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LOG CANONICAL SINGULARITIES ARE DU BOIS

JÁNOS KOLLÁR AND SÁNDOR J KOVÁCS

1. Introduction

A recurring difficulty in the Minimal Model Program (MMP) is that while log terminal singularities are quite well behaved (for instance, they are rational), *log canonical* singularities are much more complicated; they need not even be Cohen-Macaulay. The aim of this paper is to prove that, as conjectured in [Kol92, 1.13], log canonical singularities are *Du Bois*. The concept of Du Bois singularities, abbreviated as *DB*, was introduced by Steenbrink in [Ste83] as a weakening of rationality. It is not clear how to define Du Bois singularities in positive characteristic, so we work over a field of characteristic 0 throughout the paper. The precise definition is rather involved, see (1.10), but our main applications rely only on the following consequence:

Corollary 1.1. *Let X be a proper scheme of finite type over $\mathbb{C}$. If (X, Δ) is log canonical for some $\mathbb{Q}$-divisor Δ, then the natural map*

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^i(X^{\mathrm{an}}, \mathscr{O}_{X^{\mathrm{an}}}) \cong H^i(X, \mathscr{O}_X)$$

is surjective for all i.

Using [DJ74, Lemme 1], this implies the following:

Corollary 1.2. *Let $\phi : X \to B$ be a proper, flat morphism of complex varieties with B connected. Assume that for all $b \in B$ there exists a $\mathbb{Q}$-divisor D_b on X_b such that (X_b, D_b) is log canonical. Then $h^i(X_b, \mathscr{O}_{X_b})$ is independent of $b \in B$ for all i.*

Notice that we do not require that the divisors D_b form a family.

We also prove flatness of the cohomology sheaves of the relative dualizing complex of a projective family of log canonical varieties (1.8). Combining this result with a Serre duality type criterion (7.11) gives another invariance property:

Corollary 1.3. *Let $\phi : X \to B$ be a flat, projective morphism, with B connected. Assume that for all $b \in B$ there exists a $\mathbb{Q}$-divisor D_b on X_b such that (X_b, D_b) is log canonical.*

Then, if one fiber of ϕ is Cohen-Macaulay (resp. S_k for some k), then all fibers are Cohen-Macaulay (resp. S_k).

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Remark 1.3.1. The S_k case of this result answers a question posed to us by Valery Alexeev and Christopher Hacon.

For arbitrary flat, proper morphisms, the set of fibers that are Cohen-Macaulay (resp. S_k) is open, but not necessarily closed. Thus the key point of (1.3) is to show that this set is also closed.

The generalization of these results to the semi-log canonical case turns out to be not hard, but it needs some foundational work which will be presented elsewhere. The general case then implies that each connected component of the moduli space of stable log varieties parametrizes either only Cohen-Macaulay or only non-Cohen-Macaulay objects.

Let us first state a simplified version of our main theorem:

Theorem 1.4. *Let (X, Δ) be an lc pair. Then X is DB. More generally, let $W \subset X$ be a reduced, closed subscheme that is a union of log canonical centers of (X, Δ). Then W is DB.*

This settles the above mentioned conjecture [Kol92, 1.13]. For earlier results related to this conjecture, see [Kol95, §12], [Kov99, Kov00b, Ish85, Ish87a, Ish87, Sch07, KSS08, Sch08].

Actually, we prove a more general statement, which is better suited to working with log canonical centers and allows for more general applications, but it might seem a little technical for the first reading:

Theorem 1.5. *Let $f : Y \to X$ be a proper surjective morphism with connected fibers between normal varieties. Assume that there exists an effective $\mathbb{Q}$-divisor on Y such that (Y, Δ) is lc and $K_Y + \Delta \sim_{\mathbb{Q},f} 0$. Then X is DB.*

More generally, let $W \subset Y$ be a reduced, closed subscheme that is a union of log canonical centers of (Y, Δ). Then $f(W) \subset X$ is DB.

There are three, more technical results that should be of independent interest. The first is a quite flexible criterion for Du Bois singularities.

Theorem 1.6. *Let $f : Y \to X$ be a proper morphism between reduced schemes of finite type over $\mathbb{C}$. Let $W \subseteq X$ and $F := f^{-1}(W) \subset Y$ be closed reduced subschemes with ideal sheaves $\mathscr{I}_{W\subseteq X}$ and $\mathscr{I}_{F\subseteq Y}$. Assume that the natural map ϱ,*

$$\begin{tikzcd} \mathscr{I}_{W\subseteq X} \arrow[r, "\varrho"'] & \mathcal{R}f_*\mathscr{I}_{F\subseteq Y} \arrow[l, bend right, dashed, "\varrho'"'] \end{tikzcd}$$

admits a left inverse ϱ'; that is, $\varrho' \circ \varrho = \mathrm{id}_{\mathscr{I}_{W\subseteq X}}$. Then if Y, F, and W all have DB singularities, then so does X.

Remark 1.6.1. Notice that we do not require f to be birational. On the other hand the assumptions of the theorem and [Kov00a, Thm 1] imply that if $Y \setminus F$ has rational singularities, e.g., if Y is smooth, then $X \setminus W$ has rational singularities as well.

The second is a variant of the connectedness theorem [Kol92, 17.4] for not necessarily birational morphisms.

Theorem 1.7. *Let $f : Y \to X$ be a proper morphism with connected fibers between normal varieties. Assume that (Y, Δ) is lc and $K_Y + \Delta \sim_{\mathbb{Q},f} 0$. Let $Z_1, Z_2 \subset Y$*

be lc centers of (Y, Δ). *Then, for every irreducible component* $T \subset f(Z_1) \cap f(Z_2)$ *there is an lc center* Z_T *of* (Y, Δ) *such that* $Z_T \subseteq Z_1$ *and* $f(Z_T) = T$.

More precisely, let $g\colon Z_1 \cap f^{-1}(T) \to S$ *and* $\pi\colon S \to T$ *be the Stein factorization of* $Z_1 \cap f^{-1}(T) \to T$. *Then, for every irreducible component* $S_i \subset S$ *there is an lc center* Z_{S_i} *of* (Y, Δ) *such that* $g(Z_{S_i}) = S_i$.

The third is the flatness of the cohomology sheaves of the relative dualizing complex of a DB morphism:

Theorem 1.8. *Let* $\phi : X \to B$ *be a flat projective morphism such that all fibers are Du Bois. Then the cohomology sheaves* $h^i(\omega_\phi^\bullet)$ *are flat over* B, *where* $\omega_\phi^\bullet$ *denotes the relative dualizing complex of* ϕ.

Definitions and Notation 1.9. Let K be an algebraically closed field of characteristic 0. Unless otherwise stated, all objects are assumed to be defined over K, all schemes are assumed to be of finite type over K and a morphism means a morphism between schemes of finite type over K.

If $\phi : Y \to Z$ is a birational morphism, then $\mathrm{Ex}(\phi)$ will denote the *exceptional set* of ϕ. For a closed subscheme $W \subseteq X$, the ideal sheaf of W is denoted by $\mathscr{I}_{W\subseteq X}$ or, if no confusion is likely, then simply by $\mathscr{I}_W$. For a point $x \in X$, $\kappa(x)$ denotes the residue field of $\mathscr{O}_{X,x}$.

A *pair* (X, Δ) consists of a variety X and an effective $\mathbb{Q}$-divisor Δ on X. If (X, Δ) is a pair, then Δ is called a *boundary* if $\lfloor (1-\varepsilon)\Delta \rfloor = 0$ for all $0 < \varepsilon < 1$, i.e., the coefficients of all irreducible components of Δ are in the interval $[0, 1]$. For the definition of *klt, dlt*, and *lc* pairs, see [KM98], and for the definition of the *different*, Diff, see [Kol92, 16.5]. Let (X, Δ) be a pair and $f^{\mathrm{m}} : X^{\mathrm{m}} \to X$ a proper birational morphism such that $\mathrm{Ex}(f^{\mathrm{m}})$ is a divisor. Let $E = \sum a_i E_i$ be the discrepancy divisor, i.e., a linear combination of exceptional divisors such that

$$K_{X^{\mathrm{m}}} + (f^{\mathrm{m}})_*^{-1}\Delta \sim_{\mathbb{Q}} (f^{\mathrm{m}})^*(K_M + \Delta) + E$$

and let $\Delta^{\mathrm{m}} := (f^{\mathrm{m}})_*^{-1}\Delta + \sum_{a_i \leq -1} E_i$. Then $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$ is a *minimal dlt model* of (X, Δ) if it is a dlt pair, and the discrepancy of every f^{m}-exceptional divisor is at most -1. Note that if (X, Δ) is lc with a minimal dlt model $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$, then $K_{X^{\mathrm{m}}} + \Delta^{\mathrm{m}} \sim_{\mathbb{Q}} (f^{\mathrm{m}})^*(K_X + \Delta)$.

For morphisms $\phi : X \to B$ and $\vartheta : T \to B$, the symbol X_T will denote $X \times_B T$ and $\phi_T : X_T \to T$ the induced morphism. In particular, for $b \in B$ we write $X_b = \phi^{-1}(b)$. Of course, by symmetry, we also have the notation $\vartheta_X : T_X \simeq X_T \to X$, and if $\mathscr{F}$ is an $\mathscr{O}_X$-module, then $\mathscr{F}_T$ will denote the $\mathscr{O}_{X_T}$-module $\vartheta_X^*\mathscr{F}$.

For a morphism $\phi : X \to B$, the *relative dualizing complex* of ϕ (if it exists) will be denoted by $\omega_\phi^\bullet$. Recall that if ϕ is a projective morphism, then $\omega_\phi^\bullet = \phi^!\mathscr{O}_B$. In particular, for a (quasi-projective) scheme X, the *dualizing complex* of X will be denoted by $\omega_X^\bullet$.

The symbol $\simeq$ will mean isomorphism in the appropriate category. In particular, between complexes considered as objects in a derived category, it stands for a *quasi-isomorphism*.

We will use the following notation: For a functor Φ, $\mathcal{R}\Phi$ denotes its derived functor on the (appropriate) derived category and $\mathcal{R}^i\Phi := h^i \circ \mathcal{R}\Phi$, where $h^i(C^\bullet)$ is the cohomology of the complex $C^\bullet$ at the i^{th} term. Similarly, $\mathbb{H}^i_Z := h^i \circ \mathcal{R}\Gamma_Z$, where Γ_Z is the functor of cohomology with supports along a subscheme Z. Finally, $\mathcal{H}om$ stands for the sheaf-Hom functor, and $\mathcal{E}xt^i := h^i \circ \mathcal{R}\mathcal{H}om$.

1.10. **DB singularities.** Consider a complex algebraic variety X. If X is smooth and projective, its De Rham complex plays a fundamental role in understanding the geometry of X. When X is singular, an analog of the De Rham complex, introduced by Du Bois, plays a similar role.

