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ON THE "MAIN CONJECTURE" OF EQUIVARIANT IWASAWA THEORY

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The "main conjecture" of equivariant Iwasawa theory concerns the situation where

- l is a fixed odd prime number and K/k is a Galois extension of totally real number fields with $k/\mathbb{Q}$ and K/k_∞ finite, where k_∞/k is the cyclotomic $\mathbb{Z}_l$-extension (we set $G = G(K/k)$ and $\Gamma_k = G(k_\infty/k)$),
- S is a fixed finite set (which will normally be suppressed in the notation) of primes of k containing all primes which ramify in K and all archimedean primes, and M is the maximal abelian l-extension of K unramified outside S (we set $X = G(M/K)$).

It asserts that a canonical refinement $\mho = \mho_S$ of the Iwasawa module X is determined by the Iwasawa L-function $L_{K/k} = L_{K/k,S}$ of K/k. The data $\mho$ and $L_{K/k}$ (and $\mathrm{Hom}^*(R_l(G), (\mathcal{Q}^{\mathrm{c}}\Gamma_k)^\times)$ below) have been defined in [RW2, pp. 562–563, p. 568][1]; we will not repeat the definitions now (only in §1) but will briefly explain how $L_{K/k}$ should determine $\mho$.

Denote by ΛG, $\mathcal{Q}G$ the completed group ring $\mathbb{Z}_l[[G]]$ of G over $\mathbb{Z}_l$ and its total ring of fractions, respectively. The localization sequence of K-theory

$$\to K_1(\Lambda G) \to K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0T(\Lambda G) \to K_0(\mathcal{Q}G) \to$$

has $\mho$ in $K_0T(\Lambda G)$.

The reduced norms of the Wedderburn components of the semisimple algebra $\mathcal{Q}G$ induce the map

$$\mathrm{Det} : K_1(\mathcal{Q}G) \to \mathrm{Hom}^*(R_l(G), (\mathcal{Q}^{\mathrm{c}}\Gamma_k)^\times).$$

The Iwasawa L-function $L_{K/k}$ of K/k is derived from the S-truncated l-adic Artin L-functions and belongs to the above group Hom^*.

The equivariant "main conjecture" asserts that *there is a unique element* $\Theta \in K_1(\mathcal{Q}G)$ *satisfying* $\mathrm{Det}(\Theta) = L_{K/k}$ *and, moreover, that this* Θ *has* $\partial(\Theta) = \mho$. In other words, Θ is the non-abelian generalization to K/k of the pseudomeasure of an abelian field extension.

THEOREM. *If Iwasawa's μ-invariant $\mu_{K/k}$ vanishes, then the "main conjecture" of equivariant Iwasawa theory for K/k holds, up to its uniqueness statement.*

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[1]Where $\mho$ is denoted $\tilde{\mho}$.

Our motivation for formulating a "main conjecture" is discussed in [RW1, §4]: the Lifted Root Number Conjecture suggested a refinement of the Main Conjecture of classical Iwasawa theory. Of course, we should add that the paper is strongly based on classical work on totally real fields as done by, e.g., Iwasawa, Kubota Leopoldt, Greenberg, Ferrero Washington, and Mazur Wiles.

There are now much more general conjectures on non-abelian Iwasawa theory, especially in [FK][2] and [Kt1], which also have extensive bibliographies. For more recent related work see [Kt2], [Kk], [Ha1], and [Ha2].

The proof of the THEOREM depends on our previous work, which we summarize below and, in more detail, in §1.

