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ACTIONS OF $\mathbb{F}_\infty$ WHOSE II_1 FACTORS AND ORBIT EQUIVALENCE RELATIONS HAVE PRESCRIBED FUNDAMENTAL GROUP

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1. Introduction

There has recently been increasing interest in the study of rigidity properties of II_1 factors $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ and orbit equivalence relations $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$, arising from ergodic measure-preserving (m.p.) actions $\Gamma \curvearrowright X$ of countable non-amenable groups Γ on probability measure spaces (X, μ), via the *group measure space construction* of Murray and von Neumann ([23]) and its generalization in [6]. Some of the most intriguing phenomena concern the *fundamental group* $\mathcal{F}(M)$, $\mathcal{F}(\mathcal{R})$ of such II_1 factors and equivalence relations ([24]). While a lot of progress has been made in this direction, several basic questions on how these invariants depend on the isomorphism class of the group Γ and on the nature of the action $\Gamma \curvearrowright X$ remained open. Many of them are variations of Murray-von Neumann's long-standing question, on what subgroups of $\mathbb{R}_+$ may occur as fundamental groups of II_1 factors ([24]).

For instance, while any countable subgroup $\mathcal{F} \subset \mathbb{R}_+$ was shown to occur as the fundamental group of II_1 factors and equivalence relations arising from *non-free* ergodic m.p. actions in [29], the problem of whether uncountable groups $\neq \mathbb{R}_+$ can appear remained wide open. If in addition the action $\Gamma \curvearrowright X$ is required to be *free*, then much less is known: the only groups shown to appear as $\mathcal{F}(M), \mathcal{F}(\mathcal{R})$ were $\mathbb{R}_+$ itself ([24]) and the trivial group $\{1\}$ ([13], [9], [10], [7], for equivalence relations; respectively [28], [29], [30], [17] for II_1 factors). It has been speculated that $\Gamma \curvearrowright X$ free may in fact force $\mathcal{F}(M), \mathcal{F}(\mathcal{R})$ to be rational, when $\neq \mathbb{R}_+$. The case $\Gamma = \mathbb{F}_\infty$ is of particular interest, due to its universality properties and to the fact that, while $\mathcal{F}(\mathcal{R}) = \{1\}$ for any free ergodic m.p. action $\mathbb{F}_n \curvearrowright X$, $2 \leq n < \infty$ (by [9], [10]) and while some classes of $\mathbb{F}_n$-actions are known to produce factors M with $\mathcal{F}(M) = \{1\}$ ([28], [26]), there was not one single case of a free ergodic $\mathbb{F}_\infty$-action for which $\mathcal{F}(\mathcal{R}), \mathcal{F}(M)$ could be calculated.

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We solve these problems here by exhibiting a large family $\mathcal{S}$ of subgroups of $\mathbb{R}_+$, containing $\mathbb{R}_+$ itself, all of its countable subgroups, as well as uncountable subgroups with any Hausdorff dimension $\alpha \in (0,1)$, such that for each $\mathcal{F} \in \mathcal{S}$ there exist "many" free ergodic m.p. actions of $\mathbb{F}_\infty$ for which both the associated II_1 factor M and orbit equivalence relation $\mathcal{R}$ have fundamental group equal to $\mathcal{F}$. Moreover, we construct these actions so that we also have a good control on the *outer automorphism group* of M and $\mathcal{R}$. In particular, we obtain the first examples of II_1 factors with separable predual (resp. standard countable m.p. ergodic equivalence relations) having uncountable fundamental group different from $\mathbb{R}_+$.

The description of the set $\mathcal{S}$ depends on results and notation from [1], [21], which we recall for convenience. Thus, following [1] and [21], we call *ergodic measure on* $\mathbb{R}$ any σ-finite measure ν on the Borel sets of $\mathbb{R}$ satisfying the following properties, where we denote $\lambda_x(y) = x + y$.

- For all $x \in \mathbb{R}$, either $\nu \circ \lambda_x = \nu$ or $\nu \circ \lambda_x \perp \nu$.
- There exists a countable subgroup $Q \subset \mathbb{R}$ such that $\nu \circ \lambda_x = \nu$ for all $x \in Q$ and such that every Q-invariant Borel function on $\mathbb{R}$ is ν-almost everywhere constant.

For every ergodic measure ν on $\mathbb{R}$, one defines

$$H_\nu := \{x \in \mathbb{R} \mid \nu \circ \lambda_x = \nu\} .$$

The obvious ergodic measures ν are the Lebesgue measure, with $H_\nu = \mathbb{R}$, and the counting measure on a countable subgroup $H \subset \mathbb{R}$, having $H_\nu = H$. But, in [1, Section 4], ergodic measures ν are constructed such that H_ν is an uncountable subgroup of $\mathbb{R}$ with prescribed Hausdorff dimension $\alpha \in (0,1)$. We will recall this construction in the preliminary section of the paper.

The family $\mathcal{S}$ of subgroups $\mathcal{F}$ of $\mathbb{R}_+$ for which we can construct free ergodic m.p. actions of $\mathbb{F}_\infty$ whose II_1 factors and equivalence relations have fundamental group $\mathcal{F}$ will consist of all subgroups of $\mathbb{R}_+$ of the form $\exp(H_\nu)$. Thus, we prove:

Theorem 1.1. *Let ν be an ergodic measure on $\mathbb{R}$ and $\mathcal{G}$ any totally disconnected unimodular locally compact group. There exists a continuous family $(\sigma_i)_{i\in I}$ of free ergodic m.p. actions of the free group $\mathbb{F}_\infty$ on probability measure spaces, $\sigma_i : \mathbb{F}_\infty \curvearrowright (X_i, \mu_i)$, such that the associated orbit equivalence relations $\mathcal{R}_i = \mathcal{R}(\mathbb{F}_\infty \overset{\sigma_i}{\curvearrowright} X_i)$ and II_1 factors $M_i = \mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \mathbb{F}_\infty$ have the following properties.*

- *The fundamental groups are given by $\mathcal{F}(\mathcal{R}_i) = \mathcal{F}(M_i) = \exp(H_\nu)$.*
- *The outer automorphism groups are given by $\mathrm{Out}(\mathcal{R}_i) \cong \mathcal{G}$ and $\mathrm{Out}(M_i) \cong \mathrm{H}^1(\sigma_i) \rtimes \mathcal{G}$. Here $\mathrm{H}^1(\sigma_i)$ denotes the 1-cohomology group of σ_i with values in $\mathbb{T}$ and corresponds to automorphisms of M_i that are the identity on $\mathrm{L}^\infty(X_i)$.*
- *The II_1 factors M_i are not stably isomorphic and, in particular, the equivalence relations $\mathcal{R}_i$ are not stably orbit equivalent.*

In particular, the above result provides a large class of free ergodic m.p. actions of $\mathbb{F}_\infty$ whose orbit equivalence relations $\mathcal{R}$ have trivial outer automorphism group, $\mathrm{Out}(\mathcal{R}) = \{1\}$. Previous examples of free ergodic m.p. group actions $\Gamma \curvearrowright X$ with $\mathrm{Out}(\mathcal{R}(\Gamma \curvearrowright X)) = \{1\}$ were constructed in [12], [8], [22], [17], [33], [37], but for groups Γ which either are higher rank lattices ([12], [8]), contain infinite subgroups with the relative property (T) ([17], [33], [37]) or are products of word-hyperbolic groups ([22]). It was an open problem on whether actions of free groups can have this property.

There is an interesting parallel between the above result, on groups that may occur as the "symmetry groups" $\mathcal{F}(M)$, $\mathrm{Out}(M)$ of II_1 factors $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ arising from free ergodic m.p. actions of $\mathbb{F}_\infty$, and the results in [32], showing that any group-like object that occurs in Jones' theory of subfactors can be realized as "generalized symmetries" of $M = \mathrm{L}(\mathbb{F}_\infty)$. Along these lines, we conjecture that any group $\mathcal{F} \subset \mathbb{R}_+$ that can be realized as the fundamental group of a separable II_1 factor M (respectively as the fundamental group of a countable, m.p. ergodic equivalence relation $\mathcal{R}$) can in fact be realized as the fundamental group of a II_1 factor of the form $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty$ (resp. equivalence relation $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X)$), for some appropriate free ergodic m.p. action $\mathbb{F}_\infty \curvearrowright X$.

The $\mathbb{F}_\infty$-actions σ_i in Theorem 1.1 are obtained as follows, in two steps. As a first step (Sections 3 and 4), given an arbitrary countable group K, we prove the existence of m.p. actions $\mathbb{F}_\infty * K \curvearrowright (X, \mu)$ having no symmetries in the following strong sense: any partial automorphism of X that sends $\mathbb{F}_\infty$-orbits into $(\mathbb{F}_\infty * K)$-orbits is inner, i.e. belongs to the full pseudogroup of the orbit equivalence relation $\mathcal{R}(\mathbb{F}_\infty * K \curvearrowright X)$. Moreover, the restriction of the action to $\mathbb{F}_\infty$ can be taken to be ergodic and rigid. We will however not explicitly construct such $(\mathbb{F}_\infty * K)$-actions, but prove their existence through a Baire category argument. In the second step (Section 5), we put $K = \mathbb{F}_\infty$ as well and take an action $\mathbb{F}_\infty * K \curvearrowright (X, \mu)$ as above. We consider the diagonal action $\mathbb{F}_\infty * K \curvearrowright X \times Y$, where $\mathbb{F}_\infty * K$ acts on Y in an infinite measure-preserving way, through a quotient morphism $\mathbb{F}_\infty * K \to \Lambda \curvearrowright Y$. In this way, we provide an explicit construction of a continuous family of infinite measure-preserving actions of $\mathbb{F}_\infty$ such that the associated II_∞ orbit equivalence relations $\mathcal{R}_i^\infty$ are non-isomorphic and have outer automorphism group $\exp(H_\nu) \times \mathcal{G}$. The precise statement is given in Theorem 5.4 (see also Remark 5.5 for a slight generalization). We show that the restriction of $\mathcal{R}_i^\infty$ to a subset of finite measure can be implemented by a free action σ_i of $\mathbb{F}_\infty$ (see Lemma 5.2 and Remark 5.3) and that the actions σ_i satisfy the part of Theorem 1.1 concerning equivalence relations. Since the actions σ_i that we construct will be $\mathcal{HT}$, in the sense of [28], the part of Theorem 1.1 concerning the associated II_1 factors will follow automatically from the corresponding orbit equivalence statements, by results in [28] (see Section 6).

2. Notation and preliminaries

The fundamental group $\mathcal{F}(M)$ of a II_1 factor M is the subgroup of $\mathbb{R}_+$ defined by

$$\mathcal{F}(M) = \{\tau(p)/\tau(q) \mid pMp \cong qMq\},$$

where p, q are non-zero orthogonal projections in M. Similarly, the fundamental group $\mathcal{F}(\mathcal{R})$ of a countable measure-preserving (m.p.) ergodic equivalence relation $\mathcal{R}$ on the standard probability measure space (X, μ) (as defined in [6], hereafter called *II_1 equivalence relation*) is given by

$$\mathcal{F}(\mathcal{R}) = \{\mu(Y)/\mu(Z) \mid \mathcal{R}|_Y \cong \mathcal{R}|_Z\} .$$

One should point out that, while the fundamental group $\mathcal{F}(M)$ is defined for any II_1 factor M, we consider in this paper only *separable* von Neumann factors, i.e. factors with separable predual (equivalently, factors acting on separable Hilbert spaces). For a II_1 factor, this is the same as requiring M to be separable in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, given by the (unique) normalized trace τ on M, and is also equivalent to M being countably generated (as a von Neumann algebra).

