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PROOF OF ALDOUS' SPECTRAL GAP CONJECTURE

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1. Introduction

Spectral gap analysis plays an important role in the study of the convergence to equilibrium of reversible Markov chains. We begin by reviewing some well-known facts about Markov chains and their spectra. For more details we refer to [2].

1.1. **Finite state, continuous time Markov chains.** Let us consider a continuous time Markov chain $Z = (Z_t)_{t \geqslant 0}$ with finite state space S and transition rates $(q_{i,j} : i \neq j \in S)$ such that $q_{i,j} \geqslant 0$. We will always assume that the Markov chain is irreducible and satisfies

$$q_{i,j} = q_{j,i} \quad \text{for all } i \neq j.$$

Such a Markov chain is reversible with respect to the uniform distribution ν on S, which is the unique stationary distribution of the chain. The infinitesimal generator $\mathcal{L}$ of the Markov chain is defined by

$$\mathcal{L}g(i) = \sum_{j \in S} q_{i,j}(g(j) - g(i)),$$

where $g : S \to \mathbb{R}$ and $i \in S$. The matrix corresponding to the linear operator $\mathcal{L}$ is the transition matrix $Q = (q_{i,j})_{i,j}$, where $q_{i,i} := -\sum_{j \neq i} q_{i,j}$, and the corresponding quadratic form is

$$\sum_{i \in S} g(i)\mathcal{L}g(i) = \sum_{i,j \in S} q_{i,j} g(i)(g(j) - g(i)) = -\frac{1}{2} \sum_{i,j \in S} q_{i,j}(g(j) - g(i))^2.$$

Thus, $-\mathcal{L}$ is positive semi-definite and symmetric, which implies that its spectrum is of the form $\mathrm{Spec}(-\mathcal{L}) = \{\lambda_i : 0 \leqslant i \leqslant |S| - 1\}$, where

$$0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_{|S|-1}.$$

The spectral gap λ_1 is characterized as the largest constant λ such that

$$\frac{1}{2} \sum_{i,j \in S} q_{i,j}(g(j) - g(i))^2 \geqslant \lambda \sum_{i \in S} g(i)^2 \tag{1.1}$$

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for all $g : V \to \mathbb{R}$ with $\sum_i g(i) = 0$. The significance of λ_1 is its interpretation as the asymptotic rate of convergence to the stationary distribution:

$$P_i(Z_t = j) = \nu(\{j\}) + a_{i,j}e^{-\lambda_1 t} + o(e^{-\lambda_1 t}) \qquad \text{for } t \to \infty,$$

where typically $a_{i,j} \neq 0$ (and more precisely $a_{i,i} > 0$ for some i). For this reason $\frac{1}{\lambda_1}$ is sometimes referred to as the relaxation time of the Markov chain, and it is desirable to have an effective way of calculating λ_1. Aldous' conjecture relates the spectral gap of the random walk on a finite graph to that of more complicated processes on the same graph. This can be very important in applications, since generally speaking it is easier to compute or estimate (e.g. via isoperimetric inequalities) the spectral gap of the random walk than that of the other processes considered, which have much larger state spaces.

We say that the Markov chain with state space S_2 and generator $\mathcal{L}_2$ is a subprocess of the chain with state space S_1 and generator $\mathcal{L}_1$ if there is a contraction of S_1 onto S_2, i.e. if there is a surjective map $\pi : S_1 \to S_2$ such that

$$\mathcal{L}_1(f \circ \pi) = (\mathcal{L}_2 f) \circ \pi \quad \text{ for all } f : S_2 \to \mathbb{R}. \tag{1.2}$$

In this case, suppose that f is an eigenfunction of $-\mathcal{L}_2$ with eigenvalue λ. Then $-\mathcal{L}_1(f \circ \pi) = (-\mathcal{L}_2 f) \circ \pi = \lambda f \circ \pi$ and $f \circ \pi \neq 0$ for $f \neq 0$, so $f \circ \pi$ is an eigenfunction of $-\mathcal{L}_1$ with the same eigenvalue λ. Thus,

$$\mathrm{Spec}(-\mathcal{L}_2) \subset \mathrm{Spec}(-\mathcal{L}_1),$$

and, in particular, the spectral gap of the first process is smaller than or equal to that of the second process. Identity (1.2) is an example of a so-called intertwining relation; see e.g. [6] for more details on such relations and their applications.

1.2. **Random walk and interchange process on a weighted graph.** Let $G = (V, E)$ be an undirected complete graph on n vertices; without loss of generality we assume that its vertex set is $V = \{1, \ldots, n\}$. Furthermore G is a weighted graph in that we are given a collection of edge weights (or conductances) $c_{xy} \geqslant 0$, for $xy = \{x, y\} \in E$. Since we want the processes defined below to be irreducible, we will assume that the skeleton graph, i.e. the set of edges xy where $c_{xy} > 0$, is connected. If we want to stress the dependence of one of the processes described below on the underlying weighted graph, we will write $\mathcal{L}(G)$ and $\lambda_1(G)$ for its generator and gap. Finally, we note that considering complete graphs only is no loss of generality, since edges with weight 0 can be thought of as being "absent".

1.2.1. *Random walk.* The (1-particle) random walk on G is the Markov chain in which a single particle jumps from vertex $x \in V$ to $y \neq x$ at rate c_{xy}; see Figure 1.

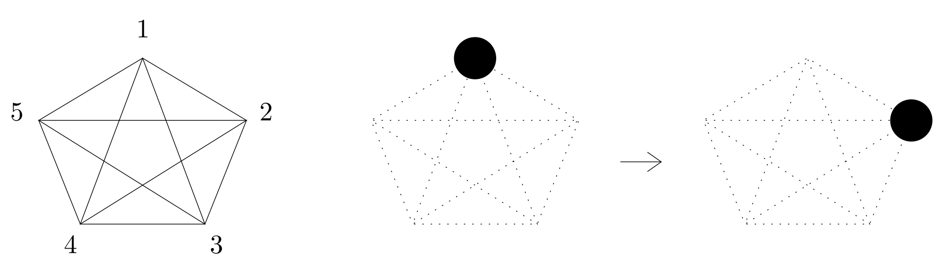

FIGURE 1. Random walk on $V = \{1, 2, 3, 4, 5\}$. The picture shows the underlying graph and a transition from state 1 to state 2.

Formally, its state space is $S^{RW} = V = \{1, 2, \dots, n\}$ and its generator is defined by

$$\mathcal{L}^{RW} f(x) = \sum_{y \neq x} c_{xy}(f(y) - f(x)), \quad \text{for } f : V \to \mathbb{R},\ x \in V.$$

By Section 1.1, $-\mathcal{L}^{RW}$ has $n = |S^{RW}|$ nonnegative eigenvalues and a positive spectral gap $\lambda_1^{RW} > 0$.

1.2.2. *Interchange process.* In the interchange process, a state is an assignment of n labeled particles to the vertices of G in such a way that each vertex is occupied by exactly one particle. The transition from a state η to a state η^{xy} (occurring with rate c_{xy}) interchanges the particles at vertices x and y; see Figure 2.

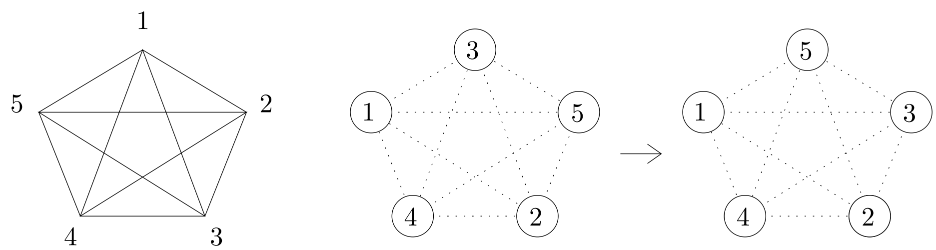

FIGURE 2. Interchange process on $V = \{1, 2, 3, 4, 5\}$. The picture shows the underlying graph and a transition from state $\eta = \left(\begin{smallmatrix}1&2&3&4&5\\5&3&1&4&2\end{smallmatrix}\right)$ to $\eta^{1,2} = \eta\tau_{1,2} = \left(\begin{smallmatrix}1&2&3&4&5\\5&3&2&4&1\end{smallmatrix}\right)$. Note that in this notation the first row refers to the labels.

For a formal definition, let $\mathcal{X}_n$ denote the set of permutations of $V = \{1, \dots, n\}$, and for $\eta \in \mathcal{X}_n$ and $xy \in E$ let $\eta^{xy} = \eta\tau_{xy}$, where $\tau_{xy} \in \mathcal{X}_n$ is the transposition of x and y. The interchange process on G is the Markov chain with state space $S^{IP} = \mathcal{X}_n$ and generator

$$\mathcal{L}^{IP} f(\eta) = \sum_{xy \in E} c_{xy}(f(\eta^{xy}) - f(\eta)), \quad \text{where } f : S^{IP} \to \mathbb{R},\ \eta \in S^{IP}.$$

We use η_x to denote the label of the particle at x, while $\xi_i = \xi_i(\eta)$ will be used to denote the position of the particle labeled i. By Section 1.1, $-\mathcal{L}^{IP}$ has $|S^{IP}| = n!$ nonnegative eigenvalues and a positive spectral gap $\lambda_1^{IP} > 0$. The random walk can be obtained as a sub-process of the interchange process by ignoring all particles apart from the one with label 1; more precisely the map $\pi : S^{IP} \to S^{RW}$, $\pi(\eta) := \xi_1(\eta)$ is a contraction in the sense of (1.2). Thus,

$$\operatorname{Spec}(-\mathcal{L}^{RW}) \subset \operatorname{Spec}(-\mathcal{L}^{IP}),$$

and in particular,

$$\lambda_1^{IP} \leqslant \lambda_1^{RW}. \tag{1.3}$$

1.3. **Main result.** Our main result states that inequality (1.3) is, in fact, an equality:

Theorem 1.1. *For all weighted graphs G, the interchange process and the random walk have the same spectral gap:*

$$\lambda_1^{IP}(G) = \lambda_1^{RW}(G). \tag{1.4}$$

A weaker form of Theorem 1.1 involving only unweighted graphs had been conjectured by Aldous around 1992, and since then it has been mostly referred to as Aldous' spectral gap conjecture in the literature. Related observations can be found in Diaconis and Shahshahani's paper [9] and in the comparison theory developed by Diaconis and Saloff-Coste [8].

