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QUANTIZED SYMPLECTIC ACTIONS AND W-ALGEBRAS

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Contents

1. Introduction

Throughout this paper $\mathbb{K}$ denotes the base field. It is assumed to be algebraically closed and of characteristic 0.

1.1. **W-algebras.** Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{K}$ and let G be the semisimple algebraic group of adjoint type with Lie algebra $\mathfrak{g}$. Fix a nonzero nilpotent element $e \in \mathfrak{g}$.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using an invariant nondegenerate symmetric form on $\mathfrak{g}$ (say, the Killing form). Let χ be the element of $\mathfrak{g}^*$ corresponding to e.

We fix an $\mathfrak{sl}_2$-triple (e, h, f). Also we fix an element $h' \in \mathfrak{g}$ satisfying the following two conditions:

- $[h', e] = 2e, [h', h] = 0$ (and then, automatically, $[h', f] = -2f$).

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- Eigenvalues of $\mathrm{ad}(h')$ on $\mathfrak{z}_{\mathfrak{g}}(e)$ are nonnegative integers.

For instance, one can take h for h'.

Consider the grading $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ associated with h', that is, $\mathfrak{g}(i) := \{\xi \in \mathfrak{g} | [h', \xi] = i\xi\}$. Define the 2-form ω_χ on $\mathfrak{g}(-1)$ by $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta]\rangle$. The kernel of ω_χ lies in $\mathfrak{z}_{\mathfrak{g}}(e)$, hence is zero. Consider an isotropic subspace $\mathfrak{y} \subset \mathfrak{g}(-1)$ and let $\mathfrak{y}^{\angle}$ denote the skew-orthogonal complement of $\mathfrak{y}$. Define two subspaces of $\mathfrak{g}$: $\mathfrak{m}_{\mathfrak{y}} := \bigoplus_{i\leqslant -2} \mathfrak{g}(i) \oplus \mathfrak{y}, \mathfrak{n}_{\mathfrak{y}} := \bigoplus_{i\leqslant -2} \mathfrak{g}(i) \oplus \mathfrak{y}^{\angle}$.

Clearly, $\mathfrak{m}_{\mathfrak{y}}, \mathfrak{n}_{\mathfrak{y}}$ are unipotent subalgebras of $\mathfrak{g}$ and $\mathfrak{m}_{\mathfrak{y}} \subset \mathfrak{n}_{\mathfrak{y}}$. Let $N_{\mathfrak{y}}$ be the connected subgroup of G with Lie algebra $\mathfrak{n}_{\mathfrak{y}}$. It is seen directly that $\chi|_{\mathfrak{m}_{\mathfrak{y}}}$ is $N_{\mathfrak{y}}$-invariant.

To simplify the notation below we write $\mathfrak{m}, \mathfrak{n}, N$ instead of $\mathfrak{m}_{\mathfrak{y}}, \mathfrak{n}_{\mathfrak{y}}, N_{\mathfrak{y}}$. Let us set $\mathfrak{m}'(= \mathfrak{m}'_{\mathfrak{y}}) := \{\xi - \langle\chi, \xi\rangle, \xi \in \mathfrak{m}\}$.

Definition 1.1.1. By the W-algebra (of finite type) associated with e we mean the algebra $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}')^N$, whose multiplication is induced from $U(\mathfrak{g})$.

Let us introduce a certain filtration on $U(\mathfrak{g})$. Firstly, there is the standard (PBW) filtration $\mathrm{F}_i^{st} U(\mathfrak{g})$. Set $U(\mathfrak{g})(i) := \{f \in U(\mathfrak{g}) | [h', f] = if\}, \mathrm{F}_k U(\mathfrak{g}) = \sum_{i+2j\leqslant k} (\mathrm{F}_j^{st} U(\mathfrak{g}) \cap U(\mathfrak{g})(i))$. Let us equip $U(\mathfrak{g}, e)$ with the induced filtration. It is referred to as the *Kazhdan* filtration. It is known (Theorem 1.1.2 below) that $U(\mathfrak{g}, e)$ does not depend on the choice of $\mathfrak{y}$ up to an isomorphism of filtered algebras. Moreover, Brundan and Goodwin proved in [BrG] that $U(\mathfrak{g}, e)$ does not depend on the choice of h' up to an isomorphism of algebras. The latter result also follows from Proposition 3.1.2 and Corollary 3.3.3 of the present paper.

The definition of $U(\mathfrak{g}, e)$ is due to Premet, [Pr1], (for Lagrangian $\mathfrak{y}$) and Gan-Ginzburg, [GG], (for general $\mathfrak{y}$). One of the main results of [Pr1], [GG] is the description of the associated graded algebra of $U(\mathfrak{g}, e)$ with respect to the Kazhdan filtration.

To state this description, let us recall the definition of the *Slodowy slice*, [S]. By definition, this is the affine subspace $S := \chi + (\mathfrak{g}/[\mathfrak{g}, f])^* \subset \mathfrak{g}^*$. Using the identification $\mathfrak{g} \cong \mathfrak{g}^*$, we see that $\mathfrak{z}_{\mathfrak{g}}(e)$ is the dual space of $(\mathfrak{g}/[\mathfrak{g}, f])^*$. So we can identify $\mathbb{K}[S]$ with the symmetric algebra $S(\mathfrak{z}_{\mathfrak{g}}(e))$. There is a unique grading on $\mathbb{K}[S]$ such that an element $\xi \in \mathfrak{z}_{\mathfrak{g}}(e) \cap \mathfrak{g}(i)$ has degree $i + 2$.

Theorem 1.1.2 ([GG], Theorem 4.1)**.** *The filtered algebra $U(\mathfrak{g}, e)$ does not depend on the choice of $\mathfrak{y}$ (up to a distinguished isomorphism) and $\mathrm{gr}\, U(\mathfrak{g}, e) \cong \mathbb{K}[S]$ as graded algebras.*

The second assertion was proved earlier by Premet for a Lagrangian $\mathfrak{y}$. Premet's approach uses a reduction to finite characteristic and is quite involved, while Gan and Ginzburg used much easier techniques. Actually Gan and Ginzburg considered the case $h' = h$, but the proofs can be generalized to the general case directly.

Another crucial result concerning W-algebras is the category equivalence theorem of Skryabin proved in the appendix to [Pr1] and then in [GG]. To state it we need the notion of a Whittaker module. Till the end of the subsection we assume that $\mathfrak{y}$ is Lagrangian.

Definition 1.1.3. We say that a $U(\mathfrak{g})$-module M is a *Whittaker module* (for e) if $\mathfrak{m}'$ acts on M by locally nilpotent endomorphisms.

Let M be a Whittaker $U(\mathfrak{g})$-module. Then $M^{\mathfrak{m}'}$ has the natural structure of a $U(\mathfrak{g}, e)$-module. Conversely, set $Q_{\mathfrak{y}} := U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}'$. This space has a natural structure of a $(U(\mathfrak{g}), U(\mathfrak{g}, e))$-bimodule. So to any $U(\mathfrak{g}, e)$-module N we may assign the $U(\mathfrak{g})$-module $\mathcal{S}(N) := Q_{\mathfrak{y}} \otimes_{U(\mathfrak{g},e)} N$.

Theorem 1.1.4. *$M \mapsto M^{\mathfrak{m}'}, N \mapsto \mathcal{S}(N)$ define quasi-inverse equivalences between the category of all Whittaker $U(\mathfrak{g})$-modules and the category of $U(\mathfrak{g}, e)$-modules.*

The study of Whittaker modules traces back to the paper of Kostant, [Ko], where the case of a principal nilpotent element was considered. In this case, $U(\mathfrak{g}, e)$ is canonically isomorphic to the center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$. Kostant's results were generalized to the case of even nilpotent elements in the thesis of Lynch, [Ly]. Some further considerations on Whittaker modules can be found in [Mo].

There are some other special cases, where the algebras $U(\mathfrak{g}, e)$ were studied in detail. In the paper [Pr2] the case when e is a minimal nilpotent element was considered. Brundan and Kleshchev, [BrKl1], [BrKl2], studied the algebras $U(\mathfrak{g}, e)$ for $\mathfrak{g} = \mathfrak{sl}_n$. Their results include the classification of irreducible finite-dimensional $U(\mathfrak{g}, e)$-modules. Their approach is based on a relation between $U(\mathfrak{g}, e)$ and a certain Hopf algebra called a *shifted Yangian*. This relation is a special feature of the $\mathfrak{sl}_n$-case.

Our approach to W-algebras is completely different. It is based on the observation that $U(\mathfrak{g}, e)$ is the invariant algebra for a certain action of G on a quantized affine symplectic variety. Our main results are presented in the next subsection.

Finally, let us mention a relation between W-algebras of finite type and affine W-algebras. The latter are certain vertex algebras arising in QFT. According to Zhu, to any vertex algebra one can assign a certain associative algebra (*Zhu algebra*) in a canonical way; see Section 2 and especially Definition 2.8 in [DSK]. The representation theory of a vertex algebra is closely related to that of its Zhu algebra; see, for example, [DSK], Proposition 2.30 for a precise statement. It turns out that the Zhu algebra of an affine W-algebra is a W-algebra of finite type, [DSK], Theorem 5.10. So the study of representations of $U(\mathfrak{g}, e)$ is important for understanding those of the corresponding affine W-algebra.

1.2. **Main results.** We use the notation of the previous subsection and assume that $\mathfrak{y}$ is a Lagrangian subspace of $\mathfrak{g}(-1)$. Set $\mathcal{U} := U(\mathfrak{g}), V := [\mathfrak{g}, f]$. We have the symplectic form $\langle \chi, [\cdot,\cdot] \rangle$ on V. So we can form the Weyl algebra $\mathbf{A}_V$ of V.

In Subsection 3.1 we introduce a certain associative filtered algebra $\mathcal{W}$ such that $\operatorname{gr} \mathcal{W} \cong \mathbb{K}[S]$. It will follow from Corollary 3.3.3 that $\mathcal{W} \cong U(\mathfrak{g}, e)$ as filtered algebras. We write $\mathbf{A}_V(\mathcal{W})$ instead of $\mathbf{A}_V \otimes \mathcal{W}$.

Our first result is the decomposition theorem. Roughly speaking, it asserts that after suitable completions the algebras $\mathcal{U}$ and $\mathbf{A}_V(\mathcal{W})$ become isomorphic. To give a precise statement we need to specify the notion of a completion.

Let $\mathcal{A}$ be an associative algebra with unit and $\mathfrak{f}$ be a Lie subalgebra of $\mathcal{A}$. Suppose that $\operatorname{ad}(\xi)$ is a locally nilpotent endomorphism of $\mathcal{A}$ for any $\xi \in \mathfrak{f}$ and that any element of $\mathcal{A}$ lies in a finite-dimensional $\operatorname{ad}(\mathfrak{f})$-module. Set $\mathcal{A}^{\wedge}_{\mathfrak{f}} := \varprojlim_{k\to\infty} \mathcal{A}/\mathcal{A}\mathfrak{f}^k$. There is a natural topological algebra structure on $\mathcal{A}^{\wedge}_{\mathfrak{f}}$ (where the kernels of the natural epimorphisms $\mathcal{A}^{\wedge}_{\mathfrak{f}} \to \mathcal{A}/\mathcal{A}\mathfrak{f}^k$ form a fundamental set of neighborhoods of 0); compare with [Gi1], Section 5. Clearly, $\mathcal{A}^{\wedge}_{\mathfrak{f}}$ is complete w.r.t. this topology. The natural map $\mathcal{A} \to \mathcal{A}^{\wedge}_{\mathfrak{f}}$ is an algebra homomorphism.

Theorem 1.2.1. *Let $\underline{\mathfrak{m}}$ denote the subspace of V equal to $\mathfrak{m}$ so that we can consider $\underline{\mathfrak{m}}$ as a commutative Lie subalgebra in $\mathbf{A}_V$. Then there is an isomorphism $\Phi : \mathcal{U}^{\wedge}_{\mathfrak{m}'} \to$*

$\mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}$ *of topological algebras such that* $\Phi(\mathcal{J}_1) = \mathcal{J}_2$*, where* $\mathcal{J}_1, \mathcal{J}_2$ *denote the kernels of the natural epimorphisms* $\mathcal{U}^\wedge_{\mathfrak{m}'} \twoheadrightarrow \mathcal{U}/\mathfrak{m}'\mathcal{U}$, $\mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}} \twoheadrightarrow \mathbf{A}_V(\mathcal{W})/\underline{\mathfrak{m}}\mathbf{A}_V(\mathcal{W})$.

We note that both natural morphisms $\mathcal{A} \to \mathcal{A}^\wedge_{\mathfrak{f}}$ (with $(\mathcal{A}, \mathfrak{f}) = (\mathcal{U}, \mathfrak{m}'), (\mathbf{A}_V(\mathcal{W}), \underline{\mathfrak{m}})$) are injective, since in both cases $\mathcal{A}$ is a free $U(\mathfrak{f})$-module.

Using Theorem 1.2.1, one can prove Theorem 1.1.4; see Proposition 3.3.4.

Our second principal result is a comparison between the sets $\mathrm{Pr}(\mathcal{U}), \mathrm{Pr}(\mathcal{W})$ of all prime ideals of the algebras $\mathcal{U}, \mathcal{W}$ whose intersection with the centers of $\mathcal{U}, \mathcal{W}$ are of codimension 1. We recall that an ideal $\mathcal{I}$ in an associative algebra $\mathcal{A}$ is said to be *prime* if $a\mathcal{A}b \not\subset \mathcal{I}$ whenever $a, b \notin \mathcal{I}$. It is known, see, for example, [J1], 7.3, that any prime ideal $\mathcal{I} \subset \mathcal{U}$ with $\operatorname{codim}_{\mathcal{Z}(\mathfrak{g})} \mathcal{Z}(\mathfrak{g}) \cap \mathcal{I} = 1$ is primitive, i.e., is the annihilator of an irreducible module.

Further, let us recall the notion of the associated variety of an ideal. Let $\mathcal{A}$ be an associative algebra with unit equipped with an increasing filtration $\mathrm{F}_i\,\mathcal{A}$ such that $\mathrm{F}_0\,\mathcal{A} = \mathbb{K}, \mathrm{F}_{-1}\,\mathcal{A} = \{0\}, \bigcup_i \mathrm{F}_i\,\mathcal{A} = \mathcal{A}$. We suppose that $[\mathrm{F}_i\,\mathcal{A}, \mathrm{F}_j\,\mathcal{A}] \subset \mathrm{F}_{i+j-1}\,\mathcal{A}$ and the associated graded algebra $\mathrm{gr}(\mathcal{A}) := \bigoplus_{i\in\mathbb{Z}} \mathrm{F}_i\,\mathcal{A}/\,\mathrm{F}_{i-1}\,\mathcal{A}$ is finitely generated. If $\mathcal{I}$ is an ideal in $\mathcal{A}$, then $\mathrm{gr}(\mathcal{I}) := \bigoplus_{i\in\mathbb{Z}}(\mathrm{F}_i\,\mathcal{A} \cap \mathcal{I})/(\mathrm{F}_{i-1}\,\mathcal{A} \cap \mathcal{I})$ is an ideal of $\mathrm{gr}(\mathcal{A})$. By the associated variety $\mathrm{V}(\mathcal{I})$ of $\mathcal{I}$ we mean the set of zeroes of $\mathrm{gr}(\mathcal{I})$ in $\mathrm{Spec}(\mathrm{gr}(\mathcal{A}))$.

The algebras $\mathcal{U}, \mathcal{W}$ are both equipped with filtrations satisfying the assumptions of the previous paragraph (we consider the standard filtration on $\mathcal{U}$). Recall that $\mathrm{V}(\mathcal{J})$ is the closure of a single nilpotent orbit for any primitive ideal $\mathcal{J}$ of $\mathcal{U}$; see [J2], 9.3, for references. To any two-sided ideal $\mathcal{J} \subset \mathcal{U}$ and an irreducible component Y of $\mathrm{V}(\mathcal{J})$ we assign its *multiplicity* $\mathrm{mult}_Y(\mathcal{J}) = \dim_{\mathrm{Quot}(S(\mathfrak{g})/\mathfrak{p})}(S(\mathfrak{g})/\,\mathrm{gr}(\mathcal{J}))_{\mathfrak{p}}$, where $\mathfrak{p}$ is the ideal of Y. For primitive $\mathcal{J}$ we write $\mathrm{mult}\,\mathcal{J}$ instead of $\mathrm{mult}_Y\,\mathcal{J}$.

The center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}$ is contained in $\mathcal{U}^{\mathfrak{m}'}$, and there is the natural homomorphism $\mathcal{U}^{\mathfrak{m}'} \to U(\mathfrak{g}, e)$. So we have the homomorphism $\iota : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{g}, e)$. It is known, see [Pr1], that $\iota : \mathcal{Z}(\mathfrak{g}) \to U(\mathfrak{g}, e) \cong \mathcal{W}$ is injective. As Premet noted in [Pr2], Section 5, footnote 2, the image of ι coincides with the center of $\mathcal{W}$.

For an associative algebra $\mathcal{A}$, by $\mathfrak{Id}(\mathcal{A})$ we denote the set of its (two-sided) ideals.

Theorem 1.2.2. *There is a map* $\mathcal{I} \mapsto \mathcal{I}^\dagger : \mathfrak{Id}(\mathcal{W}) \to \mathfrak{Id}(\mathcal{U})$ *with the following properties:*

(i) $(\mathcal{I}_1 \cap \mathcal{I}_2)^\dagger = \mathcal{I}_1^\dagger \cap \mathcal{I}_2^\dagger$.

