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CURVE COUNTING THEORIES VIA STABLE OBJECTS I. DT/PT CORRESPONDENCE

YUKINOBU TODA

1. Introduction

The purpose of this paper is to study curve counting on Calabi-Yau 3-folds via wall-crossing phenomena in the derived category. We will study the generating series of Donaldson-Thomas-type invariants without virtual fundamental cycles, i.e. the Euler characteristics of the relevant moduli spaces. The main result is to show the Euler characteristic version of the Pandharipande-Thomas conjecture [25, Conjecture 3.3], which claims the equality of the generating series of Donaldson-Thomas invariants and counting invariants of stable pairs. In a subsequent paper [28], we will apply the method used in this paper to show the transformation formula of our generating series under flops and the generalized McKay correspondence by Van den Bergh [10].

1.1. **Donaldson-Thomas invariant.** Let X be a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$; i.e., the canonical line bundle $\bigwedge^3 T_X^*$ is trivial. For a homology class $\beta \in H_2(X,\mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space which defines the DT-invariant is the classical Hilbert scheme,

$$I_n(X,\beta) = \left\{ \begin{array}{l} \text{subschemes } C \subset X, \dim C \le 1 \\ \text{with } [C]=\beta,\ \chi(\mathcal{O}_C)=n. \end{array} \right\}.$$

In other words, $I_n(X,\beta)$ is the moduli space of rank one torsion free sheaves $I \in \mathrm{Coh}(X)$ which satisfies $\det I = \mathcal{O}_X$ and

$$\mathrm{ch}(I) = (1,0,-\beta,-n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6.$$

Here we have regarded β as an element of $H^4(X,\mathbb{Z})$ via the Poincaré duality. In fact such a sheaf I is isomorphic to the ideal sheaf $I_C \subset \mathcal{O}_X$ for a subscheme $C \subset X$ with $\dim C \le 1$, $[C]=\beta$ and $\chi(\mathcal{O}_C)=n$. The moduli space $I_n(X,\beta)$ is projective and has a symmetric obstruction theory [27]. The associated virtual fundamental cycle has virtual dimension zero, and the integration along it defines the DT-invariant,

$$I_{n,\beta} = \int_{[I_n(X,\beta)^{\mathrm{vir}}]} 1 \in \mathbb{Z}.$$

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We consider the generating series,

$$\mathrm{DT}(X) = \sum_{n,\beta} I_{n,\beta} x^n y^\beta.$$

Let $\mathrm{DT}_0(X)$ be the contributions from 0-dimensional subschemes,

$$\mathrm{DT}_0(X) = \sum_n I_{n,0} x^n.$$

This is computed in [3], [21], [20],

$$\mathrm{DT}_0(X) = M(-x)^{\chi(X)}, \quad M(x) = \prod_{k \geq 1} \frac{1}{(1-x^k)^k}.$$

The reduced Donaldson-Thomas theory is defined by

$$\mathrm{DT}'(X) := \frac{\mathrm{DT}(X)}{\mathrm{DT}_0(X)} = \sum_\beta \mathrm{DT}'_\beta(X) y^\beta,$$

where $\mathrm{DT}'_\beta(X)$ is a Laurent series of x. The MNOP conjecture [24] states that $\mathrm{DT}'_\beta(X)$ is the Laurent expansion of a rational function of x invariant under $x \leftrightarrow 1/x$, and $\mathrm{DT}'(X)$ coincides with the generating series of Gromov-Witten invariants after a suitable change of variables.

1.2. **Pandharipande-Thomas theory.** Another curve counting theory via stable pairs is introduced by Pandharipande and Thomas [25] in order to give a geometric understanding of the reduced DT-theory. By definition, a stable pair (F, s) consists of a pure 1-dimensional sheaf F and a morphism $s\colon \mathcal{O}_X \to F$ with 0-dimensional cokernel. In [25], the moduli space

$$P_n(X,\beta) = \left\{ \begin{array}{l} \text{stable pairs } (F,s) \text{ with} \\ [F]=\beta,\ \chi(F)=n \end{array} \right\}$$

is shown to be a projective variety and has a symmetric obstruction theory by viewing stable pairs as two-term complexes,

$$(\cdots \to 0 \to \mathcal{O}_X \xrightarrow{s} F \to 0 \cdots) \in D^b(\mathrm{Coh}(X)). \tag{1}$$

Integrating along the virtual fundamental cycle defines the invariant,

$$P_{n,\beta} = \int_{[P_n(X,\beta)^{\mathrm{vir}}]} \in \mathbb{Z}.$$

We consider the generating series,

$$\mathrm{PT}(X) := \sum_{n,\beta} P_{n,\beta} x^n y^\beta = \sum_\beta \mathrm{PT}_\beta(X) y^\beta,$$

where $\mathrm{PT}_\beta(X)$ is a Laurent series of x. In [25, Conjecture 3.3], Pandharipande and Thomas state the following conjecture.

Conjecture 1.1 ([25, Conjecture 3.3])**.** *We have the equality of the generating series,*

$$\mathrm{DT}'(X) = \mathrm{PT}(X).$$

1.3. **Main theorem.** In this paper, we study the series,

$$\widehat{\mathrm{DT}}(X)=\sum_{n,\beta}\chi(I_n(X,\beta))x^n y^\beta,$$

where $\chi(*)$ is the topological Euler characteristic. We can similarly define the series $\widehat{\mathrm{DT}}_0(X)$, $\widehat{\mathrm{DT}}'(X)$, $\widehat{\mathrm{PT}}(X)$, which are Euler characteristic versions of $\mathrm{DT}_0(X)$, $\mathrm{DT}'(X)$, $\mathrm{PT}(X)$, respectively. The series $\widehat{\mathrm{DT}}(X)$ is closely related to $\mathrm{DT}(X)$ in the following sense.

- If $I_n(X,\beta)$ is non-singular and connected, we have
$$I_{n,\beta}=(-1)^{\dim I_n(X,\beta)}\chi(I_n(X,\beta)).$$
- In general, there is Behrend's constructible function [2],
$$\nu\colon I_n(X,\beta)\to\mathbb{Z},$$
such that $I_{n,\beta}$ is written as
$$I_{n,\beta}=\sum_{n\in\mathbb{Z}}n\chi(\nu^{-1}(n)).$$
- As for $\widehat{\mathrm{DT}}_0(X)$, we have $\widehat{\mathrm{DT}}_0(X)=M(x)^{\chi(X)}$, so it is obtained from $\mathrm{DT}_0(X)$ by $x\leftrightarrow -x$ (cf. [9]).

Our main theorem is the following.

Theorem 1.2 (Theorem 3.15)**.** *We have the equality of the generating series,*

$$\widehat{\mathrm{DT}}'(X)=\widehat{\mathrm{PT}}(X).$$

In [29, Corollary 1.4], the author showed the rationality of the series $\widehat{\mathrm{PT}}_\beta(X)$. Hence we obtain the following.

Corollary 1.3. *The series $\widehat{\mathrm{DT}}'_\beta(X)$ is the Laurent expansion of a rational function of x, invariant under $x\leftrightarrow 1/x$.*

Note that the above result is conjectured in [22, Conjecture 1.1].

1.4. **Idea of the proof of Theorem 1.2.** Our proof is based on the idea of Pandharipande and Thomas [25, Section 3] to use Joyce's wall-crossing formula [16] in the space of Bridgeland's stability conditions [7] on the triangulated category $D^b(\mathrm{Coh}(X))$. Suppose that there is a stability condition σ on $D^b(\mathrm{Coh}(X))$ such that the ideal sheaf I_C for a 1-dimensional subscheme $C\subset X$ is σ-stable. If there is a 0-dimensional subsheaf $Q\subset\mathcal{O}_C$, i.e. $\mathcal{O}_X\to\mathcal{O}_C$ is not a stable pair, then there is a distinguished triangle in $D^b(\mathrm{Coh}(X))$,

$$Q[-1]\longrightarrow I_C\longrightarrow I_{C'},\tag{2}$$

where $\mathcal{O}_{C'}=\mathcal{O}_C/Q$. Then Pandharipande and Thomas claim that we can deform stability conditions from σ to another stability condition τ, such that the sequence (2) destabilizes I_C with respect to τ. Instead if we consider the flipped sequence

$$I_{C'}\longrightarrow E\longrightarrow Q[-1],$$

then the object E should become τ-stable. The object E is isomorphic to a two-term complex (1) determined by a stable pair, so σ corresponds to the DT-theory and τ corresponds to the PT-theory. In this way, we can see that the relationship between counting invariants of σ-stable objects and τ-stable objects is relevant to

Conjecture 1.1. In principle, there should exist a wall and chamber structure on the space of stability conditions, so that the counting invariants are constant on chambers but jump at walls. The transformation formula of counting invariants under a change of stability conditions, called the *wall-crossing formula*, is studied by Joyce [16] in the case of abelian categories. As pointed out in [25, Section 3], there are two issues in applying Joyce's theory.

- We need to extend Joyce's work to stability conditions on the triangulated category $D^b(\mathrm{Coh}(X))$. However there are no known examples of Bridgeland's stability conditions on $D^b(\mathrm{Coh}(X))$ for a projective Calabi-Yau 3-fold X.
- Joyce studies the wall-crossing formula of counting invariants without virtual fundamental cycles. We need to establish a similar formula for invariants involving virtual classes, or Behrend's constructible functions.

In this paper, we deal with the first issue. The idea consists of two parts.

- Instead of working with $D^b(\mathrm{Coh}(X))$, we study the triangulated subcategory,
$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X)\rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)),$$
i.e. the smallest triangulated subcategory which contains $\mathcal{O}_X$ and $F \in \mathrm{Coh}(X)$ with $\dim \mathrm{Supp}(F) \le 1$. Note that ideal sheaves I_C and two-term complexes (1) are contained in $\mathcal{D}_X$. On the triangulated category $\mathcal{D}_X$, we are able to construct Bridgeland's stability conditions.
- Although there are stability conditions on $\mathcal{D}_X$, there are still technical difficulties to study stability conditions on $\mathcal{D}_X$ and wall-crossing phenomena. So we introduce the space of weak stability conditions on triangulated categories, which generalizes Bridgeland's stability conditions. It is easier to construct weak stability conditions than usual stability conditions, and the wall-crossing formula also becomes much more amenable.

Based on these two ideas, we can justify the discussion of Pandharipande and Thomas [25, Section 3] and give the proof of Theorem 1.2. For the application in a subsequent paper [28], we show the wall-crossing formula in the space of weak stability conditions on $\mathcal{D}_X$ under a general setting.

Here we remark that A. Bayer [1] also introduces the generalized Bridgeland stability conditions, which he calls *polynomial stability conditions*. Then he shows that DT/PT correspondence is realized as a wall-crossing phenomena in his polynomial stability conditions. However there is a slight technical issue in applying Joyce's theory to Bayer's polynomial stability conditions, which we mention in Remark 5.6. This is one of the reasons we introduce the notion of weak stability conditions.

As for the second issue, there was important progress recently. In [19], Kontsevich and Soibelman establish the wall-crossing formula for motivic Donaldson-Thomas invariants, which essentially involves virtual classes [19, Theorem 7]. Although their main result [19, Theorem 7] relies on the unsolved conjecture on Motivic Milnor fibers [19, Conjecture 4], their work is applied for numerical Donaldson-Thomas invariants once we know the l-adic version of [19, Conjecture 4], which is solved in [19, Proposition 9]. (However we still need an orientation data [19, Section 5] for the application of the result of Kontsevich and Soibelman.) If we are able to apply the work of Kontsevich and Soibelman, then Conjecture 1.1 follows from the method in this paper. T. Bridgeland [6] also recently gave a proof of

Conjecture 1.1 assuming the result of [19]. His method is different from ours and does not use any notion of stability conditions. In [17], Joyce and Song also studied the wall-crossing formula of counting invariants involving virtual classes. At the moment, as the author writes in the first version of this paper, their work applies to counting invariants of coherent sheaves, and not to those of objects in the derived category. The only issue is the derived category version of [17, Theorem 5.3]; that is, we need to show that the local moduli space of objects in the derived category is described as a critical locus of some convergent function. If the result of [17, Theorem 5.3] is extended to the case of the derived category, Conjecture 1.1 follows as well from the method in this paper.[1]

Finally we comment that Stoppa and Thomas [26] investigate DT/PT correspondence via the wall-crossing of GIT stability. Then they show the same result of Theorem 1.2 applying Joyce's theory, independently to our work. It is remarkable that they do not use Joyce's counting invariants of strictly semistable objects, which will be introduced in Proposition-Definition 5.7 in this paper.

1.5. **Content of the paper.** In Section 2, we introduce the notion of weak stability conditions on triangulated categories and study their general properties. In Section 3, we give a proof of Theorem 1.2 assuming the result in the latter sections in a general setting. In Section 4, we give a general framework to discuss the wall-crossing formula. In Section 5, we establish the wall-crossing formula of generating series. In Section 6 and Section 7, we give the proofs of several technical lemmas.

1.6. **Notation and convention.** In this paper, all the varieties are defined over $\mathbb{C}$. For a triangulated category $\mathcal{D}$, the shift functor is denoted by [1]. For a set of objects $S \subset \mathcal{D}$, we denote by $\langle S \rangle_{\mathrm{tr}} \subset \mathcal{D}$ the smallest triangulated subcategory of $\mathcal{D}$ which contains S. Also we denote by $\langle S \rangle_{\mathrm{ex}}$ the smallest extension closed subcategory of $\mathcal{D}$ which contains S. For an abelian category $\mathcal{A}$ and a set of objects $S \subset \mathcal{A}$, the subcategory $\langle S \rangle_{\mathrm{ex}} \subset \mathcal{A}$ is also defined to be the smallest extension closed subcategory of $\mathcal{A}$ which contains S. The abelian category of coherent sheaves is denoted by $\mathrm{Coh}(X)$. We say that $F \in \mathrm{Coh}(X)$ is d-dimensional if its support is d-dimensional.

2. Weak stability conditions on triangulated categories

In this section, we introduce the notion of weak stability conditions on triangulated categories, which generalizes Bridgeland's stability conditions [7].

2.1. **Slicings.** Let $\mathcal{D}$ be a triangulated category. Here we recall the notion of slicings on $\mathcal{D}$ given in [7, Section 3].

Definition 2.1. A *slicing* on $\mathcal{D}$ consists of a family of full subcategories $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$, which satisfies the following.

- For any $\phi \in \mathbb{R}$, we have $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi+1)$.
- For $E_i \in \mathcal{P}(\phi_i)$ with $\phi_1 > \phi_2$, we have $\mathrm{Hom}(E_1, E_2) = 0$.

[1] After the author wrote the first version of this paper, it was announced that the above problem on the description of the local moduli space in the derived category had been solved by Behrend and Getzler [4]. As soon as the paper [4] appears, we will be able to give a complete proof of Conjecture 1.1.

- **(Harder-Narasimhan filtration):** For any non-zero object $E \in \mathcal{D}$, we have the following collection of triangles:

(3)
$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[rr] & & E_n = E \arrow[dl] \\ & F_1 \arrow[ul, "{[1]}"] & & F_2 \arrow[ul, "{[1]}"] & & & F_n \arrow[ul, "{[1]}"] & \end{tikzcd}$$

such that $F_j \in \mathcal{P}(\phi_j)$ with $\phi_1 > \phi_2 > \cdots > \phi_n$.

We also need an additional condition called the *local finiteness*. For an interval $I \subset \mathbb{R}$, the category $\mathcal{P}(I) \subset \mathcal{D}$ is defined to be

$$\mathcal{P}(I) = \langle \mathcal{P}(\phi) : \phi \in I \rangle_{\mathrm{ex}} \subset \mathcal{D}.$$

If $I = (a, b)$ with $b - a < 1$, then the category $\mathcal{P}(I)$ is a quasi-abelian category (cf. [7, Definition 4.1]). If we have a distinguished triangle

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow A[1]$$

with $A, B, C \in \mathcal{P}(I)$, we say that i is a *strict monomorphism* and that j is a *strict epimorphism*. Then we say that $\mathcal{P}(I)$ is of *finite length* if $\mathcal{P}(I)$ is Noetherian and Artinian with respect to strict epimorphisms and strict monomorphisms, respectively. (See [7, Section 4] for the details.)

Definition 2.2. A slicing $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ is *locally finite* if there exists $\eta > 0$ such that for any $\phi \in \mathbb{R}$, the quasi-abelian category $\mathcal{P}((\phi - \eta, \phi + \eta))$ is of finite length.

The set of locally finite slicings on $\mathcal{D}$ is denoted by $\mathrm{Slice}(\mathcal{D})$. For $0 \neq E \in \mathcal{D}$ and $\mathcal{P} \in \mathrm{Slice}(\mathcal{D})$, we set $\phi_{\mathcal{P}}^{+}(E) = \phi_1$ and $\phi_{\mathcal{P}}^{-}(E) = \phi_n$, where the ϕ_i are given by the last condition of Definition 2.1. There is a generalized metric on $\mathrm{Slice}(\mathcal{D})$, given by

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi_{\mathcal{P}}^{-}(E) - \phi_{\mathcal{Q}}^{-}(E)|, |\phi_{\mathcal{P}}^{+}(E) - \phi_{\mathcal{Q}}^{+}(E)| \right\} \in [0, \infty], \tag{4}$$

for $\mathcal{P}, \mathcal{Q} \in \mathrm{Slice}(\mathcal{D})$. It is shown in [7, Section 6] that $d(\mathcal{P}, \mathcal{Q}) = d(\mathcal{Q}, \mathcal{P})$ and $d(\mathcal{P}, \mathcal{Q}) = 0$ implies $\mathcal{P} = \mathcal{Q}$.