Thus, any II_1 factor M associated to a II_1 equivalence relation $\mathcal{R}$, via the generalized group measure space construction of Feldman-Moore [6], is separable, and one clearly has $\mathcal{F}(\mathcal{R}) \subset \mathcal{F}(M)$. This inclusion may be strict; in fact, there even exist free ergodic actions for which $\mathcal{F}(\mathcal{R}) = \{1\}$ while $\mathcal{F}(M) = \mathbb{R}_+$ (cf. 6.1 in [30], based on [5]).

Suppose now that $\Lambda \curvearrowright (X, \mu)$ is a free ergodic m.p. action on a probability space (X, μ). Define $(Y, \eta) = (X \times \mathbb{Z}, \mu \times c)$, where c denotes the counting measure. We set $\Lambda^\infty := \Lambda \times \mathbb{Z}$ acting freely on (Y, η) as the product of the action $\Lambda \curvearrowright X$ and the translation action $\mathbb{Z} \curvearrowright \mathbb{Z}$. Recall in this respect that a non-singular automorphism Δ of a measure space (Y, η) is called (essentially) *free* if the set $\{y \in Y \mid \Delta(y) = y\}$ has measure zero. Also, an action of a countable group Γ on (Y, η), by non-singular automorphisms, is free if each non-trivial element of the group implements a free automorphism.

- We denote by $\mathcal{R}(\Lambda \curvearrowright X)$ the orbit equivalence relation on X, i.e. $x \sim y$ if and only if there exists $g \in \Lambda$ with $y = g \cdot x$. We also consider the amplified equivalence relation $\mathcal{R}(\Lambda^\infty \curvearrowright Y)$ and note that $(x, n) \sim (y, m)$ if and only if $x = g \cdot y$ for some $g \in \Lambda$.
- We denote by $[\Lambda]$, resp. $[\Lambda^\infty]$ the group of non-singular automorphisms Δ of (X, μ), resp. (Y, η), satisfying $\Delta(x) \sim x$ almost everywhere. Note that the automorphisms in $[\Lambda]$ and $[\Lambda^\infty]$ are measure-preserving.
- We denote by $\operatorname{Aut}(\mathcal{R}(\Lambda \curvearrowright X))$ the group of non-singular automorphisms of (X, μ) satisfying $\Delta(x) \sim \Delta(y)$ if and only if $x \sim y$. We identify automorphisms that are equal almost everywhere. We similarly consider $\operatorname{Aut}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$. An automorphism $\Delta \in \operatorname{Aut}(\mathcal{R}(\Lambda \curvearrowright X))$ is automatically measure-preserving, while an automorphism $\Delta \in \operatorname{Aut}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$ automatically scales the measure η with a positive constant that we denote by $\operatorname{mod} \Delta$.
- Note that $[\Lambda]$ and $[\Lambda^\infty]$ are normal subgroups of $\operatorname{Aut}(\mathcal{R}(\Lambda \curvearrowright X))$ and $\operatorname{Aut}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$. We denote by $\operatorname{Out}(\mathcal{R}(\Lambda \curvearrowright X))$ and $\operatorname{Out}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$ the corresponding quotient groups.
- We identify a measure-preserving automorphism Δ of X with the automorphism $\Delta \times \mathrm{id}$ of Y.
- The maps $\Delta \mapsto \Delta \times \mathrm{id}$ and $\Delta \mapsto \operatorname{mod} \Delta$ induce an exact sequence

$$e \longrightarrow \operatorname{Out}(\mathcal{R}(\Lambda \curvearrowright X)) \longrightarrow \operatorname{Out}(\mathcal{R}(\Lambda^\infty \curvearrowright Y)) \longrightarrow \mathcal{F}(\mathcal{R}(\Lambda \curvearrowright X)) \longrightarrow e \; . \tag{2.1}$$

  In particular, triviality of $\operatorname{Out}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$ is equivalent with triviality of both the outer automorphism group $\operatorname{Out}(\mathcal{R}(\Lambda \curvearrowright X))$ and the fundamental group $\mathcal{F}(\mathcal{R}(\Lambda \curvearrowright X))$.
- Suppose that $\Gamma < \Lambda$ is a subgroup and $\Gamma \curvearrowright X$ is still ergodic. We define $\operatorname{Emb}(\Gamma^\infty, \Lambda^\infty)$ as the set of non-singular automorphisms Δ of Y such that for all $g \in \Gamma^\infty$ and almost all $y \in Y$, we have $\Delta(g \cdot y) \in \Lambda^\infty \cdot \Delta(y)$. Clearly, $\operatorname{Emb}(\Gamma^\infty, \Lambda^\infty)$ is stable under composition on the left by elements of $[\Lambda^\infty]$. We say that Δ is outer if $\Delta \not\in [\Lambda^\infty]$.

In the formulation of Theorem 1.1 and in the last section of this article, we use the 1-cohomology group $\mathrm{H}^1(\sigma)$ of a free ergodic m.p. action $\sigma : \Lambda \curvearrowright X$. Recall that $\mathrm{Z}^1(\sigma)$ is the abelian Polish group of functions $\omega : \Lambda \times X \to \mathbb{T}$ satisfying $\omega(gh, x) = \omega(g, \sigma_h(x))\omega(h, x)$ almost everywhere. The subgroup $\mathrm{B}^1(\sigma)$ consists of ω having the form $\omega(g, x) = \varphi(\sigma_g(x))\overline{\varphi(x)}$. The 1-cohomology group $\mathrm{H}^1(\sigma)$ is by definition the quotient of $\mathrm{Z}^1(\sigma)$ by $\mathrm{B}^1(\sigma)$ ([35], [6]). This $\mathrm{H}^1(\sigma)$ is a stable orbit equivalence invariant ([31], [6]).

Some families of subgroups of $\mathbb{R}_+$. Since Theorem 1.1 provides the first examples of II_1 factors and equivalence relations with uncountable fundamental group different from $\mathbb{R}_+$, we are interested in the following sets of subgroups of the positive real line:

$$\mathcal{S}_{\text{factor}} := \{\mathcal{F} \subset \mathbb{R}_+ \mid \text{there exists a separable } \mathrm{II}_1 \text{ factor } M \text{ with } \mathcal{F}(M) = \mathcal{F}\} ,$$
$$\mathcal{S}_{\text{equiv}} := \{\mathcal{F} \subset \mathbb{R}_+ \mid \text{there exists a } \mathrm{II}_1 \text{ equivalence relation } \mathcal{R} \text{ with } \mathcal{F}(\mathcal{R}) = \mathcal{F}\} .$$

A consequence of our main result says that both $\mathcal{S}_{\text{factor}}$ and $\mathcal{S}_{\text{equiv}}$ contain

$$\mathcal{S} := \{\exp(H_\nu) \mid \nu \text{ is an ergodic measure on } \mathbb{R}\} .$$

In fact, in the course of the proof of Theorem 5.4, we will see that

$$\mathcal{S} \subset \mathcal{S}_{\text{centr}} \subset \mathcal{S}_{\text{factor}} \cap \mathcal{S}_{\text{equiv}} , \tag{2.2}$$

where the set $\mathcal{S}_{\text{centr}}$ is defined as follows. Suppose that (Y, η) is a standard infinite measure space and $\Lambda \curvearrowright (Y, \eta)$ an ergodic, measure-preserving action. Define $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$ as the subgroup of $\mathrm{Aut}(Y, \eta)$ consisting of automorphisms Δ that commute with the Λ-action. If $\Delta \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$, then Δ automatically scales the measure η, and we denote the scaling constant by $\mathrm{mod}\, \Delta$. Define

$$\begin{aligned}\mathcal{S}_{\text{centr}} := \{\mathcal{F} \subset \mathbb{R}_+ \mid & \text{ there exists } \Lambda \curvearrowright (Y, \eta) \text{ free ergodic m.p. action,} \\ & \text{with } \Lambda \text{ amenable and } \mathrm{mod}(\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)) = \mathcal{F} \} .\end{aligned}$$

Proposition 2.1. *If $\mathcal{F} \in \mathcal{S}_{\text{factor}}$ or $\mathcal{F} \in \mathcal{S}_{\text{equiv}}$, then $\mathcal{F}$ is a Borel subset of $\mathbb{R}_+$ and a Polishable subgroup of $\mathbb{R}_+$: $\mathcal{F}$ carries a unique Polish group topology that is compatible with its Borel structure.*

Proof. The fundamental group $\mathcal{F}(M)$, resp. $\mathcal{F}(\mathcal{R})$, of an arbitrary separable II_1 factor M, resp. countable m.p. ergodic equivalence relation $\mathcal{R}$ on a standard probability space, appears in a short exact sequence similar to (2.1). More precisely, denote by $M^\infty := \mathrm{B}(\ell^2(\mathbb{N})) \overline{\otimes} M$ the amplified II_∞ factor. The automorphism group $\mathrm{Aut}(M^\infty)$ is a Polish group and consists of trace-scaling automorphisms. The natural homomorphism $\mathrm{mod} : \mathrm{Aut}(M^\infty) \to \mathbb{R}_+$ is continuous. Therefore, the trace-preserving automorphisms form a closed subgroup of $\mathrm{Aut}(M^\infty)$, and $\mathcal{F}(M)$ appears as the image of a Polish group under an injective, continuous homomorphism. So, by a theorem of Lusin and Souslin (see e.g. [19, Theorem 15.1]), the group $\mathcal{F}(M)$ is a Borel subset of $\mathbb{R}_+$. By construction, $\mathcal{F}(M)$ is Polishable. The same reasoning can be made for $\mathcal{S}_{\text{equiv}}$. □

Remark 2.2. Proposition 2.1 provides the only known a priori restriction on the elements of $\mathcal{S}_{\text{factor}}$ and $\mathcal{S}_{\text{equiv}}$, although we do not believe that all Polishable Borel subgroups of $\mathbb{R}_+$ belong to $\mathcal{S}_{\text{factor}}$ or $\mathcal{S}_{\text{equiv}}$. On the other hand, more properties are known for the groups $\mathcal{F} \in \mathcal{S}$, but the reason for this is rather indirect. For an arbitrary ergodic measure ν, a duality argument (see [1, Thm. 3.1]) shows that the group H_ν also arises as the *eigenvalue group of a non-singular ergodic flow :* there exists a non-singular, ergodic action $(\sigma_t)_{t \in \mathbb{R}}$ on the standard infinite measure space (Y, η) such that H_ν is exactly the group of $s \in \mathbb{R}$ for which there exists a non-zero $F \in \mathrm{L}^\infty(Y,\eta)$ satisfying $\sigma_t(F) = e^{its}F$ for all $t \in \mathbb{R}$. Observe that the set of eigenvalue groups of non-singular ergodic flows coincides with the set

$$\mathcal{T} = \{H \subset \mathbb{R} \mid \text{ there exists a factor } M \text{ with separable predual such that } T(M) = H\}\ .$$

Here, $T(M)$ denotes Connes' T-invariant of the factor M ([3]). So, for all $\mathcal{F} \in \mathcal{S}$, we have $\log(\mathcal{F}) \in \mathcal{T}$; cf. [14, Section I].

Apart from being Polishable Borel subgroups of $\mathbb{R}$, the following can be said about the elements of $\mathcal{T}$.

- Every $H \in \mathcal{T}$ is a *σ-compact* subset of $\mathbb{R}$. This follows from the weak* compactness of the unit ball of $\mathrm{L}^\infty(Y,\eta)$.
- Every $H \in \mathcal{T}$ is a *saturated subgroup of* $\mathbb{R}$. Following [16, Section 2.1], a Borel set and subgroup $H \subset \mathbb{R}$ is called saturated if every bounded signed measure $\mu \in \mathrm{M}(\mathbb{R})$ satisfies $|\mu(H)| \leq \sup\{|\widehat{\mu}(t)| \mid t \in \mathbb{R}\}$, where $\widehat{\mu}$ denotes the Fourier transform of μ.

