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ON THE CASTELNUOVO-MUMFORD REGULARITY OF THE COHOMOLOGY RING OF A GROUP

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Dave Benson, in [2], conjectured that for any finite group G and any prime p the Castelnuovo-Mumford regularity of the cohomology ring, $H^*(G, \mathbb{F}_p)$, is zero. He showed that $\operatorname{reg}(H^*(G, \mathbb{F}_p)) \geq 0$ and succeeded in proving equality when the difference between the dimension and the depth is at most two.

The purpose of this paper is to prove Benson's Regularity Conjecture as a corollary of the following result.

Theorem 0.1. *If a compact Lie group G acts on a smooth manifold M with finite dimensional mod-p homology, then* $\operatorname{reg}(H^*_G(M, \mathbb{F}_p)) \leq \dim M - \dim G$.

By taking G to be finite and M to be a point and using Benson's inequality, we obtain a proof of Benson's original conjecture.

Corollary 0.2. *For any finite group G we have* $\operatorname{reg}(H^*(G, \mathbb{F}_p)) = 0$.

We record here one easily stated consequence.

Proposition 0.3. *For any non-trivial finite group, the cohomology ring $H^*(G, \mathbb{F}_p)$ is generated by elements in degree at most $|G| - 1$ and the relations between them (as a graded commutative algebra) are generated in degrees at most $2(|G| - 1)$.*

This bound is weak, although it can be improved somewhat at the cost of a more complicated formulation, but previously no such bound was known.

We go on to prove some other conjectures of Benson on the regularity of the cohomology of other classes of groups [3], specifically for compact Lie groups and virtual Poincaré duality groups.

The proof uses standard techniques in equivariant cohomology based on work of Quillen [30] and a paper of Duflot [14].

David Green and Simon King have verified Benson's Conjecture computationally for all groups of order less than 256 [20].

There is a survey of the properties of group cohomology from the point of view of commutative algebra by Benson [3].

1. Castelnuovo-Mumford regularity

We will work in the category of $\mathbb{Z}$-graded rings and modules, so maps will respect the grading, elements will be homogeneous, etc., without specific mention. For a module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, we will write $M_{\geq d} = \bigoplus_{i \geq d} M_i$ and similarly for other

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inequalities. Let k be a field and let R be a finitely generated commutative graded k-algebra in non-negative degrees with R_0 finite dimensional over k. Let $\mathfrak{m} = R_{>0}$ denote the ideal of elements in positive grading (the maximal homogeneous ideal if $R_0 = k$) and let M be a (graded) R-module. For any (homogeneous) ideal $I < R$ the I-torsion in M is $\Gamma_I(M) = \{m \in M \mid \exists n \in \mathbb{N},\ I^n m = 0\}$. The local cohomology $H^i_I(M)$ is then defined to be the ith right derived functor of $\Gamma_I(M)$. We will be interested in $H^i_{\mathfrak{m}}(M)$.

There are other equivalent definitions of local cohomology, which make some of its properties evident, for example via Čech complexes. For more information on local cohomology see [12, 9, 16, 24]. Most authors do not treat the graded case explicitly, but the first two of these sources do.

Let $a_i(R, M)$ denote the maximum degree of a non-zero element of $H^i_{\mathfrak{m}}(M)$ (possibly ∞ if unbounded or $-\infty$ if $H^i_{\mathfrak{m}}(M) = 0$). The Castelnuovo-Mumford regularity (or just regularity) of M over R is, by definition,

$$\operatorname{reg}(R, M) = \sup_i \{a_i(R, M) + i\}.$$

We refrain from defining regularity for rings that are not noetherian, in order to avoid problems with local cohomology.

Proposition 1.1. *Let R and S be commutative graded noetherian rings and let $f : R \to S$ be a homomorphism such that S is finite over R. Let M be a graded S-module; then the regularity of M is the same, whether considered as an S-module or as an R-module, that is* $\operatorname{reg}(S, M) = \operatorname{reg}(R, M)$.

Proof. This follows easily from the Independence Theorem for local cohomology ([22, 5.7] or [9, 13.1.6]), which in turn follows from the characterisation of local cohomology in terms of complexes; see [32]. $\square$

If T, say, is a graded commutative noetherian ring, such as a cohomology ring, we set $S = T^{\mathrm{ev}}$ to be the part in even degrees. Then S is a commutative noetherian ring, and for any T-module V we define $\operatorname{reg}(V) = \operatorname{reg}(S, V)$. By Proposition 1.1 we would get the same answer by calculating $\operatorname{reg}(R, V)$ for any noetherian subring R of S such that T is finitely generated over R. In particular, $\operatorname{reg}(T)$ will mean $\operatorname{reg}(T^{\mathrm{ev}}, T)$.

For example, if G is a compact Lie group and M is a G-space with $H_*(M, \mathbb{F}_p)$ finite, then the regularity of the equivariant cohomology ring $H^*_G(M, \mathbb{F}_p)$ can be calculated over $H^{\mathrm{ev}}_G(M, \mathbb{F}_p)$, over $H^{\mathrm{ev}}(G, \mathbb{F}_p)$, or indeed over any convenient noetherian subring of these over which it is still finite.

A particularly useful case is when R is a Noether normalisation of S, that is a polynomial subring of S over which S is finitely generated. These always exist and their generators are often referred to as a homogeneous system of parameters. Restricting to any normalisation will yield the same value for the regularity.

Now suppose that $R = k[x_1, \ldots, x_n]$ is a polynomial ring in which the generators have arbitrary positive degrees $|x_i| = \deg(x_i)$ (polynomial will always imply commutative for us). We set $\sigma(R) = \sum_{i=1}^n (|x_i| - 1)$.