The problem has received a lot of attention in recent years—the conjecture was stated as an open problem on David Aldous' web page [1] and in the influential monographs [2, 16]. In the meantime, various special cases have been obtained. The first class of graphs that was shown to satisfy the conjecture is the class of unweighted complete graphs (i.e. $c_{xy} = 1$ for all $xy \in E$). Diaconis and Shahshahani computed all eigenvalues of the interchange process in this case using the irreducible representations of the symmetric group [9]. Similar results were obtained for unweighted star graphs in [12]. Recently, remarkable work of Cesi pushed this algebraic approach further to obtain the conjecture for all unweighted complete multipartite graphs [4].

An alternative approach based on recursion was proposed by Handjani and Jungreis [13] (see also Koma and Nachtergaele [15] for similar results) who proved the conjecture for all weighted trees. The same ideas were recently used by Conomos and Starr [19] and Morris [18] to obtain an asymptotic version of the conjecture for boxes in the lattice $\mathbb{Z}^d$ with unweighted edges. The basic recursive approach in [13] has been recently rephrased in purely algebraic terms; see [5, Lemma 3.1].

In order to prove Theorem 1.1, we develop a general recursive approach based on the idea of network reduction; see Section 2. The method, inspired by the theory of resistive networks, allows us to reduce the proof of the theorem to the proof of an interesting comparison inequality for random transposition operators on different weighted graphs; see Theorem 2.3.

After a preliminary version [3] of this paper appeared, we learned that the same recursive strategy had been discovered around the same time independently, and from a slightly different perspective, by Dieker [10]. The comparison inequality alluded to above was conjectured to hold in both [3] and [10].

The comparison inequality will be proved in Section 3. The main idea for this proof is a decomposition of the associated matrix into a covariance matrix and a correction matrix (a Schur complement). A delicate analysis based on block decompositions corresponding to suitable cosets of the permutation group reveals that the correction matrix is nonnegative definite.

Some immediate consequences of Theorem 1.1 for other natural Markov chains associated to finite weighted graphs are discussed in Section 4.

We end this introductory section with a collection of known properties of the spectrum of the interchange process that can be deduced from the algebraic approach. We refer to [9, 12, 4] and references therein for more details. These facts are not needed in what follows and the reader may safely jump to the next section. However, we feel that the algebraic point of view provides a natural decomposition of the spectrum that is worth mentioning.

1.4. **Structure of the spectrum of $-\mathcal{L}^{IP}$.** In Section 1.2.2 we saw that

$$\mathrm{Spec}(-\mathcal{L}^{RW}) \subset \mathrm{Spec}(-\mathcal{L}^{IP}).$$

One can go a little further and show that if $0 = \lambda_0^{RW} < \lambda_1^{RW} \leqslant \cdots \leqslant \lambda_{n-1}^{RW}$ are the eigenvalues of $-\mathcal{L}^{RW}$, then for $k \geqslant 0$ and $1 \leqslant i_1 < \cdots < i_k \leqslant n-1$,

$$\lambda_{i_1}^{RW} + \cdots + \lambda_{i_k}^{RW} \in \mathrm{Spec}(-\mathcal{L}^{IP})\,. \tag{1.5}$$

The corresponding eigenfunction is the antisymmetric product of the k one-particle eigenfunctions of $\lambda_{i_1}^{RW}, \dots, \lambda_{i_k}^{RW}$. In particular, the eigenvalue

$$\lambda_1^{RW} + \cdots + \lambda_{n-1}^{RW} = \mathrm{Tr}(-\mathcal{L}^{RW}) = 2 \sum_{xy \in E} c_{xy} \tag{1.6}$$

is associated with functions that are antisymmetric in all particles, i.e. multiples of the alternating function $h(\eta) = \mathrm{sign}(\eta)$. (This also follows directly from $h(\eta^{xy}) - h(\eta) = -2h(\eta)$.) From the representation theory of the symmetric group one can compute (see below) the multiplicity of all eigenvalues of the form (1.5), and one finds that the overwhelming majority (for large n) of the spectrum of $-\mathcal{L}^{IP}$ are not of this form.

The vector space of functions $f : \mathcal{X}_n \to \mathbb{R}$ is equivalent to a direct sum $\bigoplus_\alpha \mathcal{H}_\alpha$, where α ranges over all (equivalence classes of the) irreducible representations of the symmetric group. Since the latter are in one-to-one correspondence with the partitions of n, one can identify α with a Young diagram $\alpha = (\alpha_1, \alpha_2, \dots)$, where the α_i form a nonincreasing sequence of nonnegative integers such that $\sum_i \alpha_i = n$.

Each subspace $\mathcal{H}_\alpha$ is in turn a direct sum $\mathcal{H}_\alpha = \bigoplus_{j=1}^{d_\alpha} \mathcal{H}_\alpha^j$, of subspaces $\mathcal{H}_\alpha^j$, each of dimension d_α, where the positive integer d_α is the dimension of the irreducible representation α. In particular, the numbers d_α satisfy $\sum_\alpha (d_\alpha)^2 = n!$. The subspaces $\mathcal{H}_\alpha^j$ are invariant for the action of the generator $-\mathcal{L}^{IP}$, so that $-\mathcal{L}^{IP}$ can be diagonalized within each $\mathcal{H}_\alpha^j$. Subspace $\mathcal{H}_\alpha^i$ will produce d_α eigenvalues $\lambda_k(\alpha)$, $k = 1, \dots, d_\alpha$. Some of these may coincide if the weights have suitable symmetries (for instance, if G is the complete graph with $c_{xy} = 1$ for all $xy \in E$, then they all coincide and $-\mathcal{L}^{IP}$ is a multiple of the identity matrix in each $\mathcal{H}_\alpha$; cf. [9]) but in the general weighted case they will be distinct. On the other hand, for a given α, the eigenvalues coming from $\mathcal{H}_\alpha^i$ are identical to those coming from $\mathcal{H}_\alpha^j$, for all $i, j = 1, \dots, d_\alpha$, so that each eigenvalue $\lambda_k(\alpha)$ will appear with multiplicity d_α in the spectrum of $-\mathcal{L}^{IP}$. Moreover, using known expressions for the characters of transpositions, one can compute explicitly the sum

$$\sum_{k=1}^{d_\alpha} \lambda_k(\alpha)\,,$$

for every irreducible representation α, as a function of the edge weights. For instance, when α is the partition $(n-1, 1, 0, \dots)$, which has $d_\alpha = n-1$, one obtains the relation (1.6). The trivial partition $(n, 0, \dots)$ has dimension 1 and the only eigenvalue is 0. This is the space of constant functions. Similarly, the alternating partition $(1^n, 0 \dots)$ (n ones and then all zeros), has dimension 1 and the only eigenvalue is $2\sum_{xy \in E} c_{xy}$. It can be shown that the eigenvalues of the form (1.5) come from the L-shaped partitions $\alpha = (n-k, 1^k, 0, \dots)$, each with dimension $d_\alpha = \binom{n-1}{k}$. So the total number of eigenvalues of the form (1.5) is $\sum_{k=0}^{n-1} \binom{n-1}{k}^2 = \binom{2(n-1)}{n-1}$.

Finally, using known relationships between conjugate irreducible representations (see e.g. [14, 2.1.8], [5, (2.12)]), one can show that the spectrum of $-\mathcal{L}^{IP}$ can be decomposed into pairs of eigenvalues λ, λ' such that

$$\lambda + \lambda' = 2 \sum_{xy \in E} c_{xy} ,$$

where λ, λ' are associated with conjugate Young diagrams.

2. A recursive approach based on network reduction

Given a weighted graph $G = (V, E)$ as above and a point $x \in V$, we consider the *reduced network* obtained by removing the vertex x. This gives a new graph G_x with vertex set $V_x := V \setminus \{x\}$, edge set $E_x = \{yz \in E : y, z \neq x\}$, and edge conductances $\widetilde{c}_{yz} \geqslant c_{yz}$ defined by

$$\widetilde{c}_{yz} = c_{yz} + c^{*,x}_{yz} , \qquad c^{*,x}_{yz} := \frac{c_{xy}c_{xz}}{\sum_{w \in V_x} c_{xw}} , \tag{2.1}$$

for $yz \in E_x$. We refer to G_x as the reduction of G at x or simply as the reduced graph at x. This is the general version of more familiar network reductions such as series resistance (from 3 to 2 vertices) or star-triangle transformations (from 4 to 3 vertices); see Figure 3. We refer to [11, 17, 2] for the classical probabilistic point of view on electric networks.