Note in passing that the three properties of being Polishable, σ-compact (as a subset of $\mathbb{R}$) or saturated are rather independent. Section 2.4 in [16] provides subgroups of $\mathbb{R}$ that are not saturated but nevertheless σ-compact as a subset of $\mathbb{R}$. If $K \subset \mathbb{R}$ is a compact *Kronecker set*,[1] the subgroup generated by K is a σ-compact subset of $\mathbb{R}$, saturated (see [16, Section 2.1]), but not Polishable by a Baire category argument (see e.g. [20, Theorem 3] or [1, Remark 1.2]).

All this makes it somewhat premature to set forth a plausible conjecture on what should be an abstract characterization of the groups in $\mathcal{S}_{\text{factor}}$ and $\mathcal{S}_{\text{equiv}}$. Note however that all the groups in these classes that we get in this paper are realized as fundamental groups of II_1 factors and equivalence relations arising from free ergodic m.p. actions of $\mathbb{F}_\infty$. Denoting by $\mathcal{S}_{\text{factor}}(\mathbb{F}_\infty)$, $\mathcal{S}_{\text{equiv}}(\mathbb{F}_\infty)$ the set of subgroups $\mathcal{F} \subset \mathbb{R}_+$ for which there exists a free ergodic m.p. action $\mathbb{F}_\infty \curvearrowright X$ such that $\mathcal{F}(\mathrm{L}^\infty(X) \rtimes \mathbb{F}_\infty) = \mathcal{F}$, respectively $\mathcal{F}(\mathcal{R}(\mathbb{F}_\infty \curvearrowright X)) = \mathcal{F}$, it seems very likely that one actually has $\mathcal{S}_{\text{factor}} = \mathcal{S}_{\text{equiv}} = \mathcal{S}_{\text{factor}}(\mathbb{F}_\infty) = \mathcal{S}_{\text{equiv}}(\mathbb{F}_\infty)$. This would provide a new universality property of $\mathbb{F}_\infty$, to be compared with [32].

In exactly the same way, we associate the sets $\mathcal{S}_{\text{factor}}(\Gamma)$, $\mathcal{S}_{\text{equiv}}(\Gamma)$ to any given countable group Γ. These invariants capture interesting complexity aspects of the group Γ, which we investigate in the follow-up paper [34]. For now, let us point out that by [10], if Γ has at least one ℓ^2-Betti number not equal to 0 or ∞, then $\mathcal{S}_{\text{equiv}}(\Gamma)$ consists of the trivial group $\{1\}$ only. In particular, $\mathcal{S}_{\text{equiv}}(\mathbb{F}_n) = \{\{1\}\}$, for any finite $n \geq 2$. Also, one has $\{1\} \in \mathcal{S}_{\text{factor}}(\mathbb{F}_n)$ by [28]. We expect that in fact $\mathcal{S}_{\text{factor}}(\mathbb{F}_n) = \{\{1\}\}$ as well. On the other hand, it is shown in [27], [25], by using "separability arguments" inspired by Connes' pioneering rigidity result in [4], that if Γ is an infinite conjugacy class group with the property (T), then $\mathcal{S}_{\text{factor}}(\Gamma)$, $\mathcal{S}_{\text{equiv}}(\Gamma)$ consist of countable groups only.

Ergodic measures and Hausdorff dimension. Examples of groups in $\mathcal{S}$ can be constructed as follows (see e.g. [1]). Fix a sequence $a_n \in \mathbb{N} \setminus \{0,1\}$. Every $x \in \mathbb{R}$

[1] A compact subset $K \subset \mathbb{R}$ is called a Kronecker set if for every continuous function $f : K \to S^1$ and every $\varepsilon > 0$, there exists $y \in \mathbb{R}$ such that $|f(x) - \exp(ixy)| < \varepsilon$ for all $x \in K$.

can be uniquely written as

$$x = x_0 + \sum_{n=1}^{\infty} \frac{x_n}{a_1 \cdots a_n}, \quad \text{where } x_0 \in \mathbb{Z}, x_n \in \{0, \ldots, a_n - 1\} \text{ for all } n \geq 1 ,$$

and where we follow the convention that the sequence (x_n) is not eventually given by $x_n = a_n - 1$.

Whenever $K_n \subset \{0, \ldots, a_n - 1\}$, define the generalized Cantor set

$$\mathcal{W}_K := \{x \in [0, 1) \mid x_n \in K_n \text{ for all } n \geq 1\}$$

and equip $\mathcal{W}_K$ with the probability measure λ_K arising by identifying (up to a countable set) $\mathcal{W}_K$ and $\prod_{n=1}^{\infty} K_n$, the latter being equipped with the product of the uniform probability measures.

Set $\alpha_0 = 1$ and $\alpha_n = a_1 \cdots a_n$. Define the countable dense subgroup $Q \subset \mathbb{R}$ as the union of all $\alpha_n^{-1}\mathbb{Z}$. It is straightforward to check that whenever $x \in Q$ and $\mathcal{U} \subset \mathcal{W}_K$ is a Borel set such that $x + \mathcal{U} \subset \mathcal{W}_K$, then $\lambda_K(x + \mathcal{U}) = \lambda_K(\mathcal{U})$. Therefore, there is a unique σ-finite measure ν_K on the Borel sets of $\mathbb{R}$ that is Q-invariant, supported on $\mathcal{W}_K + Q$ and such that the restriction of ν_K to $\mathcal{W}_K$ equals λ_K. Moreover, the restriction of the Q-orbit equivalence relation to $\mathcal{W}_K \subset \mathbb{R}$ transfers to the equivalence relation on $\prod_{n=1}^{\infty} K_n$ given by

$$(x_n) \sim (y_n) \text{ if and only if there exists } n_0 \text{ such that } x_n = y_n \text{ for all } n \geq n_0 .$$

It follows that ν_K is an ergodic measure on $\mathbb{R}$.

Under certain conditions, the group H_{ν_K} can be determined explicitly. For example, it follows from Theorem 4.1 in [1] that if we take $K_n = \{0, \ldots, b_n - 1\}$ such that $b_n < \frac{1}{2} a_n$ for n large enough and $\sum_{n=1}^{\infty} b_n^{-1} < \infty$, we have

$$H_{\nu_K} = \Big\{x \in \mathbb{R} \;\Big|\; \sum_{n=1}^{\infty} \frac{a_n}{b_n} \langle \alpha_{n-1} x \rangle < \infty \Big\}, \tag{2.3}$$

where $\langle x \rangle \in [0, 1)$ denotes the distance of the real number x to $\mathbb{Z}$.

If moreover $b_n \sim C_1 \rho_1^n$ and $a_n \sim C_2 \rho_2^n$ with $0 < C_1, C_2$ and $1 < \rho_1 < \rho_2$, one can prove as follows that the Hausdorff dimension of H_{ν_K} is given by $\log \rho_1 / \log \rho_2$. First of all, by [18, Ch. II, §7, Thm. V], the Hausdorff dimension of $\mathcal{W}_K$ equals $\log \rho_1 / \log \rho_2$. Defining

$$K'_n = \{0, \ldots, b_n - 1\} \cup \{a_n - b_n, \ldots, a_n - 1\} ,$$

it is clear that for every $x \in H_{\nu_K}$, we eventually have $x_n \in K'_n$. So, $H_{\nu_K} \subset Q + \mathcal{W}_{K'_n}$ and the Hausdorff dimension of H_{ν_K} is at most $\log \rho_1 / \log \rho_2$. On the other hand, for every $\rho_1^{-1} < \gamma < 1$, define b''_n as the integer part of $\gamma^n b_n$. So, $b''_n \sim C_1 (\gamma \rho_1)^n$. Set $K''_n = \{0, \ldots, b''_n - 1\}$. If $x \in \mathcal{W}_{K''}$, we have

$$\langle \alpha_{n-1} x \rangle \leq \frac{\gamma^n b_n}{a_n}$$

and so, $\mathcal{W}_{K''} \subset H_{\nu_K}$. It follows that the Hausdorff dimension of H_{ν_K} is at least $\log(\gamma \rho_1) / \log(\rho_2)$ and this for all $\rho_1^{-1} < \gamma < 1$. So, we have proven that the Hausdorff dimension of H_{ν_K} is exactly $\log \rho_1 / \log \rho_2$. Finally, $\exp(H_{\nu_K})$ has the same Hausdorff dimension as H_{ν_K}.

Rigid actions. We recall from 4.1 in [28] the definition of relative property (T) (or rigidity) for an inclusion of finite von Neumann algebras and actions of groups.

Definition 2.3. Let M be a factor of type II_1 with normalized trace τ and let $A \subset M$ be a von Neumann subalgebra. The inclusion $A \subset M$ is called *rigid* if the following property holds: for every $\varepsilon > 0$, there exists a finite subset $\mathcal{J} \subset M$ and a $\delta > 0$ such that whenever ${}_M H_M$ is a Hilbert M-M-bimodule admitting a unit vector ξ with the properties

- $\|a \cdot \xi - \xi \cdot a\| < \delta$ for all $a \in \mathcal{J}$,
- $|\langle \xi, a \cdot \xi \rangle - \tau(a)| < \delta$ and $|\langle \xi, \xi \cdot a \rangle - \tau(a)| < \delta$ for all a in the unit ball of M,

there exists a vector $\xi_0 \in H$ satisfying $\|\xi - \xi_0\| < \varepsilon$ and $a \cdot \xi_0 = \xi_0 \cdot a$ for all $a \in A$.

A free ergodic m.p. action $\Lambda \curvearrowright (X, \mu)$ is called *rigid* if the corresponding inclusion $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Lambda$ is rigid. Recall from Proposition 5.1 in [28] that if a group Γ acts outerly on a discrete abelian group H and $\Gamma \curvearrowright (\hat{H}, \mathrm{Haar})$ is the action it induces on the (dual) compact group $\hat{H}$ with its Haar measure, then $\Gamma \curvearrowright \hat{H}$ has the relative property (T) iff the pair of groups $H \subset \Gamma \ltimes H$ has the relative property (T) of Kazhdan-Margulis.

Note that, by combining Theorems 4.4 in [28] and A.1 in [25], it follows that for every rigid action $\Lambda \curvearrowright (X, \mu)$, the group $\mathrm{Out}(\mathcal{R}(\Lambda^\infty \curvearrowright Y))$ is countable. The same arguments show that if $\Lambda \curvearrowright (X, \mu)$ is a free m.p. action whose restriction to $\Gamma < \Lambda$ is ergodic and rigid, the set $\mathrm{Emb}(\Gamma^\infty, \Lambda^\infty)$ is countable modulo left composition with elements of $[\Lambda^\infty]$.

Let $\Gamma_0 < \mathrm{SL}_2(\mathbb{Z})$ be a non-amenable subgroup. By [2, Example 2 on page 62], the pair $\mathbb{Z}^2 \subset \mathbb{Z}^2 \rtimes \Gamma_0$ has the relative property (T) in the group sense. Therefore, the natural action $\Gamma_0 \curvearrowright \mathbb{T}^2$ is rigid.

Making actions freely independent. A crucial role in our construction in the next section of an action of $\mathbb{F}_\infty$ with "special properties" is played by choosing inductively the action of the n-th generator of $\mathbb{F}_\infty$ from a certain G_δ-dense subset of $\mathrm{Aut}(X, \mu)$, the existence of which uses a result in [17], [36], stated as Theorem 2.4 below, for convenience (see A.1 in [17] and [36] for details).

