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HYPERGRAPH RAMSEY NUMBERS

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1. Introduction

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that "Every large system contains a large well-organized subsystem."[1] This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to combinatorics but also to logic, analysis, number theory, and geometry. Since the publication of the seminal paper of Ramsey in 1930, this subject experienced tremendous growth and is currently among the most active areas in combinatorics.

The *Ramsey number* $r(s,n)$ is the least integer N such that every red-blue coloring of the edges of the complete graph K_N on N vertices contains a red K_s (i.e., a complete subgraph all of whose edges are colored red) or a blue K_n. Ramsey's theorem states that $r(s,n)$ exists for all s and n. Determining or estimating Ramsey numbers is one of the central problems in combinatorics; see the book *Ramsey theory* [20] for details. A classical result of Erdős and Szekeres [17], which is a quantitative version of Ramsey's theorem, implies that $r(n,n) \leq 2^{2n}$ for every positive integer n. Erdős [8] showed using probabilistic arguments that $r(n,n) > 2^{n/2}$ for $n > 2$. Over the last sixty years, there have been several improvements on these bounds (see, e.g., [6]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Off-diagonal Ramsey numbers, i.e., $r(s,n)$ with $s \neq n$, have also been intensely studied. For example, after several successive improvements, it is known (see [1], [21], [28]) that there are constants $c_1, \ldots, c_4$ such that

$$c_1 \frac{n^2}{\log n} \leq r(3,n) \leq c_2 \frac{n^2}{\log n},$$

and for fixed $s > 3$,

$$c_3 \left(\frac{n}{\log n} \right)^{(s+1)/2} \leq r(s,n) \leq c_4 \frac{n^{s-1}}{\log^{s-2} n}. \tag{1}$$

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[1] This is a paraphrase of a similar statement from [4].

(For $s = 4$, Bohman [3] recently improved the lower bound by a factor of $\log^{1/2} n$.) All logarithms in this paper are base e unless otherwise stated.

Although already for graph Ramsey numbers there are significant gaps between the lower and upper bounds, our knowledge of hypergraph Ramsey numbers is even weaker. The Ramsey number $r_k(s, n)$ is the minimum N such that every red-blue coloring of the unordered k-tuples of an N-element set contains a red set of size s or a blue set of size n, where a set is called red (blue) if all k-tuples from this set are red (blue). Erdős, Hajnal, and Rado [15] showed that there are positive constants c and c' such that

$$2^{cn^2} < r_3(n, n) < 2^{2^{c'n}}.$$

They also conjectured that $r_3(n, n) > 2^{2^{cn}}$ for some constant $c > 0$, and Erdős offered a \$500 reward for a proof. Similarly, for $k \geq 4$, there is a difference of one exponential between the known upper and lower bounds for $r_k(n, n)$, i.e.,

$$t_{k-1}(cn^2) \leq r_k(n, n) \leq t_k(c'n),$$

where the tower function $t_k(x)$ is defined by $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i(x)}$.

The study of 3-uniform hypergraphs is particularly important for our understanding of hypergraph Ramsey numbers. This is because of an ingenious construction called the stepping-up lemma due to Erdős and Hajnal (see, e.g., Chapter 4.7 in [20]). Their method allows one to construct lower bound colorings for uniformity $k + 1$ from colorings for uniformity k, effectively gaining an extra exponential each time it is applied. Unfortunately, the smallest k for which it works is $k = 3$. Therefore, proving that $r_3(n, n)$ has doubly exponential growth will allow one to close the gap between the upper and lower bounds for $r_k(n, n)$ for all uniformities k. There is some evidence that the growth rate of $r_3(n, n)$ is closer to the upper bound, namely, that with four colors instead of two this is known to be true. Erdős and Hajnal (see, e.g., [20]) constructed a 4-coloring of the triples of a set of size $2^{2^{cn}}$ which does not contain a monochromatic subset of size n. This is sharp up to the constant c. It also shows that the number of colors matters a lot in this problem and leads to the question of what happens in the intermediate case when we use three colors. The 3-color Ramsey number $r_3(n, n, n)$ is the minimum N such that every 3-coloring of the triples of an N-element set contains a monochromatic set of size n. In this case, Erdős and Hajnal have made some improvement on the lower bound 2^{cn^2} (see [14] and [5]), showing that $r_3(n, n, n) \geq 2^{cn^2 \log^2 n}$. Here, we substantially improve this bound, extending the above-mentioned stepping-up lemma of these two authors to show

Theorem 1.1. *There is a constant $c > 0$ such that*

$$r_3(n, n, n) \geq 2^{n^{c \log n}}. \tag{2}$$

For off-diagonal Ramsey numbers, a classical argument of Erdős and Rado [16] from 1952 demonstrates that

$$r_k(s, n) \leq 2^{\binom{r_{k-1}(s-1, n-1)}{k-1}}. \tag{3}$$

Together with the upper bound in (1) it gives for fixed s that $r_3(s, n) \leq 2^{\binom{r_2(s-1, n-1)}{2}} \leq 2^{c\frac{n^{2s-4}}{\log^{2s-6} n}}$. Our next result improves the exponent of this upper bound by a factor of $n^{s-2}/\text{polylog}\, n$.

Theorem 1.2. *For fixed $s \geq 4$ and sufficiently large n,*

$$\log r_3(s,n) \leq \Big(\frac{(s-3)}{(s-2)!} + o(1)\Big) n^{s-2} \log n. \tag{4}$$

Clearly, a similar improvement for off-diagonal Ramsey numbers of higher uniformity follows from this result together with (3).

Erdős and Hajnal [13] showed that $\log r_3(4,n) > cn$ using the following simple construction. They consider a random tournament on $[N] = \{1, \ldots, N\}$ and they color the triples from $[N]$ red if they form a cyclic triangle and blue otherwise. Since it is well known and easy to show that every tournament on four vertices contains at most two cyclic triangles and a random tournament on N vertices with high probability does not contain a transitive subtournament of size $c' \log N$, the resulting coloring neither has a red set of size 4 nor a blue set of size $c' \log N$. In the same paper from 1972, they suggested that $\frac{\log r_3(4,n)}{n} \to \infty$. Here we prove the following new lower bound, which implies this conjecture.

Theorem 1.3. *There is a constant $c > 0$ such that*

$$\log r_3(s,n) \geq c\, sn \log\left(\frac{n}{s} + 1\right)$$

for all $4 \leq s \leq n$.

Combining this result together with the stepping-up lemma of Erdős and Hajnal (see [20]), one can also obtain analogous improvements of lower bounds for off-diagonal Ramsey numbers for complete k-uniform hypergraphs with $k \geq 4$.

In view of our unsatisfactory knowledge of the growth rate of hypergraph Ramsey numbers, Erdős and Hajnal [13] started the investigation of the following more general problem. Fix positive integers k, s, and t. What is the smallest N such that every red-blue coloring of the k-tuples of an N-element set has either a red set of size n or has a set of size s which contains at least t blue k-tuples? Note that when $t = \binom{s}{k}$ the answer to this question is simply $r_k(n,s)$.

Let $\binom{X}{k}$ denote the collection of all k-element subsets of the set X. Define $f_k(N,s,t)$ to be the largest n for which every red-blue coloring of $\binom{[N]}{k}$ has a red n-element set or a set of size s which contains at least t blue k-tuples. Erdős and Hajnal [13] in 1972 conjectured that as t increases from 1 to $\binom{s}{k}$, $f_k(N,s,t)$ grows first like a power of N, then at a well-defined value $t = h_1^{(k)}(s)$, $f_k(N,s,t)$ grows like a power of $\log N$, i.e., $f_k(N,s,h_1^{(k)}(s)-1) > N^{c_1}$ but $f_k(N,s,h_1^{(k)}(s)) < (\log N)^{c_2}$. Then, as t increases further, at $h_2^{(k)}(s)$ the function $f_k(N,s,t)$ grows like a power of $\log\log N$, etc., and finally $f_k\,(N,s,t)$ grows like a power of $\log_{(k-2)} N$ for $h_{k-2}^{(k)}(s) \leq t \leq \binom{s}{k}$. Here $\log_{(i)} N$ is the i-fold iterated logarithm of N, which is defined by $\log_{(1)} N = \log N$ and $\log_{(j+1)} N = \log(\log_{(j)} N)$.

This problem of Erdős and Hajnal is still widely open. In [13] they started a careful investigation of $h_1^{(3)}(s)$ and made several conjectures which would determine this function. We make progress on their conjectures, computing $h_1^{(3)}(s)$ for infinitely many values of s. We also approximate $h_1^{(3)}(s)$ for all s.