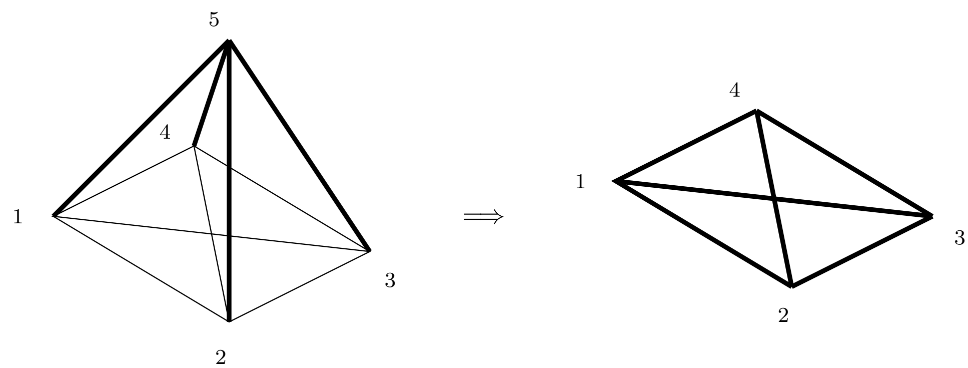

Figure 3. Reduction of a 5-vertex graph to a 4-vertex graph at $x = 5$.

2.1. **Random walk on the reduced network.**

We first show that the spectral gap of the random walk on the reduced network is not smaller than the original random walk spectral gap:

Proposition 2.1. *The spectral gaps of the random walks on a weighted graph G and the corresponding reduced graph G_x satisfy*

$$\lambda_1^{RW}(G_x) \geqslant \lambda_1^{RW}(G) .$$

Proof. We will use the shorthand notation $L = \mathcal{L}^{RW}(G)$ and $L_x = \mathcal{L}^{RW}(G_x)$ for the generators of the two random walks. We first note that, for any graph G, $\lambda_1^{RW}(G)$ can be characterized as the largest constant λ such that

$$\sum_{z \in V} (Lg(z))^2 \geqslant -\lambda \sum_{z \in V} g(z)Lg(z) \tag{2.2}$$

holds for all $g : V \to \mathbb{R}$. To see this, observe that, for any $g, h : V \to \mathbb{R}$, $\sum_{z\in V} h(z)Lg(z) = \sum_{z\in V} g(z)Lh(z)$. Thus, taking $h = Lg$, the left hand side of (2.2) coincides with the quadratic form

$$\sum_{z\in V} g(z)L^2 g(z)\,,$$

and (2.2) says that $L^2 + \lambda L$ is nonnegative definite. Taking a basis which makes L diagonal, one sees that this holds iff $\lambda \leqslant \lambda_1^{RW}(G)$.

To prove the proposition, take a function $g : V \to \mathbb{R}$ harmonic at x, i.e. such that $Lg(x) = 0$. Then

$$g(x) = \frac{\sum_{y\in V_x} c_{xy} g(y)}{\sum_{w\in V_x} c_{xw}}\,. \tag{2.3}$$

For any $z \in V_x$, from (2.3) we have

$$\begin{aligned} Lg(z) &= \sum_{y\in V_x} c_{zy}[g(y) - g(z)] + c_{zx}[g(x) - g(z)] \\ &= \sum_{y\in V_x} \left(c_{zy} + c^{*,x}_{zy}\right)[g(y) - g(z)]\,. \end{aligned}$$

In other words

$$Lg(z) = \begin{cases} L_x g(z), & z \in V_x, \\ 0, & z = x. \end{cases}$$

Applying (2.2) to this function, we have

$$\begin{aligned} \sum_{z\in V_x} (L_x g(z))^2 = \sum_{z\in V} (Lg(z))^2 &\geqslant -\lambda_1^{RW}(G)\sum_{z\in V} g(z)Lg(z) \\ &= -\lambda_1^{RW}(G)\sum_{z\in V_x} g(z)L_x g(z)\,. \end{aligned}$$

Since the function g is arbitrary on V_x, using (2.2) again, this time for the graph G_x, we obtain $\lambda_1^{RW}(G_x) \geqslant \lambda_1^{RW}(G)$. □

Proposition 2.1 generalizes the observation in [13] that if G is a graph with a vertex x of degree 1 (i.e. only one edge out of x has positive weight), then the spectral gap of the random walk cannot decrease when we cancel x and remove the only edge connecting it to the rest of G. (In that case $\widetilde{c}_{yz} = c_{yz}$ since x has degree 1.)

We end this subsection with a side remark on further relations between the generators $L = \mathcal{L}^{RW}(G)$ and $L_x = \mathcal{L}^{RW}(G_x)$. When we remove a vertex, it is interesting to compare the energy corresponding to the removed branches with the energy coming from the new conductances. The following identity can be obtained with a straightforward computation.

Lemma 2.2. *For any fixed $x \in V$ and any $g : V \to \mathbb{R}$,*

$$\sum_{y\in V_x} c_{xy}[g(y) - g(x)]^2 = \sum_{yz\in E_x} c^{*,x}_{yz}[g(y) - g(z)]^2 + \frac{1}{\sum\limits_{y\neq x} c_{xy}}\,(Lg(x))^2\,.$$

Consider the operator $\widetilde{L}_x$ defined by $\widetilde{L}_x g(x) = 0$ and $\widetilde{L}_x g(z) = L_x g(z)$ for $z \neq x$, where $g : V \to \mathbb{R}$. Then $\widetilde{L}_x$ is the generator of the random walk on $G_x \cup \{x\}$, where x is an isolated vertex. Lemma 2.2 implies that the quadratic form of $-\widetilde{L}_x$ is dominated by the quadratic form of $-L$. It follows from the Courant-Fisher min-max theorem that if $\widetilde{\lambda}_0 \leqslant \cdots \leqslant \widetilde{\lambda}_{n-1}$ denote the eigenvalues of $-\widetilde{L}_x$, then $\widetilde{\lambda}_i \leqslant \lambda_i^{RW}(G)$, $i = 0, \dots, n-1$. Note that this is not in contradiction to the result in Proposition 2.1 since, due to the isolated vertex x, one has $\widetilde{\lambda}_0 = \widetilde{\lambda}_1 = 0$ and $\widetilde{\lambda}_{k+1} = \lambda_k^{RW}(G_x)$, $k = 1, \dots, n-2$. While the bound in Proposition 2.1 will be sufficient for our purposes, it is worth pointing out that, as observed in [10], at this point standard results on interlacings can be used to prove the stronger statement

$$\lambda_j^{RW}(G) \leqslant \lambda_j^{RW}(G_x) \leqslant \lambda_{j+1}^{RW}(G)\,, \quad j = 1, \dots, n-2\,.$$

2.2. **Octopus inequality.** The following theorem summarizes the main technical ingredient we shall need. Here ν is the uniform probability measure on all permutations $\mathcal{X}_n$, and we use the notation $\nu[f] = \int f\, d\nu$. The gradient ∇ is defined by

$$\nabla_{xy} f(\eta) = f(\eta^{xy}) - f(\eta)\,.$$

Theorem 2.3. *For any weighted graph G on $|V| = n$ vertices, for every $x \in V$ and $f : \mathcal{X}_n \to \mathbb{R}$,*

$$\sum_{y \in V_x} c_{xy}\, \nu[(\nabla_{xy} f)^2] \geqslant \sum_{yz \in E_x} c_{yz}^{*,x}\, \nu[(\nabla_{yz} f)^2]\,. \tag{2.4}$$

Note that if $f(\eta) = g(\xi_1)$ is a function of one particle, then a simple computation gives

$$\nu[(\nabla_{uv} f)^2] = \frac{2}{n}(g(u) - g(v))^2\,, \qquad uv \in E\,,$$

so that this special case of Theorem 2.3 is contained in Lemma 2.2. The identity in Lemma 2.2 also shows that in this case the inequality is saturated by functions that are harmonic at x. On the other hand, the general case represents a nontrivial comparison inequality between a weighted star graph and its complement, with weights defined by (2.1). Inspired by its tentacular nature, we refer to the bound (2.4) as the *octopus* inequality. We will give a proof of Theorem 2.3 in Section 3.