The hypothesis $\mu_{K/k} = 0$, as conjectured by Iwasawa[3], allows us to work in the localization $\Lambda_\bullet G$ of ΛG obtained by inverting all central elements which are regular modulo l.[4] We then can investigate the structure of $K_1(\Lambda_\bullet G)$ via the integral logarithm $\mathbb{L} : K_1(\Lambda_\wedge G) \to T(\mathcal{Q}_\wedge G)$ ($= \mathcal{Q}_\wedge G/[\mathcal{Q}_\wedge G, \mathcal{Q}_\wedge G]$) where $\Lambda_\wedge G \stackrel{\text{def}}{=} \lim\limits_{\overleftarrow{n}} \Lambda_\bullet G/l^n \Lambda_\bullet G$ is the completion of $\Lambda_\bullet G$ and $\mathcal{Q}_\wedge G$ is the total ring of fractions of $\Lambda_\wedge G$ (see (1.D1) in §1). Proposition 1.1 restates the THEOREM by bringing in $K_1(\Lambda_\wedge G)$. However, we lose the uniqueness assertion of the conjecture; we accepted this because uniqueness would follow from $SK_1(\mathcal{Q}G) = 1$, as conjectured by Suslin (see [RW2, Remark E]). Note that $\mho$ no longer appears in Proposition 1.1. This is a consequence of the Main Conjecture of classical Iwasawa theory, proved by Wiles [Wi].

Proposition 1.1 needs to be verified only for *l-elementary extensions K/k*, i.e., those for which $G(K/k) = \langle z \rangle \times G[l]$ is a direct product of a finite cyclic group $\langle z \rangle$ of order $|z|$ prime to l and a pro-l group $G[l]$ (see Proposition 1.2).[5]

Diagram (1.D1) also yields a logarithmic interpretation of the Iwasawa L-function $L_{K/k}$, the *logarithmic pseudomeasure* $t_{K/k} \in T(\mathcal{Q}_\wedge G)$. Now $L_{K/k} \in \mathrm{Det} K_1(\Lambda_\wedge G)$ if $t_{K/k} \in T(\Lambda_\wedge G)$, by Proposition 1.3, the proof of which requires certain *torsion congruences* between abelian pseudomeasures to be verified in the special case when $G(K/k)$ has an abelian subgroup of index l. The validity of these congruences is deduced from the work of Deligne and Ribet [DR] in [RW7].

Combining the three propositions of §1 we obtain

Theorem 1. *The* THEOREM *is true if and only if $t_{K/k} \in T(\Lambda_\wedge G(K/k))$ for all l-elementary extensions K/k.*

In the special case when $G(K/k)$ has an abelian subgroup of index l this integrality of the logarithmic pseudomeasure has already been proved in [RW8], which contains most of the ideas of the present paper in embryonic form. They concern the Wall congruence (see Theorem 2), the torsion congruence (see Theorem 3), and a new ingredient, the restriction $\mathrm{Res}_G^U : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge U)$ for open subgroups $U \geq \langle z \rangle$ of G. Indeed, parts of the proof of Theorem 4, in §5, are easier versions of arguments appearing already in [RW8] but repeated here for the convenience of the reader.

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[2] It may be useful to mention that in [FK] the role of $\mho$ is played by invariants of complexes involving naturally occurring étale cohomology.

[3] I.e., the $\Lambda\Gamma$-module X has μ-invariant 0 for some open subgroup $\Gamma \leq G$ isomorphic to $\mathbb{Z}_l$.

[4] Compare also Remark 1.1 in §1.

[5] Actually, the uniqueness assertion of the "main conjecture" can also be reduced to l-elementary K/k; see [La, Chapter 4].

From now on, K/k is always l-elementary if not otherwise implied.

Let μ_Q denote the Möbius function of the partially ordered set of subgroups of the finite l-group Q. Recall that $\mu = \mu_Q$ is defined by

$$\mu(1) = 1, \qquad \mu(Q') = -\sum_{1 \leq Q'' < Q'} \mu(Q'') \quad \text{for } 1 \neq Q' \leq Q.$$

Theorem 2 (Möbius-Wall)**.** *Let A be an abelian normal open subgroup of $G = G(K/k)$ so that $Q = G/A$ is a finite l-group. If ϵ is a unit of $\Lambda_\wedge G$, then*

$$\sum_{A \leq U \leq G} \mu_Q(U/A) \mathrm{ver}_U^A (\mathrm{res}\,_G^U \epsilon) \equiv 0 \mod \mathrm{tr}_Q(\Lambda_\wedge A).$$

Here $\mathrm{ver}_U^A : \Lambda_\wedge U \to \Lambda_\wedge A$ extends the group transfer $U \to U^{\mathrm{ab}} \to A$ to a ring homomorphism between their Iwasawa algebras in the customary way[6]. Theorem 2 is proved in §2.

