are distinct characters satisfying $\omega_1 \neq \omega\omega_2$. Then*

$$e(R^{\psi}(k,\tau,\bar{\rho})/\pi R^{\psi}(k,\tau,\bar{\rho})) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. Let $U = U_0$ if $\bar{\rho}$ is irreducible and $U = U_{\omega_1}$ if not. By Corollary (1.4.7) there is a surjection $R^{\mathrm{ps}}(\bar{\mathfrak{t}}) \to R^{\psi}(k,\tau,\bar{\rho})$, which factors through $R^{\mathrm{ps}}_U(\bar{\mathfrak{t}})$ by Lemma (1.7.1). The required inequality now follows from Propositions (1.6.10) and (1.6.15). □

(1.7.3) We now consider the cases where $\bar{\rho}$ is a direct sum of two distinct characters or has scalar semi-simplification. In each case we need to study the difference between the rings $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})$ and $R^{\square,\psi}_{V_{\mathbb{F}}}(k,\tau,\bar{\rho})$.

Let J denote the kernel of the map θ of Lemma (1.5.11) and set

$$R^{\mathrm{ord}} = R^{\square,\psi}_{V_{\mathbb{F}}}/JR^{\square,\psi}_{V_{\mathbb{F}}}.$$

We remind the reader that $(R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}}))^{\mathrm{red}} = R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})/J$ is a 1-dimensional power series ring over $\mathbb{F}$ by Lemma (1.6.6).

Lemma (1.7.4). *Suppose that $\bar{\rho} \sim \omega_1 \oplus \omega_2$ where $\omega_1,\omega_2 \to \mathbb{F}^\times$ and $\omega_1\omega_2^{-1} \notin \{1,\omega,\omega^{-1}\}$. Then* $\operatorname{Spec} R^{\mathrm{ord}}$ *has two irreducible components, each of which is formally smooth over $\mathbb{F}$ and dominates* $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{t}})/J$.

Proof. Consider the functor which assigns to a local Artinian $\mathbb{F}$-algebra A with residue field $\mathbb{F}$ the set of framed deformations V_A of $V_{\mathbb{F}}$ such that V_A has a $G_{\mathbb{Q}_p}$-stable A-line $L_A \subset V_A$ (that is, a finite projective A-submodule of rank 1 with projective cokernel) on which $I_{\mathbb{Q}_p}$ acts via $\omega_1|_{I_{\mathbb{Q}_p}}$. Since $\omega_1 \neq \omega_2$, one sees easily that if L_A exists, it is unique and that this functor is representable by a quotient R_{ω_1} of $R^{\square,\psi}_{V_{\mathbb{F}}}$.

From the definitions one sees that R_{ω_1} is actually a quotient of R^{ord}. Using the fact that $H^2(G_{\mathbb{Q}_p},\omega_1\omega_2^{-1}) = 0$, one checks easily that $\operatorname{Spec} R_{\omega_1}$ is formally smooth over $\mathbb{F}$ and that it dominates $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{t}})/J$. Replacing ω_1 by ω_2, we obtain another quotient R_{ω_2} with analogous properties.

Now any prime $\mathfrak{p}$ of $\operatorname{Spec} R^{\mathrm{ord}}$ gives rise to a representation whose associated pseudo-representation is a sum of two distinct characters which are equal to ω_1 and ω_2 on inertia. Hence using Lemma (1.4.9), one sees that $\mathfrak{p}$ lies on (at least) one of

$\operatorname{Spec} R_{\omega_1}$ and $\operatorname{Spec} R_{\omega_2}$. Moreover if this representation is a non-trivial extension, then $\mathfrak{p}$ lies on exactly one of these and the map

$$\operatorname{Spec} R_{\omega_1} \coprod \operatorname{Spec} R_{\omega_2} \to \operatorname{Spec} R^{\mathrm{ord}}$$

is an isomorphism at $\mathfrak{p}$. This last claim can again be seen using Lemma (1.4.9). In particular, we see that $\operatorname{Spec} R^{\mathrm{ord}}$ has exactly two minimal primes and that the corresponding irreducible components are formally smooth over $\mathbb{F}$. □

Lemma (1.7.5). *Suppose that* $\bar{\rho} \sim \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$ *for some character* $\chi : G_{\mathbb{Q}_p} \to \mathbb{F}^\times$, *which satisfies* $\chi^2 = \psi\chi_{\mathrm{cyc}}$.

Then $\operatorname{Spec} R^{\mathrm{ord}}$ *is irreducible, generically reduced, and dominates* $R^{\mathrm{ps}}(\bar{\mathfrak{r}})/J$. *If the cocycle* $*$ *is non-zero, then the reduced ring of* R^{ord} *is formally smooth over* $\mathbb{F}$.

Proof. It suffices to consider the case where χ is the trivial character. If A is an $R^{\square,\psi}_{V_{\mathbb{F}}}$-algebra, denote by V_A the induced A-representation of $G_{\mathbb{Q}_p}$, and let $L_{V_{\mathbb{F}}}(A)$ denote the set of A-lines $L_A \subset V_A$ such that L_A is stable by $G_{\mathbb{Q}_p}$ and the inertia subgroup $I_{\mathbb{Q}_p}$ acts on L_A by $\psi\chi_{\mathrm{cyc}}$. It is clear that $L_{V_{\mathbb{F}}}$ is representable by a projective $R^{\square,\psi}_{V_{\mathbb{F}}}$-scheme, $\mathscr{L}_{V_{\mathbb{F}}}$.

Moreover, $\mathscr{L}_{V_{\mathbb{F}}}$ is formally smooth over $\mathcal{O}$. The argument for this is essentially given in [KW 2, 3.2.5]: If A is a local $\mathcal{O}$-algebra, $I \subset A$ a nilpotent ideal, and $V_{A/I}$ a framed deformation of $V_{\mathbb{F}}$ to A/I equipped with a line $L_{A/I}$ on which $I_{\mathbb{Q}_p}$ acts by $\psi\chi_{\mathrm{cyc}}$, lift the character giving the action of $G_{\mathbb{Q}_p}$ on $V_{A/I}/L_{A/I}$ to an unramified $A^\times$-valued character χ_A. Then $\psi\chi_{\mathrm{cyc}}\chi_A^{-1}$ lifts the action of $G_{\mathbb{Q}_p}$ on L_A. Consider the class in $\operatorname{Ext}^1_{A/I[G_{\mathbb{Q}_p}]}(L_{A/I}, V_{A/I}/L_{A/I})$ corresponding to $V_{A/I}$. Since $V_{\mathbb{F}}$ has trivial semi-simplification, χ_A and $\psi\chi_{\mathrm{cyc}}\chi_A^{-1}$ become trivial after $\bigotimes_A \mathbb{F}$. Since $H^2(G_{\mathbb{Q}_p}, \mathbb{F}) = 0$, the above extension lifts to a class in $\operatorname{Ext}^1_{A[G_{\mathbb{Q}_p}]}(\psi\chi_{\mathrm{cyc}}\chi_A^{-1}, \chi_A)$, which (after lifting the chosen basis of $V_{A/I}$) determines an A-valued point of $\mathscr{L}_{V_{\mathbb{F}}}$.

From the definition of $\mathscr{L}_{V_{\mathbb{F}}}$ we see that the map $\mathscr{L}_{V_{\mathbb{F}}} \to \operatorname{Spec} R^{\square,\psi}_{V_{\mathbb{F}}}$ induces a map

$$\mathscr{L}_{V_{\mathbb{F}}} \otimes_{\mathcal{O}} \mathcal{O}/\pi\mathcal{O} \to \operatorname{Spec} R^{\mathrm{ord}}. \tag{1.7.6}$$

The map (1.7.6) is an isomorphism over the generic point of $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{r}})/J$. This is easily seen using the fact that the pseudo-representation over this point is a direct sum of two *distinct* characters, as well as Lemma (1.4.9). Moreover, again using Lemma (1.4.9), one sees that (1.7.6) is surjective. In particular any minimal prime $\mathfrak{p}$ of R^{ord} lies over the generic point of $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{r}})/J$, and (1.7.6) is an isomorphism at $\mathfrak{p}$.

This shows that $\operatorname{Spec} R^{\mathrm{ord}}$ is generically reduced, dominates $\operatorname{Spec} R^{\mathrm{ps}}(\bar{\mathfrak{r}})/J$, and is irreducible provided $\mathscr{L}_{V_{\mathbb{F}}}$ is connected. To check this, it suffices to show that the fiber of $\mathscr{L}_{V_{\mathbb{F}}} \otimes_{\mathcal{O}} \mathbb{F}$ over the closed point of R^{ord} is connected. But this fiber is easily seen to be isomorphic to $\mathbb{P}^1$ if $*$ is trivial and to be a single point otherwise.

It remains to prove the final two claims in the case where $*$ is non-zero. In this case we have just seen that (1.7.6) is a closed immersion which is an isomorphism at any minimal prime of R^{ord}. Hence the kernel of the associated map of rings is contained in the nilradical of R^{ord}. □

Lemma (1.7.7). *Let* $U \subset U_0$ *be a set of closed points on* $\operatorname{Spec} R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ *whose closure is a non-empty collection of irreducible components. Set*

$$R_U = \operatorname{Im}(R^{\square,\psi}(k,\tau,\bar{\rho}) \to R^{\square,\psi}(k,\tau,\bar{\rho}) \otimes_{R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})} R^{\mathrm{ps}}_U(\bar{\mathfrak{r}})[1/p]).$$

If $V_{\mathbb{F}}$ is either a sum of two distinct characters ω_1, ω_2 with $\omega_1\omega_2^{-1} \neq \omega^{\pm 1}$ or has scalar semi-simplification, then

$$e(R_U/\pi R_U) \leqslant e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}}))e(R^{\mathrm{ord}}).$$

If $V_{\mathbb{F}} \sim \omega_1 \oplus \omega_2$ and U consists of points of type ω_1, then

$$e(R_U/\pi R_U) \leqslant e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})).$$

Proof. By Lemma (1.6.6), $R^{\mathrm{ps}}(\bar{\mathfrak{r}})/J \xrightarrow{\sim} (R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}}))^{\mathrm{red}}$ is a 1-dimensional power series ring over $\mathbb{F}$. In particular, $\mathfrak{p} = JR_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})$ is a nilpotent prime ideal. Hence we have

$$e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})) = \ell((R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi)_{\mathfrak{p}})e(R^{\mathrm{ps}}(\bar{\mathfrak{r}})/JR^{\mathrm{ps}}(\bar{\mathfrak{r}})) = \ell((R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi)_{\mathfrak{p}})$$

by Proposition (1.3.7). By Lemmas (1.7.4) and (1.7.5) the components of $\operatorname{Spec} R_U/\pi R_U$ dominate the unique component of $\operatorname{Spec} R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi$. Applying Proposition (1.3.10) and keeping in mind that R_U/JR_U is a quotient of R^{ord}, we find that

$$e(R_U/\pi R_U) \leqslant e(R^{\mathrm{ord}})\ell((R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi)_{\mathfrak{p}}) = e(R^{\mathrm{ord}})e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})).$$

Finally suppose that $V_{\mathbb{F}} \sim \omega_1 \oplus \omega_2$ and U consists of points of type ω_1. Then R_U/JR_U is a quotient of the ring R_{ω_1} introduced in the proof of Lemma (1.7.4), so we have

$$e(R_U/\pi R_U) \leqslant e(R_{\omega_1})\ell((R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi)_{\mathfrak{p}}) = e(R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_U^{\mathrm{ps}}(\bar{\mathfrak{r}})).$$

□

Proposition (1.7.8). *Suppose that $V_{\mathbb{F}} \sim \omega_1 \oplus \omega_2$ is a sum of two distinct characters with $\omega_1\omega_2^{-1} \neq \omega^{\pm 1}$. Then*

$$e(R^{\square,\psi}(k,\tau,\bar{\rho})/\pi R^{\square,\psi}(k,\tau,\bar{\rho})) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. Choose U^{irr} so that $\operatorname{Spec} R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}}) \subset \operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ is the union of the components of irreducible type. For $i = 1, 2$ choose $U_{\omega_i}^{\mathrm{red}}$ so that $\operatorname{Spec} R_{U_{\omega_i}^{\mathrm{red}}}^{\mathrm{ps}}(\bar{\mathfrak{r}}) \subset \operatorname{Spec} R_{U_0}^{\mathrm{ps}}(\bar{\mathfrak{r}})$ is the union of components of type ω_i.