Theorem 1.4. *If s is a power of 3, then*

$$h_1^{(3)}(s) = \frac{1}{4}\binom{s+1}{3} + 1.$$

Moreover, for all s,

$$h_1^{(3)}(s) = \frac{s^3}{24} + O(s \log s).$$

More precisely, we show that when s is a power of 3, then there is a red-blue coloring of the triples of an N-element set such that no subset of size s contains $\frac{1}{4}\binom{s+1}{3} + 1$ red triples, and the largest blue subset has size $O(\log N)$. On the other hand, every red-blue coloring of the triples of an N-element set has a subset of size s which contains $\frac{1}{4}\binom{s+1}{3}$ red triples or has a blue subset of size N^{c_s}. Our methods can be used to determine $h_1^{(3)}(s)$ for many other values of s, including 24 of the first 100 positive integers.

In the next section, we prove Theorem 1.2. Our lower bound on $r_3(s,n)$ appears in Section 3 and, in Section 4, we prove Theorem 1.1. In Section 5, we prove Theorem 1.4. Finally, in the last section of the paper, we make several additional remarks on related hypergraph Ramsey problems. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

2. An upper bound for $r_3(s,n)$

In this section we prove the upper bound (4) on off-diagonal hypergraph Ramsey numbers.

First we briefly discuss a classical approach to this problem by Erdős-Rado and indicate where it can be improved. Let $N = 2^{\binom{r(s-1,n-1)}{2}}$. To prove $\log_2 r_3(s,n) \le \binom{r(s-1,n-1)}{2}$, given a red-blue coloring χ of the triples from $[N]$, Erdős and Rado greedily construct a set of vertices $\{v_1, \ldots, v_{r(s-1,n-1)+1}\}$ such that for any given pair $1 \le i < j \le r(s-1,n-1)$, all triples $\{v_i, v_j, v_k\}$ with $k > j$ are of the same color, which we denote by $\chi'(v_i, v_j)$. By definition of the Ramsey number, there is either a red clique of size $s-1$ or a blue clique of size $n-1$ in coloring χ', and this clique together with $v_{r(s-1,n-1)+1}$ forms a red set of size s or a blue set of size n in coloring χ. The greedy construction of the set $\{v_1, \ldots, v_{r(s-1,n-1)+1}\}$ is as follows. First, pick an arbitrary vertex v_1 and set $S_1 = S \setminus \{v_1\}$. After having picked $\{v_1, \ldots, v_i\}$, we also have a subset S_i such that for any pair a, b with $1 \le a < b \le i$, all triples $\{v_a, v_b, w\}$ with $w \in S_i$ are the same color. Let v_{i+1} be an arbitrary vertex in S_i and set $S_{i,0} = S_i \setminus \{v_{i+1}\}$. Suppose we already constructed $S_{i,j} \subset S_{i,0}$ such that, for every $h \le j$ and $w \in S_{i,j}$, all triples $\{v_h, v_{i+1}, w\}$ have the same color. If the number of edges $\{v_{j+1}, v_{i+1}, w\}$ with $w \in S_{i,j}$ that are red is at least $|S_{i,j}|/2$, then we let

$$S_{i,j+1} = \{w : \{v_{j+1}, v_{i+1}, w\} \text{ is red and } w \in S_{i,j}\}$$

and set $\chi'(i+1, j+1) = \text{red}$; otherwise we let

$$S_{i,j+1} = \{w : \{v_{j+1}, v_{i+1}, w\} \text{ is blue and } w \in S_{i,j}\}$$

and set $\chi'(i+1, j+1) = \text{blue}$. Finally, we let $S_{i+1} = S_{i,i}$. Notice that $\{v_1, \ldots, v_{i+1}\}$ and S_{i+1} have the desired properties to continue the greedy algorithm. Also, for each edge $v_{i+1}v_{j+1}$ that we color by χ', the set $S_{i,j}$ is at most halved. So we lose a factor of at most two for each of the $\binom{r(s-1,n-1)}{2}$ edges colored by χ'.[2]

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[2]We also lose one element from S_i when we pick v_{i+1}, but this loss is rather insubstantial.

There are two ways we are able to improve on the Erdős-Rado approach. Our first improvement comes from utilizing the fact that we do not need to ensure that for every pair $i < j$, all edges $\{v_i, v_j, v_k\}$ with $k > j$ are of the same color. That is, the coloring χ' will not necessarily color every pair. Furthermore, the number of edges we color by χ' will be much smaller than the best-known estimate for $\binom{r(s-1,n-1)}{2}$, and this is how we will be able to get a smaller upper bound on $r_3(s,n)$. This idea is nicely captured using the vertex on-line Ramsey number which we next define. Consider the following game, played by two players, builder and painter: at step $i+1$ a new vertex v_{i+1} is revealed; then, for every existing vertex v_j, $j = 1, \ldots, i$, builder decides, in order, whether to draw the edge $v_j v_{i+1}$; if he does expose such an edge, painter has to color it either red or blue immediately. The *vertex on-line Ramsey number* $\tilde{r}(k,l)$ is then defined as the minimum number of edges that builder has to draw in order to force painter to create a red K_k or a blue K_l. In Lemma 2.2, we provide an upper bound on $\tilde{r}(s-1,n-1)$ which is much smaller than the best-known estimate on $\binom{r(s-1,n-1)}{2}$. Since we are losing a factor of at most two for every exposed edge, this immediately improves on the Erdős-Rado bound for $r_3(s,n)$.

A further improvement can be made by using the observation that there will not be many pairs $i < j$ for which all triples $\{v_i, v_j, v_k\}$ with $k > j$ are red. That is, we will be able to show that there are not many red edges in the coloring χ' we construct. Let $0 < \alpha \ll 1/2$. Suppose we have $\{v_1, \ldots, v_i\}$ and a set S and, for a given $j < i$, we want to find a subset $S' \subset S$ such that all triples $\{v_j, v_i, w\}$ with $w \in S'$ have the same color. Let S_{red} denote the set of $w \in S$ for which the triple $\{v_j, v_i, w\}$ is red. We pick $S' = S_{\text{red}}$ if $|S_{\text{red}}| \geq \alpha|S|$ and otherwise we pick $S' = S \setminus S_{\text{red}}$. While the size of S decreases now by a much larger factor for each red edge in χ', there are not many red edges in χ'. On the other hand, we lose very little, specifically a factor $(1-\alpha)$, for each blue edge in χ'. By picking α appropriately, we gain significantly over taking $\alpha = 1/2$ for our upper bound on off-diagonal hypergraph Ramsey numbers.

Before we proceed with the proof of our upper bound on $r_3(s,n)$, we want to discuss some other Ramsey-type numbers related to our vertex on-line Ramsey game. One variant of Ramsey numbers, which was extensively studied in the literature (e.g., [18]), is the *size Ramsey number* $\hat{r}(G_1, G_2)$, which is the minimum number of edges of a graph whose every red-blue edge-coloring contains a red G_1 or a blue G_2. Clearly, $\tilde{r}(k,\ell) \leq \hat{r}(K_k, K_l)$ since builder can choose to pick the edges of a graph which gives the size Ramsey number for (K_k, K_l). Unfortunately, it is not difficult to show that $\hat{r}(K_k, K_l) = \binom{r(k,l)}{2}$, and therefore we cannot obtain any improvement using these numbers. Another on-line Ramsey game which is quite close to ours was studied in [24]. In this game, there are two players, builder and painter, who move on the originally empty graph with an unbounded number of vertices. At each step, builder draws a new edge and painter has to color it either red or blue immediately. The *edge on-line Ramsey number* $\bar{r}(k,l)$ is then defined as the minimum number of edges that builder has to draw in order to force painter to create a red K_k or a blue K_l. A randomized version of the edge on-line Ramsey game was studied in [19]. The authors of [24] proved an upper bound for $\bar{r}(k,l)$ which is similar to our upcoming Lemma 2.2. A careful reading of their paper shows that builder first exposes edges from the first vertex to all future vertices, and so on. Thus, this builder strategy cannot be implemented when proving upper bounds for hypergraph Ramsey numbers. Moreover, it is not clear how to use the edge on-line Ramsey game to get an improvement on hypergraph Ramsey numbers. Lemma 2.2 is therefore essential for our proof giving new upper bounds for hypergraph Ramsey numbers.

Using the ideas discussed above, we next prove an upper bound on $r_3(s,n)$ which involves some parameters of the vertex on-line Ramsey game.

Theorem 2.1. *Suppose in the vertex on-line Ramsey game that builder has a strategy which ensures a red K_{s-1} or a blue K_{n-1} using at most v vertices, r red edges, and in total m edges. Then, for any $0 < \alpha \le 1/2$, the Ramsey number $r_3(s,n)$ satisfies*

$$r_3(s,n) \le (v+1)\alpha^{-r}(1-\alpha)^{r-m}. \tag{5}$$

Proof. Let $N = (v+1)\alpha^{-r}(1-\alpha)^{r-m}$ and consider a red-blue coloring χ of the triples of the set $[N]$. We wish to show that the coloring χ must contain a red set of size s or a blue set of size n.

