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NOETHER-LEFSCHETZ THEORY AND THE YAU-ZASLOW CONJECTURE

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Contents

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0. Introduction

0.1. **Yau-Zaslow conjecture.** Let S be a nonsingular projective $K3$ surface, and let

$$\beta \in \mathrm{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C})$$

be a nonzero effective curve class. The moduli space $\overline{M}_0(S, \beta)$ of genus 0 stable maps (with no marked points) has the expected dimension

$$\dim_{\mathbb{C}}^{vir} \left(\overline{M}_0(S, \beta)\right) = \int_\beta c_1(S) + \dim_{\mathbb{C}}(S) - 3 = -1.$$

Hence, the virtual class $[\overline{M}_0(S, \beta)]^{vir}$ vanishes, and the standard Gromov-Witten theory is trivial.

Curve counting on $K3$ surfaces is captured instead by the *reduced* Gromov-Witten theory constructed first via the twistor family in [6]. An algebraic construction following [1, 2] is given in [31]. Since the reduced class

$$[\overline{M}_0(S, \beta)]^{red} \in H_0(\overline{M}_0(S, \beta), \mathbb{Q})$$

has dimension 0, the reduced Gromov-Witten integrals of S,

$$R_{0,\beta}(S) = \int_{[\overline{M}_0(S,\beta)]^{red}} 1 \ \in \mathbb{Q}, \tag{1}$$

are well-defined. For deformations of S for which β remains a $(1, 1)$-class, the integrals (1) are invariant.

The second cohomology of S is a rank 22 lattice with intersection form

$$H^2(S, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1), \tag{2}$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

is the (negative) Cartan matrix. The intersection form (2) is even.

The *divisibility* $m(\beta)$ is the maximal positive integer dividing the lattice element $\beta \in H^2(S, \mathbb{Z})$. If the divisibility is 1, β is *primitive*. Elements with equal divisibility and norm are equivalent up to orthogonal transformations of $H^2(S, \mathbb{Z})$. By straightforward deformation arguments using the Torelli theorem for $K3$ surfaces, $R_{0,\beta}(S)$ depends, for effective classes, *only* on the divisibility $m(\beta)$ and the norm $\langle \beta, \beta \rangle$. We will omit the argument S in the notation.

The genus 0 BPS counts associated to $K3$ surfaces have the following definition. Let $\alpha \in \mathrm{Pic}(S)$ be a nonzero class which is both effective and primitive. The

Gromov-Witten potential $F_\alpha(v)$ for classes proportional to α is

$$F_\alpha = \sum_{m>0} R_{0,m\alpha}\ v^{m\alpha}.$$

The BPS counts $r_{0,m\alpha}$ are uniquely defined via the Aspinwall-Morrison formula,

$$F_\alpha = \sum_{m>0} r_{0,m\alpha} \ \sum_{d>0} \frac{v^{dm\alpha}}{d^3}, \tag{3}$$

for both primitive and divisible classes.

The Yau-Zaslow conjecture [36] predicts the values of the genus 0 BPS counts for the reduced Gromov-Witten theory of $K3$ surfaces. We interpret the conjecture in two parts.

Conjecture 1. *The BPS count $r_{0,\beta}$ depends upon β only through the norm $\langle\beta,\beta\rangle$.*

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory since $R_{0,\beta}$ certainly depends upon the divisibility of β. Let $r_{0,m,h}$ denote the genus 0 BPS count associated to a class β of divisibility m satisfying

$$\langle\beta,\beta\rangle = 2h-2.$$

Assuming Conjecture 1 holds, we define

$$r_{0,h} = r_{0,m,h},$$

independent[1] of m.

Conjecture 2. *The BPS counts $r_{0,h}$ are uniquely determined by*

$$\sum_{h\geq 0} r_{0,h}\ q^h = \prod_{n=1}^{\infty} (1-q^n)^{-24}. \tag{4}$$

Conjecture 2 can be written in terms of the Dedekind η function

$$\sum_{h\geq 0} r_{0,h}\ q^{h-1} = \eta(\tau)^{-24},$$

where $q = e^{2\pi i\tau}$.

The conjectures have been previously proven in very few cases. A proof of the Yau-Zaslow formula (4) for primitive classes β via Euler characteristics of compactified Jacobians following [36] can be found in [3, 7, 11]. The Yau-Zaslow formula (4) was proven via Gromov-Witten theory for primitive classes β by Bryan and Leung [6]. An early calculation by Gathmann [13] for a class β of divisibility 2 was important for the correct formulation of the conjectures. Conjectures 1 and 2 have been proven in the divisibility 2 case by Lee and Leung [26] and Wu [35]. The main result of the paper is a proof of Conjectures 1 and 2 in all cases.

Theorem 1. *The Yau-Zaslow conjecture holds for all nonzero effective classes $\beta \in \mathrm{Pic}(S)$ on a $K3$ surface S.*

[1]Independence of m holds when $2m^2$ divides $2h-2$. Otherwise, no such class β exists and $r_{0,m,h}$ is defined to vanish.

0.2. **Noether-Lefschetz theory.**

0.2.1. *Lattice polarization.* Let S be a $K3$ surface. A primitive class $L \in \mathrm{Pic}(S)$ is a *quasi-polarization* if

$$\langle L, L\rangle > 0 \quad \text{and} \quad \langle L, [C]\rangle \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power L^n of a quasi-polarization is base point free and determines a birational morphism

$$S \to \tilde{S}$$

contracting A-D-E configurations of (-2)-curves on S. Hence, every quasi-polarized $K3$ surface is algebraic.

Let Λ be a fixed rank r primitive[2] embedding

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

with signature $(1, r-1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1\rangle & \cdots & \langle v_1, v_r\rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1\rangle & \cdots & \langle v_r, v_r\rangle \end{pmatrix} .$$

The sign is chosen so that $\Delta(\Lambda) > 0$.

A Λ-*polarization* of a $K3$ surface S is a primitive embedding

$$j : \Lambda \to \mathrm{Pic}(S)$$

satisfying two properties:

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on Λ,
(ii) $\mathrm{Im}(j)$ contains a quasi-polarization.

By (ii), every Λ-polarized $K3$ surface is algebraic.

The period domain M of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric Q,

$$M \subset Q \subset \mathbb{P}\big((U^3 \oplus E_8(-1)^2) \otimes_{\mathbb{Z}} \mathbb{C}\big).$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let Γ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on Λ. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of Λ-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda / \Gamma_\Lambda.$$

We refer the reader to [10] for a detailed discussion.

[2] An embedding of lattices is primitive if the quotient is torsion free.

0.2.2. *Families.* Let X be a compact 3-dimensional complex manifold equipped with holomorphic line bundles

$$L_1, \ldots, L_r \ \to X$$

and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve.

The tuple $(X, L_1, \ldots, L_r, \pi)$ is a 1-*parameter family of nonsingular* Λ-*polarized* $K3$ *surfaces* if

(i) the fibers $(X_\xi, L_{1,\xi}, \ldots, L_{r,\xi})$ are Λ-polarized $K3$ surfaces via

$$v_i \mapsto L_{i,\xi}$$

for every $\xi \in C$,

(ii) there exists a $\lambda^\pi \in \Lambda$ which is a quasi-polarization of all fibers of π simultaneously.

The family π yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

Let $\lambda^\pi = \lambda_1^\pi v_1 + \cdots + \lambda_r^\pi v_r$. A vector $(d_1, \ldots, d_r)$ of integers is *positive* if

$$\sum_{i=1}^r \lambda_i^\pi d_i > 0.$$

If $\beta \in \mathrm{Pic}(X_\xi)$ has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then β has positive degree with respect to the quasi-polarization if and only if $(d_1, \ldots, d_r)$ is positive.

0.2.3. *Noether-Lefschetz divisors.* Noether-Lefschetz numbers are defined in [31] by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_\Lambda$. We briefly review the definition of the Noether-Lefschetz divisors.

Let $(\mathbb{L}, \iota)$ be a rank $r+1$ lattice $\mathbb{L}$ with an even symmetric bilinear form $\langle , \rangle$ and a primitive embedding

$$\iota : \Lambda \to \mathbb{L}.$$

Two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ are isomorphic if there is an isometry which restricts to the identity on Λ. The first invariant of the data $(\mathbb{L}, \iota)$ is the discriminant $\Delta \in \mathbb{Z}$ of $\mathbb{L}$.

An additional invariant of $(\mathbb{L}, \iota)$ can be obtained by considering any vector $v \in \mathbb{L}$ for which[3]

$$\mathbb{L} = \iota(\Lambda) \oplus \mathbb{Z}v. \tag{5}$$

The pairing

$$\langle v, \cdot \rangle : \Lambda \to \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be the quotient defined via the injection $\Lambda \to \Lambda^*$ obtained from the pairing $\langle , \rangle$ on Λ. The group G is abelian of order equal to the discriminant $\Delta(\Lambda)$. The image

$$\delta \in G/\pm$$

[3]Here, $\oplus$ is used just for the additive structure (not the orthogonal direct sum).

of δ_v is easily seen to be independent of v satisfying (5). The invariant δ is the *coset* of $(\mathbb{L}, \iota)$.

By elementary arguments, two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ of rank $r+1$ are isomorphic if and only if the discriminants and cosets are equal.