2.2. **Weak stability conditions.** For a triangulated category $\mathcal{D}$, let $K(\mathcal{D})$ be the Grothendieck group of $\mathcal{D}$. We fix a finitely generated free abelian group Γ together with a group homomorphism,

$$\mathrm{cl} \colon K(\mathcal{D}) \to \Gamma.$$

We also fix a filtration of Γ,

$$0 \subsetneq \Gamma_0 \subsetneq \Gamma_1 \subsetneq \cdots \subsetneq \Gamma_N = \Gamma, \tag{5}$$

such that each subquotient

$$\mathbb{H}_i := \Gamma_i / \Gamma_{i-1} \quad (0 \le i \le N)$$

is a free abelian group. We set $\mathbb{H}_i^{\vee} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{H}_i, \mathbb{C})$ and fix a norm $\|*\|_i$ on $\mathbb{H}_i \otimes_{\mathbb{Z}} \mathbb{R}$. Given an element

$$Z = \{Z_i\}_{i=0}^{N} \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee},$$

we define a map $Z\colon \Gamma \to \mathbb{C}$ as follows. For $v \in \Gamma$, there is a unique $0 \le m \le N$ such that $v \in \Gamma_m \setminus \Gamma_{m-1}$. Here we set $\Gamma_{-1} = \emptyset$. Then $Z(v)$ is defined by

$$Z(v) := Z_m([v]) \in \mathbb{C},$$

where $[v]$ is a class of v in $\mathbb{H}_m$. Using such $0 \le m \le N$, the following map is also defined:

$$\|*\|\colon \Gamma \ni v \mapsto \|v\|_m \in \mathbb{R}. \tag{6}$$

Below we often write $\mathrm{cl}(E) \in \Gamma$ as $E \in \Gamma$ when there is no confusion.

Definition 2.3. We define the set $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ to be pairs $(Z, \mathcal{P})$,

$$Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}, \quad \mathcal{P} \in \mathrm{Slice}(\mathcal{D}), \tag{7}$$

which satisfy the following axiom.

- For any non-zero $E \in \mathcal{P}(\phi)$, we have

$$Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi). \tag{8}$$

- **(Support property):** There is a constant $C > 0$ such that for any non-zero $E \in \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$, we have

$$\|E\| \le C|Z(E)|. \tag{9}$$

Remark 2.4. If $N = 0$ in (5), the set $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ coincides with the set of stability conditions on $\mathcal{D}$ introduced by Bridgeland [7], satisfying the support property. The support property is introduced by Kontsevich and Soibelman [19] to refine Bridgeland stability and introduce the notion of stability data. We will use this property to show Theorem 2.15 below.

Remark 2.5. When $N = 0$ in (5), the local finiteness condition automatically follows if the support property is satisfied. However for $N > 0$, it seems that there is no reason to conclude the local finiteness from the support property.

We call an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ a *weak stability condition* on $\mathcal{D}$. Although it is difficult to find a Bridgeland stability on the derived category of coherent sheaves on algebraic varieties, it is rather easier to find a weak one as we see below.

Example 2.6. Let X be a smooth projective variety of $\dim X = d$, ω an ample divisor on X, and $\mathcal{D} = D^b(\mathrm{Coh}(X))$. We set Γ to be the image of the Chern character map,

$$\mathrm{cl} := \mathrm{ch}\colon K(\mathcal{D}) \twoheadrightarrow \Gamma \subset H^*(X, \mathbb{Q}).$$

We choose a filtration (5) as

$$\Gamma_i := \Gamma \cap H^{\ge 2d-2i}(X, \mathbb{Q}).$$

In this case we have $\mathbb{H}_i = \Gamma \cap H^{2d-2i}(X, \mathbb{Q})$. Choose $0 < \phi_d < \phi_{d-1} < \cdots < \phi_0 < 1$ and set $Z_i \in \mathbb{H}_i^{\vee}$ to be

$$Z_i(v) = \exp(i\pi\phi_i) \int_X v \cdot \omega^i \in \mathbb{C}. \tag{10}$$

We define the slicing $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ as follows. For $0 < \phi \le 1$ with $\phi \ne \phi_i$ for any i, we set $\mathcal{P}(\phi) = \emptyset$. For $\phi = \phi_i$, set

$$\mathcal{P}(\phi_i) = \{E \in \mathrm{Coh}(X) : E \text{ is pure of } \dim \mathrm{Supp}(E) = i\}.$$

Other $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ are determined by the first condition of Definition 2.1. It is easy to check that $(\{Z_i\}_{i=0}^d, \mathcal{P})$ is an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$.

In what follows, we use the following notation. For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ and an interval $I \subset \mathbb{R}$, we set

$$C_\sigma(I) := \mathrm{Im}(\mathrm{cl}\colon \mathcal{P}(I) \to \Gamma) \subset \Gamma. \tag{11}$$

If $I \subset (a, a+1]$ for some $a \in \mathbb{R}$, we can define the *phase* of $v \in C_\sigma(I)$ by

$$\phi_\sigma(v) = \frac{1}{\pi} \arg Z(v) \in I. \tag{12}$$

2.3. **Constructions via t-structures.** In this paragraph, we give another way of constructing elements of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$, using the notion of bounded t-structures. The readers can refer to [7, Section 3] for the notion of bounded t-structures and their hearts.

Definition 2.7. Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a bounded t-structure on $\mathcal{D}$. We say that $Z \in \prod_{i=0}^N \mathbb{H}_i^\vee$ is a *weak stability function* on $\mathcal{A}$ if for any non-zero $E \in \mathcal{A}$, we have

$$Z(E) \in \mathfrak{H} := \{r \exp(i\pi\phi) : r > 0,\ 0 < \phi \le 1\}. \tag{13}$$

By (13), we can uniquely determine the argument,

$$\arg Z(E) \in (0, \pi],$$

for any $0 \neq E \in \mathcal{A}$. For an exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$, one of the following equalities holds:

$$\arg Z(F) \le \arg Z(E) \le \arg Z(G),$$
$$\arg Z(F) \ge \arg Z(E) \ge \arg Z(G).$$

Remark 2.8. When $N = 0$, a weak stability function coincides with a stability function introduced in [7, Definition 2.1]. In this case we have one of the following inequalities:

$$\arg Z(F) < \arg Z(E) < \arg Z(G),$$
$$\arg Z(F) > \arg Z(E) > \arg Z(G),$$
$$\arg Z(F) = \arg Z(E) = \arg Z(G),$$

so an inequality such as $\arg Z(F) < \arg Z(E) = \arg Z(G)$ does not occur. On the other hand, we might have such an inequality when $N > 0$, and such a function determines a weak stability condition in the sense of [14, Definition 4.1].

Definition 2.9. Let $Z \in \prod_{i=0}^N \mathbb{H}_i^\vee$ be a weak stability function on $\mathcal{A}$. We say that $0 \neq E \in \mathcal{A}$ is *Z-semistable* (resp. *stable*) if for any exact sequence $0 \to F \to E \to G \to 0$ we have

$$\arg Z(F) \le \arg Z(G) \quad (\text{resp. } \arg Z(F) < \arg Z(G)). \tag{14}$$

Remark 2.10. If $N = 0$, the condition (14) is equivalent to $\arg Z(F) \le \arg Z(E)$ (resp. $\arg Z(F) < \arg Z(E)$). However for $N > 0$, the condition (14) is not equivalent to the above condition, since we may have $\arg Z(F) < \arg Z(E) = \arg Z(G)$, as in Remark 2.8.

The notion of Harder-Narasimhan filtration is defined in a similar way to the usual stability conditions.

Definition 2.11. Let $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ be a weak stability function on $\mathcal{A}$. A *Harder-Narasimhan filtration* of an object $E \in \mathcal{A}$ is a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E,$$

such that each subquotient $F_j = E_j/E_{j-1}$ is Z-semistable with

$$\arg Z(F_1) > \arg Z(F_2) > \cdots > \arg Z(F_k).$$

A weak stability function Z is said to have the *Harder-Narasimhan property* if any object $E \in \mathcal{A}$ has a Harder-Narasimhan filtration.

The following proposition is an analogue of [7, Proposition 2.4].

Proposition 2.12. *Let $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ be a weak stability function on $\mathcal{A}$. Suppose that the following chain conditions are satisfied.*

(a) *There are no infinite sequences of subobjects in $\mathcal{A}$,*

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\arg Z(E_{j+1}) > \arg Z(E_j/E_{j+1})$ *for all j.*

(b) *There are no infinite sequences of quotients in $\mathcal{A}$,*

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \stackrel{\pi_j}{\twoheadrightarrow} E_{j+1} \twoheadrightarrow \cdots$$

with $\arg Z(\ker \pi_j) > \arg Z(E_{j+1})$ *for all j.*

Then Z has the Harder-Narasimhan property.

Proof. Although our stability condition is a weak one, the same proof of [7, Proposition 2.4] still works. Also see the proof of [14, Theorem 4.4]. □

The following proposition is an analogue of [7, Proposition 5.3], which relates weak stability conditions and weak stability functions on the hearts of bounded t-structures.

Proposition 2.13. *Giving a pair $(Z, \mathcal{P})$, where $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ and $\mathcal{P}$ is a slicing, satisfying* (8) *is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a weak stability function on its heart with the Harder-Narasimhan property.*

Proof. The proof is the same as in [7, Proposition 5.3], so we just describe how to give the correspondence. Given $Z \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$ and a slicing $\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$ satisfying (8), the category

$$\mathcal{A} = \mathcal{P}((0,1])$$

is the heart of a bounded t-structure and Z is a weak stability function on $\mathcal{A}$. Conversely suppose that $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$ and Z is a weak stability function on it. For $0 < \phi \le 1$, let $\mathcal{P}(\phi)$ be the full additive subcategory of $\mathcal{A}$, defined by

$$\mathcal{P}(\phi) = \left\{ E \in \mathcal{A} : \begin{array}{l} E \text{ is } Z\text{-semistable with} \\ Z(E) \in \mathbb{R}_{>0} \exp(i\pi\phi) \end{array} \right\}.$$

The subcategory $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ is determined by the first condition of Definition 2.1. By the Harder-Narasimhan property of Z, $\mathcal{P}$ is a slicing on $\mathcal{D}$. □

Below we write an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ as $(Z,\mathcal{P})$ with $\mathcal{P}\in \mathrm{Slice}(\mathcal{D})$, or $(Z,\mathcal{A})$ with $\mathcal{A}\subset\mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. The above proposition enables us to produce more examples of stability conditions.

Example 2.14. (i) Let X be a d-dimensional smooth projective variety, $\mathcal{D}=D^b(\mathrm{Coh}(X))$ and $\mathcal{A}=\mathrm{Coh}(X)\subset\mathcal{D}$. Then $Z=\{Z_i\}_{i=0}^d$ defined by (10) is a weak stability function on $\mathcal{A}$. An object $E\in\mathcal{A}$ is Z-semistable if and only if E is a pure sheaf; thus $\{Z_i\}_{i=0}^d$ satisfies the Harder-Narasimhan property. In this way, we can recover the slicing $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ given in Example 2.6.

(ii) Let A be a finite-dimensional $\mathbb{C}$-algebra, $\mathcal{A}=\mathrm{mod}\,A$ the category of finitely generated right A-modules and $\mathcal{D}=D^b(\mathcal{A})$. There is a finite number of simple objects $S_0,S_1,\cdots,S_N\in\mathcal{A}$ such that

$$K(\mathcal{D})=\bigoplus_{j=0}^{N}\mathbb{Z}[S_j].$$

Let $\Gamma=K(\mathcal{D})$ and $\mathrm{cl}\colon K(\mathcal{D})\to\Gamma$ be the identity map. We choose the filtration (5) to be $\Gamma_j=\bigoplus_{a=0}^{j}\mathbb{Z}[S_a]$; hence $\mathbb{H}_j=\mathbb{Z}[S_j]$. Choose $0<\phi_j\le 1$ for $0\le j\le N$ and set $Z_j\in\mathbb{H}_j^\vee$ to be

$$Z_j(r[S_j])=r\exp(i\pi\phi_j).$$

Then $\{Z_j\}_{j=0}^N$ is a weak stability function on $\mathcal{A}$. The corresponding pair $(\{Z_j\}_{j=0}^N,\mathcal{P})$ via Proposition 2.13 gives an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$.

2.4. **The space of weak stability conditions.** There is the inclusion,

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})\subset\mathrm{Slice}(\mathcal{D})\times\prod_{i=0}^{N}\mathbb{H}_i^\vee. \tag{15}$$

The generalized metric defined by (4) induces a topology on $\mathrm{Slice}(\mathcal{D})$, and we equip the set $\prod_{i=0}^N\mathbb{H}_i^\vee$ with the usual Euclidean topology. Thus we obtain the induced topology on $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ via the inclusion (15). The following theorem is a generalization of [7, Theorem 1.2], which makes each connected component of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ a complex manifold.

Theorem 2.15. *The map*

$$\Pi\colon\ \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})\ni(Z,\mathcal{P})\longmapsto Z\in\prod_{i=0}^{N}\mathbb{H}_i^\vee$$

is a local homeomorphism. In particular, each connected component of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ *is a complex manifold.*

Proof. The proof is almost the same as in [7, Theorem 7.1]. We give the outline of the proof in Section 7. □

Remark 2.16. For $\sigma=(Z,\mathcal{P})\in\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ and $0\neq E\in\mathcal{D}$, we set $\phi_\sigma^\pm(E):=\phi_\mathcal{P}^\pm(E)$. By the definition of the generalized metric (4), the maps

$$\phi_*^\pm(E)\colon\ \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})\to\mathbb{R}$$

are continuous. In particular, the subset of $\sigma\in\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ in which E is semistable is a closed subset.

Later on we will need the following lemma, which relates a family of points in $\prod_{i=0}^N\mathbb{H}_i^\vee$ and points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$. The proof will be given in Section 7.

Lemma 2.17. *Let*

$$(0,1) \ni t \longmapsto Z_t \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$$

be a continuous map, and $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. Suppose that $\sigma_t = (Z_t, \mathcal{A})$ determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$. Then $\{\sigma_t\}_{t\in(0,1)}$ is a continuous family in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$.

2.5. **Group action.** Similarly to Bridgeland's stability conditions, the space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ carries a group action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, which is a universal covering space of $\mathrm{GL}^+(2,\mathbb{R})$. Although we do not need this group action in this paper, it seems worth putting it here as an analogue of [7, Lemma 8.2].

Lemma 2.18. *The space $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ carries a right action of the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.*

Proof. Note that the group $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is identified with the set of pairs (T,f), where $f\colon \mathbb{R}\to\mathbb{R}$ is an increasing map with $f(x+1)=f(x)+1$, and $T\in \mathrm{GL}^+(2,\mathbb{R})$ such that the induced maps on $S^1=\mathbb{R}/2\mathbb{Z}=(\mathbb{R}^2\setminus\{0\})/\mathbb{R}_{>0}$ are the same. Given $\sigma=(\{Z_i\}_{i=0}^N,\mathcal{P})\in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ and $(T,f)\in\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, we set $Z_i'=T^{-1}\circ Z_i$ and $\mathcal{P}'(\phi)=\mathcal{P}(f(\phi))$. Then $\sigma'=(\{Z_i'\}_{i=0}^N,\mathcal{P}')$ gives an element of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$, and the right action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is given in this way. $\square$

We give an example on the global structure of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$.

Example 2.19. Let C be an elliptic curve and $\mathcal{D}=D^b(\mathrm{Coh}(C))$. We set Γ and the filtration $\Gamma_\bullet$ as in Example 2.6. In this case, the same proof of [7, Theorem 9.1] shows that the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ is free and transitive. Hence we have

$$\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})\cong\widetilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Remark 2.20. There is a close relationship between weak stability conditions and the polynomial stability conditions introduced by Bayer [1]. Let us fix an isomorphism

$$\Gamma \cong \bigoplus_{i=0}^{N} \mathbb{H}_i \tag{16}$$

and take a pair of the heart of a bounded t-structure $\mathcal{A}\subset\mathcal{D}$ and $Z=\{Z_i\}_{i=0}^N\in\prod_{i=0}^N\mathbb{H}_i^{\vee}$, satisfying (13). Then the polynomial function $\widetilde{Z}_m\colon\Gamma\to\mathbb{C}$, given by

$$\widetilde{Z}_m(v)=\sum_{i=0}^{N} m^i Z_i(v_i),$$

where $v_i\in\mathbb{H}_i$ is the i-th component of v, satisfies $\widetilde{Z}_m(v)\in\mathfrak{H}$ for $m\gg 0$. Hence the pair $(\widetilde{Z}_m,\mathcal{A})$ gives a polynomial stability condition if the Harder-Narasimhan property is satisfied. However the set of Z-(semi)stable objects and that of $\widetilde{Z}_m$-(semi)stable objects are different. It is easy to see that

$$Z\text{-stable}\Rightarrow\widetilde{Z}_m\text{-stable}\Rightarrow\widetilde{Z}_m\text{-semistable}\Rightarrow Z\text{-semistable}.$$

Therefore the notion of weak stability conditions is more coarse than that of polynomial stability conditions. Roughly speaking, a polynomial stability condition is an analogue of Gieseker stability, and a weak stability condition is an analogue of μ-stability.