2.3. **Reformulation of the conjecture.** We shall use the following convenient notation: As above let ν denote the uniform probability measure $\mathcal{X}_n$, ∇ the gradient, and b a generic edge, whose weight is denoted c_b. In this way $\mathcal{L}^{IP} = \sum_b c_b \nabla_b$ and the Dirichlet form $-\nu[f\mathcal{L}^{IP} f]$ is

$$\mathcal{E}(f) = \frac{1}{2}\sum_b c_b\, \nu[(\nabla_b f)^2]\,.$$

The spectral gap λ_1^{IP} is the best constant λ so that for all $f : \mathcal{X}_n \to \mathbb{R}$,

$$\mathcal{E}(f) \geqslant \lambda\, \mathrm{Var}_\nu(f)\,, \tag{2.5}$$

where $\mathrm{Var}_\nu(f) = \nu[f^2] - \nu[f]^2$ is the variance of f w.r.t. ν. In order to get some hold on the eigenvalues of the interchange process that are not eigenvalues of the random walk, we introduce the vector space

$$\begin{aligned}\mathcal{H} &= \{f : \mathcal{X}_n \to \mathbb{R} : \nu[f \mid \xi_i] = 0 \text{ for all } i \in V\}\\ &= \{f : \mathcal{X}_n \to \mathbb{R} : \nu[f \mid \eta_x] = 0 \text{ for all } x \in V\},\end{aligned}$$

where $\nu[\cdot\,|\,\xi_i]$ and $\nu[\cdot\,|\,\eta_x]$ are the conditional expectations given the position of the particle labeled i and given the label of the particle at x, respectively. The equality in the definition of $\mathcal{H}$ is a consequence of

$$\nu[\cdot\,|\,\xi_i](\eta) = \nu[\cdot\,|\,\xi_i = x] = \nu[\cdot\,|\,\eta_x = i] = \nu[\cdot\,|\,\eta_x](\eta)\,,$$

where $\eta \in \mathcal{X}_n$ is such that $\xi_i(\eta) = x$. Note that for every i,

$$\nu[\mathcal{L}^{IP} f\,|\,\xi_i = x] = \sum_{y \neq x} c_{xy}\left(\nu[f\,|\,\xi_i = y] - \nu[f\,|\,\xi_i = x]\right)\,,$$

for all $f : \mathcal{X}_n \to \mathbb{R}$ and $x \in V$. In particular, $\mathcal{H}$ is an invariant subspace for $-\mathcal{L}^{IP}$, and if $f \notin \mathcal{H}$ is an eigenfunction of $-\mathcal{L}^{IP}$ with eigenvalue λ, then $\nu[f\,|\,\xi_i] \neq 0$ for some i, and $\nu[f\,|\,\xi_i = x]$, $x \in V$, is an eigenfunction of $-\mathcal{L}^{RW}$ with the same eigenvalue λ. It follows that $\mathcal{H}$ contains all eigenfunctions corresponding to eigenvalues in $\mathrm{Spec}(-\mathcal{L}^{IP})\backslash\mathrm{Spec}(-\mathcal{L}^{RW})$. Therefore, if $\mu_1^{IP}(G)$ denotes the smallest eigenvalue of $-\mathcal{L}^{IP}$ associated to functions in $\mathcal{H}$ (i.e. the best constant λ in (2.5) restricting to functions $f \in \mathcal{H}$), then for every graph G one has

$$\lambda_1^{IP}(G) = \min\{\lambda_1^{RW}(G), \mu_1^{IP}(G)\}\,.$$

The assertion $\lambda_1^{IP}(G) = \lambda_1^{RW}(G)$ of Theorem 1.1 becomes then equivalent to

$$\mu_1^{IP}(G) \geqslant \lambda_1^{RW}(G)\,. \tag{2.6}$$

In the rest of this section we show how the network reduction idea, assuming the validity of Theorem 2.3, yields a proof of Theorem 1.1.

2.4. **Proof of Theorem 1.1.** We use the notation from the previous sections. In particular we write $\lambda_1^{RW}(G_x)$ and $\lambda_1^{IP}(G_x)$ for the spectral gaps of the random walk and the interchange process in the network reduced at x. Let us first show that Theorem 2.3 implies an estimate of $\mu_1^{IP}(G)$.

Proposition 2.4. *For an arbitrary weighted graph G*

$$\mu_1^{IP}(G) \geqslant \max_{x \in V} \lambda_1^{IP}(G_x)\,. \tag{2.7}$$

Proof. Let $f \in \mathcal{H}$ and $x \in V$. Since $\nu[f\,|\,\eta_x] = 0$, we have

$$\nu[f^2] = \mathrm{Var}_\nu(f) = \nu[\mathrm{Var}_\nu(f\,|\,\eta_x)]\,,$$

where $\mathrm{Var}_\nu(f\,|\,\eta_x)$ is the variance w.r.t. $\nu[\cdot\,|\,\eta_x]$. For a fixed value of η_x, $\nu[\cdot\,|\,\eta_x]$ is the uniform measure on the permutations on $V_x = V \setminus \{x\}$. Therefore using the spectral gap bound (2.5) on the graph G_x, we have

$$\lambda_1^{IP}(G_x)\,\mathrm{Var}_\nu(f\,|\,\eta_x) \leqslant \frac{1}{2}\sum_{b\colon b \not\ni x}(c_b + c_b^{*,x})\nu[(\nabla_b f)^2\,|\,\eta_x]\,,$$

with $c_b^{*,x}$ defined by (2.1). Taking the ν-expectation, we obtain

$$\lambda_1^{IP}(G_x)\,\nu[f^2] \leqslant \frac{1}{2}\sum_{b\colon b \not\ni x}(c_b + c_b^{*,x})\nu[(\nabla_b f)^2]\,.$$

From Theorem 2.3,

$$\sum_{b\colon b \not\ni x} c_b^{*,x}\,\nu[(\nabla_b f)^2] \leqslant \sum_{b\colon b \ni x} c_b\,\nu[(\nabla_b f)^2]\,.$$

Therefore,

$$\lambda_1^{IP}(G_x)\,\nu[f^2] \leqslant \mathcal{E}(f)\,. \tag{2.8}$$

Since $x \in V$ and $f \in \mathcal{H}$ were arbitrary, this proves that, for every $x \in V$, $\mu_1^{IP}(G) \geqslant \lambda_1^{IP}(G_x)$, establishing the inequality (2.7). $\square$

Propositions 2.1 and 2.4 allow us to conclude the proof by recursion. Indeed, note that $\lambda_1^{IP}(G) = \lambda_1^{RW}(G)$ is trivially true when $G = b$ is a single weighted edge b. (When $n = 2$, the random walk and the interchange process are the same 2-state Markov chain.) If G is a weighted graph on n vertices, we assume that $\lambda_1^{IP}(G') = \lambda_1^{RW}(G')$ holds on every weighted graph G' with $n-1$ vertices, in particular on G_x. Therefore

$$\mu_1^{IP}(G) \geqslant \max_{x\in V} \lambda_1^{IP}(G_x) = \max_{x\in V} \lambda_1^{RW}(G_x) \geqslant \lambda_1^{RW}(G),$$

where we also have used Propositions 2.1 and 2.4. Thus we have shown (2.6), which is equivalent to $\lambda_1^{IP}(G) = \lambda_1^{RW}(G)$.

3. Proof of the octopus inequality

For the proof of Theorem 2.3 we slightly change our notation as follows. We set $V = \{0, 1, \dots, n-1\}$ and $x = 0$. The only rates appearing in (2.4) are c_{0i}, so we set

$$c_i := c_{0i} \quad \text{for } 1 \leqslant i\,, \qquad c_0 := -\sum_{i\,\geqslant\,1} c_i,$$
$$\text{and} \qquad c := \sum_{1\,\leqslant\, i\,\leqslant\, n-1} c_i^2 + \sum_{1\,\leqslant\, i<j\,\leqslant\, n-1} c_i c_j\,.$$

Note that $c_0 < 0$ and

$$\sum_{i\,\geqslant\,0} c_i = 0, \quad c = -\sum_{0\,\leqslant\, i<j} c_i c_j, \quad \text{and} \quad c_{ij}^{*,0} = -\frac{c_i c_j}{c_0}. \tag{3.1}$$

Using this shorthand notation, the octopus inequality (2.4) simplifies to

$$-\sum_{0\,\leqslant\, i<j} c_i c_j \sum_\eta (f(\eta\tau_{ij}) - f(\eta))^2 \geqslant 0, \tag{3.2}$$

where τ_{ij} denotes the transposition of $i, j \in V$, i.e. $\eta\tau_{ij} = \eta^{ij}$. Thus it suffices to show that the matrix C defined by

$$C_{\eta,\eta'} = \begin{cases} c & \text{if } \eta = \eta', \\ c_i c_j & \text{if } \eta\tau_{ij} = \eta', \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

is positive semi-definite for every n and all rates $c_1, \dots, c_{n-1} \geqslant 0$.

3.1. **Decomposition of the matrix C.** In the following we write $A \geqslant B$ if the same inequality holds for the corresponding quadratic forms, i.e. if $A - B$ is positive semi-definite. Obviously, this defines a partial order and we will repeatedly use the following simple facts for square matrices A, B and a real number a:

$$A \geqslant 0\,,\ B \geqslant 0 \ \Rightarrow\ A + B \geqslant 0\,; \qquad A \geqslant 0\,,\ a \geqslant 0\ \Rightarrow aA \geqslant 0\,;$$
$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \geqslant 0 \ \Leftrightarrow\ A, B \geqslant 0\,.$$

Note that every transposition takes even to odd permutations and vice versa, so C has the block structure

$$C = \begin{pmatrix} cI & X^t \\ X & cI \end{pmatrix}, \qquad \text{where } I \text{ is the identity matrix,}$$

and we have used a basis which lists first all even permutations and then all odd permutations. We have

$$\tilde{C} := \begin{pmatrix} \frac{1}{c}X^tX & X^t \\ X & cI \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{c}}X & \sqrt{c}I \end{pmatrix}^t \begin{pmatrix} \frac{1}{\sqrt{c}}X & \sqrt{c}I \end{pmatrix} \geqslant 0,$$

since $A^tA \geqslant 0$ for any matrix A, and C and $\tilde{C}$ only differ by

$$C - \tilde{C} = \begin{pmatrix} \frac{1}{c}C' & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{where } C' = c^2 I - X^tX.$$

C' is a symmetric $(\frac{n!}{2} \times \frac{n!}{2})$-matrix, to be referred to as the correction matrix. It coincides with c times the Schur complement of the odd-odd block of C. The matrices C, C', and X only depend on the rates $c_1, \ldots, c_{n-1}$ and the system size $n = |V|$, and whenever we want to stress this dependence, we will write $C(n)$, $C'(n)$, and $X(n)$. By the above, the proof of Theorem 2.3 will be complete once we show that C' is positive semi-definite:

$$C'(n) \geqslant 0 \quad \text{for all } n \geqslant 2. \tag{3.4}$$

3.2. **Structure of the correction matrix.** It turns out that the correction matrix has a relatively simple structure: It can be written as a linear combination of matrices where the coefficients are products of rates and the matrices do not depend on the rates at all.