We greedily construct a set of vertices $\{v_1,\ldots,v_h\}$ and a graph Γ on these vertices with at most v vertices, at most r red edges, and at most m total edges across them such that for any edge $e = v_iv_j$, $i<j$ in Γ, the color of any 3-edge $\{v_i,v_j,v_k\}$ with $k>j$ is the same, say $\chi'(e)$. Moreover, this graph will contain either a red K_{s-1} or a blue K_{n-1}, which one can easily see will define a red set of size s or a blue set of size n.

We begin the construction of this set of vertices by first choosing a vertex $v_1 \in [N]$ and setting $S_1 = [N]\setminus\{v_1\}$. Given a set of vertices $\{v_1,\ldots,v_a\}$, we have a set S_a such that for each edge $e = v_iv_j$ of Γ with $i,j\le a$, the color of the 3-edge $\{v_i,v_j,w\}$ is the same for every w in S_a.

Now let v_{a+1} be a vertex in S_a. We play the vertex on-line Ramsey game, so that builder chooses the edges to be drawn according to his strategy. Painter then colors these edges. For the first edge e_1 chosen, painter looks at all triples containing this edge and a vertex from $S_a\setminus\{v_{a+1}\}$. The 2-edge is colored red in χ' if there are more than $\alpha(|S_a|-1)$ such triples that are red and is colored blue otherwise. This defines a new subset $S_{a,1}$, which are all vertices in $S_a\setminus\{v_{a+1}\}$ such that, together with edge e_1, they form a triple of color $\chi'(e_1)$. For the next drawn edge e_2, we color it red if there are more than $\alpha|S_{a,1}|$ red triples containing it and a vertex from $S_{a,1}$ and we color it blue otherwise. This will define an $S_{a,2}$, and so forth. After we have added all edges from v_{a+1}, the remaining set will be S_{a+1}. Let m_a be the number of edges $e = v_iv_a$ with $i<a$ in Γ and let r_a be the number of such edges that are red.

We now show by induction that

$$|S_a| \ge (v+1-a)\alpha^{-r+\sum_{i=1}^a r_i}(1-\alpha)^{r-m+\sum_{i=1}^a m_i-r_i}.$$

For the base case $a=1$, we have

$$|S_1| = N-1 = (v+1)\alpha^{-r}(1-\alpha)^{r-m} - 1 \ge v\alpha^{-r}(1-\alpha)^{r-m}.$$

Suppose we have proved the desired inequality for a. When we draw a vertex v_{a+1}, the size of our set S_a decreases by 1. Each time we draw an edge from v_{a+1}, the size of our set S goes down by a factor α or $1-\alpha$. Therefore,

$$\begin{aligned} |S_{a+1}| &\geq \alpha^{r_{a+1}}(1-\alpha)^{m_{a+1}-r_{a+1}}(|S_a|-1) \geq \alpha^{r_{a+1}}(1-\alpha)^{m_{a+1}-r_{a+1}}|S_a|-1 \\ &\geq (v+1-a)\alpha^{-r+\sum_{i=1}^{a+1} r_i}(1-\alpha)^{r-m+\sum_{i=1}^{a+1} m_i-r_i}-1 \\ &\geq ((v+1)-(a+1))\alpha^{-r+\sum_{i=1}^{a+1} r_i}(1-\alpha)^{r-m+\sum_{i=1}^{a+1} m_i-r_i}. \end{aligned}$$

By our assumption on the vertex on-line Ramsey game, when the constructed graph Γ contains either a red K_{s-1} or a blue K_{n-1}, this graph will have $a \leq v$ vertices, at most r red edges, and at most m total edges. Therefore at this time we have

$$|S_a| \geq (v+1-a)\alpha^{-r+\sum_{i=1}^{a} r_i}(1-\alpha)^{r-m+\sum_{i=1}^{a} m_i-r_i} \geq 1;$$

i.e., S_a is not empty. Thus a vertex from S_a, together with the red K_{s-1} or blue K_{n-1} in the edge-coloring χ' of Γ, makes either a red set of size s or a blue set of size n in coloring χ, completing the proof. □

Lemma 2.2. *In the vertex on-line Ramsey game, builder has a strategy which ensures a red K_s or a blue K_n using at most $\binom{s+n-2}{s-1}$ vertices, $(s-2)\binom{s+n-2}{s-1}+1$ red edges, and $(s+n-4)\binom{s+n-2}{s-1}+1$ total edges. In particular,*

$$\tilde{r}(s,n) \leq (s+n-4)\binom{s+n-2}{s-1}+1.$$

Proof. We are going to define a set of vertices labeled by strings and the associated set of edges to be drawn during the game as follows. The first vertex exposed will be labeled as $w_\emptyset$. Every other vertex which we expose during the game will be connected by an edge to $w_\emptyset$. Recall that immediately after the edge is exposed it is colored by painter. The first vertex which is connected to $w_\emptyset$ by a red (blue) edge is labeled w_R (w_B). Successively, we connect vertex v to w_R or w_B if and only if this vertex is already connected to $w_\emptyset$ by a red or respectively blue edge.

More generally, if we have defined $w_{a_1a_2\cdots a_p}$ with each $a_i = R$ or B and v is the first exposed vertex which is connected to $w_{a_1\cdots a_j}$ in color a_{j+1} for each $j=0,\ldots,p$, then we label v as $w_{a_1\cdots a_{p+1}}$. (When $j=0$, $w_{a_1\cdots a_j}=w_\emptyset$.) The only successively chosen vertices which we join to $w_{a_1\cdots a_{p+1}}$ by an edge will be those vertices v which are also joined to $w_{a_1\cdots a_j}$ in color a_{j+1} for each $j=0,\ldots,p$. Builder stops adding vertices and edges once painter has completed a red K_s or a blue K_n.

Suppose now that we have exposed $\binom{s+n-2}{s-1}=\binom{s+n-3}{s-2}+\binom{s+n-3}{s-1}$ vertices in total. Since $w_\emptyset$ is connected to all vertices, its degree is $\binom{s+n-2}{s-1}-1$. Thus $w_\emptyset$ is connected either to $\binom{s+n-3}{s-2}$ vertices in red or $\binom{s+n-3}{s-1}$ vertices in blue. If the former holds we look at the neighbors of w_R, which are all vertices which are labeled by a string with first letter R. Otherwise we look at neighbors of w_B. Suppose now that we are looking at the neighbors of $w_{a_1\cdots a_p}$, where r of the a_i are red and b of them are blue. Then, by our construction, $w_{a_1\cdots a_p}$ will have been joined to $\binom{s+n-r-b-2}{s-r-1}-1$ vertices. Now, either $w_{a_1\cdots a_p}$ is joined to $\binom{s+n-r-b-3}{s-r-2}$ vertices in red or $\binom{s+n-r-b-3}{s-r-1}$ vertices in blue. In the first case we look at $w_{a_1\cdots a_pR}$ and its neighbors and in the second case at $w_{a_1\cdots a_pB}$ and its neighbors. Clearly in this process we will reach a string which has either $s-1$ reds or $n-1$ blues. If $s-1$ of the a_i, say $a_{j_1+1},\ldots,a_{j_{s-1}+1}$, are R, then we know that the collection of vertices $w_{a_1\cdots a_{j_1}},\ldots,w_{a_1\cdots a_{j_{s-1}}},w_{a_1\cdots a_{j_{s-1}+1}}$ forms a red clique of size s. Similarly, if $n-1$

of the a_i are B, then we have a blue clique of size n. Therefore, using our strategy, builder wins after using at most $\binom{s+n-2}{s-1}$ vertices.

All that remains to do is to estimate how many edges builder draws. Look at the vertices in the order they were exposed. Clearly, for every vertex we can only look at the edges connecting it to preceding vertices. Notice that a vertex $w_{a_1\cdots a_p}$ is adjacent to at most p vertices which were exposed before it. Moreover, the number of red edges connecting $w_{a_1\cdots a_p}$ to vertices before it is at most the number of a_i which are R. Since all but the last vertex are labeled by strings of length at most $s+n-4$, we have at most $(s+n-4)\binom{s+n-2}{s-1}+1$ total edges. Similarly, all but the last vertex have at most $s-2$ symbols R in their string, which shows that the number of edges colored red during the game is at most $(s-2)\binom{s+n-2}{s-1}+1$. □

The following result implies (4).

