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QUANTIZATION OF QUASI-LIE BIALGEBRAS

BENJAMIN ENRIQUEZ AND GILLES HALBOUT

Let $\mathbf{k}$ be a field of characteristic 0. Unless specified otherwise, "Lie algebra", "vector space", etc., mean "Lie algebra over $\mathbf{k}$", etc.

1. Introduction and main results

The main result of this paper is the construction of quantization functors of quasi-Lie bialgebras. We first recall the situation of this problem in the theory of quantum groups.

This theory started, on the mathematical side, with the discovery by Drinfeld and Jimbo of a noncocommutative deformation $U_q\mathfrak{g}$, in the category of Hopf algebras, of the enveloping algebra of a Kac-Moody Lie algebra. This discovery was soon related to important developments in knot theory and representation theory. At the same time, Drinfeld developed a program of formal deformation of enveloping algebras in the category of Hopf algebras (such a deformation is called a quantized universal enveloping, or QUE, algebra). Any QUE algebra leads to Lie algebraic data (more precisely, a Lie bialgebra), its "classical limit", and Drinfeld asked whether a QUE algebra could be reconstructed from its classical limit; this question was solved by Etingof and Kazhdan, who constructed "quantization" functors from the category of Lie bialgebras to that of QUE algebras.

The introduction of $U_q\mathfrak{g}$ was motivated by mathematical physics, more precisely, the Wess-Zumino-Novikov-Witten model in conformal field theory. One of the mathematical incarnations of this model is the differential system obeyed by its correlation functions, the Knizhnik-Zamolodchikov (KZ) system. The relation between the KZ system and $U_q\mathfrak{g}$ was first made precise by Kohno, who expressed the monodromy representation of the KZ system using the representation theory of $U_q\mathfrak{g}$, when $\mathfrak{g}$ is finite dimensional. This theorem was then reproved by Drinfeld, who developed for this purpose the theory of quasi-Hopf algebras (QHAs). The main novel feature of this theory with respect to that of Hopf algebras is that the coassociativity axiom of the coproduct is weakened: namely the coassociativity identity $(\mathrm{id}\otimes\Delta)(\Delta(x)) = (\Delta\otimes\mathrm{id})(\Delta(x))$ for $x \in A$ is replaced by $(\mathrm{id}\otimes\Delta)(\Delta(x)) = \Phi(\Delta\otimes\mathrm{id})(\Delta(x))\Phi^{-1}$, where $\Phi \in A^{\otimes 3}$ is invertible and subject to a "pentagon" consistency condition. Similarly to Hopf algebras, QHAs lead to tensor categories via their categories of representations; finite-dimensional QHAs play an important role in the ongoing program of classification of finite tensor categories.

Drinfeld also developed a program for constructing a particular class of QHAs (the quasi-Hopf quantum universal enveloping, or QHQUE algebras) using formal

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deformation theory. Namely, he showed that such a QHA gives rise to a Lie algebraic object called a quasi-Lie bialgebra (its classical limit). This gives rise to a functor {QHQUE algebras} $\to$ {quasi-Lie bialgebras} (we recall this theory in Section 2). In [Dr4], Section 5, Drinfeld posed the problem of functorially constructing a QHQUE algebra corresponding to any given quasi-Lie bialgebra, i.e., of constructing a section Q of this functor, such that the structure maps of $Q(\mathfrak{a})$ can be expressed using the structure maps of a quasi-Lie bialgebra $\mathfrak{a}$ using universal acyclic formulas; Q is then called a quantization functor for quasi-Lie bialgebras. This is the problem which we solve in the present paper.

As in the case of Lie bialgebras, quantization functors of quasi-Lie bialgebras are best defined in the framework of "props" (these are particular symmetric monoidal categories; see [McL] and Section 3.1). In Section 3.2, we introduce a graded prop QLBA of quasi-Lie bialgebras together with its completion **QLBA**. A quantization functor of quasi-Lie bialgebras is then the same as a QHQUE algebra in **QLBA**, satisfying suitable conditions (see Proposition 3.2); there are natural notions of equivalence and twist equivalence for these QHQUE algebras. In the same way, quantization functors for QUE algebras as in [EK2] may be viewed as certain QUE algebras in the prop **LBA** of Lie bialgebras. There is a natural prop morphism **QLBA** $\to$ **LBA**, which induces a map {quantization functors for quasi-Lie bialgebras} $\to$ {quantization functors for Lie bialgebras}. Our main result is then (Theorem 4.1):

Theorem 1.1. *The map {quantization functors of quasi-Lie bialgebras}/(equivalence, twist equivalence) $\to$ {quantization functors of Lie bialgebras}/(equivalence) is a bijection.*

Together with the results of [EK1, EK2], where is constructed a map {associators over $\mathbf{k}$} $\to$ {quantization functors of Lie bialgebras}, and of [Dr3] on the existence of associators over $\mathbf{k}$, this result implies the existence of quantization functors for quasi-Lie bialgebras.

Let us explain the idea of the proof of Theorem 1.1. According to the Gerstenhaber-Schack deformation theory ([GS]), the obstruction to lifting a given deformation of a (quasi-)bialgebra A from order n to $n+1$ belongs to a cohomology group $H^4(A)$ ($\tilde{H}^4(A)$ in the quasi case), and when this obstruction vanishes, such lifts modulo equivalence (and twists in the quasi case) form an affine space over $H^3(A)$ ($\tilde{H}^3(A)$ in the quasi case). When $A = U(\mathfrak{a})$ is an enveloping algebra, these cohomology groups have been computed by Shnider and Sternberg in terms of the Chevalley-Eilenberg cohomology and cocycle groups $H^p(\mathfrak{a}, \Lambda^q(\mathfrak{a}))$ and $Z^1(\mathfrak{a}, \Lambda^q(\mathfrak{a}))$ (see [ShSt]). We develop the categorical analogue of this theory in Sections 2.4, 2.6. In particular, we construct $\mathbb{N}$-graded groups $H^\bullet_{\mathrm{(Q)LBA}} = \bigoplus_{n\geq 0} H^\bullet_{\mathrm{(Q)LBA}}[n]$, such that $H^4_{\mathrm{(Q)LBA}}[n+1]$ contains the obstruction to lifting a quantization functor of (quasi-)Lie bialgebras from order n to $n+1$, and $H^3_{\mathrm{(Q)LBA}}[n+1]$ parametrizes classes of such lifts. This viewpoint is not used in the quantization of Lie bialgebras, since the groups $H^\bullet_{\mathrm{LBA}}$ are not known (in the same way, it is not known how to construct associators using deformation theory; see Remark 2, p. 854 in [Dr3]). However, this viewpoint can be used in our context. Namely, we construct maps $H^\bullet_{\mathrm{QLBA}} \to H^\bullet_{\mathrm{LBA}}$, compatible with the map from Theorem 1.1, and prove:

Theorem 1.2. *The map $H^i_{\mathrm{QLBA}} \to H^i_{\mathrm{LBA}}$ is an isomorphism of $\mathbb{N}$-graded vector spaces, for $i \geq 0$.*

Theorem 1.1 then follows immediately (see Section 4). Note that we use Theorem 1.2 only for $i = 3, 4$. More precisely, surjectivity in Theorem 1.1 follows from the injectivity of $H^4_{\mathrm{QLBA}} \to H^4_{\mathrm{LBA}}$ and surjectivity of $H^3_{\mathrm{QLBA}} \to H^3_{\mathrm{LBA}}$ (i.e., from the vanishing of the relative H^4) and injectivity in Theorem 1.1 follows from injectivity of the latter map (i.e., from the vanishing of the relative H^3).

Let us give some details on the proof of Theorem 1.2. We have $H^i_{\mathrm{(Q)LBA}} = \bigoplus_{p+q=i,q>0} H^p_{\mathrm{(Q)LBA}}(\mathbf{id}, \Lambda^q)$, where $H^\bullet_{\mathrm{(Q)LBA}}(\mathbf{id}, \Lambda^q)$ is the cohomology of a complex $C^\bullet_{\mathrm{(Q)LBA}} = (\mathrm{(Q)LBA}(\Lambda^\bullet, \Lambda^q), [\mu, -])$. Our aim is to prove that the relative complex $\mathrm{Ker}(C^\bullet_{\mathrm{QLBA}} \to C^\bullet_{\mathrm{LBA}})$ is acyclic.

We introduce a filtration of the prop QLBA by powers of an ideal $\langle\varphi\rangle$. Our first main result is Theorem 5.1, which gives an isomorphism $\mathrm{gr}\,\mathrm{QLBA} \simeq \mathrm{LBA}_\alpha$ of the associated graded prop with an explicitly presented prop. For this, we construct a morphism $\mathrm{LBA}_\alpha \to \mathrm{gr}\,\mathrm{QLBA}$, which is clearly surjective; to prove its injectivity, we use the existence of "many" quasi-Lie bialgebras, namely, the classical twists of Lie bialgebras of the form $F(\mathfrak{c})$ (where $\mathfrak{c}$ is a Lie coalgebra) by an element $r \in \Lambda^2(\mathfrak{c})$ (in the same way, the existence of the Lie bialgebras $F(\mathfrak{c})$ is the argument underlying the structure theorem for the prop LBA; see [E, Pos]).

The associated graded (for the filtration of QLBA) of the relative complex is then the positive degree part of the complex $C^\bullet_{\mathrm{LBA}_\alpha}$, which has the form $0 \to \mathrm{LBA}_\kappa(\mathbf{1}, \Lambda^q) \to \mathrm{LBA}_\kappa(\mathbf{id}, \Lambda^q) \to \mathrm{LBA}_\kappa(\Lambda^2, \Lambda^q) \to \ldots$, where $\mathrm{LBA}_\kappa(X, Y) = \mathrm{Coker}(\mathrm{LBA}(D \otimes X, Y) \to \mathrm{LBA}(C \otimes X, Y))$ and the map corresponds to $\kappa \in \mathrm{LCA}(C, D)$. Here C, D are sums of Schur functors of positive degree, and the differential is the universal version of the differential of Lie algebra cohomology.

The proof of the acyclicity of this complex (Theorem 4.2, proof in Section 6) involves several reductions. We first show that in this complex, the spaces of cochains may be modified as follows: $\mathrm{LBA}_\kappa(\Lambda^p, \Lambda^q)$ is replaced by $\mathrm{LBA}_Z(\Lambda^p, \Lambda^q) = \mathrm{LBA}(Z \otimes \Lambda^p, \Lambda^q)$, where Z is an irreducible Schur functor, and the space of cochains is reduced to the sum of its components, where the "intermediate Schur functor between Z and Λ^q" is Z. (This notion is based on the structure theorem of LBA; see Proposition 5.1. We say that the intermediate Schur functor between X_i and Y_j in the summand appearing in the r.h.s. of (1) is Z_{ij}.) We next introduce a filtration on the complex, viewing Λ^p as a subobject of $\mathbf{id}^{\otimes p}$ and counting the number of intermediate Schur functors between the p factors $\mathbf{id}$ and Λ^q which are equal to $\mathbf{id}$. We identify the associated graded complex with a subcomplex of $0 \to \mathrm{LBA}(Z \otimes \mathbf{1} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q) \to \ldots \to \mathrm{LBA}(Z \otimes \Lambda^{p'} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q) \to \ldots$, where the differential involves Lie brackets between the components of $\mathbf{id}^{\otimes p'} \supset \Lambda^{p'}$ and of these components with Z, formed by the sums of components, where the intermediate Schur functor between a component $\mathbf{id}$ of $\mathbf{id}^{\otimes p'}$ (resp., of $\mathbf{id}^{\otimes p''} \supset \Lambda^{p''}$) and Λ^q is $\mathbf{id}$ (resp., has degree > 1) and antisymmetric with respect to $\mathfrak{S}_{p''}$. This subcomplex decomposes according to the intermediate Schur functors between the factors of $\mathbf{id}^{\otimes p''}$ and Λ^q, and these subcomplexes are obtained from the complexes $C^\bullet_{\underline{Z}''} = (0 \to \mathrm{LA}(Z \otimes \mathbf{1} \otimes (\bigotimes_{i=1}^{p''} Z''_i), \Lambda^q) \to \ldots \to \mathrm{LA}(Z \otimes \Lambda^{p'} \otimes (\bigotimes_{i=1}^{p''} Z''_i), \Lambda^q) \to \ldots)$ with the same differentials (the Z''_i are irreducible Schur functors of degree > 1) by taking tensor products with vector spaces $\mathrm{LCA}(\mathbf{id}, Z''_i)$ and taking anti-invariants under $\mathfrak{S}_{p''}$. We therefore have to show the acyclicity of the complexes $C^\bullet_{\underline{Z}''}$.

For this, we show that Λ^q may be replaced by $\mathbf{id}^{\otimes q}$, Z by $\mathbf{id}^{\otimes z}$, and $\bigotimes_{i=1}^{p''} Z_i''$ by $\mathbf{id}^{\otimes N}$ (where Z, N are suitable integers), and we express the corresponding complex as a sum of tensor products of complexes. This reduces the problem to showing the acyclicity of a complex $0 \to \mathrm{LA}(\mathbf{id}^{\otimes z} \otimes \mathbf{1} \otimes \mathbf{id}^{\otimes N}, \mathbf{id}) \to ... \to \mathrm{LA}(\mathbf{id}^{\otimes z} \otimes \Lambda^{p'} \otimes \mathbf{id}^{\otimes N}, \mathbf{id}) \to ....$ The spaces of chains are now spaces of multilinear Lie polynomials. Using Dynkin's correspondence between free Lie and free associative polynomials, we identify the complex with a complex $\mathcal{A}^\bullet_{z,N,1}$, defined in terms of associative polynomials, which we decompose as a direct sum $\bigoplus_\sigma \mathcal{A}^\bullet_\sigma$ of subcomplexes, indexed by permutations. We then identify each summand $\mathcal{A}^\bullet_\sigma$ with a tensor product of "elementary" complexes. These complexes $\mathcal{E}^\bullet_{\epsilon,\epsilon'}$ ($\epsilon, \epsilon' \in \{0,1\}$) are 1-dimensional in each degree, and are universal versions of the complexes computing the Lie algebra cohomology of a Lie algebra $\mathfrak{g}$ in $U(\mathfrak{g})$, equipped with one of its trivial, adjoint, left or right $\mathfrak{g}$-module structures. We show that two of these complexes are acyclic, using the PBW filtration of free associative algebras (when $\mathfrak{g}$ is a finite-dimensional Lie algebra, the corresponding complexes have 1-dimensional cohomology, concentrated in degree $\dim \mathfrak{g}$). As $\mathcal{E}^\bullet_{0,1}$ necessarily enters the tensor product decomposition of each subcomplex $\mathcal{A}^\bullet_\sigma$, the Künneth formula implies that each of the $\mathcal{A}^\bullet_\sigma$ is acyclic, which implies that $\mathcal{A}^\bullet_{z,N,1}$ is acyclic.

In the final section of the paper, we apply Theorem 4.2 for proving that quantization functors of quasi-Lie bialgebras are compatible with twists (Proposition 7.1). This allows us to generalize our earlier results ([EH]) on compatibility of quantization functors of Lie bialgebras with twists; see Proposition 7.2 (in [EH], this result was established for Etingof-Kazhdan quantization functors, while Proposition 7.2 applies to any quantization functor of Lie bialgebras).

2. The problem of quantization of (quasi-)Lie bialgebras

In this section, we recall the basic results of the theory of quasi-Lie bialgebras and their quantum courterparts, QHQUE algebras. The material is essentially borrowed from [Dr2, ShSt].

2.1. Quasi-Hopf algebras. A quasi-bialgebra over $\mathbf{k}$ is a set $(A, \Delta, \epsilon, \Phi)$, where A is an associative algebra with unit, $\Delta : A \to A^{\otimes 2}$ and $\epsilon : A \to \mathbf{k}$ are morphisms of algebras with unit, and $\Phi \in (A^{\otimes 3})^\times$, such that[1]

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi(\Delta \otimes \mathrm{id})(\Delta(a))\Phi^{-1}, \quad a \in A,$$
$$(\mathrm{id}^{\otimes 2} \otimes \Delta)(\Phi)(\Delta \otimes \mathrm{id}^{\otimes 2})(\Phi) = (1 \otimes \Phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi)(\Phi \otimes 1),$$
$$(\epsilon \otimes \mathrm{id})(\Delta(a)) = a = (\mathrm{id} \otimes \epsilon)(\Delta(a)), \quad a \in A,$$
$$(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})(\Phi) = 1.$$

When $\Phi = 1$, this definition reduces to that of a bialgebra.

If $F \in (A^{\otimes 2})^\times$ is such that $(\epsilon \otimes \mathrm{id})(F) = (\mathrm{id} \otimes \epsilon)(F) = 1$, we define the twist of $(A, \Delta, \epsilon, \Phi)$ by F as $(A, \tilde\Delta, \epsilon, \tilde\Phi)$, where $\tilde\Delta(a) := F\Delta(a)F^{-1}$ and

$$\tilde\Phi := (1 \otimes F)(\mathrm{id} \otimes \Delta)(F)\Phi(\Delta \otimes \mathrm{id})(F)^{-1}(F \otimes 1)^{-1}.$$

This defines an action of the group $(A^{\otimes 2})^\times$ on the set of quasi-bialgebra structures over the associative algebra A.

[1]If R is a ring with unit, we denote by $R^\times$ the group of its invertible elements.

A quasi-Hopf algebra is a quasi-bialgebra equipped with supplementary data satisfying axioms, which in the bialgebra case reduce to the existence and invertibility of an antipode (see [Dr2]).

We now introduce generalizations of the above notions in symmetric tensor categories. Let $\mathcal{C}$ be a symmetric tensor category[2] with unit object $\mathbf{1}$.

An algebra with unit in $\mathcal{C}$ is a triple (X, m, η), where $X \in \mathrm{Ob}(\mathcal{C})$, $m \in \mathcal{C}(X \otimes X, X)$, $\eta \in \mathcal{C}(\mathbf{1}, X)$, such that $m \circ (m \otimes \mathrm{id}_X) = m \circ (\mathrm{id}_X \otimes m)$ and $m \circ (\eta \otimes \mathrm{id}_X) = \mathrm{id}_X = m \circ (\mathrm{id}_X \otimes \eta)$. If (X_i, m_i, η_i) are algebras with unit in $\mathcal{C}$ $(i = 1, 2)$, then a morphism from X_1 to X_2 is an $f \in \mathcal{C}(X_1, X_2)$, such that $f \circ m_1 = m_2 \circ f^{\otimes 2}$, $\eta_2 = f \circ \eta_1$.

If (X_i, m_i, η_i), $i = 1, 2$, are algebras with unit in $\mathcal{C}$, then so is $(X_1 \otimes X_2, m, \eta_1 \otimes \eta_2)$, where $m = (m_1 \otimes m_2) \circ (\mathrm{id}_{X_1} \otimes \beta \otimes \mathrm{id}_{X_2})$, and $\beta : X_2 \otimes X_1 \to X_1 \otimes X_2$ is the commutativity constraint in $\mathcal{C}$. In particular, if X is an algebra with unit in $\mathcal{C}$, then so is $X^{\otimes n}$ for $n \geq 0$ $(X^{\otimes 0} = \mathbf{1})$.

If (X, m, η) is an algebra with unit in $\mathcal{C}$, then $\mathcal{C}(\mathbf{1}, X)$ is equipped with the structure of a $\mathbf{k}$-algebra with unit, with product $f * g := m \circ (f \otimes g)$ and unit η. Moreover, for any $Y \in \mathrm{Ob}(\mathcal{C})$, the space $\mathcal{C}(Y, X)$ is equipped with a $\mathcal{C}(\mathbf{1}, X)$-bimodule structure, where $f * x * g := m^{(2)} \circ (f \otimes x \otimes g)$, and $m^{(2)} = m \circ (m \otimes \mathrm{id}_X)$. Note that if X_1, X_2 are algebras with unit in $\mathcal{C}$, then the tensor product in $\mathcal{C}$ yields an algebra morphism $\mathcal{C}(\mathbf{1}, X_1) \otimes \mathcal{C}(\mathbf{1}, X_2) \to \mathcal{C}(\mathbf{1}, X_1 \otimes X_2)$.

A quasi-bialgebra in $\mathcal{C}$ is then defined as a set $(X, \Delta, \epsilon, \Phi)$, where (X, m, η) is an algebra in $\mathcal{C}$, $\Delta \in \mathcal{C}(X, X^{\otimes 2})$ and $\epsilon \in \mathcal{C}(X, \mathbf{1})$ are morphisms of algebras with unit in $\mathcal{C}$, and $\Phi \in \mathcal{C}(\mathbf{1}, X^{\otimes 3})^\times$, such that

$$(\mathrm{id}_X \otimes \Delta) \circ \Delta = \Phi * \big((\Delta \otimes \mathrm{id}_X) \circ \Delta\big) * \Phi^{-1},$$
$$\big((\mathrm{id}_X^{\otimes 2} \otimes \Delta) \circ \Phi\big) * \big((\Delta \otimes \mathrm{id}_X^{\otimes 2}) \circ \Phi\big) = (\eta \otimes \Phi) * \big((\mathrm{id}_X \otimes \Delta \otimes \mathrm{id}_X) \circ \Phi\big) * (\Phi \otimes \eta),$$
$$(\epsilon \otimes \mathrm{id}_X) \circ \Delta = \mathrm{id}_X = (\mathrm{id}_X \otimes \epsilon) \circ \Delta,$$
$$(\mathrm{id}_X \otimes \epsilon \otimes \mathrm{id}_X) \circ \Phi = \eta^{\otimes 2}.$$

When $\Phi = \eta^{\otimes 3}$, this definition reduces to that of a bialgebra in $\mathcal{C}$. As above, the group $\mathcal{C}(\mathbf{1}, X^{\otimes 2})^\times$ acts on the set of quasi-bialgebra structures over X by $\tilde{\Delta} = F * \Delta * F^{-1}$,

$$\tilde{\Phi} := (\eta \otimes F) * \big((\mathrm{id}_X \otimes \Delta) \circ F\big) * \Phi * \big((\Delta \otimes \mathrm{id}_X) \circ F\big)^{-1} * (F \otimes \eta)^{-1}.$$

One can similarly introduce the notion of a (quasi-)Hopf algebra in $\mathcal{C}$. Similarly to associative algebras with unit, (quasi-)bialgebras, (quasi-)Hopf algebras in $\mathcal{C}$ form a category.

When $\mathcal{C} = \mathrm{Vect}$ (the category of $\mathbf{k}$-vector spaces), one recovers the notion of an associative algebra with unit, of a (quasi-)bialgebra, and of a (quasi-)Hopf algebra over $\mathbf{k}$.

[2]We denote by $\mathcal{C}(X, Y)$ the set of morphisms between two objects of $\mathcal{C}$, and by $\mathrm{End}_{\mathcal{C}}(X)$ and $\mathrm{Aut}_{\mathcal{C}}(X)$ the sets of endomorphisms and automorphisms of an object of $\mathcal{C}$.

2.2. **Quasi-Lie bialgebras.** A quasi-Lie bialgebra over $\mathbf{k}$ is a set $(\mathfrak{a}, \delta, \varphi)$, where $\mathfrak{a}$ is a Lie algebra over $\mathbf{k}$, and $\delta : \mathfrak{a} \to \Lambda^2(\mathfrak{a})$ and $\varphi \in \Lambda^3(\mathfrak{a})$ are such that[3]

$$\delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] + [\delta(x), y \otimes 1 + 1 \otimes y], \quad x, y \in \mathfrak{a},$$
$$\frac{1}{2} \mathrm{Alt}_3(\delta \otimes \mathrm{id})(\delta(x)) = [x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \varphi], \quad x \in \mathfrak{a},$$
$$\mathrm{Alt}_4(\delta \otimes \mathrm{id} \otimes \mathrm{id})(\varphi) = 0.$$

Here $\mathrm{Alt}_k : \mathfrak{a}^{\otimes k} \to \mathfrak{a}^{\otimes k}$ is the operator $\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\sigma$.

When $\varphi = 0$, this definition reduces to that of a Lie bialgebra.

If $f \in \Lambda^2(\mathfrak{a})$, we define the twist of $(\mathfrak{a}, \delta, \varphi)$ by f as $(\mathfrak{a}, \tilde{\delta}, \tilde{\varphi})$, where

$$\tilde{\delta}(x) = \delta(x) + [x \otimes 1 + 1 \otimes x, f], \quad \tilde{\varphi} = \varphi + \frac{1}{2} \mathrm{Alt}_3((\delta \otimes \mathrm{id})(f) + [f^{12}, f^{23}]).$$

This defines an action of the additive group $\Lambda^2(\mathfrak{a})$ on the set of quasi-Lie bialgebra structures over the Lie algebra $\mathfrak{a}$.

If now $\mathcal{C}$ is a symmetric tensor category, then a quasi-Lie bialgebra in $\mathcal{C}$ is a set $(\mathbf{x}, \mu, \delta, \varphi)$, where $\mathbf{x} \in \mathrm{Ob}(\mathcal{C})$, $\mu \in \mathcal{C}(\mathbf{x}^{\otimes 2}, \mathbf{x})$, $\delta \in \mathcal{C}(\mathbf{x}, \mathbf{x}^{\otimes 2})$, $\varphi \in \mathcal{C}(\mathbf{1}, \mathbf{x}^{\otimes 3})$, such that:

$$\mu \circ \beta = -\mu, \quad \beta \circ \delta = -\delta, \quad \beta_\sigma \circ \varphi = \epsilon(\sigma)\varphi$$

for $\sigma \in S_3$ (here $\beta \in \mathrm{Aut}_{\mathcal{C}}(\mathbf{x}^{\otimes 2})$ and the morphisms $S_k \to \mathrm{Aut}_{\mathcal{C}}(\mathbf{x}^{\otimes k})$, $\sigma \mapsto \beta_\sigma$ are induced by the commutativity constraint), and

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathbf{x}}) \circ \mathrm{Alt}_3 = 0, \quad \delta \circ \mu = \mathrm{Alt}_2 \circ (\mu \otimes \mathrm{id}_{\mathbf{x}}) \circ (\mathrm{id}_{\mathbf{x}} \otimes \delta) \circ \mathrm{Alt}_2,$$
$$\mathrm{Alt}_3 \circ (\delta \otimes \mathrm{id}_{\mathbf{x}}) \circ \delta = \mathrm{Alt}_3 \circ (\mu \otimes \mathrm{id}_{\mathbf{x}}^{\otimes 2}) \circ (\mathrm{id}_{\mathbf{x}} \otimes \varphi), \quad \mathrm{Alt}_4 \circ (\delta \otimes \mathrm{id}_{\mathbf{x}}^{\otimes 3}) \circ \varphi = 0,$$

where, as above, $\mathrm{Alt}_k = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\beta_\sigma \in \mathrm{End}_{\mathcal{C}}(\mathbf{x}^{\otimes k})$.

When $\varphi = 0$ (resp., $\varphi = 0$ and $\delta = 0$), this definition reduces to that of a Lie bialgebra (resp., Lie algebra) in $\mathcal{C}$. When $\mathcal{C} = \mathrm{Vect}$, we recover the notions of (quasi-)Lie (bi)algebra over $\mathbf{k}$.

2.3. **QHQUE algebras.** Let $\hbar$ be a formal variable. Topologically free $\mathbf{k}[[\hbar]]$-modules are modules of the form $V[[\hbar]]$, where V is a vector space. We call $\mathrm{Vect}_\hbar$ the tensor category of topologically free $\mathbf{k}[[\hbar]]$-modules; then reduction mod $\hbar$ is a tensor functor $\mathrm{Vect}_\hbar \to \mathrm{Vect}$.

A QUE (quasi-)bialgebra is a (quasi-)bialgebra $(A, \Delta, \epsilon, \Phi)$ in $\mathrm{Vect}_\hbar$, whose reduction mod $\hbar$ is a universal enveloping algebra and such that $\mathrm{Alt}\,\Phi \in \hbar^2 A^{\otimes 3}$. One similarly defines (quasi-)Hopf QUE algebras. It follows from [Dr2], Theorem 1.6, that a QUE (quasi-)bialgebra is equipped with a unique (quasi-)Hopf structure.

We define $\{$(Q)HQUE algebras over $\mathbf{k}\}$ as the full subcategory of $\{$(quasi-)Hopf algebras in $\mathrm{Vect}_\hbar\}$ whose objects are the (Q)HQUE algebras.

We now define the classical limit functor $\{$(Q)HQUE algebras over $\mathbf{k}\} \to \{$(quasi-)Lie bialgebras over $\mathbf{k}\}$. If A is a (Q)HQUE algebra, then its reduction mod $\hbar$ has the form $U(\mathfrak{a})$, where $\mathfrak{a}$ is a Lie algebra. If $a \in \mathfrak{a}$ and $\tilde{a} \in A$ is any lift of a, then $\big(\hbar^{-1} \mathrm{Alt}\,\Delta(\tilde{a}) \bmod \hbar\big) \in \Lambda^2(U(\mathfrak{a}))$ in fact belongs to $\Lambda^2(\mathfrak{a})$, and depends only on a. This defines a map $\delta : \mathfrak{a} \to \Lambda^2(\mathfrak{a})$. Letting $\varphi := \big(\hbar^{-2} \mathrm{Alt}\,\Phi \bmod \hbar\big) \in \Lambda^3(U(\mathfrak{a}))$, one proves that in fact $\varphi \in \Lambda^3(\mathfrak{a})$ and that $(\mathfrak{a}, \delta, \varphi)$ is a quasi-Lie bialgebra. The assignment $A \mapsto (\mathfrak{a}, \delta, \varphi)$ is the classical limit functor

[3] We view $\Lambda^k(\mathfrak{a})$ as a subspace of $\mathfrak{a}^{\otimes k}$.

{QHQUE algebras} $\to$ {quasi-Lie bialgebras}. It restricts to a functor {Hopf QUE algebras} $\to$ {Lie bialgebras}.

If $(A, m, \tilde{\Delta}, \tilde{\Phi})$ is obtained from (A, m, Δ, Φ) by twisting by $F \in 1 + \hbar A^{\otimes 2}$, and if $\Phi, \tilde{\Phi} \in 1 + \hbar^2 A^{\otimes 3}$, then $f := -\big(\hbar \operatorname{Alt}(F) \bmod \hbar\big) \in \Lambda^2(\mathfrak{a})$ and the classical limit $(\mathfrak{a}, \tilde{\delta}, \tilde{\varphi})$ of $(A, ..., \tilde{\Phi})$ is obtained from $(\mathfrak{a}, \delta, \varphi)$ by twisting by f.