Lemma 3.1. *We have $C'(2) = 0$, $C'(3) = 0$, and*

$$C'(n) = \sum_{J \subset V : |J| = 4} -c_J A^J(n) \qquad \text{for all } n \geqslant 4, \tag{3.5}$$

where $c_J := \prod_{i \in J} c_i$ and $A^J(n)$ is defined by

$$A^J_{\eta,\eta'}(n) = \begin{cases} 2 & \text{if } \eta = \eta', \\ 2 & \text{if } \eta^{-1}\eta' \text{ is a product of 2 disjoint 2-cycles} \\ & \qquad\qquad \text{with entries from } J, \\ -1 & \text{if } \eta^{-1}\eta' \text{ is a 3-cycle with entries from } J, \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

for all even permutations $\eta, \eta' \in \mathcal{X}_V$.

Proof. We simply calculate $C'_{\eta,\eta'}$ for all even permutations η, η' using $C' = c^2I - X^tX$. For $n = 2$, $c = c_1^2$ and $X(2)$ is the (1×1)-matrix $X(2) = (-c_1^2)$, so $C'(2) = 0$. For $n = 3$, $c = c_1^2 + c_1c_2 + c_2^2$ and

$$X(3) = \begin{pmatrix} c_1c_2 & -c_2(c_1+c_2) & -c_1(c_1+c_2) \\ -c_1(c_1+c_2) & c_1c_2 & -c_2(c_1+c_2) \\ -c_2(c_1+c_2) & -c_1(c_1+c_2) & c_1c_2 \end{pmatrix},$$

where the rows are indexed by the odd permutations $(12), (01), (02)$ and the columns are indexed by the even permutations $id, (021), (012)$ in that order. This gives $C'(3) = c^2I(3) - X^t(3)X(3) = 0$.

For $n \geqslant 4$ we observe that $X_{\eta_1,\eta_2}(n) = 0$ unless η_1 and η_2 differ by a single transposition. Thus $C'_{\eta,\eta'}(n) = 0$ unless η and η' differ by a product of exactly two transpositions. Note that such a product of two transpositions can be a product of two disjoint transpositions (i.e. 2-cycles), a 3-cycle, or the identity.

(a) If $\eta^{-1}\eta'$ is a product of two disjoint 2-cycles, e.g. (01)(23), a complete list of decompositions of $\eta^{-1}\eta'$ into a product of two transpositions is $(01)(23) = (23)(01)$, so using $K := \{0,1,2,3\}$, we have

$$C'_{\eta,\eta'}(n) = -(c_0c_1c_2c_3 + c_2c_3c_0c_1) = 2(-c_K)\,.$$

(b) If $\eta^{-1}\eta'$ is a 3-cycle, e.g. (012), a complete list of decompositions of $\eta^{-1}\eta'$ into a product of two transpositions is $(012) = (01)(20) = (12)(01) = (20)(12)$, so using $K := \{0,1,2\}$, we have

$$\begin{aligned} C'_{\eta,\eta'}(n) &= -(c_0c_1c_2c_0 + c_1c_2c_0c_1 + c_2c_0c_1c_2) \\ &= c_K \sum_{i\notin K} c_i = (-1) \sum_{J\supset K,|J|=4} -c_J\,. \end{aligned}$$

(c) If $\eta^{-1}\eta' = id$, we have $\eta^{-1}\eta' = \tau^2$ for every transposition τ, so we have

$$C'_{\eta,\eta'}(n) = c^2 - \sum_{i<j}(c_ic_j)^2 = 2\sum_{J:|J|=4} -c_J.$$

Here we have used

$$\begin{aligned} &\Big(\sum_{i<j} c_ic_j\Big)^2 - \sum_{i<j}(c_ic_j)^2 \\ &\quad= 2\sum_{i<j<k}\Big(c_i^2c_jc_k + c_ic_j^2c_k + c_ic_jc_k^2\Big) + 6\sum_{i<j<k<l} c_ic_jc_kc_l \\ &\quad= 2\Big(\sum_{i<j<k} c_ic_jc_k \sum_l c_l - 4\sum_{i<j<k<l} c_ic_jc_kc_l\Big) + 6\sum_{i<j<k<l} c_ic_jc_kc_l \\ &\quad= -2\sum_{i<j<k<l} c_ic_jc_kc_l. \end{aligned}$$

Thus we have checked (3.5) entrywise. □

We already know that $C'(2) = 0$ and $C'(3) = 0$. In order to motivate the following lemmata, let us also look at $C'(4)$ and $C'(5)$: Using the shorthand notation

$$A := A^{\{0,1,2,3\}}(4) \qquad \text{and} \quad A^{(i)} := A^{\{0,1,2,3,4\}\setminus\{i\}}(5)\,, \qquad \text{for } 0 \leqslant i \leqslant 4,$$

the decomposition (3.5) of the correction matrices gives

$$\begin{aligned} &C'(4) = -c_0c_1c_2c_3A \qquad \text{and} \\ &C'(5) = -c_0c_2c_3c_4A^{(1)} - \cdots - c_0c_1c_2c_3A^{(4)} - c_1c_2c_3c_4A^{(0)}. \end{aligned}$$

For $C'(4)$ we observe that $-c_0c_1c_2c_3 \geqslant 0$, so it suffices to show that $A \geqslant 0$. For $C'(5)$ we observe that

$$-c_0c_2c_3c_4 = (c_1+c_2+c_3+c_4)c_2c_3c_4 \geqslant c_1c_2c_3c_4,$$

and similarly for the coefficients of $A^{(2)}$, $A^{(3)}$, and $A^{(4)}$. If $A^{(i)} \geqslant 0$, this implies

$$C'(5) \geqslant c_1c_2c_3c_4(A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} - A^{(0)}),$$

and since $c_1c_2c_3c_4 \geqslant 0$, we are done once we have shown that the matrix in the parentheses is ≥ 0. For general n we will need the following two lemmata. In their proofs we will repeatedly use the notation $\mathcal{X}_K$ and $\mathcal{X}_K^+$ for the set of all permutations on a set K and the set of all even permutations on K.

Lemma 3.2. *For all $n \geqslant 4$ and $J \subset V$ with $|J| = 4$,*

$$A^J(n) \geqslant 0\,. \tag{3.7}$$

Proof. Consider the block structure of $A^J(n)$ corresponding to the blocks formed by the $n!/4!$ left cosets of $\mathcal{X}_J^+$ in $\mathcal{X}_V^+$. By definition of $A^J(n)$ in (3.6), the diagonal block corresponding to the coset $\mathcal{X}_J^+$ can be identified with $A := A^{\{0,1,2,3\}}(4)$ (if J is identified with $\{0,1,2,3\}$). Furthermore $A^J_{\eta,\eta'}(n)$ only depends on $\eta^{-1}\eta'$, and thus $A^J_{\eta\sigma,\eta\sigma'}(n) = A^J_{\sigma,\sigma'}(n)$ for all $\sigma,\sigma' \in \mathcal{X}_J^+$ and $\eta \in \mathcal{X}_V^+$, which implies that all diagonal blocks of $A^J(n)$ are equal, and thus they are copies of A. Finally, $A^J_{\eta,\eta'}(n) = 0$ unless $\eta^{-1}\eta' \in \mathcal{X}_J^+$, which shows that all nondiagonal blocks of $A^J(n)$ are 0. Because of this block decomposition of $A^J(n)$ we only have to show $A \geqslant 0$.

By (3.6), $A = A^{\{0,1,2,3\}}(4)$ is a symmetric (12×12)-matrix with entries $2, -1, 0$ only. Using a computer algebra program, one can check that A has the eigenvalues 0 (with multiplicity 10) and 12 (with multiplicity 2), which implies the assertion for $n = 4$. For the sake of completeness we will also show how to obtain the spectrum of A without using a computer: The matrix A is indexed by $\mathcal{X}_4^+ := \mathcal{X}^+_{\{0,1,2,3\}}$. We note that $\mathcal{X}_4^+$ consists of the identity, 3 permutations that are a product of two disjoint 2-cycles, and 8 permutations that are 3-cycles. Furthermore $H := \{id, (01)(23), (02)(13), (03)(12)\}$ is a subgroup of $\mathcal{X}_4^+$, and we consider the decomposition of A into blocks corresponding to the 3 left cosets of H in $\mathcal{X}_4^+$. Two permutations from the same coset ηH differ by an element of H, whereas two permutations from different cosets cannot differ by an element of H, i.e. they have to differ by a 3-cycle. Thus by (3.6), A has the block structure

$$A = \begin{pmatrix} 2E_4 & (-1)E_4 & (-1)E_4 \\ (-1)E_4 & 2E_4 & (-1)E_4 \\ (-1)E_4 & (-1)E_4 & 2E_4 \end{pmatrix} = 3 \begin{pmatrix} E_4 & 0 & 0 \\ 0 & E_4 & 0 \\ 0 & 0 & E_4 \end{pmatrix} - E_{12}, \tag{3.8}$$

where $E_m \in \mathbb{R}^{m\times m}$ is the matrix with all entries equal to 1. The matrix E_n has the eigenvalues 0 (with multiplicity $n-1$) and n (with multiplicity 1), and the eigenvector corresponding to the eigenvalue n is $(1,\dots,1)$. Furthermore the two matrices in the above decomposition of A commute. This implies that A has the eigenvalues 0 (with multiplicity 10) and 12 (with multiplicity 2). $\square$

Lemma 3.3. *For all $n \geqslant 5$ and $K \subset V$ with $0 \in K$ and $|K| = 5$,*

$$B^K(n) := \sum_{J \subset K : |J| = 4} \varepsilon_J A^J(n) \geqslant 0, \tag{3.9}$$

where ε_J is the sign of $-c_J$, i.e. $\varepsilon_J = 1$ if $0 \in J$ and $\varepsilon_J = -1$ if $0 \notin J$.