Let $v_1, \ldots, v_r$ be an integral basis of Λ as before. The pairing of $\mathbb{L}$ with respect to an extended basis $v_1, \ldots, v_r, v$ is encoded in the matrix

$$\mathbb{L}_{h,d_1,\ldots,d_r} = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\ \vdots & \ddots & \vdots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\ d_1 & \cdots & d_r & 2h-2 \end{pmatrix}.$$

The discriminant is

$$\Delta(h, d_1, \ldots, d_r) = (-1)^r \det(\mathbb{L}_{h,d_1,\ldots,d_r}).$$

The coset $\delta(h, d_1, \ldots, d_r)$ is represented by the functional

$$v_i \mapsto d_i.$$

The Noether-Lefschetz divisor $P_{\Delta,\delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of Λ-polarized $K3$ surfaces S for which $(\mathrm{Pic}(S), j)$ has rank $r+1$, discriminant Δ, and coset δ. By the Hodge index theorem, $P_{\Delta,\delta}$ is empty unless $\Delta > 0$.

Let $h, d_1, \ldots, d_r$ determine a positive discriminant

$$\Delta(h, d_1, \ldots, d_r) > 0.$$

The Noether-Lefschetz divisor $D_{h,(d_1,\ldots,d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$D_{h,(d_1,\ldots,d_r)} = \sum_{\Delta,\delta} m(h, d_1, \ldots, d_r | \Delta, \delta) \cdot [P_{\Delta,\delta}],$$

where the multiplicity $m(h, d_1, \ldots, d_r | \Delta, \delta)$ is the number of elements β of the lattice $(\mathbb{L}, \iota)$ of type (Δ, δ) satisfying

$$\langle \beta, \beta \rangle = 2h-2, \quad \langle \beta, v_i \rangle = d_i. \tag{6}$$

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \ldots, d_r)$, so only finitely many divisors appear in the above sum.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ has an alternate definition. The tautological line bundle $\mathcal{O}(-1)$ is Γ-equivariant on the period domain M_Λ and descends to the *Hodge line bundle*

$$\mathcal{K} \to \mathcal{M}_\Lambda.$$

We define $D_{h,(d_1,\ldots,d_r)} = \mathcal{K}^*$. See [31] for an alternate view of degenerate intersection.

If $\Delta(h, d_1, \ldots, d_r) < 0$, the divisor $D_{h,(d_1,\ldots,d_r)}$ on $\mathcal{M}_\Lambda$ is defined to vanish by the Hodge index theorem.

0.2.4. *Noether-Lefschetz numbers.* Let Λ be a lattice of discriminant $l = \Delta(\Lambda)$, and let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. The Noether-Lefschetz number $NL^\pi_{h,d_1,\ldots,d_r}$ is the classical intersection product

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \int_C \iota_\pi^* [D_{h,(d_1,\ldots,d_r)}]. \tag{7}$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $SL_2(\mathbb{Z})$. There is a canonical representation [4] associated to Λ,

$$\rho^*_\Lambda : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{End}(\mathbb{C}[G]).$$

The full set of Noether-Lefschetz numbers $NL^\pi_{h,d_1,\ldots,d_r}$ defines a vector-valued modular form

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi^\pi_\gamma(q) v_\gamma \in \mathbb{C}[[q^{\frac{1}{2l}}]] \otimes \mathbb{C}[G],$$

of weight $\frac{22-r}{2}$ and type ρ^*_Λ by results[4] of Borcherds and Kudla-Millson [4, 25]. The Noether-Lefschetz numbers are the coefficients[5] of the components of Φ^π,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \Phi^\pi_\gamma \left[\frac{\Delta(h, d_1, \ldots, d_r)}{2l} \right],$$

where $\delta(h, d_1, \ldots, d_r) = \pm\gamma$. The modular form results significantly constrain the Noether-Lefschetz numbers.

0.2.5. *Refinements.* If $d_1, \ldots, d_r$ do not simultaneously vanish, refined Noether-Lefschetz divisors are defined. If $\Delta(h, d_1, \ldots, d_r) > 0$,

$$D_{m,h,(d_1,\ldots,d_r)} \subset D_{h,(d_1,\ldots,d_r)}$$

is defined by requiring the class $\beta \in \mathrm{Pic}(S)$ to satisfy (6) and have divisibility $m > 0$. If $\Delta(h, d_1, \ldots, d_r) = 0$, then

$$D_{m,h,(d_1,\ldots,d_r)} = D_{h,(d_1,\ldots,d_r)}$$

if $m > 0$ is the greatest common divisor of $d_1, \ldots, d_r$ and 0 otherwise.

Refined Noether-Lefschetz numbers are defined by

$$NL^\pi_{m,h,(d_1,\ldots,d_r)} = \int_C \iota^*_\pi [D_{m,h,(d_1,\ldots,d_r)}]. \tag{8}$$

In Section 2.5, the full set of Noether-Lefschetz numbers $NL^\pi_{h,(d_1,\ldots,d_r)}$ is easily shown to determine the refined numbers $NL^\pi_{m,h,(d_1,\ldots,d_r)}$.

0.3. **Three theories.** The main geometric idea in the proof is the relationship of three theories associated to a 1-parameter family

$$\pi : X \to C$$

of Λ-polarized $K3$ surfaces:

(i) the Noether-Lefschetz numbers of π,
(ii) the genus 0 Gromov-Witten invariants of X,
(iii) the genus 0 reduced Gromov-Witten invariants of the $K3$ fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin. For (ii), we view the theory in terms of the Gopakumar-Vafa invariants[6] [16, 17].

Let $n^X_{0,(d_1,\ldots,d_r)}$ denote the Gopakumar-Vafa invariant of X in genus 0 for π-vertical curve classes of degrees $d_1, \ldots, d_r$ with respect to the line bundles $L_1, \ldots, L_r$. Let $r_{0,m,h}$ denote the reduced $K3$ invariant defined in Section 0.1. The following

[4]While the results of the papers [4, 25] have considerable overlap, we will follow the point of view of Borcherds.

[5]If f is a series in q, $f[k]$ denotes the coefficient of q^k.

[6]A review of the definitions can be found in Section 2.5.

result is proven[7] in [31] by a comparison of the reduced and usual deformation theories of maps of curves to the $K3$ fibers of π.

Theorem 2. *For degrees* $(d_1, \ldots, d_r)$ *positive with respect to the quasi-polarization* λ^π,

$$n^X_{0,(d_1,\ldots,d_r)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$

0.4. **Proof of Theorem 1.** The STU model described in Section 1 is a special family of rank 2 lattice polarized $K3$ surfaces

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

The fibered $K3$ surfaces of the STU model are themselves elliptically fibered. The proof of Theorem 1 proceeds in four basic steps:

(i) The modular form [4, 25] determining the intersections of the base $\mathbb{P}^1$ with the Noether-Lefschetz divisors is calculated. For the STU model, the modular form has vector dimension 1 and is proportional to the product E_4E_6 of Eisenstein series.

(ii) Theorem 2 is used to show the 3-fold BPS counts $n^{X^{STU}}_{0,(d_1,d_2)}$, then *determine* all the reduced $K3$ invariants $r_{0,m,h}$. Strong use is made of the rank 2 lattice of the STU model.

(iii) The BPS counts $n^{X^{STU}}_{0,(d_1,d_2)}$ are calculated via mirror symmetry. Since the STU model is realized as a Calabi-Yau complete intersection in a nonsingular toric variety, the genus 0 Gromov-Witten invariants are obtained after proven mirror transformations from hypergeometric series. The Klemm-Lerche-Mayr identity, proven in Section 3, shows that the invariants $n^{X^{STU}}_{0,(d_1,d_2)}$ are themselves related to modular forms.

(iv) Theorem 1 then follows from the Harvey-Moore identity which simultaneously relates the modular structures of

$$n^{X^{STU}}_{0,(d_1,d_2)}, \quad r_{0,m,h}, \quad \text{and} \quad NL^{\pi^{STU}}_{m,h,(d_1,d_2)}$$

in the form specified by Theorem 2. D. Zagier's proof of the Harvey-Moore identity is presented in Section 4.

The strategy of proof is special to genus 0. Much less is known in higher genus. The Katz-Klemm-Vafa conjecture [21, 31] for the integral[8]

$$\int_{[\overline{M}_g(S,\beta)]^{red}} (-1)^g \lambda_g$$

is a particular generalization of the Yau-Zaslow formula to higher genera. The KKV formula does not yet appear easily approachable in Gromov-Witten theory.[9] However, a proof of the KKV formula for primitive $K3$ classes in the conjecturally equivalent theory of stable pairs in the derived category is given in [22, 34].

[7] The result of [31] is stated in the rank $r = 1$ case, but the argument is identical for arbitrary r.

[8] The integrand λ_g is the top Chern class of the Hodge bundle on $\overline{M}_g(X, \beta)$.

[9] For $g = 1$, the KKV formula follows for all classes on $K3$ surfaces from the Yau-Zaslow formula via the boundary relation for λ_1.

1. The STU model

1.1. **Overview.** The STU model[10] is a particular nonsingular projective Calabi-Yau 3-fold X equipped with a fibration

$$\pi : X \to \mathbb{P}^1. \tag{9}$$

Except for 528 points $\xi \in \mathbb{P}^1$, the fibers

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered $K3$ surfaces. The 528 singular fibers X_ξ have exactly 1 ordinary double point singularity each.

The 3-fold X is constructed as an anticanonical section of a nonsingular projective toric 4-fold Y. The Picard rank of Y is 6. The fibration (9) is obtained from a nonsingular toric fibration

$$\pi^Y : Y \to \mathbb{P}^1.$$

The image of

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(X_\xi)$$

determines a rank 2 sublattice of each fiber $\mathrm{Pic}(X_\xi)$ with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The toric data describing the construction of $X \subset Y$ and the fibration structure are explained here.