2.4. **Categorical versions of (QH)QUE algebras.** Let $(\mathbf{x}, \mu)$ be a Lie algebra in a symmetric tensor category $\mathcal{C}_0$. Assume that:

(1) for any projector $p \in \mathbb{Q}S_n$, its image $p_{\mathbf{x}} \in \operatorname{End}_{\mathcal{C}_0}(\mathbf{x}^{\otimes n})$ admits a kernel and a cokernel; this enables us to define the symmetric and exterior powers $S^n(\mathbf{x})$ and $\Lambda^n(\mathbf{x})$ as objects of $\mathcal{C}_0$;

(2) the class of objects of $\mathcal{C}_0$ contains the infinite direct sum $S(\mathbf{x}) = \bigoplus_{n\geq 0} S^n(\mathbf{x})$. Then there exists a unique bialgebra structure on $S(\mathbf{x})$ in $\mathcal{C}_0$ such that:

(a) the coproduct $\Delta_0 \colon S(\mathbf{x}) \to S(\mathbf{x})^{\otimes 2}$ is the sum of the morphisms $\frac{(a+b)!}{a!b!}(S^{a+b}(\mathbf{x}) \to \mathbf{x}^{\otimes a+b} \to \mathbf{x}^{\otimes a} \otimes \mathbf{x}^{\otimes b} \to S^a(\mathbf{x}) \otimes S^b(\mathbf{x}))$; the unit $\mathbf{1} \to S(\mathbf{x})$ and the counit $S(\mathbf{x}) \to \mathbf{1}$ are the obvious morphisms;

(b) the product $m_{\mathbf{x},\mu} : S(\mathbf{x})^{\otimes 2} \to S(\mathbf{x})$ is such that the composition $S(\mathbf{x})^{\otimes 2} \overset{m_{\mathbf{x},\mu}}{\to} S(\mathbf{x}) \to \mathbf{x}$ coincides with the sum over $a, b \geq 0$ of the morphisms $S^a(\mathbf{x}) \otimes S^b(\mathbf{x}) \to \mathbf{x}$, obtained by replacing the Lie bracket by μ in the Lie series $c_{a,b}(x_1, ..., x_a | y_1, ..., y_b)$ defined by the conditions: (i) it is multilinear in $x_1, ..., y_b$, and symmetric in each group of variables $(x_{a'})_{a'=1}^a$ and $(y_{b'})_{b'=1}^b$; (ii) $c_{a,b}(x, ..., x|y, ..., y) = a!b!c_{a,b}(x, y)$, where $c(x, y) = \log(e^x e^y) = \sum_{a,b\geq 0} c_{a,b}(x, y)$ is the decomposition of the Campbell-Baker-Hausdorff series according to the degrees in x, y.

We will call the bialgebra $(S(\mathbf{x}), m_{\mathbf{x},\mu}, \Delta_0)$ the universal enveloping (UE) algebra of $(\mathbf{x}, \mu)$ and denote it by $U(\mathbf{x})$.

Now let $\mathcal{C}$ be an $\mathbb{N}$-graded symmetric tensor category with a Lie algebra object $\mathbf{x}$ satisfying conditions (1) and (2) (with $\mathcal{C}_0$ replaced by $\mathcal{C}$). The $\mathbb{N}$-grading condition means that $\mathcal{C}(X, Y) = \bigoplus_{d\geq 0} \mathcal{C}(X, Y)[d]$, the composition and tensor product are compatible with the grading, and the commutativity constraints have degree 0. Then $\mathcal{C}$ gives rise to a completed category $\hat{\mathcal{C}}$ with $\hat{\mathcal{C}}(X, Y) = \prod_{n\geq 0} \mathcal{C}(X, Y)[d]$, categories $\mathcal{C}_{\leq d}$ with $\mathcal{C}_{\leq d}(X, Y) = \bigoplus_{i=0}^{d} \mathcal{C}(X, Y)[i]$, and functors $... \to \mathcal{C}_{\leq d} \to ... \to \mathcal{C}_{\leq 0}$.

A (QH)QUE algebra in $\hat{\mathcal{C}}$ quantizing $U(\mathbf{x})$ is a (quasi-)bialgebra structure $(m, \Delta, \epsilon, \eta)$ (resp., $(m, ..., \Phi)$) on $S(\mathbf{x})$ in $\hat{\mathcal{C}}$, whose image in $\mathcal{C}_{\leq 0}$ is $U(\mathbf{x})$. In the “quasi” case, one adds the condition that $\Phi = \eta^{\otimes 3}$ in $\mathcal{C}_{\leq 0}$, and $\operatorname{Alt}_3 \circ \Phi = 0$ in $\mathcal{C}_{\leq 1}$. Two (QH)QUE algebras $(m, \Delta, ...)$ and $(m', \Delta', ...)$ are called equivalent if they are related by some $i \in \operatorname{Ker}(\operatorname{Aut}_{\mathcal{C}}(S(\mathbf{x})) \to \operatorname{Aut}_{\mathcal{C}_0}(S(\mathbf{x})))$, i.e., $m' = i \circ m \circ (i^{-1})^{\otimes 2}$, etc. One also defines twists of QHQUE algebras as above.

One recovers the usual notion of a (QH)QUE algebra as follows. First note that $\operatorname{Vect}_\hbar$ is equivalent to the category $\operatorname{Vect}[[\hbar]]$, where objects are $\mathbf{k}$-vector spaces and $\operatorname{Vect}[[\hbar]](X, Y) = \operatorname{Vect}(X, Y)[[\hbar]]$. Let $\mathcal{C} := \operatorname{Vect}[\hbar]$, the category where objects are vector spaces and morphisms are given by $\operatorname{Vect}[\hbar](X, Y) = \operatorname{Vect}(X, Y)[\hbar]$; the grading is given by powers of $\hbar$. Then $\hat{\mathcal{C}} = \operatorname{Vect}[[\hbar]]$. A (QH)QUE algebra quantizing $(\mathfrak{a}, \mu)$ is then the same as a (QH)QUE algebra in $\operatorname{Vect}[[\hbar]]$ quantizing $U(\mathfrak{a})$.

2.5. **Deformation complexes.** In [GS], Gerstenhaber and Schack constructed a deformation theory for bialgebras. It can be summarized as follows. For (A, m, Δ) a bialgebra, there is a bicomplex $C(A) = \bigoplus_{p,q>0} C^{p,q}$, with $C^{p,q} := \operatorname{Vect}(A^{\otimes p}, A^{\otimes q})$

and differentials $d^{p,q} : C^{p,q} \to C^{p+1,q}$, $d_*^{p,q} : C^{p,q} \to C^{p,q+1}$ given by

$$(d^{p,q}\sigma)(a_1 \otimes ... \otimes a_{p+1}) := \Delta^{(q-1)}(a_1) * \sigma(a_2 \otimes ... \otimes a_{p+1})$$
$$+\sum_{i=1}^{p}(-1)^i\sigma(a_1\otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_{p+1}) + (-1)^{p+1}\sigma(a_1\otimes ... \otimes a_p) * \Delta^{(q-1)}(a_{p+1}),$$

where $*$ is the product on $A^{\otimes q}$, $\Delta^{(q-1)} : A \to A^{\otimes q}$ is the $(q-1)$th iterate of Δ, and

$$(d_*^{p,q}\sigma)(a_1 \otimes ... \otimes a_p) := (a_{1(1)} * ... * a_{p(1)}) \otimes \sigma(a_{1(2)} \otimes ... \otimes a_{p(2)})$$
$$+\sum_{j=1}^{q}(-1)^j\Delta_j(\sigma(a_1 \otimes ... \otimes a_p)) + (-1)^{q+1}\sigma(a_{1(1)} \otimes ... \otimes a_{p(1)}) \otimes (a_{1(2)} * ... * a_{p(2)}),$$

where $*$ is now the product in A, $\Delta_j = \mathrm{id}^{\otimes j-1} \otimes \Delta \otimes \mathrm{id}^{\otimes q-j-1}$, and we use Sweedler's notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$. The total cohomology is denoted by $H^\bullet(A)$. Obstruction to prolongation of a given deformation of A at a given order belongs to $H^4(A)$, and if this obstruction vanishes, the space of possible prolongations, modulo equivalences, is an affine space over $H^3(A)$ (see [ShSt], Prop. 3.6.1). In [ShSt], Shnider and Sternberg construct a theory for the deformation of a bialgebra A in the category of quasi-bialgebras. The relevant bicomplex is $\tilde{C}(A) = \bigoplus_{p\geq 0, q>0} C^{p,q}(A)$ with the same differentials as above (only one row has been added to the bicomplex). Obstructions and parametrizations of prolongations of formal deformations of A in the category of quasi-bialgebras (modulo twists and equivalences) then belong to the total cohomologies $\tilde{H}^4(A)$ and $\tilde{H}^3(A)$.

When $\mathfrak{a}$ is a Lie algebra and $A = U(\mathfrak{a})$, these cohomology groups are given by

• $H^n(A) = \bigoplus_{p+q=n; p,q>0} H^{p,q}$, where $H^{p,q} = H^q(\mathfrak{a}, \Lambda^p(\mathfrak{a}))$ if $q > 1$, and $H^{p,1} = Z^1(\mathfrak{a}, \Lambda^p(\mathfrak{a}))$;

• $\tilde{H}^n(A) = \bigoplus_{p+q=n; p>0, q\geq 0} \tilde{H}^{p,q}$, where $\tilde{H}^{p,q} = H^q(\mathfrak{a}, \Lambda^p(\mathfrak{a}))$; here $\mathfrak{a}$ acts on $\Lambda^p(\mathfrak{a})$ via the adjoint action (see [ShSt], Props. 10.9.1 and 10.9.2).

2.6. **Categorical deformation complexes.** Let $\mathcal{C}$ be an $\mathbb{N}$-graded symmetric tensor category (so $\hat{\mathcal{C}}(X, Y) = \prod_{d\geq 0} \mathcal{C}(X, Y)[d]$). Let (X, m, Δ) be a bialgebra in $\mathcal{C}_{\leq 0}$; we have a functor $\mathcal{C}_{\leq 0} \to \mathcal{C}$, so it is a bialgebra in $\mathcal{C}$. We construct a bicomplex $C(X) = \bigoplus_{p,q>0} C^{p,q}$, with $C^{p,q} := \mathcal{C}(X^{\otimes p}, X^{\otimes q})$ and differentials

$$d^{p,q}(\sigma) := m^{\otimes q} \circ \beta_{q,2} \circ (\Delta^{(q-1)} \otimes \sigma) + \sum_{i=1}^{p}(-1)^i\sigma \circ (\mathrm{id}^{\otimes i-1} \otimes m \otimes \mathrm{id}^{\otimes p-1-i})$$
$$+ (-1)^{p+1} m^{\otimes q} \circ \beta_{q,2} \circ (\sigma \otimes \Delta^{(q-1)}),$$
$$d_*^{p,q}(\sigma) := (m^{(q-1)} \otimes \sigma) \circ \beta_{2,p} \circ \Delta^{\otimes p} + \sum_{j=1}^{q}(-1)^j(\mathrm{id}_X^{\otimes j-1} \otimes \Delta \otimes \mathrm{id}_X^{\otimes q-1-j}) \circ \sigma$$
$$+ (-1)^{q+1}(\sigma \otimes m^{(p-1)}) \circ \beta_{2,p} \circ \Delta^{\otimes p},$$

where $\beta_{n,m} : (X^{\otimes n})^{\otimes m} \to (X^{\otimes m})^{\otimes n}$ is the morphism induced by the natural bijection $\{1, ..., n\} \times \{1, ..., m\} \to \{1, ..., m\} \times \{1, ..., n\}$.

These differentials have degree 0 for the $\mathbb{N}$-grading of $\mathcal{C}$, so the resulting total cohomology $H^\bullet(X)$ is $\mathbb{N}$-graded. A lifting of (m, Δ) to a bialgebra structure on X in $\mathcal{C}_{\leq n}$ being given, the obstruction to lifting it further to $\mathcal{C}_{\leq n+1}$ is a class in $H^4(X)[n + 1]$, and when this class vanishes, the set of possible lifts modulo equivalence (i.e., the natural action of $\mathrm{Ker}(\mathrm{Aut}_{\mathcal{C}}(X) \to \mathrm{Aut}_{\mathcal{C}_{\leq 0}}(X)))$ is an affine space over $H^3(X)[n+1]$.

The bicomplex controlling the deformations of (X, m, Δ) as a quasi-bialgebra in $\hat{\mathcal{C}}$ is $\tilde{C}(X) := \bigoplus_{p\geq 0, q>0} C^{p,q}$ with the same differentials. Its cohomology $\tilde{H}^\bullet(X)$ is again $\mathbb{N}$-graded. A lifting of $(m, \Delta, \eta^{\otimes 3})$ in $\mathcal{C}_{\leq n}$ being given, the obstruction to lifting it to $\mathcal{C}_{\leq n+1}$ is a class in $\tilde{H}^4(X)[n+1]$, and if this class vanishes, the set of possible lifts modulo equivalences and twists is an affine space over $\tilde{H}^3(X)[n+1]$.

If now $(\mathcal{C}_{\leq 0}, \mathbf{x}, \mu)$ is as in Section 2.4 and $(X, m, \Delta) = U(\mathbf{x})$, the cohomology groups $H^\bullet(X)$, $\tilde{H}^\bullet(X)$ may be computed as follows. Set $c(\mathbf{x}) := \bigoplus_{p,q\geq 0} c^{p,q}(\mathbf{x})$, where $c^{p,q}(\mathbf{x}) := \mathcal{C}(\Lambda^p(\mathbf{x}), \Lambda^q(\mathbf{x}))$ (we view $c^{p,q}(\mathbf{x})$ as a subspace of $\mathcal{C}(\mathbf{x}^{\otimes p}, \mathbf{x}^{\otimes q})$). According to [KS, LR], $c(\mathbf{x})$ is equipped with a structure of a $\mathbb{Z}^2$-graded Lie superalgebra, where $c^{p,q}(\mathbf{x})$ has bidegree $(p-1, q-1)$. The bracket is defined by $[a, a'] := a \diamond a' - (-1)^{(p+q)(p'+q')} a' \diamond a$ for $a \in c^{p,q}(\mathbf{x})$, $a' \in c^{p',q'}(\mathbf{x})$, where $a \diamond a' := (-1)^{(p-1)(q'-1)} \mathrm{Alt}_{q+q'-1} \circ (a \otimes \mathrm{id}_X^{\otimes q'-1}) \circ (\mathrm{id}_X^{\otimes p-1} \otimes a') \circ \mathrm{Alt}_{p+p'-1}$ if $p, q' \geq 1$ and 0 otherwise. As $\mu \in c^{2,1}(\mathbf{x})$ defines a Lie structure, we have $[\mu, \mu] = 0$. It follows that for any $q \geq 0$, $c^{\bullet,q}(\mathbf{x}) := (\mathcal{C}(\Lambda^\bullet, \Lambda^q), [\mu, -])$ is a complex; its cocycle and cohomology groups are denoted by $Z^\bullet(\mathbf{x}, \Lambda^q(\mathbf{x}))$ and $H^\bullet(\mathbf{x}, \Lambda^q(\mathbf{x}))$.

The cohomology groups $H^\bullet(U(\mathbf{x}))$ and $\tilde{H}^\bullet(U(\mathbf{x}))$ are then computed by the same formulas as in the case of $A = U(\mathfrak{a})$, replacing $\mathfrak{a}$ by $\mathbf{x}$; the gradings of these groups are induced by the grading of $\mathcal{C}$.

3. Quantization functors

Acccording to [Dr4], a quantization functor for (quasi-)Lie bialgebras is a section of the classical limit functor $\{$(Q)HQUE algebras$\} \to \{$(quasi-)Lie bialgebras$\}$, such that the structure maps of $Q(\mathfrak{a})$ may be expressed by universal acyclic formulas involving the structure maps of the (quasi-)Lie bialgebra $\mathfrak{a}$. As was observed in [EK2], the definition of quantization functors is best expressed in the setup of props, which we first recall.

3.1. **The theory of props.** Recall the definitions of the Schur categories Sch and **Sch** ([EH]). These are braided symmetric tensor categories, defined as follows. The objects of Sch are Schur functors, i.e., finitely supported families $X = (X_\rho)_\rho$ of finite-dimensional vector spaces, where $\rho \in \bigsqcup_{n\geq 0} \widehat{\mathfrak{S}}_n$ (ρ is therefore a pair (n, π_ρ), where $n \geq 0$ and π_ρ is an irreducible representation of $\mathfrak{S}_n$; n is called the degree of ρ; by convention, $\mathfrak{S}_0$ is the trivial group). The set of morphisms from X to Y is $\mathrm{Sch}(X, Y) := \bigoplus_\rho \mathrm{Vect}(X_\rho, Y_\rho)$. The direct sum of objects is $X \oplus Y = (X_\rho \oplus Y_\rho)_\rho$. Their tensor product is $X \otimes Y = (\bigoplus_{\rho', \rho''} M^\rho_{\rho'\rho''} \otimes X_{\rho'} \otimes Y_{\rho''})_\rho$, where for $\rho \in \widehat{\mathfrak{S}}_n$, $\rho' \in \widehat{\mathfrak{S}}_{n'}$, $\rho'' \in \widehat{\mathfrak{S}}_{n''}$, we set $M^\rho_{\rho'\rho''} = [\pi_\rho : \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n'} \times \mathfrak{S}_{n''}}(\pi_{\rho'} \otimes \pi_{\rho''})]$ if $n = n' + n''$ and 0 otherwise. **Sch** is defined similarly, dropping the condition that X is finitely supported.

An object X of Sch or **Sch** is called homogeneous of degree n iff $X_\rho = 0$ if the degree of ρ is $\neq n$. If X is homogeneous, we denote by $|X|$ its degree.

We have a bijection $\mathrm{Irr}(\mathrm{Sch}) \simeq \bigsqcup_{n\geq 0} \widehat{\mathfrak{S}}_n$, where Irr(Sch) is the set of equivalence classes of irreducible objects in Sch. The unit object of Sch is $\mathbf{1}$, which corresponds to the element of $\widehat{\mathfrak{S}}_0$. We also define $\mathbf{id}, S^p, \Lambda^p$ as the objects corresponding to: the element of $\widehat{\mathfrak{S}}_1$, the trivial and the signature character of $\mathfrak{S}_p$. We set $T_p := \mathbf{id}^{\otimes p}$ and $S := \bigoplus_{p\geq 0} S^p \in \mathrm{Ob}(\mathbf{Sch})$.

A prop (resp., an **Sch**-prop) is then an additive symmetric monoidal category $\mathcal{C}$, equipped with a tensor functor $\mathrm{Sch} \to \mathcal{C}$ (resp., $\mathbf{Sch} \to \mathcal{C}$), which is the identity on objects.

A prop morphism $\mathcal{C} \to \mathcal{D}$ is a tensor functor, such that the functors $\mathrm{Sch} \to \mathcal{C} \to \mathcal{D}$ and $\mathrm{Sch} \to \mathcal{D}$ coincide. An ideal I of the prop $\mathcal{C}$ is an assignment $(X,Y) \mapsto I(X,Y) \subset \mathcal{C}(X,Y)$, such that $(X,Y) \mapsto \mathcal{C}(X,Y)/I(X,Y) =: \mathcal{C}/I(X,Y)$ is a monoidal category; $\mathcal{C}/I$ is then called the quotient prop. If $f_\alpha \in \mathcal{C}(X_\alpha, Y_\alpha)$ are morphisms in $\mathcal{C}$, then the ideal $\langle f_\alpha \rangle$ is the smallest ideal I in $\mathcal{C}$ such that $f_\alpha \in I(X_\alpha, Y_\alpha)$.

Props may be presented by generators and relations. If $\underline{V} = (V_{n,m})_{n,m\geq 0}$ is a collection of vector spaces, there is a unique prop $\mathrm{Free}(\underline{V})$, such that for any prop $\mathcal{C}$, we have a bijection $\prod_{n,m} \mathrm{Vect}(V_{n,m}, \mathcal{C}(T_n, T_m)) \simeq \mathrm{Prop}(\mathrm{Free}(\underline{V}), \mathcal{C})$ (where Prop denotes the set of prop morphisms). If $\underline{R} = (R_{n,m})_{n,m\geq 0}$ is a collection of subspaces of the $\mathrm{Free}(\underline{V})(T_n, T_m)$, then the ideal generated by $\underline{V}$ with relations $\underline{R}$ is the quotient prop $\mathrm{Free}(\underline{V})/\langle \underline{R} \rangle$.

We say that $\mathcal{C}$ is a topological prop if for any $X, Y \in \mathrm{Ob}(\mathrm{Sch})$ we have a filtration $\mathcal{C}(X,Y) = \mathcal{C}^{\geq 0}(X,Y) \supset \mathcal{C}^{\geq 1}(X,Y) \supset \ldots$, complete and separated, and compatible with the prop operations (i.e., $\mathcal{C}^{\geq b}(Y,Z) \circ \mathcal{C}^{\geq a}(X,Y) \subset \mathcal{C}^{\geq a+b}(X,Z)$ and $\mathcal{C}^{\geq a}(X,Y) \otimes \mathcal{C}^{\geq a'}(X',Y') \subset \mathcal{C}^{\geq a+a'}(X \otimes X', Y \otimes Y')$); and if $\mathcal{C}(X,Y) = \mathcal{C}^{\geq v(|X|,|Y|)}(X,Y)$ for any homogeneous X, Y, where $v(x,y) \to \infty$ when $x \to \infty$, y being fixed, or $y \to \infty$, x being fixed. Such a $\mathcal{C}$ gives rise to an **Sch**-prop $\hat{\mathcal{C}}$, given by $\hat{\mathcal{C}}(X,Y) = \prod_{i,j} \mathcal{C}(X_i, Y_j)$, where $X = \bigoplus_i X_i$, $Y = \bigoplus_j Y_j$ are the homogeneous decompositions of $X, Y \in \mathrm{Ob}(\mathbf{Sch})$; $\hat{\mathcal{C}}$ is then equipped with a complete and separated filtration, compatible with the prop operations.

3.2. **The props** LBA **and** QLBA**.** The prop LBA of Lie algebras is defined by generators $\mu \in \mathrm{LBA}(\mathbf{id}^{\otimes 2}, \mathbf{id})$, $\delta \in \mathrm{LBA}(\mathbf{id}, \mathbf{id}^{\otimes 2})$ and relations $\mu \circ \beta = -\mu$, $\beta \circ \delta = -\delta$,

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ \mathrm{Alt}_3 = 0, \mathrm{Alt}_3 \circ (\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \delta = 0, \delta \circ \mu = \mathrm{Alt}_2 \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (\mathrm{id}_{\mathbf{id}} \otimes \delta) \circ \mathrm{Alt}_2 .$$

Then $\mathbf{id}$ is a Lie bialgebra in LBA, and it is an initial object in the category of props equipped with a Lie bialgebra structure on $\mathbf{id}$.

LBA is graded by $\mathbb{N}^2$, with μ, δ of degrees $(1,0)$, $(0,1)$; we denote by $(\deg_\mu, \deg_\delta)$ this grading. LBA is then $\mathbb{N}$-graded by the total degree $\deg_\mu + \deg_\delta$. If $x \in \mathrm{LBA}(X,Y)$ and X, Y, x are homogeneous, then $\deg_\mu(x) - \deg_\delta(x) = |X| - |Y|$, so $\mathrm{LBA}(X,Y) = \mathrm{LBA}^{\geq ||X|-|Y||}(X,Y)$. So the total degree completion of LBA is a topological prop. We denote by $\mathbf{LBA}$ the corresponding **Sch**-prop.

The prop QLBA of quasi-Lie bialgebras is defined by generators $\mu \in \mathrm{QLBA}(\mathbf{id}^{\otimes 2}, \mathbf{id})$, $\delta \in \mathrm{QLBA}(\mathbf{id}, \mathbf{id}^{\otimes 2})$, $\varphi \in \mathrm{QLBA}(\mathbf{1}, \mathbf{id}^{\otimes 3})$ and relations $\mu \circ \beta = -\mu$, $\beta \circ \delta = -\delta$, $\beta_\sigma \circ \varphi = \mathrm{sgn}(\sigma)\varphi$, $\sigma \in S_3$,

$$\mu \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ \mathrm{Alt}_3 = 0, \quad \delta \circ \mu = \mathrm{Alt}_2 \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (\mathrm{id}_{\mathbf{id}} \otimes \delta) \circ \mathrm{Alt}_2,$$

$$\mathrm{Alt}_3 \circ (\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \delta = \mathrm{Alt}_3 \circ (\mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes 3}) \circ (\mathrm{id}_{\mathbf{id}} \otimes \varphi), \quad \mathrm{Alt}_4 \circ (\delta \otimes \mathrm{id}_{\mathbf{id}}^{\otimes 2}) \circ \varphi = 0.$$

This prop is graded by $\{(u,v) \in \mathbb{Z}^2 \mid v \geq 0, 2u + v \geq 0\}$, with μ, δ, φ of degrees $(1,0)$, $(0,1)$, $(-1,2)$; we denote this grading by $(\deg_\mu, \deg_\delta)$. QLBA is then $\mathbb{N}$-graded by the total degree $\deg_\mu + \deg_\delta$; the generators then have degree 1. If $x \in \mathrm{QLBA}(X,Y)$ and x, X, Y are homogeneous, then $\deg_\mu(x) - \deg_\delta(x) = |X| - |Y|$, so $\mathrm{QLBA}(X,Y) = \mathrm{QLBA}^{\geq \frac{1}{3}||X|-|Y||}(X,Y)$, which implies that the total degree completion of QLBA is a topological prop. We denote by **QLBA** the corresponding **Sch**-prop. In the prop QLBA, the object **id** is equipped with a quasi-Lie bialgebra structure, and QLBA is an initial object in the category of props equipped with a quasi-Lie bialgebra structure on **id**.

We have a natural isomorphism $\mathrm{LBA} \simeq \mathrm{QLBA}/\langle\varphi\rangle$, and we denote by $x \mapsto (x \bmod \langle\varphi\rangle)$ the projection $\mathrm{QLBA} \to \mathrm{LBA}$; it is such that $\mu, \delta, \varphi \mapsto \mu, \delta, 0$.

3.3. **Quantization functors of Lie bialgebras.**

Definition 3.1. *A quantization functor (QF) of Lie bialgebras is a bialgebra structure $(m, \Delta, \eta, \epsilon)$ on S in* **LBA***, such that:*

(a) *its reduction modulo $\langle\mu, \delta\rangle$ is the standard bialgebra structure on S in* $\mathbf{Sch} = \mathbf{LBA}/\langle\mu, \delta\rangle$*;*

(b1) $(\mathbf{id}^{\otimes 2} \to S^{\otimes 2} \stackrel{m - m\circ\beta}{\to} S \to \mathbf{id}) = \mu +$ *terms with total degree* ≥ 2*, and*

(b2) $(\mathbf{id} \to S \stackrel{\Delta - \beta\circ\Delta}{\to} S^{\otimes 2} \to \mathbf{id}^{\otimes 2}) = \delta +$ *terms with total degree* ≥ 2*.*

One checks that $\mathrm{Ker}(\mathrm{Aut}_{\mathbf{LBA}}(S) \to \mathrm{Aut}_{\mathbf{Sch}}(S))$ (where the map is $\mu, \delta \mapsto 0$) acts on the set of QFs of Lie bialgebras; quantization functors which are related by this action will be called equivalent.

Now view $\hat{\mathcal{C}} := \mathbf{LBA}$ as graded by $\deg_\delta$. Then $\mathcal{C}_{\leq 0} = \mathbf{LA}$, and $(\mathbf{id}, \mu)$ is a Lie algebra in $\mathcal{C}_{\leq 0}$. We consider QUE algebras in $\hat{\mathbf{LBA}}$, quantizing $U(\mathbf{id}_{\mathbf{LA}})$ (the index emphasizes that this is a bialgebra in **LA**), and satisfying condition (b2) above. The group $\mathrm{Ker}(\mathrm{Aut}_{\mathbf{LBA}}(S) \to \mathrm{Aut}_{\mathbf{LA}}(S))$ acts on this set of QUE algebras. QUE algebras which are related by this action will be called equivalent.

Proposition 3.1. *The natural map* $\{$*quantizations of* $U(\mathbf{id}_{\mathbf{LA}})$ *satisfying (b2)*$\}/$*(equivalence)* $\to \{$*QFs of Lie bialgebras*$\}/$*(equivalence) is a bijection.*

Proof. We have:

Lemma 3.1. *If X is any Schur functor and $n \geq 0$, then the map* $\mathrm{Sch}(S^n, X) \to \mathrm{LA}(S^n, X)$ *is bijective.*

Proof. It suffices to prove this for $X = T_m$. When $m = 1$, we have $\mathrm{LA}(S^n, \mathbf{id}) = \mathrm{LA}(T_n, \mathbf{id})^{\mathfrak{S}_n}$ and $\mathrm{LA}(T_n, \mathbf{id})$ is the space $\mathcal{L}_n$ of multilinear Lie words in n letters $x_1, ..., x_n$; as an S_n-module, this is a submodule of the space $\mathcal{A}_n$ of multilinear associative words in these letters; as $\mathcal{A}_n$ identifies with the regular representation of S_n, $\mathcal{A}_n^{\mathfrak{S}_n}$ is 1-dimensional, spanned by $\sum_{\sigma \in S_n} x_{\sigma(1)} ... x_{\sigma(n)}$, which is not in $\mathcal{L}_n$ unless $n = 1$; hence

$$\mathrm{LA}(S^n, \mathbf{id}) = \mathcal{L}_n^{\mathfrak{S}_n} = \left\{ \begin{array}{ll} \mathbf{k} & \text{if } n = 1 \\ 0 & \text{else} \end{array} \right\} = \mathrm{Sch}(S^n, \mathbf{id}).$$

In general,

$$\mathrm{LA}(S^n, T_m) = \mathrm{LA}(T_n, T_m)^{\mathfrak{S}_n} = \bigoplus_{(n_1, ..., n_m) | \sum_i n_i = n} \bigotimes_{i=1}^m \mathrm{LA}(T_{n_i}, \mathbf{id})^{\mathfrak{S}_{n_i}}$$

$$= \bigoplus_{(n_1, ..., n_m) | \sum_i n_i = n} \bigotimes_{i=1}^m \mathrm{Sch}(T_{n_i}, \mathbf{id})^{\mathfrak{S}_{n_i}} = \mathrm{Sch}(S^n, T_m),$$

where the second equality uses the fact that $\mathrm{LA}(T_n, T_m)$ identifies with

$$\bigoplus_{(n_1,\ldots,n_m)|\sum_i n_i=n} \mathrm{Ind}_{\mathfrak{S}_{n_1}\times\ldots\times\mathfrak{S}_{n_m}}^{\mathfrak{S}_n} \big(\bigotimes_{i=1}^m \mathrm{LA}(T_{n_i}, \mathbf{id})\big)$$

as an $\mathfrak{S}_n$-module, and the third equality uses the already treated case $m = 1$. □

Lemma 3.2. *Let $\tilde{m} \in \mathbf{LA}(S^{\otimes 2}, S)$ be such that $(\tilde{m}, \Delta_0)$ is a bialgebra structure on S in $\mathbf{LA}$. Then for some $s \in \mathbf{k}$, $\tilde{m} = m_{\mathbf{id},s\mu}$ (see Section 2.4).*

Proof. It follows from its compatibility with Δ_0 that $\tilde{m}$ is uniquely determined by the compositions $\tilde{m}_{ab} = (S^a \otimes S^b \to S^{\otimes 2} \stackrel{\tilde{m}}{\to} S \to \mathbf{id})$. Each $\tilde{m}_{ab}$ identifies with a multilinear Lie polynomial in two sets of free variables $(x_{a'})_{a'=1}^a$, $(y_{b'})_{b'=1}^b$, symmetric in each set of variables, hence with a Lie polynomial $p_{a,b}(x, y)$ in two free variables x, y, of degree a in x and b in y. The associativity condition is translated as $p(p(x, y), z) = p(x, p(y, z))$, where $p(x, y) = \sum_{a,b\geq 0} p_{a,b}(x, y)$. According to [AT], Prop. 2.1, this implies that $p(x, y) = x + y$ or $p(x, y) = s^{-1}c(sx, sy)$ for $s \in \mathbf{k}^\times$, which implies that $\tilde{m} = m_{\mathbf{id},0}$ in the first case and $\tilde{m} = m_{\mathbf{id},s\mu}$ in the second case. □

Lemma 3.1 implies that $\mathrm{Aut}_{\mathbf{LA}}(S) \to \mathrm{Aut}_{\mathbf{Sch}}(S)$ (where the map is induced by $\mu \mapsto 0$) is bijective, so $\mathrm{Ker}(\mathrm{Aut}_{\mathbf{LBA}}(S) \to \mathrm{Aut}_{\mathbf{LA}}(S)) \to \mathrm{Ker}(\mathrm{Aut}_{\mathbf{LBA}}(S) \to \mathrm{Aut}_{\mathbf{Sch}}(S))$ is bijective.