(ii) $\mathrm{Ann}(N)^\dagger = \mathrm{Ann}(\mathcal{S}(N))$ *for any* $\mathcal{W}$*-module* N*, where, recall,* $\mathcal{S}(N) = Q_{\mathfrak{y}} \otimes_{\mathcal{W}} N$.[1]

(iii) $\iota(\mathcal{I}^\dagger \cap \mathcal{Z}(\mathfrak{g})) = \mathcal{I} \cap \iota(\mathcal{Z}(\mathfrak{g}))$ *for any* $\mathcal{I} \in \mathfrak{Id}(\mathcal{W})$.

(iv) $\mathcal{I}^\dagger \in \mathrm{Pr}(\mathcal{U})$ *provided* $\mathcal{I} \in \mathrm{Pr}(\mathcal{W})$.

(v) *For any* $\mathcal{I} \in \mathfrak{Id}(\mathcal{W})$ *we have the inclusion*

$$\mathrm{V}(\mathcal{I}^\dagger) \supset \overline{G\,\mathrm{V}(\mathcal{I})} \tag{1.1}$$

(recall that $\mathrm{Spec}(\mathrm{gr}(\mathcal{W})) = S$ *is the subvariety in* $\mathfrak{g}^*$*). In particular,* $G\chi \subset \mathrm{V}(\mathcal{I}^\dagger)$*. If* (1.1) *turns into equality, we say that an ideal* $\mathcal{I}$ *is* admissible. *The set of all admissible elements of* $\mathrm{Pr}(\mathcal{W})$ *is denoted by* $\mathrm{Pr}^a(\mathcal{W})$.

(vi) *Any ideal* $\mathcal{I}$ *of finite codimension in* $\mathcal{W}$ *is admissible.*

(vii) *Any fiber of the map* $\mathrm{Pr}^a(\mathcal{W}) \to \mathrm{Pr}(\mathcal{U}), \mathcal{I} \mapsto \mathcal{I}^\dagger$*, is finite. More precisely, for any* $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$ *there is* $\mathcal{J}_\dagger \in \mathfrak{Id}(\mathcal{W})$ *satisfying the condition that the*

[1] Note that the equality for the annihilators cannot be taken for the definition of $\mathcal{I}^\dagger$: indeed, a priori, it is not clear that $\mathrm{Ann}(N_1) = \mathrm{Ann}(N_2)$ implies $\mathrm{Ann}(\mathcal{S}(N_1)) = \mathrm{Ann}(\mathcal{S}(N_2))$.

fiber of $\mathcal{J}$ consists of all $\mathcal{I} \in \operatorname{Pr}(\mathcal{W})$ such that $\mathcal{I}$ is minimal over $\mathcal{J}_\dagger$ and $\dim \operatorname{V}(\mathcal{I}) \geqslant \dim \operatorname{V}(\mathcal{J}) - \dim G\chi$.

(viii) *Let $\mathcal{J} \in \operatorname{Pr}(\mathcal{U})$ be such that $\operatorname{V}(\mathcal{J}) = \overline{G\chi}$. Then the fiber of $\mathcal{J}$ consists of all minimal primes of $\mathcal{J}_\dagger$. In particular, there exists an $\mathcal{I} \in \operatorname{Pr}(\mathcal{W})$ of finite codimension with $\mathcal{I}^\dagger = \mathcal{J}$.*

(ix) *Let $\mathcal{J}$ be as in (viii) and $\mathcal{I}_1, \ldots, \mathcal{I}_k$ be all prime ideals of $\mathcal{W}$ such that $\mathcal{I}_j^\dagger = \mathcal{J}, j = 1, \ldots, k$. Then*

$$\operatorname{mult}(\mathcal{J}) \geqslant \sum_{j=1}^{k} \operatorname{codim}_{\mathcal{W}} \mathcal{I}_j \geqslant k \operatorname{Grk}(\mathcal{U}/\mathcal{J})^2.$$

Here $\operatorname{Grk}(\mathcal{U}/\mathcal{J})$ denotes the Goldie rank of $\mathcal{U}/\mathcal{J}$.

The most important part of this theorem is the description of the fiber of $\mathcal{J} \in \operatorname{Pr}(\mathcal{U})$ such that $\operatorname{V}(\mathcal{J}) = \overline{G\chi}$. Premet has also proved that this fiber is nonempty. A special case of this claim (when $\mathcal{J}$ has *rational infinitesimal character*) is the main result of [Pr3]. In the case $\mathfrak{g} = \mathfrak{sl}_n$ it follows from [BrKl2] that under the assumptions on $\mathcal{J}$ made above there is a unique $\mathcal{I} \in \operatorname{Pr}(\mathcal{W})$ such that $\mathcal{I}^\dagger = \mathcal{J}$. However, in general, this is not true. Assertion (ix) suggests that the first case where a counterexample is possible is $\mathfrak{g} = \mathfrak{sp}_4$, where e is subregular. As Premet communicated to me, there is indeed some $\mathcal{J}$ whose fiber consists of two elements.[2]

Let us also remark that assertion (ix) proves a part of Conjecture 6.22 from [McG]: $\operatorname{Grk}(\mathcal{U}/\mathcal{J}) \leqslant \sqrt{\operatorname{mult}(\mathcal{J})}$ for any primitive ideal $\mathcal{J} \subset \mathcal{U}$.

Finally, we study finite-dimensional representations of $\mathcal{W}$.

Theorem 1.2.3. (1) *$\mathcal{W}$ has a finite-dimensional module V. If $\mathfrak{g}$ is a classical Lie algebra, then there is a one-dimensional $\mathcal{W}$-module V.*

(2) *Let V be a $\mathcal{W}$-module. Then there is a $\mathcal{W}$-module structure on $L(\lambda) \otimes V$, where $L(\lambda)$ denotes the finite-dimensional $\mathfrak{g}$-module of highest weight λ, and the representation of $\mathcal{W}$ in $\bigoplus_\lambda L(\lambda) \otimes V$ is faithful.*

Existence of a finite-dimensional $\mathcal{W}$-module was proved earlier by Premet, [Pr3]. Existence of a one-dimensional $\mathcal{W}$-module as well as assertion (ii) was previously known for $\mathfrak{g} = \mathfrak{sl}_n$ (Brundan and Kleshchev, [BrKl2]) and for even e ([Ly]), where these claims are rather straightforward. Further, if e is minimal, then the existence of a one-dimensional module was proved by Premet in [Pr2]. Assertion 2 of the theorem answers Question 3.1 from [Pr2] in an affirmative way.

1.3. **The content of the paper.** The tool we use in the study of W-algebras is deformation quantization. Therefore this paper includes a preliminary section on deformation quantization (Section 2). It consists of two subsections. In the first one we recall standard definitions and results related to star-products and their equivalence. Subsection 2.2 recalls some facts about the Fedosov deformation quantization in the context of affine algebraic varieties.

The main part of this paper is Section 3. There we apply the machinery of Section 2 to the study of W-algebras. In Subsection 3.1 we define the filtered associative algebra $\widetilde{\mathcal{W}}$ equipped with an action of G. The algebra $\operatorname{gr}\widetilde{\mathcal{W}}$ is naturally identified

[2]In a recent preprint [Lo2] the author obtained a much more precise description of fibers of $\mathcal{J} \in \operatorname{Pr}(\mathcal{U}), \operatorname{V}(\mathcal{J}) = \overline{G\chi}$. Namely, the component group $C(e) = Z_G(e)/Z_G(e)^\circ$ acts on the set of all prime ideals of $\mathcal{W}$ of finite codimension, and any fiber is a single orbit of this action.

with the algebra of regular functions on a certain affine symplectic G-variety. We set $\mathcal{W} := \widetilde{\mathcal{W}}^G$. This is a filtered associative algebra with $\operatorname{gr}\mathcal{W} = \mathbb{K}[S]$.

Subsection 3.2 is of a very technical nature. It studies different completions of an associative filtered algebra, whose product is induced from a polynomial star-product on the associated graded algebra. The results obtained in this subsection are used in the proofs of Theorems 1.2.1 and 1.2.2.

In Subsection 3.3 we prove Theorem 1.2.1. At first, we check that certain affine symplectic formal schemes equipped with star-products and with actions of G are isomorphic (Theorem 3.3.1). Using this theorem we check that there is an isomorphism Φ between certain filtered algebras $\mathcal{U}^\heartsuit, \mathbf{A}_V(\mathcal{W})^\heartsuit$ such that $\mathcal{U} \subset \mathcal{U}^\heartsuit \subset \mathcal{U}^\wedge_{\mathfrak{m}'}$ and $\mathbf{A}_V(\mathcal{W}) \subset \mathbf{A}_V(\mathcal{W})^\heartsuit \subset \mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}$. This result yields an isomorphism of filtered algebras $\mathcal{W}$ and $U(\mathfrak{g}, e)$ (where the latter is constructed from a Lagrangian $\mathfrak{y}$). Then we use the results of Subsection 3.2 to show that the topological algebras $\mathcal{U}^\wedge_{\mathfrak{m}'}, \mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}$ are naturally identified with some completions of $\mathcal{U}^\heartsuit, \mathbf{A}_V(\mathcal{W})^\heartsuit$ and Φ induces their isomorphism. Finally, we derive Theorem 1.1.4 from Theorem 1.2.1. The key point is that a Whittaker module M is isomorphic to $\mathbb{K}[\underline{\mathfrak{m}}] \otimes M^{\mathfrak{m}'}$ as an $\mathbf{A}_V(\mathcal{W})$-module.

Subsection 3.4 is devoted to the proof of Theorem 1.2.2. At first we give two equivalent definitions of $\mathcal{I}^\dagger$. Then we prove assertions (i)-(iv) of Theorem 1.2.2. To prove the remaining assertions we construct a certain map $\mathcal{J} \mapsto \mathcal{J}_\dagger : \mathfrak{Id}(\mathcal{U}) \to \mathfrak{Id}(\mathcal{W})$ and study its properties and its relation to $\mathcal{I} \mapsto \mathcal{I}^\dagger$. Then we prove assertions (v)-(ix).

Finally, in Subsection 3.5 we prove Theorem 1.2.3. The proof of the existence of a one-dimensional representation relies on an auxiliary result proved in Subsection 3.6.

1.4. **Notation and conventions.** Let A be a vector space. By the classical part of an element $a \in A[[\hbar]], a = \sum_{i=0}^\infty a_i\hbar^i$ we mean a_0. By the classical part of a subset $\mathcal{I} \subset A[[\hbar]]$ we mean the subset of A consisting of the classical parts of all elements of $\mathcal{I}$. Finally, let B be another vector space, and $\Phi_\hbar : A[[\hbar]] \to B[[\hbar]]$ be a $\mathbb{K}[[\hbar]]$-linear map. The linear map $\Phi : A \to B$ mapping $a \in A$ to the classical part of $\Phi_\hbar(a)$ is called the classical part of $\Phi_\hbar$.

Below we present some notation used in the text.

$\mathbf{A}_V$	the Weyl algebra of a symplectic vector space V.
$\operatorname{Ann}_{\mathcal{A}}(M)$	the annihilator of an $\mathcal{A}$-module M in an algebra $\mathcal{A}$.
$\operatorname{Dim}_{\mathcal{A}}(M)$	the Gelfand-Kirillov dimension of an $\mathcal{A}$-module M.
$\operatorname{Grk}(\mathcal{A})$	the Goldie rank of a prime Noetherian algebra $\mathcal{A}$.
$\operatorname{gr}\mathcal{A}$	the associated graded algebra of a filtered algebra $\mathcal{A}$.
$\mathfrak{Id}(\mathcal{A})$	the set of all (two-sided) ideals of an algebra $\mathcal{A}$.
$\sqrt{\mathcal{J}}$	the radical of an ideal $\mathcal{J}$.
$\mathbb{K}[X]^\wedge_Y$	the completion of the algebra $\mathbb{K}[X]$ of regular functions on X w.r.t. a subvariety $Y \subset X$.
$L(\lambda)$	the irreducible finite-dimensional G-module with highest weight λ.
$\operatorname{Pr}(\mathcal{A})$	the set of all prime ideals of $\mathcal{A}$, whose intersection with the center of $\mathcal{A}$ are of codimension 1.
$R_\hbar(\mathcal{A})$	the Rees algebra of a filtered algebra $\mathcal{A}$.
$U(\mathfrak{g})$	the universal enveloping algebra of a Lie algebra $\mathfrak{g}$.

$\mathrm{V}(\mathcal{J})$	the associated variety of an ideal $\mathcal{J}$ in a filtered algebra.
V_λ	$:= L(\lambda)\otimes \mathrm{Hom}^G(L(\lambda), V)$, the isotypical component of type $L(\lambda)$ in a G-module V.
$V^{\mathfrak{m}}, V^{\mathcal{I}}$	the annihilator of a Lie algebra $\mathfrak{m}$ or a left ideal $\mathcal{I}$ of an associative algebra in a module V.
X^G	the fixed-point set for the action $G : X$.
$\mathcal{Z}(\mathfrak{g})$	the center of $U(\mathfrak{g})$.
$\mathfrak{z}_\mathfrak{g}(x)$	the centralizer of x in $\mathfrak{g}$.
$Z_G(x)$	the centralizer of $x\in\mathfrak{g}$ in G.
$\Omega^1(X)$	the space of regular 1-forms on a smooth variety X.

2. Deformation quantization

2.1. **Star-products and their equivalence.** Recall that the base field $\mathbb{K}$ is assumed to be algebraically closed and of characteristic 0. All algebras, varieties, etc., are defined over $\mathbb{K}$.

Let A be a commutative associative algebra with unit equipped with a Poisson bracket. For example, one can take for A the algebra $\mathbb{K}[X]$ of regular functions on a smooth affine symplectic variety (the symplectic form is assumed to be regular).

Definition 2.1.1. The map $* : A[[\hbar]]\otimes_{\mathbb{K}[[\hbar]]} A[[\hbar]] \to A[[\hbar]]$ is called a *star-product* if it satisfies the following conditions:

(*1) $*$ is $\mathbb{K}[[\hbar]]$-bilinear and continuous in the $\hbar$-adic topology.
(*2) $*$ is associative, equivalently, $(f*g)*h = f*(g*h)$ for all $f,g,h\in A$, and $1\in A$ is a unit for $*$.
(*3) $f*g - fg \in \hbar A[[\hbar]], f*g-g*f-\hbar\{f,g\}\in \hbar^2 A[[\hbar]]$ for all $f,g\in A$.

By (*1), a star-product is uniquely determined by its restriction to A. One may write $f*g=\sum_{i=0}^\infty D_i(f,g)\hbar^i, f,g\in A, D_i : A\otimes A\to A$. Condition (*3) is equivalent to $D_0(f,g)=fg, D_1(f,g)-D_1(g,f)=\{f,g\}$. If all D_i are bidifferential operators, then the star-product $*$ is called *differential*.

Example 2.1.2. Let $X=V$ be a finite-dimensional vector space equipped with a constant nondegenerate bivector P (i.e., nondegenerate element of $\bigwedge^2 V$). The *Moyal-Weyl* star-product on $\mathbb{K}[V][[\hbar]]$ is defined by

$$f*g = \exp(\frac{\hbar}{2}P)f(x)\otimes g(y)|_{x=y}.$$

Here P acts on $\mathbb{K}[V]\otimes\mathbb{K}[V] = S(V^*)\otimes S(V^*)$ by contraction. It is not very difficult to show that this product is associative; see, for instance, [BFFLS].

When we consider $A[[\hbar]]$ as an algebra w.r.t. the star-product, we call it a *quantum algebra*.

Definition 2.1.3. Let $*, *'$ be two star-products on A. These star-products are said to be *equivalent* if there are linear maps $T_i : A\to A, i\in\mathbb{N}$, such that the operator $T = id + \sum_{i=1}^\infty T_i\hbar^i : A[[\hbar]]\to A[[\hbar]]$ satisfies $f*' g = T(T^{-1}(f)*T^{-1}(g))$ for all $f,g\in A$. Such a T is called an equivalence between $*, *'$.

Let an algebraic group G act on A by automorphisms preserving the Poisson bracket. We say that $*$ is G-invariant if all operators D_i are G-equivariant. Now let $\mathbb{K}^\times$ act on $A, (t,a)\mapsto t.a$ by automorphisms such that $t.\{\cdot,\cdot\} = t^{-k}\{\cdot,\cdot\}$. Consider the action $\mathbb{K}^\times : A[[\hbar]]$ given by $t.\sum_{i=0}^\infty a_j\hbar^j = \sum_{j=0}^\infty t^{jk}(t.a_j)\hbar^j$. If $*$ is

$\mathbb{K}^\times$-invariant, then we say that $*$ is *homogeneous*. Clearly, $*$ is homogeneous iff the map $D_l : A \otimes A \to A$ is homogeneous of degree $-kl$.

For instance, the Moyal-Weyl star-product $*$ is invariant with respect to $\mathrm{Sp}(V)$. Now let $\tau : \mathbb{K}^\times \to \mathrm{Sp}(V)$ be a one-parameter subgroup. For $v \in V$ set $t.v = t^{-1}\tau(t)v$. Then $*$ is homogeneous w.r.t. this action of $\mathbb{K}^\times$ (for $k = 2$).

We say that a star-product $*$ on $A[[\hbar]]$ is *polynomial* if $A[\hbar]$ is a subalgebra of $A[[\hbar]]$, in other words, if for any $f, g \in A$ there is $n \in \mathbb{N}$ such that $D_i(f, g) = 0$ for all $i > n$. We are going to obtain some sufficient conditions for a star-product to be polynomial. To this end, we introduce the following definition.

Definition 2.1.4. Let $\mathcal{A}$ be an associative subalgebra with unit and H an algebraic group acting on $\mathcal{A}$ by automorphisms. By the *H-finite part* of $\mathcal{A}$ (denoted $\mathcal{A}_{H-fin}$) we mean the sum of all finite-dimensional H-submodules of $\mathcal{A}$.

It is easy to check that $\mathcal{A}_{H-fin}$ is a subalgebra of $\mathcal{A}$.

