

PROOF OF THE FUNDAMENTAL GAP CONJECTURE

BEN ANDREWS AND JULIE CLUTTERBUCK

1. INTRODUCTION

We consider Schrödinger operators of the form $-\Delta + V$ with Dirichlet boundary conditions on a compact convex domain Ω in \mathbb{R}^n . The diameter of Ω is given by $D = \sup_{x,y \in \Omega} \|y-x\|$. We assume that the potential V is semiconvex (i.e., $V+c\|x\|^2$ is convex for some c). Such an operator has an increasing sequence of eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and corresponding eigenfunctions $\{\phi_i\}_{i \geq 0}$ which vanish on $\partial\Omega$ and satisfy the equation

$$(1) \quad \Delta\phi_i - V\phi_i + \lambda_i\phi_i = 0.$$

The difference between the first two eigenvalues, $\lambda_1 - \lambda_0$, is called the *fundamental gap*. It is of importance for several reasons: In quantum mechanics it represents the ‘excitation energy’ required to reach the first excited state from the ground state; it thus determines the stability of the ground state and so is also important in statistical mechanics and quantum field theory. The spectral gap also determines the rate at which positive solutions of the heat equation approach the first eigenspace, and it is through this characterization that we will prove the conjecture of the title.

It was observed by Michiel van den Berg [vdB] that for many convex domains, $\lambda_1 - \lambda_0 \geq 3\pi^2/D^2$. This was also independently suggested by Ashbaugh and Benguria [AB] and Yau [Y2].

Gap Conjecture. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain of diameter D , and V a weakly convex potential. Then the eigenvalues of the Schrödinger operator satisfy*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

In one dimension, the conjecture is settled. If Ω is an interval of length D , the eigenvalue equation (1) becomes a boundary value problem for an ODE: The eigenfunctions $\tilde{\phi}_i$ satisfy $\tilde{\phi}_i(\pm D/2) = 0$ and

$$(2) \quad \tilde{\phi}_i'' - \tilde{V}\tilde{\phi}_i + \mu_i\tilde{\phi}_i = 0 \text{ in } (-D/2, D/2).$$

For convenience we normalize these to have $\tilde{\phi}_i'(D/2) = -1$.

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Ashbaugh and Benguria [AB] proved the conjecture for $n = 1$ under slightly different assumptions: They considered symmetric single-well potentials, so that $\tilde{V}(x) = \tilde{V}(-x)$, and \tilde{V} is nondecreasing on $[0, D/2]$.

Theorem 1.1 (Ashbaugh–Benguria). *If $n = 1$ and \tilde{V} is symmetric and single-well, then the eigenvalue gap satisfies*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2},$$

with equality attained when \tilde{V} is constant.

Later, Horváth [Ho] partly removed the symmetry assumption, allowing \tilde{V} to be a single-well potential with minimum at the midpoint of the interval. In 1994, Lavine removed the assumption on the minimum point and proved the complete conjecture [La].

Theorem 1.2 (Lavine). *The Gap Conjecture holds for $n = 1$.*

In higher dimensions, the conjecture has been proved for some highly symmetric cases: Bañuelos–Méndez-Hernández [BMH] (for $V \equiv 0$), Davis [D] and Bañuelos–Kröger [BK] proved it for $\Omega \subset \mathbb{R}^2$ symmetric with respect to the x and y axes and convex in both x and y .

Nonoptimal lower bounds have also been obtained: In an influential paper, Singer, Wong, Yau and Yau [SWYY] used gradient estimates similar to those developed by Li [Li] and Li–Yau [LY], together with the log-concavity of the first eigenfunction (proved for convex potentials by Brascamp and Lieb [BL], and also by different methods in [Ko] and in [SWYY]) to show the gap is at least $\frac{\pi^2}{4D^2}$. Yu and Zhong [YZ] improved this to $\frac{\pi^2}{D^2}$, following the work of Zhong and Yang [ZY] which similarly improved Li and Yau’s estimate for the first nontrivial eigenvalue on a compact manifold [LY]. Many other authors have used probabilistic arguments (including [BL] and [BK]). Such methods were used by Smits [S] to recover the estimate of Yu and Zhong. It seems that any further improvement requires an improved estimate on the log-concavity of the first eigenfunction. Efforts in this direction have included work by Yau [Y3] and Ling [L], who prove gap estimates depending explicitly on an upper bound for $D^2(\log \phi_0)$. The problem of nonconvex potentials has been studied by Yau [Y1]. There is also considerable literature on potentials with double wells, including examples with arbitrarily small spectral gaps (see for example [Ha]).

We should point out here that the fundamental gap for the Neumann problem (with zero potential) is simply the first nontrivial eigenvalue. On a convex domain, Payne and Weinberger [PW] proved that this is bounded below by π^2/D^2 . Zhong and Yang [ZY] later reproduced the same result as a special case, using rather different methods.

We refer to the excellent survey by Ashbaugh [A] for further discussion of the fundamental gap conjecture and its history.

Our proof of the gap conjecture combines two ingredients, both of which involve analysis of the heat equation: The first ingredient is a method for estimating the modulus of continuity of solutions of parabolic equations which was developed in our earlier papers [AC1, AC2]. In this paper we say that a function η on \mathbb{R}_+ is a

modulus of continuity for a function f on Ω if

$$(3) \quad |f(y) - f(x)| \leq 2\eta \left(\frac{\|y - x\|}{2} \right)$$

for all $x, y \in \Omega$, where $\|\cdot\|$ is the Euclidean. Here the factors of 2 are for later convenience. In [AC2, Theorem 4.1] we proved that if u is any smooth solution of the heat equation with Neumann boundary data on a convex domain Ω of diameter D in \mathbb{R}^n , then $u(\cdot, t)$ has modulus of continuity $\varphi(\cdot, t)$, where φ is the solution to the one-dimensional heat equation on $[-D/2, D/2]$ with Neumann boundary data and initial data $\varphi(z, 0) = \frac{1}{2} \text{osc } u(\cdot, 0) \text{sgn}(z)$. The proof of this uses a maximum principle argument applied to a function of $2n$ spatial variables and crucially exploits the full $2n \times 2n$ matrix of second derivatives. In that paper we were mostly concerned with the short-time application of this estimate to deduce Lipschitz bounds on u , but the estimate is also useful for large t : One can show that $|\varphi(x, t)| \leq Ce^{-\frac{\pi^2}{D^2}t}$, so the oscillation of u decays at an exponential rate, with exponent at least π^2/D^2 . But if we choose u to be the first nontrivial eigenfunction for the Neumann problem on Ω , then $u(x, t) = u(x, 0)e^{-\lambda t}$, where λ is the corresponding eigenvalue. It follows that $\lambda \geq \frac{\pi^2}{D^2}$, and we recover the sharp estimate of Payne and Weinberger [PW]. In a separate work we will apply this method to more general eigenvalue estimates on manifolds.

As might be expected from the proof in [SWYY], a second ingredient is required to apply this technique to the gap conjecture: The log-concavity of the first eigenfunction ϕ_0 . If we apply the method above using only the log-concavity of ϕ_0 , then we recover the estimate $\lambda_1 - \lambda_0 \geq \frac{\pi^2}{D^2}$, which is precisely the estimate of Yu and Zhong [YZ]. As noted above, it is clear that we must use something stronger than log-concavity to obtain a sharp result. Previous work [Y3, L] produces stronger results assuming positive lower bounds on the Hessian of $\log \phi_0$, but such estimates cannot yield sharp results.

In trying to apply the modulus of continuity argument above, we are led to a log-concavity condition involving differences of derivatives of $\log \phi_0$.

We first introduce the relevant notions in a general context: If ω is a real function of a positive real variable, and X is a vector field on a domain Ω , we say that ω is a *modulus of expansion* for X if for every $y \neq x$ in Ω we have

$$(4) \quad (X|_y - X|_x) \cdot \frac{y - x}{\|y - x\|} \geq 2\omega \left(\frac{\|y - x\|}{2} \right).$$

The factors of 2 are for later convenience. The name reflects the fact that a modulus of expansion implies a rate of increase of distances under the flow of X . We say that ω is a *modulus of contraction* for X if the sign is reversed (that is, $-\omega$ is a modulus of expansion for $-X$). If f is a semi-convex function on a domain Ω , we say that ω is a *modulus of convexity* for f if ω is a modulus of expansion for the gradient vector field ∇f of f , and ω is a *modulus of concavity* for f if ω is a modulus of contraction for ∇f . Note that f is concave (convex) if and only if zero is a modulus of concavity (convexity) for f .