Now let (m, Δ) be a QF for Lie bialgebras, and let $\tilde{m}$, $\tilde{\Delta}$ be the reductions of m, Δ mod $\langle\delta\rangle$ (these are morphisms in $\mathbf{LA}$). Lemma 3.1 then implies that $\tilde{\Delta} = \Delta_0$, which using Lemma 3.2 and $(\mathbf{id}^{\otimes 2} \to S^{\otimes 2} \stackrel{\tilde{m}-\tilde{m}\circ\beta}{\to} S \to \mathbf{id}) = \mu$ implies that $(\tilde{m}, \tilde{\Delta})$ coincides with the morphisms of $U(\mathbf{id}_{\mathbf{LA}})$. So {quantizations of $U(\mathbf{id}_{\mathbf{LA}})$ satisfying (2b)} → {QFs of Lie bialgebras} is bijective. This ends the proof of Proposition 3.1. □

Remark 1. A QF of Lie bialgebras yields a functor Q : {Lie bialgebras over $\mathbf{k}$} → {QUE algebras over $\mathbf{k}$} by $(\mathfrak{a}, \mu_\mathfrak{a}, \delta_\mathfrak{a}) \mapsto (S(\mathfrak{a})[[\hbar]]$, the maps obtained from m, Δ,... by the substitution $(\mu, \delta) \mapsto (\mu_\mathfrak{a}, \hbar\delta_\mathfrak{a}))$. It also yields a functor Q' : {Lie bialgebras over $\mathbf{k}$} → {quantized formal series Hopf algebras} by $\mathfrak{a} \mapsto (\hat{S}(\mathfrak{a})[[\hbar]]$, the maps obtained by $(\mu, \delta) \mapsto (\hbar\mu_\mathfrak{a}, \delta_\mathfrak{a}))$. Then $Q(\mathfrak{a})$ and $Q'(\mathfrak{a})$ are related by the QUE algebra-QFSH algebra correspondence ([Dr1, Gav]).

3.4. Quantization functors of quasi-Lie bialgebras.

Definition 3.2. *A QF of quasi-Lie bialgebras is a quasi-bialgebra structure $(m, \Delta, \Phi, \eta, \epsilon)$ on S in $\mathbf{QLBA}$, such that:*

(a) *its reduction modulo $\langle\mu, \delta, \varphi\rangle$ is the standard quasi-bialgebra structure on S in $\mathbf{Sch}$;*

(b1) $(\mathbf{id}^{\otimes 2} \to S^{\otimes 2} \stackrel{m-m\circ\beta}{\to} S \to \mathbf{id}) = \mu +$ *terms with* $\deg_\mu + \deg_\delta \geq 2$*;*

(b2) $(\mathbf{id} \to S \stackrel{\Delta-\beta\circ\Delta}{\to} S^{\otimes 2} \to \mathbf{id}^{\otimes 2}) = \delta +$ *terms with* $\deg_\mu + \deg_\delta \geq 2$*;*

(b3) $(\mathbf{1} \stackrel{\mathrm{Alt}_3\circ\Phi}{\to} S^{\otimes 3} \to \mathbf{id}^{\otimes 3}) = \varphi +$ *terms with* $\deg_\mu + \deg_\delta \geq 2$.

As above, one checks that $\mathrm{Ker}(\mathrm{Aut}_{\mathbf{QLBA}}(S) \to \mathrm{Aut}_{\mathbf{Sch}}(S))$ acts on the set of QFs of quasi-Lie bialgebras; quantization functors which are related by this action will be called equivalent. A QF for quasi-Lie bialgebras being fixed, one checks that its twist by an element of $\mathbf{LBA}(\mathbf{1}, S^{\otimes 2})^\times$ is again a QF of quasi-Lie bialgebras.

Now view $\hat{\mathcal{C}} = \mathbf{QLBA}$ as graded by $\deg_\delta$. Then $\mathcal{C}_{\leq 0} = \mathbf{LA}$. We consider QHQUE algebras in $\mathbf{QLBA}$ quantizing $U(\mathbf{id_{LA}})$, and satisfying conditions (b2), (b3) above. Such an element being fixed, its twist by an element of $\mathbf{QLBA}(\mathbf{1}, S^{\otimes 2})^\times$ again satisfies (b2), (b3). The group $\mathrm{Ker}(\mathrm{Aut}_{\mathbf{QLBA}}(S) \to \mathrm{Aut}_{\mathbf{LA}}(S))$ acts on this set of QHQUE algebras.

Proposition 3.2. *The natural map* $\{$*QHQUE algebras quantizing* $U(\mathbf{id_{LA}})$ *satisfying (b2), (b3)*$\}/($*twists, equivalence*$) \to \{$*QFs for quasi-Lie bialgebras*$\}/($*twists, equivalence*$)$ *is a bijection.*

Proof. The proof is similar to that of Proposition 3.1. One also has to show that for $(m, ..., \Phi)$ a QF of quasi-Lie bialgebras, $(\mathrm{Alt}_3 \circ \Phi)_1 = 0$, where the index 1 means the degree 1 part for $\deg_\delta$. Linearizing the pentagon identity satisfied by Φ, we get $(\Delta_0 \circ \mathrm{id}_{\mathbf{id}}^{\otimes 2} - \mathrm{id}_{\mathbf{id}} \otimes \Delta_0 \otimes \mathrm{id}_{\mathbf{id}} + \mathrm{id}_{\mathbf{id}}^{\otimes 2}) \circ \Phi_1 = \eta \otimes \Phi_1 + \Phi_1 \otimes \eta$. According to [Dr2], Prop. 2.2, this implies that $\mathrm{Alt}_3 \circ \Phi_1$ belongs to the image of $\mathbf{QLBA}(\mathbf{1}, \Lambda^3) \to \mathbf{QLBA}(\mathbf{1}, S^{\otimes 3})$. We now show that an element of $\mathbf{QLBA}(\mathbf{1}, \Lambda^3)$ with $\deg_\delta$ equal to 1 necessarily vanishes. More generally, $\mathrm{QLBA}(\mathbf{1}, X)_{\leq 1} = 0$ if $|X| \neq 0$, because this space is linearly spanned by compositions of μ, δ, φ, with at least one φ, as this is the only generator with source $\mathbf{1}$, and $\deg_\delta(\varphi) = 2$. It follows that $(\mathrm{Alt}_3 \circ \Phi)_1 = \mathrm{Alt}_3 \circ \Phi_1 = 0$. □

Proposition 3.3. *There is a unique map* $\{$*QFs of quasi-Lie bialgebras*$\} \to \{$*QFs of Lie bialgebras*$\}$ *such that* $(m, ..., \eta, \Phi) \mapsto (m \text{ mod } \langle\varphi\rangle, ..., \eta \text{ mod } \langle\varphi\rangle)$. *It induces a map* $\{$*QFs of quasi-Lie bialgebras*$\}/($*twists, equivalence*$) \to \{$*QFs of Lie bialgebras*$\}/($*equivalence*$)$.

Proof. $(m \text{ mod } \langle\varphi\rangle, ..., \Phi \text{ mod } \langle\varphi\rangle)$ clearly satisfies the quasi-bialgebra axioms. Moreover, $\mathbf{LBA}(\mathbf{1}, S^{\otimes 3}) = \mathbf{k}\eta^{\otimes 3}$, which implies that $(\Phi \text{ mod } \langle\varphi\rangle) = \eta^{\otimes 3}$. So $(m \text{ mod } \langle\varphi\rangle, ..., \eta \text{ mod } \langle\varphi\rangle)$ satisfies the bialgebra axioms. The fact that $(m, ..., \Phi)$ satisfies (a)-(b2) in Definition 3.2 implies that $(m \text{ mod } \langle\varphi\rangle, ..., \eta \text{ mod } \langle\varphi\rangle)$ satisfies (a)-(b2) in Definition 3.1. We also have $\mathbf{LBA}(\mathbf{1}, S^{\otimes 2}) = \mathbf{k}\eta^{\otimes 2}$, which implies that the images of twists under $x \mapsto (x \text{ mod } \langle\varphi\rangle)$ are trivial. □

Remark 2. As above, a QF of quasi-Lie bialgebras yields a functor {quasi-Lie bialgebras over $\mathbf{k}$} $\to$ {QHQUE algebras} by $(\mathfrak{a}, \mu_\mathfrak{a}, \delta_\mathfrak{a}, \varphi_\mathfrak{a}) \mapsto Q(\mathfrak{a}) := (S(\mathfrak{a})[[\hbar]]$, the morphisms derived from $m, ..., \Phi$ by the substitution $(\mu, \delta, \varphi) \mapsto (\mu_\mathfrak{a}, \hbar\delta_\mathfrak{a}, \hbar^2\varphi_\mathfrak{a}))$.

3.5. **Deformation complexes.** Recall that the props $\mathbf{LBA}$ and $\mathbf{QLBA}$ are $\mathbb{N}$-graded by $\deg_\delta$. As in Section 2.4, this gives rise to props $\mathbf{LBA}_{\leq n}$, $\mathbf{QLBA}_{\leq n}$.

Proposition 3.4. *The obstruction to lifting a QUE algebra in* $\mathbf{LBA}_{\leq n}$ *(resp., a QHQUE algebra in* $\mathbf{QLBA}_{\leq n}$*) quantizing* $U(\mathbf{id_{LA}})$ *to* $\mathbf{LBA}_{\leq n+1}$ *(resp., to* $\mathbf{QLBA}_{\leq n+1}$*) belongs to* $H^4_{\mathrm{LBA}}[n+1]$ *(resp., to* $H^4_{\mathrm{QLBA}}[n+1]$*). If this obstruction vanishes, possible lifts are parametrized, modulo equivalence (resp., and lifts), by* $H^3_{\mathrm{LBA}}[n+1]$ *(resp., by* $H^3_{\mathrm{QLBA}}[n+1]$*). Here*

$$H^i_\mathcal{C} = \bigoplus_{p+q=i; p>0, q\geq 0} H^q_\mathcal{C}(\mathbf{id}, \Lambda^p),$$

for $\mathcal{C} = \mathrm{(Q)LBA}$ *(the index* $\mathcal{C}$ *emphasizes the dependence of the groups defined in Section* 2.6 *in the ambiant symmetric monoidal category), with grading induced by* $\deg_\delta$.

In particular, the prop morphism QLBA $\to$ LBA *induces a degree zero linear map* $H^i_{\mathrm{QLBA}} \to H^i_{\mathrm{LBA}}$.

Proof. The statement follows from Section 2.6 in the case of the quantization problem in QLBA. In the case of LBA, the relevant cohomology groups are, according to Section 2.6,

$$H^i_{\mathrm{LBA}} = \bigoplus_{p+q=i, p>0, q\geq 0} H^{p,q}_{\mathrm{LBA}}, \text{ where } H^{p,q}_{\mathrm{LBA}} = \begin{cases} H^q_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p) & \text{if } q>1, \\ Z^1_{\mathrm{LBA}}(\mathbf{id}, \Lambda^q) & \text{if } q=1, \\ 0 & \text{if } q=0. \end{cases}$$

Now $C^0_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p) = \mathrm{LBA}(\mathbf{1}, \Lambda^p) = 0$ as $p>0$, which implies that $H^0_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p) = 0$ and $H^1_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p) = Z^1_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p)$. Therefore $H^{p,q}_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p) = H^q_{\mathrm{LBA}}(\mathbf{id}, \Lambda^p)$ for any $p>0, q\geq 0$, as wanted. □

4. The main result

The main result of our paper is:

Theorem 4.1. *The map* {*QFs of quasi-Lie bialgebras*}/(*twists, equivalence*) $\to$ {*QFs of Lie bialgebras*}/(*equivalence*) *(see Proposition* 3.3*) is a bijection.*

Its proof is based on

Theorem 4.2. *For any* $i \geq 0$*, the map* $H^i_{\mathrm{QLBA}} \to H^i_{\mathrm{LBA}}$ *is a graded isomorphism.*

Theorem 4.2 will be proved in Sections 5 and 6. In the rest of the present section, we prove that Theorem 4.2 implies Theorem 4.1.

Set $Q_n :=$ {bialgebra structures (m, Δ) on S in $\mathbf{LBA}_{\leq n}$, such that $m \equiv m_{\mathbf{id},\mu}$ mod $\mathbf{LBA}_{\geq 1}$, $\Delta \equiv \Delta_0$ mod $\mathbf{LBA}_{\geq 1}$, and $(\mathbf{id} \to S \overset{\Delta - \beta\circ\Delta}{\to} S^{\otimes 2} \to \mathbf{id}^{\otimes 2}) \equiv \delta$ mod $\mathbf{LBA}_{\geq 2}$}/(equivalences), $\tilde{Q}_n :=$ {quasi-bialgebra structures (m, Δ, Φ) on S in $\mathbf{QLBA}_{\leq n}$, satisfying the same conditions (replacing **LBA** by **QLBA**)}/(equivalences, twists). Here we denote by $(\mathbf{Q})\mathbf{LBA}_{\geq n+1}$ the prop ideal $\mathrm{Ker}((\mathbf{Q})\mathbf{LBA} \to (\mathbf{Q})\mathbf{LBA}_{\leq n})$ and $m_{\mathbf{id},\mu} \in \mathbf{LA}(S^{\otimes 2}, S)$ is as in Section 2.4.

We have projection maps $Q_{n+1} \overset{\pi_n}{\to} Q_n$, and the projective limit $Q_\infty := \lim_{\leftarrow} Q_n$ identifies with {QFs of Lie bialgebras}/(equivalence). Similarly, we have natural maps $\tilde{Q}_{n+1} \overset{\tilde{\pi}_n}{\to} \tilde{Q}_n$. To show that the projective limit $\tilde{Q}_\infty := \lim_{\leftarrow} \tilde{Q}_n$ identifies with {QFs of quasi-Lie bialgebras}/(equivalence, twists), we note that (a) if $(m, \Delta, \Phi) \in \tilde{Q}_\infty$, then $\mathrm{Alt}_3 \circ \Phi_1 = 0$ (see proof of Proposition 3.2), and (b) the pentagon identity then implies that $\mathrm{Alt}_3 \circ \Phi_2 \in \mathbf{QLBA}(\mathbf{1}, \Lambda^3)$, so $\mathrm{Alt}_3 \circ \Phi_2$ is proportional to φ; if now $\tilde{\delta} := \Delta - \beta \circ \Delta$, then $\mathrm{Alt}_3 \circ (\tilde{\delta} \otimes \mathrm{id}_S) \circ \tilde{\delta} = 2\,\mathrm{Alt}_3 \circ (\mathrm{id}_{S^{\otimes 3}} - \Phi * \mathrm{id}_{S^{\otimes 3}} * \Phi^{-1}) \circ \Delta^{(2)}$; restricting to **id** and retaining only the terms in degree 2, we get $\mathrm{Alt}_3 \circ (\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \delta = \mathrm{Alt}_3 \circ (\mu \otimes \mathrm{id}_{\mathbf{id}^{\otimes 2}}) \circ (\mathrm{id}_{\mathbf{id}} \otimes (\mathrm{Alt}_3 \circ \Phi_2))$, which implies, given that the same identity holds with φ in place of $\mathrm{Alt}_3 \circ \Phi_2$, the equality $\mathrm{Alt}_3 \circ \Phi_2 = \varphi$.

We also have a reduction map $\tilde{Q}_n \to Q_n$, compatible with the above projections. We will show that for any $n \geq 0$, this map is a bijection. When $n = 0$, this is clear as $\mathbf{QLBA}_{\leq 0} = \mathbf{LBA}_{\leq 0}$. Assume that we have proved that $\tilde{Q}_n \to Q_n$ is a bijection. For $q \in Q_n$, let $\tilde{q}$ be its preimage in $\tilde{Q}_n$. The obstruction to lifting q to Q_{n+1} (resp., $\tilde{q}$ to $\tilde{Q}_{n+1}$) is a class $\mathrm{obs}(q) \in H^4_{\mathrm{LBA}}[n+1]$ (resp., $\mathrm{obs}(\tilde{q}) \in H^4_{\mathrm{QLBA}}[n+1]$). These classes are mapped to each other under $H^4_{\mathrm{QLBA}}[n+1] \to H^4_{\mathrm{LBA}}[n+1]$, so Theorem 4.2 implies that $\mathrm{obs}(q) = 0$ iff $\mathrm{obs}(\tilde{q}) = 0$; in other terms, the bijection $\tilde{Q}_n \to Q_n$ restricts to a bijection between $\mathrm{image}(\tilde{Q}_{n+1} \to \tilde{Q}_n)$ and $\mathrm{image}(Q_{n+1} \to Q_n)$. To complete the proof, it remains to show that for any $q \in \mathrm{image}(Q_{n+1} \to Q_n)$, the map $\tilde{Q}_{n+1} \to Q_{n+1}$ induces a bijection $\tilde{\pi}_n^{-1}(\tilde{q}) \to \pi_n^{-1}(q)$. These fibers are affine spaces over $H^3_{\mathrm{QLBA}}[n+1]$ and $H^3_{\mathrm{LBA}}[n+1]$, and the map between them is a morphism of affine spaces, compatible with the natural map $H^3_{\mathrm{QLBA}}[n+1] \to H^3_{\mathrm{LBA}}[n+1]$. Theorem 4.2 says that this map is an isomorphism; therefore the map $\tilde{\pi}_n^{-1}(\tilde{q}) \to \pi_n^{-1}(q)$ is a bijection, as wanted.

5. Structure of the prop QLBA

In order to establish Theorem 4.2, we study the structure of QLBA.

5.1. **Products of ideals in props.** If $\mathcal{C}$ is a prop and $I_1, ..., I_n$ are ideals of $\mathcal{C}$, then the product $I_1...I_n$ is the smallest ideal containing the morphisms $f_1 * ... * f_n$, where f_i is a morphism in I_i and $*$ is $\circ$ or $\otimes$. One defines in this way the powers I^n of an ideal.

5.2. **Structure of the prop** LBA**.** Define LA (resp., LCA) as the prop generated by $\mu \in \mathrm{LA}(\mathbf{id}^{\otimes 2}, \mathbf{id})$ subject to the antisymmetry and Jacobi relation (resp., $\delta \in \mathrm{LCA}(\mathbf{id}, \mathbf{id}^{\otimes 2})$ subject to antisymmetry and the co-Jacobi relation). We have prop morphisms $\mathrm{LA}, \mathrm{LCA} \to \mathrm{LBA}$. The structure of LBA is given by

Proposition 5.1. *For $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$ finite families of objects in* Ob(Sch)*, we have an isomorphism*

$$\text{(1)}\quad \mathrm{LBA}(\bigotimes_i X_i, \bigotimes_j Y_j) \simeq \bigoplus_{(Z_{ij})\in \mathrm{Irr(Sch)}^{I\times J}} \big(\bigotimes_i \mathrm{LCA}(X_i, \bigotimes_j Z_{ij})\big) \otimes \big(\bigotimes_j \mathrm{LA}(\bigotimes_i Z_{ij}, Y_j)\big),$$

whose inverse is the direct sum of the maps $(\bigotimes_i c_i) \otimes (\bigotimes_j a_j) \mapsto (\bigotimes_j a_j) \circ \beta_{I,J} \circ (\bigotimes_i c_i)$, where $\beta_{I,J} : \bigotimes_i(\bigotimes_j Z_{ij}) \to \bigotimes_j(\bigotimes_i Z_{ij})$ is the braiding morphism.

This is proved in [E, Pos]; see also Appendix A.

5.3. **A filtration on** QLBA**.** Let $\langle\varphi\rangle$ be the prop ideal of QLBA generated by φ and denote by $\langle\varphi\rangle^n$ its nth power. For $X, Y \in \mathrm{Ob(Sch)}$, we have a decreasing filtration $\mathrm{QLBA}(X,Y) \supset \langle\varphi\rangle(X,Y) \supset \langle\varphi\rangle^2(X,Y) \supset ....$ As φ is homogeneous for the $\mathbb{Z}^2$-grading, so are the $\langle\varphi\rangle^n$, i.e., $\langle\varphi\rangle^n(X,Y) = \bigoplus_{\alpha\in\mathbb{Z}^2}\langle\varphi\rangle^n(X,Y)[\alpha]$.

Lemma 5.1. *This filtration is complete, i.e., $\bigcap_{n\geq 0}\langle\varphi\rangle^n(X,Y) = 0$.*

Proof. Observe that $\langle\varphi\rangle^n(X,Y)$ is supported in $n(-1,2) + \mathbb{N}(1,0) + \mathbb{N}(0,1) + \mathbb{N}(-1,2) \subset (2n+\mathbb{N})(0,1) + \mathbb{Z}(1,0)$. Then $\bigcap_{n\geq 0}\langle\varphi\rangle^n(X,Y)$ is supported in $\bigcap_{n\geq 0}\big((2n+\mathbb{N})(0,1) + \mathbb{Z}(1,0)\big)$, which is empty. So this intersection is zero. □

The composition of QLBA restricts to a map $\langle\varphi\rangle^m(G,H) \otimes \langle\varphi\rangle^n(F,G) \to \langle\varphi\rangle^{n+m}(F,H)$, and the tensor product restricts to $\langle\varphi\rangle^n(F,G) \otimes \langle\varphi\rangle^{n'}(F',G') \to \langle\varphi\rangle^{n+n'}(F\otimes F', G\otimes G')$, so $\mathrm{QLBA} \supset \langle\varphi\rangle \supset ...$ is a prop filtration. The associated graded prop is defined by $\mathrm{gr}\,\mathrm{QLBA}(F,G) := \bigoplus_{n\geq 0}\mathrm{gr}_n\,\mathrm{QLBA}(F,G)$, where $\mathrm{gr}_n\,\mathrm{QLBA}(F,G) = \langle\varphi\rangle^n(F,G)/\langle\varphi\rangle^{n+1}(F,G)$.

5.4. **The graded prop** LBA_α**.** Define P to be the prop with the same generators $\tilde\mu, \tilde\delta, \tilde\varphi$ as QLBA and the same relations, except for the third, which is replaced by $\mathrm{Alt}_3 \circ(\tilde\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \tilde\delta = 0$.

We now construct a prop isomorphic to P. The following general construction goes back to [EH]. For $C \in \mathrm{Ob(Sch)}$, we have a prop LBA_C defined by $\mathrm{LBA}_C(F,G) := \bigoplus_{n\geq 0} \mathrm{LBA}(F \otimes S^n(C), G)$ (the composition is induced by the coproduct $S \to S^{\otimes 2}$). For $D \in \mathrm{Ob(Sch)}$, we set $\mathrm{LBA}_{C,D}(F,G) := \bigoplus_{n\geq 0} \mathrm{LBA}(F \otimes S^n(C) \otimes D, G)$; for $\alpha \in \mathrm{LBA}(C,D)$, we have a map $\mathrm{LBA}(F \otimes S^n(C) \otimes D, G) \to \mathrm{LBA}(F \otimes S^{n+1}(C), G)$, $x \mapsto x \circ (\mathrm{id}_{F\otimes S^n(C)} \otimes \alpha) \circ (\mathrm{id}_F \otimes \Delta_{n,1})$, where $\Delta_{n,1} : S^{n+1}(C) \to S^n(C) \otimes C$ is the component $n+1 \to (n,1)$ of the coproduct $S(C) \to S(C)^{\otimes 2}$. We then have commutative diagrams

$$\begin{array}{ccc} \begin{array}{c}\mathrm{LBA}_{C,D}(F,G)\otimes \mathrm{LBA}_C(G,H) \\ \oplus\, \mathrm{LBA}_C(F,G)\otimes \mathrm{LBA}_{C,D}(G,H)\end{array} & \to & \mathrm{LBA}_{C,D}(F,H) \\ \downarrow & & \downarrow \\ \mathrm{LBA}_C(F,G)\otimes \mathrm{LBA}_C(G,H) & \to & \mathrm{LBA}_C(F,H) \end{array}$$

and

$$\begin{array}{ccc} \begin{array}{c}\mathrm{LBA}_{C,D}(F,G)\otimes \mathrm{LBA}_C(F',G') \\ \oplus\, \mathrm{LBA}_C(F,G)\otimes \mathrm{LBA}_{C,D}(F',G')\end{array} & \to & \mathrm{LBA}_{C,D}(F\otimes F',G\otimes G') \\ \downarrow & & \downarrow \\ \mathrm{LBA}_C(F,G)\otimes \mathrm{LBA}_C(F',G') & \to & \mathrm{LBA}_C(F\otimes F',G\otimes G') \end{array}$$

induced by the composition and tensor product, which implies that if

$$\mathrm{LBA}_\alpha(F,G) := \mathrm{Coker}(\mathrm{LBA}_{C,D}(F,G) \to \mathrm{LBA}_C(F,G)),$$

then we have a prop morphism $\mathrm{LBA}_C \to \mathrm{LBA}_\alpha$.

In what follows, we will set $C := \Lambda^3$, $D := \Lambda^4$, $\alpha := \mathrm{pr}_4 \circ \mathrm{Alt}_4 \circ(\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ \mathrm{inj}_3 \in \mathrm{LBA}(\Lambda^3, \Lambda^4)$, where $\mathrm{inj}_3 : \Lambda^3 \to \mathbf{id}^{\otimes 3}$ and $\mathrm{pr}_4 : \mathbf{id}^{\otimes 4} \to \Lambda^4$ are the canonical injection and projection.

Lemma 5.2. *We have a prop isomorphism* $\mathrm{LBA}_\alpha \simeq P$.

Proof. Let $\tilde P$ be the prop with generators $\tilde\mu, \tilde\delta, \tilde\varphi$ and the following relations: Lie bialgebra relations between $\tilde\mu, \tilde\delta$, and $\tilde\varphi = \frac{1}{6}\mathrm{Alt}_3 \circ \tilde\varphi$. We have a morphism $\tilde P \to \mathrm{LBA}_{\Lambda^3}$, defined by $\tilde\mu \mapsto \mu \in \mathrm{LBA}(\mathbf{id}^{\otimes 2} \otimes S^0(\Lambda^3), \mathbf{id}) \subset \mathrm{LBA}_{\Lambda^3}(\mathbf{id}^{\otimes 2}, \mathbf{id})$; $\tilde\delta \mapsto \delta \in \mathrm{LBA}(\mathbf{id} \otimes S^0(\Lambda^3), \mathbf{id}^{\otimes 2}) \subset \mathrm{LBA}_{\Lambda^3}(\mathbf{id}, \mathbf{id}^{\otimes 2})$; $\tilde\varphi \mapsto \mathrm{inj}_3 \in \mathrm{LBA}(\mathbf{1} \otimes S^1(\Lambda^3), \mathbf{id}^{\otimes 3}) \subset \mathrm{LBA}_{\Lambda^3}(\mathbf{1}, \mathbf{id}^{\otimes 3})$, as $\mathrm{inj}_3 = \frac{1}{6}\mathrm{Alt}_3 \circ \mathrm{inj}_3$. We also have a morphism $\mathrm{LBA}_{\Lambda^3} \to \tilde P$, defined by $\mathrm{LBA}_{\Lambda^3}(F,G) \supset \mathrm{LBA}(F \otimes S^n(\Lambda^3), G) \supset f \mapsto \mathrm{can}(f) \circ (\mathrm{id}_F \otimes S^n(\tilde\varphi)) \in \tilde P(F,G)$, where $\mathrm{can} : \mathrm{LBA} \to \tilde P$ is the prop morphism defined by $\mu, \delta \mapsto \tilde\mu, \tilde\delta$. One proves that these are inverse isomorphisms, which induce an isomorphism $\mathrm{LBA}_\alpha \simeq P$. □

5.5. **A graded prop morphism** $\mathrm{LBA}_\alpha \to \mathrm{gr}\,\mathrm{QLBA}$**.**

Lemma 5.3. *There is a unique prop morphism* $\mathrm{LBA}_\alpha \simeq P \to \mathrm{gr}\,\mathrm{QLBA}$*, defined by* $P(\mathbf{id}^{\otimes 2}, \mathbf{id}) \ni \tilde\mu \mapsto \mu \in \mathrm{LBA}(\mathbf{id}^{\otimes 2}, \mathbf{id}) = \mathrm{gr}_0\,\mathrm{QLBA}(\mathbf{id}^{\otimes 2}, \mathbf{id})$*,* $P(\mathbf{id}, \mathbf{id}^{\otimes 2}) \ni \tilde\delta \mapsto \delta \in \mathrm{LBA}(\mathbf{id}, \mathbf{id}^{\otimes 2}) = \mathrm{gr}_0\,\mathrm{QLBA}(\mathbf{id}, \mathbf{id}^{\otimes 2})$ *(we have* $\mathrm{QLBA}/\langle\varphi\rangle = \mathrm{LBA}$*, so* $\mathrm{gr}_0\,\mathrm{QLBA} = \mathrm{LBA}$*),* $P(\mathbf{1}, \mathbf{id}^{\otimes 3}) \ni \tilde\varphi \mapsto [\varphi] \in \mathrm{gr}_1\,\mathrm{QLBA}(\mathbf{1}, \mathbf{id}^{\otimes 3})$*.*

Proof. The images in $\mathrm{gr}_0\,\mathrm{QLBA}$ of the Jacobi relation for μ, of the cocycle relation between μ, δ, and of the quasi-co-Jacobi relation between μ, δ, φ (which hold in

$\langle\varphi\rangle^0 = \mathrm{QLBA}$) are, respectively, the Jacobi relation for $[\mu]$, the cocycle relation between $[\mu], [\delta]$ and the co-Jacobi relation for $[\delta]$. The images in $\mathrm{gr}_1\,\mathrm{QLBA}$ of the relations $\varphi = \frac{1}{6}\mathrm{Alt}_3 \circ\varphi$ and $\mathrm{Alt}_4 \circ(\delta\otimes \mathrm{id}_{\mathbf{id}}^{\otimes 2})\circ\varphi = 0$ (which hold in $\langle\varphi\rangle$) are the similar relations, with δ,φ replaced by $[\delta],[\varphi]$. It follows that we have a prop morphism $P \to \mathrm{gr}\,\mathrm{QLBA}$, $\tilde\mu, \tilde{\delta}, \tilde\varphi \mapsto [\mu], [\delta], [\varphi]$. □

Theorem 5.1. *The morphism* $\mathrm{LBA}_\alpha \to \mathrm{gr}\,\mathrm{QLBA}$ *is a prop isomorphism.*

Proof. We say that a prop morphism $\mathcal{C}\to\mathcal{D}$ is surjective (resp., injective) if the maps $\mathcal{C}(F,G)\to\mathcal{D}(F,G)$ are.