Remark 2.21. It seems that $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ is a space of limiting degeneration points of the usual space of stability conditions $\mathrm{Stab}(\mathcal{D})$. Under the isomorphism (16), the multiplicative group $\mathbb{R}_{>0}$ acts on $\Gamma^\vee := \mathrm{Hom}(\Gamma, \mathbb{C}) \cong \prod_{i=0}^{N} \mathbb{H}_i^\vee$ via

$$t \cdot (Z_0, Z_1, \cdots, Z_N) = (Z_0, tZ_1, \cdots, t^N Z_N).$$

The above action lifts to an action on $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ via $(Z, \mathcal{P}) \mapsto (t \cdot Z, \mathcal{P})$. Presumably there is a natural topology on the set

$$\overline{\mathrm{Stab}}_{\Gamma_\bullet}(\mathcal{D}) = \mathrm{Stab}(\mathcal{D}) \coprod \left(\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})/\mathbb{R}_{>0}\right),$$

which makes $\overline{\mathrm{Stab}}_{\Gamma_\bullet}(\mathcal{D})$ a complex manifold with real codimension one boundary $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})/\mathbb{R}_{>0}$. A choice of filtrations $\Gamma_\bullet$ should correspond to a choice of limiting directions.

3. Proof of the Main Theorem

In what follows, we assume that X is a smooth projective Calabi-Yau 3-fold over $\mathbb{C}$. We set $\mathrm{Coh}_{\le 1}(X)$ to be

$$\mathrm{Coh}_{\le 1}(X) := \{E \in \mathrm{Coh}(X) : \dim \mathrm{Supp}(E) \le 1\}. \tag{17}$$

In this section, we show how Theorem 1.2 is proved via wall-crossing phenomena in the space of weak stability conditions on the following triangulated category:

$$\mathcal{D}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X)\rangle_{\mathrm{tr}} \subset D^b(\mathrm{Coh}(X)). \tag{18}$$

We will construct weak stability conditions on $\mathcal{D}_X$, investigate corresponding stable objects, and show Theorem 1.2. In the proof of Theorem 1.2, we will use a wall-crossing formula which will be established under a general setting in Section 4.

3.1. Construction of a t-structure on $\mathcal{D}_X$.

We begin with constructing a t-structure on $\mathcal{D}_X$. First we recall the notion of torsion pairs and tilting.

Definition 3.1 ([11]). Let $\mathcal{A}$ be an abelian category, and $(\mathcal{T}, \mathcal{F})$ a pair of subcategories of $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a *torsion pair* if the following conditions hold:

- $\mathrm{Hom}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
- Any object $E \in \mathcal{A}$ fits into an exact sequence,

$$0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0, \tag{19}$$

  with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathcal{A}$, its *tilting* is defined by

$$\mathcal{A}^\dagger := \left\{ E \in D^b(\mathcal{A}) : \begin{array}{l} \mathcal{H}^{-1}(E) \in \mathcal{F},\ \mathcal{H}^0(E) \in \mathcal{T}, \\ \mathcal{H}^i(E) = 0 \text{ for } i \notin \{-1, 0\} \end{array} \right\}; \tag{20}$$

i.e., $\mathcal{A}^\dagger = \langle \mathcal{F}[1], \mathcal{T}\rangle_{\mathrm{ex}}$ in $D^b(\mathcal{A})$. It is known that $\mathcal{A}^\dagger$ is the heart of a bounded t-structure on $D^b(\mathcal{A})$ (cf. [11, Proposition 2.1]).

Let $\mathrm{Coh}_{\ge 2}(X)$ be the subcategory of $\mathrm{Coh}(X)$,

$$\mathrm{Coh}_{\ge 2}(X) := \{E \in \mathrm{Coh}(X) : \mathrm{Hom}(\mathrm{Coh}_{\le 1}(X), E) = 0\}. \tag{21}$$

It is easy to see that the pair

$$(\mathrm{Coh}_{\leq 1}(X), \mathrm{Coh}_{\geq 2}(X)) \tag{22}$$

is a torsion pair on $\mathrm{Coh}(X)$.

Definition 3.2. We define the abelian category $\mathrm{Coh}^{\dagger}(X)$ to be the tilting with respect to (22); i.e.,

$$\mathrm{Coh}^{\dagger}(X) = \langle \mathrm{Coh}_{\geq 2}(X)[1], \mathrm{Coh}_{\leq 1}(X) \rangle_{\mathrm{ex}}.$$

Remark 3.3. The category $\mathrm{Coh}^{\dagger}(X)$ is one of the hearts of perverse t-structures introduced by Bezrukavnikov [5] and Kashiwara [18].

Remark 3.4. It is easy to see that the subcategory $\mathrm{Coh}_{\leq 1}(X) \subset \mathrm{Coh}^{\dagger}(X)$ is closed under subobjects and quotients in $\mathrm{Coh}^{\dagger}(X)$.

The above construction induces a t-structure on $\mathcal{D}_X$.

Lemma 3.5. *The intersection $\mathcal{A}_X = \mathcal{D}_X \cap \mathrm{Coh}^{\dagger}(X)[-1]$ in $D^b(\mathrm{Coh}(X))$ is the heart of a bounded t-structure on $\mathcal{D}_X$, and $\mathcal{A}_X$ is written as*

$$\mathcal{A}_X = \langle \mathcal{O}_X, \mathrm{Coh}_{\leq 1}(X)[-1] \rangle_{\mathrm{ex}}. \tag{23}$$

Proof. Note that we have

$$D^b(\mathrm{Coh}_{\leq 1}(X)) \cap \mathrm{Coh}^{\dagger}(X)[-1] = \mathrm{Coh}_{\leq 1}(X)[-1]; \tag{24}$$

hence (24) is the heart of a bounded t-structure on $D^b(\mathrm{Coh}_{\leq 1}(X))$. For $F \in \mathrm{Coh}_{\leq 1}(X)$, we have $\mathrm{Hom}(\mathcal{O}_X, F[-1]) = 0$ and

$$\begin{aligned} \mathrm{Hom}(F[-1], \mathcal{O}_X) &\cong H^2(X, F)^{\vee} \\ &= 0, \end{aligned}$$

by the Serre duality. Then the result follows by setting $\mathcal{D} = D^b(\mathrm{Coh}(X))$, $\mathcal{D}' = D^b(\mathrm{Coh}_{\leq 1}(X))$, $\mathcal{A} = \mathrm{Coh}^{\dagger}(X)[-1]$ and $E = \mathcal{O}_X$ in Proposition 3.6 below. $\square$

We have used the following proposition, whose proof will be given in Section 6.

Proposition 3.6. *Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category and $\mathcal{A} \subset \mathcal{D}$ the heart of a bounded t-structure on $\mathcal{D}$. Take $E \in \mathcal{A}$ with $\mathrm{End}(E) = \mathbb{C}$ and a full triangulated subcategory $\mathcal{D}' \subset \mathcal{D}$, which satisfy the following conditions.*

- *The category $\mathcal{A}' := \mathcal{A} \cap \mathcal{D}'$ is the heart of a bounded t-structure on $\mathcal{D}'$, which is closed under subobjects and quotients in the abelian category $\mathcal{A}$.*
- *For any object $F \in \mathcal{A}'$, we have*

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(F, E) = 0. \tag{25}$$

Let $\mathcal{D}_E$ be the triangulated category,

$$\mathcal{D}_E := \langle E, \mathcal{D}' \rangle_{\mathrm{tr}} \subset \mathcal{D}.$$

Then $\mathcal{A}_E := \mathcal{D}_E \cap \mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}_E$, which satisfies

$$\mathcal{A}_E = \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}.$$

Remark 3.7. By Remark 3.4, the subcategory $\mathrm{Coh}_{\leq 1}(X)[-1] \subset \mathcal{A}_X$ is also closed under subobjects and quotients. In particular, $\mathcal{O}_x[-1] \in \mathcal{A}_X$ is a simple object for any closed point $x \in X$.

3.2. **Weak stability conditions on $\mathcal{D}_X$.** In this paragraph, we construct weak stability conditions on $\mathcal{D}_X$. Let $N_1(X)$, $N^1(X)$ be the abelian groups of numerical classes of curves in X, divisors in X, respectively. They are finite rank free abelian groups, and there is a perfect pairing,

$$N_1(X)_{\mathbb{R}} \times N^1(X)_{\mathbb{R}} \ni (C, D) \mapsto C \cdot D \in \mathbb{R}.$$

We denote by $\mathrm{NE}(X) \subset N_1(X)$ the numerical classes of effective curves, and by $A(X) \subset N^1(X)_{\mathbb{R}}$ the ample cone. We set

$$N_{\le 1}(X) = \mathbb{Z} \oplus N_1(X),$$

and define Γ to be

$$\Gamma = N_{\le 1}(X) \oplus \mathbb{Z}.$$

The group homomorphism $\mathrm{cl}\colon K(\mathcal{D}_X) \to \Gamma$ is defined by

$$\mathrm{cl}(E) = (\mathrm{ch}_3(E), \mathrm{ch}_2(E), \mathrm{ch}_0(E)).$$

By the definition of $\mathcal{D}_X$, it is obvious that $\mathrm{ch}_{\bullet}(E)$ has integer coefficients; thus cl is well defined. We denote by rk the projection onto the third factor,

$$\mathrm{rk}\colon \Gamma \ni (s, l, r) \mapsto r \in \mathbb{Z}.$$

Let $\Gamma_0 = \mathbb{Z}$, $\Gamma_1 = N_{\le 1}(X)$ and $\Gamma_2 = \Gamma$. This defines a filtration $\Gamma_{\bullet}$,

$$\Gamma_0 \overset{i}{\hookrightarrow} \Gamma_1 \overset{j}{\hookrightarrow} \Gamma_2 = \Gamma, \tag{26}$$

via $i(s) = (s, 0)$ and $j(s, l) = (s, l, 0)$. We have

$$\mathbb{H}_0 = \mathbb{Z}, \quad \mathbb{H}_1 = N_1(X), \quad \mathbb{H}_2 = \mathbb{Z},$$

and there is a natural isomorphism,

$$\mathbb{C} \times N^1(X)_{\mathbb{C}} \times \mathbb{C} \xrightarrow{\cong} \prod_{i=0}^{2} \mathbb{H}_i^{\vee}. \tag{27}$$

For the elements,

$$z_0, z_1 \in \mathfrak{H} \text{ with } \arg z_i \in (\pi/2, \pi), \quad \omega \in A(X),$$

the data

$$\xi = (-z_0, -i\omega, z_1) \tag{28}$$

associates the element $Z_{\xi} \in \prod_{i=0}^{2} \mathbb{H}_i^{\vee}$ via the isomorphism (27). It is written as

$$Z_{0,\xi}\colon \mathbb{H}_0 \ni s \longmapsto -sz_0,$$
$$Z_{1,\xi}\colon \mathbb{H}_1 \ni l \longmapsto -i\omega \cdot l,$$
$$Z_{2,\xi}\colon \mathbb{H}_2 \ni r \longmapsto rz_1.$$

Lemma 3.8. *The pairs*

$$\sigma_{\xi} = (Z_{\xi}, \mathcal{A}_X), \quad \xi \text{ is given by } (28), \tag{29}$$

determine points in $\mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X)$.

Proof. We check that (13) holds for any non-zero $E \in \mathcal{A}_X$. We write $\mathrm{cl}(E) = (-n, -\beta, r)$ for $n \in \mathbb{Z}$, $\beta \in N_1(X)$ and $r \in \mathbb{Z}$. By the description (23), we have either

$$r > 0, \quad \text{or} \quad r = 0,\ \beta \in \mathrm{NE}(X), \quad \text{or} \quad r = \beta = 0,\ n > 0.$$

Then (13) follows by our construction of Z_ξ. The proofs to check other properties, i.e. the Harder-Narasimhan property, the support property, and locally finiteness, are straightforward. We give the proof in Section 6. (The condition $\arg z_0 > \pi/2$ will be required to show the local finiteness, and $\arg z_1 > \pi/2$ will be required in Lemma 3.11 below. These properties will also be used in proving the Harder-Narasimhan property.) $\square$

We define the subspace $\mathcal{V}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ as follows.

Definition 3.9. We define $\mathcal{V}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ to be

$$\mathcal{V}_X := \{\sigma_\xi : \sigma_\xi \text{ is given by (29)}\}. \tag{30}$$

By Lemma 2.17, the map $\xi \mapsto \sigma_\xi$ is continuous. In particular, $\mathcal{V}_X$ is a connected subspace.

3.3. **Semistable objects of rank one.** In this paragraph, we study semistable objects in $\mathcal{A}_X$ of rank one. We first recall the notion of stable pairs.

Definition 3.10. A pair (F, s) is a *stable pair* if it satisfies the following conditions.

- $F \in \mathrm{Coh}_{\le 1}(X)$ is a pure sheaf; i.e., there is no 0-dimensional subsheaf $Q \subset F$.
- $s \colon \mathcal{O}_X \to F$ is a morphism with 0-dimensional cokernel.
- As a convention, we also call the pair $(0, 0)$ a stable pair.

We have the following lemma.

Lemma 3.11. (i) *Take $\sigma_\xi = (Z_\xi, \mathcal{P}_\xi) \in \mathcal{V}_X$ with $\mathcal{P}_\xi \in \mathrm{Slice}(\mathcal{D}_X)$, and an object $E \in \mathcal{P}_\xi((1/2, 1])$ satisfying $\mathrm{rk}(E) = 1$. Then there is an exact sequence in $\mathcal{A}_X$,*

$$0 \longrightarrow I_C \longrightarrow E \longrightarrow Q[-1] \longrightarrow 0, \tag{31}$$

where I_C is an ideal sheaf of the 1-dimensional subscheme $C \subset X$ and Q is a 0-dimensional sheaf.

(ii) *An object $E \in \mathcal{A}_X$ fits into a sequence (31) if and only if E is isomorphic to a two-term complex*

$$\cdots \to 0 \to \mathcal{O}_X \xrightarrow{s} F \to 0 \to \cdots, \tag{32}$$

with $F \in \mathrm{Coh}_{\le 1}(X)$ and s has 0-dimensional cokernel. Here $\mathcal{O}_X$ is located in degree zero and F is in degree one.

(iii) *Let $E = (\mathcal{O}_X \xrightarrow{s} F)$ be a two-term complex as in (32). Then $\mathrm{Hom}(\mathcal{O}_x[-1], E) = 0$ for all $x \in X$ if and only if (F, s) is a stable pair.*

Proof. (i) Since $E \in \langle \mathcal{O}_X, \mathrm{Coh}_{\le 1}(X)[-1] \rangle_{\mathrm{ex}}$ and $\mathrm{rk}(E) = 1$, there is a filtration in $\mathcal{A}_X$,

$$0 = E_{-1} \subset E_0 \subset E_1 \subset E_2 = E,$$

such that for $F_i = E_i / E_{i-1}$, we have

$$F_0, F_2 \in \mathrm{Coh}_{\le 1}(X)[-1], \quad F_1 = \mathcal{O}_X.$$

Suppose that F_2 is 1-dimensional, and let $F_3 \in \mathrm{Coh}_{\le 1}(X)[-1]$ be the object such that $F_3[1] \subset F_2[1]$ is the maximum 0-dimensional subsheaf of $F_2[1] \in \mathrm{Coh}_{\le 1}(X)$. We have the surjections in $\mathcal{A}_X$,

$$E \twoheadrightarrow F_2 \twoheadrightarrow F_2/F_3.$$

On the other hand, it is easy to see that

$$\mathcal{P}_\xi(1/2) = \{F[-1] : F \text{ is a pure 1-dimensional sheaf}\},$$

by noting Remark 3.7. Therefore we have $\mathrm{Hom}(E, F_2/F_3) = 0$ by the second condition of Definition 2.1. This is a contradiction; hence F_2 is 0-dimensional. This implies the existence of the sequence (31).

(ii) Obviously a two-term complex (32) fits into a sequence (31). Conversely let $E \in \mathcal{A}_X$ be an object which fits into (31). The composition $Q[-2] \to I_C \hookrightarrow \mathcal{O}_X$ becomes a zero map since

$$\begin{aligned}\mathrm{Hom}(Q[-2], \mathcal{O}_X) &\cong H^1(X, Q)^\vee \\ &= 0.\end{aligned}$$

Therefore $I_C \hookrightarrow \mathcal{O}_X$ factorizes via $I_C \to E \to \mathcal{O}_X$. Taking the cone, we obtain the distinguished triangle

$$E \longrightarrow \mathcal{O}_X \xrightarrow{s} F.$$

Since F fits into the distinguished triangle $\mathcal{O}_C \to F \to Q$, we have $F \in \mathrm{Coh}_{\le 1}(X)$. Also the cokernel of s is isomorphic to Q; hence it is 0-dimensional.

(iii) We have the exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow F[-1] \longrightarrow E \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Applying $\mathrm{Hom}(\mathcal{O}_x[-1], *)$ to the above sequence yields

$$\mathrm{Hom}(\mathcal{O}_x, F) \cong \mathrm{Hom}(\mathcal{O}_x[-1], E).$$

Hence $\mathrm{Hom}(\mathcal{O}_x[-1], E) = 0$ for all $x \in X$ is equivalent to that F is pure. $\square$

For $v \in \Gamma$ and $\sigma \in \mathcal{V}_X$, we set

$$M^v(\sigma) := \{E \in \mathcal{A}_X : E \text{ is } \sigma\text{-semistable with } \mathrm{cl}(E) = v\}.$$

The above set of objects is described as follows.

Proposition 3.12. *For $\xi = (-z_0, -i\omega, z_1)$ as in (28), let $\sigma_\xi = (Z_\xi, \mathcal{A}_X) \in \mathcal{V}_X$ be the associated weak stability condition. For an element $v = (-n, -\beta, 1) \in \Gamma$, we have the following.*

(i) *Assume that* $\arg z_0 < \arg z_1$. *Then we have*

$$M^v(\sigma_\xi) = \left\{ \begin{array}{l} \textit{ideal sheaves } I_C \subset \mathcal{O}_X \textit{ for subschemes } C \subset X \\ \textit{with } \dim C \le 1,\ [C] = \beta \textit{ and } \chi(\mathcal{O}_C) = n \end{array} \right\}.$$

(ii) *Assume that* $\arg z_0 > \arg z_1$. *Then we have*

$$M^v(\sigma_\xi) = \left\{ \begin{array}{l} \textit{two-term complexes } (\mathcal{O}_X \xrightarrow{s} F) \textit{ for} \\ \textit{stable pairs } (F, s) \textit{ with } [F] = \beta,\ \chi(F) = n \end{array} \right\}.$$

Moreover in both cases, any $E \in M^v(\sigma_\xi)$ is σ_ξ-stable.