1.2. **Toric varieties.** Let N be a lattice of rank d,

$$N \cong \mathbb{Z}^d.$$

A fan Σ in N is a collection of strongly convex rational polyhedral cones containing all faces and intersections. A toric variety V_Σ is canonically associated to Σ. The variety V_Σ is complete of dimension d if the support of Σ covers $N \otimes_{\mathbb{Z}} \mathbb{R}$. If all cones are simplicial and if all maximal cones are generated by a lattice basis, then V_Σ is nonsingular. See [8, 12, 32] for the basic properties of toric varieties.

Let Σ be a fan corresponding to a nonsingular complete toric variety. A 1-dimensional cone of Σ is a ray with a unique primitive vector. Let $\Sigma^{(1)}$ denote the set of 1-dimensional cones of Σ indexed by their primitive vectors

$$\{\rho_1, \ldots, \rho_n\}. \tag{10}$$

Let $r^1, \ldots, r^\ell$ be a basis over the integers of the module of relations among the vectors (10). We write the j^{th} relation as

$$r^j_1 \rho_1 + \ldots + r^j_n \rho_n = 0.$$

Define a torus

$$(\mathbb{C}^*)^\ell \cong \prod_{j=1}^{\ell} \mathbb{C}^*_j$$

with factors indexed by the relations.

[10] The model has been studied in physics since the 1980s. The letter S stands for the dilaton and T and U label the torus moduli in the heterotic string. The STU model was an important example for the duality between type IIA and heterotic strings formulated in [20]. The ideas developed in [18, 19, 23, 24, 30] about the STU model play an important role in our paper.

A simple description of V_Σ is obtained via a quotient construction. Let $\{z_i\}_{1\le i\le n}$ be coordinates on $\mathbb{C}^n$ corresponding to the primitives ρ_i of the rays in $\Sigma^{(1)}$. An action of $\mathbb{C}^*_j$ on $\mathbb{C}^n$ is defined by

$$\lambda_j\cdot\big(z_1,\ldots,z_n\big)=\big(\lambda_j^{r_1^j}z_1,\ldots,\lambda_j^{r_n^j}z_n\big),\quad \lambda_j\in\mathbb{C}^*_j. \tag{11}$$

In order to obtain a well-behaved quotient for the induced $(\mathbb{C}^*)^\ell$-action on $\mathbb{C}^n$, an exceptional set $Z(\Sigma)\subset\mathbb{C}^n$ consisting of a finite union of linear subspaces is excluded. The linear space defined by $\{z_i=0\,|\,i\in I\}$ is contained in $Z(\Sigma)$ if there is no single cone in Σ containing all of the primitives $\{\rho_i\}_{i\in I}$. After removing $Z(\Sigma)$, the quotient

$$V_\Sigma=\Big(\mathbb{C}^n\setminus Z(\Sigma)\Big)\Big/\big(\mathbb{C}^*\big)^\ell \tag{12}$$

yields the toric variety associated to Σ.

Since $\ell=n-d$, the complex dimension of the quotient V_Σ equals the rank d of the lattice N. The variety V_Σ is equipped with the action of the quotient torus

$$T=(\mathbb{C}^*)^n/(\mathbb{C}^*)^\ell.$$

The rank of $\mathrm{Pic}(V_\Sigma)$ is ℓ. The primitives ρ_i are in 1–to–1 correspondence with the T-invariant divisors D_i on V_Σ defined by

$$D_i=\big\{z_i=0\big\}\ \subset V_\Sigma. \tag{13}$$

Conversely, the homogeneous coordinate z_i is a section of the line bundle $\mathcal{O}(D_i)$. The anticanonical divisor class of V_Σ is determined by

$$-K_{V_\Sigma}=\sum_{i=1}^n D_i. \tag{14}$$

1.3. **The toric 4-fold Y.** The fan Σ in $\mathbb{Z}^4$ defining the toric 4-fold Y has 10 rays with primitive elements:

$$\begin{array}{lll}
\rho_1=(1,0,2,3), & \rho_2=(-1,0,2,3), & \\
\rho_3=(0,1,2,3), & \rho_4=(0,-1,2,3), & \\
\rho_5=(0,0,2,3), & \rho_6=(0,0,-1,0), & \rho_7=(0,0,0,-1),\\
\rho_8=(0,0,1,2), & \rho_9=(0,0,0,1), & \rho_{10}=(0,0,1,1).
\end{array}$$

The full fan Σ is obtained from the convex hull of the 10 primitives. By explicitly checking each of the 24 dimension-4 cones, Y is seen to be a complete nonsingular toric 4-fold.

Generators $r^1,\ldots,r^6$ of the rank 6 module of relations among the primitives can be taken to be

$$\begin{array}{cccccccccccc}
\rho_1 & +\rho_2 & & & & +4\rho_6 & +6\rho_7 & & & & =0,\\
 & & \rho_3 & +\rho_4 & & +4\rho_6 & +6\rho_7 & & & & =0,\\
 & & & & \rho_5 & +2\rho_6 & +3\rho_7 & & & & =0,\\
 & & & & & \rho_6 & +2\rho_7 & +\rho_8 & & & =0,\\
 & & & & & & +\ \rho_7 & & +\rho_9 & & =0,\\
 & & & & & \rho_6 & +\ \rho_7 & & & +\rho_{10} & =0.
\end{array}$$

By the identification (14) of $-K_Y$, the product $\prod_{i=1}^{10} z_i$ defines an anticanonical section. Hence, every product

$$\prod_{i=1}^{10} z_i^{m_i}, \quad m_i \geq 0,$$

which is homogeneous of degree $\sum_{i=1}^{10} r_i^j$ with respect to the action (11) of $\mathbb{C}_j^*$ also defines an anticanonical section. Hence,

$$\begin{aligned} &{z_1}^{12}{z_4}^{12}{z_5}^{6}{z_8}^{4}{z_9}^{2}{z_{10}}^{3}, && {z_1}^{12}{z_3}^{12}{z_5}^{6}{z_8}^{4}{z_9}^{2}{z_{10}}^{3}, \\ &{z_2}^{12}{z_4}^{12}{z_5}^{6}{z_8}^{4}{z_9}^{2}{z_{10}}^{3}, && {z_2}^{12}{z_3}^{12}{z_5}^{6}{z_8}^{4}{z_9}^{2}{z_{10}}^{3}, \\ &{z_6}^{3}z_8{z_9}^{2}, && {z_7}^{2}z_{10} \end{aligned} \tag{15}$$

are all sections of $-K_Y$.

From the definitions, we find that $Z(\Sigma)$ consists of the union of the following 11 linear spaces of dimension 2 in $\mathbb{C}^4$:

$$\begin{aligned} &I_1 = \{1,2\}, \quad I_2 = \{3,4\}, \quad I_3 = \{5,6\}, \quad I_4 = \{5,7\}, \\ &I_5 = \{5,9\}, \quad I_6 = \{6,8\}, \quad I_7 = \{6,10\}, \quad I_8 = \{7,8\}, \\ &I_9 = \{7,9\}, \quad I_{10} = \{8,10\}, \quad I_{11} = \{9,10\}\ . \end{aligned} \tag{16}$$

Recall, I_k indexes the coordinates which vanish.

A simple verification shows that the 6 sections (15) of $-K_Y$ do not have a common zero on the prequotient $\mathbb{C}^n \setminus Z(\Sigma)$. Hence, $-K_Y$ is generated by global sections on Y. A hypersurface

$$X \subset Y$$

defined by a generic section of $-K_Y$ is nonsingular by Bertini's Theorem. By adjunction, X is Calabi-Yau.

1.4. **Fibrations.** The toric variety Y admits two obvious fibrations

$$\pi^Y : Y \to \mathbb{P}^1, \quad \mu^Y : \to \mathbb{P}^1$$

given in homogeneous coordinates by

$$\pi^Y(z_1, \ldots, z_{10}) = [z_1, z_2], \quad \mu^Y(z_1, \ldots, z_{10}) = [z_3, z_4].$$

Since $Z(\Sigma)$ contains the linear spaces

$$I_1 = \{1,2\}, \quad I_2 = \{3,4\},$$

both π^Y and μ^Y are well-defined.

Consider first π^Y. The fibers of π^Y are nonsingular complete toric 3-folds defined by the fan in

$$\mathbb{Z}^3 \subset \mathbb{Z}^4, \quad (c_1, c_2, c_3) \mapsto (0, c_1, c_2, c_3)$$

determined by the primitives $\rho_3, \ldots, \rho_{10}$.

Let X be obtained from a generic section of $-K_Y$. Let

$$\pi : X \to \mathbb{P}^1$$

be the restriction $\pi^Y|_X$.

Proposition 1. *Except for* 528 *points* $\xi \in \mathbb{P}^1$*, the fibers*

$$X_\xi = \pi^{-1}(\xi)$$

are nonsingular elliptically fibered $K3$ *surfaces. The* 528 *singular fibers* X_ξ *each have exactly* 1 *ordinary double point singularity.*

Proof. Let $P_{k,k}(z_1, z_2|z_3, z_4)$ denote a bihomogeneous polynomial of degree k in (z_1, z_2) and degree k in (z_3, z_4). Let

$$F = P_{12,12}(z_1, z_2|z_3, z_4), \quad G = P_{8,8}(z_1, z_2|z_3, z_4), \quad H = P_{4,4}(z_1, z_2|z_3, z_4)$$

be bihomogeneous polynomials. Then

$$Fz_5^6 z_8^4 z_9^2 z_{10}^3, \quad Gz_5^4 z_6 z_8^3 z_9^2 z_{10}^2, \quad Hz_5^2 z_6^2 z_8^2 z_9^2 z_{10}, \quad z_6^3 z_8 z_9^2, \quad z_7^2 z_{10} \tag{17}$$

all determine sections of $-K_Y$.