Let X be a complex scheme of finite type. Based on Deligne's theory of mixed Hodge structures, Du Bois defined a filtered complex of $\mathscr{O}_X$-modules, denoted by $\underline{\Omega}^{\bullet}_X$, that agrees with the algebraic De Rham complex in a neighborhood of each smooth point, and, like the De Rham complex on smooth varieties, its analytization provides a resolution of the sheaf of locally constant functions on X [Du81]. Following Hélène Esnault's suggestion we will call $\underline{\Omega}^{\bullet}_X$ the *Deligne-Du Bois* complex.

Du Bois observed that an important class of singularities are those for which $\underline{\Omega}^{0}_X$, the zeroth graded piece of the filtered complex $\underline{\Omega}^{\bullet}_X$, takes a particularly simple form. He pointed out that singularities satisfying this condition enjoy some of the nice Hodge-theoretic properties of smooth varieties; cf. (7.8). These singularities were christened *Du Bois singularities* by Steenbrink [Ste83]. We will refer to them as *DB singularities*, and a variety with only DB singularities will be called *DB*.

The construction of the Deligne-Du Bois complex $\underline{\Omega}^{\bullet}_X$ is highly non-trivial, so we will not include it here. For a thorough treatment the interested reader should consult [PS08, II.7.3]. For alternative definitions, sufficient and equivalent criteria for DB singularities, see [Ish85, Ish87, Kov99, Kov00b, Sch07, KSS08].

Remark 1.11. Recall that the seminormalization of $\mathscr{O}_X$ is $h^0(\underline{\Omega}^0_X)$, the 0^{th} cohomology sheaf of the complex $\underline{\Omega}^0_X$, and so X is seminormal if and only if $\mathscr{O}_X \simeq h^0(\underline{\Omega}^0_X)$ by [Sai00, 5.2] (cf. [Sch06, 5.4.17], [Sch07, 4.8], and [Sch09, 5.6]). In particular, this implies that DB singularities are seminormal.

2. A criterion for Du Bois singularities

In order to prove (1.6) we first need the following abstract derived category statement.

Lemma 2.1. *Let* $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{A}', \mathsf{B}', \mathsf{C}'$ *be objects in a derived category and assume that there exists a commutative diagram in which the rows form exact triangles:*

(2.1.1)
$$\begin{tikzcd} \mathsf{A} \arrow[r, "\phi"] \arrow[d, "\alpha"'] & \mathsf{B} \arrow[r, "\psi"] \arrow[d, "\beta"'] & \mathsf{C} \arrow[r, "\zeta"] \arrow[d, "\gamma"'] & \mathsf{A}[1] \arrow[d, "\alpha[1]"] \\ \mathsf{A}' \arrow[r, "\phi'"'] & \mathsf{B}' \arrow[r, "\psi'"'] & \mathsf{C}' \arrow[r, "\zeta'"'] & \mathsf{A}'[1] \end{tikzcd}$$

Then there exist an object D*, an exact triangle,*

(2.1.2)
$$\mathsf{D} \longrightarrow \mathsf{B}' \oplus \mathsf{C} \longrightarrow \mathsf{C}' \xrightarrow{+1},$$

and a map $\delta : \mathsf{B} \to \mathsf{D}$*, such that if* λ *denotes the composition*

$$\lambda : \mathsf{D} \longrightarrow \mathsf{B}' \oplus \mathsf{C} \xrightarrow{0 \oplus \mathrm{id}_{\mathsf{C}}} \mathsf{C},$$

then $\lambda \circ \delta = \psi$ *and*

(2.1.3) *α admits a left inverse if and only if δ admits one, $\delta' : \mathsf{D} \to \mathsf{B}$ such that $\psi \circ \delta' = \lambda$, and*

(2.1.4) *α is an isomorphism if and only if δ is an isomorphism.*

Proof. Let $\eta : \mathsf{B}' \oplus \mathsf{C} \to \mathsf{C}'$ be the natural map induced by $-\psi'$ on B' and γ on C, and let D be the object that completes η to an exact triangle as in (2.1.2).

Next consider the following diagram:

$$\begin{tikzcd}
 & & \mathsf{A}'[1] \arrow[ddll, "+1"'] \arrow[drr, dashed, "\exists"] & & \\
\mathsf{C} \arrow[urr, dashed, "\exists\,\vartheta"] \arrow[d, "+1"'] & & & & \mathsf{D}[1] \arrow[llll, dashed, "+1"', "\exists"] \arrow[ddll, "+1"] \\
\mathsf{B}' \arrow[rrrr, "\psi'"'] \arrow[drr, "(-\,\mathrm{id}_{\mathsf{B}'},0)"'] & & & & \mathsf{C}' \arrow[uull, "\zeta'"'] \arrow[u] \\
 & & \mathsf{B}' \oplus \mathsf{C} \arrow[uull, "0_{\mathsf{B}'}\oplus \mathrm{id}_{\mathsf{C}}"] \arrow[urr, "\eta"'] & &
\end{tikzcd}$$

The bottom triangle $(B', C', B' \oplus C)$ is commutative with the maps indicated. The triangles with one edge common with the bottom triangle are exact triangles with the obvious maps. Then by the octahedral axiom, the maps in the top triangle, denoted by the broken arrows, exist and form an exact triangle.

Observe that it follows that the induced map $\vartheta : \mathsf{C} \to \mathsf{A}'[1]$ agrees with $\zeta' \circ (\eta\big|_{\mathsf{C}}) = \zeta' \circ \gamma$, which in turn equals $\alpha[1] \circ \zeta$ by (2.1.1).

Therefore, one has the following commutative diagram where the rows form exact triangles:

$$\begin{tikzcd}
\mathsf{A} \arrow[r] \arrow[d, "\alpha"'] & \mathsf{B} \arrow[r, "\psi"] \arrow[d, dashed, "\exists\delta"'] & \mathsf{C} \arrow[r, "\zeta"] \arrow[d, "\mathrm{id}_{\mathsf{C}}"'] & \mathsf{A}[1] \arrow[d, "\alpha[1]"'] \\
\mathsf{A}' \arrow[r] \arrow[u, bend right, "\alpha'"'] & \mathsf{D} \arrow[r, "\lambda"'] \arrow[u, dashed, bend right, "\exists\delta'"'] & \mathsf{C} \arrow[r, "\vartheta"'] \arrow[u, bend right, "\mathrm{id}_{\mathsf{C}}"'] & \mathsf{A}'[1] \arrow[u, bend right, "\alpha'[1]"']
\end{tikzcd}$$

and as $\vartheta = \alpha[1] \circ \zeta$ it follows that a δ exists that makes the diagram commutative. Now, if α admits a left inverse $\alpha' : \mathsf{A}' \to \mathsf{A}$, then $\alpha'[1] \circ \vartheta = \alpha'[1] \circ \alpha[1] \circ \zeta = \zeta = \zeta \circ \mathrm{id}_{\mathsf{C}}$, and hence δ admits a left inverse, $\delta' : \mathsf{D} \to \mathsf{B}$ and clearly $\psi \circ \delta' = \lambda$. The converse is even simpler: If $\psi \circ \delta' = \lambda$, then α' exists and it must be a left inverse. Finally, it is obvious from the diagram that α is an isomorphism if and only if δ is an isomorphism. $\square$

We are now ready to prove our DB criterion.

2.2. **Proof of** (1.6). Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
\mathscr{I}_{W \subseteq X} \arrow[r] \arrow[d, "\varrho"'] & \mathscr{O}_X \arrow[r] \arrow[d] & \mathscr{O}_W \arrow[r, "+1"] \arrow[d] & {} \\
\mathscr{R}f_* \mathscr{I}_{F \subseteq Y} \arrow[r] \arrow[u, dashed, bend right, "\varrho'"'] & \mathscr{R}f_* \mathscr{O}_Y \arrow[r] & \mathscr{R}f_* \mathscr{O}_F \arrow[r, "+1"] & {}
\end{tikzcd}$$

It follows by Lemma 2.1 that there exists an object Q, an exact triangle in the derived category of $\mathscr{O}_X$-modules,

$$\mathsf{Q} \longrightarrow \mathcal{R}f_*\mathscr{O}_Y \oplus \mathscr{O}_W \longrightarrow \mathcal{R}f_*\mathscr{O}_F \xrightarrow{+1}, \tag{2.2.1}$$

and a map $\vartheta : \mathscr{O}_X \to \mathsf{Q}$ that admits a left inverse, $\vartheta' : \mathsf{Q} \to \mathscr{O}_X$.

Now consider a similar commutative diagram with exact rows:

$$\begin{tikzcd}
\mathsf{J} \arrow[r] \arrow[d, dashed, "\psi"'] & \underline{\Omega}^0_X \arrow[r] \arrow[d, "\mu"'] & \underline{\Omega}^0_W \arrow[r, "+1"] \arrow[d, "\nu"'] & {} \\
\mathcal{R}f_*\mathsf{K} \arrow[r] & \mathcal{R}f_*\underline{\Omega}^0_Y \arrow[r] & \mathcal{R}f_*\underline{\Omega}^0_F \arrow[r, "+1"] & {}
\end{tikzcd}$$

Here J and K represent the appropriate objects in the appropriate derived categories that make the rows exact triangles. The vertical maps μ and ν exist and form a commutative square because of the basic properties of the Deligne-Du Bois complex, and their existence and compatibility imply the existence of the map ψ by the basic properties of derived categories.

It follows, again by Lemma 2.1, that there exists an object D, an exact triangle in the derived category of $\mathscr{O}_X$-modules,

$$\mathsf{D} \longrightarrow \mathcal{R}f_*\underline{\Omega}^0_Y \oplus \underline{\Omega}^0_W \longrightarrow \mathcal{R}f_*\underline{\Omega}^0_F \xrightarrow{+1}, \tag{2.2.2}$$

and a map $\delta : \underline{\Omega}^0_X \to \mathsf{D}$.

By the basic properties of exact triangles, the natural transformation $\Xi : \mathscr{O} \to \underline{\Omega}^0$ induces compatible maps between the exact triangles of (2.2.1) and (2.2.2). We would also like to show that these maps are compatible with the maps ϑ, and δ obtained from Lemma 2.1:

$$\begin{tikzcd}
\mathscr{O}_X \arrow[d, "\lambda"'] \arrow[dr, "\vartheta"'] & & & & \\
\underline{\Omega}^0_X \arrow[dr, "\delta"'] & \mathsf{Q} \arrow[r] \arrow[d, dashed, "\xi"'] \arrow[ul, bend right, "\vartheta'"'] & \mathcal{R}\pi_*\mathscr{O}_Y \oplus \mathscr{O}_W \arrow[r] \arrow[d, "\eta"'] & \mathcal{R}\pi_*\mathscr{O}_F \arrow[r, "+1"] \arrow[d, "\zeta"'] & {} \\
& \mathsf{D} \arrow[r] & \mathcal{R}\pi_*\underline{\Omega}^0_Y \oplus \underline{\Omega}^0_W \arrow[r] & \mathcal{R}\pi_*\underline{\Omega}^0_F \arrow[r, "+1"] & {}
\end{tikzcd}$$

Claim 2.2.3. Under the assumptions of the theorem the above diagram is commutative.