Using Lemmas (1.7.4) and (1.7.7) (and the notation introduced there), we find that

$$(1.7.9)\quad e(R_{U^{\mathrm{irr}}}/\pi R_{U^{\mathrm{irr}}}) \leqslant e(R^{\mathrm{ord}})e(R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})) = 2e(R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})).$$

Now let $\bar{\rho}_{\omega_1}$ be a non-trivial extension of ω_2 by ω_1 and let $\bar{\rho}_{\omega_2}$ be a non-trivial extension of ω_1 by ω_2.

Using (1.7.9) and the second part of Lemma (1.7.7), together with Proposition (1.6.15), we compute

$$\begin{aligned} e(R^{\square,\psi}(k,\tau,\bar{\rho})/\pi) &= e(R_{U_{\mathrm{irr}}}/\pi) + e(R_{U_{\omega_1}^{\mathrm{red}}}/\pi) + e(R_{U_{\omega_2}^{\mathrm{red}}}/\pi) \\ &\leqslant 2e(R_{U^{\mathrm{irr}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi) + e(R_{U_{\omega_1}^{\mathrm{red}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi) + e(R_{U_{\omega_2}^{\mathrm{red}}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi) \\ &= e(R_{U_{\omega_1}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi) + e(R_{U_{\omega_2}}^{\mathrm{ps}}(\bar{\mathfrak{r}})/\pi) \leqslant \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}_{\omega_1}) + \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}_{\omega_2}) \\ &= \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}) \end{aligned}$$

where the first two equalities follow from Proposition (1.3.4)(2) and the final equality follows from the definition of μ_{Aut}. □

Proposition (1.7.10). *Suppose that* $V_{\mathbb{F}}$ *has scalar semi-simplification, so that* $\bar\rho \sim \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \otimes \chi$. *Then we have*

$$e(R^{\square,\psi}(k,\tau,\bar\rho)/\pi R^{\square,\psi}(k,\tau,\bar\rho)) \leqslant e(R^{\mathrm{ord}})\mu'_{\mathrm{Aut}}(k,\tau,\bar\rho) = \mu_{\mathrm{Aut}}(k,\tau,\bar\rho).$$

Proof. We first remark that $\mu_{\mathrm{Aut}}(k,\tau,\bar\rho) = \mu_{p-2,s}(\bar\rho)\mu'_{\mathrm{Aut}}(k,\tau,\bar\rho)$ where $\chi|_{I_{\mathbb{Q}_p}} = \omega^s$. On the other hand $\mu_{p-2,s}(\bar\rho)$ is equal to $e(R/\pi R)$, where $\operatorname{Spec} R$ is the scheme theoretic image of the map $\mathscr{L}_{V_{\mathbb{F}}} \to \operatorname{Spec} R^{\square,\psi}_{V_{\mathbb{F}}}$ introduced in Lemma (1.7.5) (cf. [KW 2, Lem. 3.5]). The remarks following (1.7.6) show that R^{ord} differs from $R/\pi R$ by an ideal supported outside its unique minimal prime, so that $\mu_{p-2,s}(\bar\rho) = e(R^{\mathrm{ord}})$. This shows the second equality.

For the rest of the proof we may assume that χ is trivial.

Let Z be a component of $\operatorname{Spec} R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{t}})[1/p]$ of reducible type, and $U_Z \subset U_0$ a Zariski dense set of points on Z. We claim that $e(R_{U_Z}/\pi R_{U_Z}) = e(R^{\mathrm{ord}})$.

To see this, we may assume without loss of generality that Z corresponds to the pair $\{\varepsilon_1, \omega_1\}$ (of course $\omega_1 = \omega_2$ here). Consider the functor $L^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$ which to an $R^{\square,\psi}_{V_{\mathbb{F}}}$-algebra A assigns the set of lines $L_A \subset V_A$ (V_A as in Lemma (1.7.5)) such that $I_{\mathbb{Q}_p}$ acts on L_A via $\varepsilon_2\chi_{\mathrm{cyc}}^{k-1}$. Exactly as in the proof of Lemma (1.7.5), one sees that $L^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$ is representable by a projective $R^{\square,\psi}_{V_{\mathbb{F}}}$-scheme $\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$, which is formally smooth over $\mathcal{O}$. In particular, $\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$ is reduced, and its scheme theoretic image in $\operatorname{Spec} R^{\square,\psi}_{V_{\mathbb{F}}}$ is $\mathcal{O}$-flat. On the other hand, any closed point of $\operatorname{Spec} R^{\square,\psi}_{V_{\mathbb{F}}}[1/p]$ in the scheme theoretic image of $\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$ lies in $\operatorname{Spec} R_{U_Z}[1/p]$. Hence $\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}}$ is an R_{U_Z}-scheme, and the same argument as in Lemma (1.7.5) shows that the induced map

$$\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}} \otimes_{\mathcal{O}} \mathcal{O}/\pi\mathcal{O} \to \operatorname{Spec} R_{U_Z}/\pi R_{U_Z}$$

is an isomorphism at any generic point of $\operatorname{Spec} R_{U_Z}/\pi R_{U_Z}$. (In fact there is a unique such point.) Now $\varepsilon_2\chi_{\mathrm{cyc}}^{k-1}$ has trivial reduction modulo π; hence there is an isomorphism of $R^{\square,\psi}_{V_{\mathbb{F}}}$-schemes

$$\mathscr{L}^{\varepsilon_1,\varepsilon_2}_{V_{\mathbb{F}}} \otimes_{\mathcal{O}} \mathcal{O}/\pi\mathcal{O} \xrightarrow{\sim} \mathscr{L}_{V_{\mathbb{F}}} \otimes_{\mathcal{O}} \mathcal{O}/\pi\mathcal{O}. \tag{1.7.11}$$

Thus $e(R_{U_Z}/\pi R_{U_Z}) = e(R/\pi R) = e(R^{\mathrm{ord}})$.

Finally using Lemma (1.7.7) and Proposition (1.6.18), we find

$$\begin{aligned} e(R^{\square,\psi}(k,\tau,\bar\rho)/\pi R^{\square,\psi}(k,\tau,\bar\rho)) &= e(R_{U^{\mathrm{irr}}}/\pi R_{U^{\mathrm{irr}}}) + \sum_{Z\in\mathcal{C}_{\mathrm{red}}} e(R_{U_Z}/\pi R_{U_Z}) \\ &\leqslant [e(R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U^{\mathrm{irr}}}(\bar{\mathfrak{t}})) + |\mathcal{C}_{\mathrm{red}}|]e(R^{\mathrm{ord}}) \leqslant e(R^{\mathrm{ord}})\mu'_{\mathrm{Aut}}(k,\tau,\bar\rho). \end{aligned}$$

□

(1.7.12) We also have an analogue of Corollary (1.7.2), Proposition (1.7.8) and Proposition (1.7.10) for the rings $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$, which give one inequality in Conjecture (1.1.5).

Proposition (1.7.13). *Suppose that* $\bar\rho \nsim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$. *Then*

$$e(R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)) \leqslant \mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho).$$

Proof. If τ is not scalar, then the rings $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ and $R^{\square,\psi}(k,\tau,\bar\rho)$ are equal, as are the integers $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho)$ and $\mu_{\mathrm{Aut}}(k,\tau,\bar\rho)$, so in this case the result is already contained in Corollary (1.7.2), Proposition (1.7.8), and Proposition (1.7.10).

Suppose τ is scalar and that V and Π are as in Hypothesis (1.2.6), with V potentially crystalline. Then the results of [BB 1] imply that there is an injection $\sigma_{\mathrm{cr}}(k,\tau) \hookrightarrow \Pi \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. The arguments of the last two sections with $\sigma(k,\tau)$ replaced by $\sigma_{\mathrm{cr}}(k,\tau)$ now go over verbatim to prove the proposition. $\square$

Corollary (1.7.14). *Let $k \in \{2,\dots,p+1\}$, $m \in \{0,\dots,p-2\}$ and suppose τ is scalar. If $\bar\rho \nsim \left(\begin{smallmatrix}\omega\chi & *\\ 0 & \chi\end{smallmatrix}\right)$, then $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is non-trivial if and only if $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) \neq 0$, in which case*

$$e(R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)) = \mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho). \tag{1.7.15}$$

In particular, $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is formally smooth over $\mathcal{O}$ except if $k=p$ and $\bar\rho \sim \begin{pmatrix}\mu_\lambda & 0\\ 0 & \mu_{\lambda'}\end{pmatrix} \otimes \omega^m$ for some m, when

(1) *If $\lambda \neq \lambda'$, then the spectrum of $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ has two irreducible components corresponding to liftings of $\bar\rho$ which have a 1-dimensional quotient on which the action of $G_{\mathbb{Q}_p}$ is finitely ramified and lifts $\mu_\lambda\omega^m$ and $\mu_{\lambda'}\omega^m$, respectively. These two components are formally smooth and specialize to distinct components of $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\rho)/\pi$.*

(2) *If $\lambda = \lambda'$, then $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is a domain and $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\rho)/\pi$ is geometrically irreducible and generically reduced.*

Proof. It suffices to consider the case when τ is trivial. That $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is non-trivial only if $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) \neq 0$ follows from Proposition (1.7.13). Suppose that $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) > 0$.

If $\bar\rho$ is absolutely irreducible, then $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) = 1$, and we have to show that $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is non-trivial. This is a consequence of Fontaine-Laffaille theory when $k \leqslant p$ and follows from the main result of [BLZ] when $k = p+1$. (Of course when it applies, Fontaine-Laffaille theory actually gives the equality (1.7.15).)

If $\bar\rho$ is reducible, then it is easy to see that $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) \neq 0$ implies that $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is non-trivial. When $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar\rho) = 1$, this implies the other statements in the proposition, so it remains to consider the case when $k=p$ and $\bar\rho$ is semi-simple and unramified (as we are assuming τ trivial).

If $\bar\rho \sim \begin{pmatrix}\mu_\lambda & 0\\ 0 & \mu_{\lambda'}\end{pmatrix}$ with $\lambda \neq \lambda'$, then one sees that $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ has at least two components, coming from the two components of $R^{\mathrm{ps}}_{U_0}(\bar{\mathfrak{r}})$ described by Lemma (1.6.13) in this case. Since $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is pure of relative dimension 4 over $\mathcal{O}$, we have $e(R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi) \geq 2$. As Proposition (1.7.13) gives the opposite inequality, this proves (1.7.15) and shows that $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ has exactly two components each of which is formally smooth over $\mathcal{O}$. To see that these components specialize to distinct components of $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi$, note that, as in Lemma (1.7.4), if $\mathfrak{p} \subset R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi$ is a prime such that the corresponding $G_{\mathbb{Q}_p}$-representation is a non-split extension, then $\mathfrak{p}$ lies on exactly one of the two components of $\operatorname{Spec} R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$.

If $\bar\rho$ is scalar, then (1.7.15) follows from the definition of $\mu_{p-2,0}(\bar\rho)$, while the fact that $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$ is a domain follows from [KW 2, 3.2.5], the main point being that the scheme $\mathscr{L}_{V_{\mathbb{F}}}$ introduced in Lemma (1.7.5) is smooth over $\mathcal{O}$, with connected special fiber, and dominates $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)$. Finally, note that there is a surjection $R^{\mathrm{ord}} \to R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi$, which has nilpotent kernel since (1.7.6) is an isomorphism over generic points of $\operatorname{Spec} R^{\mathrm{ord}}$. Hence the claims regarding $R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar\rho)/\pi$ follow from the corresponding results for R^{ord} in Lemma (1.7.5). $\square$

(1.7.16) Of course Corollary (1.7.14) holds even when $\bar{\rho} \sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$. The most difficult case is when $k = p + 1$ and τ is trivial. In this case one can deduce the analogue of Proposition (1.7.13) starting with Lemma (1.6.8) and using the techniques of §§1.6, 1.7. This turns out to be simpler than the case of $\bar{\rho} \sim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ and arbitrary τ, because $L_{k,\tau}/\pi L_{k,\tau}$ contains no trivial Jordan-Hölder factors.

When $* \neq 0$, one can also use the techniques of [GS, §2] to compute the deformation ring explicitly. When $*$ is peu ramifié and $k = p+1$, one finds that its generic fiber is an annulus which accounts for the fact that $\mu^{\mathrm{cr}}_{\mathrm{Aut}}(k, \tau, \bar{\rho}) = 2$ in this case.

2. Modularity via the Breuil-Mézard conjecture

(2.1) Quaternionic forms. We recall some standard facts and notation from the theory of quaternionic forms. Further details may be found in [Ta 2, §1] or [Ki 2, §3].