Let X be defined by a generic linear combination of the sections (17). Since the base point free system (15) is contained in (17), X is nonsingular. We will prove that all the fibers X_ξ are nonsingular, except for finitely many with exactly 1 ordinary double point each, by an explicit study of the equations.

Since $I_7 = \{6, 10\}$, $I_{10} = \{8, 10\}$, and $I_{11} = \{9, 10\}$ are in $Z(\Sigma)$, we easily see that $X \cap D_{10} = \emptyset$ if the coefficient of $z_6^3 z_8 z_9^2$ is nonzero. Similarly,

$$X \cap D_8 = \emptyset, \quad X \cap D_9 = \emptyset.$$

Hence, using the last 3 factors of the torus $(\mathbb{C}^*)^\ell$, the coordinates z_8, z_9, and z_{10} can all be set to 1. The equation for X simplifies to

$$Fz_5^6 + Gz_5^4 z_6 + Hz_5^2 z_6^2 + \alpha z_6^3 + \beta z_7^2.$$

The coordinates z_1 and z_2 do not simultaneously vanish on Y. There are two charts to consider. By symmetry, the analysis on each is identical, so we assume $z_1 \neq 0$. Using the first factor of $(\mathbb{C}^*)^\ell$, we set $z_1 = 1$. By the same reasoning, we set $z_3 = 1$ using the second factor of $(\mathbb{C}^*)^\ell$. Since $I_3 = \{5, 6\}$ and $I_4 = \{5, 7\}$ are in $Z(\Sigma)$, either $z_5 \neq 0$ or *both* z_6 and z_7 do not vanish.

Case $z_5 \neq 0$. Using the third factor of $(\mathbb{C}^*)^\ell$ to set $z_5 = 1$, we obtain the equation

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + \alpha z_6^3 + \beta z_7^2 \tag{18}$$

in $\mathbb{C}^4$ with coordinates z_2, z_4, z_6, z_7. The map π is given by the z_2 coordinate. The partial derivative of (18) with respect to z_7 is $2\beta z_7$. Hence, if $\beta \neq 0$, all singularities of π occur when $z_7 = 0$.

We need only analyze the reduced dimension case

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + z_6^3 \tag{19}$$

with coordinates z_2, z_4, z_6. Here, α has been set to 1 by scaling the equation. We must show that all the fibers of π are nonsingular curves except for finitely many with simple nodes. We view equation (19) as defining a 1-parameter family of paths $\gamma_{z_2}(z_4)$ in the space

$$\mathcal{C} = \{\gamma_0 + \gamma_1 z_6 + \gamma_2 z_6^2 + z_6^3 \mid \gamma_0, \gamma_1, \gamma_2 \in \mathbb{C}\}$$

of cubic polynomials in the variable z_6. The coordinate of the path is z_4. The variable z_2 indexes the family of paths.

Let $\Delta \subset \mathcal{C}$ be the codimension 1 discriminant locus of cubics with double roots. The discriminant is irreducible with cuspidal singularities in codimension 2 in $\mathcal{C}$. The possible singularities of the fiber $\pi^{-1}(\lambda)$ occur only when the path $\gamma_\lambda(z_4)$ intersects Δ. The fiber $\pi^{-1}(\lambda)$ is nonsingular over such an intersection point if either

(i) γ_λ is transverse to Δ at a nonsingular point of Δ,
(ii) γ_λ is transverse to the codimension 1 tangent cone of a singular point of Δ.

The fiber $\pi^{-1}(\lambda)$ has a simple node over an intersection point of the path $\gamma_\lambda(z_4)$ with Δ if

(iii) γ_λ is tangent to Δ at a nonsingular point of Δ.

The above are all the possibilities which can occur in a generic 1-parameter family of paths in the space of cubic equations.[11] Possibility (iii) can happen only for finitely many λ and just once for each such λ.

Case $z_6 \neq 0$ *and* $z_7 \neq 0$. Using the third factor of $(\mathbb{C}^*)^\ell$ to set $z_6 = 1$, we obtain the equation

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4) + G(1, z_2|1, z_4) + \alpha + \beta z_7^2 \tag{20}$$

in $\mathbb{C}^4$ with coordinates z_2, z_4, z_5, z_7. The partial derivative of (20) with respect to z_7 is not 0 for $z_7 \neq 0$. Hence, there are no singular fibers of π on the chart.

We have proven that all the fibers X_ξ of π are nonsingular except for finitely many with exactly 1 ordinary double point each. Let X_ξ be a nonsingular fiber. Let

$$\mu : X \to \mathbb{P}^1$$

be the restriction $\mu^Y|_X$. The fibers of the product

$$(\pi, \mu) : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

are easily seen to be anticanonical sections of the nonsingular toric surface[12] W with fan in $\mathbb{Z}^2$ determined by the primitives $\rho_5, \ldots, \rho_{10}$. These anticanonical sections are elliptic curves. Since X_ξ has trivial canonical bundle by adjunction and the map

$$\mu : X_\xi \to \mathbb{P}^1$$

is dominant with elliptic fibers, we conclude that X_ξ is an elliptically fibered $K3$ surface.

The Euler characteristic of X can be calculated by toric intersection in Y,

$$\chi_{top}(X) = -480.$$

The Euler characteristic of a nonsingular $K3$ fibration over $\mathbb{P}^1$ is 48. Since each fiber singularity reduces the Euler characteristic by 1, we conclude that π has exactly 528 singular fibers. □

For emphasis, we will sometimes denote the STU model by

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1.$$

1.5. **Divisor restrictions.** The divisors D_1, D_2, D_8, D_9, and D_{10} have already been shown to restrict to the trivial class in $\text{Pic}(X_\xi)$. The divisors D_3 and D_4 restrict to the fiber class $F \in \text{Pic}(X_\xi)$ of the elliptic fibration

$$\mu : X_\xi \to \mathbb{P}^1. \tag{21}$$

Certainly $F^2 = 0$. Let $S \in \text{Pic}(X_\xi)$ denote the restriction of D_5. Toric calculations yield the products

$$F \cdot S = 1, \quad S \cdot S = -2.$$

[11] A cusp of $\pi^{-1}(\lambda)$ occurs, for example, when the path has contact order 3 at a nonsingular point of the discriminant.

[12] Since the product $(\pi^Y, \mu^Y) : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ has fibers isomorphic to the nonsingular complete (hence projective) toric surface W, the 4-fold Y is projective.

Hence, S may be viewed as the section class of the elliptic fibration (21). The divisors D_6 and D_7 restrict to classes in the rank 2 lattice generated by F and S.

The restriction of $\text{Pic}(Y)$ to each fiber X_ξ is a rank 2 lattice generated by F and S with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} .$$

We may also choose generators $L_1 = F$ and $L_2 = F + S$ with intersection form

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

1.6. **1-parameter families.** Let X be a compact 3-dimensional complex manifold equipped with two holomorphic line bundles

$$L_1,\ L_2 \to X$$

and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve.

The data (X, L_1, L_2, π) determine a *family of* Λ*-polarized* $K3$ *surfaces* if the fibers $(X_\xi, L_{1,\xi}, L_{2,\xi})$ are $K3$ surfaces with intersection form

$$\begin{pmatrix} L_{1,\xi} \cdot L_{1,\xi} & L_{2,\xi} \cdot L_{1,\xi} \\ L_{1,\xi} \cdot L_{2,\xi} & L_{2,\xi} \cdot L_{2,\xi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and there exists a simultaneous quasi-polarization. The 1-parameter family (X, L_1, L_2, π) yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of Λ-polarized $K3$ surfaces.

The construction $(X^{STU}, L_1, L_2, \pi^{STU})$ of the STU model in Sections 1.3-1.5 is almost a 1-parameter family of Λ-polarized $K3$ surfaces. The only failing is the 528 singular fibers of π^{STU}. Let

$$\epsilon : C \xrightarrow{2-1} \mathbb{P}^1$$

be a hyperelliptic curve branched over the 528 points of $\mathbb{P}^1$ corresponding to the singular fibers of π. The family

$$\epsilon^*(X^{STU}) \to C$$

has 3-fold double point singularities over the 528 nodes of the fibers of the original family. Let

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C$$

be obtained from a small resolution

$$\widetilde{X}^{STU} \to \epsilon^*(X^{STU}).$$

Let $\widetilde{L}_i \to \widetilde{X}^{STU}$ be the pull-back of L_i by ϵ. The data

$$(\widetilde{X}^{STU}, \widetilde{L}_1, \widetilde{L}_2, \widetilde{\pi}^{STU})$$

determine a 1-parameter family of Λ-polarized $K3$ surfaces; see Section 5.3 of [31]. The simultaneous quasi-polarization is obtained from the projectivity of X^{STU}.

1.7. **Gromov-Witten invariants.** Since X^{STU} is defined by an anticanonical section in a semi-positive nonsingular toric variety Y, the genus 0 Gromov-Witten invariants have been proven by Givental [14, 15, 29, 33] to be related by mirror transformation to hypergeometric solutions of the Picard-Fuchs equations of the Batyrev-Borisov mirror. By Section 5.3 of [31], the Gromov-Witten invariants of $\widetilde{X}^{STU}$ are exactly twice the Gromov-Witten invariants of X^{STU} for curve classes in the fibers.

2. Noether-Lefschetz numbers and reduced $K3$ invariants

2.1. **Refined Noether-Lefschetz numbers.** Following the notation of Section 0.2, let

$$\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$$

be primitively embedded with signature $(1, r-1)$ and integral basis $v_1, \ldots, v_r$. Let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of Λ-polarized $K3$ surfaces. Let $d_1, \ldots, d_r$ be integers which do not all vanish.