Proof. (i) Take $E \in M^v(\sigma_\xi)$ and consider the exact sequence (31). Suppose that $Q \neq 0$. Since $\arg z_0 < \arg z_1$, we have $\arg Z_\xi(I_C) > \arg Z_\xi(Q[-1])$, which contradicts the σ_ξ-semistability of E. Hence $Q = 0$ and E is isomorphic to the ideal sheaf I_C. Conversely take an object $I_C \in \mathcal{A}_X$ for a curve $C \subset X$, and an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow A \longrightarrow I_C \longrightarrow B \longrightarrow 0,$$

for non-zero $A, B \in \mathcal{A}_X$. Since $\mathcal{H}^1(I_C) = 0$, we have $\mathcal{H}^1(B) = 0$; hence we have $\mathrm{rk}(B) = 1$ and $\mathrm{rk}(A) = 0$. This implies that $A = F[-1]$ for $F \in \mathrm{Coh}_{\le 1}(X)$; thus $\arg Z_\xi(A) < \arg Z_\xi(B)$ is satisfied by $\arg z_1 > \pi/2$. Therefore I_C is σ_ξ-stable.

(ii) Take $E \in M^v(\sigma_\xi)$. Note that by Remark 3.7, the object $\mathcal{O}_x[-1] \in \mathcal{A}_X$ is σ_ξ-stable. The condition $\arg z_0 > \arg z_1$ implies that $\arg Z_\xi(\mathcal{O}_x[-1]) > \arg Z_\xi(E)$; hence we have $\mathrm{Hom}(\mathcal{O}_x[-1], E) = 0$ for any closed point $x \in X$. By Proposition 3.11 (iii), E is isomorphic to $(\mathcal{O}_X \xrightarrow{s} F)$ for a stable pair (F, s). Conversely take $E = (\mathcal{O}_X \xrightarrow{s} F)$ for a stable pair (F, s). We take an exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

for $A, B \in \mathcal{A}_X$. Suppose that $\mathrm{rk}(B) = 0$; hence $\mathrm{rk}(A) = 1$. Since there is a surjection of sheaves $\mathcal{H}^1(E) \twoheadrightarrow \mathcal{H}^1(B)$ and $\mathcal{H}^1(E)$ is 0-dimensional, we have $B = Q[-1]$ for a 0-dimensional sheaf Q. Then $\arg Z_\xi(A) < \arg Z_\xi(B)$ is satisfied by $\arg z_0 > \arg z_1$. If $\mathrm{rk}(B) > 0$, then $\mathrm{rk}(B) = 1$ and $\mathrm{rk}(A) = 0$. In this case, A is written as $G[-1]$ for $G \in \mathrm{Coh}_{\le 1}(X)$. By Proposition 3.11 (iii), G is not 0-dimensional; hence G is 1-dimensional. Then $\arg Z_\xi(A) < \arg_\xi(B)$ is satisfied by $\arg z_1 > \pi/2$; hence E is σ_ξ-stable. □

Remark 3.13. The above result is a weak stability version of the corresponding result for Bayer's polynomial stability conditions [1, Proposition 6.1.1].

3.4. **Proof of Theorem 1.2.** We show Theorem 1.2, using the wall-crossing formula in a general setting. The following theorem is a summary of the results in Section 5. (We note that the filtration on Γ in Theorem 3.14 is not necessarily given by (26).)

Theorem 3.14 (Corollary 5.13). *Let $\Gamma_\bullet$ be a filtration of $\Gamma = N_{\le 1}(X) \oplus \mathbb{Z}$ satisfying*

$$\Gamma_0 \subset \cdots \subset \Gamma_{N-1} = N_{\le 1}(X) \stackrel{i}{\hookrightarrow} \Gamma_N = \Gamma,$$

via the inclusion $i(s, l) = (s, l, 0)$, and let $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be a connected subset satisfying Assumption 4.1 *in Section* 4. *We have the following.*

- *For $\sigma = (Z, \mathcal{A}) \in \mathcal{V}$ and $v = (-n, -\beta, 1) \in \Gamma$, there is a counting invariant,*

$$\widehat{\mathrm{DT}}_{n,\beta}(\sigma) \in \mathbb{Q},$$

such that if the moduli stack of σ-semistable objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$, denoted by $\mathcal{M}^v(\sigma)$, is written as $[M/\mathbb{G}_m]$, where M is a scheme with $\mathbb{G}_m$ acting trivially, we have $\widehat{\mathrm{DT}}_{n,\beta}(\sigma) = \chi(M)$.

- *Let $\widehat{\mathrm{DT}}(\sigma)$ and $\widehat{\mathrm{DT}}_0(\sigma)$ be the series,*

$$\widehat{\mathrm{DT}}(\sigma) = \sum_{n,\beta} \widehat{\mathrm{DT}}_{n,\beta}(\sigma) x^n y^\beta,$$

$$\widehat{\mathrm{DT}}_0(\sigma) = \sum_{(n,\beta)\in\Gamma_0} \widehat{\mathrm{DT}}_{n,\beta}(\sigma) x^n y^\beta.$$

Then the quotient series

$$\widehat{\mathrm{DT}}'(\sigma) := \frac{\widehat{\mathrm{DT}}(\sigma)}{\widehat{\mathrm{DT}}_0(\sigma)}$$

is well defined and does not depend on the choice of σ*, as long as* σ *is general. (See Definition* 5.10 *for general points.)*

Let $I_n(X,\beta)$ be the moduli space of subschemes $C \subset X$ with $\dim C \le 1$ and $[C] = \beta$, $\chi(\mathcal{O}_C) = n$. Since $I_n(X,\beta)$ is a projective scheme, we can consider the generating series,

$$\widehat{\mathrm{DT}}(X) = \sum_{n,\beta} \chi(I_n(X,\beta)) x^n y^\beta,$$
$$\widehat{\mathrm{DT}}_0(X) = \sum_{n} \chi(I_n(X,0)) x^n.$$

Let $P_n(X,\beta)$ be the moduli space of stable pairs (F,s) with $[F] = \beta$, $\chi(F) = n$. In [25], it is proved that $P_n(X,\beta)$ is a fine projective moduli scheme. We consider the generating series,

$$\widehat{\mathrm{PT}}(X) = \sum_{n,\beta} \chi(P_n(X,\beta)) x^n y^\beta.$$

Applying Theorem 3.14, we obtain the Euler characteristic version of DT/PT correspondence.

Theorem 3.15. *We have the following equality of the generating series:*

$$\widehat{\mathrm{DT}}'(X) := \frac{\widehat{\mathrm{DT}}(X)}{\widehat{\mathrm{DT}}_0(X)} = \widehat{\mathrm{PT}}(X).$$

Proof. By Lemma 3.16 below, we can apply Theorem 3.14 for $\mathcal{V}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ given in Definition 3.9. Take two elements (28),

$$\xi = (-z_0, -i\omega, z_1), \quad \xi' = (-z_0', -i\omega, z_1'),$$

such that $\arg z_0 < \arg z_1$ and $\arg z_0' > \arg z_1'$. By Proposition 3.11, we have

$$\mathcal{M}^v(\sigma_\xi) = [I_n(X,\beta)/\mathbb{G}_m], \tag{33}$$
$$\mathcal{M}^v(\sigma_{\xi'}) = [P_n(X,\beta)/\mathbb{G}_m], \tag{34}$$

where $\mathbb{G}_m$ acts on $I_n(X,\beta)$ and $P_n(X,\beta)$ trivially. Here we note that any $E \in M^v(\sigma_\xi)$ or $E \in M^v(\sigma_{\xi'})$ is stable by Proposition 3.11; hence $\mathrm{Aut}(E) = \mathbb{G}_m$. The stabilizer groups $\mathbb{G}_m$ in the stacks (33), (34) are contributions of such trivial automorphisms. Applying Theorem 3.14, we have

$$\widehat{\mathrm{DT}}'(X) = \widehat{\mathrm{DT}}'(\sigma_\xi) = \widehat{\mathrm{DT}}'(\sigma_{\xi'}) = \widehat{\mathrm{PT}}(X),$$

as expected. □

We have used the following lemma, which will be proved in Section 6.

Lemma 3.16. *The subset* $\mathcal{V}_X \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ *satisfies Assumption* 4.1 *in Section* 4.

Remark 3.17. It is also possible to construct the (usual) stability conditions on $\mathcal{D}_X$. For elements

$$\alpha \in \mathbb{R}_{>0},\ B + i\omega \in N^1(X)_{\mathbb{C}},\ \gamma \in \mathfrak{H},$$

with ω ample, we set

$$Z\colon \Gamma \ni (s, l, r) \mapsto s\alpha - (B + i\omega)l + r\gamma \in \mathbb{C}.$$

Then $(Z, \mathcal{A}_X)$ satisfies (8), and it determines an element of $\mathrm{Stab}_{\Gamma}(\mathcal{D}_X)$ if $\mathrm{Im}\,\gamma > 0$. One can show that PT theory is realized as stable objects with respect to such stability conditions. On the other hand, DT theory does not appear as stable objects with respect to the above stability conditions. There might exist stability conditions in which DT theory appears as stable objects after crossing the wall $\mathrm{Im}\,\gamma = 0$. However we are unable to show the support property at the points $\mathrm{Im}\,\gamma = 0$, so the wall-crossing at such points cannot be justified. This is one of the reasons we work over the space of weak stability conditions, rather than the usual stability conditions.

4. General framework

4.1. **Moduli stacks.** In this paragraph, we give a framework to discuss the moduli problem of semistable objects in $\mathcal{D}_X$. Let us recall that there is an algebraic stack $\mathcal{M}$ locally of finite type over $\mathbb{C}$, which parameterizes $E \in D^b(\mathrm{Coh}(X))$ satisfying

$$\mathrm{Ext}^i(E, E) = 0, \quad \text{for any } i < 0. \tag{35}$$

(See [23].) Let $\mathcal{M}_0$ be the fiber at $[0] \in \mathrm{Pic}(X)$ of the following morphism:

$$\det\colon \mathcal{M} \ni E \longmapsto \det E \in \mathrm{Pic}(X).$$

For any object $E \in \mathcal{D}_X$, the corresponding $\mathbb{C}$-valued point $[E] \in \mathcal{M}$ is contained in $\mathcal{M}_0$. Let $\mathcal{A} \subset \mathcal{D}_X$ be the heart of a bounded t-structure on $\mathcal{D}_X$. We can consider the following (abstract) substack,

$$\mathcal{O}bj(\mathcal{A}) \subset \mathcal{M}_0,$$

which parameterizes objects $E \in \mathcal{A}$. The above stack decomposes as

$$\mathcal{O}bj(\mathcal{A}) = \coprod_{v \in \Gamma} \mathcal{O}bj^v(\mathcal{A}),$$

where $\mathcal{O}bj^v(\mathcal{A})$ is the stack of objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$.

4.2. **Assumption.** Here we give a framework to discuss the wall-crossing formula under a general setting. Let $\Gamma_\bullet$ be a filtration (5) on $\Gamma = N_{\le 1}(X) \oplus \mathbb{Z}$, satisfying the following:

$$\Gamma_0 \subset \cdots \subset \Gamma_{N-1} = N_{\le 1}(X) \stackrel{i}{\hookrightarrow} \Gamma_N = \Gamma, \tag{36}$$

via the inclusion $i(s, l) = (s, l, 0)$. We assume that a connected subset $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ satisfies the following assumption.

Assumption 4.1. For any $\sigma = (Z = \{Z_i\}_{i=0}^N, \mathcal{P}) \in \mathcal{V}$ with $\mathcal{A} = \mathcal{P}((0,1])$, the following conditions are satisfied.

- There is $\psi \in \mathbb{R}$ which satisfies

$$\mathcal{O}_X \in \mathcal{P}(\psi), \quad \frac{1}{2} < \psi < 1, \tag{37}$$

and $\mathcal{O}_X$ is the only object $E \in \mathcal{P}(\psi)$ with $\mathrm{cl}(E) = (0, 0, 1)$.

- For any $1 \leq j \leq N-1$, we have

$$Z_j(\mathbb{H}_j) \subset \mathbb{R} \cdot i. \tag{38}$$

- For any $v, v' \in \Gamma_0$ and any other point $\tau = (W, \mathcal{Q}) \in \mathcal{V}$, we have

$$Z(v) \in \mathbb{R}_{>0} Z(v') \quad \text{if and only if} \quad W(v) \in \mathbb{R}_{>0} W(v'). \tag{39}$$

- For any $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, the stack of objects
$$\mathcal{O}bj^v(\mathcal{A}) \subset \mathcal{M}_0$$
is an open substack of $\mathcal{M}_0$. In particular, $\mathcal{O}bj^v(\mathcal{A})$ is an algebraic stack locally of finite type over $\mathbb{C}$.
- For any $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, the stack of σ-semistable objects $E \in \mathcal{A}$ with $\mathrm{cl}(E) = v$,
$$\mathcal{M}^v(\sigma) \subset \mathcal{O}bj^v(\mathcal{A}),$$
is an open substack of finite type over $\mathbb{C}$.
- There are subsets $0 \in T \subset S \subset N_{\leq 1}(X)$, which satisfy Assumption 4.4 in the next paragraph.
- For any other point $\tau \in \mathcal{V}$, there is a good path (see Definition 4.2 below) in $\mathcal{V}$ which connects σ and τ.

The notion of good path is defined as follows.

Definition 4.2. A path $[0,1] \ni t \mapsto \sigma_t \in \mathcal{V}$ is *good* if for any $t \in (0,1)$ and $v \in \Gamma_0$ satisfying $Z_t(v) \in \mathbb{R}_{>0} Z_t(\mathcal{O}_X)$, we have

$$\arg Z_{t+\varepsilon}(v) < \arg Z_{t+\varepsilon}(\mathcal{O}_X), \quad \arg Z_{t-\varepsilon}(v) > \arg Z_{t-\varepsilon}(\mathcal{O}_X), \text{ or} \tag{40}$$

$$\arg Z_{t+\varepsilon}(v) > \arg Z_{t+\varepsilon}(\mathcal{O}_X), \quad \arg Z_{t-\varepsilon}(v) < \arg Z_{t-\varepsilon}(\mathcal{O}_X), \tag{41}$$

for $0 < \varepsilon \ll 1$.

Remark 4.3. For $\sigma = (Z, \mathcal{P}) \in \mathcal{V}$, the first condition of Assumption 4.1 implies that $\mathrm{rk}(E) \geq 0$ for any $E \in \mathcal{P}((0,1])$.

4.3. **Completions of $\mathbb{C}[N_{\leq 1}(X)]$.** Here we discuss completions of $\mathbb{C}[N_{\leq 1}(X)]$ corresponding to subsets $0 \in T \subset S \subset \mathbb{C}[N_{\leq 1}(X)]$ satisfying Assumption 4.4 below. Note that the existence of such T, S is one of the conditions of Assumption 4.1. For subsets $S_1, S_2 \subset N_{\leq 1}(X)$, we set

$$S_1 + S_2 := \{s_1 + s_2 : s_i \in S_i\} \subset N_{\leq 1}(X).$$

The sixth condition of Assumption 4.1 is stated as follows.

Assumption 4.4. In the situation of Assumption 4.1, the subsets $0 \in T \subset S \subset N_{\leq 1}(X)$ satisfy the following conditions.

- We have

$$T + T \subset T, \quad S + T \subset S. \tag{42}$$

- For any $x \in N_{\leq 1}(X)$, there are only finitely many ways to write $x = y + z$ for $y, z \in S$.
- Let $\psi \in \mathbb{R}$ be as in (37) for $\sigma \in \mathcal{V}$. Then for $I = (\psi - \varepsilon, \psi + \varepsilon)$ with $0 < \varepsilon \ll 1$, we have

$$\{(n, \beta) \in N_{\leq 1}(X) : (-n, -\beta, 1) \in C_\sigma(I)\} \subset S, \tag{43}$$

$$\{(n, \beta) \in \Gamma_0 : (-n, -\beta, r) \in C_\sigma(I), \ r = 0 \text{ or } 1\} \subset T. \tag{44}$$

- There is a family of sets $\{S_\lambda\}_{\lambda\in\Lambda}$ with $S_\lambda \subset S$ such that $S \setminus S_\lambda$ is a finite set and

$$S_\lambda + T \subset S_\lambda, \quad S = \bigcup_{\lambda\in\Lambda} (S \setminus S_\lambda).$$

Here the definition of $C_\sigma(I)$ is given in (11). An example of S and T will be given later in (87) and (88). For a possibly infinite sum,

$$f = \sum_{n,\beta} a_{n,\beta} x^n y^\beta, \quad a_{n,\beta} \in \mathbb{C},$$

its support is defined by

$$\mathrm{Supp}(f) := \{(n,\beta) \in N_{\le 1}(X) : a_{n,\beta} \neq 0\}.$$

The completions are defined as follows.