Proof. The structure of the proof is very similar to the one of the preceding lemma: We consider the block structure of $B^K(n)$ corresponding to the $n!/5!$ cosets of $\mathcal{X}_K^+$ in $\mathcal{X}_V^+$. The diagonal block corresponding to the coset $\mathcal{X}_K^+$ can be identified with $B := B^{\{0,1,2,3,4\}}(5)$ (if K is identified with $\{0,1,2,3,4\}$), and as in the proof of Lemma 3.2 we see that all diagonal blocks are equal and thus copies of B and all nondiagonal blocks are 0. Because of this block decomposition of $B^K(n)$ we only have to show $B \geqslant 0$.

$B = B^{\{0,1,2,3,4\}}(5)$ is a symmetric (60×60)-matrix with small integer entries that can be computed from (3.6). Using a computer algebra program, one can check that B has the eigenvalues 0 (with multiplicity 45) and 24 (with multiplicity 15), which implies the assertion for $n = 5$. However, the following argument allows us to obtain the spectrum of B without using a computer. Using the shorthand notation introduced before Lemma 3.2, we observe that

$$BA^{(0)} = (A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} - A^{(0)})A^{(0)} = 0. \tag{3.10}$$

Before proving (3.10), we will use it to compute the spectrum of B. Let

$$B^+ := A^{(1)} + A^{(2)} + A^{(3)} + A^{(4)} + A^{(0)} \quad \text{so that} \quad B = B^+ - 2A^{(0)}.$$

As an immediate consequence of (3.10), $(B^+ - 2A^{(0)})A^{(0)} = 0$ and

$$A^{(0)}(B^+ - 2A^{(0)}) = [(B^+ - 2A^{(0)})A^{(0)}]^t = 0,$$

i.e.

$$B^+A^{(0)} = 2(A^{(0)})^2 \quad \text{and} \quad A^{(0)}B^+ = 2(A^{(0)})^2.$$

By symmetry we get the same relations for $A^{(i)}$ instead of $A^{(0)}$, and by the proof of Lemma 3.2, $A^{(i)}$ is a symmetric matrix with eigenvalues 0 and 12 only, so $(A^{(i)})^2 = 12A^{(i)}$. Using all of these relations, we get

$$(B^+)^2 = \sum_{i=0}^{4} B^+A^{(i)} = \sum_{i=0}^{4} 2(A^{(i)})^2 = 24\sum_{i=0}^{4} A^{(i)} = 24B^+$$

and

$$\begin{aligned} B^2 &= (B^+ - 2A^{(0)})^2 = (B^+)^2 - 2B^+A^{(0)} - 2A^{(0)}B^+ + 4(A^{(0)})^2 \\ &= 24B^+ - 8(A^{(0)})^2 + 4(A^{(0)})^2 = 24B^+ - 48A^{(0)} = 24B, \end{aligned}$$

i.e. $\frac{1}{24}B$ is a projection and thus has eigenvalues $0, 1$ only. So B has eigenvalues $0, 24$ only. Since the trace of B is $60 \times (2+2+2+2-2) = 360$, the multiplicity of the eigenvalue 24 has to be $\frac{360}{24} = 15$, and the multiplicity of the eigenvalue 0 has to be 45.

We will now prove (3.10), i.e. $B = 0$ on the image of $A^{(0)}$. By the proof of Lemma 3.2, we know the block structure of $A^{(0)}$ corresponding to the cosets of $\mathcal{X}^+_{\{1,2,3,4\}}$ in $\mathcal{X}^+_{\{0,1,2,3,4\}}$: The nondiagonal blocks are 0 and the diagonal blocks are copies of A, and by (3.8), the image of A is

$$\{a1_{\eta H} + a'1_{\eta' H} + a''1_{\eta'' H} \;:\; a, a', a'' \in \mathbb{R}\,, \;\text{ such that }\; a + a' + a'' = 0\},$$

where $H = \{id, (01)(23), (02)(13), (03)(12)\}$ and ηH, $\eta' H$, $\eta'' H$ are the three distinct cosets of H in $\mathcal{X}^+_{\{0,1,2,3\}}$. As usual 1_U denotes the indicator function of a given set U; e.g. $1_{\eta H}$ is the function on $\mathcal{X}^+_{\{0,1,2,3\}}$ that takes the value 1 on ηH and the value 0 otherwise. So in particular

$$\mathrm{Im}(A) \subset \mathrm{Span}(1_{\eta H} : \eta \in \mathcal{X}^+_{\{0,1,2,3\}}),$$

and by the block structure of $A^{(0)}$ this implies

$$\mathrm{Im}(A^{(0)}) \subset \mathrm{Span}(1_{\eta H^{(0)}} : \eta \in \mathcal{X}^+_{\{0,1,2,3,4\}}),$$

where $H^{(0)} = \{id, (12)(34), (13)(24), (14)(23)\}$, and thus it suffices to show that $Bv = 0$ for every vector v of the form $v = 1_{\eta H^{(0)}}$, i.e.

$$\sum_{\sigma \in H^{(0)}} B_{\eta,\eta'\sigma} = 0 \qquad \text{for all } \ \eta, \eta' \in \mathcal{X}^+_{\{0,1,2,3,4\}}. \tag{3.11}$$

Since $B_{\eta,\eta'\sigma}$ only depends on $\eta^{-1}\eta'\sigma$, for the proof of (3.11) we may assume without loss of generality that $\eta = id$. The following observations help to reduce the number of choices of η' that have to be considered: Since η' has to be an even permutation of $\{0,1,2,3,4\}$, η' has to be id, a 3-cycle, a 5-cycle, or a product of two disjoint 2-cycles. Every 5-cycle necessarily has an entry 0, and the 3-cycle and the product of the 2-cycles may or may not contain an entry 0. This gives 6 cases altogether. Since $B = A^{(1)} + \cdots + A^{(4)} - A^{(0)}$ and $H^{(0)}$ are invariant under permutations of 1, 2, 3, 4 and in each of the above cases the permutations differ only by permuting the roles of 1, 2, 3, 4, it is sufficient to consider one permutation from each case, say $\eta' \in \{id, (123), (012), (03142), (12)(34), (02)(34)\}$. Since $(12)(34) \in idH^{(0)}$ and $(03142), (02)(34) \in (012)H^{(0)}$ (see below), we are done once we check $\sum_{\sigma \in H^{(0)}} B_{id,\eta'\sigma} = 0$ for $\eta' \in \{id, (123), (012)\}$. In each of the three cases we compute $\eta' H^{(0)}$ and check $f(\eta') := \sum_{\sigma \in \eta' H^{(0)}} B_{id,\sigma} = 0$ by considering the contributions for a fixed σ from $A^{(i)}_{id,\sigma}$ for $1 \leqslant i \leqslant 4$ and from $-A^{(0)}_{id,\sigma}$.

(a) If $\eta' = id$, we have $\eta' H^{(0)} = H^{(0)} = \{id, (12)(34), (13)(24), (14)(23)\}$. Also, id gives a contribution of $2+2+2+2-2 = 6$, and the others permutations give a contribution of -2 each, so $f(\eta') = 6 - 2 - 2 - 2 = 0$.

(b) If $\eta' = (123)$, we have $\eta' H^{(0)} = \{(123), (243), (142), (134)\}$, and each of these gives a contribution of $1 - 1 = 0$. (E.g. (123) is a 3-cycle with entries from $J = \{0,1,2,3\}$ or from $J = \{1,2,3,4\}$.) Thus $f(\eta') = 0$.