Let M be an R-module that is zero in large negative degrees and consider the minimal (graded) projective resolution of M (projective is equivalent to free in this case):

$$\cdots \to P_1 \to P_0 \to M \to 0.$$

Let $\rho_i(R, M)$ be the maximum degree of a non-zero element of $(R/\mathfrak{m}) \otimes_R P_i$ (possibly ∞ or $-\infty$), which is equal to the maximum degree of a generator of P_i. Define

$$\operatorname{Preg}(R, M) = \sup_i\{\rho_i(R, M) - i\} - \sigma(R).$$

This form of the definition first appeared in a paper of Benson [2]. The usual definition does not contain a σ-term, because all the d_i are supposed to be in degree 1, so $\sigma(R) = 0$.

Proposition 1.2. *For R a polynomial ring over a field and M a finitely generated R-module, we have* $\operatorname{Preg}(R, M) = \operatorname{reg}(R, M)$.

This is a consequence of the Local Duality Theorem (see [22, 6.3] or [12, 3.6.19]); for a proof see [2, 5.7] or [32].

We do not really need Preg for the purposes of this paper, but it lies behind many of the applications of regularity; one of these is to obtain degree bounds on the generators and relations of a ring.

Given a finitely generated graded k-algebra S in non-negative degrees with $\dim_k S_0 < \infty$, either commutative or graded commutative, and an integer N, let $\tau_N^k S$ be the k-algebra (commutative or graded commutative, the same as S) determined by the generators and relations of S that occur in degrees at most N. We will normally write just $\tau_N S$. There is a canonical map $\tau_N S \to S$, which is an isomorphism in degrees up to and including N. The next result is taken from [32].

Proposition 1.3. *Let S be a commutative or graded commutative, finitely generated, graded k-algebra in non-negative degrees with $\dim_k S_0 < \infty$. Let $R = k[x_1, \dots, x_m]$ and suppose that there is a map $f : R \to S$ such that S is finitely generated over R (e.g. if R is a Noether normalisation of S). Then:*

(1) *if $N \geq \max\{\operatorname{reg}(S) + \sigma(R),\ \deg(x_i)\}$, then $\tau_N S \to S$ is a surjection;*
(2) *if $N \geq \max\{2(\operatorname{reg}(S) + \sigma(R)),\ \operatorname{reg}(S) + \sigma(R) + 1,\ \deg(x_i)\}$, then $\tau_N S \to S$ is an isomorphism;*
(3) *if $N \geq \max\{\operatorname{reg}(S) + \sigma(R) + 1,\ \deg(x_i)\}$ and if $\tau_N S$, considered an R-module, is generated in degrees at most N, then $\tau_N S \to S$ is an isomorphism.*

Remark. Benson [2, §10] defines $\tau_N S$ and proves a version of Proposition 1.3 in the case when S is the cohomology of a group.

The other properties of regularity that we will need are summed up in the next lemma. For any graded R-module V we use $V(u)$ to denote V shifted down in degree by u.

Lemma 1.4. *Let R be a commutative noetherian k-algebra in non-negative degrees with $\dim_k R_0 < \infty$ and let V be an R-module.*

(1) $\operatorname{reg}(V(u)) = \operatorname{reg}(V) - u$.
(2) *If R is a polynomial ring, then* $\operatorname{reg}(R) = -\sigma(R)$.
(3) *If F is a graded R-module that is non-zero only in finitely many degrees, then $\operatorname{reg}(F)$ is equal to the top degree in which F is non-zero.*
(4) *If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then $\operatorname{reg}(B) \leq \max\{\operatorname{reg}(A),\ \operatorname{reg}(C)\}$, with equality if the sequence is split.*
(5) *If R' is another commutative noetherian k-algebra and V' is an R'-module, then $\operatorname{reg}(R \otimes_k R', V \otimes_k V') = \operatorname{reg}(R, V) + \operatorname{reg}(R', V')$.*

Proof. Part (1) is clear from the definitions.

Part (2) is clear if we calculate $\operatorname{Preg}(R, R)$. Alternatively use part (4) to reduce to the case of one or no variables, where there is an easy explicit injective resolution. In the former case it is $k[x, x^{-1}] \to k[x, x^{-1}]/k[x]$, which shows that $a_1(R) = -\sigma(R) - 1$ and the other $a_i(R)$ are $-\infty$.

For part (3), take a Noether normalisation P of R in which each of the variables has degree greater than the difference in degree between the top and bottom. Then the action of P on F factors through k and by Proposition 1.1 we have $\operatorname{reg}(R, F) = \operatorname{reg}(P, F) = \operatorname{reg}(k, F)$, which is equal to the top degree.

Part (4) follows from the long exact sequence for local cohomology.

Part (5) is a consequence of the Künneth Theorem for local cohomology:

$$H^*_{(I+I')}(V \otimes_k V') \cong H^*_I(V) \otimes_k H^*_{I'}(V').$$

This can be seen by using the characterisation of local cohomology in terms of Čech complexes; the tensor product of such a complex for V with one for V' is one for $V \otimes V'$. Another proof is given in [4, 2.6]. □

As an example we calculate the regularity of the cohomology ring of a finite abelian group.

Lemma 1.5. *If A is a finite abelian group, then* $\operatorname{reg}(H^*(A, \mathbb{F}_p)) = 0$.