Corollary 2.3. *The Ramsey number $r_3(s,n)$ with $4 \le s \le n$ satisfies*

$$r_3(s,n) \le 2^{\frac{(s-3)}{(s-2)!}(s+n)^{s-2}\log_2(64n/s)}. \tag{6}$$

Proof. By Lemma 2.2, in the vertex on-line Ramsey game, builder has a strategy which ensures a red K_{s-1} or a blue K_{n-1} using at most $v=\binom{s+n-4}{s-2}$ vertices, $r=(s-3)\binom{s+n-4}{s-2}+1$ red edges, and $m=(s+n-6)\binom{s+n-4}{s-2}+1$ total edges. To minimize the function $\alpha^{-r}(1-\alpha)^{r-m}$, one should take $\alpha=r/m$. Note that $(m-r)/r \le (n-3)/(s-3) \le n/s$, $v<r\le(m+1)/2\le 2m/3$ and $r \le \frac{(s-3)}{(s-2)!}(s+n)^{s-2}$. Hence, the Ramsey number $r_3(s,n)$ satisfies

$$\begin{aligned}
r_3(s,n) &\le (v+1)\left(\frac{m}{r}\right)^r\left(1-\frac{r}{m}\right)^{r-m} = (v+1)\left(\frac{m-r}{r}\right)^r\left(\frac{m-r}{m}\right)^{-m}\\
&\le (v+1)\left(\frac{n}{s}\right)^r\left(\frac{m-r}{m}\right)^{-m} = (v+1)\left(\frac{n}{s}\right)^r\left(1+\frac{r}{m-r}\right)^m\\
&\le (v+1)\left(\frac{n}{s}\right)^r\left(1+\frac{3r}{m}\right)^m \le r\left(\frac{n}{s}\right)^r e^{3r} = r\left(\frac{e^3 n}{s}\right)^r < \left(\frac{64n}{s}\right)^r\\
&\le 2^{\frac{(s-3)}{(s-2)!}(s+n)^{s-2}\log_2(64n/s)}.
\end{aligned}$$

□

Also, taking $\alpha=1/2$ in Theorem 2.1, it is worth noting that in the diagonal case our results easily imply the following theorem, which improves upon the bound $r_3(k,k)\le 2^{2^{4k}}$ due to Erdős and Rado.

Theorem 2.4.

$$\log_2\log_2 r_3(k,k) \le (2+o(1))k.$$

Our methods can also be used to study Ramsey numbers of noncomplete hypergraphs. To illustrate this, we will obtain a lower bound on $f_3(N,4,3)$, slightly improving a result of Erdős and Hajnal. Let $K_t^{(3)}$ denote the complete 3-uniform hypergraph with t vertices and let $K_t^{(3)}\setminus e$ denote the 3-uniform hypergraph with t vertices formed by removing one triple. For k-uniform hypergraphs H and G, the Ramsey number $r(H,G)$ is the minimum N such that every red-blue coloring of $\binom{[N]}{k}$ contains a red copy of H or a blue copy of G. Note that an upper bound on $f_3(N,4,3)$ is equivalent to a lower bound on the Ramsey number $r\big(K_4^{(3)}\setminus e, K_n^{(3)}\big)$

because they are inverse functions of each other. Erdős and Hajnal [13] proved the following bounds:

$$\frac{1}{2}\frac{\log_2 N}{\log_2\log_2 N} \le f_3(N,4,3) \le (2\log_2 N)+1.$$

The upper bound follows from the same coloring (discussed in the introduction) based on tournaments which gives a lower bound on $r_3(4,n)$. We will use our approach to improve the lower bound by an asymptotic factor of 2.

Proposition 2.5. *We have* $r\big(K_4^{(3)}\setminus e, K_n^{(3)}\big) \le (2en)^n$.

Sketch of proof. We apply the exact same proof technique as we did for Theorem 2.1 except that we will expose all edges. We have a coloring of the complete 3-uniform hypergraph with $N = r(K_4^{(3)}\setminus e, K_n^{(3)}) - 1$ vertices which neither contains a red $K_4^{(3)}\setminus e$ nor a blue set of size n. Note that the coloring χ' of the edges of the complete graph with vertex set $V = \{v_1,\ldots,v_{h-1}\}$ we get in the proof does not contain a pair of monochromatic red edges v_jv_i and v_jv_k with $1 \le j < i < k < h$ or $1 \le i < j < k < h$; otherwise, v_i, v_j, v_k, v_h are the vertices of a red $K_4^{(3)}\setminus e$. Therefore, the red graph in the coloring χ' is just a disjoint union of stars. Let m be the number of edges in the red graph. Note that a disjoint union of stars with m edges has an independent set of size m and forms a bipartite graph. Therefore the red graph has an independent set of size at least $\max\{m,(h-1)/2\}$. Such an independent set in the red graph is a clique in the blue graph in the coloring χ', and together with v_h makes a blue complete 3-uniform hypergraph in the coloring χ. This gives us the inequalities $m+1 < n$ and $(h-1)/2+1 < n$. With hindsight, we pick $\alpha = 1/(2n)$. By the same proof as for Theorem 2.1, this implies that

$$\begin{aligned}r(K_4^{(3)}\setminus e, n) &= N+1 \le 1+(1+h)\alpha^{-m}(1-\alpha)^{m-\binom{h-1}{2}}\\ &\le (2n)\big(2n\big)^{n-2}\left(1-\frac{1}{2n}\right)^{-h^2/2} \le (2en)^n,\end{aligned}$$

where we use that $3 \le h \le 2n-2, m \le n-2$ and that $(1-1/x)^{1-x} \le e$ for $x > 1$. □

Theorem 1.3 shows that $\log r_3(4,n) > cn\log n$ for an absolute constant c. It would also be nice to give a similar lower bound (if it is true) for $r\big(K_4^{(3)}\setminus e, K_n^{(3)}\big)$ since then we would know that $\log r\big(K_4^{(3)}\setminus e, K_n^{(3)}\big)$ has order $n\log n$.

3. A lower bound construction

The purpose of this section is to prove Theorem 1.3, which gives a new lower bound on $r_3(s,n)$. To do this, we need to recall an estimate for graph Ramsey numbers. As we already mentioned in (1), for sufficiently large n and fixed s, $r(s,n) > c\,(n/\log n)^{(s+1)/2} > n^{3/2}$. Also, for all $3 \le s \le n$ and n sufficiently large, one can easily show that $r(s,n) > (\frac{n+s}{s})^{s/3}$. (This is actually not the best lower bound for $r(s,n)$, but it is enough for our purposes.) Indeed, if $s=3$, this bound is trivial. For $s \ge 4$, consider a random red-blue edge-coloring of the complete graph on $N = (\frac{n+s}{s})^{s/3}$ vertices in which each edge is red with probability $p = \big(\frac{s}{n+s}\big)^{0.9}$. It is easy to check that the expected number of monochromatic red s-cliques and blue n-cliques in this coloring is $\binom{N}{s}p^{\binom{s}{2}} + \binom{N}{n}(1-p)^{\binom{n}{2}} < 1$. These estimates together with the next theorem clearly imply Theorem 1.3.

Theorem 3.1. *For all sufficiently large n and $4 \le s \le n$,*

$$r_3(s,n) > \big(r(s-1,n/4)-1\big)^{n/24}.$$

Proof. Let $\ell = n/4$, $r = r(s-1,\ell)-1$ and $N = r^{n/24}$. Note that, since $r(s-1,\ell) > \ell^{3/2}$ for $s \ge 4$, $\ell - 1 < r^{2/3}$. Let $\chi_1 : \binom{[r]}{2} \to \{\text{red, blue}\}$ be a red-blue edge-coloring of the complete graph on $[r]$ with no red clique of size $s-1$ and no blue clique of size ℓ. Consider a coloring $\chi_2 : \binom{[N]}{2} \to [r]$ picked uniformly at random from all r-colorings of $\binom{[N]}{2}$; i.e., each edge has probability $\frac{1}{r}$ of being a particular color independent of all other edges. Using the auxiliary colorings χ_1 and χ_2, we define the red-blue coloring $\chi : \binom{[N]}{3} \to \{\text{red, blue}\}$, where the color of a triple $\{a,b,c\}$ with $a<b<c$ is $\chi_1\big(\chi_2(a,b),\chi_2(a,c)\big)$ if $\chi_2(a,b) \ne \chi_2(a,c)$ and is blue if $\chi_2(a,b) = \chi_2(a,c)$. We next show that in coloring χ there is no red set of size s and with positive probability no blue set of size n, which implies the theorem.

First, suppose that the coloring χ contains a red set $\{u_1,\ldots,u_s\}$ of size s with $u_1 < \ldots < u_s$. Then all the colors $\chi_2(u_1,u_j)$ with $2 \le j \le s$ are distinct and form a red clique of size $s-1$ in χ_1, a contradiction.