Proposition 2.1.5. *Let A be such as above and let G be a reductive group. Suppose that A is finitely generated, there is a rational action of $G \times \mathbb{K}^\times$ on A (i.e., $A = A_{G\times\mathbb{K}^\times-fin}$) by automorphisms, and A^G has no negative graded components (w.r.t. the grading induced by $\mathbb{K}^\times$). Then $A[\hbar] = \mathcal{A}[[\hbar]]_{G\times\mathbb{K}^\times-fin}$. In particular, any homogeneous (with $k > 0$, where $t.\hbar = t^k\hbar$) G-invariant star-product on $A[[\hbar]]$ is polynomial.*

Proof. Clearly, $A[\hbar] \subset \mathcal{A}[[\hbar]]_{G\times\mathbb{K}^\times-fin}$. Let us show the other inclusion. Let λ be a highest weight of G. The isotypical component A_λ is a graded subspace of the G-module A. By [PV], Theorem 3.24, A_λ is a finitely generated A^G-module. It follows that the grading on A_λ is bounded from below. Therefore any finite-dimensional $G \times \mathbb{K}^\times$-submodule of $A[[\hbar]]$ lies in $A[\hbar]$. □

Remark 2.1.6. While working with a graded Poisson algebra $A = \bigoplus_{i\in\mathbb{Z}} A_i$ with homogeneous Poisson bracket of degree $-k, k \in \mathbb{N}$, it is convenient to modify the definitions of star-products and their equivalences. Namely, by a *star-product of degree k* we mean a map $* : A[[\hbar]] \to A[[\hbar]]$ satisfying (*1),(*2) and such that $f*g = \sum_{i=0}^\infty D_i(f, g)\hbar^{ki}$, with $D_0(f, g) = fg, D_1(f, g) - D_1(g, f) = \{f, g\}$ for $f, g \in A$. In the definition of a homogeneous star-product of degree k we set $t.\sum_{i=0}^\infty a_j\hbar^j = \sum_{j=0}^\infty t^j(t.a_j)\hbar^j$. The definition of an equivalence is modified trivially. Clearly, if $*$ is a star-product on $A[[\hbar]]$, $f * g = \sum_{i=0}^\infty D_i(f, g)\hbar^i, f, g \in A$, then the map $*^k : A[[\hbar]] \otimes_{\mathbb{K}[[\hbar]]} A[[\hbar]] \to A[[\hbar]]$ given by $f *^k g = \sum_{i=0}^\infty D_i(f, g)\hbar^{ki}$ is a star-product of degree k and vice versa. So all constructions and results concerning the usual star-products are trivially generalized to star-products of degree k.

2.2. **Fedosov quantization.** The goal of this section is to review the Fedosov approach ([Fe1],[Fe2]) to deformation quantization of smooth affine symplectic varieties. Although Fedosov studied smooth real manifolds, his approach works as well for smooth symplectic varieties and smooth formal schemes. The case of smooth symplectic varieties was considered in [Fa]; the case of formal schemes is analogous.

Let X be a smooth variety with symplectic form ω. According to Fedosov, to construct a star-product one needs to fix a symplectic connection on a variety of interest.

Definition 2.2.1. By a symplectic connection we mean a torsion-free covariant derivative $\nabla : TX \to TX \otimes \Omega^1(X)$ such that $\nabla\omega = 0$.

Proposition 2.2.2. *Let X be an affine symplectic variety, G a reductive group acting on X by symplectomorphisms. Let $\mathbb{K}^\times$ act on X by G-equivariant automorphisms such that $t.\omega = t^k\omega$ for some $k \in \mathbb{Z}$. Then there is a $G \times \mathbb{K}^\times$-invariant symplectic connection ∇ on X.*

Proof. At first, let us prove that there is some symplectic connection on X. Choose an open covering $X = \bigcup_i X_i, i = 1, \ldots, k$, such that TX_i is the trivial bundle and X_i is the principal open subset. The latter means that there are $g_i \in \mathbb{K}[X]$ s.t. $X_i = \{x \in X | g_i(x) \neq 0\}$. There is a torsion-free connection $\widetilde{\nabla}^i$ on X_i. There is a canonical way to assign a symplectic connection ∇^i to $\widetilde{\nabla}^i$; see, for instance, [Fe2], Proposition 2.5.2. Namely, ∇^i is given by the following equality:

$$\omega(\nabla^i_\xi \eta, \zeta) = \omega(\widetilde{\nabla}^i_\xi \eta, \zeta) + \frac{1}{3}\left((\widetilde{\nabla}^i_\zeta \omega)(\xi, \eta) + (\widetilde{\nabla}^i_\eta \omega)(\xi, \zeta)\right), \xi, \eta, \zeta \in H^0(X_i, TX).$$

Replacing g_i with g_i^N for sufficiently large N, we obtain that $g_i\nabla_i$ maps $H^0(X, TX) \otimes H^0(X, TX)$ to $H^0(X, TX)$. There are functions $f_i \in \mathbb{K}[X], i = 1, \ldots, k$, such that $\sum_{i=1}^k f_i g_i = 1$. Then $\sum_{i=1}^k f_i g_i \nabla_i$ is a symplectic connection on X. The $G \times \mathbb{K}^\times$-invariant component ∇_0 of the map $\nabla : H^0(X, TX) \times H^0(X, TX) \to H^0(X, TX)$ is again a symplectic connection. □

Fedosov constructed a differential star-product on $\mathbb{K}[X]$ starting with a symplectic connection ∇; see [Fe2], Section 5.2. We remark that all intermediate objects occurring in Fedosov's construction are obtained from some regular objects (such as ω, ∇ or the curvature tensor of ∇) by a recursive procedure and so are regular too. If a reductive group G acts on X by symplectomorphisms (resp., $\mathbb{K}^\times$ acts on X such that $t.\omega = t^k\omega$) and ∇ is G-invariant (resp., $\mathbb{K}^\times$-invariant), then $*$ is G-invariant, resp., homogeneous. Note also that if a Fedosov star-product is given by $f * g = \sum_{i=0}^\infty D_i(f, g)\hbar^i$, then D_i is a bidifferential operator of order at most i in each variable; see, for example, [X], Theorem 4.2.

Example 2.2.3. Let V be a symplectic vector space with constant symplectic form ω. Then the Moyal-Weyl star-product is obtained by applying the Fedosov construction to the trivial connection ∇; see Example in Subsection 5.2 of [Fe2].

Example 2.2.4. Let G be a reductive group. Choose a symplectic connection ∇ on T^*G invariant w.r.t. $G \times G \times \mathbb{K}^\times$ ($\mathbb{K}^\times$ acts on the latter by fiberwise dilations). So we get the $G \times G$-invariant homogeneous star product $*$ on $\mathbb{K}[T^*G][[\hbar]]$. Restricting this star-product to the space of G-invariants (for, say, the left action), we get the homogeneous G-invariant star-product on $\mathbb{K}[\mathfrak{g}^*][[\hbar]] = S(\mathfrak{g})[[\hbar]]$.

The following proposition was proved in [Fe2], Theorem 5.5.3, for smooth manifolds. Again, its proof can be transferred directly to the algebraic setting.

Proposition 2.2.5. *Let G be a reductive group acting on X by symplectomorphisms. Let ∇, ∇' be G-invariant symplectic connections on X. Let $*, *'$ be the Fedosov star-products constructed from ∇, ∇'. Then $*, *'$ are G-equivariantly equivalent and this equivalence is differential (in the sense that all T_i of Definition* 2.1.3 *are differential operators). If, additionally, $\mathbb{K}^\times$ acts on X by G-equivariant automorphisms such that $t.\omega = t^k\omega$ for some $k \in \mathbb{Z}$ and ∇, ∇' are $\mathbb{K}^\times$-invariant, then there is a $G \times \mathbb{K}^\times$-equivariant differential equivalence.*

Remark 2.2.6. The assertion of the previous proposition holds also when X is the completion of a smooth affine variety w.r.t. a smooth subvariety. Again, Fedosov's proof works in this situation.

3. Application to W-algebras

Throughout this section, G is a semisimple algebraic group of adjoint type and $\mathfrak{g}$ its Lie algebra. Let $e, h', h, f, \chi, \mathfrak{y} \subset \mathfrak{g}(-1), \mathfrak{m} := \mathfrak{m}_{\mathfrak{y}}, \mathfrak{n} := \mathfrak{n}_{\mathfrak{y}}, \mathfrak{m}', N, V$ have the same meaning as in Subsection 1.1. We suppose that $\mathfrak{y}$ is Lagrangian. In this case $\mathfrak{m} = \mathfrak{n}$ and $\underline{\mathfrak{m}}$ is a Lagrangian subspace of V. By S we denote the Slodowy slice $\chi + (\mathfrak{g}/[\mathfrak{g}, f])^*$.

3.1. The algebras $\widetilde{\mathcal{W}}, \mathcal{W}$.

We identify the cotangent bundle T^*G of G with $G \times \mathfrak{g}^*$ by using left-invariant sections of T^*G. The actions of G on T^*G by left (resp., right) translations are written as $g.(g_1, \alpha) = (gg_1, \alpha)$ (resp. $g.(g_1, \alpha) = (g_1 g^{-1}, g.\alpha)$).

Consider the one-parameter subgroup γ of G with $\frac{d\gamma}{dt}|_{t=0} = h'$. Define the action of $\mathbb{K}^\times$ on $T^*G \cong G \times \mathfrak{g}^*$ by $t.(g, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha)$. Let ω denote the symplectic form of T^*G. It is clear that $t.\omega = t^2\omega$.

Consider the centralizer $Q := Z_G(e, h, f)$ of e, h, f in G. Being the centralizer of a reductive subalgebra, Q is a reductive group. Set $G_0 := Q \cap Z_G(h') = Z_Q(h_0)$, where $h_0 := h' - h$. Since h_0 is a semisimple element of $\mathfrak{q}$, we see that G_0 is reductive. Consider the action of G_0 on T^*G by right translations. This action preserves ω and commutes with the G- and $\mathbb{K}^\times$-actions.

Set $X := G \times S \hookrightarrow G \times \mathfrak{g}^* \cong T^*G$. Restricting the symplectic form from T^*G, we get the 2-form on X denoted by ω'. Choose $x \in S \hookrightarrow X$. The tangent space $T_x X$ is identified with $\mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, f])^*$. Then $\omega'_x(\xi + u, \eta + v) = \langle x, [\xi, \eta]\rangle + \langle u, \eta\rangle - \langle \xi, v\rangle, \xi, \eta \in \mathfrak{g}, u, v \in (\mathfrak{g}/[\mathfrak{g}, f])^*$. By [Lo1], Lemma 2, ω' is nondegenerate. The symplectic G-variety X is a special case of *model varieties* introduced in [Lo1].

The orbit $G(1, \chi)$ and the subvariety $X \subset T^*G$ are $G_0 \times \mathbb{K}^\times$-stable. Let us choose a $G \times G_0 \times \mathbb{K}^\times$-equivariant symplectic connection ∇ on X (existing by Proposition 2.2.2 because $G \times G_0$ is reductive). Construct the star-product $f * g = \sum_{i=0}^\infty D_i(f, g)\hbar^{2i}$ of degree 2 by means of ∇. The grading on $\mathbb{K}[S]$ induced by the action of $\mathbb{K}^\times$ is positive, for all eigenvalues of $\mathrm{ad}(h')$ on $\mathfrak{z}_{\mathfrak{g}}(e)$ are nonnegative. By Proposition 2.1.5, $\mathbb{K}[X][\hbar]$ is a graded quantum subalgebra of $\mathbb{K}[X][[\hbar]]$.

Definition 3.1.1. Set $\widetilde{\mathcal{W}} := \mathbb{K}[X][\hbar]/(\hbar - 1)\mathbb{K}[X][\hbar]$. The G-algebra $\widetilde{\mathcal{W}}$ is said to be an *equivariant W-algebra*. Also set $\mathcal{W} := \widetilde{\mathcal{W}}^G$.

The algebra $\widetilde{\mathcal{W}}$ is naturally identified with $\mathbb{K}[X]$ so that the product in $\widetilde{\mathcal{W}}$ is given by $f \circ g = \sum_{i=0}^\infty D_i(f, g)$. The grading on $\mathbb{K}[X][\hbar]$ gives rise to the *Kazhdan* filtration on $\widetilde{\mathcal{W}}$. W.r.t. this filtration, we have the equalities $\mathrm{gr}\, \widetilde{\mathcal{W}} = \mathbb{K}[X], \mathrm{gr}\, \mathcal{W} = \mathbb{K}[S]$ of Poisson algebras. Proposition 2.2.5 implies that the filtered algebras $\widetilde{\mathcal{W}}, \mathcal{W}$ do not depend up to an isomorphism on the choice of ∇. Also note that G_0 acts on $\mathcal{W}, \widetilde{\mathcal{W}}$ preserving the filtrations.

Proposition 3.1.2. *The G-algebra structure on $\widetilde{\mathcal{W}}$ does not depend on the choice of h'.*

Proof. Recall that the connection ∇ is $G_0\times\mathbb{K}^\times$-invariant. Therefore it is invariant w.r.t. the actions of $\mathbb{K}^\times$ associated to both h and h'. So $*$ is homogeneous w.r.t. both gradings on $\mathbb{K}[X]$. □

Proposition 3.1.3. *The algebra $\widetilde{\mathcal{W}}$ is simple.*

Proof. Let $\mathcal{I}$ be a proper ideal of $\widetilde{\mathcal{W}}$. Then $\operatorname{gr}\mathcal{I}$ is a Poisson ideal of $\mathbb{K}[X]$. The ideal $\mathcal{I}\subset\widetilde{\mathcal{W}}$ is G-stable, for the derivation of $\widetilde{\mathcal{W}}$ induced by any $\xi\in\mathfrak{g}$ is inner. One can see this from the alternative description of $\widetilde{\mathcal{W}}$ given in Remark 3.1.4 or from Subsections 2.1, 2.2 in [Lo2]. Since $\mathcal{I}$ is G-stable, we see that $\operatorname{gr}(\mathcal{I}\cap\mathcal{W})=\operatorname{gr}\mathcal{I}\cap\mathbb{K}[S]$. But X is symplectic; hence $\mathbb{K}[X]$ has no proper Poisson ideals. It follows that $\operatorname{gr}\mathcal{I}=\mathbb{K}[X]$. The grading on $\mathbb{K}[S]$ is nonnegative and $\operatorname{gr}\mathcal{I}$ contains $1\in\mathbb{K}[S]$. So we see that $1\in\mathcal{I}\cap\mathcal{W}$, a contradiction. □

Remark 3.1.4. There is an alternative description of $\widetilde{\mathcal{W}}$. Since the only place we need this description is the proof of Proposition 3.1.3 (and even there an alternative argument can be used), we do not give all the details. Namely, let ξ_*^r denote the velocity vector field associated with $\xi\in\mathfrak{g}$ for the action of G on G by right translations (i.e., ξ_*^r is the image of ξ in the Lie algebra of vector fields on G under the homomorphism induced by the action). Set

$$\mathcal{D}(G,e)=\left(\mathcal{D}(G)/(\xi_*^r-\langle\chi,\xi\rangle,\xi\in\mathfrak{m})\right)^N.$$

This is an associative G-algebra equipped with a filtration induced from the following filtration $\mathrm{F}_k\,\mathcal{D}(G)$ on $\mathcal{D}(G)$: $\mathrm{F}_k\,\mathcal{D}(G)=\sum_{i+2j\leqslant k}(\mathrm{F}_j^{ord}\,\mathcal{D}(G)\cap\mathcal{D}(G)(i))$. Here F^{ord} stands for the filtration of $\mathcal{D}(G)$ by the order of a differential operator and $\mathcal{D}(G)(i)$ is the eigenspace of $\operatorname{ad}(h_*'^r)$ corresponding to i. Note also that the G-action on $\mathcal{D}(G)$ by left translations descends to $\mathcal{D}(G,e)$. Finally, the map $\xi\mapsto\xi_*^l$ descends to a map $\mathfrak{g}\to\mathcal{D}(G,e)$. Here ξ_*^l is the velocity vector field for the action of G on itself by left translations.

Then $\widetilde{\mathcal{W}}$ and $\mathcal{D}(G,e)$ are isomorphic as filtered associative G-algebras. To prove this one needs, at first, to check that $\operatorname{gr}\mathcal{D}(G,e)=\mathbb{K}[X]$. To this end one uses techniques analogous to those developed in [GG]; see also [Gi2]. Since the second de Rham cohomology of X is trivial, the existence of a G-equivariant isomorphism $\widetilde{\mathcal{W}}\cong\mathcal{D}(G,e)$ is derived from standard results on the equivalence of star-products; see [BeKa] or [GR], for example. We will not need an isomorphism $\widetilde{\mathcal{W}}\cong\mathcal{D}(G,e)$ in the sequel. This isomorphism yields a G-equivariant linear map $\mathfrak{g}\to\widetilde{\mathcal{W}},\xi\mapsto\widehat{H}_\xi$, such that $\xi.f=[\widehat{H}_\xi,f]$ for any $f\in\widetilde{\mathcal{W}}$. The existence of a map $\mathfrak{g}\to\widetilde{\mathcal{W}}$ with these properties can also be easily derived, see [Lo2], from the standard results on the quantization of moment maps.

3.2. **Completions.** First of all, let us recall the notion of a Rees algebra.