Proof. First observe that if Y, F, and W are all DB, then η and ζ are isomorphisms. Then it follows that ξ is an isomorphism as well. Next consider the 0^{th} cohomology sheaves of all the complexes in the diagram. From the long exact sequence of cohomology induced by exact triangles we obtain the following diagram:

$$\begin{tikzcd}
\mathscr{O}_X \arrow[d, "h^0(\lambda)"'] \arrow[dr, "h^0(\vartheta)"] & & \\
h^0(\underline{\Omega}^0_X) \arrow[dr, "h^0(\delta)"'] & h^0(\mathsf{Q}) \arrow[ul, bend right, "h^0(\vartheta')"'] \arrow[r, hook, "\nu"] \arrow[d, "h^0(\xi)", "\simeq"'] & h^0(\mathscr{R}\pi_*\mathscr{O}_Y \oplus \mathscr{O}_W) \simeq \pi_*\mathscr{O}_Y \oplus \mathscr{O}_W \arrow[d, "h^0(\eta)", "\simeq"'] \\
& h^0(\mathsf{D}) \arrow[r, hook, "\mu"'] & h^0(\mathscr{R}\pi_*\underline{\Omega}^0_Y \oplus \underline{\Omega}^0_W) \simeq \pi_*\mathscr{O}_Y \oplus \mathscr{O}_W
\end{tikzcd}$$

From the commutativity of the exact triangles we obtain that

$$h^0(\eta) \circ \nu = \mu \circ h^0(\xi).$$

Furthermore, the functoriality of the maps $h^0(\lambda)$ and $h^0(\eta)$ (they are induced by Ξ) implies that we also have

$$h^0(\eta) \circ \nu \circ h^0(\vartheta) = \mu \circ h^0(\delta) \circ h^0(\lambda).$$

Then it follows that

$$\mu \circ h^0(\xi) \circ h^0(\vartheta) = \mu \circ h^0(\delta) \circ h^0(\lambda).$$

Observe that μ is injective since $h^{-1}(\mathscr{R}\pi_*\underline{\Omega}^0_F) = 0$, and hence

$$h^0(\xi) \circ h^0(\vartheta) = h^0(\delta) \circ h^0(\lambda).$$

Finally observe that $h^0(\lambda)$ determines the entire map $\lambda : \mathscr{O}_X \to \underline{\Omega}^0_X$ by (2.2.4), and so we obtain that $\xi \circ \vartheta = \delta \circ \lambda$, as desired. □

Claim 2.2.4. Let A, B be objects in a derived category such that $h^i(\mathsf{A}) = 0$ for $i \neq 0$ and $h^j(\mathsf{B}) = 0$ for $j < 0$. Then any morphism $\alpha : \mathsf{A} \to \mathsf{B}$ is uniquely determined by $h^0(\alpha)$.

Proof. By the assumption, the morphism $\alpha : \mathsf{A} \to \mathsf{B}$ can be represented by a morphism of complexes $\widetilde{\alpha} : \widetilde{\mathsf{A}} \to \widetilde{\mathsf{B}}$, where $\mathsf{A} \simeq \widetilde{\mathsf{A}}$ such that $\widetilde{\mathsf{A}}^0 = h^0(\mathsf{A})$ and $\widetilde{\mathsf{A}}^i = 0$ for all $i \neq 0$, and $\mathsf{B} \simeq \widetilde{\mathsf{B}}$ such that $h^0(\widetilde{\mathsf{B}}) \subseteq \widetilde{\mathsf{B}}^0$. However, $\widetilde{\alpha}$ has only one non-zero term, $h^0(\alpha)$. This proves the claim. □

As ξ is an isomorphism, we obtain a map,

$$\lambda' = \vartheta' \circ \xi^{-1} \circ \delta : \underline{\Omega}^0_X \to \mathscr{O}_X.$$

By (2.2.3) $\xi \circ \vartheta = \delta \circ \lambda$, so it follows that

$$\lambda' \circ \lambda = \vartheta' \circ \xi^{-1} \circ \delta \circ \lambda = \vartheta' \circ \xi^{-1} \circ \xi \circ \vartheta = \vartheta' \circ \vartheta = \mathrm{id}_{\mathscr{O}_X}.$$

The last equality follows from the choice of ϑ'. Therefore λ' is a left inverse to λ, and so the statement follows from [Kov99, Thm. 2.3]. □

We have a similar statement for seminormality. The proof is, however, much more elementary.

Proposition 2.3. *Let $f : Y \to X$ be a proper morphism between reduced schemes of finite type over $\mathbb{C}$, $W \subseteq X$ a closed reduced subscheme, and $F := f^{-1}(W)$, equipped with the induced reduced subscheme structure. Assume that the natural map $\mathscr{I}_{W\subseteq X} \to f_*\mathscr{I}_{F\subseteq Y}$ is an isomorphism. Then if Y and W are seminormal, X is as well.*

Proof. First, not yet assuming that W or Y are seminormal, let $\mathscr{O}_W^{sn}$ be the seminormalization of $\mathscr{O}_W$ in $f_*\mathscr{O}_F$ and let $\mathscr{O}_X^{sn}$ be the seminormalization of $\mathscr{O}_X$ in $f_*\mathscr{O}_Y$. It follows from the assumption $\mathscr{I}_{W\subseteq X} \simeq f_*\mathscr{I}_{F\subseteq Y}$ that $\mathscr{O}_W^{sn} \simeq \mathscr{O}_X^{sn}\big/\mathscr{I}_{W\subseteq X}$. Now if W is seminormal (in fact it is enough if $\mathscr{O}_W$ is seminormal in $f_*\mathscr{O}_F$), then this implies that $\mathscr{O}_X\big/\mathscr{I}_{W\subseteq X} \simeq \mathscr{O}_W \simeq \mathscr{O}_W^{sn} \simeq \mathscr{O}_X^{sn}\big/\mathscr{I}_{W\subseteq X}$, and hence $\mathscr{O}_X \simeq \mathscr{O}_X^{sn}$. Finally, if Y is seminormal, then this implies that X is as well. $\square$

Corollary 2.4. *Let $g : X' \to X$ be a finite surjective morphism between normal varieties. Let $Z \subseteq X$ be a reduced (not necessarily normal) subscheme and assume that $Z' := g^{-1}(Z)_{\mathrm{red}}$ is DB. Then Z is as well.*

Remark 2.5. The special case of this statement when $Z = X$ was proved in [Kol95, 12.8.2] for X projective and in [Kov99, 2.5] in general.

Proof of (2.4). Let $\tau : g_*\mathscr{O}_{X'} \to \mathscr{O}_X$ denote the normalized trace map of $X' \to X$ and let $\mathscr{J} = \sqrt{\mathscr{I}_{Z\subseteq X} \cdot g_*\mathscr{O}_{X'}}$. Then it follows from [AM69, 5.14, 5.15] and the fact that Z is reduced, that

$$\tau(\mathscr{J}) \subseteq \sqrt{\mathscr{I}_{Z\subseteq X}} = \mathscr{I}_{Z\subseteq X}.$$

Therefore, τ gives a splitting of $\mathscr{O}_Z \to f_*\mathscr{O}_{Z'}$. The rest follows from (1.6) applied to Z with $W = \emptyset$ or directly by [Kov99, 2.4]. $\square$

3. Dlt models and twisted higher direct images of dualizing sheaves

We will frequently use the following statement in order to pass from an lc pair to its dlt model. Please recall the definition of a *boundary* and a *minimal dlt model* from (1.9).

Theorem 3.1 (Hacon)**.** *Let (X, Δ) be a pair such that X is quasi-projective, Δ is a boundary, and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Then (X, Δ) admits a $\mathbb{Q}$-factorial minimal dlt model $f^{\mathrm{m}} : (X^{\mathrm{m}}, \Delta^{\mathrm{m}}) \to (X, \Delta)$.*

Proof. Let $f : X' \to (X, \Delta)$ be a log resolution that is a composite of blow-ups of centers of codimension at least 2. Note that then there exists an effective f-exceptional divisor C such that $-C$ is f-ample. Let Δ' be a divisor such that $\Delta' - f_*^{-1}\Delta$ is f-exceptional and that

$$K_{X'} + \Delta' \sim_{\mathbb{Q}} f^*(K_X + \Delta).$$

Let $\Delta_{<1} = \Delta - \lfloor \Delta \rfloor$ denote the part of Δ with coefficients strictly less than 1, and write

$$\Delta' = f_*^{-1}\Delta_{<1} + E^+ + F - B,$$

where E^+ denotes the sum of all (not necessarily exceptional) divisors with discrepancy ≤ -1, F the sum of all f-exceptional divisors with discrepancy > -1 and ≤ 0, and B the sum of all f-exceptional divisors with discrepancy > 0. Let $E := \operatorname{red} E^+$, and notice that all of E, F, and B are effective and that all of $E^+ - E, F$, and B are f-exceptional, while $f_*^{-1}\Delta_{<1} + E + F$ contains $f_*^{-1}\Delta$.

Let H be sufficiently ample on X. Then for all $\varepsilon, \mu, \nu \in \mathbb{Q}$,

$$E + (1+\nu)F + \mu(-C + f^*H) = (1-\varepsilon\mu)E + (1+\nu)F + \mu(\varepsilon E - C + f^*H). \tag{3.1.1}$$

If $0 < \varepsilon \ll 1$, then both $-C + f^*H$ and $\varepsilon E - C + f^*H$ are ample, hence $\mathbb{Q}$-linearly equivalent to divisors H_1 and H_2 such that $\Delta' + H_1 + H_2$ has snc support. If $0 < \mu < 1$ and $0 < \nu \ll 1$, then, by the definition of E and F,

$$(X', f_*^{-1}\Delta_{<1} + (1-\varepsilon\mu)E + (1+\nu)F + \mu H_2)$$

is klt, and hence by [BCHM06] it has a ($\mathbb{Q}$-factorial) minimal model $f^{\mathrm{m}} : (X^{\mathrm{m}}, \Delta^{\mathrm{m}}_{\varepsilon,\mu,\nu}) \to X$. By (3.1.1) this is also a minimal model of the pair $(X', f_*^{-1}\Delta_{<1} + E + (1+\nu)F + \mu H_1)$, which is therefore dlt. Let Δ^{m} denote the birational transform of $f_*^{-1}\Delta_{<1} + E + F$ on X^{m}. Then we obtain that $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$ is dlt.