Next, we estimate the expected number of blue cliques of size n in coloring χ. Let $\{v_1,\ldots,v_n\}$ with $v_1 < \ldots < v_n$ be a set of n vertices. Fix for now $1 \le i \le n$. If all triples $\{v_i,v_j,v_k\}$ with $i<j<k$ are blue, then the distinct colors among the colors $\chi_2(v_i,v_j)$ for $i < j \le n$ must form a blue clique in coloring χ_1. Therefore the number of distinct colors $\chi_2(v_i,v_j)$ with $i<j\le n$ is less than ℓ. Every such subset of distinct colors is contained in at least one of the $\binom{r}{\ell-1}$ subsets of $[r]$ of size $\ell-1$. If we fix a set of $\ell-1$ colors, the probability that each of the colors $\chi_2(v_i,v_j)$ with $i<j\le n$ is one of these $\ell-1$ colors is $\left(\frac{\ell-1}{r}\right)^{n-i}$. Therefore the expected number of blue cliques of size n in coloring χ is at most

$$\begin{aligned}
\binom{N}{n}\prod_{i=1}^{n}\binom{r}{\ell-1}\left(\frac{\ell-1}{r}\right)^{n-i} &\le N^n\binom{r}{\ell-1}^n\left(\frac{\ell-1}{r}\right)^{\binom{n}{2}}\\
&\le N^n\left(\frac{er}{\ell-1}\right)^{(\ell-1)n}\left(\frac{\ell-1}{r}\right)^{\binom{n}{2}}\\
&= \left(Ne^{\ell-1}\left(\frac{\ell-1}{r}\right)^{\frac{n-1}{2}-(\ell-1)}\right)^n\\
&< \left(Ne^{\ell-1}\left(r^{-1/3}\right)^{\frac{n-1}{2}-(\ell-1)}\right)^n\\
&< \left(N^2\left(r^{-1/3}\right)^{n/4}\right)^n = 1,
\end{aligned}$$

where we use that $\ell - 1 < r^{2/3}$, $\ell = n/4$, and $N = r^{n/24} = r^{\ell/6} > e^{\ell}$. Hence, there is a coloring χ with no red set of size s and no blue set of size n. $\square$

An additional feature of our new lower bound on $r_3(s,n)$ is that it increases continuously with the growth of s and for $s=n$ coincides with the bound $r_3(n,n) \ge 2^{cn^2}$, which was given by Erdős, Hajnal, and Rado [15]. For example, for $n^{1/2} \ll$

$s \ll n$, the previously best-known bound for $r_3(s,n)$ was essentially $r_3(s,n) \geq r_3(s,s) \geq 2^{cs^2}$.

4. Bounding $r_3(n,n,n)$

We now prove the lower bound, $r_3(n,n,n) \geq 2^{n^{c\log n}}$, mentioned in the introduction. Though our method follows the stepping-up tradition of Erdős and Hajnal, it is curious to note that their own best lower bound on the problem, $r_3(n,n,n) \geq 2^{cn^2\log^2 n}$, is not proven in this manner. In Erdős and Hajnal's proof that $r_3(n,n,n,n) > 2^{2^{cn}}$, they use the stepping-up lemma starting from a 2-coloring of a complete graph with $r(n-1,n-1)-1$ vertices not containing a monochromatic clique of size $n-1$ to obtain a 4-coloring of the triples of a set of size $2^{r(n-1,n-1)-1}$ without a monochromatic set of size n. Our proof that $r_3(n,n,n) > 2^{n^{c\log n}}$ is also based on the stepping-up lemma, using essentially the following idea. We start with a 2-coloring of the complete graph on $r(\log_2 n, n-1)-1$ vertices which contains neither a monochromatic red clique of size $\log_2 n$ nor a monochromatic blue clique of size $n-1$. Then we obtain a 4-coloring of the triples of a set of size $2^{r(\log_2 n, n-1)-1} \geq 2^{n^{c\log n}}$ as in the Erdős-Hajnal proof. Next we combine two of the four color classes to obtain a 3-coloring of the triples. Finally, we carefully analyze this 3-coloring to show that it does not contain a monochromatic set of size n.

Theorem 4.1.

$$r_3(n,n,n) > 2^{r(\log_2 n, n-1)-1}.$$

Proof. Let G be a graph on $m = r(\log_2 n, n-1)-1$ vertices which contains neither a clique of size $n-1$ nor an independent set of size $\log_2 n$ and let $\bar{G}$ be the complement of G. We are going to consider the complete 3-uniform hypergraph H on the set

$$T = \{(\gamma_1, \ldots, \gamma_m) : \gamma_i = 0 \text{ or } 1\}.$$

If $\epsilon = (\gamma_1, \ldots, \gamma_m)$, $\epsilon' = (\gamma'_1, \ldots, \gamma'_m)$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma'_i\};$$

that is, $\delta(\epsilon, \epsilon')$ is the largest coordinate at which they differ. Given this, we can define an ordering on T, saying that

$$\epsilon < \epsilon' \text{ if } \gamma_i = 0, \gamma'_i = 1,$$
$$\epsilon' < \epsilon \text{ if } \gamma_i = 1, \gamma'_i = 0,$$

where $i = \delta(\epsilon, \epsilon')$. Equivalently, associate to any ϵ the number $b(\epsilon) = \sum_{i=1}^m \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ iff $b(\epsilon) < b(\epsilon')$.

We will further need the following two properties of the function δ, which one can easily prove.

(a) If $\epsilon_1 < \epsilon_2 < \epsilon_3$, then $\delta(\epsilon_1, \epsilon_2) \neq \delta(\epsilon_2, \epsilon_3)$ and
(b) if $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_p$, then $\delta(\epsilon_1, \epsilon_p) = \max_{1\leq i\leq p-1} \delta(\epsilon_i, \epsilon_{i+1})$.

In particular, these properties imply that there is a unique index i which achieves the maximum of $\delta(\epsilon_i, \epsilon_{i+1})$. Indeed, suppose that there are indices $i < i'$ such that

$$\ell = \delta(\epsilon_i, \epsilon_{i+1}) = \delta(\epsilon_{i'}, \epsilon_{i'+1}) = \max_{1\leq j\leq p-1} \delta(\epsilon_j, \epsilon_{j+1}).$$

Then, by property (b) we also have that $\ell = \delta(\epsilon_i, \epsilon_{i'}) = \delta(\epsilon_{i'}, \epsilon_{i'+1})$. This contradicts property (a) since $\epsilon_i < \epsilon_{i'} < \epsilon_{i'+1}$.

We are now ready to color the complete 3-uniform hypergraph H on the set T. If $\epsilon_1 < \epsilon_2 < \epsilon_3$, let $\delta_1 = \delta(\epsilon_1, \epsilon_2)$ and $\delta_2 = \delta(\epsilon_2, \epsilon_3)$. Note that, by property (a) above, δ_1 and δ_2 are not equal. Color the edge $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ as follows:

C_1, if $(\delta_1, \delta_2) \in e(G)$ and $\delta_1 < \delta_2$;
C_2, if $(\delta_1, \delta_2) \in e(G)$ and $\delta_1 > \delta_2$;
C_3, if $(\delta_1, \delta_2) \notin e(G)$, i.e., it is an edge in $\bar{G}$.

Suppose that C_1 contains a clique $\{\epsilon_1, \dots, \epsilon_n\}_<$ of size n. For $1 \le i \le n-1$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Note that the δ_i form a monotonically increasing sequence, that is, $\delta_1 < \delta_2 < \cdots < \delta_{n-1}$. Also, note that since, for any $1 \le i < j \le n-1$, $\{\epsilon_i, \epsilon_{i+1}, \epsilon_{j+1}\} \in C_1$, we have, by property (b) above, that $\delta(\epsilon_{i+1}, \epsilon_{j+1}) = \delta_j$, and thus $\{\delta_i, \delta_j\} \in e(G)$. Therefore, the set $\{\delta_1, \dots, \delta_{n-1}\}$ must form a clique of size $n-1$ in G. But we have chosen G so as not to contain such a clique, so we have a contradiction. A similar argument shows that C_2 also cannot contain a clique of size n.

For C_3, assume again that we have a monochromatic clique $\{\epsilon_1, \dots, \epsilon_n\}_<$ of size n, and, for $1 \le i \le n-1$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Not only can we no longer guarantee that these δ_i form a monotonic sequence, but we can no longer guarantee that they are distinct. Suppose that there are d distinct values of δ, given by $\{\Delta_1, \dots, \Delta_d\}$, where $\Delta_1 > \cdots > \Delta_d$. We will consider the subgraph of $\bar{G}$ induced by these vertices. Note that, by definition of the coloring C_3, the vertices Δ_i and Δ_j are adjacent in $\bar{G}$ if there exists $\epsilon_r < \epsilon_s < \epsilon_t$ with $\Delta_i = \delta(\epsilon_r, \epsilon_s)$ and $\Delta_j = \delta(\epsilon_s, \epsilon_t)$. We show that this set necessarily has a complete subgraph on $\log_2 n$ vertices, contradicting our assumptions on $\bar{G}$.