Lemma 1. *The Noether-Lefschetz numbers $NL^{\pi}_{h,(d_1,\ldots,d_r)}$ completely determine the refinements $NL^{\pi}_{m,h,(d_1,\ldots,d_r)}$.*

Proof. By definition, the refined Noether-Lefschetz numbers satisfy two elementary identities. The first is

$$NL^{\pi}_{h,(d_1,\ldots,d_r)} = \sum_{m=1}^{\infty} NL^{\pi}_{m,h,(d_1,\ldots,d_r)}.$$

If m does not divide all d_i, then $NL^{\pi}_{m,h,(d_1,\ldots,d_r)}$ vanishes. If m divides all d_i, then a second identity holds:

$$NL^{\pi}_{m,h,(d_1,\ldots,d_r)} = NL^{\pi}_{1,h',(d_1/m,\ldots,d_r/m)},$$

where $2h-2 = m^2(2h'-2)$.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the refined number $NL^{\pi}_{m,h,(d_1,\ldots,d_r)}$ vanishes by definition unless m is the GCD of $(d_1, \ldots, d_r)$. In the latter case,

$$NL^{\pi}_{h,(d_1,\ldots,d_r)} = NL^{\pi}_{m,h,(d_1,\ldots,d_r)}.$$

Hence the lemma is trivial in the $\Delta(h, d_1, \ldots, d_r) = 0$ case.

If $\Delta(h, d_1, \ldots, d_r) > 0$, we prove the lemma by induction on Δ. The second identity reduces us to the case where $m = 1$. The first identity determines the $m = 1$ case in terms of the Noether-Lefschetz number $NL_{h,(d_1,\ldots,d_r)}$ and refined numbers with

$$\Delta(h', d'_1, \ldots, d'_r) < \Delta(h, d_1, \ldots, d_r).$$

□

2.2. **STU model.** The resolved version of the STU model

$$\widetilde{\pi}^{STU} : \widetilde{X}^{STU} \to C$$

is lattice polarized with respect to

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The application of the results of [4, 25] to the STU model is extremely simple. Since the lattice Λ is unimodular, the corresponding representation ρ^*_{Λ} is 1-dimensional and, in fact, is the trivial representation of $\mathrm{Mp}_2(\mathbb{Z})$. The Noether-Lefschetz degrees are thus encoded by a scalar modular form of weight $\frac{22-r}{2} = 10$. The space of such forms is well known to be of dimension 1 and spanned by the product of the Eisenstein series[13]

$$E_{10}(q) = E_4(q)E_6(q) = 1 - 264 \sum_{n \geq 1} \sigma_9(n) q^n.$$

Hence, a single Noether-Lefschetz calculation determines the full series.

Lemma 2. *$NL^{\widetilde{\pi}}_{0,(0,0)} = 1056$.*

Proof. By Proposition 1, the STU model

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1$$

has 528 nodal fibers. Let S be a fiber of the resolved family $\widetilde{\pi}^{STU}$ lying over a singular fiber of π. The Picard lattice of S certainly contains

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{22}$$

spanned by L_1, L_2, and the (-2)-curve E of the small resolution. Let

$$\widetilde{\iota} : C \to \mathcal{M}_\Lambda$$

be the map to moduli. Since a class β satisfying

$$\langle \beta, \beta \rangle = -2$$

on a $K3$ surface is either effective or anti-effective, the set-theoretic intersections of $\widetilde{\iota}$ with $D_{0,(0,0)}$ correspond to fibers of $\widetilde{\pi}$, where L_1 and L_2 do not generate an ample class, precisely, the 528 fibers of $\widetilde{\pi}$ lying over the singular fibers of π.

The divisor $D_{0,(0,0)}$ has multiplicity exactly 2 at the 528 intersections with $\widetilde{\iota}$ since E and $-E$ are the only -2 classes orthogonal to L_1 and L_2. Finally, since E has normal bundle $(-1,-1)$ in $\widetilde{X}^{STU}$, the curve $\widetilde{\iota}$ is transverse to the reduced divisor $\frac{1}{2}D_{0,(0,0)}$ at the 528 intersections. We conclude that $NL^{\widetilde{\pi}}_{0,(0,0)} = 528 \cdot 2 = 1056$. □

Proposition 2. *The Noether-Lefschetz degrees of the resolved STU model are given by the equation*

$$NL^{\widetilde{\pi}}_{h,(d_1,d_2)} = -4E_4(q)E_6(q) \left[\frac{\Delta(h,d_1,d_2)}{2} \right].$$

---

[13] The Eisenstein series E_{2k} is the modular form defined by the equation

$$-\frac{B_{2k}}{4k} E_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n,$$

where B_{2n} is the $2n^{\text{th}}$ Bernoulli number and $\sigma_n(k)$ is the sum of the k^{th} powers of the divisors of n,

$$\sigma_k(n) = \sum_{i|n} i^k.$$

2.3. **BPS states.** Let $(\widetilde{X}^{STU}, \widetilde{L}_1, \widetilde{L}_2, \widetilde{\pi}^{STU})$ be the Λ-polarized STU model. The vertical classes are the kernel of the push-forward map by $\widetilde{\pi}$,

$$0 \to H_2(\widetilde{X}, \mathbb{Z})^{\widetilde{\pi}} \to H_2(\widetilde{X}, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \to 0.$$

While $\widetilde{X}$ need not be a projective variety, $\widetilde{X}$ carries a $(1,1)$-form ω_K which is Kähler on the $K3$ fibers of $\widetilde{\pi}$. The existence of a fiberwise Kähler form is sufficient to define the Gromov-Witten theory for vertical classes,

$$0 \neq \gamma \in H_2(\widetilde{X}, \mathbb{Z})^{\widetilde{\pi}}.$$

The fiberwise Kähler form ω_K is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.

Let $\overline{M}_0(\widetilde{X}, \gamma)$ be the moduli space of stable maps from connected genus 0 curves to $\widetilde{X}$. Gromov-Witten theory is defined by integration against the virtual class,

$$N^{\widetilde{X}}_{0,\gamma} = \int_{[\overline{M}_0(\widetilde{X},\gamma)]^{vir}} 1. \tag{23}$$

The expected dimension of the moduli space is 0.

The genus 0 Gromov-Witten potential $F^{\widetilde{X}}(v)$ for nonzero vertical classes is the series

$$F^{\widetilde{X}} = \sum_{0 \neq \gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}}} N^{\widetilde{X}}_{0,\gamma}\ v^{\gamma},$$

where v is the curve class variable. The BPS counts $n^{\widetilde{X}}_{0,\gamma}$ of Gopakumar and Vafa are uniquely defined by the following equation:

$$F^{\widetilde{X}} = \sum_{0 \neq \gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}}} n^{\widetilde{X}}_{0,\gamma} \sum_{d>0} \frac{v^{d\gamma}}{d^3}.$$

Conjecturally, the invariants $n^{\widetilde{X}}_{0,\gamma}$ are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on $\widetilde{X}$. We do not assume that the conjectural properties hold.

Using the Λ-polarization, we define the BPS counts

$$n^{\widetilde{X}}_{0,(d_1,d_2)} = \sum_{\gamma \in H_2(\widetilde{X},\mathbb{Z})^{\widetilde{\pi}},\ \int_\gamma \widetilde{L}_i = d_i} n^{\widetilde{X}}_{0,\gamma} \tag{24}$$

when d_1 and d_2 are not both 0.

The original STU model,

$$\pi^{STU} : X^{STU} \to \mathbb{P}^1,$$

with 528 singular fibers is a nonsingular, projective, Calabi-Yau 3-fold. Hence the Gromov-Witten invariants are well-defined. Let $n^{X}_{0,(d_1,d_2)}$ denote the fiberwise Gopakumar-Vafa invariant with degrees d_i measured by L_i. By the argument of Section 1.7,

$$n^{\widetilde{X}}_{0,(d_1,d_2)} = 2n^{X}_{0,(d_1,d_2)}$$

when d_1 and d_2 are not both 0.

2.4. **Invertibility of constraints.** Let $\mathcal{P} \subset \mathbb{Z}^2$ be the set of pairs

$$\mathcal{P} = \{ \ (d_1, d_2) \neq (0,0) \mid d_1 \geq 0, \ d_1 \geq -d_2 \ \} \ .$$

Pairs $(d_2, d_2) \in \mathcal{P}$ are certainly positive with respect to any quasi-polarization for $\widetilde{\pi}^{STU}$ since such (d_1, d_2) can be realized by linear combinations of the effective classes F and S.

Theorem 2 applied to the resolved STU model yields the equation

$$n^{\widetilde{X}}_{0,(d_1,d_2)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^{\widetilde{\pi}}_{m,h,(d_1,d_2)} \tag{25}$$

for $(d_1, d_2) \in \mathcal{P}$. The BPS states on the left side will be computed by mirror symmetry in Section 3. The refined Noether-Lefschetz degrees are determined by Lemma 1 and Proposition 2. Consequently, equation (25) provides constraints on the reduced $K3$ invariants $r_{0,m,h}$.

The integrals $r_{0,m,h}$ are very simple in case $h \leq 0$. By Lemma 2 of [31], $r_{0,m,h} = 0$ for $h < 0$,

$$r_{0,1,0} = 1,$$

and $r_{0,m,0} = 0$ otherwise.

Proposition 3. *The set of integrals* $\{r_{0,m,h}\}_{m \geq 1, h > 0}$ *is uniquely determined by the set of constraints* (25) *for* $(d_1 \geq 0, \ d_2 > 0)$ *and the integrals* $r_{0,m,h \leq 0}$.

