JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 23, Number 4, October 2010, Pages 1187–1195
S 0894-0347(2010)00673-X
Article electronically published on May 26, 2010

THE COMPLEX MONGE-AMPÈRE EQUATION ON COMPACT HERMITIAN MANIFOLDS

VALENTINO TOSATTI AND BEN WEINKOVE

1. Introduction

Let (M, g) be a compact Hermitian manifold (without boundary) of complex dimension $n \geq 2$ and write ω for the corresponding real $(1,1)$-form

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\overline{j}} dz^i \wedge d\overline{z^j}.$$

For a smooth real-valued function F on M, consider the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F \omega^n, \quad \text{with} \\ \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \sup_M \varphi = 0, \tag{1.1}$$

for a real-valued function φ.

Our main result is as follows.

Main Theorem. *Let φ be a smooth solution of the complex Monge-Ampère equation* (1.1)*. Then there are uniform C^∞ a priori estimates on φ depending only on (M, ω) and F.*

A corollary of this is that we can solve (1.1) uniquely after adding a constant to F, or, equivalently, up to scaling the volume form $e^F \omega^n$.

Corollary 1. *For every smooth real-valued function F on M there exist a unique real number b and a unique smooth real-valued function φ on M solving*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F+b} \omega^n, \quad \textit{with} \\ \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \sup_M \varphi = 0. \tag{1.2}$$

In the case of ω Kähler, that is, when $d\omega = 0$, this result is precisely the celebrated Calabi Conjecture [Ca] proved by Yau [Ya]. We note here that if ω satisfies

$$\partial\overline{\partial}\omega^k = 0, \quad \text{for } k = 1, 2 \tag{1.3}$$

Received by the editors November 11, 2009 and, in revised form, May 12, 2010.
2010 *Mathematics Subject Classification.* Primary 53C55; Secondary 32W20, 32U05.
This research is supported in part by National Science Foundation grant DMS-08-48193. The second author is also supported in part by a Sloan Foundation fellowship.

(in particular if ω is closed), then the constant b must equal

$$\log \frac{\int_M \omega^n}{\int_M e^F \omega^n}. \tag{1.4}$$

In fact, one can easily see that (1.3) implies that $\partial\overline{\partial}\omega^k = 0$ for all $1 \leq k \leq n-1$ (see for example [GL]), and integrating (1.2) over M and repeatedly using Stokes's Theorem, one sees that indeed b equals (1.4).

We mention now some special cases where the results of the Main Theorem and Corollary 1 are already known. Cherrier [Ch] gave a proof when the complex dimension is two or if ω is balanced, that is, $d(\omega^{n-1}) = 0$ (an alternative proof was very recently given in [TW]). In addition, Cherrier [Ch] dealt with the case of conformally Kähler and considered a technical assumption which is slightly weaker than balanced; see also the related work of Hanani [Ha]. Guan and Li [GL] gave a proof under the assumption (1.3). For further background we refer the reader to [TW] and the references therein.

As the reader will see in the proof below, we note that the key L^∞ bound of φ in the Main Theorem follows from combining a lemma of [Ch] with some recent estimates of the authors [TW].

Finally, we remark that one can give a geometric interpretation of (1.2) in terms of the first Chern class $c_1(M)$ of M. We denote by $\mathrm{Ric}(\omega)$ the first Chern form of the Chern connection of ω, which is a closed form cohomologous to $c_1(M)$. We then consider the real Bott-Chern space $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$ of closed real $(1,1)$-forms modulo the image of $\sqrt{-1}\partial\overline{\partial}$ acting on real functions. It has a natural surjection to the familiar space $H^{1,1}(M,\mathbb{R})$, which is an isomorphism if and only if $b_1(M) = 2h^{0,1}$ [G2] (in particular if M is Kähler). The form $\mathrm{Ric}(\omega)$ determines a class $c_1^{\mathrm{BC}}(M)$ in $H^{1,1}_{\mathrm{BC}}(M,\mathbb{R})$ which maps to the usual first Chern class $c_1(M)$ via the above surjection. Then from our Main Theorem we get the following Hermitian version of the Calabi Conjecture (see also a related question of Gauduchon [G2, IV.5]):