Since Δ_1 is the largest of the Δ_j, there is a unique index i_1 such that $\Delta_1 = \delta_{i_1}$. Note that Δ_1 is adjacent in $\bar{G}$ to all $\Delta_j, j > 1$. Indeed, every such Δ_j is of the form $\delta(\epsilon_{i'}, \epsilon_{i'+1})$ for some index $i' \ne i_1$. Suppose that $i_1 < i'$ (the other case is similar). Then $\epsilon_{i_1} < \epsilon_{i'} < \epsilon_{i'+1}$. By property (b) and the maximality of Δ_1, we have that $\Delta_1 = \delta(\epsilon_{i_1}, \epsilon_{i_1+1}) = \delta(\epsilon_{i_1}, \epsilon_{i'})$, and therefore it is connected to Δ_j in $\bar{G}$. Now, there are $(n-2)/2 = n/2 - 1$ values of j greater than i_1 or less than i_1. Let V_1 be the larger of these two intervals.

Suppose, inductively, that one has been given an interval V_{j-1} in $[n-1]$. Look at the set $\{\delta_a | a \in V_{j-1}\}$. One of these δ, say δ_{i_j}, will be the largest and, as we explain above, will be connected to every other δ_a with $a \in V_{j-1}$. There are at least $(|V_{j-1}| - 1)/2$ indices in $\{a \in V_{j-1} | a < i_j\}$ or $\{a \in V_{j-1} | a > i_j\}$. Let V_j be the larger of these two intervals, so in particular $|V_j| \ge (|V_{j-1}| - 1)/2$. By induction, it is easy to show that $|V_j| \ge \frac{n}{2^j} - 1$. Therefore, for $j \le \log_2 n - 1$, $|V_j| \ge 1$ and, hence, the set $\delta_{i_1}, \dots, \delta_{i_{\log_2 n}}$ forms a clique in $\bar{G}$, as required. This contradicts the fact that G has no independent set of this size and completes the proof. $\square$

As discussed in the beginning of Section 3, the probabilistic method demonstrates that, for $s \le n$, $r(s,n) \ge \left(\frac{n+s}{s}\right)^{c's}$. Substituting this bound with $s = \log_2 n$ into Theorem 4.1 implies the desired result $r_3(n,n,n) \ge 2^{n^{c \log n}}$.

5. A hypergraph problem of Erdős and Hajnal

In this section we prove Theorem 1.4, which determines the function $h_1^{(3)}(s)$ for infinitely many values of s. We also find a small interval containing $h_1^{(3)}(s)$ for all values of s. More precisely, for each s, we find a small interval of values for which the growth rate of $f_3(N,s,t)$ changes from a power of N to a power of $\log N$. Recall that $f_3(N,s,t)$ is the largest integer n for which every red-blue coloring of $\binom{[N]}{3}$ has a red n-element set or a set of size s with at least t blue triples. Also recall that $h_1^{(3)}(s)$ is the least t for which $f_3(N,s,t)$ stops growing like a power of N and starts growing like a power of $\log N$, i.e., $f_3(N,s,h_1^{(3)}(s)-1) > N^{c_1}$ but $f_3(N,s,h_1^{(3)}(s)) < (\log N)^{c_2}$. In the course of this section, it will be necessary to define a number of auxiliary functions. In particular, we sandwich $h_1^{(3)}(s)$ between two functions of s which are more easily computable and always close. Moreover, sometimes these functions are equal, which then allows us to determine $h_1^{(3)}(s)$. We begin by defining a function which gives a natural lower bound for $h_1^{(3)}(s)$.

Consider the minimal family $\mathcal{F}$ of 3-uniform hypergraphs defined as follows. The empty hypergraphs on 1 and 2 vertices and an edge are elements of $\mathcal{F}$. If $H,G \in \mathcal{F}$ and v is a vertex of H, then the following 3-uniform hypergraph $H(G,v)$ is in $\mathcal{F}$ as well. The vertex set of $H(G,v)$ is $(V(H)\setminus\{v\})\cup V(G)$, and its edges consist of the edges of G, the edges of H not containing v, and all triples $\{a,b,c\}$ with $a,b \in H$ and $c \in G$ for which $\{a,b,v\}$ is an edge of H. Let $g_1^{(3)}(s)$ be the maximum number of edges in an element of $\mathcal{F}$ with s vertices. Erdős and Hajnal showed that for every hypergraph $H \in \mathcal{F}$ on s vertices, every red-blue coloring of the triples of a set of size N has a red copy of H or a blue set of size N^{ϵ_s}. Therefore, by the definition of $h_1^{(3)}$, we see that $h_1^{(3)}(s) > g_1^{(3)}(s)$. Erdős and Hajnal further conjectured that this bound is tight.

Conjecture 5.1. *For all positive integers s, $h_1^{(3)}(s) = g_1^{(3)}(s)+1$.*

It was shown by Erdős and Hajnal that the function $g_1^{(3)}(s)$ may be defined recursively as follows. Put $g_1^{(3)}(1) = g_1^{(3)}(2) = 0$. Assume that $g_1^{(3)}(m)$ has already been defined for all $m < s$. Then

$$g_1^{(3)}(s) = \max_{a+b+c=s} g_1^{(3)}(a) + g_1^{(3)}(b) + g_1^{(3)}(c) + abc.$$

It is not difficult to see that the maximum is obtained when a, b, and c are as nearly equal as possible. We shall need this observation in the proof of Proposition 5.3.

To get an upper bound for $h_1^{(3)}(s)$, we must look at another function, defined as follows. Consider an edge-coloring χ of the complete graph on $[N]$ with colors I, II, III picked uniformly at random. From this coloring, we get a red-blue coloring C of the triples from $[N]$ as follows: if $a<b<c$ has (a,b) color I, (b,c) color II, and (a,c) color III, then color $\{a,b,c\}$ red; otherwise color the triple blue. With high probability, the largest blue set in the coloring C has size $O(\log N)$. Over all edge-colorings of the complete graph on $[s]$ with colors I, II, III, let $F_1(s)$ denote the maximum number of triples (a,b,c) with $1 \le a < b < c \le s$ such that (a,b) is color I, (b,c) is color II, and (a,c) is color III. In the coloring C, every set of size s has at most $F_1(s)$ red triples. Therefore, by definition, $h_1^{(3)} \le F_1(s)+1$. Since also $h_1^{(3)}(s) > g_1^{(3)}(s)$, this implies that $F_1(s) \ge g_1^{(3)}(s)$.

Erdős and Hajnal conjectured that these two functions are actually equal, which would imply $h_1^{(3)}(s) = F_1(s) + 1 = g_1^{(3)}(s) + 1$ and hence Conjecture 5.1.

Conjecture 5.2. *For all positive integers s, $F_1(s) = g_1^{(3)}(s)$.*

Erdős and Hajnal verified Conjectures 5.1 and 5.2 for $s \leq 9$.

To attack these conjectures, we use a new function which was not considered in [13]. Let $T(s)$ be the maximum number of directed triangles in all tournaments on s vertices. It is an exercise (see, e.g., [25]) to check that every tournament with n vertices of outdegrees $d_1, \ldots, d_n$ has exactly $\binom{n}{3} - \sum_{i=1}^{n} \binom{d_i}{2}$ cyclic triangles. Maximizing appropriately, this yields the following formula for $T(s)$:

$$T(s) = \begin{cases} \frac{(s+1)s(s-1)}{24} & \text{if } s \text{ is odd,} \\ \frac{(s+2)s(s-2)}{24} & \text{if } s \text{ is even.} \end{cases} \tag{7}$$

It appears that $T(s)$ and $F_1(s)$ are closely related. Indeed, given an edge-coloring of the complete graph on $[s]$ with colors I, II, III, construct the following tournament on $[s]$. If (a, b) with $a < b$ is color I or II, then direct the edge from a to b and otherwise direct the edge from b to a. Note that any triple (a, b, c) with $a < b < c$ and (a, b) color I, (b, c) color II, and (a, c) color III makes a cyclic triangle in our tournament. We therefore have $F_1(s) \leq T(s)$. Let us summarize the inequalities we have seen so far:

$$g_1^{(3)}(s) \leq h_1^{(3)}(s) - 1 \leq F_1(s) \leq T(s). \tag{8}$$

Let $d(s) = T(s) - g_1^{(3)}(s)$. We have $d(s) = 0$ if and only if all the inequalities in (8) are equalities. We call such a number s *nice*. Note that Conjectures 5.1 and 5.2 necessarily hold when s is nice. Using this fact, we now prove the first part of Theorem 1.4, showing that whenever s is a power of 3, Conjectures 5.1 and 5.2 hold.