Our log-concavity estimates amount to an explicit modulus of concavity for $\log \phi_0$. Precisely, we assume that the potential \tilde{V} is even, and that the potential V in equation (1) is ‘more convex’ than \tilde{V} , in the sense that for any $y \neq x$ in

Ω we have

$$(5) \quad (\nabla V|_y - \nabla V|_x) \cdot \frac{(y-x)}{\|y-x\|} \geq 2\tilde{V}'\left(\frac{\|y-x\|}{2}\right).$$

That is, we assume that \tilde{V}' is a modulus of convexity for V . Under this assumption we prove in Theorem 1.5 that the first eigenfunction ϕ_0 in (1) is ‘more log-concave’ than the first eigenfunction $\tilde{\phi}_0$ in (2), in the sense that

$$(6) \quad \left(\nabla \log \phi_0|_y - \nabla \log \phi_0|_x\right) \cdot \frac{y-x}{\|y-x\|} \leq 2(\log \tilde{\phi}_0)'|_{z=\frac{\|y-x\|}{2}}$$

for every $y \neq x$ in Ω . That is, we prove that $(\log \tilde{\phi}_0)'$ is a modulus of concavity for $\log \phi_0$. Note that if V is convex, then the inequality (5) holds with $\tilde{V} = 0$, and the modulus of concavity in (6) becomes $(\log \tilde{\phi}_0)'(z) = -\frac{\pi}{D} \tan\left(\frac{\pi z}{D}\right)$. We will now list the main results.

Theorem 1.3 (Fundamental gap comparison theorem). *If V and \tilde{V} are related by (5), then the spectral gap for $-\Delta + V$ on a convex domain Ω is bounded below by the spectral gap of the one-dimensional operator $-\frac{d^2}{dx^2} + \tilde{V}$ on $[-D/2, D/2]$.*

If V is convex, the gap conjecture follows since we can choose $\tilde{V} = 0$:

Corollary 1.4. *If V is weakly convex, then the spectral gap has the bound*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}.$$

In particular, this applies for zero potential. The key step in proving the gap conjecture is a sharp log-concavity estimate for the first eigenfunction:

Theorem 1.5 (Log-concavity comparison theorem). *Let ϕ_0 be the first eigenfunction of (1). If the potential V satisfies condition (5), then the log-concavity estimate (6) holds for every $y \neq x$ in Ω .*

Let us note a special case of this result which can be stated rather more elegantly: Suppose \tilde{V}' is also a modulus of convexity for \tilde{V} . It then follows from Theorem 1.5 applied with $n = 1$ and $V = \tilde{V}$ that $(\log \tilde{\phi}_0)'$ is a (sharp) modulus of concavity for $\log \tilde{\phi}_0$. This happens, for example, if \tilde{V} has nonnegative third derivatives for positive arguments: In that case $\tilde{V}'(b) - \tilde{V}'(a) \geq \tilde{V}'(\frac{b-a}{2}) - \tilde{V}'(-\frac{b-a}{2}) = 2\tilde{V}'(\frac{b-a}{2})$, so that \tilde{V} has (sharp) modulus of convexity \tilde{V}' . It also happens if \tilde{V} is the optimal choice defined by $2\tilde{V}'(z/2) = \sup\{(\nabla V(x+ze) - \nabla V(x)) \cdot e : x, x+ze \in \Omega, \|e\| = 1\}$. Thus in these cases we have:

Corollary 1.6. *Under the conditions above, if the modulus of convexity of V is bounded below by the modulus of convexity of \tilde{V} , then the modulus of concavity of $\log \phi_0$ is bounded above by the modulus of concavity of $\log \tilde{\phi}_0$.*

Throughout the computations we assume that Ω has smooth boundary and is uniformly convex (so the principal curvatures of $\partial\Omega$ have a positive lower bound), and that V is smooth. The results for the general case of convex Ω and semiconvex V follow by using a straightforward approximation argument (see for example [HP] and [He, Thm. 2.3.17]).

The paper is arranged as follows: In Section 2 we prove a general modulus of continuity estimate for solutions of Neumann heat equations with drift. In Section 3 we use this to prove Theorem 1.3, assuming the result of Theorem 1.5. The proof

of the modulus of continuity estimate provides a model for the more involved log-concavity estimate, which we prove in Section 4. Finally, in Section 5 we discuss some examples and extensions of our results.

2. MODULUS OF CONTINUITY FOR HEAT EQUATION WITH DRIFT

We prove here a generalization of [AC2, Theorem 4.1], which controls the modulus of continuity of solutions of a Neumann heat equation with drift, in terms of the modulus of contraction of the drift velocity:

Theorem 2.1. *Let Ω be a strictly convex domain of diameter D with smooth boundary in \mathbb{R}^n , and X a time-dependent vector field on Ω . Suppose $v : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a smooth solution of the equation*

$$(7) \quad \begin{aligned} \frac{\partial v}{\partial t} &= \Delta v + X \cdot \nabla v \quad \text{in } \Omega \times \mathbb{R}_+; \\ D_{\nu} v &= 0 \quad \text{in } \partial\Omega \times \mathbb{R}_+. \end{aligned}$$

Suppose that

1. $X(\cdot, t)$ has modulus of contraction $\omega(\cdot, t)$ for each $t \geq 0$, where $\omega : [0, D/2] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is smooth;
2. $v(\cdot, 0)$ has modulus of continuity φ_0 , where $\varphi_0 : [0, D/2] \rightarrow \mathbb{R}$ is smooth with $\varphi_0(0) = 0$ and $\varphi'_0(z) > 0$ for $0 \leq z \leq D/2$;
3. $\varphi : [0, D/2] \times \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies
 - (i) $\varphi(z, 0) = \varphi_0(z)$ for each $z \in [0, D/2]$;
 - (ii) $\frac{\partial \varphi}{\partial t} \geq \varphi'' + \omega \varphi'$ on $[0, D/2] \times \mathbb{R}_+$;
 - (iii) $\varphi' > 0$ on $[0, D/2] \times \mathbb{R}_+$;
 - (iv) $\varphi(0, t) \geq 0$ for each $t \geq 0$.

Then $\varphi(\cdot, t)$ is a modulus of continuity for $v(\cdot, t)$ for each $t \geq 0$.

Proof. The only difference from the situation in [AC2] is the drift term in (7). For any $\varepsilon \geq 0$, define a function Z_ε on $\bar{\Omega} \times \bar{\Omega} \times \mathbb{R}_+$ by

$$Z_\varepsilon(y, x, t) = v(y, t) - v(x, t) - 2\varphi\left(\frac{\|y - x\|}{2}, t\right) - \varepsilon e^t.$$

By assumption, $Z_\varepsilon(x, y, 0) \leq -\varepsilon$ for every $x \neq y$ in Ω , and $Z_\varepsilon(x, x, t) \leq -\varepsilon$ for every $x \in \Omega$ and $t \geq 0$. We will prove for any $\varepsilon > 0$ that Z_ε is negative on $\Omega \times \Omega \times \mathbb{R}_+$. If this is not true, then there exists a first time $t_0 > 0$ and points $x \neq y \in \bar{\Omega}$ such that $Z_\varepsilon(x, y, t_0) = 0$.

We consider two possibilities: If $y \in \partial\Omega$, then we have

$$D_{\nu_y} Z_\varepsilon = D_{\nu_y} v|_{(y,t)} - \varphi' \frac{(y-x)}{\|y-x\|} \cdot \nu_y = -\varphi' \frac{(y-x)}{\|y-x\|} \cdot \nu_y < 0,$$

where ν_y is the outward unit normal at y , and we used the Neumann condition, strict convexity of Ω (which implies $\frac{y-x}{\|y-x\|} \cdot \nu_y > 0$), and assumption 3(iii). This implies that $Z_\varepsilon(x, y - s\nu_y, t_0) > 0$ for s small, contradicting the fact that $Z_\varepsilon \leq 0$ on $\bar{\Omega} \times \bar{\Omega} \times [0, t_0]$. The case where $x \in \partial\Omega$ is similar.