Let (X, μ) be the standard non-atomic probability space. We use the terminology *non-singular partial automorphism* of (X, μ) for any non-singular isomorphism ϕ between measurable subsets $D(\phi) = X_0$ and X_1 of X. We identify partial automorphisms that are equal almost everywhere. The partial automorphisms of (X, μ) form a pseudo-group under composition. A partial automorphism ϕ with domain $D(\phi) \subset X$ is called (essentially) *free* if the set $\{x \in D(\phi) \mid \phi(x) = x\}$ has measure zero.

Theorem 2.4. *Let $\mathcal{V}$ be a countable set of free non-singular partial automorphisms of (X, μ). Define $T \subset \mathrm{Aut}(X, \mu)$ as the set of measure-preserving automorphisms θ such that every non-empty word with letters alternatingly from $\mathcal{V}$ and $\theta \mathcal{V} \theta^{-1}$ defines a free partial automorphism of (X, μ). Then, T is a G_δ-dense subset of the Polish group of m.p. automorphisms of (X, μ).*

We will use Theorem 2.4 under the following form. Set $(Y, \eta) = (X \times \mathbb{Z}, \mu \times c)$. Suppose that $\mathcal{W}$ is a countable set of non-singular partial automorphisms of Y with the property that $\sigma_n \circ \phi$ is free for all $n \in \mathbb{Z}$ and $\phi \in \mathcal{W}$. Here $(\sigma_n)_{n \in \mathbb{Z}}$

denotes the shift action $\mathbb{Z} \curvearrowright \mathbb{Z}$. Define $T \subset \mathrm{Aut}(X,\mu)$ as the set of measure-preserving automorphisms θ of (X,μ) such that every non-empty word with letters alternatingly from $\mathcal{W}$ and $\theta\mathcal{W}\theta^{-1}$ defines a free partial automorphism of Y. Then, T is a G_δ-dense subset of $\mathrm{Aut}(X,\mu)$. This statement follows by defining the partial isomorphisms $\phi_n : X \to X \times \{n\} \subset Y : \phi_n(x) = (x,n)$ and applying Theorem 2.4 to the countable set $\mathcal{V} = \{\phi_n^{-1}\phi\phi_m \mid n,m \in \mathbb{Z}, \phi \in \mathcal{W}\}$ of free partial automorphisms of (X,μ).

3. Construction of the basic $\mathbb{F}_\infty$-action

The actions $\mathbb{F}_\infty \curvearrowright X_i$ in the statement of Theorem 1.1 will (roughly) be obtained as diagonal product actions of $\mathbb{F}_\infty$, the first component of which will be referred to as the *basic action.* Its construction, which is the subject of this section, is inspired by arguments in [11], [17].

To start with, let $H = \mathbb{F}_\infty$ with free generators $a, b, g_1, g_2, \ldots$. Define as follows subgroups of H. We set $H_0 := \langle a,b\rangle$ and, for every subset $E \subset \mathbb{N} \setminus \{0\}$, $H_E := \langle a, b, g_n \mid n \in E\rangle$. We set $H_n = \langle a, b, g_1, \ldots, g_n\rangle$.

Define (X,μ) to be $\mathbb{T}^2$ with the Lebesgue measure and $H_0 \curvearrowright X$ by viewing $H_0 \subset \mathrm{SL}_2(\mathbb{Z})$ as a finite index subgroup and restricting to H_0 the canonical action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{T}^2$. Note that the action $H_0 \curvearrowright X$ is rigid and weakly mixing (see e.g. [28, Corollary 3.3.2]). Denote as before $(Y,\eta) = (X \times \mathbb{Z}, \mu \times c)$, where c is the counting measure on $\mathbb{Z}$.

Let K be any countable group and $K \curvearrowright (X,\mu)$ an arbitrary free m.p. action. Use Theorem 2.4 to conjugate $K \curvearrowright (X,\mu)$ such that the action $H_0 * K \curvearrowright (X,\mu)$ is still free.

We construct, by induction on n, a free m.p. action of $H_n * K$ on (X,μ), extending the action of $H_0 * K$ chosen above. Suppose that we are given a free m.p. action of $H_n * K$ on X extending the $(H_0 * K)$-action chosen above. Choose for every $F \subset \{1,\ldots,n\}$ a set $\mathcal{A}(F) \subset \mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$ of representatives for the outer elements in $\mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$ modulo $[(H_F * K)^\infty]$. Note that $\mathcal{A}(F)$ is countable, because $H_F \curvearrowright X$ is a rigid action. Consider the following countable set of non-singular automorphisms of Y :

$$\begin{aligned}\mathcal{V} :=& \{\sigma_g \Delta^{\pm 1} \sigma_h \mid g,h \in (H_n * K)^\infty \ ,\ \Delta \in \mathcal{A}(F) \ \text{ for some } \ F \subset \{1,\ldots,n\}\} \\ & \cup \{\sigma_k \mid k \in (H_n * K - \{e\}) \times \mathbb{Z}\} \ .\end{aligned}$$

Lemma 3.1. *The automorphism $\sigma_k \circ \Delta$ is free whenever $k \in \mathbb{Z}$ and $\Delta \in \mathcal{V}$.*

Proof. The lemma follows once we show that $\sigma_{g^{-1}} \circ \Delta$ is free whenever $g \in (H_n * K)^\infty$ and $\Delta \in \mathrm{Emb}(H_F^\infty, (H_F * K)^\infty) - [(H_F * K)^\infty]$. If this is not the case, we find $\Delta(x) = g \cdot x$ for all x in some non-negligible subset $\mathcal{U} \subset Y$. Because $H_F^\infty \curvearrowright Y$ is ergodic and Δ belongs to $\mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$, it follows that $\Delta(x) \in (H_n * K)^\infty \cdot x$ for almost all $x \in Y$. We have to prove that actually $\Delta(x) \in (H_F * K)^\infty \cdot x$ almost everywhere, in order to reach a contradiction with the outerness of Δ. Define $\Delta_1 : X \to X$ such that $\Delta(x,0) \in \{\Delta_1(x)\} \times \mathbb{Z}$ for almost all $x \in X$. Since $\Delta(x) \in (H_n * K)^\infty \cdot x$ for almost all $x \in Y$, we find $\varphi : X \to H_n * K$ such that $\Delta_1(x) = \varphi(x) \cdot x$ for almost all $x \in X$. Since $\Delta \in \mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$, it follows that $\varphi(g \cdot x) g \varphi(x)^{-1} \in H_F * K$ for almost all $x \in X$, $g \in H_F$. By weak mixing of $H_F \curvearrowright (X,\mu)$ and because an element $h \in H_n * K$ satisfying $h H_F h^{-1} \subset H_F * K$ must belong to $H_F * K$, it follows that $\varphi(x) \in H_F * K$ for almost all $x \in X$. So, we are done. $\square$

Combining Theorem 2.4, the remarks following that theorem and the previous lemma, take a free ergodic m.p. action of $\mathbb{Z} \cong g_{n+1}^{\mathbb{Z}}$ on (X, μ) such that every non-empty word with letters alternatingly from $\mathcal{V}$ and $\{\sigma_{g_{n+1}^k} \mid k \in \mathbb{Z} - \{0\}\}$ yields a free transformation of Y.

We obtained in particular a free m.p. action of $H_{n+1} * K$ extending the $(H_n * K)$-action that we started with. Continuing by induction, we obtain a free m.p. action $\sigma : H * K \curvearrowright (X, \mu)$, whose restriction to H_0 is ergodic and rigid.

4. Actions with no symmetries

The following theorem is the crucial technical result on which all results in the paper rely.

As above, $H \cong \mathbb{F}_\infty$, K is arbitrary and we defined the action $\sigma : H * K \curvearrowright (X, \mu)$. We defined the subgroup $H_E \subset H$ generated by $a, b, g_n, n \in E$ whenever $E \subset \mathbb{N} \setminus \{0\}$. All H_E act rigidly on (X, μ).

Theorem 4.1. *There exists an uncountable family $\mathcal{E}$ of infinite subsets of $\mathbb{N}$ such that*

$$\mathrm{Emb}(H_E^\infty, (H_E * K)^\infty) = [(H_E * K)^\infty]$$

for all $E \in \mathcal{E}$. In particular, $\mathrm{Out}(\mathcal{R}((H_E * K)^\infty \curvearrowright Y))$ *is trivial for all $E \in \mathcal{E}$.*

Proof. Choose an uncountable family $\mathcal{E}_1$ of infinite subsets of $\mathbb{N}$ satisfying $E \cap F$ finite whenever $E, F \in \mathcal{E}_1$ and $E \neq F$. Suppose that the theorem does not hold. Leaving out the countably many $E \in \mathcal{E}_1$ such that $\mathrm{Emb}(H_E^\infty, (H_E * K)^\infty) = [(H_E * K)^\infty]$, we find an uncountable subset $\mathcal{E} \subset \mathcal{E}_1$ and for every $E \in \mathcal{E}$ an outer $\Delta_E \in \mathrm{Emb}(H_E^\infty, (H_E * K)^\infty)$.

Recall that we fixed, for every finite subset $F \subset \mathbb{N}$, a set of representatives

$$\mathcal{A}(F) \subset \mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$$

for the outer elements in $\mathrm{Emb}(H_F^\infty, (H_F * K)^\infty)$ modulo $[(H_F * K)^\infty]$.

Step 1. *There exist $E, F \in \mathcal{E}$ such that $E \neq F$ and $\Delta_E(x) = \Delta_F(x)$ for all x in a non-negligible subset $\mathcal{U} \subset Y$.* Step 1 will follow by using relative property (T) and a separability argument. Denote $A = \mathrm{L}^\infty(X, \mu)$ and define the group measure space II_1 factors $M_0 = A \rtimes H_0$ and $M = A \rtimes (H * K)$. We write $M^\infty := \mathrm{B}(\ell^2(\mathbb{Z})) \overline{\otimes} M$ and view $\mathrm{L}^\infty(Y) = A^\infty$ as the obvious diagonal subalgebra of M^∞. Denote by Tr the natural semi-finite trace on M^∞. For every $E \in \mathcal{E}$, define $p_E \in A^\infty$ as the projection on $\Delta_E(X \times \{0\})$. Note that $\mathrm{Tr}(p_E) < \infty$. Define the embeddings $\beta_E : M_0 \to p_E M^\infty p_E$ by naturally extending the homomorphism $\beta_E : A \to A^\infty p_E : \beta_E(a) = a \circ \Delta_E^{-1}$ for all $a \in A$. Define then for all $E, F \in \mathcal{E}$ the Hilbert M_0-M_0-bimodule $\mathcal{H}_{E,F} := p_E \, \mathrm{L}^2(M^\infty) p_F$ with bimodule action given by

$$a \cdot \xi \cdot b := \beta_E(a) \xi \beta_F(b) \; .$$

Set $\xi_{E,F} = \mathrm{Tr}(p_E p_F)^{-1/2} p_E p_F$ whenever $p_E p_F$ is non-zero. As mentioned above, the inclusion $A \subset M_0$ is rigid. Set $\varepsilon = 1/2$ and take the finite subset $\mathcal{J} \subset M_0$ and $\delta > 0$ such that the conclusion of Definition 2.3 holds. The separability of the Hilbert space $\mathrm{L}^2(M^\infty)$ and the uncountability of $\mathcal{E}$ provide us with $E, F \in \mathcal{E}$, $E \neq F$ such that the vector $\xi_{E,F}$ satisfies the assumptions in Definition 2.3. So, there exists a vector $\xi_0 \in \mathcal{H}_{E,F}$ satisfying $a \cdot \xi_0 = \xi_0 \cdot a$ for all $a \in A$ and $\|\xi_{E,F} - \xi_0\| < \varepsilon = 1/2$. Taking the orthogonal projection of ξ_0 onto $\mathrm{L}^2(Y)$, we may assume that ξ_0 has its matrix coefficients in A. Since ξ_0 is non-zero, the claim of Step 1 follows.