Let S' denote the set of all non-archimedean primes of S and let $k \subseteq f \subset K$ with $[f : k] < \infty$. The pseudomeasure $\lambda_f = \lambda_{f,S'}$ of [Se] is associated to the maximal abelian S'-ramified extension $f_{S'}$ of f. We extend this notation to intermediate fields F of $f_{S'}/f_\infty$ with $[F : f_\infty] < \infty$ by defining $\lambda_{F/f} = \mathrm{defl}_{G(f_{S'}/f)}^{G(F/f)} \lambda_f$ in the sense of Lemma 4.1. Now we can state

Theorem 3. *Using notation as in Theorem 2,*

$$\sum_{A \leq U \leq G} \mu_Q(U/A) \mathrm{ver}_{U^{\mathrm{ab}}}^A (\lambda_{K^{[U,U]}/K^U}) \equiv 0 \mod \mathrm{tr}_Q(\Lambda_\wedge A).$$

This is proved in §4. It is a consequence of Theorem 5, which is stated in §3, and thus of relations between constant terms of Hilbert modular forms which can be studied by means of the q-expansion principle of Deligne and Ribet [DR].

The statement and proof of Theorem 5 in §3 depend on the methods of [RW7] which combine those of [DR] and [Se]: therefore the language of §3 continues the same compromise between that of [DR], [Se] as in [RW7] except that the Latin K of the latter becomes the calligraphic $\mathcal{K}$ of §3. We stress that it is *not* the language of the other sections of this paper (compare the first paragraph of §4).

Note that the congruence of Theorem 3 is a necessary condition for the "main conjecture" of equivariant Iwasawa theory, for it implies $\mathrm{Det}(\epsilon) = L_{K/k}$ for some $\epsilon \in K_1(\Lambda_\wedge G)$, by Proposition 1.1, and substituting this ϵ in Theorem 2 yields Theorem 3. Conversely, fusing Theorems 2 and 3 leads to a partial generalization of the proof of [RW8, Theorem] and finally to

Theorem 4. *Let $A \geq \langle z \rangle$ be an abelian normal open subgroup of $G = G(K/k)$ and let C be a central subgroup of exponent l contained in A. If $t_{K^C/k}$ is integral, i.e., $t_{K^C/k} \in T(\Lambda_\wedge G/C)$, then there exists a $\xi \in T(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{G/C} \xi = t_{K^C/k}$ and $\mathrm{Res}_G^A \xi = t_{K/K^A}$.*

Theorems 2, 3, and 4, can be viewed as generalizing [RW8]. The real strength of Theorem 4 is that it serves as a catalyst for the proof of the THEOREM in §6 by making suitable modifications in $T(\Lambda_\wedge G)$ of the element ξ provided by Theorem 4.

[6]The inverse system of $\mathbb{Z}_l$-linear extensions $\mathbb{Z}_l[U/V] \to \mathbb{Z}_l[A/V]$ of the group transfer $U/V \to A/V$, where V runs through the normal open subgroups of U contained in A, gives rise to the transfer $\Lambda U \to \Lambda A$, which is a ring homomorphism that can be localized and completed. Note that $\mathrm{ver}_U^A = \mathrm{ver}_{U^{\mathrm{ab}}}^A \mathrm{defl}_U^{U^{\mathrm{ab}}}$ factors through U^{ab}.

Section 7 contains the necessary extension of the integral logarithm to l-elementary groups. This is based on using projections to the integral logarithm for pro-l groups with unramified coefficients, which is already in [RW3]. It also discusses 'Res' and the l-elementary ingredients of the proof of Proposition 1.3 in §1.

Finally, in a short appendix, we take the opportunity to correct the proof of [RW2, Proposition 12]. In it we have referred to [RWt] where, however, Leopoldt's conjecture is assumed to hold. In the appendix we now outline an argument[7] which is not based on this conjecture.