Definition 4.5. For a subset $S \subset N_{\le 1}(X)$, the vector space $\mathbb{C}\llbracket S \rrbracket$ is defined by

$$\mathbb{C}\llbracket S \rrbracket := \left\{ f = \sum_{n,\beta} a_{n,\beta} x^n y^\beta : \mathrm{Supp}(f) \subset S \right\}.$$

Suppose that $0 \in T \subset S$ satisfy Assumption 4.4. The product on $\mathbb{C}[N_{\le 1}(X)]$ generalizes naturally to products on $\mathbb{C}\llbracket T \rrbracket$, and $\mathbb{C}\llbracket S \rrbracket$ is a $\mathbb{C}\llbracket T \rrbracket$-module with $\mathbb{C}\llbracket T \rrbracket \subset \mathbb{C}\llbracket S \rrbracket$. Let $\{f_i\}_{i\in I}$ be a possibly infinite family of elements of $\mathbb{C}\llbracket T \rrbracket$ with $f_i(0,0) = 0$. Then the infinite product of the exponential makes sense,

$$\prod_{i\in I} \exp(f_i) \in \mathbb{C}\llbracket T \rrbracket,$$

if the following condition holds for any $(n,\beta) \in N_{\le 1}(X)$:

$$\sharp\{i \in I : (n,\beta) \in \mathrm{Supp}(f_i)\} < \infty. \tag{45}$$

Also for $f \in \mathbb{C}\llbracket S \rrbracket$ and $g \in \mathbb{C}\llbracket T \rrbracket$ with $g(0,0) \neq 0$, the quotient series makes sense,

$$\frac{f}{g} \in \mathbb{C}\llbracket S \rrbracket.$$

Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be as in Assumption 4.4. We write $\lambda' \preceq \lambda$ if $S_\lambda \subset S_{\lambda'}$. For $\lambda' \preceq \lambda$, we have the surjection of $\mathbb{C}\llbracket T \rrbracket$-modules,

$$\mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket \twoheadrightarrow \mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_{\lambda'} \rrbracket.$$

In this way, we obtain the inductive system of $\mathbb{C}\llbracket T \rrbracket$-modules $\{\mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket\}_{\lambda\in\Lambda}$, and the isomorphism of $\mathbb{C}\llbracket T \rrbracket$-modules,

$$\mathbb{C}\llbracket S \rrbracket \cong \varprojlim_{\lambda\in\Lambda} \mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket. \tag{46}$$

Since each $\mathbb{C}\llbracket S \rrbracket / \mathbb{C}\llbracket S_\lambda \rrbracket$ is a finite-dimensional $\mathbb{C}$-vector space, its Euclidean topology together with the isomorphism (46) induces the topology on $\mathbb{C}\llbracket S \rrbracket$.

4.4. **Joyce invariants.** Take $\sigma \in \mathcal{V}$, $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Under Assumption 4.1, we are able to construct the $\mathbb{Q}$-valued invariant,

$$J^v(\sigma) \in \mathbb{Q},$$

such that if $\mathcal{M}^v(\sigma)$ is written as $[M/\mathbb{G}_m]$ for a scheme M with $\mathbb{G}_m$ acting on M trivially, then

$$J^v(\sigma) = \chi(M). \tag{47}$$

Here $\chi(*)$ is the topological Euler characteristic. In general, $\mathcal{M}^v(\sigma)$ includes information on the automorphisms of strictly semistable objects, and the denominator of $J^v(\sigma)$ is contributed by such non-trivial automorphisms. The invariant $J^v(\sigma)$ is introduced by D. Joyce [16], using the notion of Hall algebras. Here we briefly explain how to construct $J^v(\sigma)$.

Suppose for instance that $\mathcal{A} \subset \mathcal{D}_X$ is the heart of a bounded t-structure on $\mathcal{D}_X$, such that the stack $\mathcal{O}bj(\mathcal{A})$ is an algebraic stack locally of finite type. We denote by $\mathcal{E}x(\mathcal{A})$ the stack of short exact sequences in $\mathcal{A}$. There are morphisms of stacks,

$$p_i \colon \mathcal{E}x(\mathcal{A}) \longrightarrow \mathcal{O}bj(\mathcal{A}),$$

sending a short exact sequence

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

to objects A_i, respectively.

The $\mathbb{C}$-vector space $\mathcal{H}(\mathcal{A})$ is defined to be spanned by symbols,

$$[\mathcal{X} \stackrel{f}{\longrightarrow} \mathcal{O}bj(\mathcal{A})],$$

where $\mathcal{X}$ is an algebraic stack of finite type with affine stabilizers, and f is a morphism of stacks. The relations generated are of the form

$$[\mathcal{X} \stackrel{f}{\longrightarrow} \mathcal{O}bj(\mathcal{A})] - [\mathcal{Y} \stackrel{f|_{\mathcal{Y}}}{\longrightarrow} \mathcal{O}bj(\mathcal{A})] - [\mathcal{U} \stackrel{f|_{\mathcal{U}}}{\longrightarrow} \mathcal{O}bj(\mathcal{A})], \tag{48}$$

for a closed substack $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{U} = \mathcal{X} \setminus \mathcal{Y}$.

There is an associative product on $\mathcal{H}(\mathcal{A})$ based on Ringel-Hall algebras, defined by

$$[\mathcal{X} \stackrel{f}{\longrightarrow} \mathcal{O}bj(\mathcal{A})] * [\mathcal{Y} \stackrel{g}{\longrightarrow} \mathcal{O}bj(\mathcal{A})] = [\mathcal{Z} \stackrel{p_2 \circ h}{\longrightarrow} \mathcal{O}bj(\mathcal{A})],$$

where the morphism h fits into the Cartesian square

$$\begin{tikzcd}
\mathcal{Z} \arrow[r, "h"] \arrow[d] & \mathcal{E}x(\mathcal{A}) \arrow[r, "p_2"] \arrow[d, "(p_1, p_3)"] & \mathcal{O}bj(\mathcal{A}) \\
\mathcal{X} \times \mathcal{Y} \arrow[r, "f \times g"] & \mathcal{O}bj(\mathcal{A})^{\times 2}. &
\end{tikzcd}$$

The $*$-product is associative by [13, Theorem 5.2]. The algebra $\mathcal{H}(\mathcal{A})$ is Γ-graded,

$$\mathcal{H}(\mathcal{A}) = \bigoplus_{v \in \Gamma} \mathcal{H}_v(\mathcal{A}),$$

where $\mathcal{H}_v(\mathcal{A})$ is spanned by $[\mathcal{X} \stackrel{f}{\to} \mathcal{O}bj(\mathcal{A})]$ factoring via $\mathcal{O}bj^v(\mathcal{A}) \subset \mathcal{O}bj(\mathcal{A})$.

Let $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be a subset satisfying Assumption 4.1, and take $\sigma = (Z, \mathcal{P}) \in \mathcal{V}$ with $\mathcal{A} = \mathcal{P}((0,1])$. In Assumption 4.1, we do not assume that $\mathcal{O}bj^v(\mathcal{A})$

is algebraic for $\mathrm{rk}(v) > 1$ or $\mathrm{rk}(v) = 0$, $v \notin \Gamma_0$. Instead we can discuss as follows. Under Assumption 4.1, we are able to define the following vector spaces:

$$\mathcal{H}_0(\mathcal{A}) = \bigoplus_{v \in \Gamma_0} \mathcal{H}_v(\mathcal{A}),$$
$$\mathcal{H}_N(\mathcal{A}) = \bigoplus_{\mathrm{rk}(v)=1} \mathcal{H}_v(\mathcal{A}).$$

The similar $*$-product makes $\mathcal{H}_0(\mathcal{A})$ an associate algebra and $\mathcal{H}_N(\mathcal{A})$ a $\mathcal{H}_0(\mathcal{A})$-bimodule. We define the elements $\delta^v(\sigma)$ and $\epsilon^v(\sigma)$ as follows.

Definition 4.6. Under the above situation, take $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Suppose that $v \in C_\sigma(\phi)$ with $0 < \phi \le 1$. We define $\delta^v(\sigma)$ to be

$$\delta^v(\sigma) = [\mathcal{M}^v(\sigma) \hookrightarrow \mathcal{O}bj(\mathcal{A})] \in \mathcal{H}_*(\mathcal{A}),$$

where $* = 0$ if $v \in \Gamma_0$ and $* = N$ if $\mathrm{rk}(v) = 1$. The element $\epsilon^v(\sigma) \in \mathcal{H}_*(\mathcal{A})$ is defined to be

$$\epsilon^v(\sigma) = \sum_{\substack{v_1+\cdots+v_l=v, \\ v_i \in C_\sigma(\phi),\ 1 \le i \le l.}} \frac{(-1)^{l-1}}{l} \delta^{v_1}(\sigma) * \cdots * \delta^{v_l}(\sigma). \tag{49}$$

Remark 4.7. It is possible to define $\delta^v(\sigma)$ by the fourth condition of Assumption 4.1. Take $v_1, \cdots, v_l \in C_\sigma(\phi)$ which appear in (49). By Remark 4.3 and Lemma 5.1 below, there is $1 \le e \le l$ such that $\mathrm{rk}(v_e) = 1$ and $v_i \in \Gamma_0$ for $i \ne e$, and there is a finite number of possibilities for such v_i. Therefore (49) is a finite sum and $\epsilon^v(\sigma)$ is well defined.

There is a map (cf. [15, Theorem 4.9]),

$$\Upsilon \colon \mathcal{H}(\mathcal{A}) \longrightarrow \mathbb{Q}(t),$$

such that if G is a special algebraic group (cf. [15, Definition 2.1]) acting on a variety Y, we have

$$\Upsilon([[Y/G] \xrightarrow{f} \mathcal{O}bj(\mathcal{A})]) = P(Y,t)/P(G,t),$$

where $P(Y,t)$ is the virtual Poincaré polynomial of Y; i.e., if Y is smooth and projective, we have

$$P(Y,t) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(Y,\mathbb{C}) t^i,$$

and $P(Y,t)$ is defined for any variety Y using the motivic relation (48) for varieties.

Theorem 4.8 ([15, Section 6.2])**.** *The element*

$$(t^2-1)\Upsilon(\epsilon^v(\sigma)) \in \mathbb{Q}(t)$$

is regular at $t=1$.

The above theorem is used to define the invariant $J^v(\sigma) \in \mathbb{Q}$.

Definition 4.9. For $\sigma \in \mathcal{V}$ and $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, we define $J^v(\sigma) \in \mathbb{Q}$ as follows.

- If $v \in C_\sigma(\phi)$ for $0 < \phi \le 1$, we define
$$J^v(\sigma) := \lim_{t \to 1} (t^2-1)\Upsilon(\epsilon^v(\sigma)).$$

- If $v \in C_\sigma(\phi)$ for $1 < \phi \le 2$, we define $J^v(\sigma) := J^{-v}(\sigma)$.
- Otherwise we define $J^v(\sigma) = 0$.

Remark 4.10. Suppose that $\mathcal{M}^v(\sigma) = [M/\mathbb{G}_m]$ for a scheme M with $\mathbb{G}_m$ acting on M trivially. Then for any $\mathbb{C}$-valued point of $\mathcal{M}^v(\sigma)$, the corresponding object $E \in \mathcal{A}$ is σ-stable. Hence we have $\epsilon^v(\sigma) = \delta^v(\sigma) = ([M/\mathbb{G}_m] \to \mathcal{O}bj(\mathcal{A}))$ and

$$(t^2 - 1)\Upsilon(\epsilon^v(\sigma)) = P(M, t).$$

Therefore we obtain (47).

Under the above situation, we introduce the following generating series.

Definition 4.11. Let $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ be a subset satisfying Assumption 4.1. For $(n, \beta) \in N_{\le 1}(X)$ and $\sigma \in \mathcal{V}$, we define $\widehat{\mathrm{DT}}_{n,\beta}(\sigma)$ to be

$$\widehat{\mathrm{DT}}_{n,\beta}(\sigma) := J^{(-n,-\beta,1)}(\sigma) \in \mathbb{Q}.$$

The generating series $\widehat{\mathrm{DT}}(\sigma)$ and $\widehat{\mathrm{DT}}_0(\sigma)$ are defined by

$$\widehat{\mathrm{DT}}(\sigma) := \sum_{n,\beta} \widehat{\mathrm{DT}}_{n,\beta}(\sigma) x^n y^\beta \in \mathbb{C}\,[\![S]\!]\,,$$

$$\widehat{\mathrm{DT}}_0(\sigma) := \sum_{(n,\beta)\in\Gamma_0} \widehat{\mathrm{DT}}_{n,\beta}(\sigma) x^n y^\beta \in \mathbb{C}\,[\![T]\!]\,.$$

The above series are elements of $\mathbb{C}\,[\![S]\!]$, $\mathbb{C}\,[\![T]\!]$, respectively, by the third condition of Assumption 4.4. Since $\widehat{\mathrm{DT}}_0(\sigma) = 1 + \cdots$ by the first condition of Assumption 4.1, the following reduced series is well defined:

$$\widehat{\mathrm{DT}}'(\sigma) := \frac{\widehat{\mathrm{DT}}(\sigma)}{\widehat{\mathrm{DT}}_0(\sigma)} \in \mathbb{C}\,[\![S]\!]\,.$$

5. Wall-crossing formula

The purpose of this section is to study how $\widehat{\mathrm{DT}}(\sigma)$ varies under change of σ. First let us introduce the pairing χ on Γ,

$$\chi((s, l, r), (s', l', r')) = rs' - r's. \tag{50}$$

By the Riemann-Roch theorem and the Serre duality, for $E, F \in \mathcal{D}$ we have

$$\begin{aligned}\chi(\mathrm{cl}(E), \mathrm{cl}(F)) = \dim \mathrm{Hom}(E, F) &- \dim \mathrm{Ext}^1(E, F) \\ &+ \dim \mathrm{Ext}^1(F, E) - \dim \mathrm{Hom}(F, E).\end{aligned}$$

Below we fix a subset $\mathcal{V} \subset \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D}_X)$ satisfying Assumption 4.1.

5.1. Wall and chamber structure.

In this paragraph, we show the existence of a wall and chamber structure in a neighborhood of $\sigma \in \mathcal{V}$. For $v \in \Gamma$ and $\varepsilon > 0$, we set

$$S_{\varepsilon,v}(\sigma) := \left\{ v' \in \Gamma : \begin{array}{l} \text{there is } \phi \in \mathbb{R} \text{ such that} \\ v', v - v' \in C_\sigma((\phi - \varepsilon, \phi + \varepsilon)) \end{array} \right\}. \tag{51}$$

We show the following lemma.

Lemma 5.1. (i) *Suppose that $v \in \Gamma_0$. Then for $0 < \varepsilon \ll 1$, we have $S_{\varepsilon,v}(\sigma) \subset \Gamma_0$ and $S_{\varepsilon,v}(\sigma)$ is a finite set.*

(ii) *Suppose that $\mathrm{rk}(v) = 1$. Then for $0 < \varepsilon \ll 1$, we have $S_{\varepsilon,v}(\sigma) \cap N_{\le 1}(X) \subset \Gamma_0$ and $S_{\varepsilon,v}(\sigma)$ is a finite set.*

Proof. (i) The first assertion is obvious. The finiteness of $S_{\varepsilon,v}(\sigma)$ easily follows from the support property (9).

(ii) Suppose that $S_{\varepsilon,v}(\sigma) \neq \emptyset$. Then we have $v \in C_{\sigma}((\phi-\varepsilon, \phi+\varepsilon))$ for some $0 \le \phi < 2$. Since $\phi_{\sigma}(v) = \psi$, where ψ is given in (37), we have $\psi \in (\phi-\varepsilon, \phi+\varepsilon)$; hence $v \in C_{\sigma}((\psi-2\varepsilon, \psi+2\varepsilon))$. By choosing $\varepsilon > 0$ sufficiently small, we may assume that $(\psi-2\varepsilon, \psi+2\varepsilon) \subset (1/2, 1)$. Then if $v = v' + v''$ in $C_{\sigma}((\psi-2\varepsilon, \psi+2\varepsilon))$, we may assume that v' and v'' are written as $v' = (-n', -\beta', 1)$ and $v'' = (-n'', -\beta'', 0)$ by Remark 4.3. Then $v'' \in \Gamma_0$ follows from the second condition of Assumption 4.1, which implies the first assertion. By (43) and (44), we have $(n', \beta') \in S$ and $(n'', \beta'') \in T$. Therefore the finiteness of $S_{\varepsilon,v}(\sigma)$ follows from the second condition of Assumption 4.4. □

We have the following lemma.

Lemma 5.2. *Take $\sigma \in \mathcal{V}$, $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, and $0 < \varepsilon \ll 1$ such that $S_{\varepsilon,v}(\sigma)$ is a finite set (cf. Lemma 5.1). Let $\sigma \in U_{\varepsilon} \subset \mathrm{Stab}_{\Gamma_{\bullet}}(\mathcal{D}_X)$ be an open neighborhood of σ such that any $\tau = (W, \mathcal{Q}) \in U_{\varepsilon}$ satisfies $d(\mathcal{P}, \mathcal{Q}) < \varepsilon$. Then there are finitely many real codimension one submanifolds $\{\mathcal{W}_{\lambda}\}_{\lambda \in \Lambda}$ in U_{ε}, such that if $\sigma_1, \sigma_2 \in \mathcal{V}$ are contained in the same connected component of $U_{\varepsilon} \setminus \{\mathcal{W}_{\lambda}\}_{\lambda \in \Lambda}$, then*

$$\mathcal{M}^{v'}(\sigma_1) = \mathcal{M}^{v'}(\sigma_2),$$

for any $v' \in S_{\varepsilon,v}(\sigma)$.

Proof. We set $\Lambda = S_{\varepsilon,v}(\sigma) \times S_{\varepsilon,v}(\sigma)$. For each $\gamma = (v_1, v_2) \in \Lambda$, we define $\mathcal{W}_{\lambda}$ to be

$$\mathcal{W}_{\lambda} := \{\tau = (W, \mathcal{Q}) \in U_{\varepsilon} : W(v_1)/W(v_2) \in \mathbb{R}_{>0}\}.$$

Then it is easy to see that $\{\mathcal{W}_{\lambda}\}_{\gamma \in \Lambda}$ gives a desired set of submanifolds. (Also see the proof of [8, Proposition 9.3].) □

5.2. **Joyce's formula.** Take $\sigma \in \mathcal{V}$ and $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Our setting in this paragraph is as follows.