The second possibility is that x and y are in the interior of Ω . Then all first spatial derivatives of Z_ε (in x and y) at (x, y, t_0) vanish, and the full $2n \times 2n$ matrix

of second derivatives is nonpositive. In particular, we choose an orthonormal basis with $e_n = \frac{y-x}{\|y-x\|}$ and will use the inequalities

$$\begin{aligned} \frac{\partial^2}{\partial s^2} Z_\varepsilon(x + se_n, y - se_n, t_0)|_{s=0} &\leq 0, \quad \text{and} \\ \frac{\partial^2}{\partial s^2} Z_\varepsilon(x + se_i, y + se_i, t_0)|_{s=0} &\leq 0 \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

We compute these inequalities in terms of v : The vanishing of first derivatives implies that

$$\nabla v|_y = \nabla v|_x = \varphi' e_n.$$

Along the path $(x + se_i, y + se_i)$ the distance $\|y - x\|$ is constant, so for $i \neq n$,

$$0 \geq \frac{d^2}{ds^2} Z_\varepsilon(x + se_i, y + se_i, t_0) = D_i D_i v|_y - D_i D_i v|_x.$$

Along $(x + se_n, y - se_n)$ we have $\frac{d}{ds} \|y - x\| = -2$ and $\frac{d^2}{ds^2} \|y - x\| = 0$, so

$$0 \geq \frac{d^2}{ds^2} Z_\varepsilon(x + se_n, y - se_n, t_0) = D_n D_n v|_y - D_n D_n v|_x - 2\varphi''.$$

Summing the first inequality over $i = 1, \dots, n-1$ and adding the last gives

$$0 \geq \Delta v|_y - \Delta v|_x - 2\varphi''.$$

Finally we compute the time derivative of Z_ε at (x, y, t_0) :

$$\begin{aligned} \frac{\partial}{\partial t} Z_\varepsilon &= \Delta v|_y + X|_y \cdot \nabla v|_y \\ &\quad - \Delta v|_x - X|_x \cdot \nabla v|_x - 2\varphi_t - \varepsilon e^t \\ &\leq 2\varphi'' + \varphi' \left(X|_y - X|_x \right) \cdot \frac{y-x}{\|y-x\|} - 2\varphi_t - \varepsilon e^t \\ &< 2\varphi'' + 2\omega\varphi' - 2\varphi_t \\ &\leq 0, \end{aligned}$$

where we used the modulus of contraction of X , and the nonnegativity of φ' , as well as the differential inequality for φ assumed in the theorem. But this strict inequality is impossible since this is the first time where $Z_\varepsilon \geq 0$. This contradiction proves that $Z_\varepsilon < 0$ for every $\varepsilon > 0$, and therefore $Z_0 \leq 0$, which is the claim of the theorem. □

3. IMPROVED LOG-CONCAVITY IMPLIES THE GAP CONJECTURE

We will deduce Theorem 1.3 by applying Theorem 2.1 in the case where v is a ratio of solutions to the parabolic equation corresponding to the operator $\Delta + V$. We control the modulus of continuity of such a ratio assuming a modulus of concavity for the logarithm of the denominator:

Proposition 3.1. *Let u_1 and u_0 be two smooth solutions to the parabolic Schrödinger equation*

$$(8) \quad \begin{aligned} \frac{\partial u}{\partial t} &= \Delta u - Vu \quad \text{on } \Omega \times \mathbb{R}_+, \\ u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+, \end{aligned}$$

on a bounded strictly convex domain Ω with smooth boundary in \mathbb{R}^n , with u_0 positive on the interior of Ω . Let $v(x, t) = \frac{u_1(x, t)}{u_0(x, t)}$. Then v is smooth on $\Omega \times \mathbb{R}_+$ and satisfies the following Neumann heat equation with drift:

$$(9) \quad \begin{aligned} \frac{\partial v}{\partial t} - \Delta v - 2\nabla \log u_0 \cdot \nabla v &= 0 \quad \text{on } \Omega \times [0, \infty); \\ D_\nu v &= 0 \quad \text{on } \partial\Omega \times [0, \infty). \end{aligned}$$

Proof. The argument is essentially that of [Y3, Lemma 1.1] and [SWYY, Appendix A]: Both u_0 and u_1 are smooth on $\bar{\Omega} \times [0, \infty)$, and u_0 has negative derivative in the normal direction ν . It follows that v extends to $\bar{\Omega}$ as a smooth function (see [SWYY, Section 6]). We compute directly:

$$(10) \quad \begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t} \left(\frac{u_1}{u_0} \right) \\ &= \frac{\Delta u_1 - V u_1 - v(\Delta u_0 - V u_0)}{u_0} \\ &= \Delta v + 2\nabla \log u_0 \cdot \nabla v. \end{aligned}$$

At any point in $\partial\Omega$ we have $\frac{\partial v}{\partial t}$ and Δv bounded, $\nabla u_0 = -c\nu$ with $c > 0$, and $u_0 = 0$, so it follows from (10) that $D_\nu v = 0$. \square

We now establish the fundamental gap conjecture, assuming Theorem 1.5:

Proposition 3.2. *Inequality (6) implies Theorem 1.3.*

Proof. We use the strategy discussed in the introduction to deduce the eigenvalue bound from an exponentially decaying modulus of continuity bound, which we deduce from Theorem 2.1 and Proposition 3.1.

Suppose that u_1 is any smooth solution of (8), and let u_0 be the solution of (8) with initial data ϕ_0 . Let $v = \frac{u_1}{u_0}$ be the corresponding solution of (9). The drift velocity $X = 2\nabla \log u_0$ in equation (9) has modulus of contraction $2(\log \tilde{\phi}_0)'$, by the estimate (6).

Roughly speaking, the strategy is to apply Theorem 2.1 with $\varphi = C \frac{e^{-\mu_1 t} \tilde{\phi}_1}{e^{-\mu_0 t} \tilde{\phi}_0}$ (that is, a ratio of particular solutions for the corresponding one-dimensional heat equation) and $\omega = 2(\log \tilde{\phi}_0)'$. Condition 2 of Theorem 2.1 holds for sufficiently large C , since $v(\cdot, 0)$ is smooth, and so has bounded gradient and satisfies $|v(y, 0) - v(x, 0)| \leq K\|y - x\|$ for some large K , while $\frac{\tilde{\phi}_1}{\tilde{\phi}_0}$ is positive on $(0, D/2)$ and has positive gradient at $z = 0$, so is bounded below by cz for some small $c > 0$. The conditions 3(i)–(iv) of Theorem 2.1 also hold, except that $\varphi'(D/2, t) = 0$ and ω is not smooth at $D/2$. We handle these complications below by making slight adjustments to φ and ω .

For small $\varepsilon > 0$ we construct a smooth function $\omega_\varepsilon \geq \omega$ and corresponding functions $\varphi_{\varepsilon, \tilde{\varepsilon}}$ as follows: We first replace the Dirichlet eigenfunction $\tilde{\phi}_0$ by a Robin eigenfunction $\tilde{\phi}_{0, \varepsilon}$ with eigenvalue $\mu_{0, \varepsilon} < \mu_0$ defined by

$$(11) \quad \begin{aligned} (\tilde{\phi}_{0, \varepsilon})'' - \tilde{V} \tilde{\phi}_{0, \varepsilon} + \mu_{0, \varepsilon} \tilde{\phi}_{0, \varepsilon} &= 0 \quad \text{on } [0, D/2]; \\ \tilde{\phi}_{0, \varepsilon}(D/2) &= \varepsilon; \quad \tilde{\phi}'_{0, \varepsilon}(D/2) = -1; \\ \tilde{\phi}'_{0, \varepsilon}(0) &= 0; \quad \tilde{\phi}_{0, \varepsilon} > 0 \quad \text{on } [0, D/2]. \end{aligned}$$