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Added in proof. While this paper was with the referee, the authors learned about D. Burns's paper *On main conjectures in non-commutative Iwasawa theory and related conjectures* (December 2010; submitted for publication) as well as the one by M. Kakde, *The main conjecture of Iwasawa theory for totally real fields* (arXiv:1008.0142 [math.NT]). The first describes various applications of the THEOREM to the equivariant Tamagawa number and other conjectures; it also shows how it implies the main conjecture of non-commutative Iwasawa theory for Tate motives over compact l-adic Lie extensions of totally real fields of arbitrary finite rank [FK]. The second gives an alternative proof of the THEOREM with respect to the localized K_1 of [FK].

1. "Proof" of Theorem 1

We begin by quickly recalling some basic definitions.

(1) $\mho$: The Galois group sequence $X \rightarrowtail G(M/k) \twoheadrightarrow G$ is turned into a canonical ΛG-module sequence which is the bottom row of

$$\begin{array}{ccccccc} & & \Lambda G & = & \Lambda G & & \\ & & \psi \downarrow & & \psi' \downarrow & & \\ X & \rightarrowtail & Y & \to & \Lambda G & \overset{\mathrm{aug}}{\twoheadrightarrow} & \mathbb{Z}_l . \end{array}$$

In it Y has projective dimension ≤ 1 and ψ' has $\mathrm{aug} \circ \psi' = 0$.

Clearly, the cokernels of ψ and ψ' are torsion of finite projective dimension which permits us to set $\mho = [\operatorname{coker} \psi] - [\operatorname{coker} \psi']$ in $K_0T(\Lambda G)$; $\mho$ is independent of the choice of ψ', ψ. (See [RW1, §1] and [RW2, pp. 562–563].)

(2) $L_{K/k}$: Choosing a topological generator γ_k of Γ_k gives a $u \in 1 + l\mathbb{Z}_l$ so that $\zeta_{l^\infty}^{\gamma_k} = \zeta_{l^\infty}^{u}$. By [Ca], [DR], [Gr], and [Wi] the S-truncated l-adic Artin L-functions $L_{l,S}(s,\chi)$ of K/k for characters χ of irreducible ${\mathbb{Q}_l}^{\mathrm{c}}$-representations of G with open kernel admit a representation $L_{l,S}(1-s,\chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)}$ with $G_{\chi,S}(T) \in {\mathbb{Z}_l}^{\mathrm{c}}[[T]]$ and $H_\chi(T) = 1$ or $= \chi(\gamma_k)(1+T) - 1$ according to $\operatorname{res}^H_G \chi \neq 1$ or $= 1$.[8] Set $L_{K/k}(\chi) = \frac{G_{\chi,S}(\gamma_k-1)}{H_\chi(\gamma_k-1)}$ in $(\mathcal{Q}^{\mathrm{c}}\Gamma_k)^\times = ({\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l} \mathcal{Q}\Gamma_k)^\times$; this is independent of the choice of γ_k. (See [RW2, pp. 563, 571].)

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[7] As the referee has pointed out to us, the descent property discussed in the appendix could also be deduced from interpreting $\mho$ in terms of the compactly supported étale cohomology of $\mathbb{Z}_l(1)$.

[8] ${\mathbb{Z}_l}^{\mathrm{c}}$ denotes the integers in a fixed algebraic closure ${\mathbb{Q}_l}^{\mathrm{c}}$ of $\mathbb{Q}_l$ and $H = \ker(G \twoheadrightarrow \Gamma_k)$.

(3) $\mathrm{Hom}^*(R_l(G), (\mathcal{Q}^c\Gamma_k)^\times)$: Here $R_l(G)$ is the character ring of all the χ above and Hom^* consists of all $f : R_l(G) \to (\mathcal{Q}^c\Gamma_k)^\times$ satisfying $f(\chi^\sigma) = f(\chi)^\sigma$ for $\sigma \in G({\mathbb{Q}_l}^c/\mathbb{Q}_l)$ and $f(\chi \otimes \rho) = \rho^\sharp(f(\chi))$ for all irreducible characters ρ with $H \subset \ker \rho$, where $\rho^\sharp$ is the ${\mathbb{Q}_l}^c$-automorphism of $\mathcal{Q}^c\Gamma_k$ induced by $\rho^\sharp(\gamma_k) = \rho(\gamma_k)\gamma_k$. (Compare also [RW2, p. 558].)