For a divisor $G \subset X'$ (e.g., E, F, C, H_i) appearing above (other than Δ), let G^{m} denote its birational transform on X^{m}. By construction

$$N := K_{X^{\mathrm{m}}} + \Delta^{\mathrm{m}} + \nu F^{\mathrm{m}} + \mu H_1^{\mathrm{m}} \sim_{\mathbb{Q}} K_{X^{\mathrm{m}}} + \Delta^{\mathrm{m}}_{\varepsilon,\mu,\nu}$$

is f^{m}-nef and

$$T := K_{X^{\mathrm{m}}} + \Delta^{\mathrm{m}} + (E^+ - E)^{\mathrm{m}} - B^{\mathrm{m}} \sim_{\mathbb{Q}} (f^{\mathrm{m}})^*(K_X + \Delta)$$

is $\mathbb{Q}$-linearly f^{m}-trivial. Let

$$D^{\mathrm{m}} := \mu C^{\mathrm{m}} + (E^+ - E)^{\mathrm{m}} - \nu F^{\mathrm{m}} - B^{\mathrm{m}}.$$

Then $-D^{\mathrm{m}}$ is $\mathbb{Q}$-linearly f^{m}-equivalent to the difference $N - T$, hence it is f^{m}-nef. Since D^{m} is f-exceptional, $f^{\mathrm{m}}_*\big(D^{\mathrm{m}}\big) = 0$, so D^{m} is effective by [KM98, 3.39]. Choosing $0 < \mu \ll \nu \ll 1$ shows that both F and B disappear on X^{m}, so every f^{m}-exceptional divisor has discrepancy ≤ -1 and hence $(X^{\mathrm{m}}, \Delta^{\mathrm{m}})$ is indeed a minimal dlt model of (X, Δ) as defined in (1.9). $\square$

Theorem 3.2. *Let X be a smooth variety over $\mathbb{C}$ and $D = \sum a_i D_i$ be an effective, integral snc divisor. Let $\mathscr{L}$ be a line bundle on X such that $\mathscr{L}^m \simeq \mathscr{O}_X(D)$ for some $m > \max\{a_i\}$. Let $f : X \to S$ be a projective morphism. Then the sheaves $\mathscr{R}^i f_*\big(\omega_X \otimes \mathscr{L}\big)$ are torsion-free for all i and*

$$\mathscr{R} f_*\big(\omega_X \otimes \mathscr{L}\big) \simeq \sum_i \mathscr{R}^i f_*\big(\omega_X \otimes \mathscr{L}\big)[-i].$$

Proof. If $D = 0$, this is [Kol86a, 2.1] and [Kol86b, 3.1]. The general case can be reduced to this as follows. The isomorphism $\mathscr{L}^m \simeq \mathscr{O}_X(D)$ determines a degree m-cyclic cover $\pi : Y \to X$ with a μ_m-action, and $\omega_X \otimes \mathscr{L}$ is a μ_m-eigensubsheaf of $\pi_*\omega_Y$. In general Y has rational singularities. Let $h : Y' \to Y$ be a μ_m-equivariant resolution and $g : Y' \to S$, the composition $f \circ \pi \circ h$. Then $\mathscr{R} h_*\omega_{Y'} \simeq \omega_Y$; thus

$$\begin{aligned}\mathscr{R} f_*(\pi_*\omega_Y) &\simeq \mathscr{R} f_* \mathscr{R}\pi_*\omega_Y \simeq \mathscr{R} f_*\mathscr{R}\pi_*\mathscr{R} h_*\omega_{Y'} \\ &\simeq \mathscr{R} g_*\omega_{Y'} \simeq \sum_i \mathscr{R}^i g_*\omega_{Y'}[-i] \simeq \sum_i \mathscr{R}^i f_*\pi_*h_*\omega_{Y'}[-i] \simeq \sum_i \mathscr{R}^i f_*\left(\pi_*\omega_Y\right)[-i]\end{aligned}$$

by [Kol86b, 3.1] and because π is finite. As all of these isomorphisms are μ_m-equivariant, taking μ_m-eigensubsheaves on both sides, we obtain the desired statement. Notice that in particular we have proven that $\mathscr{R}^i f_*\big(\omega_X \otimes \mathscr{L}\big)$ is a subsheaf of $\mathscr{R}^i g_*\omega_{Y'}$ which is torsion-free by [Kol86a, 2.1]. $\square$

4. Splitting over the non-klt locus

In the following theorem we show that the DB criterion (1.6) holds in an important situation.

Theorem 4.1. *Let $f : Y \to X$ be a proper morphism with connected fibers between normal varieties. Assume that (Y, Δ) is lc and $K_Y + \Delta \sim_{\mathbb{Q},f} 0$. Set $W := f(\operatorname{nklt}(Y, \Delta))$ and assume that $W \neq X$. Let $\pi : \widetilde{Y} \to Y$ be a proper birational morphism and $F := \widetilde{f}^{-1}(W)_{\mathrm{red}}$, where $\widetilde{f} := f \circ \pi$. Then the natural map*

$$\varrho : \mathscr{I}_W \simeq \widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F)$$

has a left inverse.

Proof. First, observe that if $\tau : \widehat{Y} \to \widetilde{Y}$ is a log resolution of $(\widetilde{Y}, F)$ that factors through π, then it is enough to prove the statement for $\sigma = \pi \circ \tau$ instead of π. Indeed, let $\widehat{F} = \tau^{-1} F$, an snc divisor, and $\widehat{f} = f \circ \sigma$. Suppose that the natural map

$$\widehat{\varrho} : \mathscr{I}_W \simeq \widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \to \mathcal{R}\widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F})$$

has a left inverse, $\widehat{\delta} : \mathcal{R}\widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \to \widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F})$ such that $\widehat{\delta} \circ \widehat{\varrho} = \mathrm{id}_{\mathscr{I}_W}$. Then, as $\widehat{f} = \widetilde{f} \circ \tau$, one has that $\mathcal{R}\widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \simeq \mathcal{R}\widetilde{f}_* \mathcal{R}\tau_* \mathscr{O}_{\widehat{Y}}(-\widehat{F})$, and applying the functor $\mathcal{R}\widetilde{f}_*$ to the natural map $\overline{\varrho} : \mathscr{O}_{\widetilde{Y}}(-F) \simeq \tau_* \mathscr{O}_{\widehat{Y}}(-\widehat{F}) \to \mathcal{R}\tau_* \mathscr{O}_{\widehat{Y}}(-\widehat{F})$ shows that $\widehat{\varrho} = \mathcal{R}\widetilde{f}_*(\overline{\varrho}) \circ \varrho$:

$$\widehat{\varrho} : \mathscr{I}_W \simeq \widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \xrightarrow{\quad \varrho \quad} \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \xrightarrow{\mathcal{R}\widetilde{f}_*(\overline{\varrho})} \mathcal{R}\widehat{f}_* \mathscr{O}_{\widehat{Y}}(-\widehat{F}).$$

Therefore, $\delta = \widehat{\delta} \circ \mathcal{R}\widetilde{f}_*(\overline{\varrho})$ is a left inverse to ϱ, showing that it is indeed enough to prove the statement for σ. In particular, we may replace π with its composition with any further blow up. We will use this observation throughout the proof.

Next write

$$\pi^*(K_Y + \Delta) \sim_{\mathbb{Q}} K_{\widetilde{Y}} + E + \widetilde{\Delta} - B,$$

where E is the sum of all (not necessarily exceptional) divisors with discrepancy -1, B is an effective exceptional integral divisor, and $\lfloor \widetilde{\Delta} \rfloor = 0$. We may assume that $\widetilde{f}^{-1}\widetilde{f}(E)$ is an snc divisor. Since $B - E \geq -F$, we have natural maps

$$\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E).$$

Note that $B - E \sim_{\mathbb{Q},\widetilde{f}} K_{\widetilde{Y}} + \widetilde{\Delta}$; hence by (3.2)

$$\mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E) \cong \sum_i \mathcal{R}^i \widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E)[-i].$$

Thus we get a morphism

$$\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E) \to \widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E).$$

Note that $\pi_* \mathscr{O}_{\widetilde{Y}}(B - E) = \mathscr{I}_{\operatorname{nklt}(Y,\Delta)}$. Furthermore, for any $U \subseteq X$ open subset with preimage $U_Y := f^{-1}(U)$, a global section of $\mathscr{O}_{U_Y}$ vanishes along a fiber of f if and only if it vanishes at one point of that fiber. Thus

$$\widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(-F) = f_* \mathscr{I}_{\operatorname{nklt}(Y,\Delta)} = \widetilde{f}_* \mathscr{O}_{\widetilde{Y}}(B - E).$$

□

5. Log canonical centers

We need the following higher dimensional version of a result of Shokurov [Kol92, 12.3.1]; cf. [Fuj00].

Proposition 5.1. *Let $f : Y \to X$ be a proper morphism with connected fibers between normal varieties. Assume that (Y,Δ) is dlt and $K_Y + \Delta \sim_{\mathbb{Q},f} 0$. For an arbitrary $x \in X$ let U denote an étale local neighbourhood of $x \in X$. Then $f^{-1}(U) \cap \operatorname{nklt}(Y,\Delta)$ is either*

(5.1.1) connected, or

(5.1.2) has 2 connected components, both of which dominate U, and (Y,Δ) is plt near $f^{-1}(x)$.

Proof. We may assume that $X = U$ and then we may also assume that $\operatorname{nklt}(Y,\Delta)$ and $f^{-1}(x) \cap \operatorname{nklt}(Y,\Delta)$ have the same number of connected components.

Write $\Delta = E+\Delta'$, where $E = \operatorname{nklt}(Y,\Delta) = \lfloor\Delta\rfloor$ and (Y,Δ') is klt. Let $E = \sum E_i$ be the decomposition to a sum of the connected components. Pushing forward

$$0 \to \mathscr{O}_Y(-E) \to \mathscr{O}_Y \to \mathscr{O}_E \to 0,$$

we obtain

$$0 \to f_*\mathscr{O}_Y(-E) \to \mathscr{O}_X \to \sum_i f_*\mathscr{O}_{E_i} \to \mathscr{R}^1 f_*\mathscr{O}_Y(-E).$$

Note that $-E \sim_{\mathbb{Q},f} K_Y + \Delta'$. Hence $\mathscr{R}^1 f_*\mathscr{O}_Y(-E)$ is torsion free by (3.2) and by applying [Fuj08, 2.54] (cf. [Amb03, 3.2]) to a resolution of the dlt pair (Y,Δ).

Suppose E_1 does not dominate X. Then $f_*\mathscr{O}_{E_1}$ is a non-zero torsion sheaf; hence the induced map $f_*\mathscr{O}_{E_1} \to \mathscr{R}^1 f_*\mathscr{O}_Y(-E)$ must be zero. This implies that

$$f_*\mathscr{O}_{E_1} \subseteq \operatorname{im}\big[\mathscr{O}_X \to \sum_i f_*\mathscr{O}_{E_i}\big].$$

Since we are working locally near $x \in X$, we may assume that $(f_*\mathscr{O}_{E_1})_x \neq 0$. Observe that the natural projection map $\sum_i f_*\mathscr{O}_{E_i} \to f_*\mathscr{O}_{E_1}$ gives a splitting of the above embedding. Further observe that $\operatorname{im}[\mathscr{O}_X \to \sum_i f_*\mathscr{O}_{E_i}]$ has only one generator near x. This implies that we must have that $f_*\mathscr{O}_{E_1} = \operatorname{im}[\mathscr{O}_X \to \sum_i f_*\mathscr{O}_{E_i}]$ locally near x. In particular, there is at most one E_i that does not dominate X. Furthermore, if E_j does dominate X, then $\mathscr{O}_X \to f_*\mathscr{O}_{E_j}$ is non-zero. This again would contradict $f_*\mathscr{O}_{E_1} = \operatorname{im}\big[\mathscr{O}_X \to \sum_i f_*\mathscr{O}_{E_i}\big]$. Therefore, if E has more than one component, then they all dominate X.