Let $\mathcal{A}$ be an associative algebra with unit equipped with an increasing filtration $\mathrm{F}_i\,\mathcal{A},i\in\mathbb{Z}$, such that $\bigcup_{i\in\mathbb{Z}}\mathrm{F}_i\,\mathcal{A}=\mathcal{A},\bigcap_{i\in\mathbb{Z}}\mathrm{F}_i\,\mathcal{A}=\{0\},1\in\mathcal{A}_0$. Set $R_\hbar(\mathcal{A})=\bigoplus_{i\in\mathbb{Z}}\hbar^i\,\mathrm{F}_i\,\mathcal{A}$. This is a subalgebra in $\mathcal{A}[\hbar,\hbar^{-1}]$. We call $R_\hbar(\mathcal{A})$ the *Rees algebra* associated with $\mathcal{A}$. The group $\mathbb{K}^\times$ acts on $R_\hbar(\mathcal{A})$ by $t.\sum a_i\hbar^i=\sum t^ia_i\hbar^i$ for $a_i\in\mathrm{F}_i\,\mathcal{A}$. Note also that there are natural isomorphisms $R_\hbar(\mathcal{A})/(\hbar-1)R_\hbar(\mathcal{A})\cong\mathcal{A},R_\hbar(A)/\hbar R_\hbar(\mathcal{A})\cong\operatorname{gr}(\mathcal{A})$; see [CG], Corollary 3.2.8 for proofs.

Let us establish a relation between the sets of two-sided ideals of the algebras $\mathcal{A},R_\hbar(\mathcal{A})$. For $\mathcal{I}\in\mathfrak{Id}(\mathcal{A})$ set $R_\hbar(\mathcal{I}):=\bigoplus_{i\in\mathbb{Z}}(\mathcal{I}\cap\mathrm{F}_i\,\mathcal{A})\hbar^i$.

Definition 3.2.1. An ideal I in a $\mathbb{K}[\hbar]$-algebra B is called *$\hbar$-saturated* if $I = \hbar^{-1}I \cap B$.

The proof of the following proposition is straightforward.

Proposition 3.2.2. *The map $\mathcal{I} \mapsto R_\hbar(\mathcal{I})$ is a bijection between $\mathfrak{Id}(\mathcal{A})$ and the set of all $\mathbb{K}^\times$-stable $\hbar$-saturated ideals of $R_\hbar(\mathcal{A})$. The inverse map is given by $\mathcal{I}_\hbar \mapsto p(\mathcal{I}_\hbar)$, where p stands for the projection $R_\hbar(\mathcal{A}) \to R_\hbar(\mathcal{A})/(\hbar-1)R_\hbar(\mathcal{A}) \cong \mathcal{A}$.*

Let $\mathfrak{v}$ be a finite-dimensional graded vector space, $\mathfrak{v} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{v}(i)$, $\mathfrak{v} \neq \mathfrak{v}(0)$, and $A := S(\mathfrak{v})$. The grading on $\mathfrak{v}$ gives rise to the grading $A = \bigoplus_{i\in\mathbb{Z}} A(i)$, hence to an action $\mathbb{K}^\times : A$. Let $*$ be a polynomial star-product of degree 2 on $A[[\hbar]]$, $f * g = \sum_{i=0}^\infty D_i(f,g)\hbar^{2i}$. Suppose $D_i : A \otimes A \to A$ is a bidifferential operator of order at most i at each variable. Then we can form the associative product $\circ : A \times A \to A$, $f \circ g = \sum_{i=0}^\infty D_i(f,g)$. We denote A equipped with the corresponding algebra structure by $\mathcal{A}$. The algebra $\mathcal{A}$ is naturally filtered by the subspaces $\mathrm{F}_j\,\mathcal{A} := \bigoplus_{i\leqslant j} A(i)$ and $\operatorname{gr}\mathcal{A} \cong A$.

Clearly, the algebra $\mathcal{W}$ introduced in the previous section has the form $\mathcal{A}$ for $A = \mathbb{K}[S]$. Here are two other examples.

Example 3.2.3 (Weyl algebra)**.** Let $\mathfrak{v}$ be a symplectic vector space with symplectic form ω and a grading induced by a linear action $\mathbb{K}^\times : \mathfrak{v}, t.v = t^{-1}\theta(t)v, \theta : \mathbb{K}^\times \to \mathrm{Sp}(\mathfrak{v})$ so that $t.\omega = t^2\omega$. Let $*$ be the Moyal-Weyl star-product on $A[\hbar]$. Then the embedding $\mathfrak{v} \hookrightarrow \mathcal{A}$ induces the isomorphism $\mathbf{A}_\mathfrak{v} \to \mathcal{A}$. Indeed, from the explicit formula for the star-product given in Example 2.1.2 we see that $u\circ v - v\circ u = \omega(u,v)$, where ω denotes the symplectic form on $\mathfrak{v}$. So the inclusion $\mathfrak{v} \hookrightarrow \mathcal{A}$ does extend to an algebra homomorphism $\mathbf{A}_\mathfrak{v} \to \mathcal{A}$. This homomorphism is surjective, since $\mathfrak{v}$ generates $\mathcal{A}$, and is injective, since $\mathbf{A}_\mathfrak{v}$ is simple.

Example 3.2.4 (The universal enveloping)**.** Let us equip $\mathbb{K}[\mathfrak{g}^*][[\hbar]]$ with a star-product $*$ established in Example 2.2.4. Set $\mathfrak{v} := \mathfrak{g}$ and define the grading on $\mathfrak{v}$ by $\mathfrak{v}(2) = \mathfrak{v}$. The star-product $*$ is polynomial. The embedding $\mathfrak{g} \hookrightarrow \mathcal{A}$ induces a homomorphism $\mathcal{U} \to \mathcal{A}$. To show this we need to check that $\xi \circ \eta - \eta \circ \xi = [\xi, \eta]$ for any $\xi, \eta \in \mathfrak{g}$, where the r.h.s. is the commutator in $\mathfrak{g}$. Since $D_i(\xi,\eta)$ has degree $2-2i$ and $D_1(\xi,\eta) - D_1(\eta,\xi) = [\xi,\eta]$, we reduce the problem to checking the equality $D_2(\xi,\eta) = D_2(\eta,\xi)$. But $D_2|_{\mathfrak{g}\otimes\mathfrak{g}}$ is a G-invariant linear map $\mathfrak{g}\otimes\mathfrak{g} \to \mathbb{K}$, hence is symmetric. Using a PBW-type argument, we see that the homomorphism $\mathcal{U} \to \mathcal{A}$ we constructed is an isomorphism.

Below we will need an alternative presentation of $\mathcal{U}$. Namely, set

$$\mathfrak{v}_1 := \{\xi - \langle\chi,\xi\rangle, \xi \in \mathfrak{g}\}$$

and equip $\mathfrak{v}_1 \subset \mathfrak{g}\oplus\mathbb{K}$ with the Kazhdan grading. The algebras $\mathbb{K}[\mathfrak{v}_1^*], \mathbb{K}[\mathfrak{v}^*]$ are naturally isomorphic. Using this isomorphism, we get the star-product on $\mathbb{K}[\mathfrak{v}_1^*][\hbar]$. This star-product is homogeneous, for the star-product on $\mathbb{K}[\mathfrak{v}^*][\hbar]$ is $\mathrm{ad}(h')$-invariant. The corresponding algebra $\mathcal{A}_1$ is naturally isomorphic to the algebra $\mathcal{A}$ introduced in the previous paragraph.

By $A^\heartsuit$ we denote the subalgebra of the formal power series algebra $\mathbb{K}[[\mathfrak{v}^*]]$ consisting of all formal power series of the form $\sum_{i<n} f_i$ for some n, where f_i is a homogeneous power series of degree i. By definition, a power series f has degree i if $Df = if$ for the derivation $D : \mathbb{K}[[\mathfrak{v}^*]] \to \mathbb{K}[[\mathfrak{v}^*]]$ induced by the grading. For any $f, g \in A^\heartsuit$ we have the well-defined element $f \circ g := \sum_{i=0}^\infty D_i(f,g) \in A^\heartsuit$. The

algebra $A^\heartsuit$ w.r.t. $\circ$ is denoted by $\mathcal{A}^\heartsuit$. This algebra has a natural filtration $\mathrm{F}_i\,\mathcal{A}^\heartsuit$ such that $\mathcal{A}\cap \mathrm{F}_i\,\mathcal{A}^\heartsuit=\mathrm{F}_i\,\mathcal{A}$.

Let us describe the Rees algebras of $\mathcal{A},\mathcal{A}^\heartsuit$. The proof of the following lemma is straightforward.

Lemma 3.2.5. (1) *The map $A[\hbar]\to\mathcal{A}[\hbar^{-1},\hbar],\sum_{i,j} f_i\hbar^j\mapsto \sum_i f_i\hbar^{i+j}$, where $f_i\in A(i)$, is a $\mathbb{K}^\times$-equivariant monomorphism of $\mathbb{K}[\hbar]$-algebras. Its image coincides with $R_\hbar(\mathcal{A})$.*

(2) *The map $\mathbb{K}[[\mathfrak{v}^*,\hbar]]_{\mathbb{K}^\times-fin}\to\mathcal{A}^\heartsuit[\hbar^{-1},\hbar]$ given by $\sum_{i,j} f_i\hbar^j\mapsto\sum_{i,j} f_i\hbar^{i+j}$, where f_i is a homogeneous element of $\mathbb{K}[[\mathfrak{v}^*]]$ of degree i, is a $\mathbb{K}^\times$-equivariant monomorphism of $\mathbb{K}[\hbar]$-algebras. Its image coincides with $R_\hbar(\mathcal{A}^\heartsuit)$.*

Here $A[\hbar],\mathbb{K}[[\mathfrak{v}^*,\hbar]]_{\mathbb{K}^\times-fin}$ are regarded as quantum algebras.

The following proposition will be used in the proof of Proposition 3.4.6.

Proposition 3.2.6. *The algebra $\mathcal{A}^\heartsuit$ is a Noetherian domain.*

Proof. Clearly, $\operatorname{gr}\mathcal{A}^\heartsuit\cong\operatorname{gr}A^\heartsuit$ is naturally embedded into $\mathbb{K}[[\mathfrak{v}^*]]$. So $\operatorname{gr}\mathcal{A}^\heartsuit$ is a domain, hence $\mathcal{A}^\heartsuit$ is. Thanks to Proposition 3.2.2 and assertion 2 of Lemma 3.2.5, to prove that $\mathcal{A}^\heartsuit$ is Noetherian it is enough to check that any $\mathbb{K}^\times$-stable left (to be definite) ideal of $\mathbb{K}[[\mathfrak{v}^*,\hbar]]_{\mathbb{K}^\times-fin}$ is finitely generated. Our argument is a ramification of the proof of the Hilbert basis theorem for power series. Namely, suppose that we have already proved the analogous property for $B:=\mathbb{K}[[\mathfrak{v}^*]]_{\mathbb{K}^\times-fin}$. One checks directly $\mathbb{K}[[\mathfrak{v}^*,\hbar]]_{\mathbb{K}^\times-fin}=B[[\hbar]]_{\mathbb{K}^\times-fin}$. Let $\mathcal{I}$ be a $\mathbb{K}^\times$-stable left ideal in $B[[\hbar]]_{\mathbb{K}^\times-fin}$. For $k\in\mathbb{N}$ let I_k denote the classical part of $\hbar^{-k}(\mathcal{I}\cap\hbar^kB[[\hbar]])$. Clearly, $I_k,k\in\mathbb{N}$, is an ascending chain of $\mathbb{K}^\times$-stable ideals of B. There is $m\in\mathbb{N}$ with $I_m=I_k$ for any $k>m$. So we can choose a finite system $f_{ij}\in\hbar^j\mathcal{I},j\leqslant m$, of $\mathbb{K}^\times$-semi-invariant elements such that the classical parts of $\hbar^{-j}f_{ij}$ generate I_j for any $j\leqslant m$. Now choose $f\in\mathcal{I}$. There are $a^0_{ij}\in B$ such that $f\in\sum_{i,j}a^0_{ij}*f_{ij}+\hbar B[[\hbar]]$. Replace f with $f-\sum_{i,j}a^0_{ij}*f_{ij}\in\mathcal{I}$ and find $a^1_{ij}\in B$ with $f\in\sum_{i,j}(a^0_{ij}+\hbar a^1_{ij})*f_{ij}+\hbar^2B[[\hbar]]$, etc. So for any $f\in\mathcal{I}$ we can inductively construct $a_{ij}\in\mathbb{K}[[\mathfrak{v}^*,\hbar]]$ such that $f=\sum a_{ij}*f_{ij}$. If f is $\mathbb{K}^\times$-semi-invariant, then a_{ij} can also be chosen to be $\mathbb{K}^\times$-semi-invariant.

To check that any $\mathbb{K}^\times$-stable ideal of B is finitely generated we use an analogous argument inductively. □

For $u,v\in\mathfrak{v}(1)$ denote by $\omega_1(u,v)$ the constant term of $u\circ v-v\circ u$. Choose a maximal isotropic (w.r.t. ω_1) subspace $\mathfrak{y}\subset\mathfrak{v}(1)$ and set $\mathfrak{m}:=\mathfrak{y}\oplus\bigoplus_{i\leqslant 0}\mathfrak{v}(i)$. Further, choose a homogeneous basis $v_1,\ldots,v_n$ of $\mathfrak{v}$ such that $v_1,\ldots,v_m$ form a basis in $\mathfrak{m}$. Let d_i denote the degree of v_i.

Any element $a\in\mathcal{A}^\heartsuit$ can be written in a unique way as an infinite sum

$$\sum_{i_1\geqslant i_2\geqslant\ldots\geqslant i_l} a_{i_1,\ldots,i_l}v_{i_1}\circ\ldots\circ v_{i_l} \tag{3.1}$$

such that $\sum_{j=1}^l d_{i_j}\leqslant c$, where c depends on a. Moreover, a is contained in $\mathrm{F}_k\,\mathcal{A}^\heartsuit$ iff the sum (3.1) does not contain a monomial $v_{i_1}\circ\ldots\circ v_{i_l}$ with $\sum_{j=1}^l d_{i_j}>k$.

Let $\mathcal{I}^\heartsuit(k)$ denote the left ideal of $\mathcal{A}^\heartsuit$ consisting of all a such that any monomial in $\widetilde{a}$ has the form $v_{i_1}\circ\ldots\circ v_{i_l}$ with $v_{i_{l-k+1}}\in\mathfrak{m}$. Set $\mathcal{I}(k):=\mathcal{A}\cap\mathcal{I}^\heartsuit(k)$. We equip $\mathcal{A}$ (resp. $\mathcal{A}^\heartsuit$) with a topology, taking the left ideals $\mathcal{I}(k)$ (resp., $\mathcal{I}^\heartsuit(k)$) for a set of fundamental neighborhoods of 0. Since $[\mathfrak{m},\mathrm{F}_i\,\mathcal{A}^\heartsuit]\subset\mathrm{F}_{i-1}\,\mathcal{A}^\heartsuit$, we see that for any $a \in \mathcal{A}^\heartsuit$ and any $j \in \mathbb{N}$ there is some k such that $\mathcal{I}^\heartsuit(k) \circ a \subset \mathcal{I}^\heartsuit(j)$. Thus the inverse limits $\varprojlim \mathcal{A}/\mathcal{I}(k), \varprojlim \mathcal{A}^\heartsuit/\mathcal{I}^\heartsuit(k)$ are equipped with natural topological algebra structures. Moreover, for any $a \in \mathcal{A}^\heartsuit, k \in \mathbb{N}$, all but finitely many monomials of $\widetilde{a}$ lie in $\mathcal{I}^\heartsuit(k)$. Therefore $\mathcal{A}^\heartsuit = \mathcal{I}^\heartsuit(k) + \mathcal{A}$ for any k. It follows that the topological algebras $\varprojlim \mathcal{A}/\mathcal{I}(k), \varprojlim \mathcal{A}^\heartsuit/\mathcal{I}^\heartsuit(k)$ are naturally isomorphic. We denote this topological algebra by $\mathcal{A}^\wedge$. It is clear that $\bigcap_k \mathcal{I}^\heartsuit(k) = \{0\}$; hence the natural homomorphisms $\mathcal{A}, \mathcal{A}^\heartsuit \to \mathcal{A}^\wedge$ are injective.

Remark 3.2.7. Note that any element of $\mathbb{K}[[\mathfrak{v}^*, \hbar]]$ can be written uniquely as an infinite sum of monomials $a_{ii_1i_2\ldots i_k} v_{i_1} * \ldots * v_{i_k} \hbar^i, i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k, a_{ii_1i_2\ldots i_k} \in \mathbb{K}$. Consider the subspace $\mathcal{A}_\hbar^\wedge \subset \mathbb{K}[[\mathfrak{v}^*, \hbar]]$ consisting of all series $\sum a_{i,i_1,\ldots,i_k} v_{i_1} * \ldots * v_{i_k} \hbar^i$ satisfying the following finiteness condition:

- for any given $j \geqslant 0$ there are only finitely many tuples $(i, i_1, \ldots, i_k)$ with $a_{i,i_1,\ldots,i_k} \neq 0$ and $v_{i_{k-j}} \notin \mathfrak{m}$.

In particular, $\mathcal{A}_\hbar^\wedge \cap \mathbb{K}[[\hbar]] = \mathbb{K}[\hbar]$.

Any $v_i \notin \mathfrak{m}$ has positive degree. Therefore any element of $\mathbb{K}[[\mathfrak{v}^*, \hbar]]_{\mathbb{K}^\times - fin}$ satisfies the finiteness condition above. In other words, $\mathbb{K}[[\mathfrak{v}^*, \hbar]]_{\mathbb{K}^\times - fin} \subset \mathcal{A}_\hbar^\wedge$.

Note also that $\mathcal{A}_\hbar^\wedge/(\hbar - 1)\mathcal{A}_\hbar^\wedge$ is identified with the space of formal power series of the form $\sum a_{ii_1\ldots i_k} v_{i_1} \circ v_{i_2} \circ \ldots \circ v_{i_k}, i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k$, where $a_{ii_1\ldots i_k}$ satisfy the same finiteness condition. But the space of such series is nothing else but $\mathcal{A}^\wedge$. So $\mathcal{A}_\hbar^\wedge/(\hbar - 1)\mathcal{A}_\hbar^\wedge \cong \mathcal{A}^\wedge$.