Proof. By the Künneth Theorem in group cohomology and Lemma 1.4(5), we can reduce to the case of a cyclic group. In this case $H^{\mathrm{ev}}(A) = k[x]$, with x in degree 2, so it has regularity -1. Also $H^*(A)$ is free over $k[x]$ with basis elements in degrees 0 and 1, so it has regularity 0, by Lemma 1.4(1), (2), and (4). □

2. Reduction to elementary abelian groups

We present a proof for compact Lie groups, but the reader who is only interested in finite groups can just take all the groups to be finite (with the exception that the unitary group must, of course, remain infinite).

Fix a prime p; all cohomology will be taken with coefficients in the field $\mathbb{F}_p$. By the equivariant (or Borel) cohomology $H^*_G(X)$ we mean $H^*(EG \times_G X)$, where $EG \to BG$ is the universal principal G-bundle. Following Quillen, we will sometimes write H^*_G for $H^*(BG)$ when this makes the notation less confusing.

A basic property of equivariant cohomology is the following one ([30, 2.2]).

Lemma 2.1. *For any compact Lie group G, the $\mathbb{F}_p$-algebra H^*_G is finitely generated. If G acts on a space X with finite dimensional homology (i.e. $\dim_{\mathbb{F}_p} H^*(X) < \infty$), then $H^*_G(X)$ is finite over H^*_G.*

It follows from Proposition 1.1 that $\operatorname{reg}(H^*_G(X)) = \operatorname{reg}(H^*_G, H^*_G(X))$ (recall that we defined $\operatorname{reg}(H^*_G(X)) = \operatorname{reg}(H^{\mathrm{ev}}_G(X), H^*_G(X))$).

We also need a slight variant on another basic property.

Lemma 2.2. *Let G and H be compact Lie groups and suppose that their product $G \times H$ acts on a Hausdorff space X in such a way that H acts freely. Then $H^*_{G\times H}(X) \cong H^*_G(X/H)$ as rings or as H^*_G-modules.*

Proof. This is proved in [30, 1.12], except that there the group G does not appear. All we need to do is to replace X by $EG \times_G X$ in Quillen's formulation. □

There is also a well-known version of this result for discrete groups (cf. [11, VII, 7]).

Given a compact Lie group G, embed it in a unitary group $U = U(n)$. Let S be a maximal elementary abelian p-subgroup of U.

Given a G-space X, consider the product $X \times U$ as a $G \times S$-space, where G acts diagonally on the left and S acts on the right on the second factor only. We will write $X \times_G U$ for $G\backslash(X \times U)$.

Lemma 2.3. *If X has finite dimensional homology, then*

$$\operatorname{reg}(H^*_G(X \times U/S)) = \operatorname{reg}(H^*_S(X \times_G U)).$$

Proof. Both G and S act freely on $X \times U$. Applying Lemma 2.2 in two different ways yields

$$H^*_G(X \times U/S) \cong H^*_{G\times S}(X \times U) \cong H^*_S(X \times_G U)$$

as rings.

Notice that $X \times U$ has finite dimensional homology, by the Künneth Theorem; hence the rings are noetherian, by Lemma 2.1, and the regularity is defined. □

The next result is also due to Quillen ([30, 6.5]).

Proposition 2.4. $H^*_G(X \times U/S) \cong H^*_G(X) \otimes_k H^*(U/S)$ *as H^*_G-modules.*

In fact, what Quillen writes is $H^*_G(X \times U/S) \cong H^*_G(X) \otimes_{H^*_U} H^*_S$. But in the course of the proof he shows that $H^*_S \cong H^*_U \otimes_k H^*(U/S)$ (or just take $G = 1$ and $X = BU$ in his original formulation), so our form follows.

We now come to the main result of this section.

Proposition 2.5. *Suppose that the G-space X has finite dimensional homology. Then* $\operatorname{reg}(H^*_G(X)) = \operatorname{reg}(H^*_S(X \times_G U)) - \dim U$.

Proof. Apply Lemma 1.4(5) to the formula in Proposition 2.4 to obtain

$$\operatorname{reg}(H^*_S(X \times_G U)) = \operatorname{reg}(H^*_G(X)) + \operatorname{reg}(H^*(U/S)).$$

The last term is equal to $\dim U$, by Lemma 1.4(3). □

Notice that this proposition implies that, in order to prove Theorem 0.1 for the compact Lie group G (and the manifold M), it suffices to prove the case of the elementary abelian p-group S (and the manifold $X \times_G U$), since $\dim(X \times_G U) = \dim X + \dim U - \dim G$.

3. A topological digression

Quillen in his paper [30] uses sheaf cohomology (with coefficients in the constant sheaf $\mathbb{Z}/p$), whilst many other authors prefer singular cohomology. Here we endeavour to reconcile these two theories. None of this material is new, but it is necessary to assemble the facts from the literature carefully.

One approach is to adapt the proofs of the results of Quillen that we used in Section 2 in terms of singular cohomology. In that case we want to replace the Leray spectral sequence by the Serre spectral sequence of a Serre fibration; we need to ensure that the relevant maps are Serre fibrations.

The best way to do this seems to be to incorporate into the definition of a free action of G on X the additional condition that the fibre sequence $G \to X \to X/G$ should be a principal G-bundle, that is, that it is locally trivial. Locally triviality is transparently well-behaved with respect to the constructions that we need to make and it guarantees that we have a Serre fibration (see e.g. [19, 11.4]). In any case, this is a very weak condition since, by a theorem of Gleason [17], it is automatically satisfied provided that the action is free in the usual sense and the space X is completely regular. The details of this method are left to the reader.