We also replace $\tilde{\phi}_1$ by $\tilde{\phi}_{1,\tilde{\varepsilon}}$ with eigenvalue $\mu_{1,\tilde{\varepsilon}}$ defined by

$$(12) \quad \begin{aligned} (\tilde{\phi}_{1,\tilde{\varepsilon}})'' - \tilde{V}\tilde{\phi}_{1,\tilde{\varepsilon}} + \mu_{1,\tilde{\varepsilon}}\tilde{\phi}_{1,\tilde{\varepsilon}} &= 0 \quad \text{on } [0, D/2]; \\ \tilde{\phi}_{1,\tilde{\varepsilon}}(D/2) &= \tilde{\varepsilon}; \quad \tilde{\phi}'_{1,\tilde{\varepsilon}}(D/2) = -1; \\ \tilde{\phi}_{1,\tilde{\varepsilon}}(0) &= 0; \quad \tilde{\phi}_{1,\tilde{\varepsilon}} > 0 \quad \text{on } (0, D/2]. \end{aligned}$$

These may be constructed using the Prüfer transformation: If $\phi'' = (\tilde{V} - \mu)\phi$, then $q = \arctan(\phi'/\phi)$ satisfies a first-order ODE. Inverting this process, we let $q(z, q_0, \mu)$ be the solution of

$$(13) \quad \begin{aligned} \frac{\partial q}{\partial z} - (\tilde{V} - \mu) \cos^2 q + \sin^2 q &= 0, \quad \|z\| \leq D/2, \\ q(0, q_0, \mu) &= q_0. \end{aligned}$$

The ODE comparison theorem implies that q is strictly increasing in q_0 for all z , and strictly decreasing in μ for $z > 0$. The choice $\mu = \mu_0$ and $q_0 = 0$ corresponds to $\tilde{\phi}_0$, and so $q(D/2, 0, \mu_0) = -\pi/2$ and $q(z, 0, \mu_0) \in (-\pi/2, \pi/2)$ for $0 < z < D/2$. Similarly, the choice $\mu = \mu_1$ and $q_0 = \pi/2$ corresponds to $\tilde{\phi}_1$, and so $q(D/2, \pi/2, \mu_1) = -\pi/2$ and $q(z, \pi/2, \mu_1) \in (-\pi/2, \pi/2)$ for $0 < z < D/2$.

Since $q(D/2, 0, \mu)$ is strictly decreasing in μ and $q(D/2, 0, \mu_0) = -\pi/2$, for small $\varepsilon > 0$ there exists a unique $\mu_{0,\varepsilon} < \mu_0$ such that $q(D/2, 0, \mu_{0,\varepsilon}) = \arctan \varepsilon - \pi/2$ and $q(z, 0, \mu_{0,\varepsilon}) \in (-\pi/2, \pi/2)$ for $0 < z < D/2$. The corresponding eigenfunction $\tilde{\phi}_{0,\varepsilon}(z) = \varepsilon \exp\left(-\int_z^{D/2} \tan q(s, 0, \mu_{0,\varepsilon}) ds\right)$ satisfies conditions (11). We have $(\log \tilde{\phi}_{0,\varepsilon})'(z) = \tan q(z, 0, \mu_{0,\varepsilon}) > \tan q(z, 0, \mu_0) = (\log \tilde{\phi}_0)'(z)$ for $z \in (0, D/2]$, since μ was decreased. Thus $\omega_\varepsilon := 2(\log \tilde{\phi}_{0,\varepsilon})'$ is smooth and $\omega_\varepsilon \geq \omega$, so ω_ε is a modulus of contraction for X .

The eigenfunction $\tilde{\phi}_{1,\tilde{\varepsilon}}$ satisfying (12) is similarly produced by decreasing μ below μ_1 with $q_0 = \pi/2$.

Observe that $\arctan(\log \tilde{\phi}_{1,\tilde{\varepsilon}})'(z) = q(z, \pi/2, \mu_{1,\tilde{\varepsilon}})$ satisfies the same ODE (13) as $(\log \tilde{\phi}_{0,\varepsilon})'(z) = q(z, 0, \mu_{0,\varepsilon})$ with larger μ , and they are equal at $z = D/2$. By ODE comparison, $(\log \tilde{\phi}_{1,\tilde{\varepsilon}})'(z) > (\log \tilde{\phi}_{0,\varepsilon})'(z)$ for $0 \leq z < D/2$, and hence if $\tilde{\varepsilon} > \varepsilon > 0$, then $(\log \tilde{\phi}_{1,\tilde{\varepsilon}})'(z) > (\log \tilde{\phi}_{0,\varepsilon})'(z)$ for $0 \leq z \leq D/2$.

For any $\tilde{\varepsilon} > \varepsilon > 0$ we define a smooth function $\varphi_{\varepsilon,\tilde{\varepsilon}}$ by

$$\varphi_{\varepsilon,\tilde{\varepsilon}} = \frac{e^{-\mu_{1,\tilde{\varepsilon}}t} \tilde{\phi}_{1,\tilde{\varepsilon}}}{e^{-\mu_{0,\varepsilon}t} \tilde{\phi}_{0,\varepsilon}}.$$

Since $\varphi_{\varepsilon,\tilde{\varepsilon}}$ is a ratio of solutions of the heat equation, we have

$$\frac{\partial}{\partial t} \varphi_{\varepsilon,\tilde{\varepsilon}} = \varphi''_{\varepsilon,\tilde{\varepsilon}} + 2(\log \tilde{\phi}_{0,\varepsilon})' \varphi'_{\varepsilon,\tilde{\varepsilon}} = \varphi''_{\varepsilon,\tilde{\varepsilon}} + \omega_\varepsilon \varphi'_{\varepsilon,\tilde{\varepsilon}}.$$

The monotonicity requirement 3(iii) is satisfied, since

$$\varphi'_{\varepsilon,\tilde{\varepsilon}} = \varphi_{\varepsilon,\tilde{\varepsilon}} \left((\log \tilde{\phi}_{1,\tilde{\varepsilon}})' - (\log \tilde{\phi}_{0,\varepsilon})' \right) > 0 \quad \text{on } [0, D/2].$$

Hence Theorem 2.1 applies to prove that $C\varphi_{\varepsilon,\tilde{\varepsilon}}(\cdot, t)$ is a modulus of continuity for v for $t > 0$ if it is so at $t = 0$. Sending $\tilde{\varepsilon}$ to ε and then sending ε to zero, we have that $\varphi_{\varepsilon,\tilde{\varepsilon}}$ converges uniformly to φ , so the same conclusion holds for φ . In particular, for sufficiently large C we have

$$\text{osc } v(\cdot, t) \leq 2C \sup\{\varphi(z, t) : z \in [0, D/2]\} = 2Ce^{-(\mu_1 - \mu_0)t}.$$

Applying this with $u_1 = \phi_1 e^{-\lambda_1 t}$ gives $v = e^{-(\lambda_1 - \lambda_0)t} \frac{\phi_1}{\phi_0}$, and hence

$$e^{-(\lambda_1 - \lambda_0)t} \operatorname{osc} \frac{\phi_1}{\phi_0} \leq 2C e^{-(\mu_1 - \mu_0)t}$$

for all $t \geq 0$, which implies $\lambda_1 - \lambda_0 \geq \mu_1 - \mu_0$. This proves Theorem 1.3. \square

4. IMPROVED LOG-CONCAVITY OF THE GROUND STATE

In this section we prove the log-concavity result, Theorem 1.5. We will deduce this from the following statement about log-concavity for arbitrary positive solutions of the parabolic equation (8):

Theorem 4.1. *Let Ω be a uniformly convex open domain with diameter D in \mathbb{R}^n , and assume that \tilde{V}' is a modulus of convexity for the potential V on Ω . Let u_0 be a smooth function on $\bar{\Omega}$ which is positive on Ω and has $u_0 = 0$ and $\nabla u_0 \neq 0$ on $\partial\Omega$, and suppose $\psi_0 : [0, D/2] \rightarrow \mathbb{R}$ is a Lipschitz continuous modulus of concavity for $\log u_0$. Let $u : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be the solution of the Dirichlet problem (8) with initial data u_0 , and let $\psi \in C^0([0, D/2] \times \mathbb{R}_+) \cap C^\infty([0, D/2] \times (0, \infty))$ be a solution of*

$$(14) \quad \left. \begin{aligned} \frac{\partial \psi}{\partial t} &\geq \psi'' + 2\psi\psi' - \tilde{V}' && \text{on } [0, D/2] \times \mathbb{R}_+, \\ \psi(\cdot, 0) &= \psi_0(\cdot), \\ \psi(0, t) &= 0. \end{aligned} \right\}$$

Then $\psi(\cdot, t)$ is a modulus of concavity for $\log u(\cdot, t)$ for each $t \geq 0$.