Step 2. *There exists $\phi \in [(H_E * K)^\infty]$ such that $\phi \circ \Delta_E \in \mathcal{A}(E \cap F)$.* We have $\Delta_E(x) = \Delta_F(x)$ for all $x \in \mathcal{U}$. Since $H_0^\infty \curvearrowright Y$ is ergodic and $H_0 \subset H_E \cap H_F$, it follows that

$$\Delta_E(x) \in \left((H_E * K)^\infty \, (H_F * K)^\infty\right) \cdot \Delta_F(x)$$

for almost all $x \in Y$. Composing Δ_E and Δ_F on the left by elements in resp. $[(H_E * K)^\infty]$ and $[(H_F * K)^\infty]$, we may assume that $\Delta_E(x) = \Delta_F(x)$ for almost all $x \in Y$. But then, $\Delta_E \in \mathrm{Emb}(H_{E\cap F}^\infty, (H_{E\cap F} * K)^\infty)$. Since Δ_E is outer, we can compose Δ_E by an element of $[(H_{E\cap F} * K)^\infty]$ and assume that $\Delta_E \in \mathcal{A}(E \cap F)$. This concludes Step 2.

Step 3. *We obtain a contradiction with our choice of action $\sigma : H * K \curvearrowright X$.* Since E is infinite and $E \cap F$ finite, take $n \in E$ such that $k < n$ for all $k \in E \cap F$. Since Δ_E belongs to $\mathrm{Emb}(H_E^\infty, (H_E * K)^\infty)$, we can take $h \in (H_E * K)^\infty$ and a non-negligible subset $Y_0 \subset Y$ such that $\Delta_E(g_n \cdot x) = h \cdot \Delta_E(x)$ for all $x \in Y_0$. Suppose first that $h \in (H_n * K)^\infty$. Write $h = (h_0, m)$, where $m \in \mathbb{Z}$ and h_0 is a word with letters alternatingly in $H_{n-1} * K - \{e\}$ and $\{g_n^k \mid k \in \mathbb{Z} - \{0\}\}$. The fact that

$$\sigma_{h^{-1}} \circ \Delta_E \circ \sigma_{g_n} \circ \Delta_E^{-1}$$

is not free is a contradiction with Δ_E belonging to $\mathcal{A}(E \cap F)$ and our choice of σ_{g_n}. Suppose next that $h \notin (H_n * K)^\infty$ and take the smallest p such that $h \in (H_p * K)^\infty$. So, $p > n$ and the generator g_p appears in the reduced expression of h. Moreover,

$$\sigma_{g_n} \circ \Delta_E^{-1} \circ \sigma_{h^{-1}} \circ \Delta_E$$

is not free. This is also a contradiction with our choice of σ_{g_p}, because Δ_E belongs to $\mathcal{A}(E \cap F)$. □

Denote by σ_E the restriction of σ to $H_E * K$.

Lemma 4.2. *Let $E \subset \mathbb{N}$. There are at most countably many $F \subset \mathbb{N}$ such that σ_E and σ_F are stably orbit equivalent.*

Proof. Let $E \subset \mathbb{N}$. Suppose that $\mathcal{E}$ is an uncountable family of subsets of $\mathbb{N}$ such that σ_E and σ_F are stably orbit equivalent for all $F \in \mathcal{E}$. Take for every $F \in \mathcal{E}$, a non-singular automorphism $\Delta_F : Y \to Y$ satisfying $\Delta_F((H_E * K) \cdot x) = (H_F * K) \cdot \Delta_F(x)$ for almost all $x \in Y$. Define the II_1 factors M_0 and M as in Step 1 of the proof of Theorem 4.1. The formula $\beta_F(a) = a \circ \Delta_F^{-1}$ defines, as in the proof of Theorem 4.1, a homomorphism $\beta_F : A \to A^\infty p_F$ that naturally extends to a homomorphism $\beta_F : M_0 \to p_F M^\infty p_F$. Here $A^\infty = \mathrm{L}^\infty(Y)$ and p_F is the projection onto $\Delta_F(X \times \{0\})$. Separability and rigidity provide us with $F, F' \in \mathcal{E}$, $F \neq F'$ such that $\Delta_F(x) = \Delta_{F'}(x)$ for all x belonging to a non-negligible subset $\mathcal{U} \subset Y$. As in the proof of Theorem 4.1, we can compose $\Delta_F, \Delta_{F'}$ on the left with elements in $[(H_F * K)^\infty]$, $[(H_{F'} * K)^\infty]$ and assume that $\Delta_F(x) = \Delta_{F'}(x)$ for almost all $x \in Y$. From this it follows that $H_F * K$ and $H_{F'} * K$ are equal subgroups of $H * K$, a contradiction. □

Combining Theorem 4.1 and Lemma 4.2, we immediately get the following result.

Theorem 4.3. *Let $\sigma : K \curvearrowright (X, \mu)$ be any free, probability measure-preserving action of any countable group. There exists a continuous family $(\sigma_i)_{i \in I}$ of free ergodic m.p. actions of $\mathbb{F}_\infty * K$ on the probability spaces (X_i, μ_i) with the following* *properties:*

- *The orbit equivalence relation of* $\mathbb{F}_\infty * K \curvearrowright X_i$ *has trivial fundamental group and trivial outer automorphism group.*
- *The restriction of* σ_i *to* K *is conjugate with* σ.
- *The restriction of* σ_i *to* $\mathbb{F}_\infty$ *is ergodic and rigid.*
- *The actions* $(\sigma_i)_{i\in I}$ *are not stably orbit equivalent.*

5. Actions with prescribed symmetries

We use the following terminology and notation. When (Y,η) is a standard infinite measure space and $\Lambda \curvearrowright (Y,\eta)$ is a measure-preserving, ergodic action, we define for every $\alpha \in \mathrm{Aut}(\Lambda)$, the group $\mathrm{Aut}_\Lambda^\alpha(Y)$ of non-singular automorphisms Δ of Y satisfying $\Delta(g\cdot y) = \alpha(g)\cdot \Delta(y)$ for all $g \in \Lambda$ and almost all $y \in Y$. Note that $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda) = \mathrm{Aut}_\Lambda^{\mathrm{id}}(Y)$.

We say that $\Lambda \curvearrowright (Y,\eta)$ is induced from $\Lambda_0 \curvearrowright Y_0$ if Y_0 is a measurable subset of Y that is globally Λ_0-invariant and satisfies $\eta(g\cdot Y_0 \cap Y_0) = 0$ whenever $g \notin \Lambda_0$.

Lemma 5.1. *Suppose that we have the following data:*

- *a standard infinite measure space* (Y,η) *with a non-singular action* $\widetilde{\Lambda} \curvearrowright (Y,\eta)$ *scaling the measure* η;
- *a normal subgroup* $\Lambda \lhd \widetilde{\Lambda}$ *such that the restricted action* $\Lambda \curvearrowright (Y,\eta)$ *is measure-preserving, free and ergodic.*

We make the following additional assumption.

- *The action* $\Lambda \curvearrowright (Y,\eta)$ *defines the same orbit equivalence relation on* (Y,η) *as the free, ergodic, measure-preserving action* $\Sigma \curvearrowright (Y,\eta)$ *that moreover is profinite in the following sense: we have* $(Y,\eta) = \varprojlim (Y_n,\eta_n)$, *where the* (Y_n,η_n) *are atomic infinite measure spaces and the subalgebras* $\mathrm{L}^\infty(Y_n,\eta_n) \subset \mathrm{L}^\infty(Y,\eta)$ *are globally* Σ*-invariant.*

Note that the additional assumption is automatic when Λ *is amenable, but of course also when* $\Lambda \curvearrowright (Y,\eta)$ *is itself profinite.*

Suppose next that $\widetilde{\Gamma} \curvearrowright (X,\mu)$ *is a free m.p. action of the group* $\widetilde{\Gamma}$ *on a probability measure space* (X,μ) *and* H *is a subgroup of* $\widetilde{\Gamma}$ *with the following properties:*

- *The restricted action* $H \curvearrowright (X,\mu)$ *is ergodic and rigid.*
- *We have* $\mathrm{Emb}(H^\infty, \widetilde{\Gamma}^\infty) = [\widetilde{\Gamma}^\infty]$.

Whenever $\pi : \widetilde{\Gamma} \to \widetilde{\Lambda}$ *is a surjective homomorphism with* $H \subset \mathrm{Ker}\,\pi$, *we set* $\Gamma = \pi^{-1}(\Lambda)$ *and define* $\mathcal{R}_\pi$ *as the equivalence relation given by the orbits of the free, ergodic, (infinite) measure-preserving action* $\Gamma \curvearrowright X\times Y : g\cdot(x,y) = (g\cdot x, \pi(g)\cdot y)$.

The following results hold.

(1) *We have*

$$\mathrm{Out}(\mathcal{R}_\pi) \cong \frac{\{(g,\Delta) \mid g \in \widetilde{\Lambda}, \Delta \in \mathrm{Aut}_\Lambda^{\mathrm{Ad}\, g}(Y)\}}{\Lambda} \; .$$

In particular, we have a short exact sequence

$$e \to \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda) \to \mathrm{Out}(\mathcal{R}_\pi) \to \frac{\widetilde{\Lambda}}{\Lambda} \to e \; .$$

(2) *The equivalence relations $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$ are stably orbit equivalent if and only if $\Lambda \curvearrowright Y$ is induced from $\Lambda_i \curvearrowright Y_i$ $(i = 1, 2)$ in such a way that $\pi_1^{-1}(\Lambda_1) = \pi_2^{-1}(\Lambda_2)$ and there exists a non-singular isomorphism $\Delta_0 : Y_1 \to Y_2$ satisfying $\Delta_0(\pi_1(g) \cdot y) = \pi_2(g) \cdot \Delta_0(y)$ for all $g \in \pi_i^{-1}(\Lambda_i)$ and almost all $y \in Y_1$.*

In particular, if $\Lambda \curvearrowright Y$ can only be induced from faithful actions, stable orbit equivalence of $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$ implies $\operatorname{Ker} \pi_1 = \operatorname{Ker} \pi_2$.

Proof. Choose a surjective homomorphism $\pi : \widetilde{\Gamma} \to \widetilde{\Lambda}$ and define $\mathcal{R}_\pi$ as the orbit equivalence relation of $\Gamma \curvearrowright X \times Y$. When $g \in \widetilde{\Gamma}$ and when Δ_0 is a non-singular automorphism of (Y, η) scaling the measure η and satisfying $\Delta_0(h \cdot y) = (\pi(g) h \pi(g)^{-1}) \cdot \Delta_0(y)$ for all $h \in \Lambda$ and almost all $y \in Y$, we define the automorphism Δ of $X \times Y$ as $\Delta(x, y) = (g \cdot x, \Delta_0(y))$ and observe that Δ normalizes the action $\Gamma \curvearrowright X \times Y$. In particular, Δ is an automorphism of $\mathcal{R}_\pi$. Clearly, Δ is inner if and only if $g \in \Gamma$ and $\Delta_0(y) = \pi(g) \cdot y$ for almost all y.

To prove item (1), we have to show that every automorphism of $\mathcal{R}_\pi$ is of the form above modulo $[\Gamma]$. The main step of the proof consists in using the rigidity of $\Gamma \curvearrowright X$ and the profiniteness of $\Sigma \curvearrowright Y$ in order to prove that any automorphism $\Delta(x, y) = (\Delta_1(x, y), \Delta_2(x, y))$ of $\mathcal{R}_\pi$ is such that Δ_1 is essentially independent of the y-variable (cf. Section 6 in [28]).