An alternative approach is to show that the two cohomology theories coincide on all spaces in some class. A suitable class is that of paracompact Hausdorff locally contractible spaces, as is shown in [7, III, 1.1]. This might look more familiar if we go via Čech cohomology: it is well known that sheaf cohomology agrees with Čech cohomology on paracompact Hausdorff spaces ([18, 5.10.1]) and Čech cohomology agrees with singular cohomology on paracompact Hausdorff locally contractible spaces ([31, 6.9.5]).

The problem remains that we are usually given information about X and we need to know about $EG \times_G X$. In fact, if X is paracompact, Hausdorff, and locally contractible, then so is $EG \times_G X$, as we now explain. Note that EG and BG have the requisite properties, by construction.

Hausdorff and locally contractible follow easily from the fact that the fibre map $X \to EX \times_G X \to BG$ is locally trivial, so we concentrate on paracompactness. First we show that $EG \times X$ is paracompact. The product of two paracompact spaces need not be paracompact, but EG is σ-compact (the union of countably many compact subspaces) and the product of a paracompact space with a regular σ-compact space is paracompact, by a result of Michael [28]. That $EG \times_G X$ is paracompact follows from another result of Michael [29]: the image of a paracompact space under a proper map is paracompact.

4. The elementary abelian case

All manifolds will be smooth, and if a compact Lie group acts on one, the action will be supposed to be smooth. But otherwise the manifolds can be quite general, in particular they can be open or have boundary, and different connected components can have different dimensions (in which case the dimension will mean the maximum dimension of a component).

The reader will lose little by supposing that the manifold is closed, but the extra generality will be useful later.

Our aim is to prove the following slight generalisation of the elementary abelian case of Theorem 0.1.

Proposition 4.1. *If a finite elementary abelian p-group S acts on a smooth manifold M, then* $\operatorname{reg}(H^*_S, H^*_S(M)) \leq \dim M$. *In particular, if M has finite dimensional homology, then* $\operatorname{reg}(H^*_S(M)) \leq \dim M$.

The second part of the proposition follows from the first because $H^*_S(M)$ will be noetherian, by Lemma 2.1, so $\operatorname{reg}(H^*_S, H^*_S(M)) = \operatorname{reg}(H^*_S(M))$ by Proposition 1.1.

The remark after Proposition 2.5 shows that Proposition 2.5 and Proposition 4.1 combine to prove Theorem 0.1.

The proof relies heavily on a paper of Duflot [14]. Her work is also used by Henn, Lannes, and Schwartz [23] to prove a result with a similar flavour to regularity. It is also claimed in [23] that Duflot's results, although stated only for odd primes, are also valid at the prime 2. This is not hard to verify: Chern classes are replaced by Stiefel-Whitney classes and there is no need to find a complex structure on the normal bundles. Duflot also requires the subspaces $M_{(i)}$ defined below to have only finitely many components, but this is unnecessary. All that needs to be done is to change sum to product in the proof of [14, Proposition 5].

First observe that we can always remove the boundary of the manifold M. This is because the inclusion $M - \partial M \subseteq M$ is a homotopy equivalence (there is always a collar on ∂M, [8, V, 1]). It therefore induces an isomorphism on equivariant cohomology. The reason for doing this is that Duflot's argument depends on finding tubular neighbourhoods for certain submanifolds, and when the ambient manifold has no boundary, these always exist without preconditions ([8, VI, 2.2]).

We now summarise Duflot's results, paraphrasing slightly along the lines of [23]. With S and M as in the statement of the proposition, define $M_{(i)}$ to be the submanifold of M consisting of points with isotropy group of rank exactly i and define M_i to be the submanifold of points with isotropy group of rank at least i. Let

$$F_i = \ker(H_S^*(M) \to H_S^*(M - M_i)).$$

It is not hard to see that $M_{(i)}$ is a submanifold of $M - M_{i+1}$. Let T_i be the Thom space of the normal bundle; Theorem 1 of [14] is as follows.

Theorem 4.2. *There is an isomorphism of H_S^*-modules*

$$F_i/F_{i+1} \cong \tilde{H}_S^*(T_i).$$

Only an isomorphism of vector spaces is stated in [14], but it is clear from the construction that it is an isomorphism of H_S^*-modules.

Basic properties of Thom spaces imply that if $M_{(i),d}$, say, is the union of the components of $M_{(i)}$ with codimension d, then

$$\tilde{H}_S^*(T_{i,d}) \cong H_S^*(M_{(i),d})(-d).$$

We thus obtain an explicit filtration of $H_S^*(M)$, and the basic properties of regularity in Lemma 1.4 show that

$$\operatorname{reg}(H_S^*, H_S^*(M)) \le \max_{i,d}\{\operatorname{reg}(H_S^*, H_S^*(M_{(i),d})) + d\}.$$

Thus it suffices to show that $\operatorname{reg}(H_S^*, H_S^*(M_{(i),d})) \le \dim(M_{(i),d})$, since $\dim(M_{(i),d}) + d = \dim(M)$. In fact we will split up $M_{(i),d}$ according the isotropy group and prove the bound on each piece. Let $V \le S$ be of rank i and set $N = M_{(i),d}^V$, a union of components of $M_{(i),d}$; let W be a complement to V in S.

Because V acts trivially on N, we have $ES \times_S N \cong BV \times (EW \times_W X)$. Now $H_S^* \cong H_V^* \otimes H_W^*$ and as a module over this we have $H_S^*(N) \cong H_V^* \otimes H_W^*(N)$, the two sides of the tensor products not interacting. It follows from Lemma 1.4(5) that $\operatorname{reg}(H_S^*, H_S^*(N)) = \operatorname{reg}(H_V^*) + \operatorname{reg}(H_W^*, H_W^*(N))$.