- We choose $\varepsilon > 0$ and an open neighborhood $\sigma \in U_{\varepsilon}$ as in Lemma 5.2. By choosing $\varepsilon > 0$ sufficiently small, we may assume that any connected component $\mathcal{C} \subset U_{\varepsilon} \setminus \{\mathcal{W}_{\lambda}\}_{\lambda \in \Lambda}$ satisfies $\sigma \in \overline{\mathcal{C}}$. Below we denote by V_{ε} the connected component of $U_{\varepsilon} \cap \mathcal{V}$ which contains σ. We take two weak stability conditions $\tau, \tau' \in V_{\varepsilon}$.

The wall-crossing formula enables us to describe $J^{v}(\tau)$ in terms of $J^{v'}(\tau)$ with $v' \in S_{\varepsilon,v}(\sigma)$. The transformation coefficients are purely combinatorial. In what follows, $I \subset \mathbb{R}$ is a sufficiently small interval, i.e. $I = (a, b)$ with $0 < b - a \ll 1$. Note that for $v \in C_{\sigma}(I)$, we have (see (12))

$$\phi_{\tau}(v) \in I \pm \varepsilon.$$

Definition 5.3 ([16, Definition 4.2])**.** For non-zero $v_1, \cdots, v_l \in C_{\sigma}(I)$, we define

$$S(\{v_1, \cdots, v_l\}, \tau, \tau') \in \{0, \pm 1\}$$

as follows. If for each $i = 1, \cdots, l-1$, we have either (52) or (53),

$$\phi_{\tau}(v_i) \le \phi_{\tau}(v_{i+1}) \text{ and } \phi_{\tau'}(v_1 + \cdots + v_i) > \phi_{\tau'}(v_{i+1} + \cdots + v_l), \tag{52}$$

$$\phi_{\tau}(v_i) > \phi_{\tau}(v_{i+1}) \text{ and } \phi_{\tau'}(v_1 + \cdots + v_i) \le \phi_{\tau'}(v_{i+1} + \cdots + v_l), \tag{53}$$

then define $S(\{v_1, \cdots, v_l\}, \tau, \tau')$ to be $(-1)^r$, where r is the number of $i = 1, \cdots, l-1$ satisfying (52). Otherwise we define $S(\{v_1, \cdots, v_l\}, \tau, \tau') = 0$.

Another combinatorial coefficient is defined as follows.

Definition 5.4 ([16, Definition 4.4])**.** For non-zero $v_1, \cdots, v_l \in C_\sigma(I)$, we define

$$\begin{aligned} & U(\{v_1, \cdots, v_l\}, \tau, \tau') \\ & = \prod_{a=1}^{l''} S(\{w_i\}_{i \in \xi^{-1}(a)}, \tau, \tau') \frac{(-1)^{l''-1}}{l''} \prod_{b=1}^{l'} \frac{1}{|\psi^{-1}(b)|!}. \end{aligned} \tag{54}$$

Here ψ, ξ satisfy the following.

- ψ and ξ are non-decreasing surjective maps.
- We have

$$\phi_\tau(v_i) = \phi_\tau(v_j), \tag{55}$$

  for $1 \le i, j \le l$ with $\psi(i) = \psi(j)$.
- For $1 \le i, j \le l''$, we have

$$\phi_{\tau'}(\sum_{k \in \psi^{-1}\xi^{-1}(i)} v_k) = \phi_{\tau'}(\sum_{k \in \psi^{-1}\xi^{-1}(j)} v_k). \tag{56}$$

Also w_i for $1 \le i \le l'$ is defined as

$$w_i = \sum_{j \in \psi^{-1}(i)} v_j \in C_\sigma(I). \tag{57}$$

Note that if $S_{\varepsilon,v}(\sigma) \neq \{0\}$, then $v \in C_\sigma((\phi - \varepsilon, \phi + \varepsilon))$ for some $\phi \in \mathbb{R}$ and $S_{\varepsilon,v}(\sigma) \subset C_\sigma((\phi - \varepsilon, \phi + \varepsilon))$. By choosing $\varepsilon > 0$ sufficiently small, we can take I to be $I = (\phi - \varepsilon, \phi + \varepsilon)$ in Definition 5.3 and Definition 5.4. In our situation, Joyce's wall-crossing formula [16] is applied to show the following. (See the discussion in [29, Section 2].)

Theorem 5.5 ([16, Theorem 6.28, Equation (130)])**.** *We have*

$$\begin{aligned} J^v(\tau') = & \sum_{\substack{l \ge 1,\ v_i \in S_{\varepsilon,v}(\sigma), \\ v_1 + \cdots + v_l = v}} \ \sum_{\substack{G \text{ is a connected, simply connected oriented} \\ \text{graph with vertex } \{1, \cdots, l\},\ \overset{i}{\bullet} \to \overset{j}{\bullet} \text{ implies } i<j}} \\ & \frac{1}{2^{l-1}} U(\{v_1, \cdots, v_l\}, \tau, \tau') \prod_{\overset{i}{\bullet} \to \overset{j}{\bullet} \text{ in } G} \chi(v_i, v_j) \prod_{i=1}^{l} J^{v_i}(\tau). \end{aligned} \tag{58}$$

Remark 5.6. The property that the set of $\sigma \in \mathcal{V}$ in which a fixed $E \in \mathcal{D}_X$ is semistable is closed (cf. Remark 2.16) corresponds to the dominant condition in the sense of [16, Definition 3.16]. This property is not true for another generalized stability condition whose central charges are polynomials [1], [30]. This is one of the reasons we work over weak stability conditions rather than polynomial stability conditions.

The above theorem immediately yields the following.

Proposition-Definition 5.7. *For $v = (-n, -\beta, 0) \in \Gamma_0$, the value $J^v(\tau)$ does not depend on $\tau \in \mathcal{V}$. We define*

$$\widehat{N}_{n,\beta} := J^v(\tau), \tag{59}$$

for $\tau \in \mathcal{V}$.

Proof. Since $\mathcal{V}$ is connected by our assumption, the problem is local on $\mathcal{V}$. Noting Lemma 5.1 (i) and $\chi(*,*)=0$ on Γ_0, the formula (58) implies that $J^v(\tau)=J^v(\tau')$ for any $\tau,\tau'\in V_\varepsilon$. $\square$

5.3. **Wall-crossing formula of generating functions.** Take $\sigma=(Z,\mathcal{P})\in\mathcal{V}$ and a continuous family in $\mathcal{V}$,

$$(-\delta,\delta)\ni t\mapsto \sigma_t\in\mathcal{V},$$

with $\sigma_0=\sigma$ and $\delta>0$. By Lemma 5.2, the following limiting series makes sense:

$$\begin{aligned}\widehat{\mathrm{DT}}(\sigma_\pm)&:=\lim_{t\to\pm 0}\widehat{\mathrm{DT}}(\sigma_t)\\ &=\sum_{n,\beta}\widehat{\mathrm{DT}}_{n,\beta}(\sigma_\pm)x^n y^\beta\in\mathbb{C}\,[\![S]\!]\,,\end{aligned}$$

where S is given in Assumption 4.4. The series

$$\widehat{\mathrm{DT}}_0(\sigma_\pm):=\lim_{t\to\pm 0}\widehat{\mathrm{DT}}_0(\sigma_t)\in\mathbb{C}\,[\![T]\!]$$

is also defined. We set W to be

$$W:=\{v\in\Gamma_0 : Z(v)\in\mathbb{R}_{>0}Z(\mathcal{O}_X)\}.$$

For any $v\in W$, we assume that

$$\arg Z_t(v)>\arg Z_t(\mathcal{O}_X),\quad 0<t\ll 1, \tag{60}$$

$$\arg Z_t(v)<\arg Z_t(\mathcal{O}_X),\quad 0<-t\ll 1, \tag{61}$$

i.e. (41) happens at $t=0$. The following theorem is a generalization of the result in [29].

Theorem 5.8. *We have the following equalities of the generating series:*

$$\widehat{\mathrm{DT}}(\sigma_-)=\widehat{\mathrm{DT}}(\sigma_+)\cdot\prod_{-(n,\beta)\in W}\exp(n\widehat{N}_{n,\beta}x^n y^\beta), \tag{62}$$

$$\widehat{\mathrm{DT}}_0(\sigma_-)=\widehat{\mathrm{DT}}_0(\sigma_+)\cdot\prod_{-(n,\beta)\in W}\exp(n\widehat{N}_{n,\beta}x^n y^\beta). \tag{63}$$

Proof. We only show (62), as (63) is similarly proved. The proof goes along with the same argument of [29, Theorem 4.7]. Take $v\in\Gamma$ with $\mathrm{rk}(v)=1$ and $\varepsilon>0$ so that $S_{\varepsilon,v}(\sigma)$ is a finite set. For elements $v',v''\in S_{\varepsilon,v}(\sigma)$, we write

$$\phi_\pm(v')\le\phi_\pm(v'')$$

if $\phi_{\sigma_t}(v')\le\phi_{\sigma_t}(v'')$ holds for $0<\pm t\ll 1$. Note that for $v'\in S_{\varepsilon,v}(\sigma)$, we have

$$\phi_\pm(v')=\phi_\pm(\mathcal{O}_X)\quad\text{ if and only if }\ \mathrm{rk}(v')=1, \tag{64}$$

by (60) and (61). For $v_1,\cdots,v_l\in S_{\varepsilon,v}(\sigma)$, we can take the limit of the combinatorial coefficients,

$$\begin{aligned}S(\{v_1,\cdots,v_l\},\sigma_+,\sigma_-)&:=\lim_{t\to+0}S(\{v_1,\cdots,v_l\},\sigma_t,\sigma_{-t}),\\ U(\{v_1,\cdots,v_l\},\sigma_+,\sigma_-)&:=\lim_{t\to+0}U(\{v_1,\cdots,v_l\},\sigma_t,\sigma_{-t}).\end{aligned}$$

Step 1. (i) If $v_i \in W$ for all i, we have

$$S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-) = \begin{cases} 1, & l = 1, \\ 0, & l \geq 2. \end{cases} \tag{65}$$

(ii) Suppose that there is $1 \leq e \leq l$ such that $\mathrm{rk}(v_e) = 1$ and $v_i \in \Gamma_0$ for $i \neq e$. If $S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-) \neq 0$, then $v_i \in W$ for all $i \neq e$. Moreover in this case, we have $e = 1$ or 2 and

$$S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-) = (-1)^{l-e}. \tag{66}$$

Proof. (i) By the assumption (39), $\phi_{\sigma_t}(v) \geq \phi_{\sigma_t}(v')$ holds for $v, v' \in W$ if and only if this holds at $t = 0$. Then (65) follows easily from the definition of $S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-)$. (See the proof of [12, Theorem 4.5].)

(ii) Suppose that there is $2 \leq i < e$ such that

$$\phi_+(v_1) > \cdots > \phi_+(v_i) \leq \phi_+(v_{i+1}) \tag{67}$$

holds. By the definition of $S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-)$ and $\phi_+(v_1) > \phi_+(v_2)$, we have

$$\phi_-(v_1) \leq \phi_-(v_2 + \cdots + v_l) = \phi_-(v_e). \tag{68}$$

On the other hand, by $\phi_+(v_{i-1}) > \phi_+(v_i) \leq \phi_+(v_{i+1})$ we have

$$\begin{aligned} \phi_-(v_1 + \cdots + v_{i-1}) &\leq \phi_-(v_i + \cdots + v_l) = \phi_-(v_e), \\ \phi_-(v_1 + \cdots + v_i) &> \phi_-(v_{i+1} + \cdots + v_l) = \phi_-(v_e), \end{aligned}$$

which implies $\phi_-(v_i) > \phi_-(v_e)$ and contradicts (68). Hence a sequence (67) does not happen. Similarly a sequence

$$\phi_+(v_1) \leq \cdots \leq \phi_+(v_i) > \phi_+(v_{i+1})$$

does not happen for $2 \leq i < e$. Therefore we have two possibilities:

$$\phi_+(v_1) > \cdots > \phi_+(v_{e-1}) > \phi_+(v_e), \tag{69}$$

$$\phi_+(v_1) \leq \cdots \leq \phi_+(v_{e-1}) \leq \phi_+(v_e). \tag{70}$$

In the case of (69) (resp. (70)) the definition of $S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-)$ implies that $\phi_-(v_1) \leq \phi_-(v_e)$ (resp. $\phi_-(v_1) > \phi_-(v_e)$). Hence $v_1 \in W$, and by our assumptions (60) and (61), the inequality (70) does not happen; hence we have (69). Suppose that $e \geq 3$. Then we have $\phi_-(v_1 + v_2) \leq \phi_-(v_e)$; hence $v_1 + v_2 \in W$. Since $v_1 \in W$, we also have $v_2 \in W$, which contradicts $\phi_+(v_1) > \phi_+(v_2)$. Hence we have either $e = 1$ or $e = 2$. A similar argument also shows that $v_i \in W$ for $i > e$.

Conversely if $e = 1$ or $e = 2$ and $v_i \in W$ for $i \neq e$, it is easy to see that one of (52) or (53) holds for each i. Hence (66) holds by the definition of $S(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-)$. □

Step 2. Take $v_1, \cdots, v_l \in S_{\varepsilon,v}(\sigma)$ satisfying $\mathrm{rk}(v_e) = 1$ and $v_i \in \Gamma_0$ for $i \neq e$. Then $U(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-)$ is non-zero only if $v_i \in W$ for $i \neq e$. In this case, we have

$$U(\{v_1, \cdots, v_l\}, \sigma_+, \sigma_-) = \frac{(-1)^{l-e}}{(e-1)!(l-e)!}. \tag{71}$$

Proof. Let

$$\begin{aligned} \psi &\colon \{1, \cdots, l\} \to \{1, \cdots, l'\}, \\ \xi &\colon \{1, \cdots, l'\} \to \{1, \cdots, l''\} \end{aligned}$$

be maps which appear in a non-zero term of (54). By (56) and (64), we have $l''=1$. Also by Step 1 (ii), $\psi(e)$ is either 1 or 2, and $\psi(e)=1$ (resp. $\psi(e)=2$) is equivalent to $e=1$ (resp. $e\ge 2$) by (55) and (64), and $\psi^{-1}\psi(e)=\{e\}$ holds. By Step 1 (ii), each w_i defined by (57) is contained in W for $i\neq\psi(e)$. Noting the condition (55), we conclude that $v_i\in W$ for any $i\neq e$.

Now substituting (66) yields

$$U(\{v_1,\cdots,v_l\},\sigma_+,\sigma_-)$$
$$=\sum_{\psi\colon\{e+1,\cdots,l\}\to\{1,\cdots,l'\}}(-1)^{l'}\frac{1}{(e-1)!}\prod_{b=1}^{l'}\frac{1}{|\psi^{-1}(b)|!},$$

where ψ are non-decreasing surjective maps. For a fixed l, we have

$$\sum_{\psi\colon\{1,\cdots,l\}\to\{1,\cdots,l'\}}(-1)^{l-l'}\prod_{b=1}^{l'}\frac{1}{|\psi^{-1}(b)|!}=\frac{1}{l!},\tag{72}$$

for non-decreasing surjective maps ψ. (See [16, Proposition 4.9].) Hence we obtain (71). □

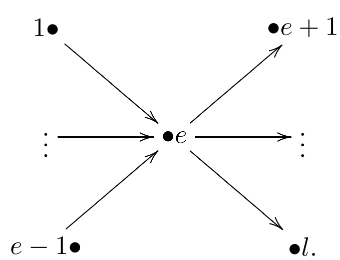
Step 3. For $v=(-n,-\beta,1)\in\Gamma$, we have the formula,

$$\widehat{\mathrm{DT}}_{n,\beta}(\sigma_-)=\sum_{\substack{n_1+\cdots+n_l=n,\\ \beta_1+\cdots+\beta_l=\beta,\\ -(n_i,\beta_i)\in W,\ 1\le i\le l-1.}}\frac{1}{(l-1)!}\prod_{i=1}^{l-1}n_i\widehat{N}_{n_i,\beta_i}\widehat{\mathrm{DT}}_{n_l,\beta_l}(\sigma_+).\tag{73}$$

Proof. We apply the formula (58) for $\tau'=\sigma_{-t}$, $\tau=\sigma_t$ for $0<t\ll 1$. Take $v_1,\cdots,v_l\in S_{\varepsilon,v}(\sigma)$ which appear in a non-zero term of (58). By Step 2, Lemma 5.1 (ii) and Remark 4.3, there is $1\le e\le l$ such that $v_i\in W$ for $i\neq e$ and $\mathrm{rk}(v_e)=1$. Let us write $v_i=(-n_i,-\beta_i,0)$ for $i\neq e$ and $v_e=(-n_e,-\beta_e,1)$. Since we have

$$\chi(v_i,v_j)=0\quad(i,j\neq e),\quad \chi(v_i,v_e)=n_i,\quad \chi(v_e,v_i)=-n_i,$$

an oriented graph G which appears in (58) is of the following form:



Hence substituting (71) to (58), we obtain

$$\widehat{\mathrm{DT}}_{n,\beta}(\sigma_-)=\sum_{1\le e\le l}\sum_{\substack{n_1+\cdots+n_l=n,\\ \beta_1+\cdots+\beta_l=\beta,\\ -(n_i,\beta_i)\in W,\ i\neq e.}}\frac{1}{2^{l-1}(e-1)!(l-e)!}\prod_{i\neq e}n_i\widehat{N}_{n_i,\beta_i}\widehat{\mathrm{DT}}_{n_e,\beta_e}(\sigma_+).$$

Noting that

$$\sum_{1\le e\le l}\frac{1}{2^{l-1}(e-1)!(l-e)!}=\frac{1}{(l-1)!},$$

we obtain the formula (73). □

Obviously (73) implies (62), as expected. □

Remark 5.9. Since $\sum_{-(n,\beta)\in W} n\widehat{N}_{n,\beta}x^n y^\beta$ belongs to $\mathbb{C}\llbracket T\rrbracket$ by (44), the formulas (62), (63) make sense.