Corollary 2. *Every representative of the first Bott-Chern class $c_1^{\mathrm{BC}}(M)$ can be represented as the first Chern form of a Hermitian metric of the form $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$.*

To see why this holds, just notice that (1.2) holds for some constant b if and only if

$$\mathrm{Ric}(\omega + \sqrt{-1}\partial\overline{\partial}\varphi) = \mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}F \tag{1.5}$$

and that by definition every form representing $c_1^{\mathrm{BC}}(M)$ can be written as $\mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}F$ for some function F. We note here that in the case $n = 2$, Corollary 2 of [TW] gives a criterion to decide which representatives of $c_1(M)$ can be written in this form.

2. Proof of the Main Theorem

By the results of [Ch], [GL], [Zh] it suffices to obtain a uniform bound of φ in the L^∞ norm. Indeed, by extending the second-order estimate on φ of Yau [Ya] (and Aubin [Au]), Cherrier [Ch] has shown, for general ω, that a uniform L^∞ bound on φ implies that the metric $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is uniformly equivalent to ω. Moreover, generalizing Yau's third-order estimate [Ya], Cherrier shows that given this one can then bound $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ in C^1. Higher-order estimates then follow from standard elliptic theory. A similar second-order estimate was also proved by Guan and Li [GL] and Zhang [Zh] for general ω and sharpened in [TW] in the cases of $n = 2$ or ω balanced. It is also possible to avoid the third-order estimate by using the Evans-Krylov theory, as in [GL] and [TW].

We remark that our L^∞ bound on φ depends only on (M, ω) and $\sup_M F$, as in Yau's estimate for the Kähler case [Ya]. In particular, the L^∞ bound does not depend on $\inf_M F$. In the course of the proof, we say that a constant is *uniform* if it depends only on the data (M, ω) and $\sup_M F$. We will often write such a constant as C, which may differ from line to line. If we say that a constant depends only on a quantity Q, then we mean that it depends only on Q, (M, ω), and $\sup_M F$.

Our goal is thus to give a uniform bound for φ. We begin with a lemma which can be found in [Ch]. For the convenience of the reader, we provide a proof. We use the notation of exterior products instead of the multilinear algebra calculations of [Ch].

Lemma 2.1. *There are uniform constants C, p_0 such that for all $p \geq p_0$ we have*

$$\int_M |\partial e^{-\frac{p}{2}\varphi}|_g^2 \omega^n \leq Cp \int_M e^{-p\varphi}\omega^n.$$

Proof. From now on we will use the shorthand notation $\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$. Let α be the $(n-1, n-1)$-form given by

$$\alpha = \sum_{k=0}^{n-1} \omega_\varphi^k \wedge \omega^{n-k-1}.$$

We compute, using the equation (1.1) and integrating by parts,

$$\begin{aligned} C\int_M e^{-p\varphi}\omega^n &\geq \int_M e^{-p\varphi}(\omega_\varphi^n - \omega^n) \\ &= \int_M e^{-p\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \wedge \alpha \\ &= p\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha + \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\alpha. \end{aligned} \tag{2.1}$$

The first term on the right-hand side of (2.1) is positive, and we are going to use part of it to deal with the second one. Notice that

$$\partial\alpha = n\sum_{k=0}^{n-2} \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega.$$