Now let us relate $\mathcal{A}^\wedge$ with completions of the form used in Theorem 1.2.1.

Lemma 3.2.8. *Let $\widetilde{\mathfrak{m}}$ be a subspace in $\mathcal{A}^\heartsuit$ such that there is a basis in $\widetilde{\mathfrak{m}}$ of the form $v_i + u_i, i = 1, \ldots, m$, with $u_i \in (\mathrm{F}_{d_i} \mathcal{A}^\heartsuit \cap \mathfrak{v}^2 A^\heartsuit) + \mathrm{F}_{d_i - 2} \mathcal{A}^\heartsuit$. Then $\mathcal{A}^\heartsuit \widetilde{\mathfrak{m}} \subset \mathcal{I}^\heartsuit(1)$ and for any $k \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $\mathcal{A}^\heartsuit \widetilde{\mathfrak{m}}^l \subset \mathcal{I}^\heartsuit(k)$.*

Proof. We can write u_i in the form $m_i + \sum_{j=1}^m a_{ij} \circ v_j$, where $a_{ij} \in \mathrm{F}_{d_i - d_j} \mathcal{A}^\heartsuit \cap \mathfrak{v} A^\heartsuit, m_i \in \mathrm{F}_{d_i - 2} \mathcal{A}^\heartsuit \cap \mathfrak{v} \subset \mathcal{A}^\heartsuit \widetilde{\mathfrak{m}}$. Thus $\mathcal{A}^\heartsuit \widetilde{\mathfrak{m}} \subset \mathcal{I}^\heartsuit(1)$. Replacing the elements $v_i + u_i$ with their suitable linear combinations, we may assume that all m_i are zero. It remains to prove that for any $k \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that $f := a_1 \circ v_{i_1} \circ \ldots \circ a_l \circ v_{i_l} \in \mathcal{I}^\heartsuit(k)$, where $v_{i_1}, \ldots, v_{i_l} \in \mathfrak{m}$ and $a_j \in \mathrm{F}_{1 - d_{i_j}} \mathcal{A}$ for all j. To this end, we perform certain steps. On each step, we get the decomposition of f into the sum of monomials of the form $b_1 \circ v_{j_1} \circ b_2 \circ \ldots \circ v_{j_q}$, where any b_j is the product of $v_i \notin \mathfrak{m}$. Suppose that $b_p \neq 1$ for some $p > q - k + 1$. Replace $v_{j_{p-1}} \circ b_p$ in this monomial with $[v_{j_{p-1}}, b_p] + b_p \circ v_{j_{p-1}}$. Then replace the old monomial with the sum of two new ones. We claim that after some steps all monomials will lie in $\mathcal{I}^\heartsuit(k)$ provided l is sufficiently large. This is deduced from $[\mathrm{F}_i \mathcal{A}^\heartsuit, \mathrm{F}_j \mathcal{A}^\heartsuit] \subset \mathrm{F}_{i+j-2} \mathcal{A}^\heartsuit$ and $F_{-kd} \mathcal{A}^\heartsuit \subset \mathcal{I}(k)^\heartsuit$, where $d := \max_{i=1}^k |d_i|$. □

Recall that any element $a \in \mathcal{A}^\heartsuit$ can be written in a unique way as an infinite sum (3.1).

Lemma 3.2.9. *Suppose that $\mathcal{A} \subset \mathcal{A}^\heartsuit$ coincides with the set of all elements a such that the sum* (3.1) *is finite (this is the case, for example, when the grading on $\mathfrak{v}$ is positive). Then the systems of subspaces $\mathcal{I}(k), \mathcal{A}\mathfrak{m}^k \subset \mathcal{A}$ are compatible; i.e., for any $k \in \mathbb{N}$ there exist k_1, k_2 such that $\mathcal{I}(k_1) \subset \mathcal{A}\mathfrak{m}^k, \mathcal{A}\mathfrak{m}^{k_2} \subset \mathcal{I}(k)$.*

Proof. It is clear that $\mathcal{I}(k) \subset \mathcal{A}\mathfrak{m}^k$. Thanks to Lemma 3.2.8, we are done. □

Now we are going to describe the behavior of ideals under completions.

Take $\mathcal{I} \in \mathfrak{Id}(\mathcal{A})$ and set $I := \operatorname{gr}(\mathcal{I})$. It is clear that I coincides with the classical part of $R_\hbar(\mathcal{I}) \subset R_\hbar(\mathcal{A}) \cong A[\hbar]$. Further, let $\overline{\mathcal{I}}_\hbar$ denote the closure of $R_\hbar(\mathcal{I})$ in $\mathbb{K}[[\mathfrak{v}^*, \hbar]]$ (w.r.t. the topology of a formal power series algebra). Then $\overline{\mathcal{I}}_\hbar$ is a $\mathbb{K}^\times$-stable closed ideal of the quantum algebra $\mathbb{K}[[\mathfrak{v}^*, \hbar]]$.

Proposition 3.2.10. (1) *The ideal $\overline{\mathcal{I}}_\hbar$ is $\hbar$-saturated.*

(2) *The classical part of $\overline{\mathcal{I}}_\hbar$ coincides with the closure $\widehat{I}$ of I in $\mathbb{K}[[\mathfrak{v}^*]]$.*

(3) *Suppose the grading on $\mathfrak{v}$ is positive. Let $\mathcal{J}$ be a closed $\hbar$-saturated, $\mathbb{K}^\times$-stable ideal of $\mathbb{K}[[\mathfrak{v}^*, \hbar]]$. Then there is a unique $\mathcal{I} \in \mathfrak{Id}(\mathcal{A})$ such that $\mathcal{J} = \overline{\mathcal{I}}_\hbar$. Moreover, in this case $\mathcal{J} \cap A[\hbar] = R_\hbar(\mathcal{I})$.*

Proof. Let $J_\hbar$ denote the ideal of the point $(0, 0)$ in the commutative algebra $A[\hbar]$. Since D_i is of order at most i at each argument for any i, we get $J_\hbar^k * J_\hbar^l \subset J_\hbar^{k+l}$.

To prove assertions (1), (2) we need an auxiliary claim.

(*) Let $a_n, n \in \mathbb{N}$, be a sequence of elements of $R_\hbar(\mathcal{I})$. Let a_n^0 denote the classical part of a_n. Suppose $a_n^0 \in J_\hbar^n$. Then, probably after replacing the sequence (a_n) with an infinite subsequence, there are sequences $(c_n), (d_n)$ of elements of $R_\hbar(\mathcal{I})$ such that $d_i \in J_\hbar^n$ and $a_n = \hbar c_n + d_n$ for all n.

Let I_0 denote the ideal of A generated by $a_n^0, n \in \mathbb{N}$. There is $l \in \mathbb{N}$ such that I_0 is generated by $a_1^0, \ldots, a_l^0$. By the Artin-Rees lemma, there is $p \in \mathbb{N}$ with $I_0 \cap J_\hbar^{i+p} \subset J_\hbar^i I_0$ for all i. So, possibly after removing some a_i, we may assume that $a_k^0 = r_k^1 a_1^0 + \ldots + r_k^l a_l^0, r_k^i \in J_\hbar^k$, for any $k > l$. Set $d_i := \sum_{j=1}^m r_i^j * a_i$. By the remark in the beginning of the proof, $d_i \in R_\hbar(\mathcal{I}) \cap J_\hbar^i$. Further, $a_i - d_i \in \hbar R_\hbar(\mathcal{I})$ and we set $c_i = \hbar^{-1}(a_i - d_i)$. So (*) is proved.

Since $R_\hbar(\mathcal{I})$ is an $\hbar$-saturated ideal, assertion 1 follows from (*) and the trivial observation that (c_n) converges provided (a_n) does.

Proceed to assertion 2. Clearly, I is dense in the classical part of $\overline{\mathcal{I}}_\hbar$. It remains to show that the latter is closed in $\mathbb{K}[[\mathfrak{v}^*]]$. Let (b_n) be a sequence of elements of $R_\hbar(\mathcal{I}), b_n = \sum_{i=0}^\infty b_n^i \hbar^i, b_n^i \in A$, such that $b_i^0 - b_j^0 \in J_\hbar^i$ for $i < j$. We need to check that, possibly after replacing b_n with an infinite subsequence, there are $c_n \in R_\hbar(\mathcal{I})$ with $(b_i + \hbar c_i) - (b_j + \hbar c_j) \in J_\hbar^i$ for $i < j$. But this follows from (*) applied to $a_n = b_n - b_{n+1}$.

Assertion 3 follows easily from Proposition 3.2.2 and the observation that $J_\hbar^n$ does not contain a nonzero element of degree less than n. □

3.3. **Decomposition theorem.** Recall that we have an action of $\widetilde{G} := G \times \mathbb{K}^\times \times G_0$ (where $G_0 = Z_G(e, h, f) \cap Z_G(h')$) on $T^*G \cong G \times \mathfrak{g}^*$ by

$$g.(g_1, \alpha) = (gg_1, \alpha), t.(g_1, \alpha) = (g\gamma(t)^{-1}, t^{-2}\gamma(t)\alpha), g_0(g_1, \alpha) = (g_1 g_0^{-1}, g_0\alpha),$$
$$g, g_1 \in G, t \in \mathbb{K}^\times, g_0 \in G_0, \alpha \in \mathfrak{g}^*.$$

The subvariety $X \subset T^*G$ is stable under the action of $\widetilde{G}$.

Set $x = (1, \chi) \in X$. Then $\widetilde{G}x = Gx$, so, in particular, $\widetilde{G}x$ is closed. The stabilizer $\widetilde{G}_x$ equals $\{(g_0\gamma(t), t, g_0), t \in \mathbb{K}^\times, g_0 \in G_0\}$. So we may (and will) identify it with $G_0 \times \mathbb{K}^\times$.

Recall the vector space $V = [\mathfrak{g}, f]$ introduced in the Introduction.

Clearly, $T_xX \subset T_x(T^*G)$ is a symplectic subspace stable w.r.t. $G_0 \times \mathbb{K}^\times$. Let us describe its skew-orthogonal complement. Identify $T_x(T^*G)$ with $\mathfrak{g} \oplus \mathfrak{g}^*$ by means of the isomorphism $T^*G \cong G \times \mathfrak{g}^*$. Then the symplectic form ω_x on $\mathfrak{g} \oplus \mathfrak{g}^*$ is given by

$$\omega_x(\xi + \alpha, \eta + \beta) = \langle \chi, [\xi, \eta] \rangle - \langle \xi, \beta \rangle + \langle \eta, \alpha \rangle, \xi, \eta \in \mathfrak{g}, \alpha, \beta \in \mathfrak{g}^*,$$

and the $\widetilde{G}_x = G_0 \times \mathbb{K}^\times$-action is given by $g_0.(\xi, \alpha) = (g_0\xi, g_0\alpha), t.(\xi, \alpha) = (\gamma(t)\xi, t^{-2}\gamma(t)\alpha)$. Under the identification $T_x(T^*G) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ we have $T_xX = \mathfrak{g} \oplus (\mathfrak{g}/[\mathfrak{g}, f])^*$. So $(T_xX)^\angle = \{(\eta, \eta.\chi), \eta \in [\mathfrak{g}, f]\}$. The projection to the first component identifies $(T_xX)^\angle$ with $V = [\mathfrak{g}, f]$ equipped with the Kostant-Kirillov symplectic form $\omega_\chi(\xi, \eta) = \langle \chi, [\xi, \eta] \rangle$, where G_0 acts naturally and $\mathbb{K}^\times$ acts by $t \mapsto \gamma(t)$. However, it will be convenient for us to identify $(T_xX)^\angle$ not with V but with V^*; the $\mathbb{K}^\times$-action on V is now given by $t.v = \gamma(t)^{-1}(v)$ (note that $V \cong V^*$ as symplectic G_0-modules but not as $\mathbb{K}^\times$-modules, since $\mathbb{K}^\times$ does not preserve the symplectic form but rather rescales it). So, finally, we get a $G_0 \times \mathbb{K}^\times$-equivariant symplectomorphism $T_x(T^*G) \to T_xX \oplus V^*$ denoted by ψ.

Consider the Fedosov star-product $*$ on $\mathbb{K}[T^*G][[\hbar]]$ corresponding to a $G \times G \times \mathbb{K}^\times$-invariant connection (here we consider the usual fiberwise action $\mathbb{K}^\times : T^*G$). Further, we equip $S(V)[[\hbar]] = \mathbb{K}[V^*][[\hbar]]$ with the Moyal-Weyl star-product. So we get $G \times G_0$-invariant homogeneous star-products of degree 2 on $\mathbb{K}[T^*G][[\hbar]], \mathbb{K}[X \times V^*][[\hbar]]$.

Consider the completions $\mathbb{K}[T^*G]^\wedge_{Gx}, \mathbb{K}[X \times V^*]^\wedge_{Gx}$ of $\mathbb{K}[T^*G], \mathbb{K}[X \times V^*]$ w.r.t. the orbits $Gx \subset T^*G, X \times V^*$. The Fedosov star-products are differential, so we get the star-products on $\mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]], \mathbb{K}[X \times V^*]^\wedge_{Gx}[[\hbar]]$. It turns out that these two algebras are $\widetilde{G}$-equivariantly isomorphic. Moreover, a more precise statement holds.

Theorem 3.3.1. *There is a $\widetilde{G}$-equivariant $\mathbb{K}[\hbar]$-algebra isomorphism*

$$\Phi_\hbar : \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]] \to \mathbb{K}[X \times V^*]^\wedge_{Gx}[[\hbar]]$$

possessing the following properties:

1. *Let $\Phi : \mathbb{K}[T^*G]^\wedge_{Gx} \to \mathbb{K}[X \times V^*]^\wedge_{Gx}$ be the classical part of $\Phi_\hbar$. Then the corresponding morphism φ of formal schemes maps x to x and $d_x\varphi : T_x(X \times V^*) \to T_x(T^*G)$ coincides with ψ^{-1}.*
2. *$\Phi_\hbar(\mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar^2]]) = \mathbb{K}[X \times V^*]^\wedge_{Gx}[[\hbar^2]]$.*

Proof. Set $H := \widetilde{G}_x$. Clearly, ψ induces an H-equivariant isomorphism of the normal spaces $T_x(T^*G)/\mathfrak{g}_*x, T_x(X \times V^*)/\mathfrak{g}_*x$. By the Luna slice theorem, [PV], §6, there is a $\widetilde{G}$-equivariant isomorphism $\Phi : \mathbb{K}[T^*G]^\wedge_{Gx} \to \mathbb{K}[X \times V^*]^\wedge_{Gx}$ with property (1). Now we can use the argument of the proof of the equivariant Darboux theorem (for example, its algebraic version, see [Kn], Theorem 5.1), to show that one can modify Φ such that the dual morphism of formal schemes becomes a symplectomorphism. This modification does not affect property (1). The existence of a $\widetilde{G}$-equivariant isomorphism $\Phi_\hbar : \mathbb{K}[T^*G]^\wedge_{Gx}[[\hbar]] \to \mathbb{K}[X \times V^*]^\wedge_{Gx}[[\hbar]]$ satisfying (1),(2) follows from Proposition 2.2.5 (where G is replaced with $G \times G_0$). □

Consider the completions $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi, \mathbb{K}[S \times V^*]^\wedge_\chi$. Clearly, $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi = (\mathbb{K}[T^*G]^\wedge_{Gx})^G$, $\mathbb{K}[S \times V^*]^\wedge_\chi = (\mathbb{K}[X \times V^*]^\wedge_{Gx})^G$. Therefore the restriction of $\Phi_\hbar$ to $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]$ is an isomorphism of the quantum algebras $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]] \to \mathbb{K}[S \times V^*]^\wedge_\chi[[\hbar]]$.

Set $\mathfrak{v}_1 := \{\xi - \langle \chi, \xi \rangle, \xi \in \mathfrak{g}\}, A_1 := S(\mathfrak{v}_1)$. Construct the algebra $\mathcal{A}_1$ as in Example 3.2.4. By that example, $\mathcal{A}_1$ satisfies the conditions of Lemma 3.2.9.

Similarly, set $\mathfrak{v}_2 := V \oplus \mathfrak{z}_\mathfrak{g}(e), A_2 := S(\mathfrak{v}_2) = S(V) \otimes \mathbb{K}[S], \mathcal{A}_2 := \mathbf{A}_V \otimes \mathcal{W}$ (see Example 3.2.3 and the construction of $\mathcal{W}$ in Subsection 3.1). Note that $\mathcal{A}_2$ satisfies the assumption of Lemma 3.2.9. Indeed, it is enough to check the analogous claim for the factor $\mathcal{W} \subset \mathcal{A}_2$. Here it follows easily from the observation that the grading on $\mathbb{K}[S]$ is positive.

So we can construct the algebras $A_1^\heartsuit, A_2^\heartsuit, \mathcal{A}_1^\heartsuit, \mathcal{A}_2^\heartsuit$ as in Subsection 3.2. Construct the left ideals $\mathcal{I}_1^\heartsuit(k), \mathcal{I}_2^\heartsuit(k)$ for the subspaces $\mathfrak{m}' \subset \mathfrak{v}_1, \underline{\mathfrak{m}} \subset \mathfrak{v}_2$. Let $\mathcal{A}_1^\wedge, \mathcal{A}_2^\wedge$ be the corresponding completions.

Corollary 3.3.2. *There is an isomorphism* $\Phi : \mathcal{A}_1^\heartsuit \to \mathcal{A}_2^\heartsuit$ *of filtered algebras having the following properties:*

(1) $\Phi(\mathcal{I}_1^\heartsuit(1)) = \mathcal{I}_2^\heartsuit(1)$.