Note that we impose no boundary condition on ψ at $(D/2, t)$.

Proof. The statement we must prove is the following:

$$Z(x, y, t) := \left(\nabla \log u|_{(y,t)} - \nabla \log u|_{(x,t)} \right) \cdot \frac{y-x}{\|y-x\|} - 2\psi \left(\frac{\|y-x\|}{2}, t \right) \leq 0$$

for all $x \neq y \in \Omega$ and $t \geq 0$. By assumption this holds for $t = 0$. We will prove this by showing negativity of a function Z_ε on $\hat{\Omega} \times [0, T]$ for any $\varepsilon > 0$ and $T \in (0, \infty)$, where

$$\hat{\Omega} = (\Omega \times \Omega) \setminus \{y = x\} \subset \mathbb{R}^{2n}$$

and Z_ε is defined by

$$\begin{aligned} Z_\varepsilon(x, y, t) &= Z(x, y, t) - \varepsilon e^{Ct} \\ &= (\nabla \log u|_y - \nabla \log u|_x) \cdot \frac{y-x}{\|y-x\|} - 2\psi \left(\frac{\|y-x\|}{2}, t \right) - \varepsilon e^{Ct}, \end{aligned}$$

for some suitably large C to be chosen (independent of ε), where we have suppressed the t -dependence of u . By the Hopf boundary point lemma, $D_\nu u(x, t) < 0$ for every $x \in \partial\Omega$ and every $t \geq 0$. The following lemmas control the behaviour of Z near the boundary:

Lemma 4.2. *Let u be as in Theorem 4.1, and $T < \infty$. Then there exists $r_1 > 0$ such that $D^2 \log u|_{(x,t)} < 0$ whenever $d(x, \partial\Omega) < r_1$ and $t \in [0, T]$, and $N \in \mathbb{R}$ such that $D^2 \log u|_{(x,t)}(v, v) \leq N\|v\|^2$ for all $x \in \Omega$ and $t \in [0, T]$.*

Proof. Let $\alpha = \inf_{\partial\Omega \times [0, T]} \|Du\| > 0$, and choose a constant K such that $|D^2u(v, v)| \leq K\|v\|^2$ at every point in $\bar{\Omega} \times [0, T]$ and $v \in \mathbb{R}^n$. If $x_0 \in \partial\Omega$, then $Du|_{x_0} = -\|Du|_{x_0}\| \nu(x_0)$, and $D^2u|_{x_0}(v, v) = h(v, v)Du|_{x_0}(\nu(x_0))$ for v tangent to $\partial\Omega$, where h is the second fundamental form of $\partial\Omega$ at x_0 (see, for example, [GT, §14.6]). Uniform convexity implies that $h(v, v) \geq \kappa\|v\|^2$ for some $\kappa > 0$. The gradient direction $e = \frac{Du}{\|Du\|}$ is smooth near x_0 , as is the projection $\pi^\perp : w \mapsto (w \cdot e)e$ and the orthogonal projection $\pi = \text{Id} - \pi^\perp$. At x_0 we have $D^2u(\pi w, \pi w) \leq -\alpha\kappa\|\pi w\|^2$. Therefore there exists $r_0 > 0$ depending on α and K such that for $x \in B_{r_0}(x_0) \cap \Omega$ and $t \in [0, T]$ we have

$$\begin{aligned} D^2u|_x(\pi w, \pi w) &\leq -\frac{\alpha\kappa}{2}\|\pi w\|^2 \quad \text{for any } w \in \mathbb{R}^n; \\ \|Du|_x\| &\geq \frac{1}{2}\|Du|_{x_0}\|; \\ 0 < u(x) &\leq 2\|Du|_{x_0}\|\|x - x_0\|. \end{aligned}$$

Therefore in this set and for any w , we have

$$\begin{aligned} D^2u(w, w) &= D^2u(\pi w + \pi^\perp w, \pi w + \pi^\perp w) \\ &\leq -\frac{1}{2}\alpha\kappa\|\pi w\|^2 + 2K\|\pi w\|\|\pi^\perp w\| + K\|\pi^\perp w\|^2 \\ &\leq -\frac{1}{4}\alpha\kappa\|\pi w\|^2 + \left(K + \frac{4K^2}{\alpha\kappa}\right)\|\pi^\perp w\|^2. \end{aligned}$$

By definition of π we have $(D_w u)^2 = \|Du\|^2\|\pi^\perp w\|^2 \geq \frac{\alpha\|Du|_{x_0}\|}{4}\|\pi^\perp w\|^2$, and also $u \leq 2\|Du|_{x_0}\|\|x - x_0\|$, so

$$\begin{aligned} D^2 \log u|_x(w, w) &= \frac{1}{u} \left(D^2u(w, w) - \frac{(D_w u)^2}{u} \right) \\ &\leq \frac{1}{u} \left(-\frac{1}{4}\alpha\kappa\|\pi w\|^2 + \left(K + \frac{4K^2}{\alpha\kappa}\right)\|\pi^\perp w\|^2 \right. \\ &\qquad \qquad \qquad \left. - \frac{\alpha}{8\|x - x_0\|}\|\pi^\perp w\|^2 \right) \\ &< 0 \end{aligned}$$

provided $\|x - x_0\| < r_1 = \min \left\{ r_0, \frac{\alpha^2\kappa}{8(K\alpha\kappa + 4K^2)} \right\}$. The lemma holds with $N = \max\{0, \sup\{D^2 \log u(z, t)(v, v) : \|v\| = 1, t \in [0, T], d(z, \partial\Omega) \geq r_1\}\}$, since $\{z \in \Omega : d(z, \partial\Omega) \geq r_1\}$ is compact. \square

Lemma 4.3. *Let u be as in Theorem 4.1, and let ψ be continuous on $[0, D/2] \times \mathbb{R}_+$ and locally Lipschitz in the first argument, with $\psi(0, t) = 0$ for each t . Then for any $T < \infty$ and $\beta > 0$ there exists an open set $U_{\beta, T} \subset \mathbb{R}^{2n}$ containing $\partial\hat{\Omega}$ such that $Z(x, y, t) < \beta$ for all $t \in [0, T]$ and $(x, y) \in U_{\beta, T} \cap \hat{\Omega}$.*

Proof. Since ψ is Lipschitz in the first argument there exists P such that $|\psi(z, t)| \leq Pz$ for all $z \in [0, D/2]$ and $t \in [0, T]$.

We construct $U_{\beta, T}$ of the form $\bigcup_{(x_0, y_0) \in \partial\hat{\Omega}} B_{r(x_0, y_0)}(x_0, y_0)$. To do this we must find $r(x_0, y_0) > 0$ for any $(x_0, y_0) \in \partial\hat{\Omega}$ such that $Z < \beta$ for $(x, y) \in B_{r(x_0, y_0)}(x_0, y_0)$. We consider several cases for (x_0, y_0) :

Case 1: $x_0 = y_0 \in \Omega$. We have by Lemma 4.2,

$$\begin{aligned} Z(x, y, t) &= \left(\nabla \log u|_{(y,t)} - \nabla \log u|_{(x,t)} \right) \cdot \frac{y-x}{\|y-x\|} - 2\psi \left(\frac{\|y-x\|}{2}, t \right) \\ &= \frac{1}{\|y-x\|} \int_0^1 D^2 \log u|_{(1-s)x+sy} (y-x, y-x) ds - 2\psi \left(\frac{\|y-x\|}{2}, t \right) \\ &\leq (N+P)\|y-x\|, \end{aligned}$$

so $Z < \beta$ provided $(x, y) \in B_r(x_0, x_0)$ with $r < \frac{\beta}{2(N+P)}$.