(2.1.1) Let F be a totally real field and D a quaternion algebra with center F which is ramified at all the infinite places of F and at a set of finite places Σ, which does not contain any primes dividing p. We fix a maximal order $\mathcal{O}_D$ of D, and for each finite place $v \notin \Sigma$ an isomorphism $(\mathcal{O}_D)_v \xrightarrow{\sim} M_2(\mathcal{O}_{F_v})$. For each finite place v of F we will denote by $\mathbf{N}(v)$ the order of the residue field at v and by $\pi_v \in F_v$ a uniformizer. We will write $\Sigma_p = \Sigma \cup \{v|p\}$.

Denote by $\mathbb{A}_F^f \subset \mathbb{A}_F$ the finite adeles, and let $U = \prod_v U_v \subset (D \otimes_F \mathbb{A}_F^f)^\times$ be a compact open subgroup contained in $\prod_v (\mathcal{O}_D)_v^\times$. We assume that if $v \in \Sigma$, then $U_v = (\mathcal{O}_D)_v^\times$ and that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ for $v|p$.

Let A be a topological $\mathbb{Z}_p$-algebra. For each $v|p$, we fix a continuous representation $\sigma_v : U_v \to \mathrm{Aut}(W_{\sigma_v})$ on a finite free A-module. Write $W_\sigma = \bigotimes_{v|p,A} W_{\sigma_v}$ and denote by $\sigma : \prod_{v|p} U_v \to \mathrm{Aut}(W_\sigma)$ the corresponding representation. We regard σ as being a representation of U by letting U_v act trivially if $v \nmid p$.

Finally, assume there exists a continuous character $\psi : (\mathbb{A}_F^f)^\times/F^\times \to A^\times$ such that for any place v of F, σ on $U_v \cap \mathcal{O}_{F_v}^\times$ is given by multiplication by ψ. Fix such a ψ, and extend the action of U on W_σ to $U(\mathbb{A}_F^f)^\times$, by letting $(\mathbb{A}_F^f)^\times$ act via ψ.[6]

Let $S_{\sigma,\psi}(U, A)$ denote the set of continuous functions

$$f : D^\times \backslash (D \otimes_F \mathbb{A}_F^f)^\times \to W_\sigma$$

such that for $g \in (D \otimes_F \mathbb{A}_F^f)^\times$ we have $f(gu) = \sigma(u)^{-1} f(g)$ for $u \in U$ and $f(gz) = \psi^{-1}(z) f(g)$ for $z \in (\mathbb{A}_F^f)^\times$. If we write $(D \otimes_F \mathbb{A}_F^f)^\times = \coprod_{i \in I} D^\times t_i U(\mathbb{A}_F^f)^\times$ for some $t_i \in (D \otimes_F \mathbb{A}_F^f)^\times$ and some finite index set I, then we have

$$S_{\sigma,\psi}(U, A) \underset{f \mapsto \{f(t_i)\}}{\xrightarrow{\sim}} \bigoplus_{i \in I} W_\sigma^{(U(\mathbb{A}_F^f)^\times \cap t_i^{-1} D^\times t_i)/F^\times}.$$

We will assume the following condition:

(2.1.2) For all $t \in (D \otimes_F \mathbb{A}_F^f)^\times$, $(U(\mathbb{A}_F^f)^\times \cap t^{-1} D^\times t)/F^\times = 1$.

This holds if U is sufficiently small. For example, if ℓ is a finite prime of F with $\ell \notin \Sigma_p$, such that for any non-trivial root of unity ζ in a quadratic extension of

[6]What is denoted here by ψ was denoted by ψ^{-1} in [Ki 2]. This corresponds to the fact that our present normalization for the Artin reciprocity map is the inverse of that in [Ki 2]. In both cases, elements of $S_{\sigma,\psi}(U, \mathcal{O})$ give rise to Galois representations with non-negative Hodge-Tate weights and determinant $\psi\chi_{\mathrm{cyc}}$.

F, $\zeta+\zeta^{-1}\neq 2\,(\ell)$, then (2.1.2) holds provided U_ℓ is contained in the subgroup of element in $\mathrm{GL}_2(\mathcal{O}_{F_\ell})$ which are upper triangular, unipotent modulo π_ℓ.

Under the condition (2.1.2), $S_{\sigma,\psi}(U,A)$ is a finite projective A-module, and the functor $W_\sigma\mapsto S_{\sigma,\psi}(U,A)$ is exact in W_σ.

(2.1.3) Let Q be a finite set of finite primes of F, such that for $v\in Q$, D is unramified at v and $v\nmid p$. Suppose that for each $v\in Q$,

$$U_v=\{g\in \mathrm{GL}_2(\mathcal{O}_{F_v}):\ g=\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\ (\pi_v)\}.$$

For $v\in Q$ fix a quotient Δ_v of $(\mathcal{O}_{F_v}/\pi_v\mathcal{O}_{F_v})^\times$ of p-power order, and write $\Delta=\prod_{v\in Q}\Delta_v$. Define a compact open subgroup $U_\Delta=\prod_v (U_\Delta)_v\subset U$ by setting $(U_\Delta)_v=U_v$ if $v\notin Q$, and $(U_\Delta)_v$ the set of $g=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in U_v$ such that ad^{-1} maps to 1 in Δ_v if $v\in Q$. Then $\Delta\xrightarrow{\sim} U/U_\Delta$ acts naturally on $S_{\sigma,\psi}(U_\Delta,A)$ via the right multiplication of U on $D^\times\backslash(D\otimes_F\mathbb{A}_F^f)^\times$. For $h\in\Delta$ we denote by $\langle h\rangle$ the corresponding operator on $S_{\sigma,\psi}(U_\Delta,A)$.

Lemma (2.1.4). *We have*

(1) *The operator* $\sum_{h\in\Delta}\langle h\rangle$ *on* $S_{\sigma,\psi}(U_\Delta,A)$ *induces an isomorphism*

$$\sum_{h\in\Delta}\langle h\rangle : S_{\sigma,\psi}(U_\Delta,A)_\Delta \xrightarrow{\sim} S_{\sigma,\psi}(U,A).$$

(2) $S_{\sigma,\psi}(U_\Delta,A)$ *is a finite projective* $A[\Delta]$*-module.*

Proof. The argument in [Ta 2, 2.3] uses duality on the space $S_{\sigma,\psi}(U_\Delta,A)$, which is not available in our level of generality. However we have the following more direct argument: For $t\in(D\otimes_F\mathbb{A}_F^f)^\times$, we have

$$(U(\mathbb{A}_F^f)^\times\cap t^{-1}D^\times t)/F^\times=(U_\Delta(\mathbb{A}_F^f)^\times\cap t^{-1}D^\times t)/F^\times.$$

Hence it suffices to show that Δ acts freely on $D^\times\backslash(D\otimes_F\mathbb{A}_F^f)^\times/U_\Delta(\mathbb{A}_F^f)^\times$. Suppose $u\in U$ fixes one of these double cosets. Then there exists $t\in(D\otimes_F\mathbb{A}_F^f)^\times$ and $v\in U_\Delta(\mathbb{A}_F^f)^\times$ such that

$$uv^{-1}\in U(\mathbb{A}_F^f)^\times\cap t^{-1}D^\times t=F^\times.$$

Hence $u\in U\cap U_\Delta(\mathbb{A}_F^f)^\times=U_\Delta$. □

(2.1.5) We now suppose that $\mathcal{O}$ and $\mathbb{F}$ are as in Proposition (1.1.1) and that $A=\mathcal{O}$. Let S be the union of the primes in Σ_p and the primes v of F such that $U_v\subset D_v^\times$ is not maximal compact. We will assume that for $v\in S\backslash\Sigma_p$, $U_v\subset\mathrm{GL}_2(\mathcal{O}_{F_v})$ is contained in the matrices whose reduction modulo π_v is upper triangular and contains those whose reduction is upper triangular and unipotent.

Let $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}=\mathcal{O}[T_v,S_v,U_{\pi_w}]_{v\notin S,w\in S\backslash\Sigma_p}$ be a commutative polynomial ring in the indicated formal variables. We consider the left action of $(D\otimes_F\mathbb{A}_F^f)^\times$ on W_σ-valued functions on $(D\otimes_F\mathbb{A}_F^f)^\times$ given by the formula $(gf)(z)=f(zg)$. Then $S_{\sigma,\psi}(U,\mathcal{O})$ becomes a $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$-module with S_v acting via the double coset $U_v\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & \pi_v \end{smallmatrix}\right)U_v$, T_v via $U_v\left(\begin{smallmatrix} \pi_v & 0 \\ 0 & 1 \end{smallmatrix}\right)U_v$ and U_{π_w} via $U_w\left(\begin{smallmatrix} \pi_w & 0 \\ 0 & 1 \end{smallmatrix}\right)U_w$. The operators T_v and S_v do not depend on the choice of π_v. We write $\mathbb{T}_{\sigma,\psi}(U,\mathcal{O})$ or simply $\mathbb{T}_{\sigma,\psi}(U)$ for the image of $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ in the endomorphisms of $S_{\sigma,\psi}(U,\mathcal{O})$.

Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$. We say that $\mathfrak{m}$ is in the support of (σ,ψ) if $S_{\sigma,\psi}(U,\mathcal{O})_\mathfrak{m}$ is non-zero. We say that $\mathfrak{m}$ is *Eisenstein* if $T_v-2\in\mathfrak{m}$ for all but finitely many primes $v\notin S$ which split in some fixed abelian extension of F.

(2.1.6) Let Q be a finite set of primes of F which is disjoint from S, and for each $v \in Q$ fix a quotient Δ_v of $(\mathcal{O}_{F_v}/\pi_v)^\times$ of p-power order. Define compact open subgroups U_Q and U_Q^- of $\prod_v (\mathcal{O}_D)_v^\times$ by setting $(U_Q)_v = (U_Q^-)_v = U_v$ if $v \notin Q$ and defining

$$(U_Q^-)_v = \{g \in \mathrm{GL}_2(\mathcal{O}_{F_v}) : g = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) (\pi_v)\}$$

and

$$(U_Q)_v = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in (U_Q^-)_v : ad^{-1} \mapsto 1 \in \Delta_v\}.$$

Fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ such that $\mathfrak{m}$ is induced by a maximal ideal of $\mathbb{T}_{\sigma,\psi}(U)$, and for $v \in Q$ the Hecke polynomial $X^2 - T_vX + \mathbf{N}(v)S_v$ has distinct roots in $\mathbb{T}^{\mathrm{univ}}_{S,\mathcal{O}}/\mathfrak{m}$. After increasing $\mathbb{F}$, we may assume that $\mathfrak{m}$ has residue field $\mathbb{F}$ and that each of these polynomials has two distinct roots $\alpha_v, \beta_v \in \mathbb{F}$.

Write $S_Q = S \cup Q$. Let $\mathfrak{m}_Q$ denote the ideal of $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$ generated by $\mathfrak{m} \cap \mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$ and the elements $U_{\pi_v} - \tilde{\alpha}_v$ for $v \in Q$, where $\tilde{\alpha}_v \in \mathcal{O}$ is any lifting of α_v.

As in (2.1.5), we denote by $\mathbb{T}_{\sigma,\psi}(U_Q)$ (resp. $\mathbb{T}_{\sigma,\psi}(U_Q^-)$) the rings of endomorphisms of $S_{\sigma,\psi}(U_Q,\mathcal{O})$ (resp. $S_{\sigma,\psi}(U_Q^-,\mathcal{O})$) generated by the elements of $T^{\mathrm{univ}}_{S_Q,\mathcal{O}}$.

Lemma (2.1.7). *The ideal $\mathfrak{m}_Q$ induces proper, maximal ideals in $\mathbb{T}_{\sigma,\psi}(U_Q)$ and $\mathbb{T}_{\sigma,\psi}(U_Q^-)$. If $\alpha_v\beta_v^{-1} \neq \mathbf{N}(v)^{\pm 1}$ for all $v \in Q$, then the natural map*

$$S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}} \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q} \tag{2.1.8}$$

is an isomorphism of $\mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$-modules.

Proof. It suffices to consider the case when Q consists of a single element. Since $\alpha_v\beta_v^{-1} \neq \mathbf{N}(v)^{\pm 1}$, the map

$$S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}^{\oplus 2} \to S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}}; \quad (f_1,f_2) \mapsto f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} f_2$$

is an isomorphism after inverting p, and a calculation using the fact that $\alpha_v \neq \beta_v$ shows that it is an isomorphism (see for example [Ki 3, 7.5]). Here the subscript $\mathfrak{m}$ on the right hand side means localization with respect to the ideal $\mathfrak{m} \cap \mathbb{T}^{\mathrm{univ}}_{S_Q,\mathcal{O}}$.