As QLBA is generated by μ,δ,φ, the prop gr QLBA is generated by their classes $[\mu],[\delta],[\varphi]$, and since the generators of $P\simeq \mathrm{LBA}_\alpha$ map to these elements, the morphism $\mathrm{LBA}_\alpha\to\mathrm{gr}\,\mathrm{QLBA}$ is surjective.

We now prove the injectivity of $\mathrm{LBA}_\alpha\to\mathrm{gr}\,\mathrm{QLBA}$. Let us first explain the main points of this proof. We construct a filtered prop morphism $\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$; composing the associated graded morphism $\mathrm{gr}\,\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$ with $\mathrm{LBA}_\alpha\to \mathrm{gr}\,\mathrm{QLBA}$, we obtain a morphism $\mathrm{LBA}_\alpha\to L(\mathbf{LCA}_{\Lambda^2})$. This morphism factors as $\mathrm{LBA}_\alpha\to L(\mathbf{LCA}_\alpha)\to L(\mathbf{LCA}_{\Lambda^2})$. The injectivity of $\mathrm{LBA}_\alpha\to L(\mathbf{LCA}_\alpha)$ is a consequence of a general argument (already used in the proof of the structure result for the prop LBA; see Appendix A), while the injectivity of the second morphism follows from that of a morphism $\mathrm{LCA}_\alpha\to\mathrm{LCA}_{\Lambda^2}$, which is a consequence of Lemma 5.6. This establishes the injectivity of $\mathrm{LBA}_\alpha\to L(\mathbf{LCA}_{\Lambda^2})$ and therefore of $\mathrm{LBA}_\alpha\to\mathrm{gr}\,\mathrm{QLBA}$. Let us now proceed with the details of the proof.

We first define the auxiliary props mentioned above. LCA_{Λ^2} is the prop with generators $\delta_{\mathrm{LCA}}:\mathbf{id}\to\mathbf{id}^{\otimes 2}$, $r:\mathbf{1}\to\mathbf{id}^{\otimes 2}$, and relations: antisymmetry and co-Jacobi for δ_{LCA}, and antisymmetry for r. Similarly, LCA_α is the prop with generators $\tilde{\tilde\delta}:\mathbf{id}\to\mathbf{id}^{\otimes 2}$ and $\tilde{\tilde\varphi}:\mathbf{1}\to\mathbf{id}^{\otimes 3}$, and relations: antisymmetry and co-Jacobi for $\tilde{\tilde\delta}$, antisymmetry for $\tilde{\tilde\varphi}$, and $\mathrm{Alt}_4\circ(\tilde{\tilde\delta}\otimes\mathrm{id}_{\mathbf{id}}^{\otimes 2})\circ\tilde{\tilde\varphi}=0$. One checks that there are unique **Sch**-props $\mathbf{LCA}_{\Lambda^2}$, $\mathbf{LCA}_\alpha$ associated to these props (for example, $\mathbf{LCA}_{\Lambda^2}(\mathbf{F},\mathbf{G})=\prod_{i,j}\mathrm{LCA}_{\Lambda^2}(F_i,G_j)$ for $\mathbf{F}=\bigoplus_i F_i$, $\mathbf{G}=\bigoplus_i G_i$). We denote by $L\in\mathrm{Ob}(\mathbf{Sch})$ the "free Lie algebra" Schur functor, i.e., if V is a vector space, then $L(V)$ is the free Lie algebra generated by V; so $L=L_1\oplus L_2\oplus ...$, where $L_1=\mathbf{id}$, $L_2=\Lambda^2$, etc.

We now define the prop morphism $\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$. The universal version of the Lie algebra bracket on $L(V)$ is an element $\mu_{\mathrm{free}}\in\mathbf{Sch}(L^{\otimes 2},L)$. The prop morphism $\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$ is then defined by $\mu\mapsto\mu_{\mathrm{free}}$, $\delta\mapsto\delta_{\mathrm{free}}+\mathrm{ad}(r)$, $\varphi\mapsto \frac{1}{2}\mathrm{Alt}_3\circ\big((\delta_{\mathrm{free}}\otimes\mathrm{id}_L)\circ\tilde r-(\mathrm{id}_L^{\otimes 2}\otimes\mu_{\mathrm{free}})\circ(\mathrm{id}_L\otimes\beta_{L,L}\otimes\mathrm{id}_L)\circ(\tilde r\otimes\tilde r)\big)$ where we identify μ_{free} with its image under $L(\mathbf{Sch})\to L(\mathbf{LCA}_{\Lambda^2})$); we identify δ_{free} with its image under $L(\mathbf{LCA})\to L(\mathbf{LCA}_{\Lambda^2})$ (see Appendix A); $\mathrm{ad}(r)\in\mathbf{LCA}_{\Lambda^2}(L,L^{\otimes 2})$ is $(\mu_{\mathrm{free}}\otimes\mathrm{id}_L+(\mathrm{id}_L\otimes\mu_{\mathrm{free}})\circ(\beta_{L,L}\otimes\mathrm{id}_L))\circ(\mathrm{id}_L\otimes\tilde r)$; $\tilde r=\mathrm{inj}_1^{\otimes 2}\circ r$, where $\mathrm{inj}_1:\mathbf{id}\to L$ is the canonical injection. This morphism is the propic version of the following construction: to a Lie coalgebra $(\mathfrak{c},\delta_{\mathfrak{c}})$ and $r_{\mathfrak{c}}\in\Lambda^2(\mathfrak{c})$, we associate the quasi-Lie bialgebra defined as the twist by $\Lambda^2(\mathrm{inj}_1^{\mathfrak{c}})(r_{\mathfrak{c}})$ of the Lie bialgebra $(L(\mathfrak{c}),\delta_{L(\mathfrak{c})})$, where $\delta_{L(\mathfrak{c})}:L(\mathfrak{c})\to L(\mathfrak{c})^{\otimes 2}$ is the unique derivation extending $\delta_{\mathfrak{c}}$ (and $\mathrm{inj}_1^{\mathfrak{c}}:\mathfrak{c}\to L(\mathfrak{c})$ is the canonical injection).

The powers of the prop ideal $\langle r\rangle$ define a filtration on the prop LCA_{Λ^2}; the associated graded prop $\mathrm{gr}\,\mathrm{LCA}_{\Lambda^2}$ is canonically isomorphic to LCA_{Λ^2}. The prop morphism $\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$ is compatible with the filtrations (as it takes φ to $\langle r\rangle$), and the associated graded morphism $\mathrm{gr}\,\mathrm{QLBA}\to L(\mathbf{LCA}_{\Lambda^2})$ is given by

$[\mu] \mapsto \mu_{\rm free}$, $[\delta] \mapsto \delta_{\rm free}$ and $[\varphi] \mapsto \frac{1}{2}\,{\rm Alt}_3 \circ (\delta_{\rm free} \otimes {\rm id}_L) \circ {\rm inj}_1^{\otimes 2} \circ r$. The composed morphism

$$\text{(2)} \qquad {\rm LBA}_\alpha \to {\rm gr}\,{\rm QLBA} \to L(\mathbf{LCA}_{\Lambda^2})$$

is then given by $\tilde\mu \mapsto \mu_{\rm free}$, $\tilde\delta \mapsto \delta_{\rm free}$ and $\tilde\varphi \mapsto \frac{1}{2}\,{\rm Alt}_3 \circ (\delta_{\rm free} \otimes {\rm id}_L) \circ {\rm inj}_1^{\otimes 2} \circ r$.

We now define two prop morphisms ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha)$ and ${\rm LCA}_\alpha \to {\rm LCA}_{\Lambda^2}$, such that the above morphism ${\rm LBA}_\alpha \to L(\mathbf{LCA}_{\Lambda^2})$ coincides with ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha) \to L(\mathbf{LCA}_{\Lambda^2})$.

First define ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha)$. There is a unique morphism ${\rm LBA} \to L({\rm LCA})$, taking μ, δ to $\mu_{\rm free}, \delta_{\rm free}$ (see Appendix A); this is the propic version of the functor $\{$Lie coalgebras$\} \to \{$Lie bialgebras$\}$, $(\mathfrak{c}, \delta_\mathfrak{c}) \mapsto (L(\mathfrak{c})$, free Lie bracket, unique cobracket extending $\delta_\mathfrak{c})$. We define ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha)$ by $\tilde\mu, \tilde\delta \mapsto \mu_{\rm free}, \delta_{\rm free}$ (we identify $\mu_{\rm free}, \delta_{\rm free}$ with their images in $L(\mathbf{LCA}_\alpha)$) and $\tilde\varphi \mapsto {\rm inj}_1^{\otimes 2} \circ \tilde{\tilde\varphi}$. This morphism is the propic version of $\{(\mathfrak{c}, \delta_\mathfrak{c}, \varphi_\mathfrak{c}) | (\mathfrak{c}, \delta_\mathfrak{c})$ is a Lie coalgebra, $\varphi_\mathfrak{c} \in \Lambda^3(\mathfrak{c}), {\rm Alt}_4 \circ (\delta_\mathfrak{c} \otimes {\rm id}_\mathfrak{c}^{\otimes 2})(\varphi_\mathfrak{c}) = 0\} \to \{(\mathfrak{a}, \delta_\mathfrak{a}, \mu_\mathfrak{a}, \varphi_\mathfrak{a}) | (\mathfrak{a}, \mu_\mathfrak{a}, \delta_\mathfrak{a})$ is a Lie bialgebra, $\varphi_\mathfrak{a} \in \Lambda^3(\mathfrak{a})$, ${\rm Alt}_4 \circ (\delta_\mathfrak{a} \otimes {\rm id}_\mathfrak{a}^{\otimes 2}) \circ \varphi_\mathfrak{a} = 0\}$, extending the above functor by $\varphi_\mathfrak{a} := \Lambda^3({\rm inj}_1^\mathfrak{c})(\varphi_\mathfrak{c})$.

We then define the morphism ${\rm LCA}_\alpha \to {\rm LCA}_{\Lambda^2}$ by $\tilde{\tilde\delta} \mapsto \delta_{\rm LCA}$, $\tilde{\tilde\varphi} \mapsto \frac{1}{2}\,{\rm Alt}_3(\delta_{\rm LCA} \otimes {\rm id}_{\mathbf{id}}) \circ r$. One checks that (2) coincides with ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha) \to L(\mathbf{LCA}_{\Lambda^2})$.

Let us prove that ${\rm LBA}_\alpha \to L(\mathbf{LCA}_\alpha)$ is injective. Using the symmetric group actions, this is equivalent to proving that for any $p, q \geq 0$, the map

$$\text{(3)} \qquad {\rm LBA}_\alpha(T_p, T_q) \to \mathbf{LCA}_\alpha(L^{\otimes p}, L^{\otimes q})$$

is injective.

Lemma 5.4. *The map* $\bigoplus_{Z \in {\rm Irr(Sch)}} {\rm LCA}_\alpha(T_p, Z) \otimes {\rm LA}(Z, T_q) \to {\rm LBA}_\alpha(T_p, T_q)$, *induced by the prop morphisms* ${\rm LCA}_\alpha, {\rm LA} \to {\rm LBA}_\alpha$ *(*$\tilde{\tilde\delta}, \tilde{\tilde\varphi} \mapsto \tilde\delta, \tilde\varphi$, $\mu \mapsto \tilde\mu$*) and by composition, is an isomorphism of vector spaces.*

Proof of Lemma. Recall that $C = \Lambda^3$, $D = \Lambda^4$, $\alpha \in {\rm LBA}(D, C)$. One may construct as above a prop ${\rm LCA}_C$ by ${\rm LCA}_C(F, G) := \bigoplus_{n \geq 0} {\rm LCA}(F \otimes S^n(C), G)$; setting ${\rm LCA}_{C,D}(F, G) := \bigoplus_{n \geq 0} {\rm LCA}(F \otimes S^n(C) \otimes D, G)$, then using the fact that $\alpha \in {\rm LCA}(D, C)$, one constructs a map ${\rm LCA}_{C,D}(F, G) \to {\rm LCA}_C(F, G)$ and one then checks that ${\rm LCA}_\alpha(F, G) = {\rm Coker}\big({\rm LCA}_{C,D}(F, G) \to {\rm LCA}_C(F, G)\big)$. For $F, G \in {\rm Ob(Sch)}$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{Z \in {\rm Irr(Sch)}} {\rm LCA}_{C,D}(F, Z) \otimes {\rm LA}(Z, G) & \overset{\simeq}{\to} & {\rm LBA}_{C,D}(F, G) \\ \downarrow & & \downarrow \\ \bigoplus_{Z \in {\rm Irr(Sch)}} {\rm LCA}_C(F, Z) \otimes {\rm LA}(Z, G) & \overset{\simeq}{\to} & {\rm LBA}_C(F, G) \end{array}$$

whose vertical cokernel is an isomorphism

$$\bigoplus_{Z \in {\rm Irr(Sch)}} {\rm LCA}_\alpha(F, Z) \otimes {\rm LA}(Z, G) \overset{\simeq}{\to} {\rm LBA}_\alpha(F, G);$$

this isomorphism coincides with the map described in the statement of the lemma. $\square$

We now consider the composite map

$$(4)\qquad \bigoplus_{Z\in \mathrm{Irr(Sch)}} \mathrm{LCA}_\alpha(T_p,Z)\otimes \mathrm{LA}(Z,T_q)\to \mathrm{LBA}_\alpha(T_p,T_q)$$
$$\to \mathbf{LCA}_\alpha(L^{\otimes p},L^{\otimes q})\to \mathbf{LCA}_\alpha(T_p,L^{\otimes q}),$$

where the first map is described in Lemma 5.4, the middle map is (3), and the last map is induced by the injection $T_p=\mathbf{id}^{\otimes p}\to L^{\otimes p}$.

Lemma 5.5. *The map* (4) *coincides with the composite map* $\bigoplus_Z \mathrm{LCA}_\alpha(T_p,Z)\otimes \mathrm{LA}(Z,T_q)\simeq \bigoplus_Z \mathrm{LCA}_\alpha(T_p,Z)\otimes \mathbf{Sch}(Z,L^{\otimes q})\to \mathbf{LCA}_\alpha(T_p,L^{\otimes q})$*, where the first map is induced by the isomorphism* $\mathrm{LA}(Z,T_q)\simeq \mathbf{Sch}(Z,L^{\otimes q})$ *and the second by composition.*

Proof. Note that the isomorphism $\mathrm{LA}(Z,T_q)\simeq \mathbf{Sch}(Z,L^{\otimes q})$ is proved in Appendix A. By using symmetric group actions, one shows that it suffices to prove the above statement with Z replaced by T_N. We have composite prop morphisms $\rho:\mathrm{LCA}_\alpha\to \mathrm{LBA}_\alpha\to L(\mathbf{LCA}_\alpha)$ and $\sigma:\mathrm{LA}\to\mathrm{LBA}_\alpha\to L(\mathbf{LCA}_\alpha)$; actually σ factors through $\mathrm{LA}\to L(\mathbf{Sch})$. The map (4) (with Z replaced by T_N) then takes $f\otimes g$ to $\sigma(g)\circ\rho(f)\circ \mathrm{inj}_1^{\otimes p}$, where $\mathrm{inj}_1:\mathbf{id}\to L$ is the canonical morphism, $f\in \mathrm{LCA}_\alpha(T_p,T_N)$, $g\in \mathrm{Sch}(T_N,T_q)$, $\rho(f)\in \mathbf{LCA}_\alpha(L^{\otimes p},L^{\otimes N})$, $\sigma(g)\in \mathbf{Sch}(L^{\otimes N},L^{\otimes q})$.

We have $\rho(f)\circ \mathrm{inj}_1^{\otimes p}=\mathrm{inj}_1^{\otimes N}\circ f$, as this property can be checked for $f=\delta_{\mathrm{LCA}},r$ and is preserved by composition and tensor products. Moreover, $\sigma(g)\circ \mathrm{inj}_1^{\otimes N}\in \mathbf{LA}(T_N,L^{\otimes N})$ is the image $\tilde g$ of g under $\mathrm{LA}(Z,T_q)\simeq \mathbf{Sch}(Z,L^{\otimes q})$. It follows that (4) coincides with $f\otimes g\mapsto \tilde g\circ f$, which was to be proved. ☐

According to Lemma A.1, the composite map $\bigoplus_Z \mathrm{LCA}_\alpha(T_p,Z)\otimes \mathrm{LA}(Z,T_q)\simeq \bigoplus_Z \mathrm{LCA}_\alpha(T_p,Z)\otimes \mathbf{Sch}(Z,L^{\otimes q})\to \mathbf{LCA}_\alpha(T_p,L^{\otimes q})$ is an isomorphism, so Lemma 5.5 implies that the composite map (4) is an isomorphism. The first map in (4) is also an isomorphism by Lemma 5.4, so the map (3) is injective, which was to be proved.

Let us now prove that $\mathrm{LCA}_\alpha\to\mathrm{LCA}_{\Lambda^2}$ is injective. For this, we outline the structure of these props. We have

$$\mathrm{LCA}_{\Lambda^2}(T_p,T_z)$$
$$=\bigoplus_{k\geq 0}\Big(\bigoplus_{\substack{z',z_1,\ldots,z_k|\\ z'+z_1+\ldots+z_k=z}} \mathrm{Ind}^{\mathfrak{S}_z}_{\mathfrak{S}_{z'}\times\mathfrak{S}_{z_1}\times\ldots\times\mathfrak{S}_{z_k}}\big(\mathrm{LCA}(T_p,T_{z'})\otimes\bigotimes_{i=1}^k \mathrm{LCA}(\Lambda^2,T_{z_i})\big)\Big)_{\mathfrak{S}_k}$$

and

$$\mathrm{LCA}_\alpha(T_p,T_z)=\bigoplus_{z\geq 0}\Big(\bigoplus_{z',z_1,\ldots,z_k|z'+z_1+\ldots+z_k=z}\mathrm{Ind}^{\mathfrak{S}_z}_{\mathfrak{S}_{z'}\times\mathfrak{S}_{z_1}\ldots\times\mathfrak{S}_{z_k}}\Big(\mathrm{LCA}(T_p,T_{z'})\otimes$$
$$\bigotimes_{i=1}^k \mathrm{Coker}\big(\mathrm{LCA}(\Lambda^4,T_{z_i})\to \mathrm{LCA}(\Lambda^3,T_{z_i})\big)\Big)\Big)_{\mathfrak{S}_k}.$$

The injectivity of $\mathrm{LCA}_\alpha\to\mathrm{LCA}_{\Lambda^2}$ is therefore equivalent to that of $\mathrm{Coker}(\mathrm{LCA}(\Lambda^4,T_z)\to \mathrm{LCA}(\Lambda^3,T_z))\to \mathrm{LCA}(\Lambda^2,T_z)$; in other terms, we have a sequence $\mathrm{LCA}(\Lambda^4,T_z)\stackrel{-\circ \mathrm{Alt}_4\circ(\delta\otimes \mathrm{id}_{\mathbf{id}}^{\otimes 2})}{\longrightarrow}\mathrm{LCA}(\Lambda^3,T_z)\stackrel{-\circ\mathrm{Alt}_3\circ(\delta\otimes \mathrm{id}_{\mathbf{id}})}{\longrightarrow}\mathrm{LCA}(\Lambda^2,T_z)$ where the composite map is zero, and we should prove that the homology vanishes.

To prove this, we will prove that the second homology of the complex

$$(5)\quad ... \to \mathrm{LA}(T_z,\Lambda^4) \stackrel{\mathrm{Alt}_3\circ(\mu\otimes\mathrm{id}_{\mathbf{id}}^{\otimes 2}))\circ -}{\longrightarrow} \mathrm{LA}(T_z,\Lambda^3) \stackrel{\mathrm{Alt}_2\circ(\mu\otimes\mathrm{id}_{\mathbf{id}}))\circ -}{\longrightarrow} \mathrm{LA}(T_z,\Lambda^2) \stackrel{\mu\circ -}{\to} \mathrm{LA}(T_z,\mathbf{id}) \to 0$$

vanishes. We will prove more generally:

Lemma 5.6. *If $z \geq 2$, the complex* (5) *is acyclic; if $z = 1$, its homology is* 1*-dimensional, concentrated in degree* 0.

Proof. Let $\mathcal{L}_z$ (resp., $\mathcal{A}_z$) be the free Lie (resp., associative) algebra with generators $x_1,...,x_z$. These spaces are both graded by $\bigoplus_{i=1}^z \mathbb{N}\delta_i$, where $|x_i| = \delta_i$. For V a vector space graded by $\bigoplus_{i=1}^z \mathbb{N}\delta_i$, and[4] $I \subset [1,z]$, we denote by V_I the part of V of degree $\sum_{i\in I}\delta_i$. We have $\mathrm{LA}(T_z,\Lambda^k) \simeq (\Lambda^k(\mathcal{L}_z))_{[1,z]}$. This isomorphism takes the complex (5) to

$$(6)\qquad ... \stackrel{\mathrm{Alt}_3\circ(\mu_{\mathcal{L}}\otimes\mathrm{id}^{\otimes 2})}{\to} (\Lambda^3(\mathcal{L}_z))_{[1,z]} \stackrel{\mathrm{Alt}_2\circ(\mu_{\mathcal{L}}\otimes\mathrm{id})}{\to} (\Lambda^2(\mathcal{L}_z))_{[1,z]} \stackrel{\mu_{\mathcal{L}}}{\to} (\mathcal{L}_z)_{[1,z]} \to 0,$$

where $\mu_{\mathcal{L}}$ is the Lie bracket of $\mathcal{L}_z$.

Let $\mathrm{Part}_k(I)$ be the set of k-partitions of a set I, i.e., of the k-tuples $(I_1,...,I_k)$ with $\bigsqcup_{i=1}^k I_i = I$. The group $\mathfrak{S}_k$ acts on $\mathrm{Part}_k([1,z])$, and we have a decomposition

$$(\Lambda^k(\mathcal{L}_z))_{[1,z]} = \bigoplus_{[(I_1,...,I_k)]\in \mathrm{Part}_k([1,z])/\mathfrak{S}_k} (\Lambda^k(\mathcal{L}_z))_{[(I_1,...,I_k)]},$$

where the summand in the r.h.s. is the space of antisymmetric tensors in $\bigoplus_{\sigma\in\mathfrak{S}_n}(\mathcal{L}_z)_{I_{\sigma(1)}}\otimes ... \otimes (\mathcal{L}_z)_{I_{\sigma(k)}}$.

We have a bijection $\{(I_1',[(I_2,...,I_k)])|I_1' \subset [2,z], [(I_2,...,I_k)] \in \mathrm{Part}_{k-1}([2,z]-I_1')/\mathfrak{S}_{k-1}\} \to \mathrm{Part}_k([1,z])$, taking $(I_1',[(I_2,...,I_k)])$ to $[(I_1'\sqcup\{1\},I_2,...,I_k)]$. The inverse bijection takes $[(I_1,...,I_k)]$ to $(I_i-\{1\},[(I_1,...,I_{i-1},I_{i+1},...,I_k)])$, where $i\in[1,k]$ is the index such that $1\in I_i$.

For $(I_1',[(I_2,...,I_k)])$ in the first set, we have an isomorphism

$$(\Lambda^k(\mathcal{L}_z))_{[(I_1'\sqcup\{1\},I_2,...,I_k)]} \simeq (\mathcal{L}_z)_{I_1'\sqcup\{1\}} \otimes (\Lambda^{k-1}(\mathcal{L}_z))_{[(I_2,...,I_k)]}$$

(whose inverse is given by Alt_k, or, up to a factor, by the sum of all cyclic permutations if k is odd, and their alternating sum if k is even), which gives rise to an isomorphism

$$(\Lambda^k(\mathcal{L}_z))_{[1,z]} \simeq \bigoplus_{(I_1',[(I_2,...,I_k)])} (\mathcal{L}_z)_{I_1'\sqcup\{1\}} \otimes (\Lambda^{k-1}(\mathcal{L}_z))_{[(I_2,...,I_k)]}$$
$$\subset (\mathcal{L}_z\otimes\Lambda^{k-1}(\mathcal{L}_z))_{[1,z]}.$$

We have a complex

$$(7)\qquad ... \to (\mathcal{L}_z\otimes\Lambda^2(\mathcal{L}_z))_{[1,z]} \to (\mathcal{L}_z\otimes\mathcal{L}_z)_{[1,z]} \to (\mathcal{L}_z)_{[1,z]} \to 0,$$

where the differential $(\mathcal{L}_z\otimes\Lambda^k(\mathcal{L}_z))_{[1,z]} \to (\mathcal{L}_z\otimes\Lambda^{k-1}(\mathcal{L}_z))_{[1,z]}$ is induced by $x_0\otimes(x_1\wedge ... \wedge x_k) \mapsto \sum_{i=1}^k (-1)^{i+1}[x_0,x_i]\otimes(x_1\wedge ...\check{x}_i ... \wedge x_k) + \sum_{1\leq i<j\leq k}(-1)^{j+1}x_0\otimes(x_1\wedge ... \wedge[x_i,x_j]\wedge ...\check{x}_j ... \wedge x_k)$. If $I,J\subset[1,z]$ are disjoint, we have $[(\mathcal{L}_z)_I,(\mathcal{L}_z)_J] \subset (\mathcal{L}_z)_{I\cup J}$, which implies that if

$$C_k := \bigoplus_{(I_1',[(I_2,...,I_k)])} (\mathcal{L}_z)_{I_1'\sqcup\{1\}} \otimes (\Lambda^{k-1}(\mathcal{L}_z))_{[(I_2,...,I_k)]},$$

[4] We set $[p,q] := \{p,p+1,\dots,q\}$.

then

$$... \to C_2 \to C_1 \to 0$$

is a subcomplex of (7), isomorphic to (6).

For $I' \subset [2,z]$, we have an isomorphism $(\mathcal{A}_z)_{I'} \to (\mathcal{L}_z)_{I'\cup\{1\}}$, given by $x_{i_1}...x_{i_s} \mapsto [[[x_1, x_{i_1}], x_{i_2}], ..., x_{i_s}]$; the inverse isomorphism is the restriction of the map $(\mathcal{A}_z)_{I'\cup\{1\}} \to (\mathcal{A}_z)_{I'}$ taking a monomial not starting with x_1 to 0, and a monomial starting with x_1 to the same monomial with the x_1 removed (see [B]).

The compatibility of these isomorphisms with the Lie bracket can be described as follows: for $I', I \subset [2,z]$ disjoint, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{A}_z)_{I'} \otimes (\mathcal{L}_z)_I & \to & (\mathcal{A}_z) \\ \downarrow & & \downarrow \\ (\mathcal{L}_z)_{I'\cup\{1\}} \otimes (\mathcal{L}_z)_I & \to & (\mathcal{L}_z)_{I\cup I'\cup\{1\}}, \end{array}$$

where the upper horizontal map is induced by the product in $\mathcal{A}_z$ ($\mathcal{L}_z$ being viewed as a subspace of $\mathcal{A}_z$) and the lower horizontal map is induced by the Lie bracket of $\mathcal{L}_z$.

We have a complex

$$(8) \qquad ... \to (\mathcal{A}_z \otimes \Lambda^2(\mathcal{L}_z))_{[2,z]} \to (\mathcal{A}_z \otimes \mathcal{L}_z)_{[2,z]} \to (\mathcal{A}_z)_{[2,z]} \to 0,$$

where the map $(\mathcal{A}_z \otimes \Lambda^k(\mathcal{L}_z))_{[2,z]} \to (\mathcal{A}_z \otimes \Lambda^{k-1}(\mathcal{L}_z))_{[2,z]}$ is induced by $x_0 \otimes (x_1 \wedge ... \wedge x_k) \mapsto \sum_{i=1}^k (-1)^{i+1} x_0 x_i \otimes x_1 \wedge ...\check{x}_i ... \wedge x_k + \sum_{1\le i<j\le k} (-1)^{j+1} x_0 \otimes x_1 \wedge ... \wedge [x_i, x_j] \wedge ...\check{x}_j ... \wedge x_k$. The isomorphisms $(\mathcal{L}_z)_{I'_1\cup\{1\}} \simeq (\mathcal{A}_z)_{I'_1}$ induce isomorphisms $C_k \simeq \bigoplus_{(I'_1,[(I_2,...,I_k)])} (\mathcal{A}_z)_{I'_1} \otimes (\Lambda^{k-1}(\mathcal{L}_z))_{[(I_2,...,I_k)]} = (\mathcal{A}_z \otimes \Lambda^{k-1}(\mathcal{L}_z))_{[2,z]}$, which are compatible with the differentials. Hence the complex (8) is isomorphic to $... \to C_2 \to C_1 \to C_0 \to 0$.

The complex (8) is the degree $\delta_2 + ... + \delta_k$ part of the complex

$$(9) \qquad ... \to \mathcal{A}_z \otimes \Lambda^2(\mathcal{L}_z) \to \mathcal{A}_z \otimes \mathcal{L}_z \to \mathcal{A}_z \to 0,$$

where the differentials are defined by the same formulas.

Define a complete increasing filtration on (9) by $\mathrm{Fil}^n(\mathcal{A}_z \otimes \Lambda^k(\mathcal{L}_z)) = (\mathcal{A}_z)_{\le n-k} \otimes \Lambda^k(\mathcal{L}_z)$, where $(\mathcal{A}_z)_{\le n}$ is the part of degree $\le n$ of $\mathcal{A}_z \simeq U(\mathcal{L}_z)$ (i.e., the span of products of $\le n$ elements of $\mathcal{L}_z$). The associated graded complex is the sum over $n \ge 0$ of complexes $\ldots \to 0 \to \Lambda^n(\mathcal{L}_z) \to ... \to S^{n-1}(\mathcal{L}_z) \otimes \mathcal{L}_z \to S^n(\mathcal{L}_z) \to 0$, which add up to the Koszul complex

$$... \to S(\mathcal{L}_z) \otimes \Lambda^2(\mathcal{L}_z) \to S(\mathcal{L}_z) \otimes \mathcal{L}_z \to S(\mathcal{L}_z) \to 0,$$

where the differential $S(\mathcal{L}_z) \otimes \Lambda^k(\mathcal{L}_z) \to S(\mathcal{L}_z) \otimes \Lambda^{k-1}(\mathcal{L}_z)$ is $f \otimes (x_1 \wedge ... \wedge x_k) \mapsto \sum_{i=1}^k (-1)^{i+1} f x_i \otimes (x_1 \wedge ...\check{x}_i ... \wedge x_k)$.

Now if V is a vector space, then the Koszul complex

$$C(V) := (... \to S(V) \otimes \Lambda^2(V) \to S(V) \otimes V \to S(V) \to 0)$$

is a sum of complexes, indexed by $\mathbb{N}$ (this index corresponds to $p+q$ in $S^p(V) \otimes \Lambda^q(V)$). It is well known that the homology of this complex is concentrated in homological degree 0 and in degree 0, where it is equal to $\mathbf{k}$. Recall a proof. One checks this directly when V is one-dimensional; we have isomorphisms $C(V \oplus W) \simeq C(V) \otimes C(W)$ of $\mathbb{N}$-graded complexes, which implies the statement when V is finite dimensional. It follows that the Koszul complex in **Sch** given by $... \to S \otimes \Lambda^2 \to S \otimes \mathbf{id} \to S \to 0$ has its homology concentrated in homological degree 0 and degree 0, where it equals $\mathbf{1}$. This implies the statement in general.