Next we compare $\widehat{\mathrm{DT}}(\sigma)$ and $\widehat{\mathrm{DT}}(\tau)$ for two weak stability conditions $\sigma, \tau \in \mathcal{V}$. We introduce the notion of general points in $\mathcal{V}$.

Definition 5.10. We say $\sigma = (Z, \mathcal{P}) \in \mathcal{V}$ is *general* if there is no $v \in \Gamma_0$ which satisfies $Z(v) \in \mathbb{R}_{>0}Z(\mathcal{O}_X)$.

For general $\sigma, \tau \in \mathcal{V}$, take a good path (cf. Definition 4.2)

$$[0,1] \ni t \mapsto \sigma_t = (Z_t, \mathcal{P}_t) \in \mathcal{V},$$

which satisfies $\sigma_0 = \sigma$ and $\sigma_1 = \tau$. For $c \in [0,1]$, let W_c be the set

$$W_c = \{v \in \Gamma_0 : Z_c(v) \in \mathbb{R}_{>0}Z_c(\mathcal{O}_X)\}.$$

For $c \in [0,1]$, we set $\epsilon(c) = 1$ (resp. $\epsilon(c) = -1$) if (40) (resp. (41)) happens at $t = c$. As a corollary of Theorem 5.8, we obtain the following.

Corollary 5.11. *We have the equalities of the generating series:*

$$\widehat{\mathrm{DT}}(\tau) = \widehat{\mathrm{DT}}(\sigma) \cdot \prod_{\substack{-(n,\beta)\in W_c, \\ c\in(0,1).}} \exp(n\widehat{N}_{n,\beta}x^n y^\beta)^{\epsilon(c)}, \tag{74}$$

$$\widehat{\mathrm{DT}}_0(\tau) = \widehat{\mathrm{DT}}_0(\sigma) \cdot \prod_{\substack{-(n,\beta)\in W_c, \\ c\in(0,1).}} \exp(n\widehat{N}_{n,\beta}x^n y^\beta)^{\epsilon(c)}. \tag{75}$$

Proof. We only show (74), as (75) is similarly proved. It is enough to show the equality (74) after the projection,

$$\pi_\lambda \colon \mathbb{C}\llbracket S\rrbracket \to \mathbb{C}\llbracket S\rrbracket / \mathbb{C}\llbracket S_\lambda\rrbracket,$$

where $\{S_\lambda\}_{\lambda\in\Lambda}$ is given in Assumption 4.4. By Lemma 5.2, there is a finite number of points

$$c_0 = 0 < c_1 < \cdots < c_{k-1} < c_k = 1,$$

such that $\pi_\lambda\widehat{\mathrm{DT}}(\sigma_t)$ is constant on $t \in (c_{i-1}, c_i)$ for $1 < i < k-1$, and constant on $[0, c_1)$, $(c_{k-1}, 1]$ since σ and τ are general. Applying Theorem 5.8 at each $t = c_i$, we obtain

$$\pi_\lambda\widehat{\mathrm{DT}}(\tau) = \pi_\lambda\widehat{\mathrm{DT}}(\sigma) \cdot \pi_\lambda \prod_{\substack{-(n,\beta)\in W_{c_i}, \\ 0<i<k.}} \exp(n\widehat{N}_{n,\beta}x^n y^\beta)^{\epsilon(c_i)}.$$

On the other hand, since $S \setminus S_\lambda$ is a finite set, there are only finitely many $c \in (0,1)$ such that

$$\pi_\lambda \prod_{-(n,\beta)\in W_c} \exp(n\widehat{N}_{n,\beta}x^n y^\beta) \neq 1.$$

Also such $c \in (0,1)$ must be equal to one of c_i, since otherwise $\pi_\lambda\widehat{\mathrm{DT}}(\sigma_t)$ is constant near $t = c$, and contradicts Theorem 5.8. Hence (74) holds after the projection. □

Remark 5.12. For $c \in [0,1]$, set $f_c = \sum_{-(n,\beta)\in W_c} n\widehat{N}_{n,\beta}x^n y^\beta$. Then $f_c \in \mathbb{C}\llbracket T\rrbracket$ and $\{f_c\}_{c\in[0,1]}$ satisfy the condition (45) since $t \mapsto \sigma_t$ is a good path. Therefore the formulas (74), (75) make sense.

Another corollary is the following.

Corollary 5.13. *The series*

$$\widehat{\mathrm{DT}}'(\sigma)=\frac{\widehat{\mathrm{DT}}(\sigma)}{\widehat{\mathrm{DT}}_0(\sigma)}\in\mathbb{C}[\![S]\!] \tag{76}$$

does not depend on general $\sigma\in\mathcal{V}$.

Proof. This follows immediately from Assumption 4.1 and Corollary 5.11. □

Remark 5.14. In the proof of Theorem 3.15, we can apply Corollary 5.11 and obtain the following formula:

$$\begin{aligned}\widehat{\mathrm{DT}}_0(X)&=\widehat{\mathrm{DT}}_0(\sigma_\xi)\\&=\prod_{n>0}\exp(n\widehat{N}_{n,0}x^n)\widehat{\mathrm{DT}}_0(\sigma_{\xi'})\\&=\prod_{n>0}\exp(n\widehat{N}_{n,0}x^n).\end{aligned}$$

On the other hand, we know that $\widehat{\mathrm{DT}}_0(X)=M(x)^{\chi(X)}$. These equalities give a calculation of $\widehat{N}_{n,0}$. An easy computation shows that

$$\widehat{N}_{n,0}=\sum_{r|n}\frac{\chi(X)}{r^2}.$$

6. Some technical lemmas

6.1. **Proof of Proposition 3.6.**

Proof. We first show the following lemma.

Lemma 6.1. (i) *For $F\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}\subset\mathcal{A}$, let $u\colon E\to F$ be a non-zero morphism in $\mathcal{A}$. Then* $\mathrm{cok}(u)\in\mathcal{A}$ *is contained in $\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$ and u is injective in $\mathcal{A}$.*

(ii) *For $F\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}\subset\mathcal{A}$ and $G\in\mathcal{A}'$, let $u\colon G\to F$ be a non-zero morphism in $\mathcal{A}$. Then* $\mathrm{cok}(u)\in\mathcal{A}$ *is contained in $\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$.*

Proof. (i) By the condition (25), there is an exact sequence in $\mathcal{A}$,

$$0\longrightarrow F_1\xrightarrow{u_1}F\xrightarrow{u_2}F_2\longrightarrow 0, \tag{77}$$

with $F_i\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$ which satisfy the following.

- The composition $u_2\circ u=0$. Hence u factorizes as $E\stackrel{u'}{\to}F_1\stackrel{u_1}{\to}F$.
- There is a surjection $F_1\stackrel{u_3}{\twoheadrightarrow}E$ in $\mathcal{A}$ such that $\ker(u_3)\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$ and the composition $u_3\circ u'$ is non-zero.

Since $\mathrm{End}(E)=\mathbb{C}$, the map u' is split injective; hence u is also injective. Also we have $\mathrm{cok}(u')\cong\ker(u_3)\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$; hence the exact sequence in $\mathcal{A}$,

$$0\longrightarrow\mathrm{cok}(u')\longrightarrow\mathrm{cok}(u)\longrightarrow F_2\longrightarrow 0,$$

shows $\mathrm{cok}(u)\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$.

(ii) We may assume that u is injective since $\mathcal{A}'$ is closed under quotients in $\mathcal{A}$. Since $F\in\langle E,\mathcal{A}'\rangle_{\mathrm{ex}}$, there is a filtration in $\mathcal{A}$,

$$A_0\subset A_1\subset\cdots A_{l-1}\subset A_l=F,$$

such that each subquotient A_i/A_{i-1} is either isomorphic to E or contained in $\mathcal{A}'$. We call the smallest such l the length of F. We show the claim by induction on l. If $l=1$, then $F \in \mathcal{A}'$ by the condition (25). Since $\mathcal{A}' \subset \mathcal{A}$ is closed under quotients, we have $\mathrm{cok}(u) \in \mathcal{A}'$. Assume that $l > 1$. Then there is an exact sequence (77) such that the lengths of the F_i are strictly smaller than l. Let G_2 be the image of the composition in $\mathcal{A}$,

$$G \xrightarrow{u} F \xrightarrow{u_2} F_2,$$

and G_1 the kernel of $G \twoheadrightarrow G_2$. We obtain the morphism of exact sequences in $\mathcal{A}$,

$$\begin{tikzcd} 0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "u"] & G_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] & F \arrow[r] & F_2 \arrow[r] & 0. \end{tikzcd}$$

Note that $G_i \in \mathcal{A}'$ and each vertical arrow is injective in $\mathcal{A}$. Hence we have the exact sequence in $\mathcal{A}$,

$$0 \longrightarrow F_1/G_1 \longrightarrow \mathrm{cok}(u) \longrightarrow F_2/G_2 \longrightarrow 0.$$

By the induction hypothesis, we have $F_i/G_i \in \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}$. Therefore we have $\mathrm{cok}(u) \in \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}$. $\square$

Proof of Proposition 3.6:

We show that $\mathcal{A}_E := \mathcal{D}_E \cap \mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}_E$ and is written as $\mathcal{A}_E = \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}$. To show this, it is enough to show that for any $F \in \mathcal{D}_E$, we have

$$\mathcal{H}^i_{\mathcal{A}}(F) \in \langle E, \mathcal{A}' \rangle_{\mathrm{ex}}, \quad i \in \mathbb{Z}. \tag{78}$$

Here we denote by

$$\mathcal{H}^i_{\mathcal{A}} \colon \mathcal{D} \ni F \longmapsto \mathcal{H}^i_{\mathcal{A}}(F) \in \mathcal{A},$$

the i-th cohomology functor with respect to the t-structure with heart $\mathcal{A}$. Noting that $\mathcal{A}'$ is the heart of a bounded t-structure on $\mathcal{D}'$, the triangulated category $\mathcal{D}_E$ is also written as $\langle E, \mathcal{A}' \rangle_{\mathrm{tr}}$. Hence any object $F \in \mathcal{D}_E$ is written as successive extensions by objects $E[i']$ and $G[i'']$ for $G \in \mathcal{A}'$ and $i', i'' \in \mathbb{Z}$. As in the proof of Lemma 6.1 (ii), we show (78) by the length of such an extension. Suppose that $F \in \mathcal{D}_E$ satisfies (78), and take a distinguished triangle

$$G \longrightarrow F \longrightarrow H \longrightarrow G[1],$$

with $G = E$ or $G \in \mathcal{A}'$. By the induction argument, it is enough to show that H satisfies (78). Taking the long exact sequence associated to $\mathcal{H}^{\bullet}_{\mathcal{A}}(\ast)$, we have $\mathcal{H}^i_{\mathcal{A}}(F) \cong \mathcal{H}^i_{\mathcal{A}}(H)$ for $i \neq -1, 0$ and the exact sequence in $\mathcal{A}$,

$$0 \to \mathcal{H}^{-1}_{\mathcal{A}}(F) \to \mathcal{H}^{-1}_{\mathcal{A}}(H) \to G \xrightarrow{u} \mathcal{H}^0_{\mathcal{A}}(F) \to \mathcal{H}^0_{\mathcal{A}}(H) \to 0.$$

If $u = 0$, then obviously H satisfies (78). Otherwise we have

$$\ker(u) \in \mathcal{A}', \quad \mathrm{im}(u) \in \mathcal{A}', \quad \mathrm{cok}(u) \in \langle E, \mathcal{A}' \rangle_{\mathrm{ex}},$$

by Lemma 6.1. Therefore H satisfies (78) also in this case. $\square$

6.2. **Proof of Lemma 3.8.** We first show the following lemma.

Lemma 6.2. *The abelian category $\mathcal{A}_X$ is Noetherian.*

Proof. We take a chain of surjections in $\mathcal{A}_X$,

$$E_0 \twoheadrightarrow E_1 \twoheadrightarrow \cdots E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots. \tag{79}$$

By Lemma 3.5, we have $\mathrm{rk}(E) \geq 0$ and $\mathrm{ch}_2(E) \cdot \omega \geq 0$ for a fixed ample divisor ω on X. Hence we may assume that $\mathrm{rk}(E_i)$ and $\mathrm{ch}_2(E_i) \cdot \omega$ are constant for all i. We have exact sequences,

$$0 \longrightarrow Q_j[-1] \longrightarrow E_j \longrightarrow E_{j+1} \longrightarrow 0, \tag{80}$$

where Q_j are 0-dimensional sheaves. The long exact sequence associated to the standard t-structure on $D^b(\mathrm{Coh}(X))$ shows that the induced morphisms $\mathcal{H}^1(E_j) \to \mathcal{H}^1(E_{j+1})$ are surjections of sheaves; hence we may assume that $\mathcal{H}^1(E_j) \cong \mathcal{H}^1(E_{j+1})$. Then the exact sequence (80) induces the sequence,

$$\mathcal{H}^0(E_0) \subset \mathcal{H}^0(E_1) \subset \cdots \subset \mathcal{H}^0(E_j) \subset \cdots \subset \mathcal{H}^0(E_0)^{\vee\vee}. \tag{81}$$

Since $\mathrm{Coh}(X)$ is Noetherian, the above sequence terminates. $\square$

Proof of Lemma 3.8:

Step 1. The pair $\sigma_\xi = (Z_\xi, \mathcal{A}_X)$ satisfies the Harder-Narasimhan property.

Proof. Let $\mathcal{F}$ be the full subcategory of $\mathrm{Coh}_{\leq 1}(X)[-1]$,

$$\mathcal{F} = \{F[-1] : F \text{ is a pure 1-dimensional sheaf}\}. \tag{82}$$

If we set $\mathcal{T} = \{T \in \mathcal{A}_X : \mathrm{Hom}(T, \mathcal{F}) = 0\}$, then the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair on $\mathcal{A}_X$. In fact for any $E \in \mathcal{A}_X$, we have the exact sequence in $\mathcal{A}_X$,

$$0 \longrightarrow Q[-1] \longrightarrow \mathcal{H}^1(E)[-1] \longrightarrow F[-1] \longrightarrow 0,$$

where Q is a 0-dimensional sheaf and $F[-1] \in \mathcal{F}$. Let T be the kernel of the surjection in $\mathcal{A}_X$, $E \to \mathcal{H}^1(E)[-1] \to F[-1]$. Since $\mathrm{Hom}(\mathcal{H}^0(E), \mathcal{F}) = 0$, we have $\mathrm{Hom}(T, \mathcal{F}) = 0$, i.e. $T \in \mathcal{T}$. The exact sequence $0 \to T \to E \to F[-1] \to 0$ gives the desired decomposition (19).

It is easy to see that any $F[-1] \in \mathcal{F}$ is Z_ξ-semistable with $\arg Z_\xi(T) > \arg Z_\xi(F[-1])$ for any non-zero $T \in \mathcal{T}$. Also applying the same argument of [30, Lemma 2.27], an object $E \in \mathcal{T}$ is Z_ξ-semistable if and only if for any exact sequence

$$0 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 0,$$

with $A, B \in \mathcal{T}$, we have $\arg Z_\xi(A) \leq \arg Z_\xi(B)$. By Lemma 6.2 and the proof of Proposition 2.12, it is enough to show that there is no infinite sequence of subobjects in $\mathcal{T}$,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1 \tag{83}$$

(cf. the proof of [30, Theorem 2.29]). Suppose that such a sequence exists. We may assume that $\mathrm{rk}(E_i)$ and $\mathrm{ch}_2(E_i) \cdot \omega$ are constant for all i; hence $E_0/E_{j+1} = Q_j[-1]$, where Q_j is a 0-dimensional sheaf. Taking the long exact sequence of cohomology, we obtain the sequence of surjections of sheaves,

$$\cdots \twoheadrightarrow Q_{j+1} \twoheadrightarrow Q_j \twoheadrightarrow \cdots \twoheadrightarrow Q_1 \twoheadrightarrow Q_0. \tag{84}$$

Since $\mathcal{H}^1(E_0)$ is 0-dimensional and we have the surjections $\mathcal{H}^1(E_0) \twoheadrightarrow Q_j$ for all j, the length of Q_j is bounded above. This implies that (84) terminates; hence (83) also terminates. $\square$

Step 2. Let $\{\mathcal{P}_\xi(\phi)\}_{\phi \in \mathbb{R}}$ be the slicing corresponding to the pair $\sigma_\xi = (Z_\xi, \mathcal{A}_X)$ via Proposition 2.13. Then $\{\mathcal{P}_\xi(\phi)\}_{\phi\in\mathbb{R}}$ is locally finite.