(4) HOM: This notation adds the further condition $f(\chi)^l \equiv \Psi(f(\psi_l\chi)) \mod l\Lambda^c\Gamma_k$ to Hom^*, with $\Lambda^c\Gamma_k = {\mathbb{Z}_l}^c \otimes_{\mathbb{Z}_l} \Lambda\Gamma_k$, ψ_l the l-th Adams operation on $R_l(G)$ (so $(\psi_l\chi)(g) = \chi(g^l)$ for $g \in G$) and Ψ the ring endomorphism of $\Lambda\Gamma_k$ induced by $\gamma_k \mapsto \gamma_k^l$. (See [RW3, p. 37].)

We are now in a position to explain the main ingredients of the proof of Theorem 1.

Proposition 1.1. *The* THEOREM *holds if and only if* $L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$.

Proof. For pro-l extensions K/k this is Theorem A in [RW3] which is reduced via the classical Main Conjecture, as in [RW3, §1], to the stronger form of Theorem B in [RW3, §6]. It is here that the logarithm $\mathbb{L}$ appears first; it is defined by the commutative square (see [RW3, Proposition 11])

$$(1.D1)\qquad \begin{array}{ccc} K_1(\Lambda_\wedge G) & \overset{\mathbb{L}}{\to} & T(\mathcal{Q}_\wedge G) \\ \mathrm{Det}\downarrow & & \mathrm{Tr}\downarrow\simeq \\ \mathrm{HOM}(R_l(G), (\Lambda^c_\wedge\Gamma_k)^\times) & \overset{\mathbf{L}}{\to} & \mathrm{Hom}^*(R_l(G), \mathcal{Q}^c_\wedge\Gamma_k), \end{array}$$

with $T(R) = R/[R,R]$ for any ring R, where $[R,R]$ is the additive subgroup generated by all Lie commutators $[a,b] = ab - ba$, $a, b \in R$, and with the isomorphism 'Tr' induced by the reduced trace of $\mathcal{Q}_\wedge G$. Above, $(\mathbf{L}f)(\chi) = \frac{1}{l}\log\frac{f(\chi)^l}{\Psi(f(\psi_l\chi))}$. Note that every $f \in \mathrm{Det}K_1(\Lambda_\wedge G)$ satisfies the extra congruence for HOM; note also that $L_{K/k} \in \mathrm{HOM}$ [RW3, pp. 34, 42].

The second basic tool in the proof of [RW3, Theorem B] is the *Wall congruence* (i.e., the special case $|Q| = l$ of Theorem 2); see [RW3, Lemma 12]. It also plays an important role later.

The generalization to arbitrary extensions K/k is carried out in [RW4, Theorems (A) and (B)] by the induction techniques there. □

The logarithm $\mathbb{L}$ is called *integral* when it takes values in $T(\Lambda_\wedge G) \subseteq T(\mathcal{Q}_\wedge G)$. We should stress that (1.D1) is available to define $\mathbb{L}$ for arbitrary groups $G = G(K/k)$. This also applies to much of [RW3], in which the pro-l hypothesis is needed only in §1, Theorem 8, Proposition 11, and §6 of [RW3] (where it is explicitly stated)[9]. The main exception is the integrality property, $\mathbb{L}(K_1(\Lambda_\wedge G)) \subseteq T(\Lambda_\wedge G)$, which nevertheless holds when G is l-elementary. This will be discussed in §7.

As a direct consequence, there is a unique element $t_{K/k} \in T(\mathcal{Q}_\wedge G)$, the *logarithmic pseudomeasure* of K/k, such that $\mathrm{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$. In the l-elementary case, by (1.D1) and (i) of Lemma 7.1 in §7, $L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$ implies $t_{K/k} \in T(\Lambda_\wedge G)$, which is the easy implication in Proposition 1.3 below.

Proposition 1.2. *If* $L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$ *holds whenever* K/k *is* l*-elementary, then* $L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$ *for arbitrary extensions* K/k.