It follows that the homology of (9) is concentrated in degree 0, where it is equal to $\mathbf{k}$; a nontrivial homology class is that of $1 \in \mathcal{A}_z$. It follows that the degree $\delta_2 + ... + \delta_z$ part of this complex is acyclic if $z \geq 2$, i.e., (8) is acyclic if $z \geq 2$. The computation of the homology of (9) is straightforward when $z = 1$. □

This ends the proof of Theorem 5.1. □

6. Comparison of cohomology groups

Theorem 4.2 means that for any $p \geq 0$ and $q > 0$, the canonical map $H^p_{\mathrm{QLBA}}(\mathbf{id}, \Lambda^q) \to H^p_{\mathrm{LBA}}(\mathbf{id}, \Lambda^q)$ is an isomorphism.

Recall that $H^\bullet_{\mathrm{(Q)LBA}}(\mathbf{id}, \Lambda^q) = H^\bullet(C^{\bullet,q}_{\mathrm{(Q)LBA}})$, where $C^{\bullet,q}_{\mathrm{(Q)LBA}} = ((\mathrm{Q})\mathrm{LBA}(\Lambda^\bullet, \Lambda^q), [\mu, -])$. If I is a prop ideal in QLBA, we also set $C^{\bullet,q}_I := I(\Lambda^\bullet, \Lambda^q)$; this is a subcomplex of $C^{\bullet,q}_{\mathrm{QLBA}}$. We then have an exact sequence of complexes $0 \to C^{\bullet,q}_{\langle\varphi\rangle} \to C^{\bullet,q}_{\mathrm{QLBA}} \to C^{\bullet,q}_{\mathrm{LBA}} \to 0$, inducing a long exact sequence in cohomology, so Theorem 4.2 will be proved if we show that $(C^{\bullet,q}_{\langle\varphi\rangle}, [\mu, -])$ is acyclic.

Observe now that we have a decreasing sequence of complexes $C^{\bullet,q}_{\langle\varphi\rangle} \supset C^{\bullet,q}_{\langle\varphi\rangle^2} \supset \dots$. All these complexes are graded by $\deg_\delta$ and in each fixed degree, the decreasing filtration is complete (i.e., for any fixed p, d, $C^{p,q}_{\langle\varphi\rangle^n}[d] = 0$ for n large enough, where $[d]$ means the part with $\deg_\delta$ equal to d). It then suffices to prove that the associated graded complex $(\bigoplus_{i>0} C^{\bullet,q}_{\langle\varphi\rangle^i}/C^{\bullet,q}_{\langle\varphi\rangle^{i+1}}$, induced differential) is acyclic. This complex coincides with $(\bigoplus_{i>0} \mathrm{LBA}^{(i)}_\alpha(\Lambda^\bullet, \Lambda^q), [\tilde\mu, -])$, where the exponent (i) means the homogeneous part with $\deg_{\tilde\varphi}$ equal to i, and $\deg_{\tilde\varphi}$ is the degree on LBA_α such that $(\tilde\mu, \tilde\delta, \tilde\varphi) \mapsto (0, 0, 1)$.

In order to prove that the latter complex is acyclic, we will prove more generally:

Theorem 6.1. *Let C, D be homogeneous Schur functors of positive degrees; let $\kappa \in \mathrm{LCA}(C, D)$. Let $\mathrm{LBA}_\kappa(X, Y) := \mathrm{Coker}\big(\mathrm{LBA}(D \otimes X, Y) \to \mathrm{LBA}(C \otimes X, Y)\big)$. Then for any $q \geq 0$, the complex $(\mathrm{LBA}_\kappa(\Lambda^p, \Lambda^q), [\mu, -])_{p\geq 0}$ is acyclic.*

Proof. Let us make this complex explicit. For $Z \in \mathrm{Irr(Sch)}$, define $\mu_Z \in \mathrm{LA}(\mathbf{id} \otimes Z, Z)$ and $\tilde\mu_Z \in \mathrm{LA}(Z \otimes \mathbf{id}, Z)$ as follows: $\mu_{T_p} \in \mathrm{LA}(\mathbf{id} \otimes T_p, T_p)$ is the universal version of $x \otimes x_1 \otimes ... \otimes x_p \mapsto \sum_{i=1}^p x_1 \otimes ... \otimes [x, x_i] \otimes ... \otimes x_p$; as it is $\mathfrak{S}_p$-equivariant, it decomposes under $\mathrm{LA}(\mathbf{id} \otimes T_p, T_p) \simeq \bigoplus_{Z,W||Z|=|W|=p} \mathrm{LA}(\mathbf{id} \otimes Z, W) \otimes \mathrm{Vect}(\pi_Z, \pi_W)$ as $\bigoplus_Z \mu_Z \otimes \mathrm{id}_{\pi_Z}$. We then set $\tilde\mu_Z := -\mu_Z \circ \beta_{Z,\mathbf{id}}$, where $\beta_{Z,\mathbf{id}} : Z \otimes \mathbf{id} \to \mathbf{id} \otimes Z$ is the braiding morphism.

Then $[\mu, -] : \mathrm{LBA}(C \otimes \Lambda^p, \Lambda^q) \to \mathrm{LBA}(C \otimes \Lambda^{p+1}, \Lambda^q)$ is the composed map $\mathrm{LBA}(C \otimes \Lambda^p, \Lambda^q) \to \mathrm{LBA}(C \otimes \Lambda^p \otimes \mathbf{id}, \Lambda^q) \to \mathrm{LBA}(C \otimes \Lambda^{p+1}, \Lambda^q)$, where the first map is $x \mapsto x \circ (\mathrm{id}_C \otimes \tilde\mu_{\Lambda^p}) - \tilde\mu_{\Lambda^q} \circ (x \otimes \mathrm{id}_{\mathbf{id}})$ and the second map is $y \mapsto y \circ \mathrm{Alt}_{p+1}$. We have a similar differential, with C replaced by D, and κ induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{LBA}(D \otimes \Lambda^p, \Lambda^q) & \to & \mathrm{LBA}(D \otimes \Lambda^p \otimes \mathbf{id}, \Lambda^q) & \to & \mathrm{LBA}(D \otimes \Lambda^{p+1}, \Lambda^q) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{LBA}(C \otimes \Lambda^p, \Lambda^q) & \to & \mathrm{LBA}(C \otimes \Lambda^p \otimes \mathbf{id}, \Lambda^q) & \to & \mathrm{LBA}(C \otimes \Lambda^{p+1}, \Lambda^q). \end{array}$$

The cokernel of this diagram is $\mathrm{LBA}_\kappa(\Lambda^p, \Lambda^q) \to \mathrm{LBA}_\kappa(\Lambda^p \otimes \mathbf{id}, \Lambda^q) \to \mathrm{LBA}_\kappa(\Lambda^{p+1}, \Lambda^q)$ and the composed map is the differential of our complex.

Recall that for $X_i, Y \in \mathrm{Ob}(\mathrm{Sch})$, $i = 1, ..., n$, we have an isomorphism $\mathrm{LBA}(X_1 \otimes ... \otimes X_n, Y) \simeq \bigoplus_{Z_1,...,Z_n \in \mathrm{Irr}(\mathrm{Sch})} \mathrm{LCA}(X_1, Z_1) \otimes ... \otimes \mathrm{LCA}(X_n, Z_n) \otimes \mathrm{LA}(Z_1 \otimes ... \otimes Z_n, Y) = \bigoplus_{Z_1,...,Z_n} \mathrm{LBA}(X_1 \otimes ... \otimes X_n, Y)_{Z_1,...,Z_n}$. The inverse isomorphism is the direct sum of the maps $c_1 \otimes ... \otimes c_n \otimes a \mapsto a \circ (c_1 \otimes ... \otimes c_n)$. If X_i is homogeneous of positive degree, $\mathrm{LCA}(X_i, \mathbf{1}) = 0$, so the above sum may be restricted by the condition $|Z_i| > 0$.

We now define a complex $0 \to C^0 \stackrel{d^{0,1}}{\to} C^1 \to ...$ as follows. The analogue of the above complex $[\mu, -] : \mathrm{LBA}(C \otimes \Lambda^p, \Lambda^q) \to \mathrm{LBA}(C \otimes \Lambda^{p+1}, \Lambda^q)$ (with C replaced by Z) admits a subcomplex, namely $C^p_{Z,q} := \bigoplus_{Z' \in \mathrm{Irr}(\mathrm{Sch})} \mathrm{LBA}(Z \otimes \Lambda^p, \Lambda^q)_{Z,Z'}$;

$$d^{p,p+1}_{Z,q} : C^p_{Z,q} \to C^{p+1}_{Z,q}$$

is then the restriction of the differential $[\mu, -]$. We then have an isomorphism between the complexes $(\mathrm{LBA}(C \otimes \Lambda^p, \Lambda^q), [\mu, -])_{p \geq 0}$ and $\bigoplus_{Z \in \mathrm{Irr}(\mathrm{Sch}), |Z|>0} \mathrm{LCA}(C, Z) \otimes (C^p_{Z,q}, d^{p,p+1}_{Z,q})_{p \geq 0}$. We have a similar isomorphism replacing C by D, and these isomorphisms are compatible with the morphisms of complexes induced by κ. Taking cokernels, we get an isomorphism of complexes

$$\begin{aligned}&(\mathrm{LBA}_\kappa(\Lambda^p, \Lambda^q), [\mu, -])_{p \geq 0} \\ &\quad \simeq \bigoplus_{Z \in \mathrm{Irr}(\mathrm{Sch}), |Z|>0} \mathrm{Coker}\big(\mathrm{LCA}(D, Z) \to \mathrm{LCA}(C, Z)\big) \otimes (C^p_{Z,q}, d^{p,p+1}_{Z,q})_{p \geq 0}.\end{aligned}$$

We now prove the acyclicity of $(C^p_{Z,q}, d^{p,p+1}_{Z,q})_{p \geq 0}$, for any $q \geq 0$ and any $Z \in \mathrm{Irr}(\mathrm{Sch})$, $|Z| > 0$. To lighten notation, we will denote it by $(C^p, d^{p,p+1})_{p \geq 0}$. We reexpress this complex as follows. View C^p as the antisymmetric part (under the action of $\mathfrak{S}_p$) of $\tilde C^p := \bigoplus_{Z_1,...,Z_p \in \mathrm{Irr}(\mathrm{Sch})} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)_{Z,Z_1,...,Z_p} \subset \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)$ (we may restrict this sum by the conditions $|Z_i| > 0$). Define

$$\tilde d^{p,p+1} : \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q) \to \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p+1}, \Lambda^q)$$

by $\tilde d^{p,p+1}(x) := x \circ (\mathrm{id}_Z \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-1}) \circ (\sum_{1 \leq i<j \leq p+1} (-1)^{i+j} \beta_{ij}) + \mu_{\Lambda^q} \circ (\mathrm{id}_{\mathbf{id}} \otimes x) \circ (\sum_{1 \leq i \leq p+1} (-1)^{i+1} \beta_i)$, where β_{ij} is the automorphism of $Z \otimes \mathbf{id}^{\otimes p+1}$, which is the universal version of $z \otimes x_1 \otimes ... \otimes x_{p+1} \mapsto z \otimes x_i \otimes x_j \otimes x_1 \otimes ... \check x_i ... \check x_j ... \otimes x_{p+1}$, and $\beta_i : Z \otimes \mathbf{id}^{\otimes p+1} \to \mathbf{id} \otimes Z \otimes \mathbf{id}^{\otimes p}$ is the universal version of $z \otimes x_1 \otimes ... \otimes x_{p+1} \mapsto x_i \otimes z \otimes x_1 \otimes ... \check x_i ... \otimes x_{p+1}$. Then $\tilde d^{p,p+1}$ restricts to $d^{p,p+1} : C^p \to C^{p+1}$.

We now introduce a filtration on C^p. Let $(\tilde C^p)^{\leq p'} \subset \tilde C^p$ be the sum of all terms where $\mathrm{card}\{i | Z_i = \mathbf{id}\} \leq p'$. This subspace is invariant under the action of $\mathfrak{S}_p$, so its totally antisymmetric part is a subspace $(C^p)^{\leq p'} \subset C^p$. □

Lemma 6.1. $d^{p,p+1}((C^p)^{\leq p'}) \subset (C^{p+1})^{\leq p'+1}$.

Proof. To prove this, we will prove that $\tilde d^{p,p+1}((\tilde C^p)^{\leq p'}) \subset (\tilde C^{p+1})^{\leq p'+1}$. If $x \in \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)_{Z,Z_1,...,Z_p}$, then $\mu_{\Lambda^q} \circ (\mathrm{id}_{\mathbf{id}} \otimes x) \circ \beta_i$ is clearly in $\mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p+1}, \Lambda^q)_{Z,Z_1,...,Z_{i-1},\mathbf{id},Z_i,...,Z_p}$. Here $\mathrm{card}\{i | Z_i = \mathbf{id}\}$ has been increased by 1. Moreover, for any $W \in \mathrm{Irr}(\mathrm{Sch})$, the image of $\mathrm{LCA}(\mathbf{id}, W) \to \mathrm{LBA}(\mathbf{id}^{\otimes 2}, W)$, $c \mapsto c \circ \mu$ lies in

$$\bigoplus_{\substack{W_1, W_2 \in \mathrm{Irr}(\mathrm{Sch}), |W_i|>0, \\ |W_1|+|W_2|=|W|+1}} \mathrm{LBA}(\mathbf{id}^{\otimes 2}, W)_{W_1,W_2}.$$

So $x \circ (\mathrm{id}_Z \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-1}) \circ \beta_{ij}$ lies in

$$\bigoplus_{\substack{W_1, W_2 \in \mathrm{Irr}(\mathrm{Sch}) \mid \\ |W_1|+|W_2|=|Z_1|+1}} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p+1}, \Lambda^q)_{Z, Z_2, \ldots, Z_i, W_1, Z_{i+1}, \ldots, Z_{j-1}, W_2, Z_j, \ldots, Z_p}. \tag{10}$$

When $(W_1, W_2) \in \mathrm{Irr}(\mathrm{Sch})$ are such that $|W_i| > 0, |W_1|+|W_2| = |Z|+1$, $\{i||W_i| = 1\}$ is ≤ 1 if $|Z_1| > 1$ and $= 2$ if $|Z_1| = 1$. So in the summands of (10), $\mathrm{card}\{i||Z_i| = 1\}$ is increased by at most 1. □

It follows that the differential $d^{p,p+1}$ is compatible with the filtration $(C^p)^{\leq 0} \subset (C^p)^{\leq 1} \subset \ldots \subset (C^p)^{\leq p} = C^p$. To prove that it is acyclic, we will prove that the associated graded complex is acyclic. For this, we first determine this associated graded complex.

For $p' + p'' = p$, let

$$\tilde{C}^{p',p''} := \bigoplus_{\substack{Z_1'', \ldots, Z_{p''}'' \in \mathrm{Irr}(\mathrm{Sch}) \mid \\ |Z_i''| > 1}} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'} \otimes \mathbf{id}^{p''}, \Lambda^q)_{Z, \mathbf{id}, \ldots, \mathbf{id}, Z_1'', \ldots, Z_{p''}''}.$$

Let $C^{p',p''}$ be the antisymmetric part of this space with respect to the action of $\mathfrak{S}_{p'} \times \mathfrak{S}_{p''}$.

Lemma 6.2. $(C^p)^{\leq p'}/(C^p)^{\leq p'-1} = C^{p',p''}$, *where* $p'' = p - p'$.

Proof. We have

$$(\tilde{C}^p)^{\leq p'}/(\tilde{C}^p)^{\leq p'-1} = \bigoplus_{\substack{Z_1, \ldots, Z_p \in \mathrm{Irr}(\mathrm{Sch}), |Z_i| > 0, \\ \mathrm{card}\{i | Z_i = \mathbf{id}\} = p'}} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)_{Z, Z_1, \ldots, Z_p}.$$

As a $\mathfrak{S}_p$-module, the r.h.s. identifies with $\mathrm{Ind}_{\mathfrak{S}_{p'} \times \mathfrak{S}_{p''}}^{\mathfrak{S}_p}(\tilde{C}^{p',p''})$. $(C^p)^{\leq p'}/(C^p)^{\leq p'-1}$ is then the $\mathfrak{S}_p$-anti-invariant part of this space, which identifies with the $\mathfrak{S}_{p'} \times \mathfrak{S}_{p''}$-anti-invariant part of $\tilde{C}^{p',p''}$, i.e., $C^{p',p''}$. The isomorphism[5]

$$\Big(\bigoplus_{(Z_1, \ldots, Z_p) || Z_i| > 0, \mathrm{card}\{i | Z_i = \mathbf{id}\} = p'} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q) \Big)^{\mathfrak{S}_p^-}$$
$$\to \Big(\bigoplus_{(Z_1'', \ldots, Z_{p''}'') || Z_i''| > 1} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)_{Z, \mathbf{id}, \ldots, \mathbf{id}, Z_1'', \ldots, Z_{p''}''} \Big)^{(\mathfrak{S}_{p'} \times \mathfrak{S}_{p''})^-}$$

is given by projection on the relevant components, and the inverse isomorphism is given by the action of $(1/p!) \sum_{\sigma \in \mathfrak{S}_p} \epsilon(\sigma)\sigma$ (or $(p'!p''!/p!) \sum_{\sigma \in \mathfrak{S}_{p',p''}} \epsilon(\sigma)\sigma$, where $\mathfrak{S}_{p',p''}$ is the set of p', p''-shuffle permutations). □

[5]For M a module over $\prod_i \mathfrak{S}_{p_i}$, we denote by $M^{(\prod_i \mathfrak{S}_{p_i})^-}$ the component of M of type $\bigotimes_i \epsilon_i$, where ϵ_i is the signature character of $\mathfrak{S}_{p_i}$.

Define

$$\tilde{d}^{p',p'+1|p''} : \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q) \to \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'+1} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q)$$

by $x \mapsto x \circ (\mathrm{id}_Z \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-1}) \circ (\sum_{1 \le i < j \le p'+1} (-1)^{i+j+1} \beta_{ij}) + x \circ (\mu_Z \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p}) \circ (\sum_{1 \le i \le p'+1} (-1)^{i+1} \beta_i)$.

Lemma 6.3. *The map $\tilde{d}^{p',p'+1|p''}$ restricts to maps $\tilde{C}^{p',p''} \to \tilde{C}^{p'+1,p''}$ and $d^{p',p'+1|p''} : C^{p',p''} \to C^{p'+1,p''}$, and the map $(C^p)^{\le p'}/(C^p)^{\le p'-1} \to (C^{p+1})^{\le p'+1}/(C^{p+1})^{\le p'}$ induced by $d^{p,p+1}$ coincides with $d^{p',p'+1|p''}$, where $p'' = p - p'$.*

For each p'', $(C^{p',p''}, d^{p',p'+1|p''})_{p' \ge 0}$ is therefore a complex (this can be checked directly); it is embedded in the similar complex, where the restrictions $|Z_i''| \ne 0, 1$ are dropped, which is the universal version of the complex computing $H^{p'}(\mathfrak{a}, Z(\mathfrak{a})^*) \otimes \Lambda^{p''}(\mathfrak{a})^* \otimes \Lambda^q(\mathfrak{a})$, where $\mathfrak{a}$ is a Lie bialgebra.

Proof. If $x \in \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q)_{Z,\mathbf{id},\ldots,\mathbf{id},Z_1'',\ldots,Z_{p''}''}$, then one checks that both $x \circ (\mathrm{id}_Z \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-1}) \circ \beta_{ij}$ and $x \circ (\mu_Z \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p}) \circ \beta_i$ lie in $\mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'+1} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q)_{Z,\mathbf{id},\ldots,\mathbf{id},Z_1'',\ldots,Z_{p''}''}$, which implies that $\tilde{d}^{p',p'+1|p''}$ induces a map $\tilde{C}^{p',p''} \to \tilde{C}^{p'+1,p''}$. The map $\tilde{d}^{p',p'+1|p''}$ maps the $\mathfrak{S}_{p'} \times \mathfrak{S}_{p''}$-antisymmetric part of $\mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p'} \otimes \mathbf{id}^{\otimes p''}, \Lambda^q)$ to its analogue with p' increased by 1, so it restricts to a map $d^{p',p'+1|p''} : C^{p',p''} \to C^{p'+1,p''}$.

Let us now show that the map $C^{p',p-p'} \to C^{p'+1,p-p'}$ induced by $d^{p,p+1} : (C^p)^{\le p'} \to (C^{p+1})^{\le p'+1}$ is $d^{p',p'+1|p-p'}$.

Let $Z_1, \ldots, Z_p \in \mathrm{Irr(Sch)}$ be such that $Z_i = \mathbf{id}$ for $i \le p'$ and $|Z_i| > 1$ if $i \ge p'+1$. Let $y \in \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p}, \Lambda^q)_{Z,Z_1,\ldots,Z_p}$ be of the form $a \circ (\mathrm{id}_Z \otimes c_1 \otimes \ldots \otimes c_p)$, where $c_i \in \mathrm{LCA}(\mathbf{id}, Z_i)$ and $a \in \mathrm{LA}(Z \otimes Z_1 \otimes \ldots \otimes Z_p, \Lambda^q)$. Let $x := y \circ (\mathrm{id}_Z \otimes (\sum_{\sigma \in \mathfrak{S}_{p'} \times \mathfrak{S}_{p''}} \epsilon(\sigma)\sigma))$ and $\tilde{x} := y \circ (\mathrm{id}_Z \otimes (\sum_{\sigma \in \mathfrak{S}_p} \epsilon(\sigma)\sigma))$. Then $x \in C^{p',p''}$, $\tilde{x} \in (\tilde{C}^p)^{\le p'}$, and x corresponds to the class of $\tilde{x}$ under $C^{p',p''} \simeq (\tilde{C}^p)^{\le p'}/(\tilde{C}^p)^{\le p'-1}$.

Let us compute $\tilde{d}^{p,p+1}(\tilde{x})$. We have

(11)
$$\tilde{x} \circ (\mathrm{id}_Z \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-2}) \circ (\sum_{i<j} (-1)^{i+j} \beta_{ij})$$
$$= \sum_{1 \le i < j \le p} (-1)^{i+j} \sum_{\sigma \in \mathfrak{S}_p} \epsilon(\sigma) a \circ (\mathrm{id}_Z \otimes \beta_\sigma) \circ \big(\mathrm{id}_Z \otimes (c_{\sigma(1)} \circ \mu) \otimes c_{\sigma(2)} \otimes \ldots \otimes c_{\sigma(p)}\big) \circ \beta_{ij},$$

where $\beta_\sigma : Z_{\sigma(1)} \otimes \ldots \otimes Z_{\sigma(p)} \to Z_1 \otimes \ldots \otimes Z_p$ is the braiding map.

We now use the fact that if $c \in \mathrm{LCA}(\mathbf{id}, Z)$, then

$$c \circ \mu = \mu_Z \circ (\mathrm{id}_{\mathbf{id}} \otimes c) + \tilde{\mu}_Z \circ (c \otimes \mathrm{id}_{\mathbf{id}}) + \kappa(c), \tag{12}$$

where $\kappa(c) \in \mathrm{LBA}(\mathbf{id}^{\otimes 2}, Z)$ is such that:

- $\kappa(c) \in \bigoplus_{|W_1|,|W_2|>1} \mathrm{LBA}(\mathbf{id}^{\otimes 2}, Z)_{W_1,W_2}$ if $|Z| > 1$,
- $\kappa(c) = -c \circ \mu$ if $Z = \mathbf{id}$.

(12) is proved as follows: it is obvious when $Z = \mathbf{id}$; we first prove it when $Z = T_p$ ($p > 0$) and $c = (\delta \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p-2}) \circ ... \circ \delta$ (iterating the use of the cocycle identity); as this element generates the $\mathfrak{S}_p$-module $\mathrm{LCA}(\mathbf{id}, T_p)$, this implies the identity when $Z = T_p$. The case of $Z \in \mathrm{Irr}(\mathrm{Sch})$, $|Z| = p$ is derived from there by taking isotypic components under the action of $\mathfrak{S}_p$.

When $|Z_{\sigma(1)}| > 1$, the contribution of $\kappa(c_{\sigma(1)})$ to (11) belongs to $(\tilde{C}^{p+1})^{\leq p'}$. The class of (11) in $(\tilde{C}^{p+1})^{\leq p'+1}/(\tilde{C}^{p+1})^{\leq p'}$ is then the same as that of

(13)
$$\sum_{1\leq i<j\leq p+1} (-1)^{i+j} \sum_{\sigma\in\mathfrak{S}_p} a\circ(\mathrm{id}_Z \otimes\beta_\sigma) \circ (\mathrm{id}_Z \otimes(\mu_{Z_{\sigma(1)}} \circ (\mathrm{id}_{\mathbf{id}} \otimes c_{\sigma(1)})) \otimes c_{\sigma(2)} \otimes ... \otimes c_{\sigma(p)}) \circ \beta_{ij}$$
$$+ \sum_{1\leq i<j\leq p+1} (-1)^{i+j} \sum_{\sigma\in\mathfrak{S}_p} a \circ (\mathrm{id}_Z \otimes\beta_\sigma)\circ(\mathrm{id}_Z \otimes(\tilde{\mu}_{Z_{\sigma(1)}} \circ (c_{\sigma(1)} \otimes \mathrm{id}_{\mathbf{id}})) \otimes c_{\sigma(2)} \otimes ... \otimes c_{\sigma(p)}) \circ \beta_{ij}$$
$$+ \sum_{1\leq i<j\leq p+1} (-1)^{i+j+1} \sum_{\sigma\in\mathfrak{S}_p|\sigma(1)\in[1,p']} \epsilon(\sigma) a \circ (\mathrm{id}_Z \otimes\beta_\sigma)\circ (\mathrm{id}_Z \otimes(c_{\sigma(1)} \circ \mu) \otimes c_{\sigma(2)} \otimes ... \otimes c_{\sigma(p)}) \circ \beta_{ij}.$$

The first line may be rewritten as follows. Let $\alpha_j \in \mathfrak{S}_p$ be the cycle $\alpha_j(1) = 2$,..., $\alpha_j(j-2) = j-1$, $\alpha(j-1) = 1$, $\alpha_j(j) = j$,..., $\alpha_j(p) = p$. In terms of $\tau := \sigma \circ \alpha_j$, this line can be expressed as

$$\sum_{j\in[1,p+1]} \sum_{i<j} \sum_{\tau\in\mathfrak{S}_p} (-1)^i \epsilon(\tau) a\circ(\mathrm{id}_Z \otimes\beta_\tau)\circ(\mathrm{id}_Z \otimes c_{\tau(1)}\otimes...\otimes(\mu_{Z_{\tau(j-1)}}\circ(\mathrm{id}_{\mathbf{id}} \otimes c_{\tau(j-1)})) \otimes ... \otimes c_{\tau(p)}) \circ \gamma_{ij},$$

where $\gamma_{ij} \in \mathrm{Aut}(Z\otimes\mathbf{id}^{\otimes p})$ is the categorical version of $z\otimes x_1 \otimes ... \otimes x_{p+1} \mapsto z\otimes x_1 \otimes ... \check{x}_i ... \otimes x_{j-1} \otimes x_i \otimes x_{j+1} \otimes ... \otimes x_{p+1}$. In the same way, one shows that the second line has the same expression, with the condition $i < j$ replaced by $i > j$ and γ_{ij} the categorical version of $z\otimes x_1\otimes...\otimes x_{p+1} \mapsto z\otimes x_1\otimes...\otimes x_{j-1}\otimes x_i\otimes x_{j+1}\otimes...\check{x}_i...\otimes x_{p+1}$.

Adding up these lines, and using the identity

$$\mu_W \circ (\mathrm{id}_{\mathbf{id}} \otimes a) = \sum_{\alpha=1}^{k} a \circ (\mathrm{id}_{W_1} \otimes ... \otimes \mu_{W_\alpha} \otimes ... \otimes \mathrm{id}_{W_k}) \circ \beta_\alpha,$$

in $\mathrm{LA}(\mathbf{id} \otimes W_1 \otimes ... \otimes W_k, W)$, where $a \in \mathrm{LA}(W_1 \otimes ... \otimes W_k, W)$ and β_α is the braiding $\mathbf{id} \otimes W_1 \otimes ... \otimes W_k \to W_1 \otimes ... \otimes W_{\alpha-1} \otimes \mathbf{id} \otimes W_\alpha \otimes ... \otimes W_k$, we express the contribution of (11) as (last line of (13)) $+\mu_{\Lambda^q} \circ (\mathrm{id}_{\mathbf{id}} \otimes\tilde{x}) \circ (\sum_{i=1}^{p+1}(-1)^i\beta_i) + \tilde{x} \circ (\mu_Z \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p}) \circ (\sum_{i=1}^{p+1}(-1)^{i+1}\beta_i)$.

The class of $\tilde{d}^{p,p+1}(\tilde{x})$ in $(\tilde{C}^{p+1})^{\leq p'+1}/(\tilde{C}^{p+1})^{\leq p}$ is therefore the same as that of (last line of (13)) $+\tilde{x} \circ (\mu_Z \otimes \mathrm{id}_{\mathbf{id}}^{\otimes p}) \circ (\sum_{i=1}^{p+1}(-1)^{p+1}\beta_i)$. To evaluate its image in $C^{p'+1,p''}$, we apply the projection of $\bigoplus_{Z_1,...,Z_{p+1}} \mathrm{LBA}(Z \otimes \mathbf{id}^{\otimes p+1}, \Lambda^q)_{Z,Z_1,...,Z_{p+1}}$ on the sum of the components with $Z_1 = ... = Z_{p'+1} = \mathbf{id}$, $|Z_{p'+1}|, ..., |Z_{p+1}| > 1$ along the other components.

We have $\tilde x\circ(\mu_Z\otimes\mathrm{id}_{\mathbf{id}}^{\otimes p})\circ\beta_i=\sum_{\sigma\in\mathfrak{S}_p}\epsilon(\sigma)a\circ(\mathrm{id}_Z\otimes\beta_\sigma)\circ(\mu_Z\otimes c_{\sigma(1)}\otimes...\otimes c_{\sigma(p)})\circ\beta_i$, and the summand corresponding to σ belongs to

$$\mathrm{LBA}(Z\otimes\mathbf{id}^{\otimes p},\Lambda^q)_{Z,Z_{\sigma(1)},...,Z_{\sigma(i-1)},\mathbf{id},Z_{\sigma(i)},...,Z_{\sigma(p)}};$$

the projection is the identity on the components such that $i\in[1,p'+1]$ and $\sigma\in\mathfrak{S}_{p'}\times\mathfrak{S}_{p''}$ and zero on the other ones. So the projection of $\tilde x\circ(\mu_Z\otimes\mathrm{id}_{\mathbf{id}}^{\otimes p})\circ(\sum_{i=1}^{p+1}(-1)^{i+1}\beta_i)$ is $x\circ(\mu_Z\otimes\mathrm{id}_{\mathbf{id}}^{\otimes p})\circ(\sum_{i=1}^{p'+1}(-1)^{i+1}\beta_i)$.