(2) *The systems of subspaces* $\Phi(\mathcal{I}_1^\heartsuit(k)), \mathcal{I}_2^\heartsuit(k)$ *are compatible (in the sense of Lemma* 3.2.9*).*

Proof. Note that $\Phi_\hbar$ maps $\mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]_{\mathbb{K}^\times-fin}$ to $\mathbb{K}[S\times V^*]^\wedge_\chi[[\hbar]]_{\mathbb{K}^\times-fin}$. By Lemma 3.2.5, $R_\hbar(\mathcal{A}_1^\heartsuit) = \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]]_{\mathbb{K}^\times-fin}, R_\hbar(\mathcal{A}_2^\heartsuit) = \mathbb{K}[S\times V^*]^\wedge_\chi[[\hbar]]_{\mathbb{K}^\times-fin}$. For Φ we take the isomorphism $\mathcal{A}_1^\heartsuit \to \mathcal{A}_2^\heartsuit$ induced by $\Phi_\hbar$. By conditions (1),(2) of Theorem 3.3.1, if $v \in \mathfrak{v}_1(i)$, then $\Phi(v) - \psi(v) \in \mathrm{F}_{i-2}\,\mathcal{A}_2^\heartsuit + (\mathrm{F}_i\,\mathcal{A}^\heartsuit \cap \mathfrak{v}_2^2 S(\mathfrak{v}_2))$. Conditions (1),(2) are now deduced from Lemma 3.2.8 applied to the pairs $\mathfrak{m}', \Phi^{-1}(\underline{\mathfrak{m}}) \subset \mathcal{A}_2^\heartsuit$ and $\Phi(\mathfrak{m}'), \underline{\mathfrak{m}} \subset \mathcal{A}_2^\heartsuit$. □

Corollary 3.3.3. *There is an isomorphism of filtered algebras* $\Phi_0 : U(\mathfrak{g}, e) \to \mathcal{W}$.

Proof. In the notation of Corollary 3.3.2, there are isomorphisms of filtered algebras

$$U(\mathfrak{g}, e) \cong (\mathcal{A}_1^\heartsuit/\mathcal{I}_1^\heartsuit(1))^{\mathcal{I}_1^\heartsuit(1)} \cong (\mathcal{A}_2^\heartsuit/\mathcal{I}_2^\heartsuit(1))^{\mathcal{I}_2^\heartsuit(1)} \cong \mathcal{W}$$

(where the middle one is induced by Φ). □

Proof of Theorem 1.2.1. By Lemma 3.2.9, $\mathcal{A}_1^\wedge \cong \mathcal{U}^\wedge_{\mathfrak{m}'}, \mathcal{A}_2^\wedge \cong \mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}$ (isomorphisms of topological algebras). By condition (2) of Corollary 3.3.2, Φ can be extended to an isomorphism of topological algebras $\mathcal{A}_1^\wedge \to \mathcal{A}_2^\wedge$. The equality $\Phi(\mathcal{J}_1) = \mathcal{J}_2$ stems from (1) of Corollary 3.3.2, for $\mathcal{J}_i$ is the closure of $\mathcal{I}_i^\heartsuit(1), i = 1, 2$. □

Let us choose a $\mathbb{K}^\times$-stable Lagrangian subspace $\underline{\mathfrak{m}}^* \subset V$ complementary to $\underline{\mathfrak{m}}$. Choose a homogeneous basis $q_1, \ldots, q_k$ of $\underline{\mathfrak{m}}$ and let $p_1, \ldots, p_k$ be the dual basis of $\underline{\mathfrak{m}}^*$. Let $e_1, \ldots, e_k, e_1', \ldots, e_k'$ denote the degrees of $q_1, \ldots, q_k, p_1, \ldots, p_k$. Below we use the following notation. We write $\mathbf{i}, \mathbf{j}$ for the multi-indices $\mathbf{i} = (i_1, \ldots, i_k), \mathbf{j} = (j_1, \ldots, j_k), i_l, j_l \geqslant 0$. Set $p^\mathbf{i} := p_1^{i_1} \ldots p_k^{i_k}, q^\mathbf{j} := q_1^{j_1} \ldots q_k^{j_k}$ (the products are taken w.r.t. $\circ$). Any element of $\mathbf{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}$ is uniquely represented in the form

$$a = \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i},\mathbf{j}} p^\mathbf{i} q^\mathbf{j}, a_{\mathbf{i},\mathbf{j}} \in \mathcal{W}, \tag{3.2}$$

where for fixed $\mathbf{j}$ only finitely many coefficients $a_{\mathbf{i},\mathbf{j}}$ are nonzero.

Let M be a Whittaker $\mathfrak{g}$-module. The representation of $\mathcal{U}$ in M is uniquely extended to a continuous (w.r.t. the discrete topology on M) representation of $\mathcal{U}^\wedge_{\mathfrak{m}'}$. By Theorem 1.2.1, we obtain the representation of $\mathbf{A}_V(\mathcal{W})$ in M such that $\underline{\mathfrak{m}}$ acts on M by nilpotent endomorphisms and $M^{\mathfrak{m}'} = M^{\mathcal{J}_1} = M^{\mathcal{J}_2} = M^{\underline{\mathfrak{m}}}$.

Proposition 3.3.4. *The* $\mathbf{A}_V(\mathcal{W})$*-modules* M *and* $M^{\mathfrak{m}'} \otimes \mathbb{K}[\underline{\mathfrak{m}}]$ *are isomorphic. Here* $\mathbf{A}_V$ *acts on* $\mathbb{K}[\underline{\mathfrak{m}}]$ *as the algebra of differential operators.*

Proof. Recall that $V = \underline{\mathfrak{m}} \oplus \underline{\mathfrak{m}}^*$. We have the natural homomorphism $\iota : S(\underline{\mathfrak{m}}^*) \otimes M^{\underline{\mathfrak{m}}} \to M$ of $\mathcal{W} \otimes \mathbf{A}_V$-modules, $p_1 \ldots p_k \otimes v \mapsto p_1 \ldots . p_k . v$. The map $V \to \operatorname{End}(M)$ extends to a representation of the Heisenberg Lie algebra associated to V. The Lagrangian subspace $\underline{\mathfrak{m}} \subset V$ acts on M by locally nilpotent endomorphisms. Now the representation theory of Heisenberg Lie algebras implies that ι is an isomorphism; a proof (in a more general situation) is given, for example, in [Ka], Theorem 3.5. □

For example, $Q_{\mathfrak{y}} = \mathcal{U}/\mathcal{U}\mathfrak{m}' \cong \mathcal{U}^\wedge_{\mathfrak{m}'}/\mathcal{J}_1$ is a Whittaker module. Applying Corollary 3.3.3, we can identify $\mathcal{U}(\mathfrak{g}, e) \cong (Q_{\mathfrak{y}})^{\mathfrak{m}'}$ with $\mathcal{W}$, so we obtain the $(\mathcal{U}, \mathcal{W})$-bimodule structure on $Q_{\mathfrak{y}}$. By Proposition 3.3.4, $Q_{\mathfrak{y}}$ is isomorphic to $\mathbb{K}[\underline{\mathfrak{m}}] \otimes \mathcal{W}$ as a $(\mathbf{A}_V(\mathcal{W}), \mathcal{W})$-bimodule.

This observation together with Proposition 3.3.4 implies Theorem 1.1.4. Note that Theorem 1.1.4 automatically implies the equivalence of the categories of finitely generated modules. Indeed, finitely generated is the same as Noetherian and the latter is a purely categorical notion.

Proposition 3.3.5. *Let M be a finitely generated $\mathcal{W}$-module. Then* $\operatorname{Dim}_{\mathcal{U}}(\mathcal{S}(M)) = \operatorname{Dim}_{\mathcal{W}}(M) + \dim \mathfrak{m}$.

Proof. We identify $\mathcal{A}_1^\heartsuit, \mathcal{A}_2^\heartsuit$ by means of Φ and write $\mathcal{A}^\heartsuit, \mathcal{I}^\heartsuit$ instead of $\mathcal{A}_j^\heartsuit, \mathcal{I}_j^\heartsuit(1)$, $j = 1, 2$. There is a finite-dimensional subspace $M_0 \subset M$ generating the $\mathcal{A}^\heartsuit$-module $\mathcal{S}(M) \cong \mathbb{K}[\underline{\mathfrak{m}}] \otimes M$. It is easy to see that $\mathrm{F}_j \mathcal{A}^\heartsuit \subset \mathrm{F}_j \mathcal{A}_i + \mathcal{I}^\heartsuit, i = 1, 2$, for any j. Therefore $\dim(\mathrm{F}_j \mathcal{A}_i) M_0 = \dim(\mathrm{F}_j \mathcal{A}^\heartsuit) M_0, i = 1, 2$. Hence it remains to prove that

$$\operatorname{Dim}_{\mathcal{U}}(\mathcal{S}(M)) = \lim_{n \to \infty} \frac{\ln \dim(\mathrm{F}_n \mathcal{A}_1) M_0}{\ln n}, \tag{3.3}$$

$$\operatorname{Dim}_{\mathbf{A}_V(\mathcal{W})}(\mathcal{S}(M)) = \lim_{n \to \infty} \frac{\ln \dim(\mathrm{F}_n \mathcal{A}_2) M_0}{\ln n}. \tag{3.4}$$

To prove (3.3) we note that the elements $v_{i_1} \circ \ldots \circ v_{i_k}$, where $i_1 \geqslant i_2 \geqslant \ldots \geqslant i_k, v_{i_k} \notin \mathfrak{m}'$, and $\sum_{j=1}^k d_{i_j} \leqslant i$, form a basis of $\mathrm{F}_i \mathcal{A}_1 / (\mathrm{F}_i \mathcal{A}_1 \cap \mathcal{I}_1^\heartsuit(1))$. Note that all degrees d_{i_j} are positive. Therefore $\mathrm{F}_i \mathcal{U} + (\mathcal{I}_1^\heartsuit(1) \cap \mathcal{U}) \subset \mathrm{F}_i^{st} \mathcal{U} + (\mathcal{I}_1^\heartsuit(1) \cap \mathcal{U}) \subset \mathrm{F}_{d_1 i} \mathcal{U} + (\mathcal{I}_1^\heartsuit(1) \cap \mathcal{U})$ (here $\mathrm{F}_\bullet^{st}$ denotes the standard filtration on $\mathcal{U}$). Therefore $(\mathrm{F}_i \mathcal{U}) M_0 \subset (\mathrm{F}_i^{st} \mathcal{U}) M_0 \subset (\mathrm{F}_{d_1 i} \mathcal{U}) M_0$, hence (3.3).

The proof of (3.4) is completely analogous. □

3.4. **Correspondence between ideals.** To simplify the notation we write $\mathcal{A}(\mathcal{W})$, $\mathcal{A}(\mathcal{W})^\wedge, \mathcal{U}^\wedge$ instead of $\mathcal{A}_V(\mathcal{W}), \mathcal{A}_V(\mathcal{W})^\wedge_{\underline{\mathfrak{m}}}, \mathcal{U}^\wedge_{\mathfrak{m}'}$. Fix an isomorphism $\Phi_\hbar$ satisfying the assumptions of Theorem 3.3.1 and the isomorphism $\Phi : \mathcal{U}^\wedge \to \mathbf{A}(\mathcal{W})^\wedge$ constructed in the proof of Theorem 1.2.1.

For $\mathcal{I} \in \mathfrak{Id}(\mathcal{W})$ define the ideal $\mathbf{A}(\mathcal{I})^\wedge \subset \mathbf{A}(\mathcal{W})^\wedge$ by

$$\mathbf{A}(\mathcal{I})^\wedge = \varprojlim (\mathcal{I} + \mathbf{A}(\mathcal{W})\underline{\mathfrak{m}}^k)/\mathbf{A}(\mathcal{W})\underline{\mathfrak{m}}^k.$$

Alternatively, $\mathbf{A}(\mathcal{I})^\wedge$ is the closure of $\mathbf{A}(\mathcal{I}) := \mathbf{A}(\mathcal{W})\mathcal{I}$ (or $\mathbf{A}(\mathcal{W})^\wedge \mathcal{I}$) in $\mathbf{A}(\mathcal{W})^\wedge$. Clearly, $\mathbf{A}(\mathcal{I})^\wedge$ consists of all elements of the form (3.2) with $a_{\mathbf{i},\mathbf{j}} \in \mathcal{I}$.

Set $\mathcal{I}^\dagger := \mathcal{U} \cap \Phi^{-1}(\mathbf{A}(\mathcal{I})^\wedge)$. This is an ideal of $\mathcal{U}$.

Below we will need an alternative construction of $\mathcal{I}^\dagger$. By Lemma 3.2.5, there is a natural identification $A_i[\hbar] \cong R_\hbar(\mathcal{A}_i)$. So we can consider $R_\hbar(\mathcal{I})$ as an ideal in the quantum algebra $A_2[\hbar]$. Consider the closure $\overline{\mathcal{I}}_\hbar$ of $R_\hbar(\mathcal{I})$ in $\mathbb{K}[S]^\wedge_\chi[[\hbar]]$.

The ideal $\mathbb{K}[[V^*,\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar \subset \mathbb{K}[[\mathfrak{v}_2^*,\hbar]]$ is closed, $\mathbb{K}^\times$-stable and $\hbar$-saturated. By Proposition 3.2.2, there is a unique ideal $\mathcal{I}^\ddagger \subset \mathcal{U}$ such that $R_\hbar(\mathcal{I}^\ddagger) = R_\hbar(\mathcal{U}) \cap \Phi_\hbar^{-1}(\mathbb{K}[[V^*,\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar)$.

Proposition 3.4.1. $\mathcal{I}^\dagger = \mathcal{I}^\ddagger$.

Proof. Set $\overline{\mathcal{J}}_\hbar := \mathbb{K}[[V^*,\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar$. Let $\mathcal{A}_{2\hbar}^\wedge \subset \mathbb{K}[[\mathfrak{v}_2^*,\hbar]]$ be such as in Remark 3.2.7. Recall that we have the following commutative diagram:

$$\begin{array}{ccccccc} R_\hbar(\mathcal{A}_2) & \longrightarrow & R_\hbar(\mathcal{A}_2^\heartsuit) & \longrightarrow & \mathcal{A}_{2\hbar}^\wedge & \longrightarrow & \mathbb{K}[[\mathfrak{v}_2^*,\hbar]] \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{A}_2 & \longrightarrow & \mathcal{A}_2^\heartsuit & \longrightarrow & \mathcal{A}_2^\wedge & & \end{array}$$

Here all horizontal arrows are natural embeddings. All vertical arrows are quotients by the ideals generated by $\hbar - 1$; we denote these epimorphisms by π.

Any element $a \in \mathbb{K}[[\mathfrak{v}_2^*,\hbar]]$ is uniquely represented as a series

$$a = \sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i},\mathbf{j}} p^{*\mathbf{i}} * q^{*\mathbf{j}}, \quad a_{\mathbf{i},\mathbf{j}} \in \mathbb{K}[S]^\wedge_\chi[[\hbar]], \tag{3.5}$$

where $p^{*\mathbf{i}}, q^{*\mathbf{j}}$ have an analogous meaning to $p^{\mathbf{i}}, q^{\mathbf{j}}$, but the product is taken w.r.t. $*$ instead of $\circ$.

By Remark 3.2.7, an element (3.5) lies in $\mathcal{A}_{2\hbar}^\wedge$ iff all coefficients $a_{\mathbf{i},\mathbf{j}}$ lie in $R_\hbar(\mathcal{W}) = \mathbb{K}[S][\hbar]$ and for given $\mathbf{j}$ only finitely many of them are nonzero. Therefore $\pi(\mathcal{A}_{2\hbar}^\wedge \cap \overline{\mathcal{J}}_\hbar) = \mathbf{A}(\mathcal{I})^\wedge$.

So $\Phi(\mathcal{I}^\ddagger) = \pi\left(\Phi_\hbar\left(R_\hbar(\mathcal{U})\right) \cap \overline{\mathcal{J}}_\hbar\right) \subset \Phi(\mathcal{U}) \cap \pi(\mathcal{A}_{2\hbar}^\wedge \cap \overline{\mathcal{J}}_\hbar) = \Phi(\mathcal{U}) \cap \mathbf{A}(\mathcal{I})^\wedge = \Phi(\mathcal{I}^\dagger)$; hence $\mathcal{I}^\ddagger \subset \mathcal{I}^\dagger$. To verify the opposite inclusion we need to check that $R_\hbar(\Phi(\mathcal{U}) \cap \mathbf{A}(\mathcal{I})^\wedge) \subset \overline{\mathcal{J}}_\hbar$. Since $\Phi(\mathcal{U}) \subset \mathcal{A}_2^\heartsuit$ by the construction of Φ, it is enough to show that $R_\hbar(\mathcal{A}_2^\heartsuit \cap \mathbf{A}(\mathcal{I})^\wedge) \subset R_\hbar(\mathcal{A}_2^\heartsuit) \cap \overline{\mathcal{J}}_\hbar$. Both these ideals of $R_\hbar(\mathcal{A}_2^\heartsuit)$ are $\mathbb{K}^\times$-stable and $\hbar$-saturated. Thanks to Proposition 3.2.2, we need to check only that

$$\mathcal{A}_2^\heartsuit \cap \mathbf{A}(\mathcal{I})^\wedge = \pi(R_\hbar(\mathcal{A}_2^\heartsuit) \cap \overline{\mathcal{J}}_\hbar). \tag{3.6}$$

The l.h.s. of (3.6) consists of all infinite sums of the form $\sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i},\mathbf{j}} p^{\mathbf{i}} q^{\mathbf{j}}$, where $a_{\mathbf{i},\mathbf{j}} \in \mathcal{I}$ and there is c such that $a_{\mathbf{i},\mathbf{j}} \in \mathrm{F}_{c-\sum_{l=1}^k (e'_l i_l + e_l j_l)}\mathcal{W}$. The ideal $R_\hbar(\mathcal{A}_2^\heartsuit) \cap \overline{\mathcal{J}}_\hbar$ consists of all infinite sums $\sum_{\mathbf{i},\mathbf{j}} a_{\mathbf{i},\mathbf{j}} p^{*\mathbf{i}} * q^{*\mathbf{j}}$, where $a_{\mathbf{i},\mathbf{j}}$ is a homogeneous element of $R_\hbar(\mathcal{I})$ of degree, say, $e_{\mathbf{i},\mathbf{j}}$ and the set $\{e_{\mathbf{i},\mathbf{j}} + \sum_{l=1}^k (e'_l i_l + e_l j_l) | a_{\mathbf{i},\mathbf{j}} \neq 0\}$ is finite. Since $e_{\mathbf{i},\mathbf{j}} \geqslant 0$, (3.6) follows. □

Proof of assertion (i)-(iv) of Theorem 1.2.2. (i) is obvious. To prove (ii) let us note that $\mathbf{A}(\mathrm{Ann}_{\mathcal{W}}(N))^\wedge = \mathrm{Ann}_{\mathbf{A}(\mathcal{W})^\wedge}(\mathbb{K}[\mathfrak{m}] \otimes N)$ and use Proposition 3.3.4.