Since $X^2 - T_vX + \mathbf{N}(v)S_v$ has distinct roots in $\mathbb{F}$, by Hensel's lemma it has two distinct roots $A_v, B_v \in \mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$, lifting α_v and β_v, respectively. Then

$$(U_{\pi_v} - B_v)(f_1 + \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} f_2) = (U_{\pi_v} - B_v)(f_1 + B_vf_2),$$

and since $\alpha_v \neq \beta_v$, $U_{\pi_v} - B_v$ induces an automorphism of $S_{\sigma,\psi}(U_Q^-,\mathcal{O})_{\mathfrak{m}_Q}$. This shows that (2.1.8) is a surjection between finite free $\mathcal{O}$-modules of the same rank and hence is an isomorphism. $\square$

(2.2) Global patching and multiplicities. We now carry out the Taylor-Wiles style patching argument (as modified in [Di] and [Ki 2]), which allows us to relate the local deformation rings studied in §1, with patched Hecke algebras.

Keeping the notation above, we denote by $G_{F,S}$ the Galois group of the maximal extension of F, which is unramified outside S. For each finite prime v we denote by G_{F_v} the absolute Galois group of F_v, and we fix a map $G_{F_v} \to G_{F,S}$ induced by the inclusion of an algebraic closure of F into an algebraic closure of F_v. We also fix a continuous absolutely irreducible representation

$$\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbb{F})$$

such that $\det\bar{\rho}$ is equal to the reduction of $\psi\chi_{\mathrm{cyc}}$ modulo π. Write $V_{\mathbb{F}}$ for the underlying $\mathbb{F}$-vector space of $\bar{\rho}$ and fix a basis for $V_{\mathbb{F}}$.

For $v \in \Sigma_p$, we denote by $R_v^{\square}$ the universal framed deformation $\mathcal{O}$-algebra of $\bar{\rho}|_{G_{F_v}}$ (considered with the chosen basis for $V_{\mathbb{F}}$) and by $R_v^{\square,\psi}$ the quotient of $R_v^{\square}$ corresponding to deformations with determinant $\psi\chi_{\mathrm{cyc}}$. We set $R_{\Sigma_p}^{\square,\psi} = \widehat{\bigotimes}_{\mathcal{O}} R_v^{\square,\psi}$, where in the tensor product v runs over the elements of Σ_p.

When $\bar{\rho}$ is absolutely irreducible, we denote by $R_{F,S}^{\psi}$ the quotient of the universal deformation $\mathcal{O}$-algebra of $\bar{\rho}$, corresponding to deformations with determinant $\psi\chi_{\mathrm{cyc}}$. We denote by $R_{F,S}^{\square,\psi}$ the complete local $\mathcal{O}$-algebra representing the functor which assigns to a local Artinian $\mathcal{O}$-algebra A the set of isomorphism classes of tuples $\{V_A, \beta_v\}_{v\in\Sigma_p}$, where V_A is a deformation of $V_{\mathbb{F}}$ to A having determinant $\psi\chi_{\mathrm{cyc}}$ and β_v is a lifting of the chosen basis of $V_{\mathbb{F}}$ to an A-basis of V_A. For $v \in \Sigma_p$, the functor $\{V_A, \beta_w\}_{w\in\Sigma_p} \mapsto \{V_A, \beta_v\}$ induces the structure of an $R_v^{\square,\psi}$-algebra on $R_{F,S}^{\square,\psi}$.

We now assume the following conditions hold.

(1) $\bar{\rho}$ is unramified outside Σ_p and has odd determinant.
(2) The restriction of $\bar{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.
(3) If $p = 5$ and $\bar{\rho}$ has projective image isomorphic to $\mathrm{PGL}_2(\mathbb{F}_5)$, then the kernel of $\mathrm{proj}\,\bar{\rho}$ does not fix $F(\zeta_5)$. This condition holds if $[F(\zeta_5) : F] = 4$.
(4) If $v \in S \setminus \Sigma_p$, then $(1 - \mathbf{N}(v)) \in \mathbb{F}^{\times}$, and the ratio of the eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$ is not in $\{1, \mathbf{N}(v), \mathbf{N}(v)^{-1}\}$. Here, Frob_v denotes an arithmetic Frobenius at v.

In applications to modularity the following lemma will allow us to reduce to situations where condition (4) holds.

Lemma (2.2.1). *Suppose $\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Then there exists a prime v where $\bar{\rho}$ is unramified and such that $(1 - \mathbf{N}(v)) \in \mathbb{F}^{\times}$ and the ratio of the eigenvalues of $\bar{\rho}(\mathrm{Frob}_v)$ is not in $\{1, \mathbf{N}(v), \mathbf{N}(v)^{-1}\}$.*

Proof. Denote by ω the mod p cyclotomic character of $G_{F,S}$. Let G denote the image of $G_{F,S}$ under $\bar{\rho} \oplus \omega$. By [DDT, Lem. 4.11] there exists $g \in G$ such that $\omega(g) \neq 1$ and the ratio of the eigenvalues of $\bar{\rho}(g)$ is not $\omega(g)^{\pm 1}$. That is

$$(\mathrm{tr}\bar{\rho}(g))^2/\det(\bar{\rho}(g)) \neq (1 + \omega(g))^2/\omega(g). \qquad (2.2.2)$$

If $\bar{\rho}(g)$ is not of the form zu with $z \in \mathrm{GL}_2(\mathbb{F})$ central and u unipotent, then we are done. Suppose that $\bar{\rho}(g) = zu$. After replacing g by $g^{|\mathbb{F}|}$, we may assume that $\bar{\rho}(g)$ is central.

If $g' \in G$ is any element satisfying (2.2.2), such that $\bar{\rho}(g')$ has distinct eigenvalues, then gg' has the same property and ω is non-trivial on one of g' and gg'. Hence we may assume that if $\bar{\rho}(g')$ has distinct eigenvalues, then their ratio is $\omega(g')^{\pm 1}$. In particular, if $g' \in \ker\omega$, then the eigenvalues of $\bar{\rho}(g')$ are equal. Hence the image of $\ker\omega$ under the projectivization of $\bar{\rho}$ is a p-group, so $\bar{\rho}(\ker\omega)$ is contained in $\mathbb{F}^{\times} \cdot \mathcal{U}$ for some unipotent subgroup $\mathcal{U}$ of $\mathrm{GL}_2(\mathbb{F})$.

If $\bar{\rho}(\ker\omega)$ contains non-trivial unipotent elements, then its normalizer is contained in a Borel subgroup and so is $\bar{\rho}(G)$. Otherwise $\bar{\rho}(\ker\omega)$ is central and $\bar{\rho}(G)$ is abelian. In either case $\bar{\rho}$ cannot be absolutely irreducible, contradicting our assumptions. □

(2.2.3) Suppose now that $\mathfrak{m} \subset \mathbb{T}_{S,\mathcal{O}}^{\mathrm{univ}}$ is as in (2.1.5) and that $\mathfrak{m}$ is non-Eisenstein, with associated representation $\bar{\rho}$. That is, if $v \notin S$ and $\mathrm{Frob}_v \in G_{F,S}$ is an arithmetic

Frobenius, then $\bar\rho(\mathrm{Frob}_v)$ has trace equal to the image of T_v in $\mathbb{F}$. We will also assume that $\mathfrak{m}$ is chosen so that if $v \in S\backslash\Sigma_p$, then U_{π_v} modulo $\mathfrak{m}$ is equal to one of the eigenvalues of $\bar\rho(\mathrm{Frob}_v)$.

Finally, we assume that $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \neq 0$. That is, there is an eigenform $f \in S_{\sigma,\psi}(U,\mathcal{O})$ whose associated $G_{F,S}$-representation reduces to $\bar\rho$ and such that for $v \in S\backslash\Sigma_p$, the eigenvalue of U_{π_v} on f reduces to the chosen eigenvalue of $\bar\rho(\mathrm{Frob}_v)$. Note that (2.2)(4) implies that if an f satisfying the first condition exists, then the associated automorphic representation of $D^\times$ is spherical at $v \in S\backslash\Sigma_p$ and $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \neq 0$. Note also that for such an $\mathfrak{m}$, $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ is a (not necessarily free) $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$-module of rank 1.

There is a map $R^{\psi}_{F,S} \to \mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$ such that for $v \notin S$ the trace of Frob_v on the tautological $R^{\psi}_{F,S}$-representation of $G_{F,S}$ maps to T_v. This map is surjective, for example by Hensel's lemma, as in the proof of Lemma (2.1.7).

As in [Ki 2, 2.3.5], we have

Proposition (2.2.4). *Set $g = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar\rho(1)) - [F:\mathbb{Q}] + |\Sigma_p| - 1$. For each positive integer n, there exists a finite set of primes Q_n of F, which is disjoint from S and such that*

1. *If $v \in Q_n$, then $\mathbf{N}(v) = 1(p^n)$ and $\bar\rho(\mathrm{Frob}_v)$ has distinct eigenvalues.*
2. *$|Q_n| = \dim_{\mathbb{F}} H^1(G_{F,S}, \mathrm{ad}^0\bar\rho(1))$. If $S_{Q_n} = S \cup Q_n$, then as an $R^{\square,\psi}_{\Sigma_p}$-algebra $R^{\square,\psi}_{F,S_{Q_n}}$ is topologically generated by g elements. In particular $g \geq 0$.*

(2.2.5) For $n \geq 1$ fix a set Q_n as in Proposition (2.2.4). Let Δ_v be the maximal p-quotient of $(\mathcal{O}_{F_v}/\pi_v)^\times$, and let $\Delta_{Q_n} = \prod_{v\in Q_n} \Delta_v$. For each $v \in Q_n$ we fix a choice of zero of the polynomial $X^2 - T_vX + \mathbf{N}(v)S_v$ in $\mathbb{F}$ (increasing $\mathbb{F}$ if necessary), and we denote by $\mathfrak{m}_{Q_n} \in \mathbb{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$ the corresponding maximal ideal. We apply the discussion of (2.1.5) and (2.1.6) to each of these Q_n.

There is a surjective map of $\mathcal{O}$-algebras $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$ such that for $v \notin S_{Q_n}$, the trace of Frob_v on the tautological $R^{\psi}_{F,S_{Q_n}}$-representation of $G_{F,S_{Q_n}}$ maps to T_v. We regard $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ as an $R^{\psi}_{F,S_{Q_n}}$-module via this map. Moreover $R^{\psi}_{F,S_{Q_n}}$ has a natural structure of $\mathcal{O}[\Delta_{Q_n}]$-algebra so that the induced $\mathcal{O}[\Delta_{Q_n}]$-structure on $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ is the one given by Lemma (2.1.4), [Ta 2, 1.3, 2.1]. By Lemma (2.1.4), $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ is a finite free $\mathcal{O}[\Delta_{Q_n}]$-module, whose rank does not depend on n. Denote this rank by r. Following [Ki 2], set $j = 4|\Sigma_p| - 1$, $h = |Q_n|$, and $d = [F:\mathbb{Q}] + 3|\Sigma_p|$. Then $g = h + j - d$. We fix surjections

$$\mathcal{O}[\![y_1,\dots,y_h]\!] \to \mathcal{O}[\Delta_{Q_n}]. \tag{2.2.6}$$

The map $R^{\psi}_{F,S_{Q_n}} \to R^{\square,\psi}_{F,S_{Q_n}}$ is formally smooth of relative dimension j. We extend the maps (2.2.6) to maps

$$\mathcal{O}[\![y_1,\dots,y_{h+j}]\!] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{2.2.7}$$

in such a way that $R^{\square,\psi}_{F,S_{Q_n}}$ is identified with $R^{\psi}_{F,S_{Q_n}}[\![y_{h+1},\dots,y_{h+j}]\!]$. We also fix surjections of $R^{\square,\psi}_{\Sigma_p}$-algebras

$$R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!] \to R^{\square,\psi}_{F,S_{Q_n}} \tag{2.2.8}$$

and a lifting of the maps in (2.2.7) to maps

$$\mathcal{O}[\![y_1.\dots,y_{h+j}]\!] \to R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!].$$

(2.2.9) For $n \geq 0$, set

$$M_n = R^{\square,\psi}_{F,S_{Q_n}} \otimes_{R^{\psi}_{F,S_{Q_n}}} S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}},$$

where $S_{Q_0} = S$. Then $M_0 \xrightarrow{\sim} M_n/(y_1,\dots,y_h)$ by Lemmas (2.1.4) and (2.1.7). We regard M_n as an $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-module via the map $R^{\psi}_{F,S_{Q_n}} \to \mathbb{T}_{\sigma,\psi}(U_{Q_n})_{\mathfrak{m}_{Q_n}}$ introduced above and (2.2.8).