Proof. A certain subset of the linear equations with $d_2 > 0$ will be shown to be upper triangular in the variables $r_{0,m,h}$. Picard rank 2 is crucial for the argument.

Let us fix in advance the values of $m \geq 1$ and $h > 0$. We proceed by induction on m assuming the reduced invariants $r_{0,m',h}$ have already been determined for all $m' < m$. The assumption is vacuous when $m = 1$. We can also assume that $r_{0,m,h'}$ has been determined inductively for $h' < h$. If $2h - 2$ is not divisible by $2m^2$, then we have $r_{0,m,h} = 0$, so we can further assume that

$$2h - 2 = m^2(2s - 2)$$

for an integer $s > 0$.

Consider equation (25) for $(d_1, d_2) = (m(s-1), m)$. Certainly

$$NL^{\widetilde{\pi}}_{m',h',(m(s-1),m)} = 0$$

unless m' divides m. By the Hodge index theorem, we must have

$$\Delta(h', m(s-1), m) = 2 - 2h' + m^2(2s-2) \geq 0 \tag{26}$$

if $NL^{\widetilde{\pi}}_{m,h',(m(s-1),m)} \neq 0$. Inequality (26) implies that $h' \leq h$.

Therefore, the constraint (25) takes the form

$$n^{\widetilde{X}}_{0,(m(s-1),m)} = r_{0,m,h} NL^{\widetilde{\pi}}_{m,h,(m(s-1),m)} + \dots,$$

where the dots represent terms involving $r_{0,m',h'}$ with either

$$m' < m \quad \text{ or } \quad m' = m, \ h' < h.$$

The leading coefficient is given by

$$NL^{\widetilde{\pi}}_{m,h,(m(s-1),m)} = NL^{\widetilde{\pi}}_{h,(m(s-1),m)} = -4.$$

As the system is upper-triangular, we can invert to solve for $r_{0,m,h}$. □

2.5. **Proof of the Yau-Zaslow conjecture.** By Proposition 3, we need only show that the answer for $r_{0,m,h}$ predicted by the Yau-Zaslow conjecture satisfies the constraints (25) for all pairs $(d_1 \geq 0,\ d_2 > 0)$.

Let X^{STU} be the original Calabi-Yau 3-fold of the STU model. Let

$$D_2^3 F^X = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3\ N^X_{0,(d_1,d_2)}\ q_1^{d_1} q_2^{d_2} \tag{27}$$

be the third derivative[14] of the genus 0 Gromov-Witten series for π-vertical classes in $\mathcal{P}$.

We can calculate $D_2^3 F^X$ by the constraint (25) *assuming the validity of the Yau-Zaslow conjecture*,

$$D_2^3 F^X = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3\ c(d_1,d_2)\ \frac{q_1^{d_1} q_2^{d_2}}{1-q_1^{d_1} q_2^{d_2}}, \tag{28}$$

where $c(k,l)$ is the coefficient of q^{kl} in

$$-2\frac{E_4(q)E_6(q)}{\eta^{24}(q)}.$$

Proposition 4. *The Yau-Zaslow conjecture is implied by the identity*

$$\sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2} = \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3\ c(d_1,d_2)\ \frac{q_1^{d_1} q_2^{d_2}}{1-q_1^{d_1} q_2^{d_2}}.$$

Proof. The $q_1^{d_1} q_2^{d_2}$ coefficient of the above identity is simply d_2^3 times the constraint (25). Since we only require the constraints in case

$$(d_1 \geq 0,\ d_2 > 0) \in \mathcal{P},$$

the identity implies all the constraints we need. □

The remainder of the paper is devoted to the proof of Proposition 4. The genus 0 Gromov-Witten invariants of X are related, after a mirror transformation, to hypergeometric solutions of the associated Picard-Fuchs system of differential equations. Hence, Proposition 4 amounts to a subtle identity among special functions.

3. Mirror transform

3.1. **Picard-Fuchs.** Let $\pi : X \to \mathbb{P}^1$ be the STU model. Let

$$\delta_0 \in H^*(X,\mathbb{C})$$

denote the identity class. A basis of $H^2(X,\mathbb{C})$ is obtained from the restriction of the toric divisors of Y discussed in Section 1.5,

$$\delta_1 = 2D_1 + 2D_3 + D_5, \quad \delta_2 = D_3, \quad \delta_3 = D_1.$$

Recall, δ_3 vanishes on the fibers of π. Let $\{\delta_j\}$ be a full basis of $H^*(X,\mathbb{C})$ extending the above selections.

Let u_1, u_2, u_3 be the canonical coordinates for the mirror family with respect to the divisor basis $\delta_1, \delta_2, \delta_3$. Let

$$\theta_i = u_i \frac{\partial}{\partial u_i}.$$

[14] $D_2 = q_2 \frac{d}{dq_2}$.

The Picard-Fuchs system associated to the mirror of X^{STU} is:

$$\begin{aligned} \mathcal{L}_1 &= \theta_1\,(\theta_1 - 2\,\theta_2 - 2\,\theta_3) - 12\,(6\,\theta_1 - 5)\,(6\,\theta_1 - 1)\,u_1, \\ \mathcal{L}_2 &= {\theta_2}^2 - (2\,\theta_2 + 2\,\theta_3 - \theta_1 - 2)\,(2\,\theta_2 + 2\,\theta_3 - \theta_1 - 1)\,u_2, \\ \mathcal{L}_3 &= {\theta_3}^2 - (2\,\theta_2 + 2\,\theta_3 - \theta_1 - 2)\,(2\,\theta_2 + 2\,\theta_3 - \theta_1 - 1)\,u_3\,. \end{aligned} \tag{29}$$

The system is obtained canonically from the Batyrev-Borisov construction; see [9] for the formalism.

3.2. **Solutions.** A fundamental solution to the Picard-Fuchs system can be written in terms of GKZ hypergeometric series,

$$\varpi \in H^*(X,\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\log(u_1), \log(u_2), \log(u_3)][[u_1, u_2, u_3]]. \tag{30}$$

Let $\varpi(u,\delta_j)$ be the corresponding coefficient of (30). Then

$$\mathcal{L}_i\ \varpi(u,\delta_j) = 0.$$

The standard normalization of ϖ satisfies two important properties:

(i) The δ_0 coefficient is the unique solution

$$\varpi(u,\delta_0) = 1 + O(u)$$

holomorphic at $u = 0$.

(ii) For $1 \leq i \leq 3$,

$$\varpi(u,\delta_i) = \frac{\varpi(u,\delta_0)}{2\pi i}\log(u_i) + O(u)$$

are the logarithmic solutions.

Let T_1, T_2, T_3 be coordinates on $H^2(X,\mathbb{C})$ with respect to the basis δ. The mirror transformation is defined by

$$T_i = \frac{\varpi(u,\delta_i)}{\varpi(u,\delta_0)} = \frac{1}{2\pi i}\log(u_i) + O(u)$$

for $1 \leq i \leq 3$.

The mirror transformation relates the genus 0 Gromov-Witten theory of X to the Picard-Fuchs system for the mirror family. For anticanonical hypersurfaces in toric varieties, a proof is given in [15].

3.3. **Mirror transform for $q_3 = 0$.** We introduce two modular parameters

$$\tau_1 = T_1, \quad \tau_2 = T_1 + T_2\,. \tag{31}$$

For $i = 1$ and 2, let

$$\widehat{q}_i = \exp(2\pi i \tau_i),$$

and let $q_3 = \exp(2\pi i T_3)$.

Our first step is to find a modular expression for the mirror map and the period $\varpi(u,\delta_0)$ to leading order in q_3. We prove two formulas discovered by Klemm, Lerche, and Mayr in [24].

Lemma 3. *We have*

$$u_1 = \frac{2(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)}{j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)}} + O(q_3),$$

$$u_2 = \frac{(j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2) - \mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2) - \mu)})^2}{4j(\widehat{q}_1)j(\widehat{q}_2)(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu)^2} + O(q_3),$$

where $\mu = 1728$ *and*

$$j(q) = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + O(q^2) \tag{32}$$

is the normalized j *function.*

Lemma 4. $\mathrm{Lim}_{q_3 \to 0}\ \varpi(u, \delta_0) = E_4(\widehat{q}_1)^{\frac{1}{4}} E_4(\widehat{q}_2)^{\frac{1}{4}}$.

Proof. We prove Lemmas 3 and 4 together. The first step is to perform the following change of variables:

$$u_1 = z_1, \qquad u_2 = \frac{z_2}{2}\left(1 + \sqrt{1-4z_3}\right), \qquad u_3 = \frac{z_2}{2}\left(1 - \sqrt{1-4z_3}\right),$$

with the inverse change

$$z_1 = u_1, \quad z_2 = u_2 + u_3, \quad z_3 = \frac{u_2 u_3}{(u_2+u_3)^2}.$$

In the new variables, the limit $u_3 \to 0$ becomes the limit $z_3 \to 0$.