Since $\partial\omega$ is a fixed tensor, there is a constant C so that for any $\varepsilon > 0$ and any k we have the elementary pointwise inequality

$$\begin{aligned} &\left| \frac{\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\omega \wedge \omega_\varphi^k \wedge \omega^{n-k-2}}{\omega^n} \right| \\ &\qquad \leq \frac{C}{\varepsilon}\frac{\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}}{\omega^n} + \varepsilon C\frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n} \end{aligned} \tag{2.2}$$

that the reader can verify by choosing local coordinates at a point that make ω the identity and ω_φ diagonal. Applying (2.2), we have for any $\varepsilon > 0$ and any p,

$$\begin{aligned} -\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi\wedge\partial\alpha &= -n\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi\wedge\omega_\varphi^k\wedge\omega^{n-k-2}\wedge\partial\omega \\ &\leq \frac{C}{\varepsilon}\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega_\varphi^k\wedge\omega^{n-k-1} \\ &\quad+\varepsilon C\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k\wedge\omega^{n-k}. \end{aligned}$$

Now if we choose $p_0/2 \geq C/\varepsilon$, we see that if $0 < \varepsilon \leq 1$, then for $p \geq p_0$,

$$\begin{aligned} -\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi\wedge\partial\alpha &\leq \frac{p}{2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\alpha + C\int_M e^{-p\varphi}\omega^n \\ &\quad+\varepsilon C\sum_{k=1}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k\wedge\omega^{n-k}. \end{aligned}$$

Combining this with (2.1), we see that for any $0 < \varepsilon < 1$ there exists p_0 depending only on ε such that for $p \geq p_0$,

$$\frac{p}{2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\alpha \leq C\int_M e^{-p\varphi}\omega^n + \varepsilon C\sum_{k=1}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k\wedge\omega^{n-k}. \tag{2.3}$$

We now claim the following. There exist uniform constants $C_2, \ldots, C_n$ and ε_0 such that for all ε with $0 < \varepsilon \leq \varepsilon_0$, there exists a constant p_0 depending only on ε such that for all $p \geq p_0$ we have for $i = 2, \ldots, n$,

$$\frac{p}{2^{i-1}}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\alpha \leq C_i\int_M e^{-p\varphi}\omega^n + \varepsilon C_i\sum_{k=1}^{n-i}\int_M e^{-p\varphi}\omega_\varphi^k\wedge\omega^{n-k}. \tag{2.4}$$

Given the claim, the lemma follows. Indeed once we have the statement with $i = n$, then, fixing $\varepsilon = \varepsilon_0$, we have for $p \geq p_0$,

$$\begin{aligned} \int_M |\partial e^{-\frac{p}{2}\varphi}|_g^2\omega^n &= \frac{np^2}{4}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\omega^{n-1} \\ &\leq \frac{np^2}{4}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi\wedge\overline{\partial}\varphi\wedge\alpha \\ &\leq n2^{n-3}C_n p\int_M e^{-p\varphi}\omega^n, \end{aligned}$$

as required.

We will prove the claim by induction on i. By (2.3) we have already proved the statement for $i = 2$. So we assume the induction statement (2.4) for i and prove it for $i+1$. We compute

$$
\begin{aligned}
\varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k} &= \varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi}\omega_\varphi^{k-1} \wedge \omega^{n-k+1} \\
&\quad + \varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&= A_1 + A_2,
\end{aligned} \tag{2.5}
$$

where

$$
A_1 = \varepsilon C_i \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k},
$$

$$
A_2 = \varepsilon C_i \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}.
$$

The term A_1 is already acceptable for the induction. For A_2 we integrate by parts to obtain

$$
\begin{aligned}
A_2 &= \varepsilon C_i p \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1} \\
&\quad + \varepsilon C_i \sum_{k=1}^{n-i-1} k \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k-1} \wedge \partial\omega \\
&\quad + \varepsilon C_i \sum_{k=0}^{n-i-1} (n-k-1) \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega \\
&= B_1 + B_2 + B_3,
\end{aligned} \tag{2.6}
$$

where

$$
\begin{aligned}
B_1 &= \varepsilon C_i p \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1} \\
B_2 &= \varepsilon C_i \sum_{k=0}^{n-i-2} (k+1) \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega \\
B_3 &= \varepsilon C_i \sum_{k=0}^{n-i-1} (n-k-1) \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega.
\end{aligned}
$$