Let us compute the projection of the last line of (13). The term in this line corresponding to i,j,σ belongs to

$$\mathrm{LBA}(Z\otimes\mathbf{id}^{\otimes p},\Lambda^q)_{Z,Z_{\sigma(2)},...,Z_{\sigma(i)},\mathbf{id},Z_{\sigma(i+1)},...,Z_{\sigma(j-1)},\mathbf{id},Z_{\sigma(j)},...,Z_{\sigma(p)}}.$$

The projection is then the identity on the terms such that $i,j\in[1,p'+1]$ and $\sigma\in\mathfrak{S}_{p'}\times\mathfrak{S}_{p''}$ and zero on the other ones. The projection of the last line of (13) is therefore $x\circ(\mathrm{id}_Z\otimes\mu\otimes\mathrm{id}_{\mathbf{id}}^{\otimes p-1})\circ(\sum_{1\le i<j\le p'+1}(-1)^{i+j+1}\beta_{ij})$.

The projection of $\tilde d^{p,p+1}(\tilde x)$ is then the sum of these projections, i.e., $\tilde d^{p',p'+1|p''}(x)$. □

The associated graded complex of the complex $(C^p,d^{p,p+1})_{p\ge0}$ is therefore $\bigoplus_{p''\ge0}(C^{p',p''},d^{p',p'+1|p''})_{p'\ge0}$.

We now prove that for each $p''\ge0$, the complex $(C^{p',p''},d^{p',p'+1|p''})_{p'\ge0}$ is acyclic.

For $\underline{Z}''=(Z''_1,...,Z''_{p''})\in\mathrm{Irr}(\mathrm{Sch})$, let

$$\tilde d^{p',p'+1}_{\underline{Z}''}:\mathrm{LA}(Z\otimes\mathbf{id}^{\otimes p'}\otimes(\bigotimes_i Z''_i),\Lambda^q)\to\mathrm{LA}(Z\otimes\mathbf{id}^{\otimes p'+1}\otimes(\bigotimes_i Z''_i),\Lambda^q)$$

be defined by the same formula as $\tilde d^{p',p'+1|p''}$, replacing $\mathrm{id}_{\mathbf{id}}^{\otimes p-1}$, $\mathrm{id}_{\mathbf{id}}^{\otimes p}$ by $\mathrm{id}_{\mathbf{id}}^{\otimes p'-1}\otimes\mathrm{id}_{\bigotimes_i Z''_i}$, $\mathrm{id}_{\mathbf{id}}^{\otimes p'}\otimes\mathrm{id}_{\bigotimes_i Z''_i}$. Let $C^{p'}_{\underline{Z}''}$ be the antisymmetric part of $\mathrm{LA}(Z\otimes\mathbf{id}^{\otimes p'}\otimes(\bigotimes_i Z''_i),\Lambda^q)$ (with respect to the action of $\mathfrak{S}_{p'}$). Then $\tilde d^{p',p'+1}_{\underline{Z}''}$ restricts to a differential $d^{p',p'+1}_{\underline{Z}''}:C^{p'}_{\underline{Z}''}\to C^{p'+1}_{\underline{Z}''}$; moreover, we have an isomorphism between $(C^{p',p''},d^{p',p'+1|p''})_{p'\ge0}$ and the antisymmetric part (with respect to the action of $\mathfrak{S}_{p''}$) of

$$\bigoplus_{\substack{Z''_1,...,Z''_{p''}\in\mathrm{Irr}(\mathrm{Sch}),\\|Z''_i|\neq0,1}}\bigotimes_{i=1}^{p''}\mathrm{LA}(\mathbf{id},Z''_i)\otimes(C^{p'}_{\underline{Z}''},d^{p',p'+1}_{\underline{Z}''})_{p'\ge0}.$$

Since the differential of this complex is $\mathfrak{S}_{p''}$-equivariant, it suffices to prove that each complex $(C^{p'}_{\underline{Z}''},d^{p',p'+1}_{\underline{Z}''})_{p'\ge0}$ is acyclic.

Let $z:=|Z|$, $N:=\sum_i|Z''_i|$; let

$$\tilde d^{p',p'+1}_{z,N,q}:\mathrm{LA}(\mathbf{id}^{\otimes z}\otimes\mathbf{id}^{\otimes p'}\otimes\mathbf{id}^{\otimes N},\mathbf{id}^{\otimes q})\to\mathrm{LA}(\mathbf{id}^{\otimes z}\otimes\mathbf{id}^{\otimes p'+1}\otimes\mathbf{id}^{\otimes N},\mathbf{id}^{\otimes q})$$

be defined by the same formula as $\tilde d^{p',p'+1}_{\underline{Z}''}$, replacing $\bigotimes_i Z''_i$ by $\mathbf{id}^{\otimes N}$ and μ_Z by $\mu_{\mathbf{id}^{\otimes z}}$. Let $C^{p'}_{z,N,q}$ be the antisymmetric part of $\mathrm{LA}(\mathbf{id}^{\otimes z}\otimes\mathbf{id}^{\otimes p'}\otimes\mathbf{id}^{\otimes N},\mathbf{id}^{\otimes q})$ (with respect to the action of $\mathfrak{S}_{p'}$). Then $\tilde d^{p',p'+1}_{z,N,q}$ restricts to a differential $d^{p',p'+1}_{z,N,q}:C^{p'}_{z,N,q}\to C^{p'+1}_{z,N,q}$. The complex $(C^{p'}_{z,N,q},d^{p',p'+1}_{z,N,q})_{p'\ge0}$ is equipped with a natural action of $\mathfrak{S}_z \times \prod_i \mathfrak{S}_{|Z''_i|} \times \mathfrak{S}_q$, and $(C^{p'}_{\underline{Z}''}, d^{p',p'+1}_{\underline{Z}''})$ is an isotypic component of this action. It suffices therefore to prove that $(C^{p'}_{z,N,q}, d^{p',p'+1}_{z,N,q})_{p'\geq 0}$ is acyclic.

In what follows, we denote by $\mathcal{L}(u_1,\ldots,u_s)$ (resp., $\mathcal{A}(u_1,...,u_s)$) the free Lie (resp., associative) algebra generated by $u_1,...,u_p$. These spaces are graded by $\bigoplus_{i\in[1,p]}\mathbb{N}\delta_i$ and for $S\subset[1,p]$, we denote by $\mathcal{L}(u_1,\ldots,u_s)_S$, $\mathcal{A}(u_1,\ldots,u_s)_S$ the subspaces of degree $\bigoplus_{i\in S}\delta_i$. In the case of two sets of generating variables $(u_1,\ldots,u_s)$ and $(v_1,...,v_t)$, the spaces are graded by[6] $\bigoplus_{i\in[1,s]\coprod[1,t]}\mathbb{N}\delta_i$ and we use the same notation for homogeneous subspaces.

Lemma 6.4. *We have an isomorphism of complexes*

$$C^\bullet_{z,N,q} \simeq \bigoplus_{\substack{\sqcup_{\alpha=1}^q I_\alpha=[1,z],\\ \sqcup_{\alpha=1}^q J_\alpha=z+[1,N]}} \bigotimes_{\alpha=1}^{q} C^\bullet_{|I_\alpha|,|J_\alpha|,1}. \tag{14}$$

Proof. Identify $C^{p'}_{z,N,q}$ with $\big(\mathcal{L}(a_1,...,a_{z+N},x_1,...,x_{p'})^{\otimes q}\big)^{\mathfrak{S}^-_{p'}}_{[1,z+N]\coprod[1,p']}$, which is the part of the qth tensor power of $\mathcal{L}(a_1,...,x_{p'})$, multilinear in $a_1,...,a_{z+N},x_1,...,x_{p'}$, and antisymmetric in $x_1,...,x_{p'}$ ($a_1,...,a_z$ correspond to the z factors of $\mathbf{id}^{\otimes z}$, $a_{z+1},...,a_{z+N}$ to the N factors of $\mathbf{id}^{\otimes N}$, and $x_1,...,x_{p'}$ to the p' factors of $\mathbf{id}^{\otimes p'}$). The differential $d^{p',p'+1}_{z,N,q}$ can then be expressed as

$$\begin{aligned}(15)\qquad & F(a_1,...,a_{z+N},x_1,...,x_{p'}) \mapsto \\ & \sum_{1\leq i<j\leq p'+1}(-1)^{i+j+1}F(a_1,...,a_{z+N},[x_i,x_j],x_1,...\check{x}_i...\check{x}_j...,x_{p'+1}) \\ & + \sum_{1\leq i\leq p'+1}\sum_{z'=1}^{z}(-1)^{i+1}F(a_1,...,[x_i,a_{z'}],...,a_{z+N},x_1,...\check{x}_i...,x_{p'+1}).\end{aligned}$$

On the other hand, we have an isomorphism

$$\mathrm{LA}(\mathbf{id}^{\otimes N},\mathbf{id}^{\otimes q}) \simeq \bigoplus_{I_1\sqcup...\sqcup I_q=[1,N]} \bigotimes_{\alpha=1}^{q} \mathrm{LA}(\mathbf{id}^{\otimes|I_\alpha|},\mathbf{id}),$$

with its inverse given by the sum of the maps $a_1\otimes...\otimes a_q \mapsto (a_1\otimes...\otimes a_q)\circ\beta_{I_1,...,I_q}$, where $\beta_{I_1,\ldots,I_q}:\mathbf{id}^{\otimes N}\to\bigotimes_\alpha \mathbf{id}^{\otimes|I_\alpha|}$ is the braiding induced by the maps $[1,N]\to\coprod_\alpha[1,|I_\alpha|]$, taking I_α to $[1,|I_\alpha|]\times\{\alpha\}$ by preserving the order. Analyzing the action of $\mathfrak{S}_N$ on the set of q-compositions of $[1,N]$, we derive an isomorphism $\mathrm{LA}(\Lambda^N,\mathbf{id}^{\otimes q})\simeq\bigoplus_{N_1+...+N_q=N}\bigotimes_{\alpha=1}^q \mathrm{LA}(\Lambda^{N_\alpha},\mathbf{id})$, with inverse given by the direct sum of the maps $a_1\otimes...\otimes a_q\mapsto(a_1\otimes...\otimes a_q)\circ\beta_{N_1,...,N_q}$, where

[6] For $A_1,\ldots,A_s\subset\mathbb{N}$, we set $\coprod_i A_i := \bigcup_i(A_i\times\{i\})\subset\mathbb{N}\times\{1,\ldots,s\}$.

$\beta_{N_1,\dots,N_q}:\Lambda^N\to\Lambda^{N_1}\otimes\dots\otimes\Lambda^{N_q}$ is the composite Schur morphism $\Lambda^N\hookrightarrow\mathbf{id}^{\otimes N}\simeq\bigotimes_\alpha\mathbf{id}^{\otimes N_\alpha}\twoheadrightarrow\bigotimes_\alpha\Lambda^{N_\alpha}$. One proves similarly that we have an isomorphism

$$\mathrm{LA}(\mathbf{id}^{\otimes z}\otimes\Lambda^{p'}\otimes\mathbf{id}^{\otimes N},\mathbf{id}^{\otimes q})\simeq \bigoplus_{\bigsqcup_\alpha I_\alpha=[1,z],\bigsqcup_\alpha J_\alpha=z+[1,N],\sum_\alpha p_\alpha=p'}\bigotimes_{\alpha=1}^{q}\mathrm{LA}(\mathbf{id}^{\otimes|I_\alpha|}\otimes\Lambda^{p_\alpha}\otimes\mathbf{id}^{\otimes|J_\alpha|},\mathbf{id}),$$

with its inverse induced by the direct sum of the maps $\bigotimes_\alpha a_\alpha\mapsto(\bigotimes_\alpha a_\alpha)\circ\beta_{(I_\alpha),(J_\alpha),(p_\alpha)}$, where $\beta_{(I_\alpha),(J_\alpha),(p_\alpha)}:\mathbf{id}^{\otimes z}\otimes\Lambda^{p'}\otimes\mathbf{id}^{\otimes N}\to\bigotimes_\alpha(\mathbf{id}^{\otimes|I_\alpha|}\otimes\Lambda^{p_\alpha}\otimes\mathbf{id}^{\otimes|J_\alpha|})$ is constructed from the above Schur morphisms. It follows that we have the isomorphism (14) between graded vector spaces. Let us show that it is compatible with differentials.

For $\bigsqcup_\alpha I_\alpha=[1,z]$, $\bigsqcup_\alpha J_\alpha=z+[1,N]$, $\sum_\alpha p_\alpha=p'$, the map $\bigotimes_{\alpha=1}^q C^{p_\alpha}_{|I_\alpha|,|J_\alpha|,1}\to C^{p'}_{z,N,q}$ identifies with the map

$$\bigotimes_\alpha\mathcal{L}(a_1,\dots,x_{p'})^{\mathfrak{S}^-_{p_\alpha}}_{(I_\alpha\cup J_\alpha)\coprod(p_1+\dots+p_{\alpha-1}+[1,p_\alpha])}\to\big(\mathcal{L}(a_1,\dots,x_{p'})^{\otimes q}\big)^{\mathfrak{S}^-_{p'}}_{[1,z+N]\coprod[1,p']},$$

$\bigotimes_\alpha F_\alpha\mapsto\sum_{\sigma\in\mathfrak{S}_{p_1,\dots,p_\alpha}}\epsilon(\sigma)\sigma*(\bigotimes_\alpha F_\alpha)$, where $\mathfrak{S}_{p_1,\dots,p_\alpha}$ is the set of shuffle permutations of $\mathfrak{S}_{p'}$ (preserving the order of the elements of $[1,p_1]$, $p_1+[1,p_2]$, etc.) and $*$ is the permutation action on $x_1,\dots,x_{p'}$. The projection $C^{p'}_{z,N,q}\to\bigotimes_\alpha C^{p_\alpha}_{|I_\alpha|,|J_\alpha|,1}$ to the component indexed by $((I_\alpha),(J_\alpha))_\alpha$ along the other components can then be described as follows: the composite map

$$\begin{aligned}\big(\mathcal{L}(a_1,\dots,x_{p'})^{\otimes q}\big)_{[1,z+N]\coprod[1,p']}&\simeq\bigoplus_{\substack{\sqcup I_\alpha=[1,z],\\ \sqcup J_\alpha=z+[1,N],\\ \sqcup P_\alpha=[1,p']}}\bigotimes_\alpha\mathcal{L}(a_1,\dots,x_{p'})_{(I_\alpha\cup J_\alpha)\coprod P_\alpha}\\ &\to\bigotimes_\alpha\mathcal{L}(a_1,\dots,x_{p'})_{(I_\alpha\cup J_\alpha)\coprod(p_1+\dots+p_{\alpha-1}+[1,p_\alpha])}\end{aligned}$$

(where the second map is the projection along all other components) restricts to

$$\big(\mathcal{L}(a_1,\dots,x_{p'})^{\otimes q}\big)^{\mathfrak{S}^-_{p'}}_{[1,z+N]\coprod[1,p']}\to\bigotimes_\alpha\mathcal{L}(a_1,\dots,x_{p'})^{\mathfrak{S}^-_{p_\alpha}}_{(I_\alpha\cup J_\alpha)\coprod(p_1+\dots+p_{\alpha-1}+[1,p_\alpha])},$$

which identifies with the projection $C^{p'}_{z,N,q}\to\bigotimes_\alpha C^{p_\alpha}_{|I_\alpha|,|J_\alpha|,1}$.

Extending formula (15) defining $d^{p',p'+1}_{z,N,q}$, we define a map

$$\begin{aligned}\tilde d^{p',p'+1}_{z,N,q}:\big(\mathcal{L}(a_1,\dots,x_{p'})^{\otimes q}\big)&_{[1,z+N]\coprod[1,p']}\\ &\to\big(\mathcal{L}(a_1,\dots,a_{z+N},x_1,\dots,x_{p'+1})^{\otimes q}\big)_{[1,z+N]\coprod[1,p'+1]}.\end{aligned}$$

It follows that the map $d^{p',p'+1}_{z,N,q} : C^{p'}_{z,N,q} \to C^{p'+1}_{z,N,q}$ may be identified with the composite map[7]

$$\bigoplus_{\substack{\bigsqcup_\alpha I_\alpha=[1,z]\\ \bigsqcup_\alpha J_\alpha=z+[1,N]\\ \sum_\alpha p_\alpha=p'}} \Big(\bigotimes_{\alpha=1}^{q}\mathcal{L}(a_1,...,x_{p'})^{\mathfrak{S}^-_{p_\alpha}}_{I_\alpha,J_\alpha,p_1+...+p_{\alpha-1}+[1,p_\alpha]}\Big)$$

$$\overset{\oplus(\sum_{\sigma\in\mathfrak{S}_{p_1,...,p_n}}\epsilon(\sigma)\sigma)}{\to} \big(\mathcal{L}(a_1,...,x_{p'})^{\otimes q}\big)_{[1,z+N]\coprod[1,p']}$$

$$\overset{\tilde d^{p',p'+1}_{z,N,q}}{\to} \big(\mathcal{L}(a_1,...,x_{p'+1})^{\otimes q}\big)_{[1,z+N]\coprod[1,p'+1]}$$

$$\simeq \bigoplus_{\substack{\bigsqcup_\alpha \tilde I_\alpha=[1,z]\\ \bigsqcup_\alpha \tilde J_\alpha=z+[1,N]\\ \bigsqcup_\alpha \tilde P_\alpha=[1,p'+1]}} \Big(\bigotimes_{\alpha=1}^{q}\mathcal{L}(a_1,...,x_{p'+1})_{\tilde I_\alpha,\tilde J_\alpha,\tilde P_\alpha}\Big)$$

$$\twoheadrightarrow \bigoplus_{\substack{\bigsqcup_\alpha \tilde I_\alpha=[1,z]\\ \bigsqcup_\alpha \tilde J_\alpha=z+[1,N]\\ \sum_\alpha \tilde p_\alpha=p'+1}} \Big(\bigotimes_{\alpha=1}^{q}\mathcal{L}(a_1,...,x_{p'+1})_{\tilde I_\alpha,\tilde J_\alpha,\tilde p_1+...+\tilde p_{\alpha-1}+[1,\tilde p_\alpha]}\Big)$$

We have a decomposition $\tilde d^{p',p'+1}_{z,N,q} = \sum_{1\le i<j\le p'} \tilde d^{ij} + \sum_{i=1}^{p'+1}\sum_{z'=1}^{z} \tilde d^{iz'}$. Then:

• $\tilde d^{ij}$ takes the summand indexed by $((I_\alpha)_\alpha,(J_\alpha)_\alpha,(P_\alpha)_\alpha)$ to the summand indexed by $((I_\alpha)_\alpha,(J_\alpha)_\alpha,(\tilde P^{ij}_\alpha)_\alpha)$, where $(\tilde P^{ij}_\alpha)_\alpha$ is the partition of $[1,p'+1]$ given by $\tilde P^{ij}_\alpha = \big((P_\alpha\cap[2,i])-1\big)\cup(P_\alpha\cap[i+1,j-1])\cup\big((P_\alpha\cap[j,p])+1\big)$ if $1\notin P_\alpha$, and the union of the same set with $\{i,j\}$ if $1\in P_\alpha$ (all these unions are disjoint);

• $\tilde d^{iz'}$ takes the summand indexed by $((I_\alpha)_\alpha,(J_\alpha)_\alpha,(P_\alpha)_\alpha)$ to the summand indexed by $((I_\alpha)_\alpha,(J_\alpha)_\alpha,(\tilde P^{i,z'}_\alpha)_\alpha)$, where $(\tilde P^{i,z'}_\alpha)_\alpha$ is the partition of $[1,p'+1]$ given by $\tilde P^{i,z'}_\alpha = (P_\alpha\cap[1,i-1])\cup\big((P_\alpha\cap[i,p'])+1\big)$ if $z'\notin I_\alpha$, and the union of the same set with $\{i\}$ if $z'\in I_\alpha$ (all these unions are disjoint).

As the partitions $(I_\alpha)_\alpha$ and $(J_\alpha)_\alpha$ of $[1,z]$ and $z+[1,N]$ are not modified, (14) is a decomposition of complexes. If $(\tilde P_\alpha)_\alpha$ is one of the partitions $(\tilde P^{ij}_\alpha)_\alpha$ or $(\tilde P^{iz'}_\alpha)_\alpha$, then the sequence $(|\tilde P_\alpha|)_{\alpha=1,...,q}$ has the form $(p_\alpha+\delta_{\alpha\beta})_\alpha$, where $\beta\in[1,q]$ and $p_\alpha=|P_\alpha|$.

Fix $\beta\in[1,q]$ and set $p^\beta_\alpha := p_\alpha+\delta_{\alpha\beta}$. The partition $(\tilde P_\alpha)_\alpha$ coincides with $(p^\beta_1+...+p^\beta_{\alpha-1}+[1,p^\beta_\alpha])_\alpha$ if:

(a) $P_\alpha = 1+p_1+...+p_{\alpha-1}+[1,p_\alpha]$ if $\alpha<\beta$, $P_\beta = \big(1+p_1+...+p_{\beta-1}+[1,p_\beta-1]\big)\sqcup\{1\}$, $P_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ if $\alpha>\beta$ and $p_1+...+p_{\beta-1}+1\le i<j\le p_1+...+p_\beta+1$; in that case, $(\tilde P^{ij}_\alpha)_\alpha$ is given by $\tilde P^{ij}_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for $\alpha<\beta$, $\tilde P^{ij}_\beta = p_1+...+p_{\beta-1}+[1,p_\beta+1]$, and $\tilde P^{ij}_\alpha = 1+p_1+...+p_{\alpha-1}+[1,p_\alpha]$ if $\alpha>\beta$. In particular, $i<j$ belong to $\tilde P^{ij}_\beta$;

(b) $P_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for any α, $p_1+...+p_{\beta-1}+1\le i\le p_1+...+p_\beta+1$ and $z'\in I_\beta$; in that case, $(\tilde P^{iz'}_\alpha)_\alpha$ is given by $\tilde P^{iz'}_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for $\alpha<\beta$, $\tilde P^{iz'}_\beta = p_1+...+p_{\beta-1}+[1,p_\beta+1]$ and $\tilde P^{iz'}_\beta = 1+p_1+...+p_{\beta-1}+[1,p_\beta]$ for $\alpha>\beta$. In particular, $i\in\tilde P^{iz}_\beta$ and $z'\in I_\beta$.

[7]We lighten the notation by writing $X_{I_\alpha,J_\alpha,P_\alpha}$ instead to $X_{(I_\alpha\cup J_\alpha)\coprod P_\alpha}$.

Now let $\bigotimes_\alpha F_\alpha(a_1,...,x_{p'})$ belong to $\bigotimes_\alpha \mathcal{L}(a_1,...,x_{p'})^{\mathfrak{S}^-_{p_\alpha}}_{I_\alpha,J_\alpha,p_1+...+p_{\alpha-1}+[1,p_\alpha]}$. The image of this element in $\mathcal{L}(a_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,z+N]\coprod[1,p']}$ is $(\sum_{\sigma\in\mathfrak{S}_{p_1,\dots,p_q}}\epsilon(\sigma)\sigma) * (\bigotimes_\alpha F_\alpha)$. Let us apply $d^{p',p'+1}$ to this element, and let us project the result to $\bigoplus_{\beta=1}^q \bigotimes_\alpha \mathcal{L}(a_1,...,x_{p'+1})^{\mathfrak{S}^-_{p^\beta_\alpha}}_{I_\alpha,J_\alpha,p^\beta_1+...+p^\beta_{\alpha-1}+[1,p^\beta_\alpha]}$.

According to what we have seen, the nontrivial contributions to the summand indexed by β are:

• for $i<j$ in $p_1+...+p_{\beta-1}+[1,p_\beta+1]$, the projection of $\tilde{d}^{ij}(\epsilon(\sigma)\sigma*(\bigotimes_\alpha F_\alpha))$, where σ is the shuffle permutation taking the $p_1+...+p_{\alpha-1}+[1,p_\alpha]$ to P_α described in (a) above;

• for $i\in p_1+...+p_{\beta-1}+[1,p_\beta+1]$ and $z'\in I_\beta$, the projection of $\tilde{d}^{iz'}(\bigotimes_\alpha F_\alpha)$, where $\tilde{d}^{iz'}$ is the summand of $\tilde{d}^{p',p'+1}_{z,N,q}$ corresponding to (i,z').

Let

$$d_\beta^{p_\beta,p_\beta+1}: \mathcal{L}(a_1,...,x_{p'})^{\mathfrak{S}^-_{p_\beta}}_{I_\beta,J_\beta,p_1+...+p_{\beta-1}+[1,p_\beta]} \to \mathcal{L}(a_1,...,x_{p'+1})^{\mathfrak{S}^-_{p_\beta+1}}_{I_\beta,J_\beta,p_1+...+p_{\beta-1}+[1,p_\beta+1]}$$

be the differential of the complex $C^\bullet_{|I_\beta|,|J_\beta|,1}$ and let d^{ij}_β, $d^{iz'}_\beta$ be its components. We have $\tilde{d}^{ij}(\sigma*(\bigotimes_\alpha F_\alpha)) = F_1\otimes...\otimes d^{ij}_\beta(F_\beta)\otimes...\otimes F_q$ (to prove this equality, note that the x_1 present in the βth factor of $\sigma*(\bigotimes_\alpha F_\alpha)$ gets replaced by $[x_i,x_j]$ on both sides; the signs coincide since the "usual" indices of the variables x_i,x_j are shifts of i,j by the same quantity, and this does not alter $(-1)^{i+j+1}$), while $\epsilon(\sigma)=(-1)^{p_1+...+p_{\beta-1}}$; on the other hand, $\tilde{d}^{iz'}(\bigotimes_\alpha F_\alpha) = (-1)^{p_1+...+p_{\beta-1}}F_1\otimes...\otimes d^{iz'}_\beta(F_\beta)\otimes...\otimes F_q$ (here the sign is due to the fact that the index of x_i is, in the usual ordering, $i-(p_1+...+p_{\beta-1})$). It follows that the contribution to the summand indexed by β is $(-1)^{p_1+...+p_{\beta-1}}F_1\otimes...\otimes d^{p_\beta,p_\beta+1}_\beta(F_\beta)\otimes...\otimes F_q$. So the projection of $d^{p',p'+1}((\sum_{\sigma\in\mathfrak{S}_{p_1,\dots,p_q}}\epsilon(\sigma)\sigma)*(\bigotimes_\alpha F_\alpha))$ is

$$(\sum_\beta (-1)^{p_1+...+p_{\beta-1}}\,\mathrm{id}\otimes...\otimes d^{p_\beta,p_\beta+1}\otimes...\otimes \mathrm{id})(\bigotimes_\alpha F_\alpha),$$

as was to be proved. □

As $z\neq 0$, for each partition $(I_1,...,I_q)$ of $[1,z]$, there exists i such that $|I_i|\neq 0$. So renaming $|I_i|,|J_i|$ by z,N, it suffices to prove that if $z\neq 0$, then $C^\bullet_{z,N,1}$ is acyclic.

Recall that $C^{p'}_{z,N,1}\simeq \mathcal{L}(a_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,z+N]\coprod[1,p']}$ and $d^{p',p'+1}_{z,N,1}$ is given by (15). On the other hand, the map $a\mapsto \mathrm{ad}(a)(a_1)$ gives rise to an isomorphism

$$\mathcal{A}^{p'}_{z,N,1} := \mathcal{A}(a_2,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[2,z+N]\coprod[1,p']} \simeq C^{p'}_{z,N,1},$$

where $\mathcal{A}(u_1,...,u_n)$ is the free associative algebra generated by $u_1,\dots,u_n$ and $\mathrm{ad}: \mathcal{A}(u_1,...,u_s)\to \mathrm{End}(\mathcal{L}(u_1,...,u_s))$ is the algebra morphism derived from the adjoint action of $\mathcal{L}(u_1,...,u_s)$ on itself. The differential $d^{p',p'+1}_{z,N,1}: \mathcal{A}^{p'}_{z,N,1}\to \mathcal{A}^{p'+1}_{z,N,1}$ is given by

$$Q(a_2,...,a_{z+N},x_1,...,x_{p'}) \mapsto \sum_{1\le i<j\le p'+1}(-1)^{i+j+1}Q(a_2,...,a_{z+N},[x_i,x_j],x_1,...\check{x}_i...\check{x}_j...,x_{p'+1})$$

$$+\sum_{i=1}^{p'+1}(-1)^{i+1}\Big(Q(a_2,...,a_{z+N},x_1,...\check{x}_i...,x_{p'+1})x_i + \sum_{z'=2}^{z}Q(a_2,...,[x_i,a_{z'}],...,a_{z+N},x_1,...\check{x}_i...,x_{p'+1})\Big), \tag{16}$$

as $\mathrm{ad}(a)([x,a_1]) = \mathrm{ad}(ax)(a_1)$, for any $x\in\mathcal{L}(a_1,...,x_{p'})$ and $a\in\mathcal{A}(a_1,...,x_{p'})$.

We have an isomorphism

$$\mathcal{A}^{p'}_{z,N,1} \simeq \bigoplus_{\sigma\in\mathrm{Perm}(\{2,...,z+N\})} \mathcal{A}^{p'}_{\sigma},$$

where $\mathcal{A}^{p'}_{\sigma} := (\mathcal{A}(x_1,...,x_{p'})^{\otimes z+N})^{\mathfrak{S}^-_{p'}}_{[1,p']}$, whose inverse is the direct sum of the maps induced by

$$\bigotimes_{\alpha=1}^{z+N} Q_\alpha(x_1,...,x_{p'}) \mapsto Q_1(x_1,...,x_{p'})a_{\sigma(2)}Q_2(x_1,...,x_{p'})a_{\sigma(3)}...a_{\sigma(z+N)}Q_{z+N}(x_1,...,x_{p'}).$$

The explicit formula (16) shows that if $Q(a_1,\dots,x_{p'})$ is a multilinear monomial, then the image of Q by the extension of $d^{p',p'+1}_{z,N,1}$ given by the same formula is a linear combination of monomials, where the a_i appear in the same order as in $Q(a_1,...,x_{p'})$. It follows that for each $\sigma\in\mathrm{Perm}(\{2,...,z+N\})$, $\mathcal{A}^\bullet_\sigma$ is a subcomplex of $\mathcal{A}^\bullet_{z,N,1}$, and that we have a direct sum decomposition of the complex $\mathcal{A}^\bullet_{z,N,1}$,

$$\mathcal{A}^\bullet_{z,N,1} \simeq \bigoplus_\sigma \mathcal{A}^\bullet_\sigma. \tag{17}$$

The acyclicity of $\mathcal{A}^\bullet_{z,N,1}$ is then a consequence of that of each subcomplex $\mathcal{A}^\bullet_\sigma$, which we now prove. Let us fix $\sigma\in\mathrm{Perm}(\{2,...,z+N'\})$. There is a unique linear map

$$\tilde{d}^{p',p'+1}_\sigma : (\mathcal{A}(x_1,...,x_{p'})^{\otimes z+N})_{[1,p']} \to (\mathcal{A}(x_1,...,x_{p'+1})^{\otimes z+N})_{[1,p'+1]},$$

given by

$$\tilde{Q}(x_1,...,x_{p'}) \mapsto \sum_{1\le i<j\le p'+1}(-1)^{i+j+1}\tilde{Q}([x_i,x_j],x_1,...\check{x}_i...\check{x}_j...,x_{p'+1})$$

$$+\sum_{i=1}^{p'+1}(-1)^{i+1}\Big(\tilde{Q}(x_1,...\check{x}_i...,x_{p'+1})\big(\sum_{\substack{\alpha\in[\sigma^{-1}([2,z])-1]\\ \sqcup\{z+N\}}} x_i^{(\alpha)}\big) - \big(\sum_{\alpha\in\sigma^{-1}([2,z])} x_i^{(\alpha)}\big)\tilde{Q}(x_1,...\check{x}_i...,x_{p'+1})\Big),$$

where $f^{(\alpha)} = 1^{\otimes\alpha-1}\otimes f\otimes 1^{\otimes z+N-\alpha}$. If we set $\epsilon_1=0$, $\epsilon_{z+N+1}=1$, and

$$\epsilon_\alpha = 1 \Leftrightarrow \sigma(\alpha)\in[2,z], \quad \epsilon_\alpha = 0 \Leftrightarrow \sigma(\alpha)\in z+[1,N], \tag{18}$$

for $\alpha \in [2, z+N]$, then this map is

(19)

$$\tilde{Q}(x_1, ..., x_{p'}) \mapsto \sum_{1\leq i<i\leq p'+1} (-1)^{i+j+1}\tilde{Q}([x_i, x_j], x_1 ... \check{x}_i ... \check{x}_j ..., x_{p'+1})$$
$$+ \sum_{\alpha\in[1,z+N]} \sum_{i=1}^{p'+1} (-1)^{i+1}\big(\epsilon_{\alpha+1}\tilde{Q}(x_1, ... \check{x}_i ..., x_{p'+1})x_i^{(\alpha)} - \epsilon_\alpha x_i^{(\alpha)}\tilde{Q}(x_1, ... \check{x}_i ..., x_{p'+1})\big).$$

The map $\tilde{d}_\sigma^{p',p'+1}$ then restricts to a linear map between the subspaces of totally antisymmetric tensors (under the actions of $\mathfrak{S}_{p'}$ on the left side and $\mathfrak{S}_{p'+1}$ on the right side), which coincides with $d_\sigma^{p',p'+1}$.