Fix a filtration by $\mathbb{F}$-subspaces

$$0 = L_0 \subset L_1 \subset \dots \subset L_s = W_\sigma \otimes_{\mathcal{O}} \mathbb{F} = W_{\bar\sigma}$$

on $W_{\bar\sigma}$ such that L_i is $\mathrm{GL}_2(\mathbb{Z}_p)$-stable, and for $i = 0,1,\dots,s-1$, $\sigma_i = L_{i+1}/L_i$ is absolutely irreducible. This induces a filtration on $S_{\sigma,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}} \otimes_{\mathcal{O}} \mathbb{F}$ whose associated graded pieces are the finite free $\mathbb{F}[\Delta_{Q_n}]$-modules $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$. We denote by

$$0 = M_n^0 \subset M_n^1 \subset \dots M_n^s = M_n \otimes_{\mathcal{O}} \mathbb{F}$$

the induced filtration in $M_n \otimes_{\mathcal{O}} \mathbb{F}$, obtained by extension of scalars.

For $n \geq 1$ let

$$\mathfrak{c}_n = (\pi^n, (y_1+1)^{p^n}-1,\dots,(y_h+1)^{p^n}-1, y_{h+1}^{p^n},\dots,y_{h+j}^{p^n}) \subset \mathcal{O}[\![y_1,\dots,y_{h+j}]\!].$$

The proof of [Ki 2, 3.3.1] (which is of course based on the argument of Taylor-Wiles) shows that, after replacing the sequence $\{Q_n\}_{n\geq 1}$ by a subsequence, we may assume that there exist maps of $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-modules

$$f_n : M_{n+1}/\mathfrak{c}_{n+1}M_{n+1} \to M_n/\mathfrak{c}_n M_n$$

which reduce modulo $(y_1,\dots,y_h)+\mathfrak{c}_n$ to the identity on $M_0/\mathfrak{c}_n M_0$. Moreover, the same finiteness argument as in *loc. cit* implies that, if we give $M_n/(\mathfrak{c}_n,\pi)M_n$ the filtration induced by that on $M_n \otimes_{\mathcal{O}} \mathbb{F}$, then we may assume that f_n modulo π is compatible with filtrations.

Passing to the limit over n, we obtain a map of $R^{\square,\psi}_{\Sigma_p}[\![x_1,\dots,x_g]\!]$-modules

$$\varprojlim_n M_n/\mathfrak{c}_n M_n =: M_\infty \to M_\infty/(y_1,\dots,y_h)M_\infty \xrightarrow{\sim} M_0.$$

Since M_n is a finite free $\mathcal{O}[\Delta_{Q_n}][\![y_{h+1},\dots,y_{h+j}]\!]$-module, $M_n/\mathfrak{c}_n M_n$ is a finite free $\mathcal{O}[\![y_1,\dots,y_{h+j}]\!]/\mathfrak{c}_n$-module, and M_∞ is a finite free $\mathcal{O}[\![y_1,\dots,y_{h+j}]\!]$-module. Moreover, $M_\infty \otimes_{\mathcal{O}} \mathbb{F}$ has a filtration

$$0 = M_\infty^0 \subset M_\infty^1 \subset \dots M_\infty^s = M_\infty \otimes_{\mathcal{O}} \mathbb{F}$$

and since M_n^i/M_n^{i-1} is a finite free $\mathbb{F}[\Delta_{Q_n}][\![y_{h+1},\dots,y_{h+j}]\!]$-module, $M_\infty^i/M_\infty^{i-1}$ is a finite free $\mathbb{F}[\![y_1,\dots,y_{h+j}]\!]$-module for $i = 1,\dots,s$.

(2.2.10) We now assume that p splits in F, so that $F_v = \mathbb{Q}_p$ for $v|p$.[7] We also assume that W_{σ_v} has the form

$$\sigma(k_v,\tau_v) \otimes \det{}^{w_v} = \mathrm{Sym}^{k_v-2}\mathcal{O}_{F_v}^2 \otimes \sigma(\tau_v) \otimes \det{}^{w_v}$$

where $k_v \geq 2$, w_v is an integer and $\tau_v : I_v \to \mathrm{GL}_2(E)$ is a representation with open kernel, of Galois type. Here $I_v \subset G_{F_v}$ denotes the inertia subgroup, and $\sigma(\tau_v)$ is associated to τ_v by the local Langlands correspondence in the sense explained in

[7]Note that this implies that the condition (2.2)(3) is automatic.

(1.1.2). The existence of the character ψ such that $\sigma_v|_{\mathcal{O}_{F_v}^\times} = \psi|_{\mathcal{O}_{F_v}^\times}$ for $v|p$ implies that $k_v + 2w_v$ is independent of v.

For each $v \in \Sigma_p$ we now define a quotient $\bar{R}_v^{\square,\psi}$ of $R_v^{\square,\psi}$ such that the action of $R_v^{\square,\psi}$ on each M_n factors through $\bar{R}_v^{\square,\psi}$. If $v|p$, let $R_v^{\square}(\bar{\rho}\otimes\omega^{-w_v})$ denote the universal framed deformation $\mathcal{O}$-algebra of $\bar{\rho}|_{G_{F_v}}\otimes\omega^{-w_v}$, and let $\psi_v = \psi\cdot\chi_{\mathrm{cyc}}^{-2w_v}$. Then $R_v^{\square}(\bar{\rho}\otimes\omega^{-w_v})$ has a quotient denoted $R^{\square,\psi_v}(k_v,\tau_v,\bar{\rho}\otimes\omega^{-w_v})$ in Proposition (1.1.1), and $\bar{R}_v^{\square,\psi}$ is the quotient of $R_v^{\square}$ corresponding to $R^{\square,\psi_v}(k_v,\tau_v,\bar{\rho}\otimes\omega^{-w_v})$ via the isomorphism $R_v^{\square} \xrightarrow{\sim} R_v^{\square}(\bar{\rho}\otimes\omega^{-w_v})$, induced by twisting by $\chi_{\mathrm{cyc}}^{-w_v}$.

That the action of $R_v^{\square,\psi}$ on M_n factors through $\bar{R}_v^{\square,\psi}$ follows from the fact that the p-adic Galois representations attached to Hilbert modular eigenforms are compatible with the local Langlands correspondence at p [Ki 1],[8] as well as the compatibility of the local and global Jacquet-Langlands correspondences.

For $v \in \Sigma$ let $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ be the unramified character such that $\gamma_v^2 = \psi|_{G_{F_v}}$ and $\bar{\rho}|_{G_{F_v}}$ is an extension of γ_v by $\gamma_v(1)$. By [Ki 2, 2.6.7] there is a quotient $\bar{R}_v^{\square,\psi}$ of $R_v^{\square,\psi}$ which is a domain of dimension 4, with $\bar{R}_v^{\square,\psi}[1/p]$ formally smooth over E and such that for any finite extension E'/E a map $x : R_v^{\square,\psi} \to E'$ factors through $\bar{R}_v^{\square,\psi}$ if and only if the corresponding representation V_x is an extension of γ_v by $\gamma_v(1)$. Again, the fact that the action of $R_v^{\square,\psi}$ on M_n factors through $\bar{R}_v^{\square,\psi}$ is a consequence of the compatibility between the local and global Langlands and Jacquet-Langlands correspondences.

We set $\bar{R}_{\Sigma_p}^{\square,\psi} = \widehat{\bigotimes}_{\mathcal{O}} \bar{R}_v^{\square,\psi}$ where v runs over Σ_p. The relative dimension over $\mathcal{O}$ of $\bar{R}_v^{\square,\psi}$ is $3 + [F_v : \mathbb{Q}_p] = 4$ if $v|p$, and it is 3 if $v \nmid p$. In particular $\bar{R}_{\Sigma_p}^{\square,\psi}$ has relative dimension $[F:\mathbb{Q}_p] + 3|\Sigma_p|$ over $\mathcal{O}$.

The following lemma shows that to prove a modularity lifting theorem, we are reduced to showing that M_∞ is a faithful $\bar{R}_\infty = \bar{R}_{\Sigma_p}^{\square,\psi}[\![x_1,\dots,x_g]\!]$-module, or to a question on Hilbert-Samuel multiplicities.

Lemma (2.2.11). *The following conditions are equivalent.*

1. *M_∞ is a faithful $\bar{R}_\infty$-module.*
2. *M_∞ is a faithful $\bar{R}_\infty$-module which has rank 1 at all generic points of R_∞.*
3. *$e(\bar{R}_\infty/\pi\bar{R}_\infty) = e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$.*
4. *$e(\bar{R}_\infty/\pi\bar{R}_\infty) \leqslant e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$.*

Moreover, if these conditions hold and $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ is a deformation of $\bar{\rho}$ such that for $v \in \Sigma_p$, $\rho|_{I_v}$ is an extension of γ_v by $\gamma_v(1)$ if $v \nmid p$ and $\rho|_{G_{F_v}}$ is potentially semi-stable of type (k,τ_v,ψ) if $v|p$, then ρ is modular and arises from an eigenform in $S_{\sigma,\psi}(U,\mathcal{O})\otimes_{\mathcal{O}} E$.

Proof. Write $\mathcal{O}[\![\Delta_\infty]\!] = \mathcal{O}[\![y_1,\dots,y_{h+j}]\!]$, and denote by $\mathbb{T}_\infty$ the image of $\bar{R}_\infty$ in $\mathrm{End}_{\mathcal{O}[\![\Delta_\infty]\!]}(M_\infty)$. Then $\mathbb{T}_\infty$ is a finite, torsion free $\mathcal{O}[\![\Delta_\infty]\!]$-module, and hence all its components have relative dimension $h+j$ over $\mathrm{Spec}\,\mathcal{O}$. Hence, if Z is such a component, then Z surjects onto $\mathrm{Spec}\,\mathcal{O}[\![\Delta_\infty]\!]$. This implies that the rank of $M_\infty|_Z$ is at most one, since otherwise $M_0 = M_\infty\otimes_{\mathcal{O}[\![\Delta_\infty]\!]}\mathcal{O}$ would have a fiber of dimension >1 over some point of $\mathrm{Spec}\,R_{F,S}^{\psi}[1/p]$, and $S_{\sigma,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ would have rank >1 over some generic point of $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}}$, which is impossible as remarked in (2.2.3). Thus, if

[8]More precisely, this is proved in [Ki 1] for eigenforms such that the associated mod p Galois representation is absolutely irreducible.

M_∞ is a faithful $\mathbb{T}_\infty$-module, its rank is exactly one on each irreducible component of $\operatorname{Spec}\mathbb{T}_\infty$.

This shows the equivalence of (1) and (2). Moreover, since $\dim \bar{R}_\infty = \dim \mathbb{T}_\infty$, we have

$$e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty) = e(M_\infty/\pi M_\infty, \mathbb{T}_\infty/\pi\mathbb{T}_\infty) = e(\mathbb{T}_\infty/\pi\mathbb{T}_\infty),$$

where the second equality follows from Corollary (1.3.5).

Since $\bar{R}_\infty$ is reduced and pure of relative dimension $d+g = h+j$ over $\mathcal{O}$, the inclusion $\operatorname{Spec}\mathbb{T}_\infty \hookrightarrow \operatorname{Spec}\bar{R}_\infty$ identifies $\operatorname{Spec}\mathbb{T}_\infty$ with a union of irreducible components of $\operatorname{Spec}\bar{R}_\infty$, and we have $e(\mathbb{T}_\infty/\pi\mathbb{T}_\infty) \leqslant e(\bar{R}_\infty/\pi\bar{R}_\infty)$ with equality if and only if the above inclusion is an isomorphism. This shows that (1), (3) and (4) are equivalent.