The statement of Lemma 3 in the variables z_i remains unchanged to first order in q_3. We will prove

$$\begin{aligned} z_1 &= \frac{2\left(j(\widehat{q}_1) + j(\widehat{q}_2) - \mu\right)}{j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_1)-\mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2)-\mu)}} + O(q_3), \\ z_2 &= \frac{(j(\widehat{q}_1)j(\widehat{q}_2) + \sqrt{j(\widehat{q}_1)(j(\widehat{q}_2)-\mu)}\sqrt{j(\widehat{q}_2)(j(\widehat{q}_2)-\mu)})^2}{4j(\widehat{q}_1)j(\widehat{q}_2)(j(\widehat{q}_1)+j(\widehat{q}_2)-\mu)^2} + O(q_3)\,. \end{aligned}$$

The Picard-Fuchs differential operators (29) can be rewritten as

$$\begin{aligned} \mathcal{L}_1'(z) &= \mathcal{L}_1(u), \\ z_2\sqrt{1-4z_3}\ \mathcal{L}_2'(z) &= \mathcal{L}_2(u) - \mathcal{L}_3(u), \\ z_2\sqrt{1-4z_3}\ \mathcal{L}_3'(z) &= u_3\mathcal{L}_2(u) - u_2\mathcal{L}_3(u), \end{aligned}$$

with

$$\begin{aligned} \mathcal{L}_1' &= \theta_1\left(\theta_1 - 2\,\theta_2\right) - 12\ (6\,\theta_1 - 5)\,(6\,\theta_1 - 1)\,z_1, \\ \mathcal{L}_2' &= \theta_2\left(\theta_2 - 2\,\theta_3\right) - (2\,\theta_2 - \theta_1 - 2)\,(2\,\theta_2 - \theta_1 - 1)\,z_2, \\ \mathcal{L}_3' &= {\theta_3}^2 - (2\,\theta_3 - \theta_2 - 2)\,(2\,\theta_3 - \theta_2 - 1)\,z_3, \end{aligned}$$

where now $\theta_i = z_i \frac{d}{dz_i}$. Since $\mathcal{L}_3'(z) \to 0$ in the limit $z_3 \to 0$, we need only focus on $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$.

Next, we transform $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$ to new variables y_1, y_2, y_3 via the change

$$\begin{aligned} z_1 &= \frac{2\left(y_1 + y_2 - \mu\right)}{y_1y_2 + \sqrt{y_1(y_1-\mu)}\sqrt{y_2(y_2-\mu)}}, \\ z_2 &= \frac{(y_1y_2 + \sqrt{y_1(y_1-\mu)}\sqrt{y_2(y_2-\mu)})^2}{4y_1y_2(y_1+y_2-\mu)^2}, \\ z_3 &= y_3. \end{aligned}$$

We obtain

$$\begin{aligned}\mathcal{L}_1'' &= y_1^2 y_2 (y_1-\mu)\partial_{y_1}^2 + y_1 y_2 (y_1 - \frac{\mu}{2})\partial_{y_1} - y_1 y_2^2 (y_2-\mu)\partial_{y_2}^2 \\ &\quad - y_1 y_2 (y_2 - \frac{\mu}{2})\partial_{y_2} + 60(y_1-y_2), \\ \mathcal{L}_2'' &= -y_1^2(y_1-\mu)\,\partial_{y_1}^2 + y_1(\frac{\mu}{2} - y_1)\partial_{y_1} + y_2^2(y_2-\mu)\,\partial_{y_2}^2 + y_2(y_2-\frac{\mu}{2})\partial_{y_2} \\ &\quad - 2y_1 y_3 (y_1-\mu)\partial_{y_1}\partial_{y_3} + 2y_2 y_3 (y_2-\mu)\partial_{y_2}\partial_{y_3}.\end{aligned}$$

In the limit $y_3 \to 0$, the second line on the right for $\mathcal{L}_2''$ vanishes. We can combine $\mathcal{L}_1''$ and $\mathcal{L}_2''$ to obtain the following simple forms:

$$\begin{aligned}\mathcal{L}_1'' + y_1 \lim_{y_3\to 0} \mathcal{L}_2'' &= (y_1-y_2)\left(60 - \left(y_1 - \frac{\mu}{2}\right) y_1\,\partial_{y_1} - (y_1-\mu)\; y_1^2\,\partial_{y_1}^2\right), \\ \mathcal{L}_1'' + y_2 \lim_{y_3\to 0} \mathcal{L}_2'' &= (y_1-y_2)\left(60 - \left(y_2 - \frac{\mu}{2}\right) y_2\,\partial_{y_2} - (y_2-\mu)\; y_2^2\,\partial_{y_2}^2\right).\end{aligned}$$

The solution $\varpi(y,\delta_0)_{y_3=0}$ therefore satisfies the differential equation

$$\mathcal{L} = (y-\mu)\,y^2\partial_y^2 + \left(y - \frac{\mu}{2}\right) y\,\partial_y - 60 \tag{33}$$

in both y_1 and y_2.

Changing (33) to the variable $t = \frac{1728}{y}$ yields

$$\mathcal{L} = t(1-t)\partial_t^2 + (1-\tfrac{3}{2}t)\partial_t - \tfrac{5}{144}\ ,$$

which by comparing with the general hypergeometric differential operator

$$\mathcal{L} = t(1-t)\partial_t^2 + (c-(1+a+b)t)\partial_t - ab$$

is identified with the system

$${}_2F_1(a,b;c;t) = {}_2F_1(\frac{1}{12},\frac{5}{12};1;t(\tau)).$$

According to the results of Klein and Fricke as reviewed in [37], we have a unique (up to scaling) solution g_0 to (33) locally analytic at $y=\infty$. The solution can be written as

$$g_0(j(\tau)) = (E_4)^{\frac{1}{4}}(\tau), \qquad\qquad y(\tau) = j(\tau)\ .$$

Moreover, the inverse is

$$\tau(y) = \frac{g_1(y)}{2\pi i g_0(y)},$$

where g_1 is a logarithmic solution at $y=\infty$ of $\mathcal{L}$, unique up to normalization and addition of g_0.

Transformation of the solution $\varpi(u,\delta_0)$ is seen to be analytic in a neighborhood of $t_1=t_2=0$. We conclude that

$$\varpi(u,\delta_0)_{u_3=0} = E_4^{\frac{1}{4}}(\tau_1)E_4^{\frac{1}{4}}(\tau_2).$$

By comparing the first few coefficients of the actual solutions $\varpi(u,\delta_i)$ in the $u_3\to 0$ limit, we can uniquely identify

$$\tau_1(u) = T_1(u), \qquad \tau_2(u) = T_1(u) + T_2(u)\ .$$

Hence, Lemma 4 is established. Lemma 3 is proven by transforming back to the u_1 and u_2 variables. □

Restricted to a $K3$ fiber of $\pi : X \to \mathbb{P}^1$, we have

$$\delta_1 = 2F + S, \quad \delta_2 = F.$$

The coordinates $2\pi i\tau_1$ and $2\pi i\tau_2$ correspond to the divisor basis

$$L_2 = F + S, \quad L_1 = F$$

of the $K3$ fiber. Since the variables q_1 and q_2 of Section 2 measure degrees against L_1 and L_2, we see that

$$\widehat{q}_1 = q_2 \quad \text{and} \quad \widehat{q}_2 = q_1$$

for the fiber geometry.

3.4. **B-model.** The mirror transformation results of Section 3.3 together with a B-model calculation of the periods will be used to prove the following result discovered by Klemm, Mayr, and Lerche [24].

Proposition 5. *We have*

$$2 + \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2} = 2\frac{E_4(q_1)E_6(q_1)}{\eta^{24}(q_1)}\frac{E_4(q_2)}{j(q_1) - j(q_2)}.$$

The left side of Proposition 5 is the left side of Proposition 4 with an added degree 0 constant 2.

Proof. We will use following universal expression for the Gromov-Witten invariants of X in terms of the periods of the mirror:

$$2 + \sum_{(d_1,d_2)\in\mathcal{P}} d_2^3 N^X_{0,(d_1,d_2)} q_1^{d_1} q_2^{d_2}$$
$$= \lim_{q_3\to 0} \frac{1}{\varpi(u(T),\delta_0)^2} \sum_{i,j,k=1}^{3} \frac{\partial u_i}{\partial \tau_1}\frac{\partial u_j}{\partial \tau_1}\frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)),$$

where the $Y_{i,j,k}$ are the Yukawa couplings of the mirror family; see [9, 24].

The periods $Y_{i,j,k}$ can be explicitly computed via Griffith transversality [24] and greatly simplify in the $q_3 \to 0$ limit. We tabulate the results below:

$$Y_{111} = \frac{8(1-\tilde{u}_1)}{\tilde{u}_1^3\Delta_1}, \qquad Y_{133} = \frac{2\tilde{u}_1(1-\tilde{u}_1)}{\tilde{u}_3\Delta_1},$$
$$Y_{112} = \frac{2(1-\tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_2 - \tilde{u}_3)}{\tilde{u}_1^2\tilde{u}_2\Delta_1}, \qquad Y_{222} = \frac{(1-2\tilde{u}_1)\,A_2}{2\tilde{u}_2^2\Delta_1\Delta_2},$$
$$Y_{113} = \frac{2(1-\tilde{u}_1)^2 + \tilde{u}_1^2(\tilde{u}_3 - \tilde{u}_2)}{\tilde{u}_1^2\tilde{u}_3\Delta_1}, \qquad Y_{223} = \frac{(1-2\tilde{u}_1)\,A_3}{2\tilde{u}_3\tilde{u}_2\Delta_1\Delta_2},$$
$$Y_{122} = \frac{2\tilde{u}_1(1-\tilde{u}_1)}{\tilde{u}_2\Delta_1}, \qquad Y_{233} = \frac{(1-2\tilde{u}_1)\,A_2}{2\tilde{u}_3\tilde{u}_2\Delta_1\Delta_2},$$
$$Y_{123} = \frac{(1-\tilde{u}_1)\left((1-\tilde{u}_1)^2 - (\tilde{u}_2+\tilde{u}_3)\tilde{u}_1^2\right)}{\tilde{u}_1\tilde{u}_2\tilde{u}_3\Delta_1}, \quad Y_{333} = \frac{(1-2\tilde{u}_1)\,A_3}{2\tilde{u}_3^2\Delta_1\Delta_2}.$$

Here, we have introduced the variables

$$\tilde{u}_1 = 432u_1, \quad \tilde{u}_2 = 4u_2, \quad \tilde{u}_3 = 4u_3$$

and the discriminant loci

$$\begin{aligned}\Delta_1 &= (1-\tilde u_1)^4 - 2(\tilde u_2+\tilde u_3)\tilde u_1^2(1-\tilde u_1)^2 + (\tilde u_2-\tilde u_3)^2\tilde u_1^4,\\ \Delta_2 &= (1-\tilde u_2-\tilde u_3)^2 - 4\tilde u_2\tilde u_3.\end{aligned} \tag{34}$$

The quantities A_2 and A_3 are defined by

$$\begin{aligned}A_2 &= (1+\tilde u_2-\tilde u_3)(1-\tilde u_1)^2 + \tilde u_1^2(1-\tilde u_3-3\,\tilde u_2)(\tilde u_2-\tilde u_3),\\ A_3 &= (1+\tilde u_3-\tilde u_2)(1-\tilde u_1)^2 + \tilde u_1^2(1-\tilde u_2-3\,\tilde u_3)(\tilde u_3-\tilde u_2).\end{aligned} \tag{35}$$

The normalizations of the Yukawa couplings $Y_{i,j,k}$ are fixed by the classical intersections.