For $\epsilon, \epsilon' \in \{0, 1\}$, define the "elementary" complexes $\mathcal{E}^\bullet_{\epsilon,\epsilon'}$ as follows. We set $\mathcal{E}^{p'}_{\epsilon,\epsilon'} := \mathcal{A}(x_1, ..., x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,p']}$, and define $d^{p',p'+1}_{\epsilon,\epsilon'} : \mathcal{E}^{p'}_{\epsilon,\epsilon'} \to \mathcal{E}^{p'+1}_{\epsilon,\epsilon'}$ by

(20)

$$(d^{p',p'+1}_{\epsilon,\epsilon'}E)(x_1, ..., x_{p'+1}) := \sum_{1\leq i<j\leq p'+1} (-1)^{i+j+1}E([x_i, x_j], x_1, ... \check{x}_i ... \check{x}_j ..., x_{p'+1})$$
$$+ \epsilon \sum_{i=1}^{p'+1} (-1)^i x_i E(x_1, ... \check{x}_i ..., x_{p'+1}) + \epsilon' \sum_{i=1}^{p'+1} (-1)^{i+1} E(x_1, ... \check{x}_i ..., x_{p'+1})x_i.$$

Lemma 6.5. *For $\epsilon, \epsilon' \in \{0, 1\}$, $\mathcal{E}^\bullet_{\epsilon,\epsilon'} := (\mathcal{E}^{p'}_{\epsilon,\epsilon'}, d^{p',p'+1}_{\epsilon,\epsilon'})_{p'\geq 0}$ is a complex.*

Proof. Note first that for any $p' \geq 0$, $\mathcal{E}^{p'}_{\epsilon,\epsilon'}$ is 1-dimensional, spanned by $e_{p'}(x_1, ..., x_{p'}) := \sum_{\sigma\in\mathfrak{S}_{p'}} \epsilon(\sigma)x_{\sigma(1)} ... x_{\sigma(p')}$.

If $\mathfrak{g}$ is a Lie algebra, let $U(\mathfrak{g})_{\epsilon,\epsilon'}$ be the universal enveloping algebra of $\mathfrak{g}$, equipped with the trivial $\mathfrak{g}$-module structure if $(\epsilon, \epsilon') = (0, 0)$, the left (resp., right) regular $\mathfrak{g}$-module structure if $(\epsilon, \epsilon') = (1, 0)$ (resp., $(0, 1)$), and the adjoint $\mathfrak{g}$-module structure if $(\epsilon, \epsilon') = (1, 1)$. Let $(C^{p'}_{\epsilon,\epsilon'}(\mathfrak{g}), d^{p',p'+1}_{\epsilon,\epsilon'}(\mathfrak{g}))$ be the cochain complex computing the cohomology of $\mathfrak{g}$ in these modules. We have $C^{p'}_{\epsilon,\epsilon'}(\mathfrak{g}) = \mathrm{Hom}(\Lambda^{p'}(\mathfrak{g}), U(\mathfrak{g}))$. There is a unique linear map $\mathcal{E}^{p'}_{\epsilon,\epsilon'} \to C^{p'}_{\epsilon,\epsilon'}(\mathfrak{g})$, taking $e_{p'}$ to the composite map $\Lambda^{p'}(\mathfrak{g}) \to \mathfrak{g}^{\otimes p'} \to U(\mathfrak{g})$, where the last map is the product map, and one checks that the diagram

$$\begin{array}{ccc} \mathcal{E}^{p'}_{\epsilon,\epsilon'} & \overset{d^{p',p'+1}_{\epsilon,\epsilon'}}{\rightarrow} & \mathcal{E}^{p'+1}_{\epsilon,\epsilon'} \\ \downarrow & & \downarrow \\ C^{p'}_{\epsilon,\epsilon'}(\mathfrak{g}) & \overset{d^{p',p'+1}_{\epsilon,\epsilon'}(\mathfrak{g})}{\rightarrow} & C^{p'+1}_{\epsilon,\epsilon'}(\mathfrak{g}) \end{array}$$

commutes. Since $C^\bullet_{\epsilon,\epsilon'}(\mathfrak{g})$ is a complex, and there exists a Lie algebra $\mathfrak{g}$ such that the morphisms $\mathcal{E}^{p'}_{\epsilon,\epsilon'} \to C^{p'}_{\epsilon,\epsilon'}(\mathfrak{g})$ are injective (for example, $\mathfrak{g}$ is a free Lie algebra with countably many generators), $\mathcal{E}^\bullet_{\epsilon,\epsilon'}$ is also a complex. □

Lemma 6.6. *We have an isomorphism of complexes $\mathcal{A}^\bullet_\sigma \simeq \mathcal{E}^\bullet_{0,\epsilon_2} \otimes \mathcal{E}^\bullet_{\epsilon_2,\epsilon_3} \otimes ... \otimes \mathcal{E}^\bullet_{\epsilon_{z+N},1}$, where $(\epsilon_2, ..., \epsilon_{z+N})$ is as in* (18).

Proof. The proof is parallel to that of Lemma 6.4. Let us set

$$\mathcal{A}^{p'} := \mathcal{A}_\sigma^{p'} = \big(\mathcal{A}(x_1,...,x_{p'})^{\otimes z+N}\big)_{[1,p']}^{\mathfrak{S}_{p'}^-}, \quad \tilde{\mathcal{A}}^{p'} := \big(\mathcal{A}(x_1,...,x_{p'})^{\otimes z+N}\big)_{[1,p']};$$

if $p_1+...+p_{z+N}=p'$, set

$$\mathcal{A}_{p_1,...,p_{z+N}} := \bigotimes_{\alpha=1}^{z+N} \mathcal{A}(x_1,...,x_{p'})_{p_1+...+p_{\alpha-1}+[1,p_\alpha]}^{\mathfrak{S}_{p_\alpha}^-}$$

and if $\bigsqcup_{\alpha=1}^{z+N} P_\alpha = [1,p']$, set

$$\tilde{\mathcal{A}}_{P_1,...,P_{z+N}} := \bigotimes_{\alpha=1}^{z+N} \mathcal{A}(x_1,...,x_{p'})_{P_\alpha}.$$

We have a decomposition

$$\tilde{\mathcal{A}}^{p'} = \bigoplus_{\bigsqcup_\alpha P_\alpha=[1,p']} \tilde{\mathcal{A}}_{P_1,...,P_{z+N}}.$$

We will define the support of an element x of $\tilde{\mathcal{A}}^{p'}$ as the set of partitions $(P_1,...,P_{z+N})$ of $[1,p']$ such that the component $x_{(P_1,...,P_{z+N})}$ is nonzero. We also have natural morphisms $\mathcal{A}_{p_1,...,p_{z+N}} \to \mathcal{A}^{p'}$, given by $x \mapsto (\sum_{\sigma\in\mathfrak{S}_{p_1,...,p_{z+N}}} \epsilon(\sigma)\sigma) * x$, where $*$ is the permutation action of $\mathfrak{S}_{p'}$ on $x_1,...,x_{p'}$. The direct sum of these morphisms gives rise to an isomorphism

$$\bigoplus_{p_1+...+p_{z+N}=p'} \mathcal{A}_{p_1,...,p_{z+N}} \simeq \mathcal{A}^{p'}.$$

As the l.h.s. identifies with $\bigoplus_{p_1+...+p_{z+N}=p'} \mathcal{E}_{0,\epsilon_1}^{p_1} \otimes ... \otimes \mathcal{E}_{\epsilon_{z+N},1}^{p_{z+N}}$, we obtain the identification $\mathcal{A}_\sigma^\bullet \simeq \bigotimes_{\alpha=1}^{z+N} \mathcal{E}_{\epsilon_\alpha,\epsilon_{\alpha+1}}^\bullet$ at the level of graded vector spaces. We now show that this identification is compatible with the differentials.

The composite map

$$\bigoplus_{\sum_\alpha p_\alpha=p'} \mathcal{A}_{p_1,...,p_{z+N}} \simeq \mathcal{A}^{p'} \overset{\text{can}}{\hookrightarrow} \tilde{\mathcal{A}}^{p'} \overset{\pi}{\twoheadrightarrow} \bigoplus_{\sum_\alpha p_\alpha=p'} \tilde{\mathcal{A}}_{[1,p_1],p_1+[1,p_2],...,p_1+...+p_{z+N-1}+[1,p_{z+N}]},$$

where the last map is the projection along the components indexed by the other (nonconsecutive) partitions, is the canonical inclusion map. It follows that the map $(\bigotimes_\alpha \mathcal{E}_{\epsilon_\alpha,\epsilon_{\alpha+1}}^\bullet)^{p'} \to (\bigotimes_\alpha \mathcal{E}_{\epsilon_\alpha,\epsilon_{\alpha+1}}^\bullet)^{p'+1}$ may be identified with the composite map

$$\bigoplus_{\sum_\alpha p_\alpha=p'} \mathcal{A}_{p_1,...,p_{z+N}} \simeq \mathcal{A}^{p'} \hookrightarrow \tilde{\mathcal{A}}^{p'} \overset{\tilde{d}_\sigma^{p',p'+1}}{\to} \tilde{\mathcal{A}}^{p'+1} \twoheadrightarrow \bigoplus_{\sum_\alpha \tilde{p}_\alpha=p'+1} \tilde{\mathcal{A}}_{[1,\tilde{p}_1],...,\tilde{p}_1+...+\tilde{p}_{z+N-1}+[1,\tilde{p}_{z+N}]}.$$

Now let $Q_\alpha \in \mathcal{E}_{\epsilon_\alpha,\epsilon_{\alpha+1}}^{p_\alpha} \simeq \mathcal{A}(x_1,...,x_{p'})_{p_1+...+p_{\alpha-1}+[1,p_\alpha]}^{\mathfrak{S}_{p_\alpha}^-}$ and $Q := \bigotimes_\alpha Q_\alpha \in \mathcal{A}_{p_1,...,p_{z+N}}$. The image of this element in $\tilde{\mathcal{A}}^{p'}$ is $(\sum_{\sigma\in\mathfrak{S}_{p_1,...,p_{z+N}}} \epsilon(\sigma)\sigma) * Q$. The summand $\epsilon(\sigma)\sigma * Q$ belongs to $\tilde{\mathcal{A}}_{P_1^\sigma,...,P_{z+N}^\sigma}$, where $P_\alpha^\sigma := \sigma(p_1+...+p_{\alpha-1}+[1,p_\alpha])$.

Decompose $\tilde d_\sigma^{p',p'+1} : \tilde{\mathcal{A}}^{p'} \to \tilde{\mathcal{A}}^{p'+1}$ as a sum $\tilde d_\sigma^{p',p'+1} = \sum_{1\leq i<j\leq p'+1} \tilde d^{ij} + \sum_{i=1}^{p'+1}\sum_{\alpha\in[1,z+N]} \tilde d^{i\alpha}$. If $\bigsqcup_\alpha P_\alpha = [1,p']$, then $\tilde d^{ij}(\tilde{\mathcal{A}}_{P_1,...,P_\alpha}) \subset \tilde{\mathcal{A}}_{P_1^{ij},...,P_{z+N}^{ij}}$ and $\tilde d^{i\alpha}(\tilde{\mathcal{A}}_{P_1,...,P_{z+N}}) \subset \mathcal{A}_{P_1^{i\alpha},...,P_{z+N}^{i\alpha}}$, where

• $(P_1^{ij},...,P_{z+N}^{ij})$ is given by $P_{ij}^\alpha := \big((P_\alpha\cap[2,i])-1\big)\sqcup(P_\alpha\cap[i+1,j-1])\sqcup\big((P_\alpha\cap[j,p'])+1\big)$ if $1\notin P_\alpha$, and the union of the same set with $\{i,j\}$ if $1\in P_\alpha$;

• $(P_1^{i\alpha},...,P_{z+N}^{i\alpha})$ is given by $P_\gamma^{i\alpha} = (P_\gamma\cap[1,i-1])\sqcup\big((P_\gamma\cap[i,p'])+1\big)$ if $\gamma\neq\alpha$, and the union of the same set with $\{i\}$ if $\gamma=\alpha$.

Note that the sequences $(|P_1^{ij}|,...,|P_{z+N}^{ij}|)$ and $(|P_1^{i\alpha}|,...,|P_{z+N}^{i\alpha}|)$ are necessarily of the form $(p_1^\beta,...,p_{z+N}^\beta) := (p_1+\delta_{1\beta},...,p_{z+N}+\delta_{z+N,\beta})$, where $\beta\in[1,z+N]$ is the index such that $1\in P_\alpha$ in the first case, and α in the second case. Then:

(a) for any i,j $(1\leq i<j\leq p'+1)$ and any $\beta\in[1,z+N]$, $(P_1^{ij},...,P_{z+N}^{ij})$ coincides with $([1,p_1^\beta],...,p_1^\beta+...+p_{z+N-1}^\beta+[1,p_{z+N}^\beta])$ if $P_\alpha = 1+p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for $\alpha<\beta$, $P_\beta = (1+p_1+...+p_{\beta-1}+[1,p_\beta-1])\sqcup\{1\}$, and $P_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for $\beta>\alpha$, and $i,j\in p_1+...+p_{\beta-1}+[1,p_\beta+1]$;

(b) for $i\in[1,p'+1]$ and $\alpha\in[1,z+N]$, $(P_1^{i\alpha},...,P_{z+N}^{i\alpha})$ coincides with $([1,p_1^\alpha],..., p_1^\alpha+...+p_{z+N-1}^\alpha+[1,p_{z+N}^\alpha])$ if $P_\alpha = p_1+...+p_{\alpha-1}+[1,p_\alpha]$ for any α and $i\in p_1+...+p_{\alpha-1}+[1,p_{\alpha+1}]$.

If i,j are such that $1\leq i<j\leq p'+1$, then the condition on $\sigma\in\mathfrak{S}_{p_1,...,p_{z+N}}$ for the support $\tilde d^{ij}(\epsilon(\sigma)\sigma * Q)$ to consist in a consecutive partition of $[1,p'+1]$ is therefore: there exists $\beta\in[1,z+N]$ such that $i,j\in p_1+...+p_{\beta-1}+[1,p_\beta+1]$, and σ is the shuffle permutation taking $[1,p_1],p_1+[1,p_2],...,p_1+...+p_{z+N-1}+[1,p_{z+N}]$ to the partition described in (a) above.

If $i\in[1,p'+1]$ and $\alpha\in[1,z+N]$, then the condition on $\sigma\in\mathfrak{S}_{p_1,...,p_{z+N}}$ for the support of $\tilde d^{i\alpha}(\epsilon(\sigma)\sigma*Q)$ to consist in a consecutive partition of $[1,p'+1]$ is therefore: $\sigma=\mathrm{id}$ and $i\in p_1+...+p_{\alpha-1}+[1,p_\alpha+1]$.

In the first case, we have $\epsilon(\sigma) = (-1)^{p_1+...+p_{\beta-1}}$ and $\pi\circ\tilde d^{ij}(Q) = Q_1\otimes...\otimes\tilde d_\beta^{ij}(Q_\beta)\otimes...\otimes Q_{z+N}$; in the second case, $\pi\circ\tilde d^{i\beta}(Q)$ is $(-1)^{p_1+...+p_{\beta-1}}Q_1\otimes...\otimes\tilde d^i_{\epsilon_\beta,\epsilon_{\beta+1}}(Q_\beta)\otimes...\otimes Q_{z+N}$. Here

$$\tilde d^{p_\beta,p_{\beta+1}}_{\epsilon_\beta,\epsilon_{\beta+1}} : \mathcal{A}(x_1,...,x_{p'})_{p_1+...+p_{\beta-1}+[1,p_\beta]} \to \mathcal{A}(x_1,...,x_{p'+1})_{p_1+...+p_{\beta-1}+[1,p_{\beta+1}]}$$

is decomposed as

$$\tilde d^{p_\beta,p_{\beta+1}}_{\epsilon_\beta,\epsilon_{\beta+1}} = \sum_{p_1+...+p_{\beta-1}+1\leq i<j\leq p_1+...+p_\beta+1} \tilde d^{ij}_\beta + \sum_{i\in p_1+...+p_{\beta-1}+[1,p_\beta+1]} \tilde d^i_{\epsilon_\beta,\epsilon_{\beta+1}}.$$

Then

$$\pi\circ\tilde d_\sigma^{p',p'+1}\circ\mathrm{can}(Q) = \sum_{\beta=1}^{z+N}(-1)^{p_1+...+p_{\beta-1}}Q_1\otimes...\otimes\tilde d^{p_\beta,p_\beta+1}_{\epsilon_\beta,\epsilon_{\beta+1}}(Q_\beta)\otimes...\otimes Q_{z+N},$$

which proves our claim. □

Proposition 6.1. *The complexes $\mathcal{E}^\bullet_{0,1}$ and $\mathcal{E}^\bullet_{1,0}$ are acyclic; moreover, for $\epsilon\in\{0,1\}$, $H^{p'}(\mathcal{E}^\bullet_{\epsilon,\epsilon})$ is zero for any $p'\neq0$ and $\mathbf{k}$ for $p'=0$.*

Proof. If $u_1,...,u_n$ are free variables, let $\mathbf{k} = \mathcal{A}_{\leq0}(u_1,...,u_n)\subset...\subset\mathcal{A}_{\leq i}(u_1,...,u_n)$ $\subset ... \subset \mathcal{A}(u_1,...,u_n)$ be the increasing PBW filtration of $\mathcal{A}(u_1,...,u_n)$, induced by its identification with $U(\mathcal{L}(u_1,...,u_n))$. The symmetrization isomorphism $\mathcal{A}(u_1,...,u_n) \simeq S(\mathcal{L}(u_1,...,u_n))$ identifies $\mathcal{A}_{\leq i}(u_1,...,u_n)$ with

$\bigoplus_{i'\leq i} S^{i'}(\mathcal{L}(u_1,...,u_n))$. The graded space associated to this filtration is the free Poisson algebra $\mathcal{P}(u_1,...,u_n) = S(\mathcal{L}(u_1,...,u_n))$; its degree i part is $\mathcal{P}[i](u_1,...,u_n) = S^i(\mathcal{L}(u_1,...,u_n))$.

Define a filtration on $\mathcal{E}^\bullet_{\epsilon,\epsilon'}$ by $F_u(\mathcal{E}^{p'}_{\epsilon,\epsilon'}) := \mathcal{A}_{\leq u}(x_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,p']}$ for $u \geq 0$. If $E(x_1,...,x_{p'}) \in \mathcal{A}_{\leq u}(x_1,...,x_{p'})_{[1,p']}$, then:

$$E([x_i,x_j],x_1,...\check{x}_i...\check{x}_j...,x_{p'+1}) \in \mathcal{A}_{\leq u}(x_1,...,x_{p'+1})_{[1,p'+1]};$$
$$x_i E(x_1,...\check{x}_i...,x_{p'+1}), E(x_1,...\check{x}_i...,x_{p'+1})x_i \in \mathcal{A}_{\leq u+1}(x_1,...,x_{p'+1})_{[1,p'+1]},$$

while $[x_i, E(x_1,...\check{x}_i...,x_{p'+1})] \in \mathcal{A}_{\leq u}(x_1,...,x_{p'+1})_{[1,p'+1]}$. It follows that for $\epsilon \in \{0,1\}$, we have

$$d^{p',p'+1}_{\epsilon,\epsilon}(F_u(\mathcal{E}^{p'}_{\epsilon,\epsilon})) \subset F_u(\mathcal{E}^{p'+1}_{\epsilon,\epsilon}),$$

while for $\epsilon \neq \epsilon'$ in $\{0,1\}$,

$$d^{p',p'+1}_{\epsilon,\epsilon'}(F_u(\mathcal{E}^{p'}_{\epsilon,\epsilon'})) \subset F_{u+1}(\mathcal{E}^{p'+1}_{\epsilon,\epsilon'}).$$

The associated graded complex is $\mathcal{P}^\bullet_{\epsilon,\epsilon'}$, where

$$\mathcal{P}^{p'}_{\epsilon,\epsilon'} = \mathcal{P}(x_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,p']} = \bigoplus_{u\geq 0} \mathcal{P}[u](x_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,p']}, \tag{21}$$

with differential

$$\mathrm{gr}\, d^{p',p'+1}_{\epsilon,\epsilon'} : \mathcal{P}^{p'}_{\epsilon,\epsilon'} \to \mathcal{P}^{p'+1}_{\epsilon,\epsilon'}$$

given by

$$(\mathrm{gr}\, d^{p',p'+1}_{\epsilon,\epsilon} P)(x_1,...,x_{p'+1}) := \sum_{1\leq i<j\leq p'+1} (-1)^{i+j+1} P(\{x_i,x_j\},x_1,...\check{x}_i...\check{x}_j...,x_{p'+1}) + \epsilon \sum_{i=1}^{p'+1} (-1)^i \{x_i, P(x_1,...\check{x}_i...,x_{p'+1})\}$$

for $\epsilon \in \{0,1\}$,

$$(\mathrm{gr}\, d^{p',p'+1}_{0,1} P)(x_1,...,x_{p'+1}) := \sum_{i=1}^{p'+1} (-1)^{i+1} x_i P(x_1,...\check{x}_i...,x_{p'+1}),$$

and $\mathrm{gr}\, d^{p',p'+1}_{1,0} = -\,\mathrm{gr}\, d^{p',p'+1}_{0,1}$ (when $\epsilon' = \epsilon$, the commutators give rise to brackets in the associated graded differential, while if $\epsilon \neq \epsilon'$, the only part of the differential with nontrivial contribution to the associated graded differential is the second line of (20)). The differentials $\mathrm{gr}\, d^{p',p'+1}_{\epsilon,\epsilon}$ have degree 0, and the differentials $\mathrm{gr}\, d^{p',p'+1}_{\epsilon,\epsilon'}$ have degree 1 (if $\epsilon' \neq \epsilon$) with respect to the $\mathbb{N}$-grading on $\mathcal{P}^\bullet_{\epsilon,\epsilon'}$ induced by (21). We therefore have direct sum decompositions of complexes

$$\mathcal{P}^\bullet_{\epsilon,\epsilon} = \bigoplus_{u\in\mathbb{Z}} \mathcal{P}^\bullet_{\epsilon,\epsilon}[u], \quad \mathcal{P}^\bullet_{\epsilon,\epsilon'} = \bigoplus_{u\in\mathbb{N}} \mathcal{P}^\bullet_{\epsilon,\epsilon'}\{u\} \text{ (if } \epsilon' \neq \epsilon),$$

where for any ϵ,ϵ', we set $\mathcal{P}^{p'}_{\epsilon,\epsilon'}[u] := \mathcal{P}[u](x_1,...,x_{p'})^{\mathfrak{S}^-_{p'}}_{[1,p']}$ and $\mathcal{P}^{p'}_{\epsilon,\epsilon'}\{u\} = \mathcal{P}^{p'}_{\epsilon,\epsilon'}[u+p']$.

Lemma 6.7. *For $n, u \geq 0$, $\mathcal{P}^n_{\epsilon,\epsilon'}[u]$ have the following values:*[8] • *if $n = 2m$, $\mathcal{P}^{2m}_{\epsilon,\epsilon'}[m]$ is 1-dimensional, spanned by*

$$p_{2m}(x_1, ..., x_{2m}) := \sum_{\sigma \in \mathfrak{S}_{2,...,2}} \epsilon(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}\}...\{x_{\sigma(2m-1)}, x_{\sigma(2m)}\}$$

and $\mathcal{P}^{2m}_{\epsilon,\epsilon'}[u] = 0$ for $u \neq m$;

• *if $n = 2m + 1$, $\mathcal{P}^{2m+1}_{\epsilon,\epsilon'}[m + 1]$ is 1-dimensional, spanned by*

$$p_{2m+1}(x_1, ..., x_{2m+1}) := \sum_{\sigma \in \mathfrak{S}_{1,2,...,2}} \epsilon(\sigma)x_{\sigma(1)}\{x_{\sigma(2)}, x_{\sigma(3)}\}...\{x_{\sigma(2m)}, x_{\sigma(2m+1)}\},$$

and $\mathcal{P}^{2m+1}_{\epsilon,\epsilon'}[u] = 0$ for $u \neq m + 1$.

Proof of Lemma. As the category of $\mathfrak{S}_n$-modules is semisimple, the $\mathfrak{S}_n$-modules $\mathcal{A}(x_1, ..., x_n)_{[1,n]}$ and $\mathcal{P}(x_1, ..., x_n)_{[1,n]}$ are equivalent. It follows that $\mathcal{P}(x_1, ..., x_n)^{\mathfrak{S}_n^-}_{[1,n]}$ is 1-dimensional. Since this space is equal to $\bigoplus_{u \geq 0} \mathcal{P}[u](x_1, ..., x_n)^{\mathfrak{S}_n^-}_{[1,n]}$, it follows that exactly one of these summands is 1-dimensional, and the others are zero. It then remains to prove that $p_n \in \mathcal{P}^n_{\epsilon,\epsilon'}[[(n + 1)/2]]$ and $p_n \neq 0$, where $[x]$ is the integral part of x.

If $n = 2m$, we have

$$p_{2m}(x_1, ..., x_{2m}) = 2^{-m} \sum_{\sigma \in \mathfrak{S}_{2m}} \epsilon(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}\}...\{x_{\sigma(2m-1)}, x_{\sigma(2m)}\},$$

so p_{2m} is $\mathfrak{S}_n$-anti-invariant, so $p_{2m} \in \mathcal{P}^{2m}_{\epsilon,\epsilon'}[m]$; and if Γ is the set of $\sigma \in \mathfrak{S}_{2m}$, such that $\sigma(1) < \sigma(3) < ... < \sigma(2m-1)$ and $\sigma(2i+1) < \sigma(2i+2)$ for $i = 0, ..., m-1$ (this identifies with the set of partitions of $[1, 2m]$ into subsets of cardinality 2, modulo a permutation of the subsets), we have

$$p_{2m}(x_1, ..., x_{2m}) = m! \sum_{\sigma \in \Gamma} \epsilon(\sigma)\{x_{\sigma(1)}, x_{\sigma(2)}\}...\{x_{\sigma(2m-1)}, x_{\sigma(2m)}\},$$

and as the summands in this expression are linearly independant, $p_{2m} \neq 0$.

If $n = 2m + 1$, we have similarly

$$p_{2m+1}(x_1, ..., x_{2m+1}) = 2^{-m} \sum_{\sigma \in \mathfrak{S}_{2m+1}} \epsilon(\sigma)x_{\sigma(1)}\{x_{\sigma(2)}, x_{\sigma(3)}\}...\{x_{\sigma(2m)}, x_{\sigma(2m+1)}\},$$

which implies that p_{2m+1} is $\mathfrak{S}_n$-anti-invariant, and

$$p_{2m+1}(x_1, ..., x_{2m+1}) = m! \sum_{\sigma \in \Gamma} \epsilon(\sigma)x_{\sigma(1)}\{x_{\sigma(2)}, x_{\sigma(3)}\}...\{x_{\sigma(2m)}, x_{\sigma(2m+1)}\},$$

where Γ is the set of permutations $\sigma \in \mathfrak{S}_n$ such that $\sigma(2) < \sigma(4) < ... < \sigma(2m)$ and $\sigma(2i) < \sigma(2i + 1)$ for $i = 1, ..., m$, which implies that p_{2m+1} is nonzero, as the summands in this expression are linearly independent. □

End of proof of Proposition 6.1. For $u \in \mathbb{Z}$, the complex $\mathcal{P}^\bullet_{0,1}\{u\}$ is $0 \to \mathcal{P}^0_{0,1}[u] \to \mathcal{P}^1_{0,1}[u + 1] \to ....$ For $u > 0$, the groups of this complex are all zero, so $\mathcal{P}^\bullet_{0,1}\{u\}$ is acyclic. For $u \leq 0$, this complex is $0 \to ... \to 0 \to \mathcal{P}^{2m}_{0,1}[m] \to \mathcal{P}^{2m+1}_{0,1}[m + 1] \to 0 \to ...$, where $m = -u$. The nontrivial map in this complex is $p_{2m} \mapsto \mathrm{gr}\, d^{2m,2m+1}_{0,1}(p_{2m}) = p_{2m+1}$, which is an isomorphism, so $\mathcal{P}^\bullet_{0,1}\{u\}$ is acyclic. It

[8] Recall that for $n_1 + ... + n_k = n$, $\mathfrak{S}_{n_1,...,n_k} = \{\sigma | \forall i \in [1, k], \sigma(n_1 + ... + n_{i-1} + 1) < ... < \sigma(n_1 + ... + n_i)\} \subset \mathfrak{S}_n$ is the set of $(n_1, ..., n_k)$-shuffle permutations.

follows that $\mathcal{P}^\bullet_{0,1}$ is acyclic. As the differential of $\mathcal{P}^\bullet_{1,0}$ is the negative of that of $\mathcal{P}^\bullet_{0,1}$, $\mathcal{P}^\bullet_{1,0}$ is acyclic as well.

Let $\epsilon \in \{0,1\}$ and $u \in \mathbb{N}$. The complex $\mathcal{P}^\bullet_{\epsilon,\epsilon}[u]$ is $0 \to \mathcal{P}^0_{\epsilon,\epsilon}[u] \to \mathcal{P}^1_{\epsilon,\epsilon}[u] \to ...$; if $i = u$, this complex is $0 \to \mathbf{k} \to 0 \to 0 \to ...$, whose cohomology is 1-dimensional, concentrated in degree 0; if $u > 0$, this complex is $0 \to ... \to 0 \to \mathcal{P}^{2u-1}_{\epsilon,\epsilon}[u] \to \mathcal{P}^{2u}_{\epsilon,\epsilon}[u] \to 0 \to 0...$; the nontrivial map in this complex is $p_{2u-1} \mapsto \operatorname{gr} d^{2u-1,2u}_{\epsilon,\epsilon}(p_{2u-1}) = up_{2u}$ if $\epsilon = 0$ and $-up_{2u}$ if $\epsilon = 1$. As this is an isomorphism in both cases, $\mathcal{P}^\bullet_{\epsilon,\epsilon}[u]$ is acyclic for $u > 0$. It follows that the cohomology of $\mathcal{P}^\bullet_{\epsilon,\epsilon}$ is 1-dimensional, concentrated in degree 0.