Until now the statement and the proof could have been done birationally, but for the rest we use the MMP repeatedly. Note that the proof is a bit messier than [Kol92, 12.3.1], since we do not have the full termination of MMP.

First we run the $(Y,(1-\varepsilon)E+\Delta')$-MMP; cf. [BCHM06, 1.3.2]. Every step is numerically $K_Y+\Delta$-trivial; hence, by the usual connectedness (cf. [KM98, 5.48]) the E_i stay disjoint. At some point, we must encounter a Fano-contraction $\gamma : (Y^*,(1-\varepsilon)E^*+\Delta^*) \to S$, where E^* is ample on the general fiber. As we established above, every connected component of E^* dominates S. We may assume that E^* is disconnected, as otherwise we are done.

Since the relative Picard number of Y is 1, every connected component of E^* is relatively ample. As E^* is disconnected, all fibers are 1 dimensional. As γ is a Fano-contraction, the generic fiber is $\mathbb{P}^1$, and so E^* can have at most, and hence exactly, two connected components, E_1^* and E_2^*. Since the fibration is numerically

$K_{Y^*}+\Delta^*$-trivial, it follows that the intersection product of either E_i^* with any fiber is 1. In other words, the E_i^* are sections of γ. Since they are also relatively ample, it follows that every fiber is irreducible, and so outside a codimension 2 set on the base, $\gamma : Y^* \to S$ is a $\mathbb{P}^1$-bundle with two disjoint sections. It also follows that Δ^* does not intersect the general fiber; hence $\Delta^* = \gamma^*\Delta_S$ for some $\Delta_S \subset S$. Then since the E_i^* are sections, we have that $(E_i^*, \Delta^*\big|_{E_i^*}) \simeq (S, \Delta_S)$.

We need to prove that $(Y^*, E_1^* + E_2^* + \Delta^*)$ is plt, and for that it is enough to show that $(E_i^*, \Delta^*\big|_{E_i^*})$ is klt for $i = 1, 2$. By the above observation, all we need to prove then is that (S, Δ_S) is klt. Since γ is a $\mathbb{P}^1$-bundle (in codimension 1) with 2 disjoint sections, we have that $K_{Y^*} + E_1^* + E_2^* \sim \gamma^* K_S$ and then that $K_{Y^*} + E_1^* + E_2^* + \Delta^* \sim_{\mathbb{Q}} \gamma^*(K_S + \Delta_S)$. Now we may apply [Kol92, 20.3.3] to a general section of Y^* mapping to S to get that (S, Δ_S) is klt. □

We are now ready to prove our main connectivity theorem.

5.2. **Proof of** (1.7). We may assume that f is surjective, and we replace (Y, Δ) by a $\mathbb{Q}$-factorial dlt model by (3.1). If $Z_1 = Y$, then $Z_2 \subseteq Z_1$, and if $f(Z_2) = X$, then $Z_1 \subseteq Z_1$ satisfies the requirement. Hence we may assume that (Y, Δ) is dlt, $Z_1, Z_2 \subset \lfloor \Delta \rfloor$ are divisors, and Z_2 is disjoint from the generic fiber of f. Then, by localizing at a generic point of $f(Z_1) \cap f(Z_2)$ we reduce to the case when $x := f(Z_1) \cap f(Z_2)$ is a closed point.

By working in a suitable étale neighborhood of x, we may also assume that $Z_1 \cap f^{-1}(x)$ is geometrically connected. Thus it is sufficient to prove that $Z_1 \cap f^{-1}(x)$ contains an lc center.

Since we are now assuming that Z_2 does not dominate X, it follows from (5.1) that $f^{-1}(x) \cap \lfloor \Delta \rfloor$ is connected, and hence there are irreducible divisors

$$V_1 := Z_2, V_2, \dots, V_{m-1}, V_m := Z_1 \quad \text{with} \quad V_i \subset \lfloor \Delta \rfloor$$

such that $f^{-1}(x) \cap V_i \cap V_{i+1} \neq \emptyset$ for $i = 1, \dots, m-1$. By working in the étale topology on X, we may also assume that each $f^{-1}(x) \cap V_i$ is connected.

Next, we prove by induction on i that

$$W_i := V_i \cap \bigcap_{j<i} f^{-1}\big(f(V_j)\big) \tag{5.2.1}$$

contains an lc center of (Y, Δ).

For $i = 1$ the statement of (5.2.1) follows from the fact that $V_1 = Z_2$ is an lc center of (Y, Δ). Next we go from i to $i+1$. Consider $\big(V_i, \mathrm{Diff}_{V_i}(\Delta - V_i)\big)$. Note that every irreducible component of $V_i \cap V_{i+1}$ is an lc center of $\big(V_i, \mathrm{Diff}_{V_i}(\Delta - V_i)\big)$ and by induction and adjunction W_i contains an lc center of $(V_i, \mathrm{Diff}_{V_i}(\Delta - V_i))$. Thus, by induction on the dimension, replacing Y by V_i, Z_1 by $V_i \cap V_{i+1}$, and Z_2 by the lc center contained in W_i, we conclude that $f^{-1}(f(W_i)) \cap V_i \cap V_{i+1}$ contains an lc center U_i of $(V_i, \mathrm{Diff}_{V_i}(\Delta - V_i))$. By inversion of adjunction, U_i is also an lc center of (Y, Δ), and it is contained in W_{i+1}.

At the end we obtain that

$$W_m = Z_1 \cap f^{-1}\big(f(Z_2)\big) \cap f^{-1}\big(f(V_2)\big) \cap \dots \cap f^{-1}\big(f(V_{m-1})\big) \tag{5.2.2}$$

contains an lc center of (Y, Δ). Observe that W_m contains $Z_1 \cap f^{-1}(x)$ and is contained in $Z_1 \cap f^{-1}\big(f(Z_2)\big)$. These two are the same; hence we are done.

Remark 5.2.3. The statement of (1.7) is stronger than that which has been previously known [Kaw97, 1.5], [Amb03, 4.8], [Fuj08, 3.45]. The usual claim in a similar situation has been that every irreducible component of $f(Z_1) \cap f(Z_2)$ is dominated by an lc center, whose precise location was not known.

It would also be interesting to find a proof of (1.7) without using the MMP.

Definition 5.3. Let X be a normal scheme. A *minimal quasi-log canonical structure*, or simply a *minimal qlc structure*, on X is a proper surjective morphism $f : (Y, \Delta) \to X$ where

(5.3.1) (Y, Δ) is a log canonical pair,
(5.3.2) Δ is effective,
(5.3.3) $\mathscr{O}_X \simeq f_*\mathscr{O}_Y$, and
(5.3.4) $K_Y + \Delta \sim_{f,\mathbb{Q}} 0$.

Remark 5.3.5. This definition is similar to Ambro's definition of a quasi-log variety [Amb03, 4.1], [Fuj08, 3.29]. The main difference here, underscored by the word "minimal" in the definition, is the additional assumption (5.3.4).

One should also note that what Fujino calls a quasi-log variety is essentially X together with a qlc stratification which we define next.

Definition 5.4. Let X be a normal scheme and assume that it admits a minimal qlc structure $f : (Y, \Delta) \to X$. We define the *qlc stratification* of X *with respect to* f or simply the f*-qlc stratification* in the following way: Let $\mathcal{H}_Y$ denote the set containing all the lc centers of (Y, Δ), including the components of Δ and Y itself. For each $Z \in \mathcal{H}_Y$ let

$$W_Z := f(Z) \setminus \bigcup_{\substack{Z' \in \mathcal{H}_Y \\ f(Z) \not\subseteq f(Z')}} f(Z').$$

Further, let

$$\mathcal{H}_{X,f} = \{W_Z | Z \in \mathcal{H}_Y\}.$$

Notice that it is possible that $W_Z = W_{Z'}$ for some $Z \neq Z'$, but in $\mathcal{H}_{X,f}$ they are only counted once. Then

$$X = \coprod_{W \in \mathcal{H}_{X,f}} W$$

will be called the *qlc stratification* of X *with respect to* f, and the strata will be called the f*-qlc strata.* Note that by construction each f-qlc stratum is reduced.

Definition 5.5. Let X_i be varieties that admit minimal qlc structures, $f_i : (Y_i, \Delta_i) \to X_i$, and let $W_i = \bigcup_{j=1}^{r_i} W_{i,j}$ be unions of some f-qlc strata on X_i for $i = 1, 2$. Assume that there exists a morphism $\alpha : W_1 \to W_2$. Then we will say that α is a *qlc stratified morphism* if for every f-qlc stratum $W_{2,j}$, its preimage $\alpha^{-1}W_{2,j}$ is equal to a disjoint union of f-qlc strata $\bigcup_\alpha W_{1,j_\alpha}$ for an appropriate set of α's.

Using our new terminology we have the following important consequence of (1.7).

Corollary 5.6. *Let X be a normal variety with a minimal qlc structure, $f : (Y, \Delta) \to X$. Then the closure of any union of some f-qlc strata is also a union of some f-qlc strata.*

Proof. It is enough to prove this for the closure of a single f-qlc stratum. By definition, the difference between the closure and the f-qlc stratum is a union of intersections of that single stratum with the images of lc centers. By (1.7) this is covered by a union of f-qlc strata. □

In (1.11) we observed that DB singularities are seminormal, so it follows from Theorem 6.2 that the closure of any union of f-qlc strata is seminormal. On the other hand it also follows from the somewhat simpler (5.6) and similar results from [Fuj08].