Suppose that the conditions (1)–(4) hold. Then ρ induces a map $\mathbb{T}_\infty = \bar{R}_\infty \to \mathcal{O}$, which kills the ideal $(y_1,\dots,y_{h+j})$, and hence a map $\xi : \mathbb{T}_\infty/(y_1,\dots,y_{h+j})[1/p] \to E$. Since M_∞ has positive rank on all components of $\mathbb{T}_\infty$, the fiber of M_0 over the closed point of $\mathbb{T}_\infty/(y_1,\dots,y_{h+j})[1/p]$ corresponding to ξ is non-empty, and ξ induces a map $\mathbb{T}_{\sigma,\psi}(U)_{\mathfrak{m}} \to E$, which corresponds to the required eigenform in $S_{\sigma,\psi}(U,\mathcal{O}) \otimes_{\mathcal{O}} E$. □

(2.2.12) Our next task is to compute $e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty)$. For $i = 1,\dots,s$, write σ_i for the representation L_i/L_{i-1}. Thus σ_i has the form $\sigma_i = \bigotimes_{v|p}\sigma_{n_{i,v},m_{i,v}}$ where $(n_{i,v},m_{i,v}) \in \{0,1,\dots,p-1\}\times\{0,1,\dots,p-2\}$ and $\sigma_{n_{i,v},m_{i,v}}$ is an irreducible constituent of $W_{\sigma_v}/\pi W_{\sigma_v}$.

Lemma (2.2.13). *For $i = 1,\dots,s$ and each $v|p$, the action of $R_v^\square$ on M_n^i/M_n^{i-1} factors through $R_{\mathrm{cr}}^{\square,\psi_{i,v}}(n_{i,v}+2, (\tilde{\omega}^{m_{i,v}})^{\oplus 2}, \bar{\rho})$, where $\tilde{\omega}$ denotes the Teichmüller lift of ω and $\psi_{i,v} : G_{F_v} \to \mathcal{O}^\times$ is any character which is given by $\chi_{\mathrm{cyc}}^{n_{i,v}}\tilde{\omega}^{2m_{i,v}}$ on inertia and has reduction equal to $\psi|_{G_{F_v}}$.*

Proof. Fix i and a prime $v_0|p$. Let

$$\tilde{\sigma}_{i,v_0} = \operatorname{Sym}^{n_{i,v_0}}\mathcal{O}^2 \otimes \tilde{\omega}^{m_{i,v_0}} \circ \det.$$

For $v \neq v_0$ let $\tilde{\sigma}_{i,v}$ be a representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ of the form $\operatorname{Sym}^{n_{i,v_0}}\mathcal{O}^2 \otimes \tilde{\sigma}_{i,v}^{\mathrm{sm}}$ where $\tilde{\sigma}_{i,v}^{\mathrm{sm}}$ is a smooth representation of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ on a finite free $\mathcal{O}$-module, such that $\tilde{\sigma}_{i,v}^{\mathrm{sm}}$ has an $\mathcal{O}^\times$-valued central character, and $\tilde{\sigma}_{i,v} \otimes_{\mathcal{O}} \mathbb{F}$ admits $\sigma_{n_{i,v},m_{i,v}}$ as a Jordan-Hölder factor. A representation $\tilde{\sigma}_{i,v}^{\mathrm{sm}}$ satisfying these conditions exists, since $\operatorname{Hom}_{\mathbb{F}}(\operatorname{Sym}^{n_{i,v}}\mathbb{F}^2 \otimes \det^{m_{i,v}}, \operatorname{Sym}^{n_{i,v_0}}\mathbb{F}^2)$ is a smooth $\mathrm{GL}_2(\mathcal{O}_{F_v})$-representation and can therefore be embedded into a sum of a finite number of copies of the space of smooth $\mathbb{F}$-valued functions on $\mathrm{GL}_2(\mathcal{O}_{F_v})$. Alternatively, one can exhibit such a $\tilde{\sigma}_{i,v}^{\mathrm{sm}}$ explicitly [CDT, Lem. 3.1.1]. Let $\tilde{\sigma}_i = \bigotimes_{v|p}\tilde{\sigma}_{i,v}$.

Next we choose a continuous character $\tilde{\psi} : (\mathbb{A}_F^f)^\times/F^\times \to \mathcal{O}^\times$, such that $\tilde{\psi} = \psi$ modulo π, and a compact open subgroup $\tilde{U} = \prod_v \tilde{U}_v \subset U_{Q_n}$ such that $\tilde{U}_v$ is maximal compact for $v \in \Sigma_p$ and for all v the restriction $\tilde{\sigma}_{i,v}|_{\tilde{U}_v\cap\mathcal{O}_{F_v}^\times}$ is given by multiplication by $\tilde{\psi}$.

Let $\tilde{S}$ be the union of the primes in S_{Q_n} and the primes where $\tilde{U}_v$ is not maximal. Denote by $\tilde{\mathfrak{m}}$ the maximal ideal in $\mathbb{T}_{\tilde{S},\mathcal{O}}^{\mathrm{univ}}$ corresponding to $\bar{\rho}$. Then

$$M_n^i/M_n^{i-1} = R_{F,S_{Q_n}}^{\square,\psi} \otimes_{R_{F,S_{Q_n}}^{\psi}} S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$$

is a subquotient of $R^{\square,\tilde{\psi}}_{F,\tilde{S}} \otimes_{R^{\tilde{\psi}}_{F,\tilde{S}}} S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U},\mathcal{O})_{\tilde{\mathfrak{m}}} \otimes_{\mathcal{O}} \mathbb{F}$. As remarked in (2.2.10), on the latter module the action of $R^{\square}_{v_0}$ factors through $R^{\square,\tilde{\psi}}_{\mathrm{cr}}(n_{i,v_0},(\tilde{\omega}^{m_{i,v_0}})^{\oplus 2},\bar{\rho})$.

This proves the lemma with $\psi_{i,v_0} = \tilde{\psi}|_{G_{F_v}}$. However the reduction modulo π of $R^{\square,\psi_{i,v_0}}_{\mathrm{cr}}(n_{i,v_0}+2,(\tilde{\omega}^{m_{i,v_0}})^{\oplus 2},\bar{\rho})$ is independent of the character ψ_{i,v_0}, satisfying the conditions of the lemma. □

Lemma (2.2.14). *Let $v|p$ and fix $\gamma \in G_{F_v}$ mapping to Frob_p^{-1} and such that $\omega(\gamma) = 1$. Suppose that $\bar{\rho}|_{G_{F_v}} \sim \left(\begin{smallmatrix} \mu_\lambda & 0 \\ 0 & \mu_{\lambda'} \end{smallmatrix}\right) \otimes \omega^{m_v}$ so that if $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}} \neq 0$, then $\sigma_{i,v} = \sigma_{p-2,m_v}$. Then there exists an endomorphism T_v of $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$ which commutes with the action of $R^{\psi}_{F,S_{Q_n}}$ and such that $T_v^2 - \mathrm{tr}(\gamma)T_v + \det(\gamma) = 0$, where $\mathrm{tr}(\gamma),\det(\gamma) \in R^{\psi}_{F,S_{Q_n}}$ denote the trace and determinant of γ on the universal $G_{F,S}$-representation over $R^{\psi}_{F,S_{Q_n}}$.*

Proof. This is a consequence of the calculations of [Ge, §4.4]; however we sketch the construction.

For any $v|p$ there is a natural extension of the action of $\mathrm{GL}_2(\mathcal{O}_{F_v})$ on $\sigma_{i,v}$ to an action of the semi-group $\mathrm{GL}_2(F_v) \cap M_2(\mathcal{O}_{F_v})$. Namely this semi-group acts naturally on $\mathbb{F}^2$ and its symmetric powers, while the character det can be extended to $\mathrm{GL}_2(F_v) \cap M_2(\mathcal{O}_{F_v})$ by sending an element of determinant p to 1. Then the double coset $U_v \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) U_v$ defines an operator T_v on $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$.

Now fix $v = v_0$ satisfying the conditions of the lemma. To see that T_v satisfies $T_v^2 - \mathrm{tr}(\gamma)T_v + \det(\gamma) = 0$, we give another description of this operator.

Let $\tau_{i,v_0} = \tilde{\omega}^{m_{v_0}} \oplus \tilde{\omega}^{m_{v_0}-1}$ and $\tilde{\sigma}_{i,v_0} = \sigma(\tau_{i,v_0})$, so

$$(\tilde{\sigma}_{i,v_0} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \sim \sigma_{1,m_{v_0}-1} \oplus \sigma_{p-2,m_{v_0}}$$

and $\tilde{\sigma}_{i,v_0} \otimes_{\mathcal{O}} \mathbb{F}$ has σ_{i,v_0} as a Jordan-Hölder factor [CDT, Lem. 3.1.1]. For $v \neq v_0$ let $\tilde{\sigma}_{i,v}$ be a smooth representation such that $\tilde{\sigma}_{i,v} \otimes_{\mathcal{O}} \mathbb{F}$ has $\sigma_{i,v}$ as a Jordan-Hölder factor. Set $\tilde{\sigma}_i = \bigotimes_{v|p} \tilde{\sigma}_{i,v}$, and choose $\tilde{\psi}$, $\tilde{U} \subset U_{Q_n}$, and $\tilde{S}$ compatible with $\tilde{\sigma}_i$ as in Lemma (2.2.13).

Then $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$ is a subquotient of $S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U},\mathcal{O})_{\tilde{\mathfrak{m}}} \otimes_{\mathcal{O}} \mathbb{F}$. On the other hand, there is an operator U_v on $S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U},\mathcal{O})_{\tilde{\mathfrak{m}}}$ corresponding to $I_1 \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) I_1$, where $I_1 \subset \mathrm{GL}_2(\mathcal{O}_{F_v})$ denotes the subgroup of elements whose reduction has the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$.

Now any lifting of $\bar{\rho}|_{G_{F_{v_0}}}$ of type $(2,\tau_{i,v_0},\psi)$ is reducible. One way to see this is to note that $(\tilde{\sigma}_{i,v_0} \otimes_{\mathcal{O}} \mathbb{F})^{\mathrm{ss}} \sim \sigma_{1,m_{v_0}-1} \oplus \sigma_{p-2,m_{v_0}}$ so that $\mu_{\mathrm{Aut}}(2,\tilde{\tau}_{i,v_0},\bar{\rho}) = 2$. The bound on the Hilbert-Samuel multiplicity of the corresponding deformation ring (1.7.13) now implies that the only liftings are the ones described in Lemma (1.6.13). (This is completely analogous to the argument in Corollary (1.7.14).)

It follows that any eigenform in $S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U},\mathcal{O})_{\tilde{\mathfrak{m}}}$ is nearly ordinary at v_0. Hence $U_{v_0}^2 - \mathrm{tr}(\gamma)U_{v_0} + \det(\gamma) = 0$ modulo π, where we now consider $\mathrm{tr}(\gamma),\det(\gamma) \in R^{\tilde{\psi}}_{F,\tilde{S}}$. The calculations of [Ge, §4.4] show that the induced action of U_{v_0} on $S_{\tilde{\sigma}_i,\tilde{\psi}}(\tilde{U},\mathbb{F})_{\tilde{\mathfrak{m}}}$ leaves $S_{\sigma_i,\psi}(U_{Q_n},\mathbb{F})_{\mathfrak{m}_{Q_n}}$ stable and induces T_{v_0} on the latter space. □

Proposition (2.2.15). *The $\bar{R}_\infty$-module $M^i_\infty/M^{i-1}_\infty$ is non-zero if and only if for each $v|p$ we have $\mu_{n_{i,v},m_{i,v}}(\bar{\rho}|_{G_{F_v}}) \neq 0$. If this condition holds for all $v|p$ and if for* *each $v|p$ we have $\bar\rho|_{G_{F_v}} \not\sim \left(\begin{smallmatrix}\omega\chi & * \\ 0 & \chi\end{smallmatrix}\right)$ for any character $\chi : G_{F_v} \to \mathbb{F}^\times$, then*

$$e(M^i_\infty/M^{i-1}_\infty, \bar R_\infty/\pi\bar R_\infty) \geq e_\Sigma \prod_{v|p} \mu_{n_{i,v},m_{i,v}}(\bar\rho|_{G_{F_v}}) := e_{\Sigma_p} \tag{2.2.16}$$

where

$$e_\Sigma := \prod_{v\in\Sigma} e(\bar R_v^{\square,\psi}/\pi\bar R_v^{\square,\psi}).$$

Proof. The first statement follows from results of Gee [Ge, §4.4], and we may assume that both sides of (2.2.16) are non-zero.