But W acts freely on N, and thus Lemma 2.2 tells us that $H_W^*(N) \cong H^*(N/W)$, which vanishes in degrees higher than $\dim N$. It follows from Lemma 1.4(3) that $\operatorname{reg}(H_W^*, H_W^*(N)) \le \dim N \le \dim M_{(i),d}$.

This completes the proofs of Proposition 4.1 and Theorem 0.1.

Remark. It really is necessary to require the space M in the theorems to be a manifold, or at least to impose some significant restrictions on it. The following example was described to us by Jean Lannes [27].

Let $p = 2$ and let G be an elementary abelian 2-group of rank 2. Let G act on a set Z of six points in such a way that each subgroup of rank 1 appears as an isotropy group. Consider the suspension SZ: then $\operatorname{reg}(H_G^*(SZ)) = 3$.

This means that for a G-space X the regularity $\operatorname{reg}(H^*_G(X))$ can be regarded as an obstruction to the possibility of X being G-homotopy equivalent to a manifold of given dimension. It is left to the reader to verify that in the example above SZ is G-homotopy equivalent to a 3-manifold; so the obstruction is sharp in this case.

5. Discrete groups

Let G be a Lie group with finitely many components, not necessarily compact. There is a maximal compact subgroup $K \leq G$ and $G/K \cong \mathbb{R}^c$ as smooth manifolds, where $c = c(G) = \dim G - \dim K$ (see e.g. [26, XV, 3]).

Proposition 5.1. *Let G be a Lie group with finitely many components and let $\Gamma < G$ be a discrete subgroup. Suppose that H^*_Γ is noetherian. Then* $\operatorname{reg}(H^*_\Gamma) \leq c(G)$.

Proof. As in the proof of Lemma 2.3 we can use Lemma 2.2 twice to obtain

$$H^*_\Gamma \cong H^*_\Gamma(G/K) \cong H^*_{\Gamma \times K}(G) \cong H^*_K(\Gamma\backslash G).$$

So $\operatorname{reg}(H^*_\Gamma) \leq \dim G - \dim K = c(G)$, by Theorem 0.1. □

Now suppose that Γ is a discrete group with a normal subgroup Γ' of finite index and that Γ/Γ' acts on a manifold M in such a way that the natural map $H^*_\Gamma \to H^*_{\Gamma/\Gamma'}(M)$ is an isomorphism. This applies to a great many classes of groups (see e.g. [23, after II, 2.3]). We know from Theorem 0.1 that $\operatorname{reg}(H^*_\Gamma) \leq \dim M$.

Proposition 5.2. *Suppose that a discrete group Γ acts properly discontinuously with finite stabilisers on a contractible manifold X and that there is a normal subgroup Γ' of Γ of finite index, torsion free and with $H^*_{\Gamma'}$ finite dimensional. Then* $\operatorname{reg}(H^*_\Gamma) \leq \dim X$.

Proof. Take $M = X/\Gamma'$ in the above discussion. Then $H^*(M) \cong H^*(\Gamma')$, so it is finite dimensional. □

Remark. If X/Γ' is a closed manifold, then Γ' will be a Poincaré duality group of dimension $\dim X$, by [11, VIII, 10, Example 1].

6. Fusion systems

It is now easy to deal with the cohomology of a fusion system in the sense of [10]. This includes block cohomology.

Proposition 6.1. *If $\mathcal{F}$ is a saturated fusion system on a finite p-group, then* $\operatorname{reg}(H^*(\mathcal{F})) \leq 0$.

Proof. The cohomology $H^*(\mathcal{F})$ is a subring of the cohomology of the underlying group $H^*(S)$. Furthermore $H^*(\mathcal{F})$ is noetherian, $H^*(S)$ is noetherian over it, and $H^*(\mathcal{F})$ is a summand of $H^*(S)$ as an $H^*(\mathcal{F})$-module, by [10, 5.2, 5.5].

This is what we need to see that $\operatorname{reg}(H^*(\mathcal{F})) \leq \operatorname{reg}(H^*(S)) \leq 0$. □

7. Lower bounds

For a finite group G there is a Greenlees spectral sequence [21, 1] $H^{s,t}_{\mathfrak{m}}(H^*(BG)) \Rightarrow H_{s+t}(BG)$. Since $H_0(BG) \cong k \neq 0$, we must have $H^{s,-s}(H^*(BG)) \neq 0$ for some s. This implies that $a_s(H^*(BG)) \geq -s$ and thus $\operatorname{reg}(H^*(BG)) \geq 0$, as was proved by Benson ([4, 4.2]). Combining this with Theorem 0.1 for M a point, we obtain Corollary 0.2.

When G is a compact Lie group, there is a question of orientability if G is not connected and $\operatorname{char} k \neq 2$. The group G acts on itself by conjugation, hence on $H^{\dim G}(G, \mathbb{Z}) \cong \mathbb{Z}$. This action induces a homomorphism $\epsilon : \pi_0(G) \to \{\pm 1\}$, which we may also regard as a 1-dimensional module for G. We say that G is orientable if ϵ is trivial. Otherwise let $H = \ker(\epsilon)$, in which case

$$H^*(BH, k) \cong H^*(BG, k[G/H]) \cong H^*(BG) \oplus H^*(BG, \epsilon).$$

Thus $\operatorname{reg}(H^*(BG, \epsilon)) \leq -\dim H = -\dim G$, by Theorem 0.1.

It is not always true that $\operatorname{reg}(H^*(BG)) = -\dim G$ (consider $O(2)$), so we consider $H^*(BG, \epsilon)$ instead.