Proof. This is [RW4, Theorem (C)]. □

[9] $\mu = 0$ is not needed for Theorem 9 of [RW3], but only for its corollary.

Proposition 1.3. *If K/k is l-elementary, then '$L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$' and '$t_{K/k} \in T(\Lambda_\wedge G)$' are equivalent.*

Proof. It remains to discuss the implication '$t_{K/k} \in T(\Lambda_\wedge G) \implies L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$', which is proved in [RW5], [RW7] for pro-l extensions K/k. Indeed, [RW5, Proposition 2.4] yields a unique torsion element $w \in \mathrm{HOM}(R_l(G), (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times)$ deflating to 1 on applying $\mathrm{defl}_G^{G^{\mathrm{ab}}}$ such that $wL_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$; this is extended to the l-elementary case in (i) of Lemma 7.3. Then, provided that $t_{K/k} \in T(\Lambda_\wedge G)$, [RW5, Theorem, p. 1096] reduces the question of whether $L_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$ to the analogous one for (all intermediate) Galois extensions K/k with $G = G(K/k)$ having an abelian subgroup of index l: this carries over to the l-elementary case without change. In order to deduce that $w = 1$, we may thus assume that G has an abelian subgroup A of index l and then apply the extension in Lemma 7.3(ii) of [RW5, Proposition 3.2] to l-elementary groups to obtain the equivalence

$$(1.1) \qquad w = 1 \iff \mathrm{ver}(\lambda_{K^{[G,G]}/k}) \equiv \lambda_{K/K^A} \mod \mathrm{tr}_{G/A}(\Lambda_\wedge A).$$

Here, $\lambda_{F/f}$ is the pseudomeasure for the extension F/f and $\mathrm{ver} : \Lambda_\wedge G^{\mathrm{ab}} \to \Lambda_\wedge A$ is induced by the group transfer $G^{\mathrm{ab}} \to A$. Note that the proof of [RW5, Proposition 3.2] depends on the Wall congruence mentioned above. This *torsion congruence* (the right-hand side of (1.1)) is proved in [RW7] (or §3) by interpreting the methods of [DR] as in [Se].

We point out right away that the validity of the torsion congruence persists for l-elementary extensions in Theorem 3 because every open subgroup of index l in G contains $\langle z\rangle$; since $K \subset L_{S'}^+$ (in the notation of the first paragraph of §4), we can specialize the group $H_{S'}^+$ there to our A. □

Theorem 1 is now a direct consequence of the three propositions. □

We close this section by displaying a commutative diagram which was the motivation for the definition of $\mathrm{Res}_G^U : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge U)$ for open subgroups U of G and which will be important in the discussions in §§5 and 6:

$$(1.\mathrm{D}2) \qquad \begin{array}{ccccc} K_1(\Lambda_\wedge G) & \stackrel{\mathbb{L}}{\to} & T(\mathcal{Q}_\wedge G) & \stackrel{\mathrm{Tr}}{\to} & \mathrm{Hom}^*(R_l(G), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k) \\ \mathrm{res}_G^U \downarrow & & \mathrm{Res}_G^U \downarrow & & \mathrm{Res}_G^U \downarrow \\ K_1(\Lambda_\wedge U) & \stackrel{\mathbb{L}}{\to} & T(\mathcal{Q}_\wedge U) & \stackrel{\mathrm{Tr}}{\to} & \mathrm{Hom}^*(R_l(U), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_{K^U}). \end{array}$$

Provided that G is pro-l, the actual formula for $\mathrm{Res}_G^U(f)$, $f \in \mathrm{Hom}^*(R_l(G), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k)$, is in [RW8, §1 and Appendix]; its extension to pairs (G, U) with l-elementary G and open $U \le G$ containing $\langle z\rangle$ is again in §7; see Lemma 7.2.