This implies that $\mathcal{E}^\bullet_{\epsilon,\epsilon'}$ is acyclic for $\epsilon \neq \epsilon'$, and that the cohomology of $\mathcal{E}^\bullet_{\epsilon,\epsilon}$ is concentrated in degree 0. As $d^{0,1}_{\epsilon,\epsilon} = 0$, we have in degree 0, $H^0(\mathcal{E}^\bullet_{\epsilon,\epsilon}) = \mathcal{E}^0_{\epsilon,\epsilon} \simeq \mathbf{k}$. □

Remark 3. If $\mathfrak{g}$ is a Lie algebra, we have natural maps

$$H^\bullet(\mathcal{E}^\bullet_{\epsilon,\epsilon'}) \to H^\bullet(\mathfrak{g}, U(\mathfrak{g})_{\epsilon,\epsilon'}). \tag{22}$$

When $(\epsilon, \epsilon') = (0,1)$ and $\mathfrak{g}$ is finite dimensional, then $H^n(\mathfrak{g}, U(\mathfrak{g})_{0,1}) = \mathbf{k}$ if $n = \dim \mathfrak{g}$, and $= 0$ otherwise. Indeed, if $C^n(\mathfrak{g}) := \Lambda^n(\mathfrak{g}) \otimes U(\mathfrak{g})$, then the differential $d^{n,n+1}_{\mathfrak{g}} : C^n(\mathfrak{g}) \to C^{n+1}(\mathfrak{g})$ is given by $\omega \otimes x \mapsto \delta(\omega) \otimes x + \sum_{\alpha=1}^{\dim \mathfrak{g}} (\omega \wedge e^\alpha) \otimes (e_\alpha x)$, where $(e^\alpha)_\alpha, (e_\alpha)_\alpha$ are dual bases of $\mathfrak{g}^*$ and $\mathfrak{g}$ and $\delta : \Lambda^n(\mathfrak{g}^*) \to \Lambda^{n+1}(\mathfrak{g}^*)$ is induced by the Lie coalgebra structure of $\mathfrak{g}^*$. For $i \in \mathbb{Z}$, set $F_i(C^n(\mathfrak{g})) := \Lambda^n(\mathfrak{g}^*) \otimes U(\mathfrak{g})_{\leq n+i}$ (where the last term is the subspace of elements of degree $\leq n + i$ for the PBW filtration). Then $d^{n,n+1}(F_i(C^n(\mathfrak{g}))) \subset F_i(C^{n+1}(\mathfrak{g}))$, so $... \subset F_i(C^n(\mathfrak{g})) \subset ... \subset C^n(\mathfrak{g})$ is a complete filtration of $C^n(\mathfrak{g})$. The associated graded complex is $\tilde{C}^n(\mathfrak{g}) := \Lambda^n(\mathfrak{g}^*) \otimes S(\mathfrak{g})$, with differential $\tilde{d}^{n,n+1} : \tilde{C}^n(\mathfrak{g}) \to \tilde{C}^{n+1}(\mathfrak{g})$, $\omega \otimes x \mapsto \sum_{\alpha=1}^{\dim \mathfrak{g}} (\omega \wedge e^\alpha) \otimes (e_\alpha x)$. This complex only depends on the vector space structure of $\mathfrak{g}$; if we denote it by $\tilde{C}^\bullet(\mathfrak{g})$, then we have an isomorphism $\tilde{C}^\bullet(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \simeq \tilde{C}^\bullet(\mathfrak{g}_1) \otimes \tilde{C}^\bullet(\mathfrak{g}_2)$, so $C^\bullet(\mathfrak{g}) \simeq C^\bullet(\mathbf{k})^{\otimes \dim \mathfrak{g}}$. As the cohomology of $\tilde{C}^\bullet(\mathbf{k})$ is 1-dimensional, concentrated in degree 1, the cohomology of $\tilde{C}^\bullet(\mathfrak{g})$ is 1-dimensional, concentrated in degree $\dim \mathfrak{g}$. It follows that $C^\bullet(\mathfrak{g})$ is acyclic in every degree $\neq \dim \mathfrak{g}$, and its cohomology in degree $\dim \mathfrak{g}$ has dimension ≤ 1. If $\omega \in \Lambda^{\dim \mathfrak{g}}(\mathfrak{g})$ is nonzero, then $\omega \otimes 1 \in C^n(\mathfrak{g})$ is a nontrivial cocycle, so the cohomology of $C^\bullet(\mathfrak{g})$ coincides with that of $\tilde{C}^\bullet(\mathfrak{g})$. As $U(\mathfrak{g})_{1,0} \simeq U(\mathfrak{g})_{0,1}$ (using the antipode), we have $H^\bullet(\mathfrak{g}, U(\mathfrak{g})_{1,0}) \simeq H^\bullet(\mathfrak{g}, U(\mathfrak{g})_{0,1})$.

When $\epsilon \neq \epsilon'$, (22) is the zero map. If $\epsilon = \epsilon'$, then the map $\mathbf{k} = H^0(\mathcal{E}^\bullet_{\epsilon,\epsilon}) \to H^0(\mathfrak{g}, U(\mathfrak{g})_{\epsilon,\epsilon})$ takes 1 to the class of $1 \in U(\mathfrak{g})_{\epsilon,\epsilon}$ (which is invariant, both under the trivial and the adjoint actions of $\mathfrak{g}$ on $U(\mathfrak{g})$).

End of proof of Theorem 6.1. One of the pairs $(0, \epsilon_2), (\epsilon_2, \epsilon_3), ..., (\epsilon_{z+N}, 1)$ necessarily coincides with $(0,1)$; call it $(\epsilon_i, \epsilon_{i+1})$. According to Proposition 6.1, the corresponding complex $\mathcal{E}^\bullet_{\epsilon_i,\epsilon_{i+1}}$ is then acyclic. Lemma 6.6 and the Künneth formula then imply that $\mathcal{A}^\bullet_\sigma$ is acyclic. This being valid for any σ, the decomposition (17) then implies that $\mathcal{A}^\bullet_{z,N,1}$ is acyclic, as claimed. □

7. Compatibility of quantization functors with twists

In this section, we prove the compatibility of quantization functors of quasi-Lie bialgebras with twists; we derive from there the compatibility of quantization functors of Lie bialgebras with twists (a result which was obtained in [EH] in the case of Etingof-Kazhdan quantization functors).

7.1. **Twists of quasi-Lie bialgebras.** Let QLBA_f be the prop with the same generators as QLBA with the additional $f \in \mathrm{QLBA}_f(\Lambda^2, \mathbf{id})$, and the same relations. This prop is $\mathbb{N}^2$-graded if we extend the degree $(\deg_\mu, \deg_\delta)$ of the generators of QLBA by $f \mapsto (0,1)$.

We then have $\mathrm{QLBA}_f(X,Y) = \mathrm{QLBA}(S(\Lambda^2) \otimes X, Y)$. The filtration of QLBA_f induced by the degree $\deg_\mu + \deg_\delta$ is such that

$$\mathrm{QLBA}_f^{\geq n}(X,Y) = \bigoplus_{k \geq 0} \mathrm{QLBA}^{\geq n-k}(S^k(\Lambda^2) \otimes X, Y).$$

It follows that $\mathrm{QLBA}_f(X,Y) \subset \mathrm{QLBA}_f^{\geq v_f(|X|,|Y|)}(X,Y)$, where $v_f(|X|,|Y|) = \inf\{v(|X|+2k, |Y|) + k, k \geq 0\}$ and $v(|X|,|Y|) = \frac{1}{3}||X| - |Y||$. As $v_f(|X|,|Y|) \geq v(|X|,|Y|)$, QLBA_f gives rise to a topological prop $\mathbf{QLBA}_f$.

We have two prop morphisms $\kappa_i : \mathrm{QLBA} \to \mathrm{QLBA}_f$, defined by

$$\kappa_1 : \mu, \delta, \varphi \mapsto \mu, \delta, \varphi,$$
$$\kappa_2 : \mu \mapsto \mu, \quad \delta \mapsto \delta + \mathrm{Alt}_2 \circ (\mathrm{id}_{\mathbf{id}} \otimes \mu) \circ (f \otimes \mathrm{id}_{\mathbf{id}}),$$
$$\varphi \mapsto \varphi + \frac{1}{2} \mathrm{Alt}_3 \circ \big((\delta \otimes \mathrm{id}_{\mathbf{id}}) \circ f + (\mathrm{id}_{\mathbf{id}} \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (f \otimes f) \big);$$

this is the universal version of the operation of twisting of a quasi-Lie bialgebra structure. The prop morphisms κ_i extend to topological props.

Let $(m, \Delta, \Phi, \eta, \epsilon)$ be a quantization functor for quasi-Lie bialgebras; so this is a quasi-bialgebra structure on S in **QLBA** as in Definition 3.2. For $i = 1, 2$, set $(m_i, \Delta_i, \Phi_i, \eta_i, \epsilon_i) := \kappa_i(m, \Delta, \Phi, \eta, \epsilon)$. Then $(m_i, \Delta_i, \Phi_i, \eta_i, \epsilon_i)$ are quasi-bialgebra structures on S in $\mathbf{QLBA}_f$.

Proposition 7.1. *The quasi-bialgebra structures* $(m_i, \Delta_i, \Phi_i, \eta_i, \epsilon_i)$ *on* S *in* **QLBA** *are related by equivalence and twist.*

This implies that the quantization functors of quasi-Lie bialgebras take quasi-Lie bialgebras related by a classical twist to quasi-Hopf QUE algebras related by a quantum twist.

Proof. The prop $\mathbf{QLBA}_f$ is graded by $\deg_\delta$. We have a prop morphism $\kappa_0 : \mathbf{QLBA}_f \to \mathbf{QLBA}_f/\langle f\rangle \simeq \mathbf{QLBA}$; this morphism has degree 0 for $\deg_\delta$.

Lemma 7.1. *The linear map* $H^p_{\mathrm{QLBA}_f}(\mathbf{id}, \Lambda^q) \to H^p_{\mathrm{QLBA}}(\mathbf{id}, \Lambda^q)$ *induced by* κ_0 *is an isomorphism.*

Proof of Lemma. The complexes computing these cohomology groups are $C^\bullet_{\mathcal{C}} = (\mathcal{C}(\Lambda^\bullet, \Lambda^q), [\mu, -])$ for $\mathcal{C} = \mathrm{QLBA}_f, \mathrm{QLBA}$. As $\kappa_0(\mu) = \mu$, κ_0 induces a morphism $C^\bullet_{\mathrm{QLBA}_f} \to C^\bullet_{\mathrm{QLBA}}$. We will show that the relative complex $\mathrm{Ker}(C^\bullet_{\mathrm{QLBA}_f} \to C^\bullet_{\mathrm{QLBA}})$ is acyclic, which implies the statement of the lemma.

If I is a prop ideal in $C^\bullet_{\mathrm{QLBA}_f}$, we set $C^\bullet_I := (I(\Lambda^\bullet, \Lambda^q), [\mu, -])$. The relative complex is then $C^\bullet_{\langle\varphi\rangle_f}$, where $\langle\varphi\rangle_f$ is the prop ideal of QLBA_f generated by φ. This complex is graded by $\deg_\delta$. As before, we have a filtration $C^\bullet_{\langle\varphi\rangle_f} \supset C^\bullet_{\langle\varphi\rangle_f^2} \supset \ldots$, which is total in each degree. It suffices therefore to prove that the associated graded complex is acyclic. We now compute this graded complex.

Recall that if $\langle\varphi\rangle$ is the ideal of QLBA generated by φ, then $\langle\varphi\rangle^0/\langle\varphi\rangle^1(X,Y) = \mathrm{LBA}(X,Y)$, and if $k>0$, then $\langle\varphi\rangle^k/\langle\varphi\rangle^{k+1}(X,Y) = \mathrm{LBA}_\alpha(X,Y)\{k\} = \mathrm{Coker}\big(X\otimes S^{k-1}(\Lambda^3)\otimes\Lambda^4, Y) \to \mathrm{LBA}(X\otimes S^k(\Lambda^3),Y)\big)$, where $\{k\}$ means the grading $(\tilde\mu,\tilde\delta,\tilde\varphi)\mapsto(0,0,1)$ on LBA_α.

Under the identification $\mathrm{QLBA}_f(X,Y)\simeq \mathrm{QLBA}(X\otimes S(\Lambda^2),Y)$, $\langle\varphi\rangle^k_f(X,Y)$ identifies with $\langle\varphi\rangle^k(X\otimes S(\Lambda^2),Y)$. Then if $k>0$, we have $\langle\varphi\rangle^k_f/\langle\varphi\rangle^{k+1}_f(X,Y)\simeq \langle\varphi\rangle^k/\langle\varphi\rangle^{k+1}(X\otimes S(\Lambda^2),Y)\simeq \mathrm{Coker}\big(\mathrm{LBA}(X\otimes S(\Lambda^2)\otimes S^{k-1}(\Lambda^3)\otimes\Lambda^4,Y)\to \mathrm{LBA}(X\otimes S^k(\Lambda^3),Y)\big)$.

It follows that the complex $C^\bullet_{\langle\varphi\rangle^k_f}/C^\bullet_{\langle\varphi\rangle^{k+1}_f}$ identifies with $\mathrm{Coker}\big(\mathrm{LBA}(\Lambda^\bullet\otimes S(\Lambda^2)\otimes S^{k-1}(\Lambda^3)\otimes\Lambda^4,\Lambda^q)\to \mathrm{LBA}(\Lambda^\bullet\otimes S^k(\Lambda^3),\Lambda^q)\big)$, equipped with the differential induced by $[\mu,-]$. Theorem 6.1 then implies that this complex is acyclic, as wanted. □

As in Section 2.4, $\deg_\delta$ gives rise to props $\mathbf{QLBA}_{\le n}$, such that $\mathbf{QLBA}_{\le 0}=\mathbf{LA}$. The morphisms κ_i $(i=1,2)$ have degree 0 for $\deg_\delta$.

We define Q^f_n (resp., Q_n) as {quasi-bialgebra structures on S in $(\mathbf{QLBA}_f)_{\le n}$ (resp., $\mathbf{QLBA}_{\le n}$) quantizing $U(\mathbf{id_{LA}})$}/(equivalence, twists). The prop morphisms κ_i then induce maps $\kappa^n_i : Q_n\to Q^f_n$ for $i=1,2$.

Lemma 7.2. $\kappa^n_1=\kappa^n_2$.

Proof of Lemma. Since κ_0 induces $\kappa^n_0: Q^f_n\to Q_n$, and since $\kappa_0\circ\kappa_i=\mathrm{id}$, we have $\kappa^n_0\circ\kappa^n_i=\mathrm{id}$ for $i=1,2$. We also have a commutative diagram

$$\begin{array}{ccccc} Q_n & \overset{\kappa^n_i}{\rightrightarrows} & Q^f_n & \overset{\kappa^n_0}{\to} & Q_n \\ \pi\downarrow & & \pi_f\downarrow & & \downarrow\pi \\ Q_{n-1} & \overset{\kappa^{n-1}_i}{\rightrightarrows} & Q^f_{n-1} & \overset{\kappa^{n-1}_0}{\to} & Q_{n-1}. \end{array}$$

Moreover, $q_f\in\pi_f(Q^f_n)$ (resp., $q\in\pi(Q_n)$) being fixed, $\pi_f^{-1}(q_f)$ (resp., $\pi^{-1}(q)$) is an affine space over $H^3_{\mathrm{QLBA}_f}[n]$ (resp., over $H^3_{\mathrm{QLBA}}[n]$), and if $q:=\kappa^{n-1}_0(q_f)$, then the map $\pi_f^{-1}(q_f)\overset{\kappa_0}{\to}\pi^{-1}(q)$ is a morphism of affine spaces, compatible with the map $H^3_{\mathrm{QLBA}_f}[n]\to H^3_{\mathrm{QLBA}}[n]$ induced by κ_0. Lemma 7.1, together with the formulas of Proposition 3.4, then imply that this map is bijective. It follows that the restriction of κ^n_0 to a nonempty fiber of π_f is injective.

We now prove the statement by induction over n. Let us assume that $\kappa^{n-1}_1=\kappa^{n-1}_2$. Let $\tilde q\in Q_n$ and let $\tilde q_i:=\kappa^n_i(\tilde q)$. Then $\pi_f(\tilde q_1)=\kappa^{n-1}_1(\pi(\tilde q))=\kappa^{n-1}_2(\pi(\tilde q))=\pi_f(\tilde q_2)$ as $\kappa^{n-1}_1=\kappa^{n-1}_2$. It follows that $\tilde q_1,\tilde q_2$ belong to the same fiber of π_f. Now $\kappa^n_0(\tilde q_1)=\tilde q=\kappa^n_0(\tilde q_2)$. The injectivity of the restriction of κ^n_0 to fibers of π_f then implies that $\tilde q_1=\tilde q_2$; hence $\kappa^n_1=\kappa^n_2$. □

End of proof of Proposition 7.1. Now set $Q_\infty:=\lim_\leftarrow Q_n$, $Q^f_\infty:=\lim_\leftarrow Q^f_n$. These sets identify with {quasi-bialgebra structures on S in $\mathbf{QLBA}$ (resp., $\mathbf{QLBA}_f$) quantizing $U(\mathbf{id_{LA}})$}/(equivalence, twists). The morphisms κ_i induce maps $\kappa^\infty_i: Q_\infty\to Q^f_\infty$, and as $\kappa^\infty_i=\lim_\leftarrow\kappa^n_i$, we have $\kappa^\infty_1=\kappa^\infty_2$. As (class of (m_i,Δ,Φ_i)) $=\kappa^\infty_i$(class of (m,Δ,Φ)), the classes of (m_1,Δ_1,Φ_1) and (m_2,Δ_2,Φ_2) modulo equivalence and twists are the same. □

7.2. **Twists of Lie bialgebras.** If (A, m, Δ) is a bialgebra in a symmetric tensor category $\mathcal{C}$, then a twist for A is an element $F \in \mathcal{C}(\mathbf{1}, A^{\otimes 2})^\times$, such that $(\epsilon \otimes \mathrm{id}_A) \circ F = (\mathrm{id}_A \otimes \epsilon) \circ F = \eta$ and $(F \otimes \eta) * \big((\Delta \otimes \mathrm{id}_A) \circ F\big) = (\eta \otimes F) * \big((\mathrm{id}_A \otimes \Delta) \circ F\big)$. Then $(A, m, F * \Delta * F^{-1})$ is again a bialgebra in $\mathcal{C}$, called the twist of A by F.

Let LBA_f be the prop of pairs $(\mathfrak{a}, f)$, where $\mathfrak{a}$ is a Lie bialgebra and $f \in \Lambda^2(\mathfrak{a})$ is a Lie bialgebra twist (see [EH]); it has the same generators as LBA with the additional $f \in \mathrm{LBA}_f(\mathbf{1}, \Lambda^2)$ and the same relations with the additional $\mathrm{Alt}_3 \circ \big((\delta \otimes \mathrm{id}_{\mathbf{id}} + (\mathrm{id}_{\mathbf{id}} \otimes \mu \otimes \mathrm{id}_{\mathbf{id}}) \circ (f \otimes f))\big) = 0$. This prop is $\mathbb{N}^2$-graded if we extend $(\deg_\mu, \deg_\delta)$ of the generators of LBA by $f \mapsto (0,1)$, and it gives rise to a topologial prop $\mathbf{LBA}_f$.

We then have prop morphisms $\kappa_i^0 : \mathrm{LBA} \to \mathrm{LBA}_f$ $(i = 1, 2)$, given by $\kappa_1^0 : \mu, \delta \mapsto \mu, \delta$ and $\kappa_2^0 : \mu, \delta \mapsto \mu, \delta + \mathrm{Alt}_2 \circ (\mathrm{id}_{\mathbf{id}} \otimes \mu) \circ (f \otimes \mathrm{id}_{\mathbf{id}})$, which extend to topological props.

Let $(m, \Delta, \eta, \epsilon)$ be a QF of Lie bialgebras; this is in particular a bialgebra structure on S in $\mathbf{LBA}$. We set $(m_i, \Delta_i, \eta_i, \epsilon_i) := \kappa_i^0(m, \Delta, \eta, \epsilon)$; these are bialgebra structures on S in $\mathbf{LBA}_f$.

The following statement was proved in [EH] when (m, Δ) is an Etingof-Kazhdan quantization functor.

Proposition 7.2. *The bialgebra structures $(m_i, \Delta_i, \eta_i, \epsilon_i)$ are related by equivalence and a bialgebra twist.*

Proof. Let $\pi : \mathrm{QLBA} \to \mathrm{QLBA}/\langle \varphi \rangle \simeq \mathrm{LBA}$ be the canoncial morphism and $\pi_f : \mathrm{QLBA}_f \to \mathrm{LBA}_f$ be the morphism defined by $\mu, \delta, f, \varphi \mapsto \mu, \delta, f, 0$. We have commutative diagrams for $i = 1, 2$

$$\begin{array}{ccc} \mathrm{QLBA} & \overset{\kappa_i}{\rightrightarrows} & \mathrm{QLBA}_f \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_f}\downarrow \\ \mathrm{LBA} & \overset{\kappa_i^0}{\rightrightarrows} & \mathrm{LBA}_f \end{array}$$

which extend to topological props. According to Theorem 4.1, the bialgebra structure (m, Δ) on S in $\mathbf{LBA}$ may be lifted to a quasi-bialgebra structure $(\tilde{m}, \tilde{\Delta}, \Phi)$ on S in $\mathbf{QLBA}$, so $\pi(\tilde{m}, \tilde{\Delta}, \Phi) = (m, \Delta, \eta^{\otimes 3})$. According to Proposition 7.1, the quasi-bialgebra structures $(\tilde{m}_i, \tilde{\Delta}_i, \Phi_i) := \kappa_i(\tilde{m}, \tilde{\Delta}, \Phi)$ $(i = 1, 2)$ on S in $\mathbf{QLBA}_f$ are related by equivalence and twist, i.e., for some $\tilde{F} \in \mathbf{QLBA}_f(\mathbf{1}, S^{\otimes 2})^\times$, $(\tilde{m}_2, \tilde{\Delta}_2, \Phi_2)$ is equivalent to $(\tilde{m}_1, \tilde{F} * \tilde{\Delta}_1 * \tilde{F}^{-1}, (\eta \otimes \tilde{F}) * \big((\mathrm{id}_S \otimes \tilde{\Delta}_1) \circ \tilde{F}\big) * \Phi_1 * \big((\tilde{\Delta}_1 \otimes \mathrm{id}_S) \circ \tilde{F}\big)^{-1} * (\tilde{F} \otimes \eta)^{-1})$. As $\pi_f \circ \kappa_i = \kappa_i^0 \circ \pi$, $\pi_f(\tilde{m}_i, \tilde{\Delta}_i, \Phi_i) = (m_i, \Delta_i, \eta^{\otimes 3})$, so applying π_f to the above equivalence, we obtain (with $F := \pi_f(\tilde{F})$) that (m_2, Δ_2) is equivalent to $(m_1, F * \Delta_1 * F^{-1})$ and that $(\tilde{F} \otimes \eta) * ((\tilde{\Delta}_1 \otimes \mathrm{id}_S) \circ \tilde{F}\big) = (\eta \otimes F) * \big((\mathrm{id}_S \otimes \Delta_1) \circ F\big)$, as wanted. □

Appendix A. Structure of the prop LBA

The following structure theorem of the prop LBA was proved in [E, Pos]. We reformulate here this proof using the language of props. In [EH], we derived Proposition 5.1 from Theorem A.1 below.

Theorem A.1. *If* $F, G \in \mathrm{Ob}(\mathrm{Sch})$*, then the map* $\bigoplus_{Z \in \mathrm{Irr}(\mathrm{Sch})} \mathrm{LCA}(F, Z) \otimes \mathrm{LA}(Z, G) \to \mathrm{LBA}(F, G)$ *induced by composition and the prop morphisms* $\mathrm{LCA} \to \mathrm{LA}$, $\mathrm{LBA} \to \mathrm{LA}$ *is a linear isomorphism.*

Proof. It suffices to prove this when $F, G \in \mathrm{Irr}(\mathrm{Sch})$, and then (using the action of $\mathfrak{S}_n, \mathfrak{S}_m$) for $F = T_n$, $G = T_m$. Using the cocycle relation and the isomorphism of the l.h.s. with $\bigoplus_{z\geq 0}(\mathrm{LCA}(T_n, T_z) \otimes \mathrm{LA}(T_z, T_m))_{\mathfrak{S}_z}$, one proves that the morphism is surjective. We now prove that it is injective. We have

$$\begin{aligned}\mathrm{LBA}(T_n, T_m) &= \bigoplus_{a,b\geq 0 | a-b=n-m} \mathrm{LBA}(T_n, T_m)[a, b] \\ &= \bigoplus_{z\geq \min(n,m)} \mathrm{LBA}(T_n, T_m)[z-m, z-n],\end{aligned}$$

and the morphism is the direct sum over $z \geq \min(n, m)$ of the maps

$$\bigoplus_{Z\in \mathrm{Irr}(\mathrm{Sch})\,||Z|=z} \mathrm{LCA}(T_n, Z) \otimes \mathrm{LA}(Z, T_m) \to \mathrm{LBA}(T_n, T_m)[z-m, z-n].$$

It remains to show that each of the maps is injective.

There is a unique morphism $\mathrm{LBA} \to L(\mathbf{LCA})$ (the generators of LBA are μ, δ, and the generator of LCA is δ_{LCA}), taking μ to $\mu_{\mathrm{free}} : L^{\otimes 2} \to L$ and δ to the unique $\delta_{\mathrm{free}} : L \to L^{\otimes 2}$, such that $\mathbf{id} \to L \stackrel{\delta_{\mathrm{free}}}{\to} L^{\otimes 2}$ is $\delta_{\mathrm{LCA}} : \mathbf{id} \stackrel{\delta_{\mathrm{LCA}}}{\to} \mathbf{id}^{\otimes 2} \to L^{\otimes 2}$ and $\delta \circ \mu_{\mathrm{free}} = ((\mu_{\mathrm{free}} \otimes \mathrm{id}_L) \circ (\mathrm{id}_L \otimes \beta_{L,L}) + \mathrm{id}_L \otimes \mu_{\mathrm{free}}) \circ (\delta_{\mathrm{free}} \otimes \mathrm{id}_L) + (\mu_{\mathrm{free}} \otimes \mathrm{id}_L + (\mathrm{id}_L \otimes \mu_{\mathrm{free}}) \circ (\beta_{L,L} \otimes \mathrm{id}_L)) \circ (\mathrm{id}_L \otimes \delta_{\mathrm{free}})$. The prop LCA is $\mathbb{Z}$-graded, with $\deg \delta_{\mathrm{LCA}} = 1$. Then the morphism $\mathrm{LBA} \to L(\mathbf{LCA})$ is compatible with the morphism $\mathbb{Z}^2 \to \mathbb{Z}$, $(1,0) \mapsto 0$, $(0,1) \mapsto 1$.

We then have maps $\mathrm{LBA}(T_n, T_m) \to L(\mathbf{LCA})(T_n, T_m) = \mathbf{LCA}(L^{\otimes n}, L^{\otimes m}) \to \mathbf{LCA}(T_n, L^{\otimes m})$, where the last map is induced by $\mathbf{id} \to L$, which restrict to $\mathrm{LBA}(T_n, T_m)[z-m, z-n] \to \mathbf{LCA}(T_n, L^{\otimes m})[z-n] = \mathrm{LCA}(T_n, (L^{\otimes m})_z)$, where the index z denotes the (Schur functor) degree z part. □

Lemma A.1. *If X is any prop and $F \in \mathrm{Ob}(\mathrm{Sch})$, we have an isomorphism $X(F, (L^{\otimes m})_z) \simeq \bigoplus_{Z\in \mathrm{Irr}(\mathrm{Sch}), |Z|=z} X(F, Z) \otimes \mathrm{LA}(Z, T_m)$.*

Proof of Lemma. We have isomorphisms $\mathrm{LA}(T_z, \mathbf{id}) \simeq$ multilinear part of the free Lie algebra in z ordered generators $\simeq \mathrm{Sch}(T_z, L_z)$. So if $|Z| = z$, $\mathrm{LA}(Z, \mathbf{id}) \simeq \mathrm{Sch}(Z, L_z)$, which may be expressed as $L_z = \bigoplus_{|Z|=z} \mathrm{LA}(Z, \mathbf{id}) \otimes Z$.

So

$$\begin{aligned}X(F, (L^{\otimes m})_z) &= \bigoplus_{z_1+\ldots+z_m=z} X(F, \bigotimes_{i=1}^{m} L_{z_i}) \\ &= \bigoplus_{|Z_1|+\ldots+|Z_m|=z} X(F, \bigotimes_{i=1}^{m} Z_i) \otimes (\bigotimes_{i=1}^{m} \mathrm{LA}(Z_i, \mathbf{id})) \\ &= \bigoplus_{|Z_1|+\ldots+|Z_m|=z, |Z|=z} X(F, Z) \otimes \mathrm{Sch}(Z, \bigotimes_{i=1}^{m} Z_i) \otimes (\bigotimes_{i=1}^{m} \mathrm{LA}(Z_i, \mathbf{id})) \\ &= \bigoplus_{|Z|=z} X(F, Z) \otimes \mathrm{LA}(Z, T_m),\end{aligned}$$

where the last equality follows from

$$\mathrm{LA}(Z, T_m) = \bigoplus_{Z_1,\ldots,Z_m\in \mathrm{Irr}(\mathrm{Sch})\,|\,\sum_i |Z_i|=z} \mathrm{Sch}(Z, \bigotimes_i Z_i) \otimes \bigotimes_i \mathrm{LA}(Z_i, \mathbf{id}),$$

for $Z \in \mathrm{Ob}(\mathrm{Sch})$ (see [EH]). □

End of proof of Theorem. We have constructed a map $\mathrm{LBA}(T_n, T_m)[z-m, z-n] \to \bigoplus_{Z\in\mathrm{Irr(Sch)},|Z|=z} \mathrm{LCA}(T_n, Z) \otimes \mathrm{LA}(Z, T_m)$, and one proves that is a section of the morphism $\bigoplus_{Z\in\mathrm{Irr(Sch)}\,||Z|=z} \mathrm{LCA}(T_n, Z)\otimes\mathrm{LA}(Z, T_m) \to \mathrm{LBA}(T_n, T_m)[z-m, z-n]$, which is therefore injective. □

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Institut de Recherche Matématique Avancée (CNRS) et Université de Strasbourg, 7, rue René Descartes, F-67084 Strasbourg, France

E-mail address: enriquez@math.u-strasbg.fr

Institut de Mathématiques, Université Montpellier 2, Place E. Bataillon, F-34095 Montpellier, France

E-mail address: ghalbout@math.univ-montp2.fr