We can prove another conjecture of Benson ([3, 14.9]).

Theorem 7.1. *For any compact Lie group G we have $\operatorname{reg}(H^*(BG, \epsilon)) = -\dim G$.*

Proof. The relevant Greenlees spectral sequence appears in [5]:

$$H^{s,t}_{\mathfrak{m}}(H^*(BG)) \Rightarrow H_{s+t+\dim G}(BG, \epsilon).$$

If G is orientable, then the same argument as before proves the result. If G is not orientable, then we would like a spectral sequence that starts with cohomology with coefficients (like the one we will use for discrete groups later), but this does not appear explicitly in the literature, so we use a trick instead.

Let G act on $U(1)$ via ϵ by sending an element to its inverse. Then the semi-direct product $F = U(1) \rtimes G$ is orientable, so $\operatorname{reg}(H^*(BF)) = -\dim F = -\dim G - 1$.

Note that $H^*(BH) = H^*(BH)_+ \oplus H^*(BH)_-$, where $+$ and $-$ indicate the eigenspaces under the action of G/H. Also $H^*(BH)_+ \cong H^*(BG)$ and $H^*(BH)_- \cong H^*(BG, \epsilon)$.

From the short exact sequence $U(1) \times H \to F \to C_2$ we obtain

$$H^*(BF) \cong H^*(BU(1) \times BH)^{C_2} \cong (k[x] \otimes H^*(BH))^{C_2},$$

where $\deg x = 2$ and C_2 sends x to $\pm x$.

Thus

$$H^*(BF) \cong (k[x^2] \otimes H^*(BH)_+) \oplus (xk[x^2] \otimes H^*(BH)_-),$$

and so, using Lemma 1.4,

$$\operatorname{reg}(H^*(BF)) = \max\{\operatorname{reg}(H^*(BH)_+) - 3, \operatorname{reg}(H^*(BH)_-) - 1\}.$$

But $\operatorname{reg}(H^*(BH)_+) \leq \operatorname{reg}(H^*(BH)) = -\dim G$, and $\operatorname{reg}(H^*(BF)) = -\dim G - 1$. This forces $-\dim G - 1 = \operatorname{reg}(H^*(BH)_-) - 1$ and thus $-\dim G = \operatorname{reg}(H^*(BG, \epsilon))$. $\square$

For discrete groups Γ that are virtual Poincaré duality groups over k there is also an orientation ϵ given by the action of Γ on $H^d(B\Gamma, k\Gamma) \cong k$, where $d = \operatorname{vcd}_k \Gamma$. In [6] there is constructed a Greenlees spectral sequence $H^{s,t}_{\mathfrak{m}}(H^*(B\Gamma, M)) \Rightarrow H_{s+t-d}(B\Gamma, M \otimes \epsilon)$.

This time we can just let $M = \epsilon^{-1}$ and use the same argument as for finite groups to show that $\operatorname{reg}(H^*(B\Gamma, \epsilon^{-1})) \geq d$.

If there is a contractible manifold X on which Γ acts with finite isotropy groups in such a way that X/Γ is compact, then ϵ corresponds to the action of Γ on $H^{\dim X}_c(X, \mathbb{Z})$ ([11, VIII, 7.5 and Exercise 4]) so it factors through ± 1.

Just as for compact Lie groups, $\operatorname{reg}(H^*(B\Gamma, \epsilon)) \leq \operatorname{reg}(H^*(B\Gamma))$ (at least if the image of ϵ is finite). By Proposition 5.2 and the remark afterwards, if Γ is virtually torsion free, then $\operatorname{reg}(H^*(B\Gamma)) \leq \dim X = \operatorname{vcd} \Gamma$.

We have proved part of another conjecture of Benson ([3, 14.10]), originally made for any orientable virtual Poincaré duality group over k.

Theorem 7.2. *Let Γ be a discrete group that is a virtual Poincaré duality group by virtue of being virtually torsion free and of acting properly discontinuously on a manifold without boundary in such a way that the isotropy groups are finite and the quotient is compact. Then* $\operatorname{reg}(H^*(B\Gamma, \epsilon)) = \operatorname{vcd}_k \Gamma$.

8. Characteristic zero

All our results remain valid if the field k has characteristic zero, with a few minor changes. Of course, this is only of interest for infinite compact Lie groups. The main difference is that the maximal elementary abelian p-subgroup S of $U(n)$ must be replaced by the maximal torus T.

In Lemma 2.2 the action of H can be allowed to have finite stabilisers. The conclusion of Proposition 2.5 becomes $\operatorname{reg}(H^*_G(X)) = \operatorname{reg}(H^*_T(X \times_G U)) - \dim U + \dim T$. Duflot also treats the case of $H^*_T(M)$; the submanifold M_i is now the set of points for which the isotropy groups have dimension i. In the argument at the end of Section 4, N must now be the part of $M_{(i),d}$ consisting of all the points for which the identity component of the isotropy group is V. The complement W to V no longer acts freely on N, but it does act with finite stabilizers, so the new version of Lemma 2.2 applies.

9. The Strong Conjecture

It was remarked by Benson that the bound on the $a_i(H^*_G)$ could be improved by 1 in all except the top case, the one in which i is equal to the dimension of H^*_G ([2, before 3.4]). (In fact, he phrased this in terms of the existence of a very strongly quasi-regular sequence, but the two are equivalent, by [2, 4.7].) He gave a proof when the difference between the dimension and the depth is at most two ([2, 9.11]). The general case was dubbed the Strong Conjecture by Nick Kuhn [25].