Proof. Since $\mathcal{A}_X$ is Noetherian, it is enough to show that there is $\eta > 0$ such that $\mathcal{P}_\xi((\phi-\eta, \phi+\eta))$ is Artinian for any $\phi \in \mathbb{R}$ with respect to strict monomorphisms. Let $\phi_i = \frac{1}{\pi}\arg z_i \in (1/2, 1)$. By the construction of Z_ξ, there is a non-zero object in $\mathcal{P}_\xi(\phi)$ only if $\phi \in \{1/2, \phi_0, \phi_1\}$. Therefore it is enough to show that $\mathcal{P}_\xi(1/2)$ and $\langle \mathcal{P}_\xi(\phi_0), \mathcal{P}_\xi(\phi_1)\rangle_{\mathrm{ex}}$ are Artinian. It is easy to see that $\mathcal{P}_\xi(1/2)$ coincides with $\mathcal{F}$, where $\mathcal{F}$ is given by (82). Suppose that there is an infinite sequence of strict monomorphisms in $\mathcal{P}_\xi(1/2)$,

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1. \tag{85}$$

Since each E_j is a 1-dimensional sheaf, we have $\mathrm{ch}_2(E_{j+1})\cdot\omega \le \mathrm{ch}_2(E_j)\cdot\omega$ for an ample divisor ω, and $\mathrm{ch}_2(E_j)\cdot\omega = 0$ if and only if $E_j = 0$. Therefore (85) terminates. The Artinian condition of $\langle \mathcal{P}_\xi(\phi_0), \mathcal{P}_\xi(\phi_1)\rangle_{\mathrm{ex}}$ follows from the same argument to show the termination of (83). □

Step 3. The pair $\sigma_\xi = (Z_\xi, \mathcal{P}_\xi)$ satisfies the support property (9).

Proof. Let $E \in \mathcal{A}_X$ be a non-zero object with $\mathrm{cl}(E) = (-n, -\beta, r)$. We introduce the usual Euclidean norm on $\mathbb{H}_0 \otimes \mathbb{R} = \mathbb{R}$ and $\mathbb{H}_2 \otimes \mathbb{R} = \mathbb{R}$. We have

$$\frac{\|E\|}{|Z(E)|} = \begin{cases} |z_1|, & r > 0, \\ \frac{\|\beta\|}{\omega\cdot\beta}, & r = 0,\ \beta \in \mathrm{NE}(X), \\ |z_0|, & r = \beta = 0,\ n > 0. \end{cases}$$

Since β is effective or zero, the above description immediately implies the support property. □

6.3. **Proof of Lemma 3.16.**

Proof. The first, second, third and last conditions are obviously satisfied. We check the other three conditions. Recall the heart of a bounded t-structure $\mathcal{A}_X \subset \mathcal{D}_X$ given in (23).

Step 1. For $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$, the stack of objects

$$\mathcal{O}bj^v(\mathcal{A}_X) \subset \mathcal{M}_0$$

is an open substack of $\mathcal{M}_0$.

Proof. If $\mathrm{rk}(v) = 0$, then $\mathcal{O}bj^v(\mathcal{A})$ is the stack of coherent sheaves $E \in \mathrm{Coh}_{\le 1}(X)$ of numerical type v, and the result is well known. Suppose that $\mathrm{rk}(v) = 1$ and let $\mathcal{E} \in D^b(X \times S)$ be an S-valued point of $\mathcal{M}_0$. Suppose that for each $s \in S$, the object $\mathcal{E}_s := \mathbf{L}i_s^*\mathcal{E}$ satisfies $\mathrm{cl}(\mathcal{E}_s) = v$, where $i_s \colon X \times \{s\} \hookrightarrow X \times S$ is the inclusion. We show that the locus

$$S^\circ := \{s' \in S : \mathcal{E}_{s'} \in \mathcal{A}_X\}$$

is open in S. We may assume that S is irreducible and $S^\circ \ne \emptyset$. Note that the stack of objects in $E \in \mathrm{Coh}^\dagger(X)[-1]$ (cf. (20)) with $\det(E) = \mathcal{O}_X$ is open in $\mathcal{M}_0$ (cf. [30, Lemma 3.14]); hence we may assume that $\mathcal{E}_{s'} \in \mathrm{Coh}^\dagger(X)[-1]$ for all $s' \in S$. As in the proof of Lemma 3.11 (i), any object $E \in \mathcal{A}_X$ with $\mathrm{rk}(E) = 1$ is given by an extension,

$$I_C \longrightarrow E \longrightarrow F[-1],$$

where I_C is the ideal sheaf of $C \subset X$ with $\dim C \le 1$ and $F \in \mathrm{Coh}_{\le 1}(X)$. Therefore an object $E \in \mathrm{Coh}^{\dagger}(X)[-1]$ with $\det(E) = \mathcal{O}_X$ and $\mathrm{rk}(E) = 1$ is contained in $\mathcal{A}_X$ if and only if $\mathcal{H}^0(E)$ is torsion free. First we show the case that S is a smooth curve. We have the spectral sequence,

$$E_2^{p,q} = \mathcal{T}or_{-p}^{\mathcal{O}_{X\times S}}(\mathcal{H}^q(\mathcal{E}), \mathcal{O}_{X\times\{s\}}) \Rightarrow \mathcal{H}^{p+q}(\mathcal{E}_s).$$

Since $E_2^{p,q} = 0$ for $p \le -2$ or $p \ge 1$, the above spectral sequence degenerates at E_2-terms. Therefore $E_2^{-1,0} = 0$, and this implies that $\mathcal{H}^0(\mathcal{E})$ is flat over S, and we have the exact sequence,

$$0 \longrightarrow \mathcal{H}^0(\mathcal{E})_s \longrightarrow \mathcal{H}^0(\mathcal{E}_s) \longrightarrow \mathcal{T}or_1^{\mathcal{O}_{X\times S}}(\mathcal{H}^1(\mathcal{E}), \mathcal{O}_{X\times\{s\}}) \longrightarrow 0.$$

Since $\mathcal{H}^0(\mathcal{E}_s)$ is torsion free by $\mathcal{E}_s \in \mathcal{A}_X$, the sheaf $\mathcal{H}^0(\mathcal{E})_s$ is also torsion free by the above exact sequence. Since $\mathcal{H}^0(\mathcal{E})$ is flat over S, there is an open neighborhood $s \in U$ such that $\mathcal{H}^0(\mathcal{E})_{s'}$ is torsion free for $s' \in U$. By the generic flatness, we have $\mathcal{H}^0(\mathcal{E}_{s'}) \cong \mathcal{H}^0(\mathcal{E})_{s'}$ for $s \neq s' \in U$ by shrinking U if necessary. Therefore $U \subset S^{\circ}$ and S° is open in S.

In general, we can show the openness of S° as follows. Let $V \subset S$ be an open subset on which $\mathcal{H}^j(\mathcal{E})$ is flat for all j. Then we have $\mathcal{H}^j(\mathcal{E}) = 0$ unless $j = 0, 1$, $\mathcal{H}^0(\mathcal{E}_{s'}) = \mathcal{H}^0(\mathcal{E})_{s'}$ for any $s' \in V$, and $S^{\circ} \cap V$ is open in V. Also by the result for smooth curves, we know that S° is dense in S in the Zariski topology; hence $S^{\circ} \cap V$ is non-empty. We apply the same argument for the object $\mathbf{L}i^*\mathcal{E}$, where i is the inclusion,

$$i \colon S \setminus (S^{\circ} \cap V) \hookrightarrow S.$$

By Noetherian induction, we conclude that S° is open in S.

□

Step 2. Take $\sigma_{\xi} = (Z_{\xi}, \mathcal{A}_X) \in \mathcal{V}_X$ and $v \in \Gamma$ with $\mathrm{rk}(v) = 1$ or $v \in \Gamma_0$. Then the substack

$$\mathcal{M}^v(\sigma_{\xi}) \subset \mathcal{O}bj^v(\mathcal{A}_X)$$

is an open substack and it is of finite type over $\mathbb{C}$.

Proof. If $v \in \Gamma_0$, then $\mathcal{M}^v(\sigma_{\xi})$ is the moduli stack of 0-dimensional sheaves, and the result is well known. Suppose that $\mathrm{rk}(v) = 1$. By Step 1 and the argument of [31, Theorem 3.20], it is enough to show the boundedness of σ_{ξ}-semistable objects of numerical type v. For a σ_{ξ}-semistable object $E \in \mathcal{A}_X$, consider the exact sequence (31). For an effective class $\beta \in N_1(X)$, we set $m(\beta)$ as

$$m(\beta) = \inf\{\mathrm{ch}_3(\mathcal{O}_C) : \dim C = 1 \text{ with } [C] = \beta\}. \tag{86}$$

It is well known that $m(\beta) > -\infty$ (cf. [30, Lemma 3.10]); hence the length of Q in (31) is bounded above. Since the set of ideal sheaves with a fixed numerical class is bounded, the object E is contained in a bounded family. □

Step 3. There are subsets $0 \in T \subset S \subset N_{\le 1}(X)$ which satisfy Assumption 4.4.

Proof. We set S and T to be

$$S := \{(n, \beta) \in N_{\le 1}(X) : \beta \ge 0,\ n \ge m(\beta)\}, \tag{87}$$

$$T := \{(n, \beta) \in N_{\le 1}(X) : \beta \ge 0,\ n \ge 0\}. \tag{88}$$

Here $\beta \ge 0$ means β is effective or zero, $m(\beta)$ is given in (86) when β is effective, and $m(0) = 0$. We show that T, S satisfy Assumption 4.4. The first condition is obvious. The second condition follows easily from the fact that any effective class in $N_1(X)$ can be written in finitely many ways as a sum of effective classes. The third one follows from the existence of the exact sequence (31). As for the last one, let Λ be the set of pairs (k, β') of $k \in \mathbb{Z}$ and an effective class $\beta' \in N_1(X)$. For $\lambda = (k, \beta')$, we set

$$S_\lambda = \{(n, \beta) \in S : n \geq k \text{ if } \beta \leq \beta'\}.$$

Here $\beta \leq \beta'$ means that $\beta' - \beta$ is effective or zero. Then $\{S_\lambda\}_{\lambda \in \Lambda}$ gives the desired family. $\square$

Assumption 4.1 has been checked by Step 1, Step 2 and Step 3. $\square$

Remark 6.3. If $v \notin \Gamma_0$, then $\mathcal{M}^v(\sigma)$ is not necessarily of finite type. For instance, consider the class $v = (0, 2[C], 0)$ for a curve $C \subset X$. Then $\mathcal{M}^v(\sigma)$ contains objects $\mathcal{O}_C(D) \oplus \mathcal{O}_C(-D)$ for arbitrary divisors $D \subset C$. Thus $\mathcal{M}^v(\sigma)$ is not of finite type.

7. Some results on weak stability conditions

7.1. **Outline of the proof of Theorem 2.15.**

Proof. We first note that if two elements of $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$, $\sigma = (Z, \mathcal{P})$ and $\tau = (W, \mathcal{Q})$ satisfy $d(\mathcal{P}, \mathcal{Q}) < 1$, then $\sigma = \tau$. (See [7, Lemma 6.4] for the proof.) In particular the map Π is locally injective; hence it is enough to show that Π is locally surjective. For $\sigma = (\{Z_i\}_{i=0}^N, \mathcal{P})$, let us take a σ-semistable object $E \in \mathcal{D}$ with $\mathrm{cl}(E) \in \Gamma_m \setminus \Gamma_{m-1}$. For $\{W_i\}_{i=0}^N \in \prod_{i=0}^N \mathbb{H}_i^\vee$, the support property (9) implies that

$$\left| 1 - \frac{W_m([E])}{Z_m([E])} \right| \leq C \cdot (W_m - Z_m) \left(\frac{[E]}{\|[E]\|} \right), \tag{89}$$

for a constant $C > 0$. For any $0 < \varepsilon \ll 1$, we can find an open neighborhood $\{Z_i\}_{i=0}^N \in U_\varepsilon \subset \prod_{i=0}^N \mathbb{H}_i^\vee$ such that the RHS of (89) is less than $\sin \pi\varepsilon$ for any $\{W_i\}_{i=0}^N \in U_\varepsilon$. In particular $W_m([E]) \neq 0$ for such $\{W_i\}_{i=0}^N$, and we have

$$|\arg W_m([E]) - \arg Z_m([E])| < \pi\varepsilon. \tag{90}$$

The above condition (90) is enough to apply the same proof of [7, Theorem 7.1] to show the existence of $\mathcal{Q} \in \mathrm{Slice}(\mathcal{D})$ satisfying $d(\mathcal{P}, \mathcal{Q}) < \varepsilon$, and (8) holds for the pair $\tau = (\{W_i\}_{i=0}^N, \mathcal{Q})$. Here we just describe how to construct $\mathcal{Q}$ and leave the details to the reader to check that the proof of [7, Theorem 7.1] works in our situation. For $\phi \in \mathbb{R}$ and $a, b \in \mathbb{R}$, a quasi-abelian category $\mathcal{P}((a, b))$ is called thin and envelops ϕ if $b - a < 1 - 2\varepsilon$ and $a + \varepsilon \leq \phi \leq b - \varepsilon$. Then $W = \{W_i\}_{i=0}^N$ determines a map,

$$W \colon \mathcal{P}((a, b)) \ni E \longmapsto \arg W(E) \in (\pi(a - \varepsilon), \pi(b + \varepsilon)).$$

The subcategory $\mathcal{Q}(\phi) \subset \mathcal{D}$ is defined by W-semistable objects $E \in \mathcal{P}((a, b))$ with phase ϕ; i.e., $E \in \mathcal{Q}(\phi)$ if and only if for any exact sequence in $\mathcal{P}((a, b))$,

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0,$$

we have $\arg W(F) \leq \arg W(G)$. The same proof of [7, Theorem 7.1] shows that $\mathcal{Q}(\phi)$ does not depend on a, b, and determines a desired slicing on $\mathcal{D}$. It is enough to check that $\tau = (\{W_i\}_{i=0}^N, \mathcal{Q})$ satisfies the support property. Let $F \in \mathcal{D}$ be a

τ-semistable object with $\mathrm{cl}(F) \in \Gamma_m \setminus \Gamma_{m-1}$. Since (89) is less than $\sin \pi\varepsilon$ and $d(\mathcal{P}, \mathcal{Q}) < \varepsilon$, it is easy to see that

$$\|[F]\|_m \leq \frac{1 - \sin \pi\varepsilon}{\cos 2\pi\varepsilon} C |W_m([F])|.$$

Hence τ satisfies the support property (9), and Π is surjective on U_ε. $\square$

7.2. **Proof of Lemma 2.17.** This follows from a stronger lemma below, by setting $\mathcal{F} = 0$ there. We will need this stronger version in the next paper [28].

Lemma 7.1. *Let $\mathcal{A}$ be the heart of a bounded t-structure on $\mathcal{D}$, and $(\mathcal{T}, \mathcal{F})$ a torsion pair on $\mathcal{A}$. Let $\mathcal{B} = \langle \mathcal{F}[1], \mathcal{T} \rangle_{\mathrm{ex}}$ be the associated tilting. Let*

$$[0, 1) \ni t \longmapsto Z_t \in \prod_{i=0}^{N} \mathbb{H}_i^{\vee}$$

be a continuous map such that $\sigma_t = (Z_t, \mathcal{A})$ for $0 < t < 1$ and $\sigma_0 = (Z_0, \mathcal{B})$ determine points in $\mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$. Then we have $\lim_{t \to 0} \sigma_t = \sigma_0$.

Proof. By Theorem 2.15, we have a continuous family of points $\sigma'_t = (Z_t, \mathcal{Q}_t) \in \mathrm{Stab}_{\Gamma_\bullet}(\mathcal{D})$ for $0 \leq t \ll 1$ such that $\sigma'_0 = \sigma_0$. It is enough to show that $\sigma'_t = \sigma_t$ for such a t, i.e. $\mathcal{Q}_t((0, 1]) = \mathcal{A}$. This follows from the inclusion

$$\mathcal{Q}_t((0, 1]) \subset \mathcal{A}, \tag{91}$$

since both are hearts of bounded t-structures on $\mathcal{A}$. To check (91), first note that any object $E \in \mathcal{F}[1]$ is contained in $\mathcal{Q}_0(1)$, since otherwise $\mathrm{Im}\, Z_t(E) > 0$ for $0 < t \ll 1$, contradicting that Z_t is a weak stability function on $\mathcal{A}$ for such a t. Hence we have

$$\mathcal{Q}_0((0, 1)) \subset \mathcal{T} \subset \mathcal{A}. \tag{92}$$

Next take $0 < \phi \leq 1$ and a quasi-abelian category $\mathcal{Q}_0((a, b))$ which is thin and envelops ϕ. (See the proof of Theorem 2.15.) As in the proof of Theorem 2.15, objects of $\mathcal{Q}_t(\phi)$ consist of Z_t-semistable objects in $\mathcal{Q}_0((a, b))$. If $a < 1$, then we have $\mathcal{Q}_t(\phi) \subset \mathcal{Q}_0((a, b)) \subset \mathcal{A}$ by (92). Suppose $a \geq 1$, and take $E \in \mathcal{Q}_t(\phi)$. Noting that $\mathcal{F}[1] \subset \mathcal{Q}_0(1)$ and (92), we have

$$\mathcal{H}_{\mathcal{A}}^{-1}(E)[1] \in \mathcal{Q}_0([1, a)), \quad \mathcal{H}_{\mathcal{A}}^{0}(E) \in \mathcal{Q}_0((b, 1)).$$

Therefore the following sequence is an exact sequence in $\mathcal{Q}_0((a, b))$,

$$\mathcal{H}_{\mathcal{A}}^{-1}(E)[1] \longrightarrow E \longrightarrow \mathcal{H}_{\mathcal{A}}^{0}(E).$$

If $\mathcal{H}_{\mathcal{A}}^{-1}(E) \neq 0$, then $\mathrm{Im}\, Z_t(\mathcal{H}_{\mathcal{A}}^{-1}(E)[1]) < 0$ for $0 < t \ll 1$ which contradicts Z_t-semistability of E. Hence $\mathcal{H}_{\mathcal{A}}^{-1}(E) = 0$, i.e. $E \in \mathcal{A}$ and (91) holds. $\square$

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Institute for the Physics and Mathematics of the Universe (IPMU), University of Tokyo, Kashiwano-ha 5-1-5, Kashiwa City, Chiba 277-8582, Japan

E-mail address: `toda-914@pj9.so-net.ne.jp`