Choosing ε_0 such that $\varepsilon_0 C_i < 2^{-i-1}$, we have for $\varepsilon \leq \varepsilon_0$ and $p \geq p_0$,

$$
B_1 \leq \frac{p}{2^{i+1}} \int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha. \tag{2.7}
$$

For the terms B_2 and B_3 we use again (2.2) to obtain

$$
\begin{aligned}
B_2 + B_3 &\leq nC_iC \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1} \\
&\quad + \varepsilon^2 nC_iC \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k}.
\end{aligned} \tag{2.8}
$$

Notice that the second term on the right-hand side of (2.8) is acceptable for the induction. Moreover, we may assume that $p_0 \geq 2^{i+1} n C_i C$ and thus for $p \geq p_0$,

$$\begin{aligned} B_2 + B_3 \leq{}& \frac{p}{2^{i+1}} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \\ &+ \varepsilon^2 n C_i C \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k}. \end{aligned} \tag{2.9}$$

Combining the inductive hypothesis (2.4) with (2.5), (2.6), (2.7), (2.9), we obtain for $p \geq p_0$,

$$\frac{p}{2^i} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \leq C_{i+1} \int_M e^{-p\varphi} \omega^n + \varepsilon C_{i+1} \sum_{k=1}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k},$$

completing the inductive step. This finishes the proof of the claim and thus the lemma. □

We now complete the proof of the Main Theorem. Using Lemma 2.1 and the Sobolev inequality, we have for $\beta = \frac{n}{n-1} > 1$,

$$\begin{aligned} \left(\int_M e^{-p\beta\varphi} \omega^n \right)^{1/\beta} &\leq C \left(\int_M |\partial e^{-\frac{p}{2}\varphi}|^2 \omega^n + \int_M e^{-p\varphi} \omega^n \right) \\ &\leq Cp \int_M e^{-p\varphi} \omega^n, \end{aligned}$$

for all $p \geq p_0$. Thus

$$\|e^{-\varphi}\|_{L^{p\beta}} \leq C^{1/p} p^{1/p} \|e^{-\varphi}\|_{L^p}.$$

Since this holds for all $p \geq p_0$, we can iterate this estimate in a standard way to obtain

$$\|e^{-\varphi}\|_{L^\infty} \leq C \|e^{-\varphi}\|_{L^{p_0}},$$

which is equivalent to

$$e^{-p_0 \inf_M \varphi} \leq C \int_M e^{-p_0\varphi} \omega^n.$$

We now make use of a result from [TW]:

Lemma 2.2. *Let f be a smooth function on (M, ω). Write $d\mu = \omega^n / \int_M \omega^n$. If there exists a constant C_1 such that*

$$e^{-\inf_M f} \leq e^{C_1} \int_M e^{-f} d\mu, \tag{2.10}$$

then

$$|\{f \leq \inf_M f + C_1 + 1\}| \geq \frac{e^{-C_1}}{4}, \tag{2.11}$$

where $|\cdot|$ denotes the volume of the set with respect to $d\mu$.

Proof. See [TW, Lemma 3.2]. □

Applying this lemma to $f = p_0\varphi$, we see that there exist uniform constants C, $\delta > 0$ so that

$$|\{\varphi \leq \inf_M \varphi + C\}| \geq \delta. \tag{2.12}$$

We remark that, in [TW], the bound (2.12) is established whenever one has the improved second-order estimate,

$$\mathrm{tr}_{\omega}\omega_{\varphi} \leq Ce^{A(\varphi - \inf_M \varphi)}, \tag{2.13}$$

for uniform A and C. It is shown in [TW] that (2.13) holds if $n = 2$ or ω is balanced.