Let us prove (iii). Choose $z \in \mathcal{Z}(\mathfrak{g})$. By definition, we have $\mathcal{I} \cap \iota(\mathcal{Z}(\mathfrak{g})) = \mathrm{Ann}_{\iota(\mathcal{Z}(\mathfrak{g}))}(\mathcal{W}/\mathcal{I})$, $\mathcal{S}(\mathcal{W}/\mathcal{I}) = Q_\mathfrak{y}/Q_\mathfrak{y}\mathcal{I}$; hence $\mathcal{I} \cap \iota(\mathcal{Z}(\mathfrak{g})) \subset \mathrm{Ann}_{\iota(\mathcal{Z}(\mathfrak{g}))}(Q_\mathfrak{y}/Q_\mathfrak{y}\mathcal{I})$. By (ii), $\mathcal{I}^\dagger = \mathrm{Ann}(Q_\mathfrak{y}/\mathcal{I}Q_\mathfrak{y})$; hence $\mathcal{I}^\dagger \cap \mathcal{Z}(\mathfrak{g}) = \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}(Q_\mathfrak{y}/Q_\mathfrak{y}\mathcal{I})$. Thanks to Theorem 1.1.4, $\mathcal{W}/\mathcal{I} \cong (Q_\mathfrak{y}/Q_\mathfrak{y}\mathcal{I})^{\mathfrak{m}'}$. Therefore $\mathcal{I}^\dagger \cap \mathcal{Z}(\mathfrak{g}) \subset \mathrm{Ann}_{\mathcal{Z}(\mathfrak{g})}\mathcal{W}/\mathcal{I}$. Clearly, z and $\iota(z)$ coincide on $Q_\mathfrak{y}/\mathcal{I}Q_\mathfrak{y}, \mathcal{W}/\mathcal{I}$. This implies (iii).

Proceed to (iv). First of all, let us check that $\mathbf{A}(\mathcal{I})^\wedge$ is prime. Indeed, let $\mathcal{J}_1, \mathcal{J}_2$ be ideals of $\mathbf{A}(\mathcal{W})^\wedge$ such that $\mathbf{A}(\mathcal{I})^\wedge \subset \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_1\mathcal{J}_2 \subset \mathbf{A}(\mathcal{I})^\wedge$. Replacing $\mathcal{J}_1, \mathcal{J}_2$ with their closures, we may assume that $\mathcal{J}_1, \mathcal{J}_2$ are closed. It is easy to show (compare with Lemma 3.4.3 below) that $\mathcal{J}_i = \mathbf{A}(\mathcal{I}_i)^\wedge, i = 1,2$, for a unique ideal $\mathcal{I}_i$ of $\mathcal{W}$. We get $\mathcal{I}_1\mathcal{I}_2 \subset \mathcal{W}\cap \mathbf{A}(\mathcal{I})^\wedge = \mathcal{I}$. It follows that $\mathcal{I}_1 = \mathcal{I}$ or $\mathcal{I}_2 = \mathcal{I}$.

Let $a, b \in \mathcal{U}$ be such that $a\mathcal{U}b \subset \mathcal{I}^\dagger$. But $\Phi(a)\mathbf{A}(\mathcal{W})^\wedge\Phi(b)$ lies in the closure of $\Phi(a\mathcal{U}b)$ in $\mathbf{A}(\mathcal{W})^\wedge$. Therefore $\Phi(a)\mathbf{A}(\mathcal{W})^\wedge\Phi(b) \subset \mathbf{A}(\mathcal{I})^\wedge$. So $a \in \mathcal{I}^\dagger$ or $b \in \mathcal{I}^\dagger$. In other words, $\mathcal{I}^\dagger$ is prime. By (iii), $\mathcal{I}^\dagger \subset \operatorname{Pr}(\mathcal{U})$. □

Now let us construct a map $\mathfrak{Id}(\mathcal{U}) \to \mathfrak{Id}(\mathcal{W})$. Recall, Lemma 3.2.5, that the algebras $A_1[\hbar]$ and $R_\hbar(\mathcal{U})$ are naturally identified. Choose $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$. By $\overline{\mathcal{J}}_\hbar$ we denote the closure of $R_\hbar(\mathcal{J})$ in $\mathbb{K}[[\mathfrak{v}_1^*,\hbar]]$. By assertion 1 of Proposition 3.2.10, the ideal $\overline{\mathcal{J}}_\hbar$ is $\hbar$-saturated. Therefore $\Phi_\hbar(\overline{\mathcal{J}}_\hbar)\cap \mathbb{K}[S]^\wedge_\chi[[\hbar]]$ is $\mathbb{K}^\times$-stable and $\hbar$-saturated. By assertion 3 of Proposition 3.2.10, there is a uniquely determined element $\mathcal{J}_\dagger \in \mathfrak{Id}(\mathcal{W})$ such that $R_\hbar(\mathcal{J}_\dagger) = \Phi_\hbar(\overline{\mathcal{J}}_\hbar)\cap R_\hbar(\mathcal{W})$. Note that $(\mathcal{J}_1)_\dagger \subset (\mathcal{J}_2)_\dagger$ provided $\mathcal{J}_1 \subset \mathcal{J}_2$.

The following proposition is the main property of the map $\mathcal{J} \mapsto \mathcal{J}_\dagger$.

Proposition 3.4.2. *Let $\mathcal{J} \in \mathfrak{Id}(\mathcal{U})$. Then $\operatorname{gr}\mathcal{J}_\dagger = (\operatorname{gr}\mathcal{J} + I(S))/I(S)$, where $I(S)$ denotes the ideal in $S(\mathfrak{g}) \cong \mathbb{K}[\mathfrak{g}^*]$ consisting of all functions vanishing on S.*

We note that $\operatorname{gr}\mathcal{J}$ does not depend on whether we consider the Kazhdan filtration or the standard one. This stems easily from the observation that $\mathcal{J}$ is $\operatorname{ad}(h')$-stable.

Lemma 3.4.3. *Any closed $\hbar$-saturated $\mathbb{K}^\times$-stable ideal $\overline{\mathcal{J}}_\hbar$ of the quantum algebra $\mathbb{K}[[\mathfrak{v}_2^*,\hbar]]$ has the form $\mathbb{K}[[V^*,\hbar]]\widehat{\otimes}_{\mathbb{K}[[\hbar]]}\overline{\mathcal{I}}_\hbar$ for $\overline{\mathcal{I}}_\hbar := \overline{\mathcal{J}}_\hbar \cap \mathbb{K}[S]^\wedge_\chi[[\hbar]]$.*

Proof. Indeed, fix an element $a = \sum_{\mathbf{i},\mathbf{j}} p^{*\mathbf{i}} * q^{*\mathbf{j}}a_{\mathbf{i},\mathbf{j}} \in \overline{\mathcal{J}}_\hbar$. Note that

$$\hbar^{-2}p_l*(q_l*a-a*q_l) = \sum_{\mathbf{i},\mathbf{j}} i_l p^{*\mathbf{i}}*q^{*\mathbf{j}}a_{\mathbf{i},\mathbf{j}},\ \hbar^{-2}q_k*(p_k*a-a*p_k) = \sum_{\mathbf{i},\mathbf{j}} j_l p^{*\mathbf{i}}*q^{*\mathbf{j}}a_{\mathbf{i},\mathbf{j}}.$$

Since $\overline{\mathcal{J}}_\hbar$ is closed and $\hbar$-saturated, we see that $p^{\mathbf{i}}*q^{\mathbf{j}}a_{\mathbf{i},\mathbf{j}} \in \overline{\mathcal{J}}_\hbar$. Now it is easy to see that $a_{\mathbf{i},\mathbf{j}} \in \overline{\mathcal{J}}_\hbar$, hence the claim. □

Proof of Proposition 3.4.2. For $\overline{\mathcal{J}}_\hbar$ take the closure of $\Phi_\hbar(R_\hbar(\mathcal{J}))$ in $\mathbb{K}[[\mathfrak{v}_2^*,\hbar]]$. Let $J^\wedge \subset \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi, I^\wedge \subset \mathbb{K}[S]^\wedge_\chi$ denote the classical parts of $\overline{\mathcal{J}}_\hbar, \overline{\mathcal{I}}_\hbar$. Clearly, $J^\wedge = \mathbb{K}[[V^*]]\widehat{\otimes} I^\wedge$. It follows that $I^\wedge = (J^\wedge + I(S)^\wedge)/I(S)^\wedge$, where $I(S)^\wedge := V\mathbb{K}[[\mathfrak{v}_2^*]]$. Note that $\operatorname{gr}\mathcal{J}_\dagger$ is dense in $I^\wedge$ and $\operatorname{gr}\mathcal{J}$ is dense in $J^\wedge$. Therefore $(\operatorname{gr}\mathcal{J}+I(S))/I(S)$ is dense in $I^\wedge$ too. Analogously to assertion 3 of Proposition 3.2.10, we get $\operatorname{gr}\mathcal{J}_\dagger = (\operatorname{gr}\mathcal{J} + I(S))/I(S)$. □

Now let us relate the maps $\mathcal{I} \mapsto \mathcal{I}^\dagger, \mathcal{J} \mapsto \mathcal{J}_\dagger$.

Proposition 3.4.4. *$\mathcal{I} \supset (\mathcal{I}^\dagger)_\dagger$ and $\mathcal{J} \subset (\mathcal{J}_\dagger)^\dagger$ for any $\mathcal{I} \in \mathfrak{Id}(\mathcal{W}), \mathcal{J} \in \mathfrak{Id}(\mathcal{U})$.*

Proof. Thanks to Proposition 3.4.1, we need to prove that $\mathcal{I} \supset (\mathcal{I}^\ddagger)_\dagger, \mathcal{J} \subset (\mathcal{J}_\dagger)^\ddagger$. The first inclusion stems from assertion 3 of Proposition 3.2.10, while the second one follows from Lemma 3.4.3. □

Proof of assertions (v),(vi). Assertion (v) follows from Proposition 3.4.2 and the first inclusion of Proposition 3.4.4. To prove assertion (vi) (also proved in [Pr2], Theorem 3.1) consider a faithful finite-dimensional $\mathcal{W}/\mathcal{I}$-module M. By Proposition 3.3.5, $\operatorname{Dim}_\mathcal{U}(\mathcal{S}(M)) = \dim \underline{\mathfrak{m}}$. By [J1], 10.7 and assertion (ii), $\dim \operatorname{V}(\mathcal{I}^\dagger) \leqslant 2\operatorname{Dim}\mathcal{S}(M) = \dim G\chi$. □

Corollary 3.4.5. *Let M be a finitely generated $\mathcal{W}$-module. Then*

$$2\operatorname{Dim}_{\mathcal{W}} M \geqslant \operatorname{Dim}_{\mathcal{W}}(\mathcal{W}/\operatorname{Ann}(M)).$$

Proof. By Proposition 3.3.5, $\operatorname{Dim}_{\mathcal{U}}\mathcal{S}(M) = \operatorname{Dim}_{\mathcal{W}} M + \dim\mathfrak{m}$. Assertion (iii) of Theorem 1.2.1 implies that $\operatorname{Ann}(\mathcal{S}(M)) = \operatorname{Ann}(M)^\dagger$. By assertion (v), $\operatorname{Dim}_{\mathcal{U}}\mathcal{U}/\operatorname{Ann}(M)^\dagger \geqslant \operatorname{Dim}_{\mathcal{W}}(\mathcal{W}/\operatorname{Ann}(M)) + \dim V$. Finally, we apply the fact that $2\operatorname{Dim}_{\mathcal{U}}(\mathcal{S}(M)) \geqslant \operatorname{Dim}_{\mathcal{U}}(\mathcal{U}/\operatorname{Ann}(\mathcal{S}(M)))$; see [J1], 10.7. □

Proof of assertions (vii),(viii). By assertion (v), $G\chi \subset \mathrm{V}(\mathcal{I}^\dagger)$ for any $\mathcal{I} \in \operatorname{Pr}(\mathcal{W})$. Now let $\mathcal{J} \in \operatorname{Pr}(\mathcal{U})$ be such that $G\chi \subset \mathrm{V}(\mathcal{J})$. By Proposition 3.4.4, the equality $\mathcal{J} = \mathcal{I}^\dagger$ implies $\mathcal{J}_\dagger \subset \mathcal{I}$.

The Slodowy slice S intersects transversally any G-orbit, whose closure contains $G\chi$. Moreover, the description of the Poisson structure on S given in [GG], Section 3, implies that any Poisson subvariety S_0 of S is the union of irreducible components of the intersections of S with G-stable subvarieties of $\mathfrak{g}^*$. Therefore for any such S_0 we have the equality $\dim\overline{GS_0} = \dim S_0 + \dim G\chi$.

Let $\mathcal{I}$ be a prime ideal of $\mathcal{W}$ containing $\mathcal{J}_\dagger$. Then $\mathrm{V}(\mathcal{J}_\dagger) \supset \mathrm{V}(\mathcal{I})$. By Proposition 3.4.2, $\mathrm{V}(\mathcal{J}_\dagger) = \mathrm{V}(\mathcal{J}) \cap S$. Therefore $\dim\mathrm{V}(\mathcal{I}) \leqslant \dim\mathrm{V}(\mathcal{J}_\dagger) = \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$.

Now let $\mathcal{I}$ be an admissible element of $\operatorname{Pr}(\mathcal{W})$ such that $\mathcal{I}^\dagger = \mathcal{J}$. Then $\dim\mathrm{V}(\mathcal{J}) = \dim\mathrm{V}(I^\dagger) = \dim\mathrm{V}(\mathcal{I}) + \dim G\chi$. Let us check that $\mathcal{I}$ is a minimal prime ideal containing $\mathcal{J}_\dagger$. Assume the converse; i.e., assume there is $\mathcal{I}_0 \in \operatorname{Pr}(\mathcal{W})$ with $\mathcal{I} \supset \mathcal{I}_0 \supset \mathcal{J}_\dagger$. Then $\mathcal{I}_0^\dagger \subset \mathcal{I}^\dagger = \mathcal{J}$ and $\dim\mathrm{V}(\mathcal{J}_\dagger) \geqslant \dim\mathrm{V}(\mathcal{I}_0) \geqslant \dim\mathrm{V}(\mathcal{I})$; hence $\dim\mathrm{V}(\mathcal{I}_0) = \dim\mathrm{V}(\mathcal{I})$. Applying [BoKr], Corollary 3.6, we see that $\mathcal{I}_0 = \mathcal{I}$.

Conversely, let $\mathcal{I}$ be a minimal prime ideal of $\mathcal{W}$ containing $\mathcal{J}_\dagger$ and such that $\dim\mathrm{V}(\mathcal{I}) \geqslant \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$. Let us show that $\mathcal{I}$ is admissible and $\mathcal{I}^\dagger = \mathcal{J}$. First of all, let us check that $\mathcal{J} = (\mathcal{J}_\dagger)^\dagger$. Indeed, by Proposition 3.4.4, $\mathcal{J} \subset (\mathcal{J}_\dagger)^\dagger$. On the other hand, $\dim\mathrm{V}(\mathcal{J}) = \dim\mathrm{V}(\mathcal{J}_\dagger) + \dim G\chi \leqslant \dim\mathrm{V}((\mathcal{J}_\dagger)^\dagger)$. By Corollary 3.6 from [BoKr], $\mathcal{J} = (\mathcal{J}_\dagger)^\dagger$. Therefore $\mathcal{J} \subset \mathcal{I}^\dagger$. Since $\dim\mathrm{V}(\mathcal{I}) \geqslant \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$, we see that $\dim\mathrm{V}(\mathcal{J}) \geqslant \dim\mathrm{V}(\mathcal{I}^\dagger) \geqslant \dim\mathrm{V}(\mathcal{I}) + \dim G\chi \geqslant \dim\mathrm{V}(\mathcal{J})$. Applying [BoKr], Corollary 3.6, again, we get $\mathcal{I}^\dagger = \mathcal{J}$.

To complete the proof we need to check that there is $\mathcal{I} \in \operatorname{Pr}(\mathcal{W})$ with $\mathcal{I} \supset \mathcal{J}_\dagger, \dim\mathrm{V}(\mathcal{I}) = \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$. Let $\mathcal{I}_1, \ldots, \mathcal{I}_k$ be all minimal prime ideals of $\mathcal{W}$ containing $\mathcal{J}_\dagger$. Then $\sqrt{\mathcal{J}_\dagger} = \bigcap_{i=1}^k \mathcal{I}_i$. It follows that $\mathrm{V}(J_\dagger) = \bigcup_{i=1}^k \mathrm{V}(\mathcal{I}_i)$. Since $\dim\mathrm{V}(\mathcal{J}_\dagger) = \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$, we see that $\dim\mathrm{V}(\mathcal{I}_j) = \dim\mathrm{V}(\mathcal{J}) - \dim G\chi$ for some j. □

The following proposition will be used in the proof of assertion (ix). In particular, it proves Conjecture 3.1 (3) from [Pr2].