The leading behavior of the mirror map for u_1, u_2 is obtained by rewriting Lemma 3 in terms of $E_4(\tau_i)$ and $E_6(\tau_i)$ as

$$\begin{aligned}u_1 &= \frac{1}{864}\left(1-\frac{E_6(\tau_1)\,E_6(\tau_2)}{E_4(\tau_1)^{\frac32}\,E_4(\tau_2)^{\frac32}}\right)+\mathcal{O}(q_3)\ ,\\ u_2 &= \frac{\left(E_4(\tau_1)^3-E_6(\tau_1)^2\right)\left(E_4(\tau_2)^3-E_6(\tau_2)^2\right)}{4\left(E_4(\tau_1)^{\frac32}\,E_4(\tau_2)^{\frac32}-E_6(\tau_1)\,E_6(\tau_2)\right)^2}+\mathcal{O}(q_3).\end{aligned} \tag{36}$$

Denote the leading behavior of the last mirror map by

$$u_3 = q_3 f_3(\widehat q_1,\widehat q_2)+\mathcal{O}(q_3^2)\ . \tag{37}$$

The derivatives of the mirror maps with respect to T_2 are easily evaluated using the standard identities

$$\begin{aligned}q\frac{d}{dq}E_2 &= \tfrac{1}{12}(E_2^2-E_4),\\ q\frac{d}{dq}E_4 &= \tfrac13(E_2E_4-E_6),\\ q\frac{d}{dq}E_6 &= \tfrac12(E_2E_6-E_4^2),\\ q\frac{d}{dq}j &= -j\frac{E_6}{E_4}.\end{aligned}$$

We find, to leading order in q_3,

$$\begin{aligned}\frac{\partial u_1}{\partial\tau_1} &= \frac{E_6(\tau_2)\left(E_4(\tau_1)^3-E_6(\tau_1)^2\right)}{1728\,E_4(\tau_2)^{\frac32}\,E_4(\tau_1)^{\frac52}},\\ \frac{\partial u_2}{\partial\tau_1} &= \frac{\sqrt{E_4(\tau_1)}\left(E_4(\tau_2)^3-E_6(\tau_2)^2\right)\left(-\left(E_4(\tau_1)^{\frac32}\,E_6(\tau_2)\right)+E_4(\tau_2)^{\frac32}\,E_6(\tau_1)\right)\left(E_4(\tau_1)^3-E_6(\tau_1)^2\right)}{4\left(E_4(\tau_2)^{\frac32}\,E_4(\tau_1)^{\frac32}-E_6(\tau_2)\,E_6(\tau_1)\right)^3}.\end{aligned}$$

The derivative $\frac{\partial u_3}{\partial\tau_1}$ can be written to this order as

$$\frac{\partial u_3}{\partial\tau_1} = \frac{u_3}{f_3(\widehat q_1,\widehat q_2)}\frac{\partial}{\partial\tau_1}f_3(\widehat q_1,\widehat q_2)+O(u_3^2)\ . \tag{38}$$

There are many simplifications in the limit $u_3\to 0$. First the triple couplings

$$Y_{133},\quad Y_{233},\quad Y_{333}$$

do not have enough inverse powers of u_3 and therefore do not contribute by the vanishing (38). Second, the surviving $Y_{i,j,k}$ simplify in the limit. We evaluate

$$\lim_{q_3\to 0} \frac{1}{\varpi(u(T),\delta_0)^2} \sum_{i,j,k=1}^{3} \frac{\partial u_i}{\partial \tau_1}\frac{\partial u_j}{\partial \tau_1}\frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)) = -2\,\frac{E_4(\tau_2)\,E_4(\tau_1)\,E_6(\tau_2)\,\Big(E_4(\tau_1)^3 - E_6(\tau_1)^2\Big)}{E_4(\tau_2)^3\,E_6(\tau_1)^2 - E_4(\tau_1)^3\,E_6(\tau_2)^2}\,. \tag{39}$$

The possible linear dependence on $f_3(\widehat{q}_1, \widehat{q}_2)$ drops out as claimed in [24]! Using the standard identities

$$j = \frac{E_4^3}{\eta^{24}}, \quad \eta^{24} = E_4^3 - E_6^2,$$

we obtain the right side of Proposition 5. □

4. The Harvey-Moore identity

4.1. **Proof of Proposition 4.** After evaluating the left side via Proposition 5 and dividing by 2, Proposition 4 amounts to a modular form identity. Let

$$f(\tau) = \frac{E_4(\tau)E_6(\tau)}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c(n)q^n,$$

where $q = \exp(2\pi i\tau)$. Then, we must prove

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \frac{q_1}{q_1 - q_2} + E_4(\tau_2) - \sum_{d,k,\ell>0} \ell^3 c(k\ell)\; q_1^{kd} q_2^{\ell d}. \tag{40}$$

Equation (40) is the Harvey-Moore identity conjectured in [18].

4.2. **Zagier's proof of the Harvey-Moore identity.** The Harvey-Moore identity implies Proposition 4 and concludes the proof of the Yau-Zaslow conjecture. We present here Zagier's argument from [38].

Let $S_k \subset M_k \subset M_k^!$ denote the spaces of cusp forms, modular forms, and weakly holomorphic[15] modular forms for $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Certainly

$$f(\tau) \in M_{-2}^!.$$

For each $n \geq 0$, there is a unique function $F_n \in M_4^!$ satisfying

$$F_n(\tau) = q^{-n} + \mathcal{O}(q)$$

as $\Im(\tau) \to \infty$. Uniqueness follows from the vanishing of S_4. Existence follows by writing $F_n(\tau)$ as $E_4(\tau)$ times a polynomial in $j(\tau)$,

$$F_0 = E_4, \quad F_1 = E_4(j - 984), \quad F_2 = E_4(j^2 - 1728j + 393768) \;\ldots\; .$$

We draw several consequences:

(i) $F_1|T_n = n^3 F_n$ for all $n \geq 1$, where T_n is the n^{th} Hecke operator in weight 4. Indeed, T_n sends $M_4^!$ to itself and, by standard formulas for the action of T_n on Fourier expansions, T_n sends $q^{-1} + \mathcal{O}(q)$ to $n^3 q^{-n} + \mathcal{O}(q)$.

[15]Holomorphic except for a possible pole at infinity.

(ii) $F_1 = -f'''$, where the prime denotes differentiation by

$$\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

We see that f''' lies in $M_4^!$ by the $k = 4$ case of Bol's identity,

$$\frac{d^{k-1}}{d\tau^{k-1}}(f|_{2-k}\gamma) = \left(\frac{d^{k-1}f}{d\tau^{k-1}}\right)|_k\gamma \quad \forall\gamma\in\Gamma.$$

Since the Fourier expansion of f''' begins as $-q^{-1} + \mathcal{O}(q)$, the claim is proven.

(iii) For $\Im(\tau_1) > \max_{\gamma\in\Gamma}\Im(\gamma\tau_2)$,

$$\frac{f(\tau_1)E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \sum_{n=0}^{\infty} F_n(\tau_2)q_1^n.$$

Let $L(\tau_1,\tau_2)$ denote the left side of (4.2). We see that $L(\tau_1,\tau_2)$ is a meromorphic modular form in τ_2 with a simple pole of residue $-\frac{1}{2\pi i}$ at $\tau_2 = \tau_1$ (since $j' = -E_4^2E_6/\eta^{24}$) and no poles outside $\Gamma\tau_1$. Moreover, $L(\tau_1,\tau_2)$ tends to 0 as $\Im(\tau_2)\to\infty$. These properties characterize $L(\tau_1,\tau_2)$ uniquely and show that the n^{th} Fourier coefficient with respect to τ_1 for $\Im(\tau_1)\to\infty$ has the properties characterizing $F_n(\tau_2)$.

Combining (i) and (ii) with the formula for the action of T_n on Fourier expansions, we obtain

$$\begin{aligned} F_n(\tau) &= (-n^{-3}f''')|T_n = n^{-3}\left(q^{-1} - \sum_{m=1}^{\infty} m^3c(m)\ q^m\right)|T_n \\ &= q^{-n} - \sum_{\substack{k,\ell,d>0\\ kd=n}} \ell^3c(k\ell)\ q^{\ell d} \end{aligned} \tag{41}$$

for $n > 0$. The Harvey-Moore identity follows from (41) and (iii) together with the equality $F_0 = E_4$. □

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