Following [9], given an integer ℓ and a graded R-module M, we define the regularity at and below level ℓ to be

$$\operatorname{reg}^{\leq \ell}(R, M) = \sup_{i \leq \ell} \{a_i(R, M) + i\}.$$

Similarly, for a polynomial ring R, we set

$$\operatorname{Preg}^{\geq \ell}(R, M) = \sup_{i \geq \ell} \{\rho_i(R, M) - i\}.$$

The proof that $\operatorname{Preg}(M) = \operatorname{reg}(M)$ works by downward induction on the terms of the minimal projective resolution of M, so we still have

$$\operatorname{Preg}^{\geq \ell}(R, M) = \operatorname{reg}^{\leq n-\ell}(R, M),$$

where n is the dimension of R, or equivalently the number of polynomial variables. In particular, $\operatorname{Preg}^{\geq 1}(R, M) = \operatorname{reg}^{\leq n-1}(R, M)$.

The Strong Conjecture can now be written in the following form.

Conjecture. *For G any compact Lie group,* $\operatorname{reg}^{\leq \dim H^*_G - 1}(H^*_G) \leq -\dim G - 1$.

(Kuhn's formulation was for finite groups and included the bound on $a_{\dim H^*_G}(H^*_G)$ from the ordinary conjecture.)

Green's calculations [20] have also verified this conjecture for all groups of order less than 256.

Our methods only yield a partial result in this direction.

Proposition 9.1. *For G any finite group,* $\operatorname{reg}^{\leq \dim H_G^* - 1}(H_G^*) \leq -\dim G - 1$ *except, perhaps, if there exists an elementary abelian p-subgroup of G that*

(1) *is maximal by inclusion (amongst the elementary abelian p-subgroups of G),*
(2) *is not maximal by rank,*
(3) *has rank that exceeds the depth of H_G^* by at least 2.*

David Green has informed us that of the 2-groups of order less than 256 there is only one that does not satisfy the hypotheses of the proposition. It has order 128 and maximal elementary abelian subgroups of orders 3 and 4. The depth of its cohomology is 1. (It is number 780 in the Small Groups Library.)

Examining our proof of the Regularity Conjecture, we see that it all depends on the fact, used in the proof of Proposition 4.1, that $H^*(N)$ vanishes in degrees above $\dim N$, where $N = M^V_{(i),d}$. If no connected component of N is closed, then $H^{\dim N}(N) = 0$, so we can improve this bound by 1.

In the case of H_G^* we are applying Duflot's filtration to the right action of S on $G\backslash U$. For ease of notation we will reflect this and consider S acting on the left of U/G.

Let $E \leq S$; then

$$(U/G)^E = \{uG \mid EuG = uG\} = \{uG \mid E^u \leq G\}.$$

Given a $u \in U$ such that $E^u \leq G$, its connected component in $(U/G)^E$ is $C_U(E)uG$, since $C_U(E)$ is a product of $U(n_i)$, one for each isotypic component of the representation of E corresponding to the inclusion of E^u in U, and these are connected.

Let us write X for this connected component and set $i = \operatorname{rk} E$. Then $X_{(i)} = X - \bigcup_{E \lneqq F \leq S} X^F$. The only way that a fixed-point submanifold X^F can separate is if it has codimension 1 and the group swaps the two sides. The quotient space will still be connected. In this way we see that if $X_{(i)}$ is non-empty, then $S\backslash X_{(i)}$ is connected. Also $S\backslash X_{(i)}$ is closed if and only if $X = X_{(i)}$, i.e. every point has isotropy group exactly E.

Lemma 9.2. *Let SuE be a point of $S\backslash(U/G)^E_{(i)}$. Then its connected component is closed if and only if E^u is maximal by inclusion amongst the elementary abelian p-subgroups of G.*

Proof. If there is a subgroup F with $E \lneqq F \leq S$ that fixes cuG for some $c \in C_U(E)$, then $E^u \lneqq F^{cu} \leq G$.

Conversely, if $E^u \lneqq F' \leq G$, then $E \lneqq {}^uF' \leq C_U(E)$. Since $C_U(E)$ is connected and S is a maximal elementary abelian p-subgroup, ${}^uF'$ must be conjugate to a subgroup of S, i.e. there is a $c \in C_U(E)$ such that $E \lneqq^{cu} F' \leq S$ and ${}^{cu}F'$ fixes cuG. $\square$

Now we consider the Greenlees spectral sequence [21]

$$H^{s,t}_{\mathfrak{m}}(H^*(BG)) \Rightarrow H_{s+t}(BG),$$

in particular the $E^{s,-s}$ diagonal. It was shown by Benson (in [1, proof of 4.1]) that it is the $E_2^{n,-n}$-term with $n = \dim H_G^*$ that survives to give $H_0(BG) \cong k$. (This is the only place in the argument where we require G to be finite, and it is probably not necessary.) Any other terms on this diagonal must fail to survive. We have shown that such a term can only possibly occur for some s which is equal to the rank of a maximal elementary p-subgroup of G. Any differential leaving such a term lands in a 0-group, by the original Regularity Conjecture, now Corollary 0.2. The hypotheses of the proposition imply that any differential arriving in an $E_r^{s,-s}$-term with $s < n$ must have departed from some $E_r^{s-r,-s+r-1}$ with $s-r$ less than the depth of H_G^*. But the local cohomology $H_{\mathfrak{m}}^i$ vanishes for i less than the depth ([22, 3.10], [12, 3.5.7]).

This shows that no $E_2^{s,-s}$-term can be 0 for $s < n$, proving the proposition.