Case 2: $x_0 \in \partial\Omega, y_0 \in \Omega$. In this case $u(y_0) > 0$, so there exists $r_2 > 0$ and $L > 0$ such that $\|D \log u|_y\| \leq L$ for $\|y - y_0\| < r_2$. Let $\alpha_0 = \|Du|_{x_0}\| > 0$ and $\gamma = -\frac{y_0-x_0}{\|y_0-x_0\|} \cdot \nu(x_0)$ (this is positive by strict convexity). Then $Du(x_0) = -\alpha_0 \nu(x_0)$, so $Du|_{x_0} \cdot \frac{y_0-x_0}{\|y_0-x_0\|} = \gamma \alpha_0$. In addition, $Du|_x \cdot \frac{y-x}{\|y-x\|}$ is smooth in x and y near (x_0, y_0) . Therefore $Du|_x \cdot \frac{y-x}{\|y-x\|} \geq \frac{1}{2} \gamma \alpha_0$ and $0 < u(x) \leq 2\alpha_0 \|x - x_0\|$ for $x, y \in \Omega$ with $\max\{\|y - y_0\|, \|x - x_0\|\} < r_3$, for some $r_3 \in (0, r_2]$. This implies that

$$\begin{aligned} Z(x, y, t) &= \left(\nabla \log u|_{(y,t)} - \nabla \log u|_{(x,t)} \right) \cdot \frac{y-x}{\|y-x\|} - 2\psi \left(\frac{\|y-x\|}{2}, t \right) \\ &\leq L - \frac{\gamma}{4\|x-x_0\|} + PD, \end{aligned}$$

where D is diameter, so $Z < 0$ provided $\|y - y_0\| < r_3$ and $\|x - x_0\| < \min\{r_3, \frac{\gamma}{4(L+PD)}\}$.

Case 3: $x_0 \in \Omega, y_0 \in \partial\Omega$. This is similar to Case 2.

Case 4: $x_0, y_0 \in \partial\Omega, x_0 \neq y_0$. Here both x and y can be handled in the same way as x in Case 2.

Case 5: $x_0 = y_0 \in \partial\Omega$. Then letting $z_s = (1-s)x + sy$ we write as in Case 1,

$$Z(x, y, t) = \frac{1}{\|y-x\|} \int_0^1 D^2 \log u|_{z_s} (y-x, y-x) ds - 2\psi \left(\frac{\|y-x\|}{2}, t \right).$$

The first term is negative for $\max\{\|x - x_0\|, \|y - x_0\|\} < r_1$ by Lemma 4.2, so $Z(x, y, t) < \beta$ provided $\max\{\|x - x_0\|, \|y - x_0\|\} < \min\{r_1, \frac{\beta}{P}\}$.

□

Now we proceed with the proof of Theorem 4.1. Fix $T < \infty$ and $\varepsilon > 0$. By assumption we have $Z_\varepsilon(x, y, 0) \leq -\varepsilon e^{Ct} < 0$ for all $x \neq y$ in Ω . Off the compact set $(\hat{\Omega} \setminus U_{\varepsilon/2, T}) \times [0, T]$ we have $Z_\varepsilon \leq -\frac{1}{2}\varepsilon$, by Lemma 4.3. Therefore if Z_ε is not negative on $\hat{\Omega} \times [0, T]$, then there exists a first time $t_0 > 0$ and points $(x, y) \in \hat{\Omega} \setminus U_{\varepsilon/2, T}$ such that $Z_\varepsilon < 0$ on $\hat{\Omega} \times [0, t_0)$, but $Z_\varepsilon(x, y, t_0) = 0$. In particular x and y are in the interior of Ω , and $y \neq x$. We choose an orthonormal basis $\{e_i\}$ for \mathbb{R}^n with $e_n = \frac{y-x}{\|y-x\|}$, and note the following: The first spatial derivatives of Z_ε in x and y

vanish, so

$$(15a) \quad \begin{aligned} 0 &= \frac{\partial}{\partial s} Z_\varepsilon(x + se_i, y, t_0)|_{s=0}, \quad 1 \leq i < n \\ &= -D_i D_n \log u|_x - \frac{D_i \log u|_y - D_i \log u|_x}{\|y - x\|}; \end{aligned}$$

$$(15b) \quad \begin{aligned} 0 &= \frac{\partial}{\partial s} Z_\varepsilon(x, y + se_i, t_0)|_{s=0}, \quad 1 \leq i < n \\ &= D_i D_n \log u|_y + \frac{D_i \log u|_y - D_i \log u|_x}{\|y - x\|}; \end{aligned}$$

while

$$(15c) \quad 0 = \frac{\partial}{\partial s} Z_\varepsilon(x + se_n, y, t_0)|_{s=0} = -D_n D_n \log u|_x + \psi';$$

$$(15d) \quad 0 = \frac{\partial}{\partial s} Z_\varepsilon(x, y + se_n, t_0)|_{s=0} = D_n D_n \log u|_y - \psi'.$$

The second derivative of $Z_\varepsilon(\cdot, \cdot, t_0)$ along any path is nonpositive at (x, y) . In particular, $y - x$ is constant along $(x + se_i, y + se_i)$ for $1 \leq i < n$, so

$$(16a) \quad \begin{aligned} 0 &\geq \frac{\partial^2}{\partial s^2} Z_\varepsilon(x + se_i, y + se_i, t_0)|_{s=0} \\ &= D_i D_i D_n \log u|_y - D_i D_i D_n \log u|_x. \end{aligned}$$

Along the path $(x + se_n, y - se_n)$ we have $\frac{y-x}{\|y-x\|}$ constant, $\frac{d}{ds}\|y-x\| = -2$, and $\frac{d^2}{ds^2}\|y-x\| = 0$. This gives

$$(16b) \quad \begin{aligned} 0 &\geq \frac{\partial^2}{\partial s^2} Z_\varepsilon(x + se_n, y - se_n, t_0)|_{s=0} \\ &= D_n D_n D_n \log u|_y - D_n D_n D_n \log u|_x - 2\psi''. \end{aligned}$$

Finally, since $Z_\varepsilon(x, y, t) < Z_\varepsilon(x, y, t_0)$ for $t < t_0$, we have $\frac{\partial Z_\varepsilon}{\partial t}(x, y, t_0) \geq 0$. We have $\frac{\partial}{\partial t} \log u = \Delta \log u + \|D \log u\|^2 - V$ by (8), and differentiation gives

$$\frac{\partial}{\partial t} \nabla \log u = \Delta \nabla \log u + \sum_{k=1}^n 2D_k \log u D_k \nabla \log u - \nabla V.$$

This gives the following inequality:

$$\begin{aligned}
 0 &\leq \frac{\partial}{\partial t} Z_\varepsilon \\
 &= \left(\Delta \nabla \log u|_y + \sum_{k=1}^n 2D_k \log u D_k \nabla \log u|_y - \nabla V|_y \right. \\
 &\quad \left. - \Delta \nabla \log u|_x - \sum_{k=1}^n 2D_k \log u D_k \nabla \log u|_x + \nabla V|_x \right) \cdot \frac{y-x}{\|y-x\|} \\
 &\quad - 2 \frac{\partial \psi}{\partial t} - C\varepsilon e^{Ct} \\
 &= D_n D_n D_n \log u|_y - D_n D_n D_n \log u|_x \\
 &\quad + \sum_{i=1}^{n-1} (D_i D_i D_n \log u|_y - D_i D_i D_n \log u|_x) \\
 &\quad + 2 (D_n \log u|_y D_n D_n \log u|_y - D_n \log u|_x D_n D_n \log u|_x) \\
 &\quad + 2 \sum_{i=1}^{n-2} (D_i \log u|_y D_i D_n \log u|_y - D_i \log u|_x D_i D_n \log u|_x) \\
 &\quad - (\nabla V|_y - \nabla V|_x) \cdot \frac{y-x}{\|y-x\|} - 2 \frac{\partial \psi}{\partial t} - C\varepsilon e^{Ct}.
 \end{aligned}$$