(c) If $\eta' = (012)$, we have $\eta' H^{(0)} = \{(012), (02)(34), (03142), (04132)\}$. Also, (012) gives the contribution $-1-1$, (02)(34) gives 2, and the 5-cycles do not contribute, so $f(\eta') = -2 + 2 = 0$. □

3.3. **Proof of Theorem 2.3.** In Subsection 3.1 we have seen that Theorem 2.3 follows once we have shown that the correction matrix is positive semi-definite. This can now be obtained from the results of Subsection 3.2 concerning the structure of the correction matrix:

Lemma 3.4. *For every $n \geqslant 2$ we have $C'(n) \geqslant 0$.*

Proof. We already have seen that $C'(n) = 0$ for $n = 2, 3$ and $C'(4) = -c_0c_1c_2c_3A \geqslant 0$. For $n \geqslant 5$ we use the variable J for a subset $J \subset V$ with $|J| = 4$ and K for a subset $K \subset V$ with $0 \in K$ and $|K| = 5$. The lemma follows from the two inequalities in

$$\begin{aligned} 0 &\leqslant \sum_K \frac{|c_K|}{|c_0|} \sum_{J \subset K} \varepsilon_J A^J(n) = \sum_J \Big(\sum_{K \supset J} \frac{\varepsilon_J |c_K|}{|c_0|} \Big) A^J(n) \\ &\leqslant \sum_J (-c_J) A^J(n) = C'(n). \end{aligned}$$

The first inequality is an immediate consequence of (3.9), and the second follows from (3.7) once we have checked that

$$\sum_{K \supset J} \frac{\varepsilon_J |c_K|}{|c_0|} \leqslant -\, c_J\,, \qquad \text{for all } \ J\,.$$

If $0 \notin J$, the only set $K \supset J$ containing 0 is $K = J \cup \{0\}$ and we get

$$\sum_{K \supset J} \frac{\varepsilon_J |c_K|}{|c_0|} = \frac{\varepsilon_J |c_{J\cup\{0\}}|}{|c_0|} = \varepsilon_J |c_J| = -c_J.$$

If $0 \in J$, the sets $K \supset J$ containing 0 are of the form $K = J \cup \{i\}$ with $i \notin J$ and we get

$$\sum_{K \supset J} \frac{\varepsilon_J |c_K|}{|c_0|} = \sum_{i \notin J} \frac{\varepsilon_J |c_{J\cup\{i\}}|}{|c_0|} = -c_J \frac{\sum_{i\notin J} |c_i|}{|c_0|} \leqslant -c_J$$

since $-c_J \geqslant 0$ in this case, and

$$\sum_{i \notin J} |c_i| \leqslant \sum_{i>0} |c_i| = \sum_{i>0} c_i = -c_0 = |c_0|.$$

□

4. Related Markov chains on weighted graphs

Here we discuss several stochastic processes that can be associated in a natural way to weighted graphs. Each of them is an irreducible, symmetric Markov chain as in Section 1.1; in particular each one is reversible with respect to the uniform distribution on the corresponding state space and has a strictly positive sepectral gap. Furthermore, all of them are sub-processes of the interchange process in the sense of (1.2), which allows us to obtain estimates on their spectral gaps as simple corollaries of Theorem 1.1. In all examples let $G = (V, E)$ be the complete graph on n vertices—w.l.o.g. we assume that $V = \{1, \dots, n\}$—and let $c_{xy} \geqslant 0$ be given edge weights such that the corresponding skeleton graph is connected; see Section 1.2.

4.1. **Exclusion processes.**

4.1.1. *Symmetric exclusion process.* In the k-particle exclusion process a state is an assignment of k indistinguishable particles to k of the n vertices of G. Here $k \in \{1, \dots, n-1\}$ is a fixed number, which is often omitted in our notation. The transition from a state ζ to a state ζ^{xy} (occurring with rate c_{xy}) is possible only if in ζ one of the positions x, y is occupied and the other is empty. In this transition, the particle at the occupied site jumps to the empty site; see Figure 4. We note that the 1-particle exclusion process is the same as the random walk. Formally,

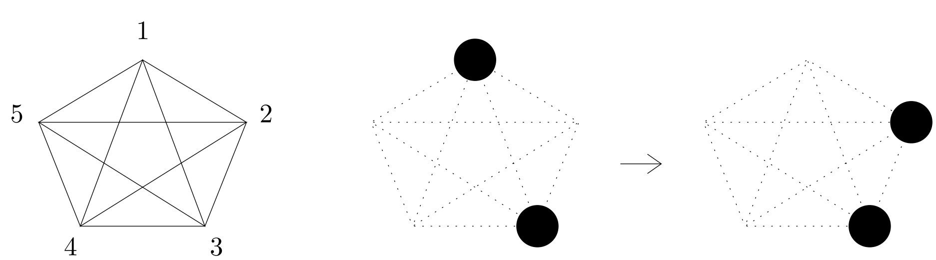

FIGURE 4. 2-particle exclusion process on the graph $V = \{1, 2, 3, 4, 5\}$. The picture shows the underlying graph and a transition from $\zeta = \{1, 3\}$ to $\zeta^{1,2} = \{2, 3\}$.

the k-particle exclusion process is defined to be the Markov chain with state space $S^{EP} = \{\zeta \subset V : |\zeta| = k\}$ and generator

$$\mathcal{L}^{EP} f(\zeta) = \sum_{xy \in E} c_{xy}(f(\zeta^{xy}) - f(\zeta)), \quad \text{where } f : S^{EP} \to \mathbb{R},\, \zeta \in S^{EP}.$$

Here $\zeta^{xy} = \zeta$ if $xy \subset \zeta$ or $xy \subset \zeta^c$ and

$$\zeta^{xy} = \begin{cases} (\zeta \setminus \{y\}) \cup \{x\} & \text{if } y \in \zeta \text{ and } x \notin \zeta, \\ (\zeta \setminus \{x\}) \cup \{y\} & \text{if } x \in \zeta \text{ and } y \notin \zeta. \end{cases}$$

By Section 1.1, $-\mathcal{L}^{EP}$ has $|S^{EP}| = \binom{n}{k}$ nonnegative eigenvalues and a positive spectral gap $\lambda_1^{EP} > 0$. The k-particle exclusion process can be obtained as a sub-process of the interchange process by declaring the sites occupied by particles 1 through k to be occupied and the other vertices to be empty; more precisely $\pi : S^{IP} \to S^{EP}$, $\pi(\eta) = \{\xi_1(\eta), \dots, \xi_k(\eta)\}$ is a contraction in the sense of (1.2), which gives $\mathrm{Spec}(-\mathcal{L}^{EP}) \subset \mathrm{Spec}(-\mathcal{L}^{IP})$. In order to compare the exclusion process to the random walk, let $f : V \to \mathbb{R}$ be an eigenfunction of $-\mathcal{L}^{RW}$ with eigenvalue λ and define $g : S^{EP} \to \mathbb{R}$ by $g(\zeta) = \sum_{x \in \zeta} f(x)$. Note that if g is constant, then f must be constant. Therefore, $g \not\equiv 0$ (since otherwise f is constant and thus $f \equiv 0$), and $\sum_{x,y \in \zeta, x \neq y} c_{xy}(f(y) - f(x)) = 0$ implies

$$\begin{aligned} (-\mathcal{L}^{EP} g)(\zeta) &= -\sum_{x \in \zeta, y \notin \zeta} c_{xy}(g(\zeta^{xy}) - g(\zeta)) \\ &= -\sum_{x \in \zeta, y \notin \zeta} c_{xy}(f(y) - f(x)) = -\sum_{x \in \zeta, y \neq x} c_{xy}(f(y) - f(x)) \\ &= \sum_{x \in \zeta} (-\mathcal{L}^{RW} f)(x) = \lambda \sum_{x \in \zeta} f(x) = \lambda g(\zeta)\,, \end{aligned}$$

i.e. g is an eigenfunction of $-\mathcal{L}^{EP}$ with eigenvalue λ. This gives

$$\mathrm{Spec}(-\mathcal{L}^{RW}) \subset \mathrm{Spec}(-\mathcal{L}^{EP}) \subset \mathrm{Spec}(-\mathcal{L}^{IP})\,.$$

As a corollary of Theorem 1.1 one has that, for an arbitrary number of particles $k = 1, \dots, n-1$, for every graph G,

$$\lambda_1^{EP}(G) = \lambda_1^{RW}(G). \tag{4.1}$$

4.1.2. *Colored exclusion process.* In the colored exclusion process there are $r \geqslant 2$ types of particles ($n_i \geqslant 1$ of type i such that $n_1 + \cdots + n_r = n$), where particles of the same type (or color) are indistinguishable. A state is an assignment of these particles to the vertices of G so that every vertex is occupied by exactly one particle, and in the transition from a state α to a state α^{xy} particles at sites x and y interchange their positions; see Figure 5. Formally, the colored exclusion process is the Markov chain on the state space S^{CEP}, which is the set of partitions $\alpha = (\alpha_1, \dots, \alpha_r)$ of V such that $|\alpha_i| = n_i$, and the generator is defined by

$$\mathcal{L}^{CEP} f(\alpha) = \sum_{xy \in E} c_{xy}(f(\alpha^{xy}) - f(\alpha)), \qquad f : S^{CEP} \to \mathbb{R}, \alpha \in S^{CEP}.$$

Here $\alpha^{xy} = \alpha$ if $x, y \in \alpha_i$ for some i, and if $x \in \alpha_i$ and $y \in \alpha_j$ for $i \neq j$, we have $\alpha^{xy} = (\alpha_1^{xy}, \dots, \alpha_r^{xy})$, where $\alpha_i^{xy} = (\alpha_i \setminus \{x\}) \cup \{y\}$, $\alpha_j^{xy} = (\alpha_j \setminus \{y\}) \cup \{x\}$, and $\alpha_k^{xy} = \alpha_k$ for all $k \neq i, j$. By Section 1.1, $-\mathcal{L}^{CEP}$ has $|S^{CEP}| = \binom{n}{n_1, \dots, n_r}$ nonnegative eigenvalues and a positive spectral gap $\lambda_1^{CEP} > 0$. The n_1-particle