Remark 1.1. For the convenience of the reader, it should perhaps be added that $\Lambda_\bullet G = \Sigma^{-1}\Lambda G$ with $\Sigma = \Lambda\Gamma \setminus l \cdot \Lambda\Gamma$ for any central open subgroup $\Gamma \simeq \mathbb{Z}_l$ of an arbitrary $G = G(K/k)$. Note that $\Sigma^{-1}\Lambda\Gamma$ has the unique maximal ideal $l\Sigma^{-1}\Lambda\Gamma$. So it suffices to show that every element $c \in \Sigma^{-1}\Lambda G$, which is (left) regular modulo l, is a unit of $\Sigma^{-1}\Lambda G$. For this consider right multiplication $\Sigma^{-1}\Lambda G \xrightarrow{\cdot c} \Sigma^{-1}\Lambda G$ by c. Since $\Sigma^{-1}\Lambda G/l$ is a finite-dimensional $\Sigma^{-1}\Lambda\Gamma/l$-vector space, $c \mod l$ has a (left) inverse in $\Sigma^{-1}\Lambda G/l$, and hence c has a left inverse b in $\Sigma^{-1}\Lambda G$ by Nakayama's lemma. Since $b \mod l$ is now also (left) regular modulo l, the same argument provides $a \in \Sigma^{-1}\Lambda G$ with $ab = 1$. Then $a = abc = c$, so c is a unit.

2. Proof of Theorem 2

Fix a set of coset representatives r_q of A in G, whence $G = \dot{\bigcup}_{q\in Q} r_q A$, $q = r_q A$, and $r_{q_1} r_{q_2} = r_{q_1 q_2} a_{q_1,q_2}$ with $a_{q_1,q_2} \in A$ a 2-cocycle, so $a_{q_1,q_2q_3} a_{q_2,q_3} = a_{q_1q_2,q_3} a_{q_1,q_2}^{q_3}$. Further, let $\Sigma = \mathrm{Sym}(Q)$ denote the symmetric group on the elements of Q. It carries the natural (right) Q-action

$$\pi^q(q_1) = \pi(q_1 q^{-1})q, \quad q, q_1 \in Q,\, \pi \in \Sigma,$$

satisfying $(\pi_1\pi_2)^q = \pi_1^q \pi_2^q$. For $V \le Q$, the set of fixed points Σ^V of V in Σ is thus a subgroup of Σ. Note that $\pi \in \Sigma^V$ has $\pi(qv) = \pi(q)v$ for all $q \in Q$, $v \in V$.

Lemma 2.1. *Let U be a subgroup of G containing A, set $V = U/A$, and fix a section $\kappa : Q/V \to Q$, so $(\kappa s)V = sV$ for $s \in Q/V$. Let $\epsilon = \sum_{q\in Q} r_q e_q$, with $e_q \in \Lambda_\wedge A$, be a unit in $\Lambda_\wedge G = \bigoplus_{q\in Q} r_q \cdot \Lambda_\wedge A$. Then*

$$\mathrm{ver}_U^A \mathrm{res}_G^U \epsilon = \sum_{\pi\in\Sigma^V} \mathrm{sgn}(\pi) \prod_{q\in Q} a_{\pi(q)q^{-1},q} \prod_{s\in Q/V} \mathrm{ver}_U^A(e^{\kappa(s)}_{\pi(\kappa s)\kappa(s)^{-1}}) \ .$$

Proof. Writing $\Lambda_\wedge G = \bigoplus_{s\in Q/V} r_{\kappa(s)} \Lambda_\wedge U$, then

$$\epsilon r_{\kappa(s_1)} = \sum_{s_2\in Q/V} r_{\kappa(s_2)} \Big(\sum_{v\in V} r_v a^{-1}_{\kappa(s_2),v} a_{\kappa(s_2)v\kappa(s_1)^{-1},\kappa(s_1)} \cdot e^{\kappa(s_1)}_{\kappa(s_2)v\kappa(s_1)^{-1}} \Big)$$