Proposition 5.3. *If s is a power of 3, then*

$$g_1^{(3)}(s) = h_1^{(3)}(s) - 1 = F_1(s) = T(s) = \frac{1}{4}\binom{s+1}{3}.$$

Proof. We easily see that $s = 1$ is nice. By induction, the proposition follows from checking that if s is odd and nice, then so is $3s$. Since, by definition, $g_1^{(3)}(3s) = s^3 + 3g_1^{(3)}(s)$, we indeed have

$$d(3s) = T(3s) - g_1^{(3)}(3s) = \frac{1}{4}\binom{3s+1}{3} - 3g_1^{(3)}(s) - s^3 = \frac{3}{4}\binom{s+1}{3} - 3g_1^{(3)}(s) = 3d(s).$$

□

The computation in the proof of the proposition above shows that if $s = 6x + 3$ with x a nonnegative integer, then $d(s) = 3d(2x + 1)$. One can check the other cases of $s \pmod 6$ rather easily.

Lemma 5.4. *If x is a positive integer, then*

$$\begin{aligned}
d(6x-2) &= 2d(2x-1)+d(2x),\\
d(6x-1) &= d(2x-1)+2d(2x)+x,\\
d(6x) &= 3d(2x),\\
d(6x+1) &= 2d(2x)+d(2x+1)+x,\\
d(6x+2) &= d(2x)+2d(2x+1),\\
d(6x+3) &= 3d(2x+1).
\end{aligned}$$

Note that from this lemma, we can easily determine which values of s are nice. In particular, the nice positive integers up to 100 are

$$1, 2, 3, 4, 6, 8, 9, 10, 12, 18, 24, 26, 27, 28, 30, 36, 54, 72, 78, 80, 81, 82, 84, 90.$$

Lemma 5.4 now allows us to prove the second part of Theorem 1.4.

Proposition 5.5. *For all positive integers s,*

$$h_1^{(3)}(s) = \frac{s^3}{24} + O(s \log s).$$

Proof. Let $D(s) = d(s) - cs \log s$ with c a sufficiently large constant. Using induction on s and the recursive formula for $d(s)$ depending on $s \pmod 6$ in Lemma 5.4, we get that $D(s)$ is negative for $s > 1$. Indeed, assuming $s = 6x+1$ with x a positive integer (the other five cases are handled similarly), we get

$$\begin{aligned}
D(s) &= d(6x+1) - c(6x+1)\log(6x+1)\\
&= 2d(2x) + d(2x+1) + x - c(6x+1)\log(6x+1)\\
&< 2d(2x) + d(2x+1) - c(6x+1)\log(2x+1)\\
&< 2D(2x) + D(2x+1) < 0.
\end{aligned}$$

Since $d(s) = T(s) - g_1^{(3)}(s)$, this implies that $T(s)$ and $g_1^{(3)}(s)$ are always within $cs \log s$ of one another. Using the formula for $T(s)$, we therefore have that $h_1^{(3)}(s)$ always lies in an interval of length $O(s \log s)$ around $s^3/24$. $\square$

In their attempt to determine $h_1^{(3)}(s)$, Erdős and Hajnal consider yet another function. Consider a coloring of the edges of the complete graph on s vertices labeled $1, \ldots, s$ by two colors I and II which maximizes the number of triangles (a, b, c) with $1 \le a < b < c \le s$ such that (a, b) and (b, c) have color I, and (a, c) has color II. Denote this maximum by $F_2(s)$. Trivially, $F_2(s) \ge F_1(s)$. Erdős and Hajnal thought that "perhaps $F_2(s) = F_1(s)$". As we will show, this is indeed the case for some values of s, but it is not true in general. For example, it is false already for $s = 5$ and $s = 7$. Moreover, we precisely determine $F_2(s)$ for all values of s.

Proposition 5.6. *For all positive integers s, $F_2(s) = T(s)$.*

Proof. We first show that $T(s) \ge F_2(s)$. Indeed, from a two-coloring with colors I and II of the edges of the complete graph with vertices $1, \ldots, s$, we get a tournament on s vertices as follows: if (a, b) with $a < b$ is color I, then orient the edge from a to b; otherwise (a, b) is color II and orient the edge from b to a. Any triangle (a, b, c) with $a < b < c$ with (a, b) and (b, c) color I and (a, c) color II is a cyclic triangle in the tournament, and the inequality $T(s) \ge F_2(s)$ follows.

We next show that actually $T(s) = F_2(s)$. Consider the two-coloring of the edges of the complete graph on s vertices, where (a, b) is color II if and only if $b - a$ is even. A simple calculation shows that the number of triangles (a, b, c) with $a < b < c$ with (a, b) and (b, c) color I and (a, c) color II in this coloring is precisely the formula (7) for $T(s)$. Assume s is even (the case s is odd can be treated similarly). For fixed a and c with $c - a$ even, the number of such triangles containing edge (a, c) is $\lfloor \frac{c-a}{2} \rfloor$. Letting $c = a + 2i$, we thus have

$$F_2(s) \geq \sum_{a=1}^{s} \sum_{1 \leq i \leq \lfloor \frac{s-a}{2} \rfloor} i = \sum_{a=1}^{s} \binom{\lfloor \frac{s-a}{2} \rfloor + 1}{2} = \sum_{j=1}^{s/2} 2\binom{j}{2} = 2\binom{\frac{s}{2}+1}{3} = T(s),$$

and hence $F_2(s) = T(s)$. □

6. Odds and ends

6.1. **Polynomial versus exponential Ramsey numbers.** As we discussed in Section 2, the Ramsey number of $K_4^{(3)} \setminus e$ versus $K_n^{(3)}$ is at least exponential in n. The hypergraph $K_4^{(3)} \setminus e$ is a special case of the following construction. Given an arbitrary graph G, let H_G be the 3-uniform hypergraph whose vertices are the vertices of G plus an auxiliary vertex v. The edges of H_G are all triples obtained by taking the union of an edge of G with vertex v. For example, by taking G to be the triangle, we obtain $K_4^{(3)} \setminus e$. It appears that the Ramsey numbers $r\big(H_G, K_n^{(3)}\big)$ have a very different behavior depending on the bipartiteness of G.

Proposition 6.1. *If G is a bipartite graph, then there is a constant $c = c(G)$ such that $r(H_G, K_n^{(3)}) \leq n^c$. On the other hand, for nonbipartite G, $r(H_G, K_n^{(3)}) \geq 2^{c'n}$ for an absolute constant $c' > 0$.*

Proof. Let G be a bipartite graph with t vertices. The classical result of Kövari, Sós, and Turán [23] states that a graph with N vertices and at least $N^{2-1/t}$ edges contains the complete bipartite graph $K_{t,t}$ with two parts of size t. Therefore, any 3-uniform hypergraph of order N which contains a vertex of degree at least $N^{2-1/t}$ contains also a copy of $H_{K_{t,t}}$ and hence also H_G. Consider a red-blue edge-coloring C of the complete 3-uniform hypergraph on $N = (3n)^{2t}$ vertices, and let m denote the number of red edges in C. If $m \geq N^{3-1/t}$, then there is a vertex whose red degree is at least $3m/N \geq N^{2-1/t}$, which by the above remark gives a red copy of H_G. Otherwise, $m < N^{3-1/t}$ and we can use a well-known Turán-type bound to find a large blue set in coloring C. Indeed, it is well known (see, e.g., Chapter 3, Exercise 3 in [2]) that a 3-uniform hypergraph with N vertices and $m \geq N$ edges has an independent set (i.e., set with no edges) of size at least $\frac{N^{3/2}}{3m^{1/2}}$. Thus, the hypergraph of red edges has an independent set of size at least

$$\frac{N^{3/2}}{3m^{1/2}} > \frac{N^{3/2}}{3(N^{3-1/t})^{1/2}} = \frac{1}{3}N^{1/(2t)} = n,$$

which clearly is a blue set.

To prove the second part of this proposition, we use a construction of Erdős and Hajnal mentioned in the introduction. Suppose that G is not bipartite, so it contains an odd cycle with vertices $\{v_1, \ldots, v_{2i+1}\}$ and edges $\{v_j, v_{j+1}\}$ for $1 \leq j \leq 2i + 1$, where $v_{2i+2} := v_1$. We start with a tournament T on $[N]$ with $N = 2^{c'n}$ which contains no transitive tournament of order n. As we already mentioned, for sufficiently small c', a random tournament has this property with high probability. Color the triples from $[N]$ red if they form a cyclic triangle in T and blue otherwise. Clearly, this coloring does not contain a blue set of size n. Suppose it contains a red copy of H_G. This implies that T contains $2i+2$ vertices $v, u_1, \dots, u_{2i+1}$ such that all the triples (v, u_j, u_{j+1}) form a cyclic triangle. Then, the edges (v, u_j) and (v, u_{j+1}) have opposite orientation (one edge oriented towards v and the other oriented from v). Coloring the vertices u_j by 0 or 1 depending on the direction of edge (v, u_j) gives a proper 2-coloring of an odd cycle, a contradiction. □

6.2. **Discrepancy in hypergraphs.** Despite the fact that Erdős [12] (see also the book [5]) believed that $r_3(n,n)$ is closer to $2^{2^{cn}}$, together with Hajnal [14] they discovered the following interesting fact about hypergraphs, which may indicate the opposite. They proved that there are $c, \epsilon > 0$ such that every 2-coloring of the triples of an N-set contains a set of size $s > c(\log N)^{1/2}$ which contains at least $(1/2+\epsilon)\binom{s}{3}$ 3-sets in one color. That is, the set of size s deviates from having density 1/2 in each color by at least some fixed positive constant. Erdős further remarks that he would begin to doubt that $r_3(n,n)$ is double-exponential in n if one can prove that in any 2-coloring of the triples of the N-set, it contains some set of size $s = c(\eta)(\log N)^{\epsilon}$ for which at least $(1-\eta)\binom{s}{3}$ triples have the same color. We prove the following result, which demonstrates this if we allow ϵ to decrease with η.