10. Consequences

Benson explains various consequences of the cohomology of a group having regularity zero, notably in terms of the existence of quasi-regular sequences and in terms of Hilbert polynomials. An application of the Regularity Conjecture to estimating central detection numbers in the cohomology of finite groups is given by Kuhn [25].

We will point out just two other applications. The first is that the method of the proof can be used to give an explicit formula for the additive structure of $H_G^*(X)$ in terms of the cohomology of the strata of $X \times_G U$ under the action of S. This is best expressed in terms of Hilbert series. For any graded vector space over k that is finite dimensional in each degree, set $\mathcal{H}(M,t) = \sum_i (\dim M_i)t^i$.

From Proposition 2.5 we see that

$$\mathcal{H}(H_S^*(X \times_G U), t) = \frac{\mathcal{H}(H_G^*(X), t) \cdot \mathcal{H}(H_S^*, t)}{\mathcal{H}(H_U^*, t)}.$$

The filtration of Duflot yields (cf. [14, Theorem 2])

$$\mathcal{H}(H_S^*(X \times_G U), t) = \sum_{E \le S,\, d} \mathcal{H}(H^*((X \times_G U)^E_{\mathrm{rk}\, E,\, d}/S), t) \cdot \mathcal{H}(H^*E, t) \cdot t^d.$$

Combining these, we obtain an expression for the Hilbert series of $H_G^*(X)$.

Proposition 10.1. *If $H_G^*(X)$ is noetherian, then*

$$\mathcal{H}(H_G^*(X), t) = \prod_{i=1}^{n} \frac{1-t^e}{1-t^{2i}} \cdot \sum_{E \le S,\, d} \frac{\mathcal{H}(H^*((X \times_G U)^E_{\mathrm{rk}\, E,\, d}/S), t) \cdot t^d}{(1-t^e)^{\mathrm{rk}\, E}},$$

where e is 1 in finite characteristic and 2 in characteristic 0.

In the case when X is a point, we saw in Section 9 that the action of S on $G\backslash U$ can be described in terms of representation theory, but it is not easy to calculate the cohomology of the pieces.

Our second application is to the degrees of generators and relations for the cohomology of a group. Knowing a bound for these is extremely important for computer calculations; see [13].

Proposition 1.3 bounds the degrees of the generators and relations once we have found some polynomial ring R over which H_G^* is finite. If G can be embedded in the unitary group $U(n)$, then $H_{U(n)}^*$ is such a ring, by a result of Venkov ([30, 2.2], [34]). But $H_{U(n)}^* \cong k[c_1, \ldots, c_n]$, where the degree of c_i is $2i$. The bound in Proposition 1.3(1) becomes $\max\{n^2 - \dim G,\, 2n\}$. If the corresponding representation is a sum of irreducibles of dimensions n_i, then the embedding factors through one into $\times_i U(n_i)$ and we obtain the bound $\max_i\{\sum_j n_j^2 - \dim G,\, 2n_i\}$.

It is possible to refine these bounds further, but the method of computer calculation of cohomology by constructing a projective resolution lends itself to finding a good ring R in any case, so we just record some attractive formulations.

Proposition 10.2. *If the finite group G is not cyclic and if it has a faithful complex representation of dimension n, then H^*_G is generated as a ring by elements of degree at most n^2. If this representation is a sum of irreducibles of dimensions n_i, then the bound becomes $\sum_i n_i^2$. For any finite group G the cohomology is generated by elements of degree at most $|G|-1$.*

In all cases the bound for the relations is twice this.

Proof. Because, in the first part of the proposition, we assume that the group is not cyclic, we are not embedding it in just one copy of $U(1)$ and it is easy to see that the first of the terms over which we take the maximum is always the largest. It is bounded by $\sum_i n_i^2$ by the discussion above.

One uneconomical way to obtain a faithful representation is to take one copy of every irreducible representation except the trivial one. By elementary character theory this leads to the bound $|G|-1$, which is easily checked to be valid for cyclic groups too. □

It is interesting that Burt Totaro considers the Chow ring of the classifying space of an algebraic group over a field in [33] and obtains essentially the same bounds for the degrees of the generators.

Remark. Part (3) of Proposition 1.3 suggests an algorithm for determining when, in a computer calculation of $H^*(G)$, a sufficiently high degree N has been attained such that $\tau_N(H^*(G)) \cong H^*(G)$. This is based on algorithms in [2] and [13]. For finite groups G is as follows.

(1) Compute up to a sufficiently high degree that it is possible to find a set of homogeneous elements $\{\tilde{x}_i\}$ such that $H^*(G)$ must be finite over the ring generated by the $\{\tilde{x}_i\}$. It is known how to do this by considering the restrictions to elementary abelian p-subgroups [13]. Let $R = k[x_1, \dots, x_n]$ be the corresponding polynomial ring and let $f : R \to H^*(G)$ be given by $x_i \mapsto \tilde{x}_i$. Calculate $\sigma(R) = \sum(\deg(\tilde{x}_i) - 1)$.
(2) Compute further so that $N \geq \sigma(R) + 1$.
(3) Calculate $\tau_N(H^*(G))$ and determine whether it is generated as an R-module in degrees at most N. If it is, then $\tau_N(H^*(G)) \cong H^*(G)$ and the computation is complete. If not, then compute to a higher value of N and repeat.
(4) The process is guaranteed to terminate with $N \leq 2\sigma(R)$, except in the trivial case when $\sigma(R) = 0$, where it stops with $N = 1$.

In the presence of the Strong Conjecture it suffices to take $N \geq \sigma(R)$ in (2).

The condition in (3) is necessary, so the procedure does not take us further past degree $\sigma(R) + 1$ than necessary.

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