Set $A = \mathrm{L}^\infty(X, \mu)$ and $B = \mathrm{L}^\infty(Y, \eta)$. Denote $M = (A \overline{\otimes} B) \rtimes \Gamma$ and note that M is a II_∞ factor. Write $\mathcal{M} = (A \rtimes \Gamma) \overline{\otimes} (B \rtimes \Lambda)$. The map

$$(a \otimes b) u_g \mapsto (a u_g) \otimes (b u_{\pi(g)})$$

extends to a trace-preserving embedding $M \subset \mathcal{M}$. Let $\Sigma \curvearrowright (Y, \eta)$ be a profinite action with $\Sigma \cdot y = \Lambda \cdot y$ for almost all $y \in Y$. Identify $B \rtimes \Lambda$ with $B \rtimes \Sigma$. Take $(Y, \eta) = \varprojlim (Y_n, \eta_n)$ in such a way that the subalgebras $B_n := \mathrm{L}^\infty(Y_n, \eta_n) \subset B$ are globally Σ-invariant. Denote by $E_n : B \rtimes \Sigma \to B_n \rtimes \Sigma$ the unique trace-preserving conditional expectation.

Let Δ be an automorphism of $\mathcal{R}_\pi$. Write $\Delta(x, y) = (\Delta_1(x, y), \Delta_2(x, y))$. Let p be a minimal projection in $B_0 = \mathrm{L}^\infty(Y_0, \eta_0)$ and rescale η such that p has trace 1. So, p projects onto a measurable subset $Z \subset Y$ with $\eta(Z) = 1$. Denote by p_1 the projection onto $\Delta(X \times Z)$. Note that p and p_1 are finite projections in the II_∞ factor M. The automorphism Δ yields a *-isomorphism of II_1 factors

$$\Psi : p_1 M p_1 \to pMp \quad \text{satisfying} \quad \Psi(p_1(A \overline{\otimes} B)) = p(A \overline{\otimes} B) \quad \text{and}$$
$$\Psi(f)(x, y) = f(\Delta(x, y))$$

for almost all $(x, y) \in X \times Z$ whenever $f \in p_1(A \overline{\otimes} B)$.

Set $D = \Psi(p_1(A \otimes 1))$. Then, $D \subset pMp$ is a rigid inclusion and hence, also the inclusion $D \subset p\mathcal{M}p$ is rigid. Since $\|(\mathrm{id} \otimes E_n)(a) - a\|_2 \to 0$ for all $a \in p\mathcal{M}p$, it follows that $\|(\mathrm{id} \otimes E_n)(d) - d\|_2 \to 0$ uniformly on the unit ball of D. Lemma 5.6 below allows us to take n and a projection q onto a subset $\mathcal{U} \subset X \times Z$ of measure strictly greater than $\frac{3}{4}$ such that $Dq \subset (A \overline{\otimes} B_n) q$. Denote by $\rho_n : Y \to Y_n$ the quotient maps given by $(Y, \eta) = \varprojlim (Y_n, \eta_n)$. We get measurable subsets $\mathcal{U}_1 \subset X \times Y_n$ and $X_1 \subset X$ as well as a quotient map of measure spaces $\alpha : \mathcal{U}_1 \to X_1$ such that $\Delta_1(x, y) = \alpha(x, \rho_n(y))$ for almost all $(x, y) \in \mathcal{U}$.

Since also $(A \overline{\otimes} B_n) p \subset pMp$ is rigid, we use Ψ^{-1} and similarly find for every $\varepsilon > 0$, an $m \in \mathbb{N}$ and a projection q' in $(A \overline{\otimes} B) p$ with $\tau(q') \geq 1 - \varepsilon$, such that

$$(A \overline{\otimes} B_n) q' \subset \Psi(p_1(A \overline{\otimes} B_m)) q' \; .$$

This means that, after making $\mathcal{U}$ slightly smaller but still with measure greater than $\frac{3}{4}$, the quotient map $\alpha : \mathcal{U}_1 \to X_1$ may be assumed to be countable-to-one.

Partitioning Y in subsets on which ρ_n is constant, we find a non-negligible subset $Z_0 \subset Z$ and a measurable subset $\mathcal{V} \subset X \times Z_0$ with $(\mu \times \eta)(\mathcal{V}) \geq \frac{3}{4}\eta(Z_0)$ such that $\Delta_1(x, y) = \alpha(x)$ for almost all $(x, y) \in \mathcal{V}$, where $\alpha : X_2 \subset X \to X_1 \to X$ is a countable-to-one quotient map. Whenever $x \in X$, define $\mathcal{V}_x = \{y \in Z_0 \mid (x, y) \in \mathcal{V}\}$. Let $\mathcal{W} \subset X$ be the necessarily non-negligible subset of $x \in X$ with $\eta(\mathcal{V}_x) > \frac{2}{3}\eta(Z_0)$. Whenever $g \in H$, $x \in \mathcal{W}$ and $g \cdot x \in \mathcal{W}$, the set $\mathcal{V}_{g \cdot x} \cap \mathcal{V}_x$ has measure at least $\frac{1}{3}\eta(Z_0)$. So, we can take $y \in \mathcal{V}_{g \cdot x} \cap \mathcal{V}_x$ and hence

$$\alpha(g \cdot x) = \Delta_1(g \cdot x, y) = \Delta_1(g \cdot (x, y)) \in \Gamma \cdot \Delta_1(x, y) = \Gamma \cdot \alpha(x) \ .$$

Since α is countable-to-one, we can further restrict $\alpha|_{\mathcal{W}}$ to a non-negligible partial automorphism of (X, μ) with the property that $\alpha(g \cdot x) \in \Gamma \cdot \alpha(x)$ whenever $g \in H$ and both x and $g \cdot x$ belong to the domain of α. By our assumption on the action $\widetilde{\Gamma} \curvearrowright (X, \mu)$, it follows that $\alpha(x) \in \widetilde{\Gamma} \cdot x$ for all x in the domain of α. So, we can compose the automorphism Δ by an automorphism of the form $(x, y) \mapsto (g \cdot x, \pi(g) \cdot y)$, where $g \in \widetilde{\Gamma}$ and we may assume now that $\Delta(x, y) \in \{x\} \times Y$ for all (x, y) in some non-negligible subset $\mathcal{U}$ of $X \times Y$. We prove that, modulo $[\Gamma]$, such a Δ has the form $\Delta(x, y) = (x, \Delta_0(y))$, where $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$.

Since the action $\Gamma \curvearrowright X \times Y$ is ergodic, choose a measurable map $\varphi : X \times Y \to \Gamma$ such that $\varphi(x, y) \cdot (x, y) \in \mathcal{U}$ for almost all $(x, y) \in X \times Y$. Define the measurable map

$$\psi : X \times Y \to X \times Y : \psi(x, y) = \varphi(x, y)^{-1} \cdot \Delta(\varphi(x, y) \cdot (x, y)) \ .$$

Note that $\psi(x, y) \in \{x\} \times Y$ for almost all (x, y) and that ψ preserves $\mathcal{R}_\pi$ in the sense that $(x, y)\mathcal{R}_\pi(x', y')$ if and only if $\psi(x, y)\mathcal{R}_\pi\psi(x', y')$. It follows that $\psi(g \cdot (x, y)) = g \cdot \psi(x, y)$ for all $g \in \Gamma$ and almost all (x, y). Also note that $X \times Y$ can be partitioned into subsets $X \times Y = \bigsqcup_m Z_m$ such that the restriction of ψ to Z_m is a non-singular partial automorphism of $X \times Y$.

We claim that ψ is a non-singular automorphism of $X \times Y$. First assume that ψ is not essentially injective. We get a partial automorphism θ of $X \times Y$ with domain and range being disjoint subsets of $X \times Y$ and $\psi(\theta(x, y)) = \psi(x, y)$ for all $(x, y) \in D(\theta)$. But then, $\theta(x, y)\mathcal{R}_\pi(x, y)$ for almost all $(x, y) \in D(\theta)$. Making the domain of θ smaller, we find $g \in \Gamma - \{e\}$ and a non-negligible subset $\mathcal{V} \subset X \times Y$ such that $\psi(g \cdot (x, y)) = \psi(x, y)$ for all $(x, y) \in \mathcal{V}$. Since $\psi(g \cdot (x, y)) = g \cdot \psi(x, y)$, we reached a contradiction with the essential freeness of $\Gamma \curvearrowright X$. Since the range of ψ is Γ-invariant, ψ is also essentially surjective.

It follows that $\psi(x, y) = (x, \psi_x(y))$, where ψ_x is, for almost every $x \in X$, a non-singular automorphism of (Y, η). Since $\psi(g \cdot (x, y)) = g \cdot \psi(x, y)$, it follows that $\psi_{g \cdot x}(\pi(g) \cdot y) = \pi(g) \cdot \psi_x(y)$. In particular, $\psi_{g \cdot x} = \psi_x$ for all $g \in \mathrm{Ker}\, \pi$ and almost all $x \in X$. Since $\mathrm{Ker}\, \pi$ acts ergodically on (X, μ), it follows that almost every ψ_x equals almost everywhere Δ_0, where $\Delta_0 \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$. This concludes the proof of item (1).

We now prove item (2). Let $\pi_1, \pi_2 : \widetilde{\Gamma} \to \widetilde{\Lambda}$ be surjective homomorphisms such that $H \subset \mathrm{Ker}\, \pi_i$ for $i = 1, 2$.

Suppose first that $\Lambda \curvearrowright Y$ is induced from $\Lambda_i \curvearrowright Y_i$ in such a way that the conditions in item (2) hold. Write $\Gamma_0 = \pi_i^{-1}(\Lambda_i)$. For $i = 1, 2$, we have the actions

$$\sigma_i : \Gamma_0 \curvearrowright X \times Y_i : g \cdot (x, y) = (g \cdot x, \pi_i(g) \cdot y) \ .$$

The map $(x,y) \mapsto (x,\Delta_0(y))$ conjugates σ_1 and σ_2. Moreover, the reduction of $\mathcal{R}_{\pi_i}$ to $X \times Y_i$ is precisely given by the orbit equivalence relation of σ_i. So, these reductions are orbit equivalent and hence, $\mathcal{R}_{\pi_1}$ is stably orbit equivalent with $\mathcal{R}_{\pi_2}$.

Suppose next that Δ is a non-singular automorphism of $X \times Y$ defining a stable orbit equivalence of $\mathcal{R}_{\pi_1}$ and $\mathcal{R}_{\pi_2}$. Write $\Gamma_i = \pi_i^{-1}(\Lambda)$. The same rigidity vs. profiniteness argument as above tells us that we can compose Δ with an automorphism of $\mathcal{R}_{\pi_2}$ and assume that $\Delta(x,y) \in \{x\} \times Y$ for all (x,y) in some non-negligible subset $\mathcal{U}$ of $X \times Y$. Note that writing $\mathcal{V} = \Delta(\mathcal{U})$, we also have $\Delta^{-1}(x,y) \in \{x\} \times Y$ for all (x,y) in $\mathcal{V}$.

Let $\mathcal{U}_0$ be the set of $y \in Y$ such that $\mathcal{U}^y := \{x \in X \mid (x,y) \in \mathcal{U}\}$ is non-negligible. Define Λ_1 as the subgroup of Λ generated by $g \in \Lambda$ satisfying $\eta(g\cdot\mathcal{U}_0 \cap \mathcal{U}_0) > 0$. Set $Y_1 = \bigcup_{g \in \Lambda_1} g \cdot \mathcal{U}_0$. By construction, $\Lambda \curvearrowright Y$ is induced from $\Lambda_1 \curvearrowright Y_1$. Therefore, the latter is ergodic, and it also follows that the action $\pi_1^{-1}(\Lambda_1) \curvearrowright X \times Y_1$ is ergodic.