Proposition 5.7 ([Amb03], [Fuj08, §3])**.** *Let X be a normal variety that admits a minimal qlc structure, $f:(Y,\Delta)\to X$. Then each f-qlc stratum is normal, and the closure of any union of f-qlc strata is seminormal.*

Proof. Let T be the closure of a union of some f-qlc strata. Then by Corollary 5.6 and [Fuj08, 3.39(i)] (cf. [Amb03, 4.4]) the qlc centers of T are exactly the f-qlc strata (of X) that lie inside T. It follows by [Fuj08, 3.33] that T is seminormal and by [Fuj08, 3.44] (cf. [Amb03, 4.7]) that each f-qlc stratum is normal. □

Corollary 5.8. *Let X be a normal variety with a minimal qlc structure, $f:(Y,\Delta)\to X$. Then the support of the conductor subscheme of the closure of any union of f-qlc strata is contained in a smaller dimensional union of f-qlc strata.*

Proof. As individual f-qlc strata are normal, it follows that the conductor subscheme is contained in the part of the closure that was subtracted in (5.4). By (1.7) this is a union of f-qlc strata, and as it does not contain any (maximal) component of the original union, the dimension of each contributing strata has to be strictly smaller. □

6. Log canonical singularities are Du Bois

Lemma 6.1. *Let X be a normal variety and $f:(Y,\Delta)\to X$ a minimal qlc structure on X. Let $W\in\mathcal{H}_{X,f}$ be a qlc stratum of X and $\overline{W}$ its closure in X. Then there exist a normal variety $\widehat{W}$ with a minimal qlc structure $g:(Z,\Sigma)\to\widehat{W}$ such that $g(\operatorname{nklt}(Z,\Sigma))\neq\widehat{W}$ and a finite surjective qlc stratified morphism $\widehat{W}\to\overline{W}$.*

Proof. We will repeat the following procedure until all the desired conditions are satisfied.

Iteration: Note that we may replace (Y,Δ) by a $\mathbb{Q}$-factorial dlt model by (3.1). Recall that in that case $\overline{W}$ is the union of some f-qlc strata by (5.6). If $\overline{W}=X$ and $f(\operatorname{nklt}(Y,\Delta))\neq X=\overline{W}$, then by choosing $(Z,\Sigma)=(Y,\Delta)$, $g=f$, and $\widehat{W}=X$, the desired conditions are satisfied. Otherwise, there exists an irreducible component $E\subseteq\lfloor\Delta\rfloor$ such that $\overline{W}\subseteq f(E)$. Consider the Stein factorization of $f\big|_E$:

$$f\big|_E : E \xrightarrow{f_E} G \xrightarrow{\sigma} f(E)\ .$$

Observe that then $f_E:(E,\operatorname{Diff}_E\Delta)\to G$ is a minimal qlc structure, G is normal, and σ is finite. Let $W_1=\sigma^{-1}(W)$ denote the preimage of W, and $\overline{W}_1$ its closure in G. By (1.7) the f_E-qlc stratification of G is just the preimage of the restriction of the f-qlc stratification of X to $f(E)$, so the induced morphism $\overline{W}_1\to\overline{W}$ is a qlc stratified morphism. As long as $\overline{W}\neq f(E)$ or $f(\operatorname{nklt}(E,\operatorname{Diff}_E\Delta))=f(E)$, we may go back to the beginning and repeat our procedure with X replaced with G and

W replaced with $\sigma^{-1}(W)$ without changing the induced qlc structure on W. By noetherian induction this process must end, and then we will have $\overline{W} = f(E)$ and $f(\mathrm{nklt}(E, \mathrm{Diff}_E \Delta)) \neq f(E)$. Then $f_E : (E, \mathrm{Diff}_E \Delta) \to G$ and $\sigma : \widehat{W} := G \to \overline{W}$ satisfy the desired conditions. $\square$

Theorem 1.5 is implied by the following.

Theorem 6.2. *If X admits a minimal qlc structure, $f : (Y, \Delta) \to X$, then the closure of any union of f-qlc strata is DB.*

Proof. Let $T \subseteq X$ be a union of f-qlc strata. By (5.6) we know that $\overline{T}$, the closure of T in X, is also a union of f-qlc strata, so by replacing T with $\overline{T}$ we may assume that T is closed. Let $\widetilde{T}$ denote the normalization of T. We have that $T = \bigcup_{W \in \mathcal{J}} W$ for some $\mathcal{J} \subseteq \mathcal{H}_{X,f}$, so T is seminormal by (5.7). For $W \in \mathcal{J}$, we will denote the closure of W in X by $\overline{W}$. Note that by definition $\overline{W}$ is contained in T. In order to prove that T is DB, we will apply a double induction the following manner:

- *induction on* $\dim X$*:* Assume that the statement holds if X is replaced with a smaller dimensional variety admitting a minimal qlc structure.
- *induction on* $\dim T$*:* Assume that the statement holds if X is fixed and T is replaced with a smaller dimensional subvariety of X which is also a union of f-qlc strata.

First assume that $X \neq T$. Then $\overline{W}$ must also be a proper subvariety of X for any $W \in \mathcal{J}$. Then by (6.1) for each $W \in \mathcal{J}$ there exists a normal variety $\widehat{W}$ with a minimal qlc structure and a finite surjective qlc stratified morphism $\sigma : \widehat{W} \to \overline{W}$. By induction on $\dim X$ we obtain that $\widehat{W}$ is DB. Then by (2.4) it follows that the normalization of $\overline{W}$ is DB as well. Note that $\widehat{W}$ is normal, but may not be the normalization of $\overline{W}$, however σ factors through the normalization morphism.

Let $\mathcal{J}' \subseteq \mathcal{J}$ be a subset such that $T = \bigcup_{W \in \mathcal{J}'} \overline{W}$ and $\overline{W} \not\subseteq \overline{W'}$ for any $W, W' \in \mathcal{J}'$. Then let $\widehat{T} := \coprod_{W \in \mathcal{J}'} \widehat{W}$ and let $\widehat{\tau} : \widehat{T} \to T$ be the natural morphism. Observe that as the $\widehat{W}$ have DB singularities, so does $\widehat{T}$, and then by (2.4) it follows that for the normalization of T, $\tau : \widetilde{T} \to T$, $\widetilde{T}$ is DB as well. Next let $Z \subset T$ be the conductor subscheme of T and let $\widetilde{Z}$ be its preimage in $\widetilde{T}$. Then since T is seminormal, both Z and $\widetilde{Z}$ are reduced and

$$\mathscr{I}_{Z \subseteq T} = \tau_* \mathscr{I}_{\widetilde{Z} \subseteq \widetilde{T}}. \tag{6.2.1}$$

Claim 6.2.2. Let $\Gamma \subseteq T$ be a reduced subscheme that contains the conductor Z and let $\widetilde{\Gamma}$ be its preimage in $\widetilde{T}$. Then $\mathscr{I}_{\Gamma \subseteq T} \subseteq \mathscr{O}_T \subseteq \tau_* \mathscr{O}_{\widetilde{T}}$ is also a $\tau_* \mathscr{O}_{\widetilde{T}}$ ideal, i.e., $\mathscr{I}_{\Gamma \subseteq T} = \mathscr{I}_{\Gamma \subseteq T} \cdot \tau_* \mathscr{O}_{\widetilde{T}}$. In particular,

$$\mathscr{I}_{\Gamma \subseteq T} = \tau_* \mathscr{I}_{\widetilde{\Gamma} \subseteq \widetilde{T}}. \tag{6.2.3}$$

Proof. If $\mathscr{J} = \mathscr{I}_{\Gamma \subseteq T}$ is a $\tau_* \mathscr{O}_{\widetilde{T}}$ ideal, then (6.2.3) follows, so it is enough to prove the first statement. Let $\mathscr{I} = \mathscr{I}_{Z \subseteq T}$. Clearly, $\mathscr{J} \cdot \tau_* \mathscr{O}_{\widetilde{T}} \subseteq \mathscr{I} \cdot \tau_* \mathscr{O}_{\widetilde{T}} = \mathscr{I} \subseteq \mathscr{O}_T$. Then $\mathscr{J} \cdot \tau_* \mathscr{O}_{\widetilde{T}} \subseteq \mathscr{O}_T \cap \sqrt{\mathscr{J} \cdot \tau_* \mathscr{O}_{\widetilde{T}}}$, which is equal to $\sqrt{\mathscr{J}}$ by [AM69, 5.14]. In turn, $\sqrt{\mathscr{J}} = \mathscr{J}$ by assumption, so we have that $\mathscr{J} \cdot \tau_* \mathscr{O}_{\widetilde{T}} \subseteq \mathscr{J}$. $\square$

By (5.8) Z is contained in a union of f-qlc strata whose dimension is smaller then $\dim T$. Replace Z by this union and $\widetilde{Z}$ by its reduced preimage on $\widetilde{T}$. Then Z is DB by induction on $\dim T$. In the sequel we are only going to use one property of Z that followed from being the conductor, namely the equality in (6.2.1). However, by (6.2.2) this remains true for the new choice of Z. Next let $\widehat{Z} = (\widehat{\tau}^{-1}Z)_{\mathrm{red}} \subset \widehat{T}$ be the reduced preimage of Z (as well as of $\widetilde{Z}$) in $\widehat{T}$. The following diagram shows the connections between the various objects we have defined so far:

$$\begin{tikzcd}
\text{normal, admits minimal qlc structure} \arrow[r, dotted, bend right] & \coprod \widehat{W} = \widehat{T} \supset \widehat{Z} \arrow[dd, "\hat{\tau}"'] \arrow[dr] & & \text{preimage of } Z \arrow[ll, dotted] \\
\text{finite} \arrow[r, dotted] & {} & \widetilde{T} \supset \widetilde{Z} \arrow[dl, "\tau"'] & \text{preimage of } Z \arrow[l, dotted] \\
& X \supseteq \bigcup \overline{W} = T \supset Z & \text{normalization of } T \arrow[ul, dotted] & \text{the smaller dimensional union of } f\text{-qlc strata containing the conductor of } T \arrow[ll, dotted]
\end{tikzcd}$$

As we replaced the conductor with a union of f-qlc strata it was contained in and as each $\widehat{W}$ admits a minimal qlc structure compatible with the part of the minimal qlc structureof X that lies in T, it follows that $\widehat{Z}$ is also a union of qlc strata on $\widehat{T}$ and the morphism $\widehat{Z} \to \widetilde{Z}$ is a qlc stratified morphism. Then since $\dim \widehat{T} < \dim X$, by replacing X with $\widehat{T}$ shows that $\widehat{Z}$ is DB by induction on $\dim X$. In turn this implies that $\widetilde{Z}$ is DB by (2.4).

Therefore, by now we have proved that $\widetilde{T}$, Z, and $\widetilde{Z}$ all have DB singularities, so by using (6.2.2) and (1.6) we conclude that T is DB as well.

Now assume that $X = T$ and hence $X = T = \widetilde{T}$. Let $f : (Y, \Delta) \to X$ be a minimal qlc structure and $W = f(\mathrm{nklt}(Y, \Delta))$. By (6.1) we may assume that $W \neq X$ by replacing X by a finite cover. Note that by (2.4) it is enough to prove that this finite cover is DB.

Then let $\pi : \widetilde{Y} \to Y$ be a log resolution and $F := (f \circ \pi)^{-1}(W)$, an snc divisor. By (4.1) the natural map $\varrho : \mathscr{I}_W = f_*\mathscr{O}_{\widetilde{Y}}(-F) \to \mathcal{R}f_*\mathscr{O}_{\widetilde{Y}}(-F)$ has a left inverse. Finally, (1.6) implies that $T = X$ is DB. $\square$

Definition 6.3. Let $\phi : X \to B$ be a flat morphism. We say that ϕ is a *DB family* if X_b is DB for all $b \in B$.