For $i = 1, \dots, s$ and each $v|p$ choose a character $\psi_{i,v}$ as in Lemma (2.2.13) and let $\bar R^{\square,\psi}_{v,i} = R^{\square,\psi_{i,v}}_{\mathrm{cr}}(n_{i,v}+2, (\tilde\omega^{m_{i,v}})^{\oplus 2}, \bar\rho)/\pi$. The notation is justified since $\bar R^{\square,\psi}_{v,i}$ depends only on ψ and not on $\psi_{i,v}$. Let C denote the set of primes $v|p$ such that $\bar\rho|_{G_{F_v}} \sim \begin{pmatrix}\mu_\lambda & 0 \\ 0 & \mu_{\lambda'}\end{pmatrix} \otimes \omega^{m_v}$, with $\lambda \neq \lambda'$. For $v \in C$ we set

$$\tilde R^{\square,\psi}_{v,i} = \bar R^{\square,\psi}_{v,i}[X]/(X^2 - \mathrm{tr}(\gamma)X + \det(\gamma))$$

where γ is an element as in Lemma (2.2.14). By Lemmas (2.2.14) and (2.2.13), M^i_n/M^i_{n-1} has a natural structure of an $\tilde R^{\square,\psi}_{v,i}$-module, with X acting via the operator T_v. Note that by Hensel's lemma $\tilde R^{\square,\psi}_{v,i}$ is a semi-local ring with two maximal ideals generated by the radical of $\bar R^{\square,\psi}_{v,i}$ and one of $(X-\lambda), (X-\lambda')$. In fact Corollary (1.7.14) shows that $\tilde R^{\square,\psi}_{v,i}$ is the normalization of $\bar R^{\square,\psi}_{v,i}$.

For $v \in \Sigma$ write $\bar R^{\square,\psi}_{v,i} = \bar R^{\square,\psi}_v/\pi$. Set $\bar R^{\square,\psi}_{\Sigma_p,i} = \widehat\bigotimes_{v\in\Sigma_p} \bar R^{\square,\psi}_{v,i}$ and

$$\tilde R^{\square,\psi}_{\Sigma_p,i} = \widehat{\bigotimes}_{v\in\Sigma_p, v\notin C} \bar R^{\square,\psi}_{v,i} \widehat{\bigotimes}_{v\in C} \tilde R^{\square,\psi}_{v,i}.$$

Write $\bar R^i_\infty = \bar R^{\square,\psi}_{\Sigma_p,i}[\![x_1,\dots,x_g]\!]$ and $\tilde R^i_\infty = \tilde R^{\square,\psi}_{\Sigma_p,i}[\![x_1,\dots,x_g]\!]$. Replacing the M_n by a subsequence, we may assume that for $v \in C$, the maps $M^i_n/\mathfrak{c}_n \to M^i_{n-1}/\mathfrak{c}_{n-1}$ induced by f_n mod π are compatible with the action of T_v on M^i_n/M^{i-1}_n and M^i_{n-1}/M^{i-1}_{n-1}. Then $M^i_\infty/M^{i-1}_\infty$ is naturally an $\tilde R^i_\infty$-module.

By Corollary (1.7.14), for $v|p$, $v \notin C$, the spectrum of $\bar R^{\square,\psi}_{v,i}$ is geometrically irreducible and generically reduced. In particular, $\tilde R^i_\infty$ is generically reduced and has $2^{|C|}$ connected components, indexed by the choice of a component of $\tilde R^{\square,\psi}_{v,i}$ for each $v \in C$. By [Ge, Thm. 4.4.12], the support of M^i_0/M^{i-1}_0 meets each of these connected components. On the other hand, since $M^i_\infty/M^{i-1}_\infty$ is flat over $\mathbb{F}[\![\Delta_\infty]\!]$, the image of $\tilde R^i_\infty$ in $\mathrm{End}_{\mathbb{F}[\![\Delta_\infty]\!]} M^i_\infty/M^{i-1}_\infty$ has dimension $h+j$. It follows that the support of $M^i_\infty/M^{i-1}_\infty$ consists of all the components of $\tilde R^i_\infty$. In particular, the support of $M^i_\infty/M^{i-1}_\infty$ as an $\bar R^i_\infty$-module is all of $\mathrm{Spec}\,\bar R^i_\infty$. As $\bar R^i_\infty$ is generically reduced, using Propositions (1.3.7) and (1.3.8), we find

$$\begin{aligned} e(M^i_\infty/M^{i-1}_\infty, \bar R_\infty/\pi\bar R_\infty) &= e(M^i_\infty/M^{i-1}_\infty, \bar R^i_\infty) \\ &\geq e(\bar R^i_\infty) = e(\bar R^{\square,\psi}_{\Sigma_p,i}) = \prod_{v\in\Sigma_p} e(\bar R^{\square,\psi}_{v,i}) = e_{\Sigma_p}, \end{aligned}$$

where the final equality follows from Corollary (1.7.14). □

Corollary (2.2.17). *Suppose that for each* $v|p$, $\sigma(k_v, \tau_v)$ *satisfies Hypothesis* (1.2.6) *and* $\bar\rho|_{G_{F_v}} \nsim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ *for any character* $\chi : G_{F_v} \to \mathbb{F}^\times$. *Then* M_∞ *is a faithful* $\bar R_\infty$-*module, and any* $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ *as in Lemma* (2.2.11) *is modular.*

Proof. Using (1.3.9) together with Corollary (1.7.2), Proposition (1.7.8) and Proposition (1.7.10), we have

$$e(\bar R_\infty/\pi\bar R_\infty) = e_\Sigma \prod_{v|p} e(\bar R_v^{\square,\psi}/\pi\bar R_v^{\square,\psi}) \leqslant e_\Sigma \prod_{v|p} \mu_{\mathrm{Aut}}(k_v, \tau_v, \bar\rho|_{G_{F_v}} \otimes \omega^{-w_v}).$$

On the other hand, Proposition (2.2.15) yields

$$\begin{aligned} e(M_\infty/\pi M_\infty, \bar R_\infty/\pi\bar R_\infty) &= \sum_{i=1}^s e(M_\infty^i/M_\infty^{i-1}, \bar R_\infty/\pi\bar R_\infty) \\ &\geq \sum_{i=1}^s e_\Sigma \prod_{v|p} \mu_{n_{i,v},m_{i,v}}(\bar\rho|_{G_{F_v}}) = \sum_{i=1}^s e_\Sigma \prod_{v|p} \mu_{n_{i,v},m_{i,v}-w_v}(\bar\rho|_{G_{F_v}} \otimes \omega^{-w_v}) \\ &= e_\Sigma \prod_{v|p} \mu_{\mathrm{Aut}}(k_v, \tau_v, \bar\rho|_{G_{F_v}} \otimes \omega^{-w_v}). \end{aligned}$$

Hence

$$e(\bar R_\infty/\pi\bar R_\infty) \leqslant e(M_\infty/\pi M_\infty, \bar R_\infty/\pi\bar R_\infty),$$

and the corollary follows from Lemma (2.2.11). □

Theorem (2.2.18). *Let* F *be a totally real field where* p *is totally split, and let*

$$\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$$

be a continuous representation. Suppose that

(1) *For* $v|p$, $\rho|_{G_{F_v}}$ *becomes semi-stable over an abelian extension of* F_v *and has distinct Hodge-Tate weights.*
(2) $\bar\rho : G_{F,S} \xrightarrow{\rho} \mathrm{GL}_2(\mathcal{O}) \to \mathrm{GL}_2(\mathbb{F})$ *is modular and* $\bar\rho|_{F(\zeta_p)}$ *is absolutely irreducible.*
(3) *For* $v|p$, $\bar\rho|_{G_{F_v}} \nsim \left(\begin{smallmatrix} \omega\chi & * \\ 0 & \chi \end{smallmatrix}\right)$ *for any character* $\chi : G_{F_v} \to \mathbb{F}^\times$.

Then ρ *is modular.*

Proof. After making a quadratic base change, we may assume that $[F : \mathbb{Q}]$ is even. Let D be the totally definite quaternion algebra over F which is split at all the finite places of F.

For $v|p$ suppose that $\rho|_{G_{F_v}}$ has Hodge-Tate weights $k_v - 1 + w_v, w_v$ with $k_v \geq 2$, and type τ_v. Let $\sigma_v = \sigma(k_v, \tau_v) \otimes \det^{w_v}$, and set $\sigma = \bigotimes_{v|p} L_v$, where $L_v \subset \sigma_v$ is a $\mathrm{GL}_2(\mathcal{O}_{F_v})$-stable lattice. By Corollary (1.7.2), Proposition (1.7.8) and (1.7.9), the existence of ρ implies that for all $v|p$, $\mu_{\mathrm{Aut}}(k_v, \tau_v, \bar\rho \otimes \omega^{-w_v}) \neq 0$. It follows from the result of Gee [Ge, Thm. 4.4.12] already used above that $\bar\rho$ arises from an eigenform in $S_{\sigma,\psi}(U, \mathcal{O})$, where $\psi = (\det\rho)\chi_{\mathrm{cyc}}^{-1}$, and $U \subset (D \otimes \mathbb{A}_F^f)^\times$ is an appropriately chosen compact open subgroup.

The theorem now follows from Corollary (2.2.17) using the same base change arguments as in [Ki 2, 3.5]. Note that the relevant results on raising and lowering the level at $v \nmid p$ can be deduced from the case where all the W_{σ_v} are of the form $\mathrm{Sym}^{k_v-2} \otimes \det^{m_v}$ where $2 \leqslant k_v \leqslant p+1$, and in this case one has the relevant version of Ihara's lemma (see [Ki 2, 3.1.8, 3.1.10]). That the open subgroup U can be chosen so that S satisfies (2.2)(4) follows from Lemma (2.2.1). □

(2.3) The Breuil-Mézard conjecture. To end the paper, we explain how to deduce the Breuil-Mézard conjecture from the results of the previous section.

Lemma (2.3.1). *In the situation of Corollary* (2.2.17)*, for each $v|p$ we have*

$$e(\bar{R}_v^{\square,\psi}/\pi\bar{R}_v^{\square,\psi}) = \mu_{\mathrm{Aut}}(k_v,\tau_v,\bar{\rho}|_{G_{F_v}}\otimes\omega^{-w_v}).$$

Proof. As remarked in the proof of Corollary (2.2.17), we have

$$e(\bar{R}_v^{\square,\psi}/\pi\bar{R}_v^{\square,\psi}) \leqslant \mu_{\mathrm{Aut}}(k_v,\tau_v,\bar{\rho}|_{G_{F_v}}\otimes\omega^{-w_v}).$$

If this inequality were strict, then the argument of Corollary (2.2.17) would yield

$$e(\bar{R}_\infty/\pi\bar{R}_\infty) < e(M_\infty/\pi M_\infty, \bar{R}_\infty/\pi\bar{R}_\infty),$$

which contradicts Lemma (2.2.11). □

Corollary (2.3.2). *Let $\bar{\rho}: G_{\mathbb{Q}_p}\to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation, $k\geq 2$ an integer and $\tau: I_{\mathbb{Q}_p}\to\mathrm{GL}_2(E)$ of Galois type. Suppose that $\bar{\rho}\nsim\left(\begin{smallmatrix}\omega\chi & *\\ 0 & \chi\end{smallmatrix}\right)$ for any character χ and that if $\bar{\rho}$ has scalar semi-simplification, then it is scalar.*

If the pair (k,τ) satisfies the condition of Hypothesis (1.2.6) *(for example τ is abelian), then*

$$e(R^{\square,\psi}(k,\tau,\bar{\rho})/\pi R^{\square,\psi}(k,\tau,\bar{\rho})) = \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. By Lemma (2.3.1) it suffices to show that there exist a totally real field F, in which p splits, a finite set of primes S of F, and a modular representation $\bar{\rho}_F: G_{F,S}\to\mathrm{GL}_2(\mathbb{F})$ satisfying the conditions (1)–(4) of (2.2), such that $\bar{\rho}_F|_{G_{F_v}}\nsim\left(\begin{smallmatrix}\omega\chi & *\\ 0 & \chi\end{smallmatrix}\right)$ for all primes $v|p$ of F and $\bar{\rho}_F|_{G_{F_w}}\sim\bar{\rho}$ for some $w|p$. When $\bar{\rho}$ is semi-simple, such a $\bar{\rho}_F$ can easily be constructed using CM forms, and this proves the result for semi-simple $\bar{\rho}$.