We also define $\mathcal{V}_0$ as the set of $y \in Y$ such that $\mathcal{V}^y$ is non-negligible. This leads to a similar construction of $\Lambda_2 \curvearrowright Y_2$ from which $\Lambda \curvearrowright Y$ is induced.

We claim that $\pi_1^{-1}(\Lambda_1) = \pi_2^{-1}(\Lambda_2)$. By symmetry, it is sufficient to take $g \in \pi_1^{-1}(\Lambda_1)$ and to prove that $\pi_2(g) \in \Lambda_2$. It is even sufficient to start with $g \in \Gamma_1$ such that $\pi_1(g)^{-1} \cdot \mathcal{U}_0 \cap \mathcal{U}_0$ is non-negligible. This means that $(gH)^{-1} \overset{\sigma_1}{\cdot} \mathcal{U} \cap \mathcal{U}$ is non-negligible. Since Δ maps $\mathcal{R}_{\pi_1}$ onto $\mathcal{R}_{\pi_2}$ and satisfies $\Delta(x,y) \in \{x\} \times Y$ for all $(x,y) \in \mathcal{U}$, it follows that $g \in \Gamma_2$ and that $(gH)^{-1} \overset{\sigma_2}{\cdot} \mathcal{V} \cap \mathcal{V}$ is non-negligible. Hence, $\pi_2(g)^{-1} \cdot \mathcal{V}_0 \cap \mathcal{V}_0$ is non-negligible, proving that $\pi_2(g) \in \Lambda_2$.

We set $\Gamma_0 = \pi_1^{-1}(\Lambda_1) = \pi_2^{-1}(\Lambda_2)$. By ergodicity of $\Gamma_0 \curvearrowright X \times Y_1$, we can choose a measurable map $\varphi : X \times Y_1 \to \Gamma_0$ such that $\varphi(x,y) \cdot (x,y) \in \mathcal{U}$ for almost all $(x,y) \in X \times Y_1$. Write

$$\psi : X \times Y_1 \to X \times Y_2 : \psi(x,y) = \varphi(x,y)^{-1} \cdot \Delta(\varphi(x,y) \cdot (x,y)) \; .$$

In exactly the same way as in the proof of item (1), it follows that $\psi(x,y) = (x,\Delta_0(y))$, where Δ_0 is a non-singular isomorphism of Y_1 onto Y_2 satisfying $\Delta_0(\pi_1(g) \cdot y) = \pi_2(g) \cdot \Delta_0(y)$ for all $g \in \Gamma_0$ and almost all $y \in Y$. □

As we will see below, Lemma 5.1 allows us to construct free, ergodic, measure-preserving actions of groups on *infinite* measure spaces, whose orbit equivalence relation has prescribed fundamental group and, to a certain extent, prescribed outer automorphism group. In order to achieve similar results with *probability measure-preserving* actions, we will need the following lemma.

Lemma 5.2. *Suppose that we are in the setup of Lemma* 5.1*.*

(1) *Suppose that $\Lambda \curvearrowright Y$ is induced from $\Lambda_1 \curvearrowright Y_1$ with $\eta(Y_1) < \infty$. Set $\Gamma_1 = \pi^{-1}(\Lambda_1)$. Then, the orbit equivalence relation of the free, ergodic, probability measure-preserving action $\Gamma_1 \curvearrowright X \times Y_1$ coincides with the reduction of $\mathcal{R}_\pi$ to $X \times Y_1$.*

(2) *Suppose that $\widetilde{\Gamma} = H * K$, where $H \cong \mathbb{F}_\infty \cong K$. Let $Y_1 \subset Y$ be any measurable subset with $0 < \eta(Y_1) < \infty$. Then, the restriction $\mathcal{R}$ of $\mathcal{R}_\pi$ to $X \times Y_1$ is treeable with infinite cost. Hence, by Corollary 1.2 in* [15]*, there exists a free, ergodic, probability measure-preserving action $\mathbb{F}_\infty \curvearrowright X \times Y_1$ such that $\mathcal{R} = \mathcal{R}(\mathbb{F}_\infty \curvearrowright X \times Y_1)$.*

Remark 5.3. In [34, Lemma 3.6], we consider the more special kind of diagonal action $\Gamma * \Lambda \curvearrowright X \times Y$, given an amenable group Λ acting on the infinite measure space (Y,η) and using the canonical quotient morphism $\Gamma * \Lambda \to \Lambda$. In that case, it is easy to see that the restriction of the orbit equivalence relation to a subset of finite measure can be implemented by a free action of a group. See [34, Lemma 3.6] for a precise statement.

Proof of Lemma 5.2. The first part of the statement is obvious.

Since $\Gamma \cong \mathbb{F}_\infty$ acts freely on $X \times Y$, the orbit equivalence relation $\mathcal{R}_\pi$ is treeable. By Proposition II.6 in [10], also $\mathcal{R}$ is treeable. Now, $H \cong \mathbb{F}_\infty \cong K$ and Γ is a normal subgroup of $H * K$ containing H. It follows that Γ freely splits off H and that we can choose free generators $g_n, n \in \mathbb{N}$ of H that can be completed into a free generating set of $\Gamma \cong \mathbb{F}_\infty$. Lemma II.8 in [10] provides a treeing for $\mathcal{R}$, starting from the treeing for $\mathcal{R}(\Gamma \curvearrowright X \times Y)$ given by the set of free generators of Γ. This treeing contains for every $n \in \mathbb{N}$, the automorphisms $(x, y) \mapsto (g_n \cdot x, y)$ of $X \times Y_1$. Hence, the cost of $\mathcal{R}$ is infinite. $\square$

Recall from the introduction the notion of an *ergodic measure* on the Borel sets of $\mathbb{R}$, as well as the associated subgroup $H_\nu \subset \mathbb{R}$. As explained in the preliminaries, the group H_ν can be uncountable (without being $\mathbb{R}$) and with prescribed Hausdorff dimension.

Theorem 5.4. *Let ν be an ergodic measure on $\mathbb{R}$ with associated subgroup $H_\nu \subset \mathbb{R}$. Let $\mathcal{G}$ be any totally disconnected unimodular locally compact group. There exists a continuous family $(\sigma_i)_{i \in I}$ of free, ergodic, probability measure-preserving actions $\sigma_i : \mathbb{F}_\infty \curvearrowright (X_i, \mu_i)$ with the following properties.*

- *The fundamental group of $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X_i)$ equals $\exp(H_\nu)$.*
- *The outer automorphism group of $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X_i)$ is isomorphic with $\mathcal{G}$.*
- *The actions $(\sigma_i)_{i \in I}$ are not stably orbit equivalent.*
- $\mathrm{L}^\infty(X_i)$ *is an $\mathcal{HT}$ Cartan subalgebra of* $\mathrm{L}^\infty(X_i) \rtimes_{\sigma_i} \mathbb{F}_\infty$ *in the sense of* [28, Definition 6.1].

Proof. We apply Lemma 5.1 with $\widetilde{\Lambda} = \Lambda$. Below, we construct a free ergodic m.p. action of Λ on an infinite measure space (Y, η), $\Lambda \curvearrowright (Y, \eta)$, such that

- the homomorphism mod : $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda) \to \mathbb{R}_+$ has image $\exp(H_\nu)$ and kernel isomorphic with $\mathcal{G}$,
- the action $\Lambda \curvearrowright (Y, \eta)$ is orbit equivalent to a profinite action.

Recall here that $\mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$ denotes the group of non-singular automorphisms of Y that commute with the action $\Lambda \curvearrowright (Y, \eta)$. Such automorphisms Δ automatically scale the measure η and the scaling factor is denoted by $\operatorname{mod} \Delta$.

We then take $\Gamma = H * K$, where $H \cong \mathbb{F}_\infty \cong K$. Using Theorem 4.1, we choose a free m.p. Γ-action on a probability space, $\Gamma \curvearrowright (X, \mu)$, such that H acts ergodically and rigidly, and such that $\mathrm{Emb}(H^\infty, \Gamma^\infty) = [\Gamma^\infty]$. Because of Lemmas 5.1 and 5.2, any surjective homomorphism $\pi : \Gamma \to \Lambda$ with $H \subset \operatorname{Ker} \pi$ will provide us with an action of $\mathbb{F}_\infty$. Since we can choose continuously many different $\operatorname{Ker} \pi$, we actually get continuously many non-stably orbit equivalent actions of this kind.

We now construct $\Lambda \curvearrowright (Y, \eta)$ as the product of free, ergodic, infinite measure-preserving actions $\Lambda_1 \curvearrowright (Y_1, \eta_1)$ and $\Lambda_2 \curvearrowright (Y_2, \eta_2)$, in such a way that

- Λ_1 is amenable and $\Lambda_2 \curvearrowright (Y_2, \eta_2)$ is profinite,
- the homomorphism mod : $\mathrm{Centr}_{\mathrm{Aut}\, Y_1}(\Lambda_1) \to \exp(H_\nu)$ is an isomorphism of groups,
- the elements of $\mathrm{Centr}_{\mathrm{Aut}\, Y_2}(\Lambda_2)$ are measure-preserving and form a group isomorphic with $\mathcal{G}$.

By ergodicity of $\Lambda_i \curvearrowright (Y_i, \eta_i)$, any $\Delta \in \mathrm{Centr}_{\mathrm{Aut}\, Y}(\Lambda)$ has the form $\Delta = \Delta_1 \times \Delta_2$, so that we can safely construct both actions separately. By amenability of Λ_1, the action $\Lambda_1 \curvearrowright (Y_1, \eta_1)$ is orbit equivalent with a profinite action.

Part 1. Construction of the action $\Lambda_1 \curvearrowright (Y_1, \eta_1)$.

Fix a countable amenable group G, different from $\{e\}$ and having infinite conjugacy classes.

Define for every $0 < t < 1$, the probability space

$$(Z_t, \eta_t) := \Big(\{0,1\}, \frac{1}{1+t}\delta_0 + \frac{t}{1+t}\delta_1\Big)^G .$$

The group $G \times G$ acts on the set G by left-right translation and so, we define the amenable group

$$G_t := \Big(\frac{\mathbb{Z}}{2\mathbb{Z}}\Big)^{(G)} \rtimes (G \times G) ,$$

acting freely on (Z_t, η_t) by non-singular automorphisms. Note however that these automorphisms do not preserve η_t.

Let ν be an ergodic measure on $\mathbb{R}$ and $Q \subset \mathbb{R}$ a countable subgroup such that $\nu \circ \lambda_x = \nu$ for all $x \in Q$ and such that every Q-invariant Borel function on $\mathbb{R}$ is ν-almost everywhere constant. Set $\mathcal{J} = Q \cap (0,1)$. Define the amenable group

$$\Lambda = \bigoplus_{t \in \mathcal{J}} G_t \quad \text{and} \quad (Y_0, \eta_0) = \prod_{t \in \mathcal{J}} (Z_t, \eta_t) .$$

Note that Λ acts on (Y_0, η_0) by non-singular automorphisms.

Define $(Y, \eta) = (Y_0 \times \mathbb{R}, \mu_0 \times \nu_1)$, where ν_1 is given by $d\nu_1(x) = \exp(x) d\nu(x)$. Denote by $\omega : \Lambda \times Y_1 \to Q$ the logarithm of the Radon-Nikodym 1-cocycle. It follows that the action

$$\Lambda \curvearrowright Y : g \cdot (y, x) = (g \cdot y, x - \omega(g, y))$$

preserves the infinite measure η. The action $\Lambda \curvearrowright Y$ is free.