In the first line we use the second derivative inequality (16b), and in the second we use (16a). In the third line we rewrite the second derivatives of $\log u$ using (15c) and (15d), while in the fourth we use (15a) and (15b). In the last line we use the modulus of convexity assumption (5). This yields

$$\begin{aligned}
 0 &\leq 2\psi'' + 2\psi' (D_n \log u|_y - D_n \log u|_x) \\
 &\quad - \frac{2}{\|y-x\|} \sum_{i=1}^{n-1} |D_i \log u|_y - D_i \log u|_x|^2 - 2\tilde{V}' - 2 \frac{\partial \psi}{\partial t} - C\varepsilon e^{Ct} \\
 &\leq 2\psi'' + 2\psi' (2\psi + \varepsilon e^{Ct}) - 2\tilde{V}' - 2 \frac{\partial \psi}{\partial t} - C\varepsilon e^{Ct} \\
 &< 2 \left(\psi'' + 2\psi\psi' - \tilde{V}' - \frac{\partial \psi}{\partial t} \right) \leq 0.
 \end{aligned}$$

Here the second inequality is obtained by discarding the negative term involving $D_i \log u$, and using $Z_\varepsilon = D_n \log u|_y - D_n \log u|_x - 2\psi - \varepsilon e^{Ct} = 0$. The last inequality is from equation (14), and the strict inequality is produced by choosing $C > 2 \sup_{[0, D/2] \times [0, T]} \psi'$ (this is independent of ε , as required). The result contradicts the assumption that Z_ε does not remain negative. Therefore $Z_\varepsilon < 0$ on $\hat{\Omega} \times [0, T]$ for every $\varepsilon > 0$ and every $T < \infty$. It follows that $Z \leq 0$ on $\hat{\Omega} \times \mathbb{R}_+$, and the theorem is proved. \square

We next consider the long-term behaviour of solutions of (14):

Corollary 4.4. *Under the conditions of Theorem 4.1, there exists a smooth function $\psi : [0, D/2) \times \mathbb{R}_+$, decreasing in the second argument, such that*

$$\left(\nabla \log u|_{(y,t)} - \nabla \log u|_{(x,t)} \right) \cdot \frac{y-x}{\|y-x\|} \leq 2\psi \left(\frac{\|y-x\|}{2}, t \right)$$

for all $x, y \in \Omega$ with $x \neq y$ and all $t \geq 0$, and $\lim_{t \rightarrow \infty} \psi(z, t) = (\log \tilde{\phi}_0)'(z)$ for every $z \in [0, D/2)$.

Proof. We construct a sequence of functions ψ_k satisfying the assumptions of Theorem 4.1 and decreasing monotonically towards the (nonsmooth) function ψ as $k \rightarrow \infty$. These are constructed by solving the equality case of (14) with $\psi(D/2, t) = -k$ for suitable initial data.

The first step is to construct suitable initial functions $\psi_{k,0}$. To do this we observe that there are many stationary solutions, corresponding to solutions of the eigenvalue equation via the Prüfer transformation: Stationary solutions satisfy $0 = \psi'' + 2\psi\psi' - \tilde{V}' = (\psi' + \psi^2 - \tilde{V})'$, and hence

$$(17) \quad \psi' + \psi^2 = \tilde{V} - \mu$$

for some μ . We consider solutions ψ_μ^L and $\psi_{k,\mu}^R$ of (17) satisfying $\psi_\mu^L(0) = 0$ and $\psi_{k,\mu}^R(D/2) = -k$. These relate to the Robin eigenfunctions $\tilde{\phi}_{0,\varepsilon}$ satisfying equation (11), which were constructed in the proof of Proposition 3.2, with $\varepsilon = 1/k$: In particular, $\psi_{\mu_{0,1/k}}^L = \psi_{k,\mu_{0,1/k}}^R = (\log \tilde{\phi}_{0,1/k})'$. The ODE comparison theorem applied to equation (17) implies that ψ_μ^L is strictly decreasing in μ for $0 < z \leq D/2$, and $\psi_{k,\mu}^R$ is strictly increasing in μ for $0 \leq z < D/2$. In particular, $\psi_{k,\mu}^R > (\log \tilde{\phi}_{0,1/k})'$ for $\mu > \mu_{0,1/k}$ and $0 \leq z < D/2$, and $\psi_\mu^L > (\log \tilde{\phi}_{0,1/k})'$ for $\mu < \mu_{0,1/k}$ and $0 < z \leq D/2$. ODE comparison also gives upper bounds: If $\lambda_+^2 \geq \sup \tilde{V} - \mu$ and $\lambda_-^2 \geq \mu - \inf \tilde{V}$, then

$$\begin{aligned} \psi_\mu^L(z) &\leq \lambda_+ \tanh(\lambda_+ z), \quad \text{and} \\ \psi_{k,\mu}^R(z) &\leq \frac{\lambda_- \tan(\lambda_- (D/2 - z)) - k}{1 + \frac{k}{\lambda_-} \tan(\lambda_- (D/2 - z))} \text{ if } z > \frac{D}{2} - \frac{\frac{\pi}{2} + \arctan\left(\frac{k}{\lambda_-}\right)}{\lambda_-}. \end{aligned}$$

The upper and lower bounds imply the existence of a supersolution $\psi_{k,s}^+ = \min\{\psi_{\mu_{0,1/k}-s}^L, \psi_{k,\mu_{0,1/k}+s}^R\}$ for any $s \geq 0$. Comparison gives lower bounds on these for large s : For $s > \max\{\mu_{0,1/k} - \inf \tilde{V}, \sup \tilde{V} - \mu_{0,1/k}\}$,

$$(18) \quad \psi_{k,s}^+(z) \geq \begin{cases} \tilde{\lambda}_+ \tanh(\tilde{\lambda}_+ z), & 0 \leq z \leq z_0, \\ \frac{\tilde{\lambda}_- \tan(\tilde{\lambda}_- (D/2 - z)) - k}{1 + \frac{k}{\tilde{\lambda}_-} \tan(\tilde{\lambda}_- (D/2 - z))}, & z_0 \leq z \leq \frac{D}{2}, \end{cases}$$

where $\tilde{\lambda}_+ = \sqrt{s + \inf \tilde{V} - \mu_{0,1/k}}$, $\tilde{\lambda}_- = \sqrt{s + \mu_{0,1/k} - \sup \tilde{V}}$, and we take $z_0 > \frac{D}{2} - \tilde{\lambda}_-^{-1} \left(\frac{\pi}{2} + \arctan\left(\frac{k}{\tilde{\lambda}_-}\right)\right)$ to make the two cases equal.

Lemma 4.5. *For u_0 as in Theorem 4.1, for each k there exists $s(k) \geq 0$ such that $\psi_{k,s}^+$ is a modulus of concavity for $\log u_0$.*

Proof. By Lemma 4.2, we have for all $(y, x) \in \tilde{\Omega}$,

$$\left(\nabla \log u_0|_y - \nabla \log u_0|_x\right) \cdot \frac{y - x}{\|y - x\|} \leq N\|y - x\| \leq 2\lambda_1 \tanh\left(\frac{\lambda_1 \|y - x\|}{2}\right),$$

where λ_1 is such that $ND \leq 2\lambda_1 \tanh(\lambda_1 D/2)$. Now apply Lemma 4.3 with $\psi(z) = -\frac{6kz}{D}$ and $\beta = k$ to find an open set $U \subset \mathbb{R}^2$ containing $\partial\hat{\Omega}$ such that

$$\left(\nabla \log u_0|_y - \nabla \log u_0|_x\right) \cdot \frac{y-x}{\|y-x\|} \leq -\frac{6k\|y-x\|}{D} + k$$

for $(x, y) \in \hat{\Omega} \cap U$. In particular, there exists $\delta > 0$ such that U contains all points $(x, y) \in \hat{\Omega}$ with $\|y-x\| \geq D-\delta$. Decreasing δ so that $\delta < D/3$ if necessary, we have that for all $(x, y) \in \hat{\Omega}$ with $\|y-x\| \geq D-\delta$,

$$\begin{aligned} \left(\nabla \log u_0|_y - \nabla \log u_0|_x\right) \cdot \frac{y-x}{\|y-x\|} &\leq -\frac{6k(D-\delta)}{D} + k \\ &\leq -2k \\ &\leq 2 \frac{\lambda_2 \tan\left(\lambda_2\left(\frac{D-\|y-x\|}{2}\right)\right) - k}{1 + \frac{k}{\lambda_2} \tan\left(\lambda_2\left(\frac{D-\|y-x\|}{2}\right)\right)} \end{aligned}$$

for any $\lambda_2 > 0$, provided $\frac{D-\|y-x\|}{2} < \frac{\frac{\pi}{2} + \arctan\left(\frac{k}{\lambda_2}\right)}{\lambda_2}$. We choose λ_2 so large that $\frac{\pi}{2} + \arctan\left(\frac{k}{\lambda_2}\right) < \delta\lambda_2$. Then by (18) we have for all $(x, y) \in \hat{\Omega}$,

$$\left(\nabla \log u_0|_y - \nabla \log u_0|_x\right) \cdot \frac{y-x}{\|y-x\|} \leq \psi_{k,s}^+ \left(\frac{\|y-x\|}{2}\right)$$

provided $s \geq \max\{\lambda_1^2 + \mu_{0,1/k} - \inf \tilde{V}, \lambda_2^2 + \sup \tilde{V} - \mu_{0,1/k}\}$. □