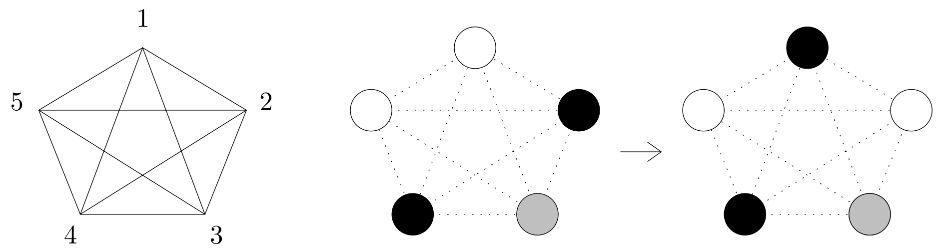

FIGURE 5. Colored exclusion process on $V = \{1, 2, 3, 4, 5\}$ with 3 types of particles ($n_1 = 2, n_2 = 1, n_3 = 2$). The picture shows the underlying graph and a transition from $\alpha = (\{2, 4\}, \{3\}, \{1, 5\})$ to $\alpha^{1,2} = (\{1, 4\}, \{3\}, \{2, 5\})$.

exclusion process is a sub-process of the colored exclusion process (by declaring all sites occupied by particles of type $2, \ldots, r$ to be empty), which in turn is a sub-process of the interchange process (by declaring particles $1, \ldots, n_1$ to be of type 1, ... and particles $n - n_r + 1, \ldots, n$ to be of type r). The definitions of the corresponding contractions are obvious. This gives, for the given choice of parameters $n_1, \ldots, n_r$,

$$\mathrm{Spec}(-\mathcal{L}^{EP}) \subset \mathrm{Spec}(-\mathcal{L}^{CEP}) \subset \mathrm{Spec}(-\mathcal{L}^{IP})\,.$$

From Theorem 1.1 and (4.1) it follows that for any choice of the parameters $r, n_1, \ldots, n_r$, for any graph G,

$$\lambda_1^{CEP}(G) = \lambda_1^{RW}(G)\,. \tag{4.2}$$

4.2. **Cycles and matchings.** We turn to examples of processes with a gap that is in general strictly larger than that of the random walk. We note that the processes defined here are examples from a general class of processes, obtained as the evolution of certain subgraphs of the complete graph when the labels undergo the interchange process on G. For all these processes one has analogous estimates for the spectral gap.

4.2.1. *Cycle process.* The states of the cycle process are n-cycles, where $n = |V|$. In order to avoid a trivial situation, we assume $n \geqslant 4$. One could think of a rubber band that at certain points is pinned to the vertices of G. The transition from γ to γ^{xy} (occurring with rate c_{xy}) can be obtained by taking the point of the rubber band pinned to x from x to y and the point pinned to y from y to x; see Figure 6. Formally, an n-cycle in G is a set of edges $\gamma \subset E$ forming a subgraph of G isomorphic to $\{\{1, 2\}, \ldots, \{n - 1, n\}, \{n, 1\}\}$. The cycle process is the Markov chain with state space S^{CP}, the set of all n-cycles of G, and generator

$$\mathcal{L}^{CP} f(\gamma) = \sum_{xy \in E} c_{xy}(f(\gamma^{xy}) - f(\gamma)), \quad \text{ where } f : S^{CP} \to \mathbb{R},\, \gamma \in S^{CP}.$$

Here $\gamma^{xy} = \{b^{xy} : b \in \gamma\}$, where for an edge $b \in E$ we define $b^{xy} = b$ if $x, y \notin b$ or $b = xy$, and $b^{xy} = xz$ if $b = yz$ ($z \neq x, y$). Thus, $-\mathcal{L}^{CP}$ has $|S^{CP}| = \frac{(n-1)!}{2}$ nonnegative eigenvalues and a positive spectral gap $\lambda_1^{CP} > 0$. The cycle process can be obtained from the interchange process by pinning a cycle on the particles labeled $1, \ldots, n, 1$ in that order, i.e. $\pi : S^{IP} \to S^{CP}$, $\pi(\eta) = \{\xi_1(\eta)\xi_2(\eta), \ldots, \xi_{n-1}(\eta)\xi_n(\eta), \xi_n(\eta)\xi_1(\eta)\}$ is

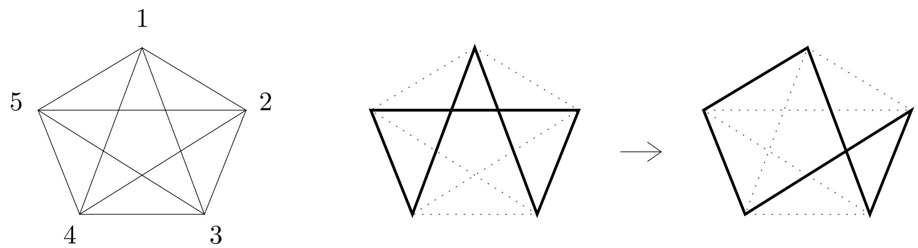

FIGURE 6. Cycle process on $V = \{1, 2, 3, 4, 5\}$.
The picture shows the underlying graph and a transition
from $\gamma = \{\{1, 3\}, \{3, 2\}, \{2, 5\}, \{5, 4\}, \{4, 1\}\}$
to $\gamma^{1,2} = \{\{1, 3\}, \{3, 2\}, \{2, 4\}, \{4, 5\}, \{5, 1\}\}$.

a contraction. This gives $\mathrm{Spec}(-\mathcal{L}^{CP}) \subset \mathrm{Spec}(-\mathcal{L}^{IP})$, and thus, by Theorem 1.1, on any graph G,

$$\lambda_1^{CP}(G) \geqslant \lambda_1^{RW}(G) \,. \tag{4.3}$$

It is possible to see that in general the inequality is strict, e.g. for $n = 4$ and $c_{ij} = 1$ for all $ij \in E$ one can compute $\lambda_1^{RW} = 4$ and $\lambda_1^{CP} = 6$.

4.2.2. *Matching process.* Set $n = 2k$, $k \geqslant 2$. A perfect matching of n elements is given by k disjoint edges of the complete graph on $V = \{1, \dots, n\}$. A perfect matching configuration is thus denoted $\zeta = \{b_1, \dots, b_k\}$, where the edges $b_i \in E$ are such that $b_i \cap b_j = \emptyset$. The state space of the matching process, denoted S^{MP}, is the set of all possible perfect matchings ζ. The transition $\zeta \to \zeta^{xy}$, occurring with rate c_{xy}, is described as follows: if $xy \in \zeta$, then nothing happens, and $\zeta^{xy} = \zeta$; if $xy \notin \zeta$, then there are uniquely determined points $u, v \in V$ such that $xu, yv \in \zeta$, and ζ^{xy} coincides with ζ except that xu, yv are removed while yu, xv are added; see Figure 7. The generator is then given by

$$\mathcal{L}^{MP} f(\zeta) = \sum_{xy \in E} c_{xy}(f(\zeta^{xy}) - f(\zeta)), \quad \text{ where } f : S^{MP} \to \mathbb{R},\, \zeta \in S^{MP} \,.$$

Thus, $-\mathcal{L}^{MP}$ has $|S^{MP}| = \frac{(2k)!}{2^k k!}$ nonnegative eigenvalues and a positive spectral gap $\lambda^{MP} > 0$. The matching process is a sub-process of the interchange process in the sense of (1.2). Indeed, given a permutation $\eta \in \mathcal{X}_n$, one can obtain a perfect

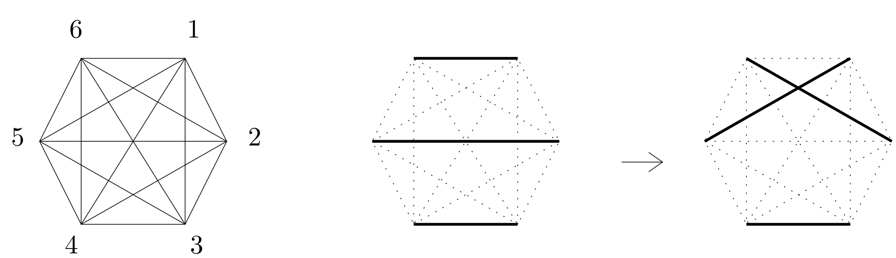

FIGURE 7. Matching process on $V = \{1, 2, 3, 4, 5, 6\}$. The picture shows the underlying graph and a transition from $\zeta = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}$ to $\zeta^{1,2} = \{\{2, 6\}, \{1, 5\}, \{3, 4\}\}$.

matching by setting

$$\zeta = \pi(\eta) = \{\xi_1(\eta)\xi_{k+1}(\eta), \dots, \xi_k(\eta)\xi_{2k}(\eta)\},$$

and the map $\pi : S^{IP} \to S^{MP}$ defines the desired contraction. This shows that $\mathrm{Spec}(-\mathcal{L}^{MP}) \subset \mathrm{Spec}(-\mathcal{L}^{IP})$, which implies, by Theorem 1.1, that for all graphs G,

$$\lambda_1^{MP}(G) \geqslant \lambda_1^{RW}(G). \tag{4.4}$$

It is known that for the unweighted complete graph $c_{xy} \equiv 1$ the inequality (4.4) is strict. We refer to [7] for a complete description of $\mathrm{Spec}(-\mathcal{L}^{MP})$ in this special case and note that for $k = 2, n = 4$ the two eigenvalues are 6 and 4, respectively.

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