Definition 6.4. Let $\phi : X \to B$ be a flat morphism. We say that ϕ is a *family with potentially lc fibers* if for all closed points $b \in B$ there exists an effective $\mathbb{Q}$-divisor $D_b \subset X_b$ such that (X_b, D_b) is log canonical.

Definition 6.5 ([KM98, 7.1])**.** Let X be a normal variety, $D \subset X$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier, and $\phi : X \to B$ a non-constant morphism to a smooth curve B. We say that ϕ is a *log canonical morphism*, or an *lc morphism*, if $(X, D + X_b)$ is lc for all closed points $b \in B$.

Remark 6.6. Notice that for a family with potentially lc fibers it is not required that the divisors D_b also form a family over B. On the other hand, if $\phi : X \to B$ is a family with potentially lc fibers, B is a smooth curve and there exists an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier and $D\big|_{X_b} = D_b$, then ϕ is an lc morphism by inversion of adjunction [Kaw07].

Further, observe that if $\phi : (X, D) \to B$ is an lc morphism, then for any $b \in B$, choosing $(Y, \Delta) = (X, D + X_b)$ and $f : (Y, \Delta) \to X$, the identity of X gives an f-qlc stratification of X such that X_b is a union of f-qlc strata. In particular, it follows by (6.2) that X_b is DB. Note that if X_b is reducible, then (1.5) would not suffice here.

Corollary 6.7. *Let $\phi : X \to B$ be either a family with potentially lc fibers or an lc morphism. Then ϕ is a DB family.*

Proof. Follows directly from (6.2). $\square$

7. Invariance of cohomology for DB morphisms

The following notation will be used throughout this section.

Notation 7.1. Let $\pi : \mathbb{P}^N_B \to B$ be a projective N-space over B, $\iota : X \hookrightarrow \mathbb{P}^N_B$ a closed embedding, and $\phi := \pi \circ \iota$. Further, let $\mathscr{O}_\pi(1)$ be a relatively ample line bundle on $\mathbb{P}^N_B$, and denote by $\omega^\bullet_\phi$ the relative dualizing complex $\phi^! \mathscr{O}_B$ and by $h^{-i}(\omega^\bullet_\phi)$ its $-i^{\text{th}}$ cohomology sheaf. We will also use the notation $\omega_\phi := h^{-n}(\omega^\bullet_\phi)$, where $n = \dim(X/B)$. Naturally, these definitions automatically apply for π in place of ϕ by choosing $\iota = \mathrm{id}_{\mathbb{P}^N_B}$.

Lemma 7.2. *Let $b \in B$. Then*

$$h^{-i}(\omega^\bullet_\phi) \simeq \mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi) \quad \text{and} \quad h^{-i}(\omega^\bullet_{X_b}) \simeq \mathscr{E}xt^{N-i}_{\mathbb{P}^N_b}(\mathscr{O}_{X_b}, \omega_{\mathbb{P}^N_b}).$$

In particular, $h^{-i}(\omega^\bullet_\phi) = 0$ and $h^{-i}(\omega^\bullet_{X_b}) = 0$ if $i < 0$ or $i > N$.

Proof. By Grothendieck duality ([Har66, VII.3.3]; cf. [Har77, III.7.5]),

$$h^{-i}(\omega^\bullet_\phi) \simeq h^{-i}(\mathscr{R}\mathscr{H}om_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega^\bullet_\pi)) \simeq h^{-i}(\mathscr{R}\mathscr{H}om_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi)[N]) \simeq \mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi).$$

The same argument obviously implies the equivalent statement for $h^{-i}(\omega^\bullet_{X_b})$.

Furthermore, clearly $\mathscr{E}xt^j_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi) = 0$ and $\mathscr{E}xt^j_{\mathbb{P}^N_b}(\mathscr{O}_{X_b}, \omega_{\mathbb{P}^N_b}) = 0$ if $j < 0$ and hence $h^{-i}(\omega^\bullet_\phi) = 0$ and $h^{-i}(\omega^\bullet_{X_b}) = 0$ if $i > N$. Since $\mathbb{P}^N_b$ is smooth and thus all the local rings are regular, it also follows that $\mathscr{E}xt^j_{\mathbb{P}^N_b}(\mathscr{O}_{X_b}, \omega_{\mathbb{P}^N_b}) = 0$ if $j > N$ and hence $h^{-i}(\omega^\bullet_{X_b}) = 0$ if $i < 0$.

Next, consider the restriction map [AK80, 1.8],

$$\varrho^{-i}_b : \mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi)\big|_{X_b} \to \mathscr{E}xt^{N-i}_{\mathbb{P}^N_b}(\mathscr{O}_{X_b}, \omega_{\mathbb{P}^N_b}).$$

We have just observed that the target of the map is 0 if $i < 0$. In particular, ϱ^{-i}_b is surjective in that range. Then by [AK80, 1.9] ϱ^{-i}_b is an isomorphism, and therefore $h^{-i}(\omega^\bullet_\phi) = 0$ if $i < 0$. $\square$

Lemma 7.3. *Let $\mathscr{F}$ be a coherent sheaf on X, $i \in \mathbb{N}$, and assume that $\mathscr{R}^i\pi_*(\mathscr{F}(-q))$ is locally free for $q \gg 0$. Then*

$$\pi_* \mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{F}, \omega_\pi(q)) \simeq \mathscr{H}om_B(\mathscr{R}^i\pi_*(\mathscr{F}(-q)), \mathscr{O}_B) \qquad \text{for } q \gg 0.$$

Proof. Let $q \gg 0$ and $U \subseteq B$ be an affine open set such that $\mathscr{R}^i\pi_*(\mathscr{F}(-q))\big|_U$ is free. Then by [Har77, III.6.7] and [Har66, III.5.2],

$$\begin{aligned} H^0(\pi^{-1}(U), \mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{F}, \omega_\pi(q))) &\simeq \mathrm{Ext}^{N-i}_{\mathbb{P}^N_U}(\mathscr{F}_U(-q), \omega_{\pi_U}) \\ &\simeq \mathrm{Hom}_U(\mathscr{R}^i\pi_*\mathscr{F}(-q)\big|_U, \mathscr{O}_U) \simeq H^0(U, \mathscr{H}om_B(\mathscr{R}^i\pi_*\mathscr{F}(-q), \mathscr{O}_B)). \quad \square \end{aligned}$$

The following statement and its consequences will be needed in the proof of (7.9). It is likely known to experts, but we could not find an appropriate reference.

Lemma 7.4. *Let Z be a complex scheme of finite type and $\phi_\bullet : Z_\bullet \to Z$ be a hyperresolution. Let $\pi : W \to Z$ be a morphism such that $\psi_\bullet : W_\bullet := W \times_Z Z_\bullet \to W$ is also a hyperresolution. Let $\pi_i : W_i \to Z_i$ be the morphisms induced by π, and assume that the natural transformation $\mathcal{L}\pi^*\mathcal{R}\phi_{\bullet\,*} \to \mathcal{R}\psi_{\bullet\,*}\mathcal{L}\pi_\bullet^*$ induces an isomorphism*

$$\mathcal{L}\pi^*\mathcal{R}\phi_{\bullet\,*}\mathscr{O}_{Z_\bullet} \simeq \mathcal{R}\psi_{\bullet\,*}\mathcal{L}\pi_\bullet^*\mathscr{O}_{Z_\bullet}.$$

Then

$$\mathcal{L}\pi^*\underline{\Omega}^0_Z \simeq \underline{\Omega}^0_W.$$

In particular, if Z has only DB singularities, then W has only DB singularities.

Remark 7.4.1. See [Du81, GNPP88, PS08, KS09] for details on hyperresolutions.

Corollary 7.5. *Let Z be a complex scheme of finite type with only DB singularities and let $\widetilde{Z} \to Z$ be a smooth morphism. Then $\widetilde{Z}$ has DB singularities.* □

Corollary 7.6. *Let Z be a complex scheme of finite type with only DB singularities and let $H \subseteq Z$ be a general member of a basepoint free linear system. Then H has DB singularities.* □

Corollary 7.7. *Let Z be a complex scheme of finite type with only DB singularities and $\mathscr{M}$ be a semi-ample line bundle on Z. Let $\pi : W \to Z$ be the cyclic cover associated to a general section of $\mathscr{M}^m$ for some $m \gg 0$; cf.* [KM98, 2.50]. *Then W has only DB singularities.*

Proof. One can easily prove that π satisfies the conditions of (7.4) or argue as follows: By (7.5) the total space M of $\mathscr{M}$ has DB singularities, and then the statement follows by (7.4) applied to the embedding $W \subseteq M$ [Kol95, 9.4]. □

Proof of (7.4). The hyperresolutions $\phi_\bullet$ and $\psi_\bullet$ fit into the commutative diagram

$$\begin{tikzcd} Z_\bullet \arrow[d, "\phi_\bullet"'] & W_\bullet \arrow[l, "\pi_\bullet"'] \arrow[d, "\psi_\bullet"] \\ Z & W \arrow[l, "\pi"] \end{tikzcd}$$

We also obtain the following representations of the Deligne-Du Bois complexes of Z and W:

$$\underline{\Omega}^0_Z \simeq \mathcal{R}\phi_{\bullet\,*}\mathscr{O}_{Z_\bullet} \quad \text{and} \quad \underline{\Omega}^0_W \simeq \mathcal{R}\psi_{\bullet\,*}\mathscr{O}_{W_\bullet}.$$

Then by assumption

$$\mathcal{L}\pi^*\underline{\Omega}^0_Z \simeq \mathcal{L}\pi^*\mathcal{R}\phi_{\bullet\,*}\mathscr{O}_{Z_\bullet} \simeq \mathcal{R}\psi_{\bullet\,*}\mathcal{L}\pi_\bullet^*\mathscr{O}_{Z_\bullet} \simeq \mathcal{R}\psi_{\bullet\,*}\mathscr{O}_{W_\bullet} \simeq \underline{\Omega}^0_W.$$

□

We will also need the base-change theorem of Du Bois and Jarraud [DJ74, Théorème] (cf. [Du81, 4.6]):

Theorem 7.8. *Let $\phi : X \to B$ be a projective DB family. Then $\mathcal{R}^i\phi_*\mathscr{O}_X$ is locally free of finite rank and compatible with arbitrary base change for all i.* □

The next theorem is our main flatness and base change result.