Proposition 3.4.6. *Let $\mathcal{I} \subset \mathcal{W}$ be a primitive (or prime) ideal of finite codimension. Then* $\operatorname{Grk}(\mathcal{U}/\mathcal{I}^\dagger) \leqslant \operatorname{Grk}(\mathcal{W}/\mathcal{I}) = (\dim\mathcal{W}/\mathcal{I})^{1/2}$.

Proof. Clearly, $\mathcal{W}/\mathcal{I}$ is a matrix algebra; hence $\operatorname{Grk}(\mathcal{W}/\mathcal{I}) = (\dim\mathcal{W}/\mathcal{I})^{1/2}$. Suppose we have found some Noetherian domain $\underline{\mathcal{A}}'$ such that there exists an embedding $\mathcal{U}/\mathcal{I}^\dagger \hookrightarrow \underline{\mathcal{A}}' \otimes \mathcal{W}/\mathcal{I}$. The Goldie rank of the last algebra coincides with that of $\mathcal{W}/\mathcal{I}$; see, for example, [McCR], Example 2.11(iii). Applying Theorem 1 from [W], we get $\operatorname{Grk}(\mathcal{U}/\mathcal{I}^\dagger) \leqslant \operatorname{Grk}(\underline{\mathcal{A}}' \otimes \mathcal{W}/\mathcal{I})$. So the goal is to construct $\underline{\mathcal{A}}'$.

Set $\underline{A} := S(V), \underline{\mathcal{A}} := \mathbf{A}_V$ so that $\mathcal{A}_2 = \underline{\mathcal{A}} \otimes \mathcal{W}$. Construct $\underline{\mathcal{A}}^\heartsuit$ from $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}}^\wedge$ from $\underline{\mathcal{A}}, \underline{\mathfrak{m}}$ as in Subsection 3.2. By Proposition 3.2.6, $\underline{\mathcal{A}}^\heartsuit$ is a Noetherian domain.

Recall that Φ induces the embedding $\mathcal{U} \hookrightarrow \mathcal{A}_2^\heartsuit$ and $\mathcal{I}^\dagger = \mathcal{U} \cap \Phi^{-1}(\mathbf{A}(\mathcal{I})^\wedge)$. So we need to check that $\mathcal{A}_2^\heartsuit/(\mathcal{A}_2^\heartsuit \cap \mathbf{A}(\mathcal{I})^\wedge) \cong \underline{\mathcal{A}}^\heartsuit \otimes \mathcal{W}/\mathcal{I}$.

An element (3.2) lies in $\mathcal{A}_2^\heartsuit$ iff there is a c with

$$a_{\mathbf{i},\mathbf{j}} \in \mathrm{F}_{c-\sum_{l=1}^k (i_l e_l' + j_l e_l)} \mathcal{W}.$$

We need to check that the image of $\mathcal{A}_2^\heartsuit$ under the natural epimorphism $\rho : \mathcal{A}_2^\wedge \to \mathcal{A}_2^\wedge/\mathbf{A}(\mathcal{I})^\wedge \cong \underline{\mathcal{A}}^\wedge \otimes \mathcal{W}/\mathcal{I}$ coincides with $\underline{\mathcal{A}}^\heartsuit \otimes \mathcal{W}/\mathcal{I}$. Recall that the filtration on $\mathcal{W}$ is nonnegative. Therefore the sum $\sum_{l=1}^k (i_l e_l' + j_l e_l)$ with $a_{\mathbf{i},\mathbf{j}} \neq 0$ is bounded from above for any $a \in \mathcal{A}_2^\heartsuit$. Hence the inclusion $\rho(\mathcal{A}_2^\heartsuit) \subset \underline{\mathcal{A}}^\heartsuit \otimes \mathcal{W}/\mathcal{I}$. To prove the opposite inclusion note that there is $n \in \mathbb{N}$ with $\mathrm{F}_n \mathcal{W} + \mathcal{I} = \mathcal{W}$. So for any $b \in \underline{\mathcal{A}}^\heartsuit \otimes \mathcal{W}/\mathcal{I}$ the inverse image $\rho^{-1}(b)$ contains an element of "finite degree", that is, from $\mathcal{A}_2^\heartsuit$. □

Proof of assertion (ix). By Proposition 3.4.2, $\mathrm{mult}(\mathcal{J}) = \mathrm{codim}_{\mathcal{W}} \mathcal{J}_\dagger$. The inverse image of $\mathcal{J}$ in $\mathrm{Pr}^a(\mathcal{W})$ consists of all prime ideals $\mathcal{I}_1, \ldots, \mathcal{I}_k$ of $\mathcal{W}$ containing $\mathcal{J}_\dagger$ (the minimality condition and the dimension estimate hold automatically). By Proposition 3.4.6, $\mathrm{codim}_{\mathcal{W}} \mathcal{I}_j \geqslant \mathrm{Grk}(\mathcal{U}/\mathcal{J})^2$. Since, obviously, $\mathrm{codim}_{\mathcal{W}} \mathcal{J}_\dagger \geqslant \sum_{i=1}^k \mathrm{codim}_{\mathcal{W}} \mathcal{I}_j$, we are done. □

3.5. **Finite-dimensional representations.** To prove Theorem 1.2.3 we need the following construction.

Let $\mathcal{A}$ be a simple filtered associative algebra equipped with an action of G by filtration-preserving automorphisms. Assume, in addition, that $\mathrm{gr}\,\mathcal{A}$ is finitely generated. For example, $\mathcal{A} = \widetilde{\mathcal{W}}$ satisfies these conditions. Let V be an $\mathcal{A}^G$-module. Consider the $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathcal{A}^G} V$. Since $\mathcal{A}$ is simple, this module is faithful. So we have the faithful G-invariant representation of $\mathcal{A}^G$ in $\mathcal{A} \otimes_{\mathcal{A}^G} V = \bigoplus_\lambda \mathcal{A}_\lambda \otimes_{\mathcal{A}^G} V$.

Proof of Theorem 1.2.3. There is $\mathcal{J} \in \mathrm{Pr}(\mathcal{U})$ with $\mathrm{V}(\mathcal{J}) = \overline{G\chi}$; see [J2], 9.12, for references. From assertion (viii) of Theorem 1.2.2 we deduce that $\mathcal{W}$ has a nontrivial finite-dimensional irreducible module, say V (this result was obtained previously by Premet, [Pr3]).

Now suppose $\mathfrak{g}$ is classical. We will see in the next subsection that there is an ideal $\mathcal{J} \subset \mathcal{U}$ s.t. $\overline{G\chi}$ is a component of $\mathrm{V}(\mathcal{J})$ and $\mathrm{mult}_{\overline{G\chi}} \mathcal{J} = 1$. Now the existence of an ideal of codimension 1 (and so existence of a one-dimensional $\mathcal{W}$-module) follows from Proposition 3.4.2 applied to $\mathcal{J}$.

Proceed to assertion (2). It follows from the discussion preceding the proof that it is enough to show that there is a G-equivariant isomorphism of right $\mathcal{W}$-modules $\widetilde{\mathcal{W}}_\lambda, \mathbb{K}[G]_\lambda \otimes \mathcal{W}$. Since $\mathbb{K}[X] = \mathbb{K}[G] \otimes \mathbb{K}[S]$, we get $\mathbb{K}[X]_\lambda = \mathbb{K}[G]_\lambda \otimes \mathbb{K}[S]$. The previous equality gives rise to a $G \times \mathbb{K}^\times$-equivariant embedding $\mathbb{K}[G]_\lambda \hookrightarrow \mathbb{K}[X]_\lambda$. There is a G-equivariant embedding $\mathbb{K}[G]_\lambda \hookrightarrow \widetilde{\mathcal{W}}_\lambda$ lifting $\mathbb{K}[G]_\lambda \hookrightarrow \mathbb{K}[X]_\lambda$. The embedding $\mathbb{K}[G]_\lambda \hookrightarrow \widetilde{\mathcal{W}}_\lambda$ is extended to a G-equivariant map $\psi : \mathbb{K}[G]_\lambda \otimes \mathcal{W} \to \widetilde{\mathcal{W}}_\lambda$. The map $\mathrm{gr}\,\psi$ is an isomorphism. Since the filtrations on both $\mathbb{K}[G]_\lambda \otimes \mathcal{W}, \widetilde{\mathcal{W}}$ are bounded from below, we see that ψ is an isomorphism. □

The following conjecture implies the existence of a 1-dimensional G-module without restrictions on $\mathfrak{g}$.

Conjecture 3.5.1. *Let* $\mathfrak{g}$ *be exceptional. Then there is* $\mathcal{J} \in \mathrm{Pr}(\mathcal{U})$ *with* $\mathrm{V}(\mathcal{J}) = \overline{G\chi}$ *and* $\mathrm{mult}(\mathcal{J}) = 1$.

3.6. **Existence of an ideal of multiplicity 1.** Below $\mathfrak{g}=\mathfrak{sl}_n,\mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$. Set $G=\mathrm{SL}_n,\mathrm{O}_n,\mathrm{Sp}_{2n}$ for $\mathfrak{g}=\mathfrak{sl}_n,\mathfrak{so}_n,\mathfrak{sp}_{2n}$, respectively (we remark that for $\mathfrak{g}=\mathfrak{so}_n$ we need a disconnected group).

To complete the proof of Theorem 1.2.3 it remains to check that there is a primitive ideal $\mathcal{J}\subset\mathcal{U}$ such that $\overline{G^\circ\chi}$ is a component in $\mathrm{V}(\mathcal{J})$ and $\mathrm{mult}_{\overline{G^\circ\chi}}(\mathcal{J})=1$. We will show more, namely that there is a $\mathcal{J}$ with $\mathrm{gr}\,\mathcal{U}/\mathcal{J}=\mathbb{K}[\overline{G\chi}]$ (here we consider the filtration on $\mathcal{U}/\mathcal{J}$ coming from the PBW filtration).

To do this we will need to recall a construction of Kraft and Procesi, [KP1],[KP2].

Let $\widetilde{V}$ be a symplectic vector space, $\widetilde{G}$ a reductive group acting on $\widetilde{V}$ by linear symplectomorphisms, and let $\mathbf{A}$ be the Weyl algebra of $\widetilde{V}^*$ equipped with the standard filtration $\mathrm{F}_i\,\mathbf{A}$. Let ω denote the symplectic form on $\widetilde{V}$. There is a moment map $\widetilde{\mu}:\widetilde{V}\to\widetilde{\mathfrak{g}}^*$ given by $\langle\widetilde{\mu}(v),\xi\rangle=\frac{1}{2}\omega(\xi v,v),\xi\in\widetilde{\mathfrak{g}},v\in\widetilde{V}$. The corresponding comoment map $\widetilde{\mathfrak{g}}\to\mathbb{K}[\widetilde{V}],\xi\mapsto H_\xi$, is the composition of the homomorphism $\widetilde{\mathfrak{g}}\to\mathfrak{sp}(\widetilde{V})$ corresponding to the action $\widetilde{G}:\widetilde{V}$ and a natural identification $\mathfrak{sp}(\widetilde{V})\cong S^2(\widetilde{V}^*)\subset\mathbb{K}[\widetilde{V}]$. There is a unique $\mathrm{Sp}(\widetilde{V})$-equivariant embedding $S^2(\widetilde{V}^*)\hookrightarrow\mathbf{A}$ lifting the natural embedding $S^2(\widetilde{V}^*)\subset\mathbb{K}[\widetilde{V}]$. The corresponding map $\widetilde{\mathfrak{g}}\to\mathbf{A},\xi\mapsto\widehat{H}_\xi$, is a quantum comoment map; i.e., $[\widehat{H}_\xi,f]=\xi.f$ for any $f\in\mathbf{A}$, where in the r.h.s. ξ denotes the derivation induced by the G-action.

In our situation, $\widetilde{G}:=G\times\widetilde{G}_0$, where $\widetilde{G}_0$ is a certain reductive group. Kraft and Procesi essentially constructed (in [KP1], Section 3, for $\mathfrak{g}=\mathfrak{sl}_n$ and in [KP2], Section 5, for $\mathfrak{g}=\mathfrak{so}_n,\mathfrak{sp}_{2n}$) a symplectic vector space $\widetilde{V}$ equipped with a $\widetilde{G}$-action (both depend on χ) having the following properties (here $\mu,\widetilde{\mu}_0$ are the moment maps for the G- and $\widetilde{G}_0$-action, so $\widetilde{\mu}=(\mu,\widetilde{\mu}_0)$):

- (A) As a scheme, $\widetilde{\mu}_0^{-1}(0)$ is reduced of dimension $\dim\widetilde{V}-\dim\widetilde{\mathfrak{g}}_0$.
- (B) The image of the restriction of μ to $\widetilde{\mu}_0^{-1}(0)$ coincides with $\overline{G\chi}$. The induced morphism $\widetilde{\mu}_0^{-1}(0)\to\overline{G\chi}$ is the quotient morphism for the $\widetilde{G}_0$-action.

The quantum comoment map $\widetilde{\mathfrak{g}}\to\mathbf{A}$ gives rise to an algebra homomorphism

$$\mathcal{U}\to(\mathbf{A}/\mathbf{A}\widetilde{\mathfrak{g}}_0)^{\widetilde{\mathfrak{g}}_0}.$$

Here we embed $\widetilde{\mathfrak{g}}_0$ into $\mathbf{A}$ by means of the quantum comoment map. We are going to check that this homomorphism is surjective and that its kernel $\mathcal{J}$ has the required properties. To do this we need the following lemma, which seems to be pretty standard.

Lemma 3.6.1. *Let $\mathcal{I}$ denote the left ideal in $\mathbf{A}$ generated by $\widehat{H}_\xi,\xi\in\widetilde{\mathfrak{g}}_0$. Then $\mathrm{gr}\,\mathcal{I}$ is generated by $H_\xi,\xi\in\widetilde{\mathfrak{g}}_0$.*

Proof. Choose a basis $\xi_1,\ldots,\xi_n$ in $\widetilde{\mathfrak{g}}_0$. Consider the free module $\mathbb{K}[\widetilde{V}]^{\oplus n}$ and let $e_1,\ldots,e_n$ denote its tautological basis. Consider the $\mathbb{K}[\widetilde{V}]$-module homomorphism $\theta:\mathbb{K}[\widetilde{V}]^{\oplus n}\to\mathbb{K}[\widetilde{V}]$ given by $e_i\mapsto H_{\xi_i},i=1,\ldots,n$. It follows from (A) that the H_{ξ_i} form a regular sequence in $\mathbb{K}[\widetilde{V}]$. In particular, $\ker\theta$ is generated by the elements $\Theta_{ij}:=H_{\xi_i}e_j-H_{\xi_j}e_i$.

We need to check that if $\widehat{f}_1,\ldots,\widehat{f}_n\in\mathrm{F}_l\,\mathbf{A}$ are such that $\sum_{i=1}^n\widehat{f}_i\widehat{H}_{\xi_i}\in\mathrm{F}_k\,\mathbf{A}$ for $k<l+2$, then there are $\widehat{f}'_i\in\mathrm{F}_{k-2}\,\mathbf{A}$ with $\sum_{i=1}^n\widehat{f}_i\widehat{H}_{\xi_i}=\sum_{i=1}^n\widehat{f}'_i\widehat{H}_{\xi_i}$. Let f_i denote the image of $\widehat{f}_i$ in $S^l\widetilde{V}^*$. Then $\sum_{i=1}^n f_iH_{\xi_i}=0$. It follows that $\sum_{i=1}^n f_ie_i\in\ker\theta$, so there are $g_{ij}\in S^{l-2}\widetilde{V}^*$ such that $\sum_{i=1}^n f_ie_i=\sum_{ij}g_{ij}\Theta_{ij}$. Let $\widehat{\theta}$ denote the homomorphism $\mathbf{A}^{\oplus n}\to\mathbf{A}$ of left $\mathbf{A}$-modules given by $e_i\mapsto\widehat{H}_{\xi_i}$ and set $\widehat{\Theta}_{ij}:=$

$\widehat{H}_{\xi_i}e_j-\widehat{H}_{\xi_j}e_i$. Lift g_{ij} to some elements $\widehat{g}_{ij}\in \mathrm{F}_{l-2}\,\mathbf{A}$. Then the vector $\sum_{i=1}^n \widehat{f}_i e_i - \sum_{i,j}\widehat{g}_{ij}\widehat{\Theta}_{ij}$ maps to $\sum_{i=1}^n \widehat{f}_i\widehat{H}_{\xi_i}$ but all components of this vector lie in $\mathrm{F}_{l-1}\,\mathbf{A}$. Repeating this procedure, we get our claim. □

It follows from the lemma that $\operatorname{gr}\mathbf{A}/\mathcal{I}=\mathbb{K}[\widetilde{\mu}_0^{-1}(0)]$. Since the group $\widetilde{G}_0$ is reductive, we see that $\operatorname{gr}\left((\mathbf{A}/\mathcal{I})^{\widetilde{\mathfrak{g}}_0}\right)=\mathbb{K}[\widetilde{\mu}_0^{-1}(0)]^{\widetilde{\mathfrak{g}}_0}$. From (B) it follows that the natural homomorphism $\mathcal{U}\to(\mathbf{A}/\mathcal{I})^{\widetilde{\mathfrak{g}}_0}$ is surjective and its kernel $\mathcal{J}$ has the required properties.

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