Proposition 6.2. *For $\eta > 0$ and all positive integers r and k, there is a constant $\beta = \beta(r,k,\eta) > 0$ such that every r-coloring of the k-tuples of an N-element set has a subset of size $s > (\log N)^{\beta}$ which contains more than $(1-\eta)\binom{s}{k}$ k-sets in one color.*

These results can be conveniently restated in terms of another function introduced by Erdős in [12]. Denote by $F^{(k)}(N,\alpha)$ the largest integer for which it is possible to split the k-tuples of an N-element set S into two classes so that for every $X \subset S$ with $|X| \geq F^{(k)}(N,\alpha)$, each class contains more than $\alpha\binom{|X|}{k}$ k-tuples of X. Note that $F^{(k)}(N,0)$ is essentially the inverse function of the usual Ramsey function $r_k(n,n)$. It is easy to show that for $0 \leq \alpha < 1/2$,

$$c(\alpha)\log N < F^{(2)}(N,\alpha) < c'(\alpha)\log N.$$

As Erdős points out, for $k \geq 3$ the function $F^{(k)}(N,\alpha)$ is not well understood. If $\alpha = 1/2 - \epsilon$ for sufficiently small $\epsilon > 0$, then the result of Erdős and Hajnal from the previous paragraph (for general k) demonstrates

$$c_k(\epsilon)\,(\log N)^{1/(k-1)} < F^{(k)}(N,\alpha) < c'_k(\epsilon)\,(\log N)^{1/(k-1).}$$

On the other hand, since $F^{(k)}(N,0)$ is the inverse function of $r_k(n,n)$, then the old conjecture of Erdős, Hajnal, and Rado would imply that

$$c_1 \log_{(k-1)} N < F^{(k)}(N,0) < c_2 \log_{(k-1)} N,$$

where we recall that $\log_{(t)} N$ denotes the t times iterated logarithm function. Assuming the conjecture, as α increases from 0 to 1/2, $F^{(k)}(N,\alpha)$ increases from $\log_{(k-1)} N$ to $(\log N)^{(1/(k-1)}$. Erdős [5] asked (and offered a \$500 cash reward) if the change in $F^{(k)}(N,\alpha)$ occurs continuously, or are there jumps? He suspected the only jump occurs at $\alpha = 0$. If α is bounded away from 0, Proposition 6.2

demonstrates that $F^{(k)}(N,\alpha)$ already grows as some power of $\log N$. That is, for each $\alpha > 0$ and k, there are $c, \epsilon > 0$ such that $F^{(k)}(N,\alpha) > c(\log N)^{\epsilon}$.

We will deduce Proposition 6.2 from a result about the r-color Ramsey number of a certain k-uniform hypergraph with n vertices and edge density almost one. The Ramsey number $r(H;r)$ of a k-uniform hypergraph H is the minimum N such that every r-edge-coloring of the k-tuples of an N-element set contains a monochromatic copy of H. The blow-up $K_\ell^{(k)}(n)$ is the k-uniform hypergraph whose vertex set consists of ℓ parts of size n and whose edges are all k-tuples that have their vertices in some k different parts. Note that $K_\ell^{(k)}(n)$ has ℓn vertices and $\binom{\ell}{k}n^k \geq \left(1-\binom{k}{2}/\ell\right)\binom{\ell n}{k}$ edges. In particular, as ℓ grows with k fixed, the edge density of $K_\ell^{(k)}(n)$ goes to 1. Therefore, Theorem 6.2 is a corollary of the following result.

Proposition 6.3. *For all positive integers r,k,ℓ, there is a constant $c = c(r,k,\ell)$ such that*

$$r\big(K_\ell^{(k)}(n);r\big) \leq e^{cn^\ell}.$$

Proof. Consider an r-coloring of $\binom{[N]}{k}$ with $N = e^{cn^{\ell-1}}$ and $c = \left(2r\cdot\binom{t}{\ell}\right)^{\ell-1}$, where t is the r-color Ramsey number $r(K_\ell^{(k)};r)$. The proof uses a simple trick which appears in [11] and later in [22]. By definition, every vertex subset of size t contains a monochromatic set of size ℓ. Since each monochromatic set of size ℓ is contained in $\binom{N-\ell}{t-\ell}$ subsets of size t, the number of monochromatic sets of size ℓ is at least

$$\binom{N}{t}\Big/\binom{N-\ell}{t-\ell} = \binom{t}{\ell}^{-1}\binom{N}{\ell}.$$

By the pigeonhole principle, there is a color $1 \leq i \leq r$ for which there are at least $\frac{1}{r}\binom{t}{\ell}^{-1}\binom{N}{\ell}$ monochromatic sets of size ℓ in color i. Define the ℓ-uniform hypergraph G with vertex set $[N]$ whose edges consist of the monochromatic sets of size ℓ in color i in our r-coloring. We have just shown that hypergraph G with N vertices has at least $\frac{1}{r}\binom{t}{\ell}^{-1}\binom{N}{\ell} \geq \epsilon\frac{N^\ell}{\ell!}$ edges with $\epsilon = \frac{1}{2r}\binom{t}{\ell}^{-1}$. A standard extremal lemma for hypergraphs (see, e.g., [9], [26]) demonstrates that any ℓ-uniform hypergraph with N vertices and at least $\epsilon\frac{N^\ell}{\ell!}$ edges with $(\ln N)^{-1/(\ell-1)} \leq \epsilon \leq \ell^{-3}$ contains a complete ℓ-uniform ℓ-partite hypergraph with parts of size $\lfloor\epsilon(\ln N)^{1/(\ell-1)}\rfloor$. (An l-uniform hypergraph is *l-partite* if there is a partition of the vertex set into l parts such that each edge has exactly one vertex in each part.) In particular, G contains a complete ℓ-uniform ℓ-partite hypergraph with parts of size $\lfloor\epsilon(\ln N)^{1/(\ell-1)}\rfloor = n$, where we use that $\epsilon = c^{-1/(\ell-1)}$. The vertices of this complete ℓ-uniform ℓ-partite hypergraph with n vertices in each part in G are the vertices of a monochromatic $K_\ell^{(k)}(n)$ in color i, completing the proof. $\square$

Finally we want to mention another problem of Erdős related to the growth of Ramsey numbers of complete 3-uniform hypergraphs. Erdős [10] (see also [12] and [5]) asked the following problem.

Question 6.4. Suppose $|S| = N$ and the triples from S are split into two classes. Does there exist a pair of subsets $A, B \subset S$ with $|A| = |B| \geq c(\log N)^{1/2}$ such that all triples from $A \cup B$ that hit both A and B are in the same class?

Erdős showed that the answer is yes under the weaker assumption that only the triples with two vertices in A and one vertex in B must be monochromatic. Although this question is still open, we would like to mention that the answer to it is no if the triples of S are split into four classes instead of two. Indeed, in [7], we found a 3-uniform hypergraph C_n on n vertices which is much sparser than the complete hypergraph $K_n^{(3)}$ and whose four-color Ramsey number satisfies $r(C_n; 4) > 2^{2^{c_1 n}}$. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices and let C_n be the 3-uniform hypergraph on V whose edge set is given by $\{v_i, v_{i+1}, v_j\}$ for all $1 \leq i, j \leq n$. (Note that when $i = n$, we consider $i + 1$ to be equal to 1.) When n is even, the vertices of C_n can be partitioned into two subsets A and B (with $v_i \in A$ if and only if i is even) of size $n/2$ such that all edges of C_n hit both A and B. Thus, a four-coloring of the triples of $[N]$ with $N = 2^{2^{c_1 n}}$ and with no monochromatic copy of C_n also does not contain a pair $A, B \subset [N]$ with $|A| = |B| = \frac{1}{2c_1} \log \log N$ such that all triples that hit both A and B are in the same class.

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