It remains to consider the case where $\bar{\rho}$ is a non-trivial extension of ω_2 by ω_1 for distinct characters $\omega_1,\omega_2: G_{\mathbb{Q}_p}\to\mathbb{F}^\times$ satisfying $\omega_1\omega_2^{-1}\neq\omega$. Let $\bar{\mathfrak{t}}$ denote the pseudo-representation associated to $\bar{\rho}$. By Lemma (1.7.1) there is a map $R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{t}})\to R^{\square,\psi}(k,\tau,\bar{\rho})$ which is a surjection by Corollary (1.4.4) and which is injective by Corollary (1.4.5) and the definition of U_{ω_1}. Hence we have

$$e(R^{\square,\psi}(k,\tau,\bar{\rho})/\pi R^{\square,\psi}(k,\tau,\bar{\rho})) = e(R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{t}})/\pi R^{\mathrm{ps}}_{U_{\omega_1}}(\bar{\mathfrak{t}}))\leqslant\mu_{\mathrm{Aut}}(k,\tau,\bar{\rho})$$

by Proposition (1.6.15). If this inequality were strict, then the argument of Proposition (1.7.8) would show that

$$e(R^{\square,\psi}(k,\tau,\bar{\rho}^{\mathrm{ss}})/\pi R^{\square,\psi}(k,\tau,\bar{\rho}^{\mathrm{ss}})) < \mu_{\mathrm{Aut}}(k,\tau,\bar{\rho}^{\mathrm{ss}}),$$

contradicting what we have already proved for semi-simple $\bar{\rho}$. □

(2.3.3) We also have the following variant which establishes cases of Conjecture (1.1.5).

Corollary (2.3.4). *With the notation and assumptions of Corollary* (2.3.2)*, we have*

$$e(R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar{\rho})/\pi R^{\square,\psi}_{\mathrm{cr}}(k,\tau,\bar{\rho})) = \mu^{\mathrm{cr}}_{\mathrm{Aut}}(k,\tau,\bar{\rho}).$$

Proof. To prove this, we modify the constructions of (2.2) for $v|p$, by taking W_{σ_v} to be $\mathrm{Sym}^{k_v-2}\mathcal{O}^2\otimes\sigma_{\mathrm{cr}}(\tau_v)\otimes\det^{w_v}$ and taking $\bar{R}_v^{\square,\psi}$ to be the quotient of $R_v^{\square,\psi}$ corresponding to the quotient $R^{\square,\psi_v}_{\mathrm{cr}}(k_v,\tau_v,\bar{\rho}\otimes\omega^{-w_v})$ of $R_v^{\square}(\bar{\rho}\otimes\omega^{-w_v})$. The arguments of (2.2) go over verbatim to establish an analogue of Corollary (2.2.17) in this context. The only difference is that in the proof of Corollary (2.2.17) one invokes Proposition (1.7.13) instead of Corollary (1.7.2), Proposition (1.7.8) and Proposition (1.7.10). The arguments of Lemma (2.3.1) and Corollary (2.3.2) now go over verbatim to prove the corollary. $\square$

Addendum to [Ki 1]

(A.1) Let $K/\mathbb{Q}_p$ be a finite extension, $\bar{K}$ an algebraic closure of K, and $G_K = \mathrm{Gal}(\bar{K}/K)$. We denote by $I_K \subset G_K$ the inertia subgroup.

Let $\mathbb{F}/\mathbb{F}_p$ be a finite extension and $V_{\mathbb{F}}$ a finite dimensional $\mathbb{F}$-vector space of dimension d. Let $E/\mathbb{Q}_p$ be a finite extension and $\tau : I_K \to \mathrm{GL}_d(E)$ a 2-dimensional representation with open kernel which extends to a representation of the Weil group W_K. We also fix a p-adic Hodge type $\mathbf{v}$. The notion of a p-adic Hodge type is defined in [Ki 2, §2.6]. In particular $\mathbf{v}$ specifies a collection of Hodge-Tate weights but, in fact, it gives slightly more information.

Fix a basis of $V_{\mathbb{F}}$ and let $R^{\square}_{V_{\mathbb{F}}}$ denote the universal framed deformation ring. In [Ki 2] we showed that there was a quotient $(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ of $R^{\square}_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathcal{O}_E$ which parameterized potentially semi-stable liftings of $V_{\mathbb{F}}$ of type τ and p-adic Hodge type $\mathbf{v}$. The main point of this addendum is to give a more precise description of the local structure of $(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ when $d = 2$, which we assume from now on. More precisely we show

Theorem (A.2). *The ring $(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}}$ is reduced with formally smooth irreducible components.*

Proof. If every lifting of $V_{\mathbb{F}}$ of type τ and p-adic Hodge type $\mathbf{v}$ is potentially crystalline, then this follows from [Ki 2, Thm. 3.3.8], which in fact shows that these rings are formally smooth. Thus we may assume that $V_{\mathbb{F}}$ has a potentially semi-stable lifting of type $(\tau, \mathbf{v})$ which is not potentially crystalline. This is only possible if $\tau \sim \eta^{\oplus 2}|_{I_K}$ where η is a continuous character of W_K. We may assume that η extends to a character of G_K. Twisting by η^{-1}, we may assume that τ is trivial.

We now define two groupoids on the category of E-algebras, as in [Ki 1, 3.1.1, 3.1.4]. Let $K_0 \subset K$ denote the maximal absolutely unramified subfield. We denote by $\mathfrak{Mod}_{\varphi,N}$ the groupoid whose value on an E-algebra A is the category of finite projective $K_0 \otimes_{\mathbb{Q}_p} A$-module D_A of rank 2, equipped with an isomorphism $1 \otimes \varphi : \varphi^*(D_A) \xrightarrow{\sim} D_A$ and a nilpotent endomorphism $N : D_A \to D_A$ such that $p\varphi N = N\varphi$. Here we extend φ to $K_0 \otimes_{\mathbb{Q}_p} A$, by A-linearity.

To define the second groupoid, fix a finite free $K_0 \otimes_{\mathbb{Q}_p} E$-module D_E of rank 2, and for A an E-algebra write $D_A = D_E \otimes_E A$. We denote by $X_{\varphi,N}$ the groupoid such that the objects of $X_{\varphi,N}(A)$ consist of a φ-semi-linear isomorphism $D_A \xrightarrow{\sim} D_A$ and a nilpotent endomorphism $N : D_A \to D_A$ such that $p\varphi N = N\varphi$.

We then have morphisms of groupoids on E-algebras

$$(R^{\square}_{V_{\mathbb{F}}}[1/p])^{\tau,\mathbf{v}} \to \mathfrak{Mod}_{\varphi,N} \leftarrow X_{\varphi,N}$$

where both maps are formally smooth by [Ki 1, 3.1.5, 3.2.1, 3.3.1]. The groupoid $X_{\varphi,N}$ is representable by an E-scheme of finite type, and the theorem follows from the following $\square$

Lemma (A.3). *$X_{\varphi,N}$ is reduced with smooth irreducible components.*

Proof. Let D_A be in $\mathfrak{Mod}_{\varphi,N}(A)$ and denote by $C^\bullet(A)$ the complex

$$D_A \stackrel{(N,1-\varphi)}{\longrightarrow} D_A \oplus D_A \stackrel{(p\varphi-1,N)}{\longrightarrow} D_A,$$

concentrated in degrees $0,1,2$. Suppose that A is a local E-algebra with maximal ideal $\mathfrak{m}_A$, and let $I \subset A$ be an ideal with $I\mathfrak{m}_A = 0$. Let $D_{A/I}$ be in $\mathfrak{Mod}_{\varphi,N}(A/I)$, and choose a φ-semi-linear lift $\tilde{\varphi} : D_A \xrightarrow{\sim} D_A$ of $\varphi|_{D_{A/I}}$ and a linear lift $\tilde{N} : D_A \to D_A$ of $N|_{D_{A/I}}$. Then $h = \tilde{N} - p\tilde{\varphi}\tilde{N}\tilde{\varphi}^{-1} \in D_A$ gives rise to a class $[h] \in H^2(C^\bullet(D_{A/I}))$ which does not depend on the choice of $\tilde{N}$ and $\tilde{\varphi}$. The computations of [Ki 1, 3.1.2] show that $[h]$ is the obstruction to lifting $D_{A/I}$ to an object of $\mathfrak{Mod}_{\varphi,N}(A)$.

In our present situation, when $d = 2$, the E-dimension of $H^2(C^\bullet(D_{A/I}))$ is at most 1. A standard argument now shows that the complete local ring $\widehat{\mathcal{O}}_x$ at a closed point $x \in X_{\varphi,N}$ is a quotient of a power series ring over $\kappa(x)$ by at most one relation. In particular, since $X_{\varphi,N}$ is generically reduced by [Ki 1, 3.1.5, 3.1.6], it is reduced.

We now define two smooth closed subspaces of $X_{\varphi,N}$. The first, denoted $(X_{\varphi,N})^{\mathrm{cr}}$, corresponds to those objects D_A such that $N = 0$. This is obviously smooth of dimension $4[K_0 : \mathbb{Q}_p]$. To define the second subspace, let T_φ denote the $\mathbb{Q}_p$-torus

$$T_\varphi = \mathrm{coker}(\mathrm{Res}_{K_0/\mathbb{Q}_p}\mathbb{G}_m \stackrel{1-\varphi}{\longrightarrow} \mathrm{Res}_{K_0/\mathbb{Q}_p}\mathbb{G}_m).$$

For any $\mathbb{Q}_p$-algebra A, by a $K_0 \otimes_{\mathbb{Q}_p} A$-line, we mean a free $K_0 \otimes_{\mathbb{Q}_p} A$-module which is everywhere of rank 1. Given such a module L, a φ-semi-linear isomorphism $\varphi_L : L \xrightarrow{\sim} L$ gives rise to an element of $[\varphi_L] \in T_\varphi(A)$. Denote by $(X_{\varphi,N})^{\mathrm{st}}$ the groupoid which assigns to an E-algebra A the following data: A pair of $K_0 \otimes_{\mathbb{Q}_p} A$-lines $L_1, L_2 \subset D_A$ such that $D_A = L_1 \oplus L_2$, φ-semi-linear isomorphisms $\varphi_{L_i} : L_i \xrightarrow{\sim} L_i$, for $i = 1, 2$, such that $[\varphi_{L_1}][\varphi_{L_2}^{-1}] = p \in T_\varphi(A)$, and a map $N : L_1 \to L_2$ such that $p\varphi_{L_2}N = N\varphi_{L_1}$. Note that given $\varphi_{L_1}, \varphi_{L_2}$ with the above property, the space of possible maps N is locally free of rank 1 over A.

One sees easily that $(X_{\varphi,N})^{\mathrm{st}}$ is represented by a formally smooth E-scheme of dimension $4[K_0 : \mathbb{Q}_p]$. Define a morphism $(X_{\varphi,N})^{\mathrm{st}} \to X_{\varphi,N}$, by setting $\varphi = \varphi_{L_1} \oplus \varphi_{L_2}$ and extending N to D_A by setting $N|_{L_2} = 0$. We claim that this is a closed embedding. To see this, it suffices to check that the morphism induces an injection on points in finite local E-algebras. This follows from the fact that one can recover L_1, L_2 from φ on D_A, using the slope decomposition.

It is easy to check that any closed point of $X_{\varphi,N}$ lies on $(X_{\varphi,N})^{\mathrm{cr}}$ or $(X_{\varphi,N})^{\mathrm{st}}$. Hence $X_{\varphi,N}$ is the union of these two formally smooth, closed subspaces. □

(A.4) Erratum to [Ki 1, 2.6.1]**.** We end this appendix with a correction to [Ki 1, §2.6]. We are grateful to G. Pappas for pointing out this inaccuracy. We freely use the notion of [Ki 1] in this paragraph, and all references are to that paper.

The K_A-modules in (3) and (4) of Lemma 2.6.1 in [Ki 1] need not be projective as claimed. The mistake is that the final map in the displayed equation at the end of the proof of that lemma is not well defined in general. The lemma is used only in the proof of Corollary 2.6.2. The statements of the lemma are correct in the situation of the corollary, and in fact its proof may be rearranged to derive these statements without using them as an input: The final two paragraphs of the proof show that $\mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)) \otimes_A B$ is projective over

$K \otimes_{\mathbb{Q}_p} B$ for any A-algebra B which is finite and local over E. This implies that $\mathrm{Fil}^i\varphi^*(\mathfrak{M}_A)/(E(u)\varphi^*(\mathfrak{M}_A) \cap \mathrm{Fil}^i\varphi^*(\mathfrak{M}_A))$ is projective over K_A, as required, and $A^{\mathrm{st},\mathbf{v}}$ can now be defined as in the first paragraph of the proof.

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