Now we proceed to construct the solutions ψ_k : We define

$$s(k) = \inf\{s \geq 0 : \psi_{k,s}^+ \text{ is a modulus of concavity for } \log u_0\}.$$

Then we choose the initial data as follows:

$$\psi_{k,0} = \min\left\{\psi_{j,s(j)}^+ : 1 \leq j \leq k\right\}.$$

Note that $\psi_{k,0}$ is pointwise nonincreasing in k . Now define $\psi_k(z, t)$ by

$$(19) \quad \left. \begin{aligned} \frac{\partial \psi_k}{\partial t} &= \psi_k'' + 2\psi_k \psi_k' - \tilde{V}' \quad \text{on } [0, D/2] \times (0, \infty), \\ \psi_k(\cdot, 0) &= \psi_{k,0}(\cdot), \\ \psi_k(0, t) &= 0, \\ \psi_k(D/2, t) &= -k. \end{aligned} \right\}$$

The solution ψ_k remains between the supersolution $\psi_{k,0}$ and the subsolution $(\log \tilde{\phi}_{0,1/k})'$ and is decreasing in t and in k . The inequality $\frac{\partial \psi_k}{\partial t} \leq 0$ implies bounds on ψ_k' (since the barriers bound ψ_k' at the endpoints), and it follows that ψ_k exists and is Lipschitz in the first argument and continuous on $[0, D/2] \times \mathbb{R}_+$, and smooth on $[0, D/2] \times (0, \infty)$. By the strong maximum principle, $\psi_k(z, t)$ is strictly decreasing in t on $(0, D/2) \times \mathbb{R}_+$ unless $\psi_k = (\log \tilde{\phi}_{0,1/k})'$. It follows that for each k , $\psi_k(\cdot, t)$ converges in C^∞ to $(\log \tilde{\phi}_{0,1/k})'$ as $t \rightarrow \infty$. By Theorem 4.1, $\psi_{k,t}$ is a modulus of concavity for $\log u(\cdot, t)$ for each $t > 0$ and each $k \in \mathbb{N}$, and therefore $\psi(\cdot, t) = \lim_{k \rightarrow \infty} \psi_k(\cdot, t) = \min_{k \in \mathbb{N}} \psi_k(\cdot, t)$ is also. Since ψ_k is bounded independently of k in C^q for every q on compact subsets of $[0, D/2] \times (0, \infty)$, ψ is a smooth

solution of

$$(20) \quad \left. \begin{aligned} \frac{\partial \psi}{\partial t} &= \psi'' + 2\psi\psi' - \tilde{V}' \quad \text{on } [0, D/2) \times (0, \infty), \\ \psi(0, t) &= 0, \\ \lim_{z \rightarrow D/2} \psi(z, t) &= -\infty, \end{aligned} \right\}$$

and $\psi(\cdot, t)$ converges in C^∞ to $(\log \tilde{\phi}_0)'$ as $t \rightarrow \infty$ on compact subsets of $[0, D/2)$. This completes the proof. \square

Proof of Theorem 1.5. Theorem 1.5 follows by applying Corollary 4.4 to the particular solution $u(x, t) = \phi_0(x)e^{-\lambda_1 t}$ of (8). In this case, $\nabla \log u$ is independent of t . Therefore $\psi(\cdot, t)$ is a modulus of concavity for $\log u_0$ for every t , and hence $(\log \tilde{\phi}_0)' = \inf_{t \geq 0} \psi(\cdot, t)$ is also. \square

5. EXAMPLES AND EXTENSIONS

When V is convex, Theorem 1.3 gives Corollary 1.4, since then 0 is a modulus of convexity for V , so we can choose $\tilde{V} = 0$ and the first two eigenfunctions are $\tilde{\phi}_0(z) = \cos(\frac{\pi z}{D})$ with $\mu_0 = \frac{\pi^2}{D^2}$, and $\tilde{\phi}_1(z) = \cos(\frac{2\pi z}{D})$ with $\mu_1 = \frac{4\pi^2}{D^2}$, and hence $\lambda_1 - \lambda_0 \geq \mu_1 - \mu_0 = \frac{3\pi^2}{D^2}$.

A positive lower bound on the Hessian of the potential implies stronger estimates on the gap since a particular modulus of convexity follows:

$$\begin{aligned} (\nabla V|_y - \nabla V|_x) \cdot \frac{y-x}{\|y-x\|} &= \frac{1}{\|y-x\|} \int_0^1 D^2V|_{(1-s)x+sy}(y-x, y-x) ds \\ &\geq K\|y-x\| = 2\tilde{V}'\left(\frac{\|y-x\|}{2}\right), \end{aligned}$$

where $\tilde{V}(z) = \frac{K}{2}z^2$ is the harmonic oscillator potential. Thus the fundamental gap on Ω is bounded below by that of the harmonic oscillator on the interval $[-D/2, D/2]$. This provides a sharp version of [Y3, Theorem 1.2]. The result also extends to entire domains: Suppose V is a potential on \mathbb{R}^n with $D^2V \geq K\delta_{ij}$. Then the eigenvalues λ_i on $B_R(0)$ converge to those on \mathbb{R}^n , so the fundamental gap for this potential on \mathbb{R}^n is no less than that of the one-dimensional harmonic oscillator, which is $\sqrt{2K}$. Equality will hold in the limit of potentials of the form $V = \frac{K}{2}(x_1^2 + n \sum_{i=2}^n x_i^2)$ as $n \rightarrow \infty$.

Our result also gives useful consequences for nonconvex potentials: Consider a one-dimensional potential \tilde{V} which is even and has nonnegative third derivatives. A useful example is the double-well potential $\tilde{V}(z) = -az^2 + bz^4$ with $a, b > 0$. Now consider higher-dimensional potentials of the form

$$V(x) = \tilde{V}(\|x\|) + c \sum_{i=2}^n x_i^2.$$

For large c these are double-well potentials which agree with \tilde{V} along the x_1 -axis. One can check directly that \tilde{V}' is a modulus of convexity for V for any $c \geq 0$, and it follows that the fundamental gap for such a potential on a convex domain Ω of diameter D is bounded below by that of \tilde{V} on the interval $[-D/2, D/2]$. Again, the result is sharp, with equality holding in the limit as $c \rightarrow \infty$ in the case where Ω contains the interval $(-D/2, D/2)$.

An extension of our methods yields sharp results for domains in constant curvature spaces, implying that the spectral gap for a convex domain with convex potential is at least as large as the gap for a one-dimensional comparison problem (Equation (1) in [Kr] with Dirichlet boundary conditions), with equality in the limit of domains which are small neighbourhoods of a geodesic segment. We will provide the details elsewhere.

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MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA AND MATHEMATICAL SCIENCES CENTER, TSINGHUA UNIVERSITY, BEIJING, 100084, PEOPLES REPUBLIC OF CHINA AND MORNINGSIDE CENTER OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, PEOPLES REPUBLIC OF CHINA

E-mail address: Ben.Andrews@anu.edu.au

CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, AUSTRALIAN NATIONAL UNIVERSITY, ACT 0200, AUSTRALIA

E-mail address: Julie.Clutterbuck@anu.edu.au