Theorem 7.9. *Let $\phi : X \to B$ be a projective DB family and $\mathscr{L}$ be a relatively ample line bundle on X. Then*

(7.9.1) *the sheaves $h^{-i}(\omega^{\bullet}_{\phi})$ are flat over B for all i,*

(7.9.2) *the sheaves $\phi_*(h^{-i}(\omega^{\bullet}_{\phi})\otimes\mathscr{L}^q)$ are locally free and compatible with arbitrary base change for all i and for all $q \gg 0$, and*

(7.9.3) *for any base change morphism $\vartheta : T \to B$ and for all i,*

$$\big(h^{-i}(\omega^{\bullet}_{\phi})\big)_T \simeq h^{-i}(\omega^{\bullet}_{\phi_T}).$$

Remark 7.9.4. For a coherent sheaf $\mathscr{F}$ on X, the pushforward $\phi_*\mathscr{F}$ being compatible with arbitrary base change means that for any morphism $\vartheta : T \to B$,

$$\big(\phi_*\mathscr{F}\big)_T \simeq \big(\phi_T\big)_*\mathscr{F}_T.$$

In particular, (7.9.2) implies that for any $\vartheta : T \to B$,

$$\big(\phi_*(h^{-i}(\omega^{\bullet}_{\phi})\otimes\mathscr{L}^q)\big)_T \simeq \big(\phi_T\big)_*\Big(\big(h^{-i}(\omega^{\bullet}_{\phi})\big)_T\otimes\mathscr{L}^q_T\Big).$$

Combined with (7.9.3), this means that for any $\vartheta : T \to B$,

$$\big(\phi_*(h^{-i}(\omega^{\bullet}_{\phi})\otimes\mathscr{L}^q)\big)_T \simeq \big(\phi_T\big)_*(h^{-i}(\omega^{\bullet}_{\phi_T})\otimes\mathscr{L}^q_T). \tag{7.9.5}$$

Proof of (7.9). We may asssume that $B = \operatorname{Spec} R$ is affine. By definition, $\mathscr{L}^m$ is relatively generated by global sections for all $m \gg 0$. For a given $m \in \mathbb{N}$, choose a general section $\vartheta \in H^0(X, \mathscr{L}^m)$ and consider the $\mathscr{O}_X$-algebra

$$\mathscr{A}_m = \bigoplus_{j=0}^{m-1}\mathscr{L}^{-j} \simeq \bigoplus_{j=0}^{\infty}\mathscr{L}^{-j}t^j\Big/\big(t^m-\vartheta\big)$$

as in [KM98, 2.50]. Let $Y^m := \operatorname{Spec}_X \mathscr{A}_m$ and let $\sigma : Y^m \to X$ be the induced finite morphism. Then

$$\mathscr{R}^i(\phi\circ\sigma)_*\mathscr{O}_{Y^m} \simeq \mathscr{R}^i\phi_*(\sigma_*\mathscr{O}_{Y^m}) \simeq \mathscr{R}^i\phi_*\mathscr{A}_m \simeq \bigoplus_{j=0}^{m-1}\mathscr{R}^i\phi_*\mathscr{L}^{-j}$$

for all i and all $b \in B$. Note that by construction, this direct sum decomposition is compatible with arbitrary base change. By (7.7), $\phi\circ\sigma$ is again a DB family and hence $\mathscr{R}^i(\phi\circ\sigma)_*\mathscr{O}_{Y^m}$ is locally free and compatible with arbitrary base change by (7.8). Since ϕ is flat and $\mathscr{L}$ is locally free, it follows that then $\mathscr{R}^i\phi_*\mathscr{L}^{-j}$ is locally free and compatible with arbitrary base change for all i and for all $j > 0$. Then taking $\mathscr{F} = \mathscr{O}_X$ and applying [Har77, III.6.7], (7.2), and (7.3), we obtain that

$$\phi_*(h^{-i}(\omega^{\bullet}_{\phi})\otimes\mathscr{L}^q) \simeq \mathscr{H}om_B(\mathscr{R}^i\phi_*\mathscr{L}^{-q}, \mathscr{O}_B) \qquad \text{for } q \gg 0. \tag{7.9.6}$$

This proves (7.9.2), and then (7.9.1) follows easily by an argument similar to the one used to prove the equivalence of (i) and (ii) in the proof of [Har77, III.9.9].

To prove (7.9.3) we will use induction on i. Notice that it follows trivially for $i < 0$ (and $i > N$, but we will not use that fact) by (7.2), so the start of the induction is covered. Consider the pull back map,

$$\varrho^{-i}_T : \underbrace{\big(\mathscr{E}xt^{N-i}_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi)\big)_T}_{\left(h^{-i}(\omega^{\bullet}_{\phi})\right)_T} \to \underbrace{\mathscr{E}xt^{N-i}_{\mathbb{P}^N_T}(\mathscr{O}_{X_T}, \omega_{\mathbb{P}^N_T})}_{h^{-i}(\omega^{\bullet}_{\phi_T})}.$$

By the inductive hypothesis ϱ_T^{-j} is an isomorphism, and $\mathcal{E}xt^{N-j}_{\mathbb{P}^N_B}(\mathscr{O}_X, \omega_\pi) \simeq h^{-j}(\omega_\phi^\bullet)$ is flat over B by (7.9.1). Then by [AK80, 1.9], $\varrho_T^{-(j+1)}$ is also an isomorphism. This proves (7.9.3). $\square$

Lemma 7.10. *Let X be a subscheme of $\mathbb{P}^N$, $\mathscr{F}$ a coherent sheaf on X and $\mathscr{N}$ a fixed line bundle on $\mathbb{P}^N$. Then $\mathscr{F}$ is S_k at x if and only if $\mathcal{E}xt^j_{\mathbb{P}^N}(\mathscr{F}, \mathscr{N})_x = 0$ for all $j > N - k$.*

Proof. Since $\mathscr{O}_{\mathbb{P}^N,x}$ is a regular local ring,

$$d := \operatorname{depth}_{\mathscr{O}_{X,x}} \mathscr{F}_x = \operatorname{depth}_{\mathscr{O}_{\mathbb{P}^N,x}} \mathscr{F}_x = N - \operatorname{proj\,dim}_{\mathscr{O}_{\mathbb{P}^N,x}} \mathscr{F}_x.$$

Therefore, $d \geq k$ if and only if $\operatorname{Ext}^j_{\mathscr{O}_{\mathbb{P}^N,x}}(\mathscr{F}_x, \mathscr{N}_x) = 0$ for all $j > N - k$. $\square$

Using our results in this section we obtain a criterion for Serre's S_k condition, analogous to [KM98, 5.72], in the relative setting.

Theorem 7.11. *Let $\phi : X \to B$ be a projective DB family, $x \in X$ and $b = \phi(x)$. Then X_b is S_k at x if and only if*

$$h^{-i}(\omega_\phi^\bullet)_x = 0 \qquad \textit{for } i < k.$$

Proof. Let $\mathscr{F} = \mathscr{O}_{X_b}$, $j = N - i$ and $\mathscr{N} = \omega_{\mathbb{P}^N_b}$. Then (7.3) and (7.10) imply that X_b is S_k at x if and only if $h^{-i}(\omega_{X_b}^\bullet)_x = 0$ for $i < k$. Then the statement follows from (7.9.3) and Nakayama's lemma. $\square$

The following result asserts the invariance of the S_k property in DB families:

Theorem 7.12. *Let $\phi : X \to B$ be a projective DB family and $U \subseteq X$ be an open subset. Assume that B is connected and the general fiber $U_{b_{gen}}$ of $\phi\big|_U$ is S_k. Then all fibers U_b of $\phi\big|_U$ are S_k.*

Proof. Suppose that the fiber U_b of $\phi\big|_U$ is not S_k. Then by (7.11) there exists an $i < k$ such that $h^{-i}(\omega_\phi^\bullet)_x \neq 0$ for some $x \in U_b$. Let Z be an irreducible component of $\operatorname{supp} h^{-i}(\omega_\phi^\bullet)$ such that $Z \cap U_b \neq \emptyset$. It follows that $Z \cap U$ is dense in Z. By (7.9.1) $h^{-i}(\omega_\phi^\bullet)$ is flat over B, and thus Z and then also $Z \cap U$ dominate B. However, that implies that $Z \cap U_{b_{\rm gen}} \neq \emptyset$, contradicting the assumption that $U_{b_{\rm gen}}$ is S_k, and hence the proof is complete. $\square$

As mentioned in the Introduction, our main application is the following.

Corollary 7.13. *Let $\phi : X \to B$ be a projective family with potentially lc fibers or a projective lc morphism and let $U \subseteq X$ be an open subset. Assume that B is connected and the general fiber $U_{b_{gen}}$ of $\phi\big|_U$ is S_k (resp. CM). Then all fibers U_b of $\phi\big|_U$ are S_k (resp. CM).*

Proof. Follows directly from (6.2) and (7.12). $\square$

The following example shows that the equivalent statement does not hold in mixed characteristic.

Example 7.14 (Schröer)**.** Let S be an ordinary Enriques surface in characteristic 2 (see [CD89, p. 77] for the definition of ordinary). Then S is liftable to characteristic 0 by [CD89, 1.4.1]. Let $\eta : Y \to \operatorname{Spec} R$ be a family of Enriques surfaces such that the special fiber is isomorphic to S and the general fiber is an Enriques surface of characteristic 0.

Let $\zeta : Z \to \operatorname{Spec} R$ be the family of the projectivized cones over the members of the family η. I.e., for any $t \in \operatorname{Spec} R$, Z_t is the projectivized cone over Y_t. Since $K_{Y_t} \equiv 0$ for all $t \in \operatorname{Spec} R$, we obtain that ζ is both a projective family with potentially lc fibers and a projective lc morphism. By the choice of η, the dimension of the cohomology group $H^1(Y_b, \mathscr{O}_{Y_b})$ jumps: it is 0 on the general fiber and 1 on the special fiber. Consequently, by (7.15), the general fiber of ζ is CM, but the special fiber is not.

Recall the following CM condition used in the above example:

Lemma 7.15. *Let E be a smooth projective variety over a field of arbitrary characteristic and let Z be the cone over E. Then Z is CM if and only if $h^i(E, \mathscr{O}_E(m)) = 0$ for $0 < i < \dim E$ and $m \in \mathbb{Z}$.*

Proof. See [Kol08, Ex. 71] and [Kov99, 3.3]. □

The most natural statement along these lines would be if we did not have to assume the existence of the projective compactification of the family $U \to B$. This is related to the following conjecture, which is an interesting and natural problem on its own:

Conjecture 7.16. *Let $\psi : U \to B$ be an affine, finite type lc morphism. Then there exists a base change morphism $\vartheta : T \to B$ and a projective lc morphism $\phi : X \to T$ such that $U_T \subseteq X$ and $\psi_T = \phi\big|_{U_T}$.*

We expect that (7.16) should follow from an argument using MMP techniques, but it might require parts that are at this time still open, such as the abundance conjecture. On the other hand, (7.16) would clearly imply the following strengthening of (7.13):

Conjecture-Corollary 7.17. *Let $\psi : U \to B$ be a finite type lc morphism. Assume that B is connected and the general fiber of ψ is S_k (resp. CM). Then all fibers are S_k (resp. CM).*

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Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, New Jersey 08544-1000

E-mail address: `kollar@math.princeton.edu`

Department of Mathematics, University of Washington, Seattle, Washington 98195-4350

E-mail address: `skovacs@uw.edu`