The L^∞ bound on φ, and hence the Main Theorem, now follow from the arguments of [TW]. However, we include an outline of these arguments for the reader's convenience. Recall that, from [G1], if (M, ω) is a compact Hermitian manifold, then there exists a unique smooth function $u : M \to \mathbb{R}$ with $\sup_M u = 0$ such that the metric $\omega_{\mathrm{G}} = e^u\omega$ is *Gauduchon*, that is, it satisfies

$$\partial\overline{\partial}(\omega_{\mathrm{G}}^{n-1}) = 0. \tag{2.14}$$

Writing Δ_{G} for the complex Laplacian associated to ω_{G} (which differs from the Levi-Civita Laplacian in general), we have the following lemma (cf. [TW, Lemma 3.4]).

Lemma 2.3. *Let M be a compact complex manifold of complex dimension n with a Gauduchon metric ω_{G}. If ψ is a smooth nonnegative function on M with*

$$\Delta_{\mathrm{G}}\psi \geq -C_0,$$

then there exist constants C_1 and C_2 depending only on (M, ω_{G}) and C_0 such that

$$\int_M |\partial\psi^{\frac{p+1}{2}}|^2_{\omega_{\mathrm{G}}}\omega_{\mathrm{G}}^n \leq C_1 p \int_M \psi^p \omega_{\mathrm{G}}^n \quad \textit{for all } p \geq 1 \tag{2.15}$$

and

$$\sup_M \psi \leq C_2 \max\left\{\int_M \psi\, \omega_{\mathrm{G}}^n, 1\right\}. \tag{2.16}$$

Proof. Compute for $p \geq 1$,

$$\begin{aligned}
\int_M |\partial\psi^{\frac{p+1}{2}}|^2_{\omega_{\mathrm{G}}}\omega_{\mathrm{G}}^n &= \frac{n(p+1)^2}{4}\int_M \sqrt{-1}\psi^{p-1}\partial\psi \wedge \overline{\partial}\psi \wedge \omega_{\mathrm{G}}^{n-1} \\
&= \frac{n(p+1)^2}{4p}\int_M \sqrt{-1}\partial\psi^p \wedge \overline{\partial}\psi \wedge \omega_{\mathrm{G}}^{n-1} \\
&= \frac{(p+1)^2}{4p}\int_M \psi^p(-\Delta_{\mathrm{G}}\psi)\omega_{\mathrm{G}}^n \\
&\quad + \frac{n(p+1)}{4p}\int_M \sqrt{-1}\,\overline{\partial}\psi^{p+1} \wedge \partial\omega_{\mathrm{G}}^{n-1} \\
&\leq C\frac{(p+1)^2}{4p}\int_M \psi^p\omega_{\mathrm{G}}^n,
\end{aligned}$$

thus establishing (2.15). The inequality (2.16) then follows by a standard iteration argument, using the Sobolev inequality for the metric ω_{G}. Indeed, writing $q = p+1$ and $\beta = \frac{n}{n-1}$, we obtain for $q \geq 2$,

$$\left(\int_M \psi^{q\beta}\omega_{\mathrm{G}}^n\right)^{1/\beta} \leq Cq \max\left\{\int_M \psi^q\omega_{\mathrm{G}}^n, 1\right\}.$$

By repeatedly replacing q by $q\beta$ and iterating, we have, after setting $q=2$,

$$\sup_M \psi \leq C \max\left\{ \left(\int_M \psi^2 \omega_{\mathrm{G}}^n \right)^{1/2}, 1 \right\} \leq C \max\left\{ \left(\sup_M \psi \right)^{1/2} \left(\int_M \psi\, \omega_{\mathrm{G}}^n \right)^{1/2}, 1 \right\},$$

and (2.16) follows. □

We apply Lemma 2.3 to the function $\psi = \varphi - \inf_M \varphi$, which satisfies $\Delta_{\mathrm{G}}\psi = e^{-u}\Delta\psi > -C$, where Δ is the complex Laplacian with respect to ω. In light of (2.16), once we bound the L^1 norm of ψ, the Main Theorem follows. Denoting by $\underline{\psi}$ the average of ψ with respect to ω_{G}^n, we obtain from the Poincaré inequality and (2.15) with $p=1$

$$\|\psi - \underline{\psi}\|_{L^2} \leq C \left(\int_M |\partial\psi|^2_{\omega_{\mathrm{G}}} \omega_{\mathrm{G}}^n \right)^{1/2} \leq C\|\psi\|_{L^1}^{1/2}. \tag{2.17}$$