Consider the subgroup $\mathcal{L}$ of Λ defined by

$$\mathcal{L} = \bigoplus_{t \in \mathcal{J}} (G \times G) .$$

Observe that $\mathcal{L} \curvearrowright (Y_0, \eta_0)$ preserves the measure η_0 and is ergodic. Therefore, a Λ-invariant measurable function F on $Y = Y_0 \times \mathbb{R}$ is η-almost everywhere equal to a function only depending on the $\mathbb{R}$-variable. Since the Radon-Nikodym 1-cocycle attains all values in Q, the ergodicity of ν implies that F is η-almost everywhere constant.

To conclude part 1, it suffices to prove that every non-singular automorphism Δ of (Y, η) that commutes with the Λ-action, is of the form $\Delta_z : (y, x) \mapsto (y, x + z)$ for some $z \in H_\nu$. Write $\Delta(y, x) = (\Delta_1(y, x), \Delta_2(y, x))$. It follows that Δ_2 is $\mathcal{L}$-invariant and hence essentially independent of the Y_0-variable. Hence, $\Delta_2(y, x) = \beta(x)$, η-almost everywhere, for some non-singular automorphism β of $(\mathbb{R}, \nu)$. So, for ν-almost every $x \in \mathbb{R}$, we find a non-singular automorphism α_x of (Y_0, η_0) such that $\Delta(y, x) = (\alpha_x(y), \beta(x))$ almost everywhere. But then, almost every α_x commutes with the $\mathcal{L}$-action on (Y_0, η_0). As in the proof of [33, Theorem 5.4], such an α_x is the identity almost everywhere and we get $\Delta(y, x) = (y, \beta(x))$ almost everywhere. Hence, β is a non-singular automorphism of $(\mathbb{R}, \nu)$ commuting with the Q-action. But then the function $\mathbb{R} \to \mathbb{R} : x \mapsto \beta(x) - x$ is Q-invariant and so, by our assumptions on ν, ν-almost everywhere constant. Hence, $\beta(x) = x + z$ for some $z \in \mathbb{R}$ and ν-almost every $x \in \mathbb{R}$. The non-singularity of β implies that $z \in H_\nu$. We have proven that $\Delta(y, x) = (y, x + z)$ almost everywhere.

We denote as $\Lambda_1 \curvearrowright (Y_1, \eta_1)$ the action $\Lambda \curvearrowright (Y, \eta)$ constructed above.

Part 2. Construction of the action $\Lambda_2 \curvearrowright (Y_2, \eta_2)$.

Choose a countable dense subgroup $\Lambda \subset \mathcal{G}$ acting on $\mathcal{G}$ by left multiplication. Equip $\mathcal{G}$ with its Haar measure. Let $\mathcal{K}_n \subset \mathcal{G}$ be a decreasing sequence of compact open subgroups of $\mathcal{G}$ with trivial intersection. Defining $Y_n = \mathcal{G}/\mathcal{K}_n$, the profiniteness of $\Lambda \curvearrowright \mathcal{G}$ follows, with $\Lambda_n = \Lambda$ for all n.

We leave it as an exercise to prove that every non-singular automorphism Δ of $\mathcal{G}$ commuting with the Λ-action is given by right multiplication with an element in $\mathcal{G}$. We denote as $\Lambda_2 \curvearrowright (Y_2, \eta_2)$ the action $\Lambda \curvearrowright (\mathcal{G}, \mathrm{Haar})$ constructed above. $\square$

Remark 5.5. The unimodularity assumption on $\mathcal{G}$ in Theorem 5.4 is not essential. If $\mathcal{G}$ is no longer unimodular, the image of $\mathcal{G}$ under the modular function will be part of the fundamental group, and the outer automorphism group will be given by the kernel of the modular function.

Since in Lemma 5.1, the action $\Lambda \curvearrowright (Y, \eta)$ is only assumed to be orbit equivalent with a profinite action, locally compact groups other than the totally disconnected ones can be covered in Theorem 5.4. In particular, any second countable, locally compact, abelian group arises, since any countable dense subgroup is still abelian and hence, amenable.

The following lemma was used in the proof of Lemma 5.1. Its proof is very similar to arguments in A.1 of [28]. We include the details for completeness.

Lemma 5.6. *Let (A, τ) be an abelian von Neumann algebra with faithful normal tracial state τ and $0 < \varepsilon < 1$. Let $B, C \subset A$ be von Neumann subalgebras. If $\|E_C(b) - b\|_2 \leq \varepsilon$ for all b in the unit ball of B, there exists a projection $q \in A$ with $\tau(q) \geq 1 - 28\sqrt{\varepsilon}$ such that $Bq \subset Cq$.*

Proof. Consider the basic construction $\langle A, e_C \rangle$ of the inclusion $C \subset A$, equipped with its natural semi-finite trace Tr. Denote

$$\mathcal{K} = \operatorname{conv}\{u e_C u^* \mid u \in \mathcal{U}(B)\} .$$

By our assumption, $\|x - e_C\|_{2,\mathrm{Tr}} \leq 2\varepsilon$ for all $x \in \mathcal{K}$. Let x be the element of minimal $\|\cdot\|_{2,\mathrm{Tr}}$ in the weak closure of $\mathcal{K}$. Then, $\|x - e_C\|_{2,\mathrm{Tr}} \leq 2\varepsilon$, $0 \leq x \leq 1$, $\mathrm{Tr}(x) \leq 1$ and $x \in B' \cap \langle A, e_C \rangle$. Denote by p the spectral projection $p = \chi_{[1/2,1]}(x)$. It follows that p is a projection in $B' \cap \langle A, e_C \rangle$ such that $\|p - e_C\|_{2,\mathrm{Tr}} \leq 4\varepsilon$.

Denote by $\mathcal{E}_C$ the center-valued weight from $\langle A, e_C \rangle^+$ to the extended positive part of C. Note that $\mathrm{Tr} = \tau \circ \mathcal{E}_C$.

The image of the projection p is a Hilbert B-C-subbimodule K of $\mathrm{L}^2(A, \tau)$. As a right C-module, the isomorphism class of K is determined by $\mathcal{E}_C(p)$. We have $\|\mathcal{E}_C(p) - 1\|_2 \leq 4\varepsilon$. If we view (C, τ) as $\mathrm{L}^\infty(X, \mu)$, the function $\mathcal{E}_C(p)$ takes values in $\mathbb{N} \cup \{+\infty\}$. Defining z as the largest projection in C such that $\mathcal{E}_C(p)z = z$, it follows that $\|1 - z\|_2 \leq 4\varepsilon$. Then,

$$\|p(1 - z)\|_{2,\mathrm{Tr}}^2 = \tau(\mathcal{E}_C(p)(1 - z)) = \tau(\mathcal{E}_C(p) - z) = \tau(\mathcal{E}_C(p) - 1) + \tau(1 - z) .$$

So, $\|p(1 - z)\|_{2,\mathrm{Tr}}^2 \leq 8\varepsilon$ and hence, $\|p - pz\|_{2,\mathrm{Tr}} \leq 3\sqrt{\varepsilon}$. But then,

$$\|pz - e_C\|_{2,\mathrm{Tr}} \leq \|pz - p\|_{2,\mathrm{Tr}} + \|p - e_C\|_{2,\mathrm{Tr}} \leq 3\sqrt{\varepsilon} + 4\varepsilon \leq 7\sqrt{\varepsilon} .$$

Replacing p by pz, we have found a projection p in $B' \cap \langle A, e_C\rangle$ such that $\|p - e_C\|_{2,\mathrm{Tr}} \leq 7\sqrt{\varepsilon}$ and such that $\mathcal{E}_C(p) = z$, where z is a projection in C satisfying $\|1 - z\|_2 \leq 4\varepsilon$.

As a C-module, the image K of p is then isomorphic to $\mathrm{L}^2(Cz)$. This provides us with $v \in \mathrm{L}^2(A)z$ such that K is the L^2-closure of vC and such that $E_C(v^*v) = z$. Hence, $p = ve_Cv^*$. Denote by q the support projection of v. Because K is also a B-module, we have $Bq \subset Cq$. It remains to prove that $\tau(q) \geq 1 - 28\sqrt{\varepsilon}$.

We have $\|ve_Cv^* - e_C\|_{2,\mathrm{Tr}} \leq 7\sqrt{\varepsilon}$, meaning that $1+\tau(z)-2\|E_C(v)\|_2^2 \leq 49\varepsilon$. Since $\tau(1-z) \leq 4\varepsilon$, we get $1 - \|E_C(v)\|_2^2 \leq 27\varepsilon$. The left-hand side equals $\|v - E_C(v)\|_2^2$ and so $\|v - E_C(v)\|_2 \leq 6\sqrt{\varepsilon}$. Also $\|v\|_2 \leq 1$, implying that $\|v^*v - E_C(v)^*E_C(v)\|_1 \leq 12\sqrt{\varepsilon}$. Applying E_C and using $E_C(v^*v) = z$, we conclude that $\|v^*v - z\|_1 \leq 24\sqrt{\varepsilon}$. But then, since $q \leq z$, we have $\tau(z - q) \leq 24\sqrt{\varepsilon}$. We also have $\tau(1-z) \leq 4\varepsilon \leq 4\sqrt{\varepsilon}$, finally leading to $\tau(q) \geq 1 - 28\sqrt{\varepsilon}$. $\square$

6. The II_1 factor case

The part of Theorem 1.1 pertaining to the group measure space II_1 factors $M_i = \mathrm{L}^\infty(X) \rtimes_{\sigma_i} \mathbb{F}_\infty$ now follows immediately from Theorem 5.4 and the results in [28] about uniqueness of $\mathcal{HT}$ Cartan subalgebras. Indeed, by [28, Theorem 6.2], if M_1 and M_2 are II_1 factors with $\mathcal{HT}$ Cartan subalgebras $A_i \subset M_i$ and corresponding equivalence relations $\mathcal{R}_i$, any *-isomorphism $\pi : M_1 \to M_2$ can be composed with an inner automorphism $\mathrm{Ad}u$ of M_2 so that $\mathrm{Ad}(u)(\pi(A_1)) = A_2$, thus implementing an orbit equivalence of $\mathcal{R}_1$ and $\mathcal{R}_2$. A similar result holds for stable isomorphisms. This shows in particular that if $A \subset M$ is an $\mathcal{HT}$ Cartan subalgebra with corresponding equivalence relation $\mathcal{R}$, then $\mathcal{F}(M) = \mathcal{F}(\mathcal{R})$ and $\mathrm{Out}(M) \cong \mathrm{H}^1(\sigma) \rtimes \mathrm{Out}(\mathcal{R})$.

Note however that for the actions $\sigma_i : \mathbb{F}_\infty \curvearrowright X_i$ constructed in the proof of Theorem 5.4, neither $\mathrm{H}^1(\sigma_i)$ nor $\mathrm{Out}(M_i)$ are Polish groups, because the orbit equivalence relation $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X_i)$ is not strongly ergodic and the II_1 factor M_i has property (Γ). Indeed, $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X_i)$ arises as the restriction to a set of finite measure of the orbit equivalence relation of a diagonal action $\mathbb{F}_\infty \curvearrowright X \times Y_1 \times Y_2$, where $\mathbb{F}_\infty$ acts on Y_1 through an amenable quotient Λ_1 of $\mathbb{F}_\infty$. Since the hyperfinite relation $\mathcal{R}(\Lambda_1 \curvearrowright Y_1)$ is not strongly ergodic, neither is $\mathcal{R}(\mathbb{F}_\infty \curvearrowright X_i)$.

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