In (2.17) and the following we are using L^q norms with respect to the volume form ω_{G}^n, which are equivalent to L^q norms with respect to $d\mu$. Using (2.12), we see that the set $S := \{\psi \leq C\}$ satisfies $|S|_{\mathrm{G}} \geq \delta$ for a uniform $\delta > 0$, where $|\cdot|_{\mathrm{G}}$ denotes the volume of a set with respect to ω_{G}^n. Hence

$$\frac{\delta}{\int_M \omega_{\mathrm{G}}^n} \int_M \psi\omega_{\mathrm{G}}^n = \delta\underline{\psi} \leq \int_S \underline{\psi}\omega_{\mathrm{G}}^n \leq \int_S (|\psi - \underline{\psi}| + C)\omega_{\mathrm{G}}^n \leq \int_M |\psi - \underline{\psi}|\omega_{\mathrm{G}}^n + C.$$

Then,

$$\|\psi\|_{L^1} \leq C(\|\psi - \underline{\psi}\|_{L^1} + 1) \leq C(\|\psi - \underline{\psi}\|_{L^2} + 1) \leq C(\|\psi\|_{L^1}^{1/2} + 1),$$

which shows that ψ is uniformly bounded in L^1. This completes the proof of the Main Theorem.

Finally we mention that Corollary 1 follows from the argument of Cherrier [Ch], which uses results from [De], or for another proof, see [TW, Corollary 1].

Acknowledgements

The authors thank S.-T. Yau for many useful discussions over the last few years on the complex Monge-Ampère equation. The authors also express their gratitude to D. H. Phong for his support, encouragement, and helpful suggestions. In addition the authors thank G. Székelyhidi for some helpful discussions.

References

[Au] Aubin, T. *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95. MR494932 (81d:53047)

[Ca] Calabi, E. *On Kähler manifolds with vanishing canonical class*, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pp. 78–89. Princeton University Press, Princeton, N. J., 1957. MR0085583 (19:62b)

[Ch] Cherrier, P. *Équations de Monge-Ampère sur les variétés Hermitiennes compactes*, Bull. Sc. Math (2) **111** (1987), 343–385. MR921559 (89d:58131)

[De] Delanoë, P. *Équations du type de Monge-Ampère sur les variétés Riemanniennes compactes, II*, J. Functional Anal. **41** (1981), 341–353. MR619957 (83f:53028b)

[G1] Gauduchon, P. *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris **285** (1977), 387–390. MR0470920 (57:10664)

[G2] Gauduchon, P. *La 1-forme de torsion d'une variètè hermitienne compacte*, Math. Ann. **267** (1984), no. 4, 495–518. MR742896 (87a:53101)

[GL] Guan, B., Li, Q. *Complex Monge-Ampère equations and totally real submanifolds*, Adv. Math. (2010), doi:10.1016/j.aim.2010.03.019

[Ha] Hanani, A. *Équations du type de Monge-Ampère sur les variétés hermitiennes compactes*, J. Funct. Anal. **137** (1996), no. 1, 49–75. MR1383012 (97c:32018)

[TW] Tosatti, V., Weinkove, B. *Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds*, preprint, arXiv:0909.4496, to appear in Asian J. Math. 2010.

[Ya] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), no.3, 339–411. MR480350 (81d:53045)

[Zh] Zhang, X. *A priori estimates for complex Monge-Ampère equation on Hermitian manifolds*, Int. Math. Res. Not. **2010**, Art. ID rnq029, 23 pp.

Department of Mathematics, Columbia University, 2990 Broadway, New York, New York 10027

E-mail address: tosatti@math.columbia.edu

Department of Mathematics, University of California San Diego, 9500 Gilman Drive #0112, La Jolla, California 92093

E-mail address: weinkove@math.ucsd.edu