(with the term in parentheses in $\Lambda_\wedge U$). The ring homomorphism $\mathrm{ver}_U^A : \Lambda_\wedge U \to \Lambda_\wedge A$ induces the map $\mathrm{ver}_U^A : K_1(\Lambda_\wedge U) \to K_1(\Lambda_\wedge A) \overset{\det}{=} (\Lambda_\wedge A)^\times$, and we compute $\mathrm{ver}_U^A \mathrm{res}_G^U \epsilon$ by applying ver_U^A to the matrix of the action of ϵ on the right $\Lambda_\wedge U$-module $\Lambda_\wedge G$ to get

$$\begin{aligned}
&\mathrm{ver}_U^A \mathrm{res}_G^U \epsilon \\
&= \sum_{\sigma\in \mathrm{Sym}(Q/V)} \mathrm{sgn}(\sigma) \prod_{s\in Q/V} \Big(\sum_{v\in V} \mathrm{ver}_U^A (r_v a^{-1}_{\kappa(\sigma s),v} a_{\kappa(\sigma s)v\kappa(s)^{-1},\kappa(s)} \cdot e^{\kappa(s)}_{\kappa(\sigma s)v\kappa(s)^{-1}}) \Big) \\
&= \sum_{\sigma\in \mathrm{Sym}(Q/V)} \mathrm{sgn}(\sigma) \sum_f \prod_{s\in Q/V} \mathrm{ver}_U^A \Big(r_{f(s)} a^{-1}_{\kappa(\sigma s),f(s)} a_{\kappa(\sigma s)f(s)\kappa(s)^{-1},\kappa(s)} \cdot e^{\kappa(s)}_{\kappa(\sigma s)f(s)\kappa(s)^{-1}} \Big)
\end{aligned}$$

where f varies over all functions $Q/V \to V$; hence

$$\begin{aligned}
\text{(2.a)} \qquad &\mathrm{ver}_U^A \mathrm{res}_G^U \epsilon \\
&= \sum_{\sigma\in \mathrm{Sym}(Q/V)} \mathrm{sgn}(\sigma) \sum_f \Big(\prod_{s\in Q/V} \prod_{v\in V} a_{f(s),v} a^{-v}_{\kappa(\sigma s),f(s)} a^v_{\kappa(\sigma s)f(s)\kappa(s)^{-1},\kappa(s)} \Big) \\
&\quad \times \prod_{s\in Q/V} \mathrm{ver}_U^A (e^{\kappa(s)}_{\kappa(\sigma s)f(s)\kappa(s)^{-1}})
\end{aligned}$$

because $\mathrm{ver}_U^A(r_{f(s)}) = \prod_{v\in V} a_{f(s),v}$ and $\mathrm{ver}_U^A a = \prod_{v\in V} a^v$ for $a \in A$.

We next simplify the above double product for a fixed $f : Q/V \to V$ to get

$$\text{(2.b)} \qquad \textstyle\prod_{s\in Q/V} \prod_{v\in V} a_{\kappa(\sigma s)f(s)\kappa(s)^{-1},\kappa(s)v}.$$

Namely, the cocycle relation for the triple $(\kappa(\sigma s), f(s), v)$,

$$a_{f(s),v} a^{-v}_{\kappa(\sigma s),f(s)} = a^{-1}_{\kappa(\sigma s),f(s)v} a_{\kappa(\sigma s)f(s),v},$$

turns the double product into

$$\prod_{s\in Q/V} \prod_{v\in V} a^{-1}_{\kappa(\sigma s),f(s)v} a_{\kappa(\sigma s)f(s),v} a^v_{\kappa(\sigma s)f(s)\kappa(s)^{-1},\kappa(s)},$$

and from the triple $(\kappa(\sigma s)f(s)\kappa(s)^{-1}, \kappa(s), v)$ it then becomes

$$\prod_{s\in Q/V}\prod_{v\in V} a^{-1}_{\kappa(\sigma s), f(s)v}\, a_{\kappa(s),v}\, a_{\kappa(\sigma s)f(s)\kappa(s)^{-1},\kappa(s)v}.$$

Now the substitutions $v \rightsquigarrow f(s)^{-1}v$, $s \rightsquigarrow \sigma^{-1}(s)$ yield

$$\prod_{s\in Q/V}\prod_{v\in V} a_{\kappa(\sigma s), f(s)v} = \prod_{s\in Q/V}\prod_{v\in V} a_{\kappa(s),v}$$

confirming (2.b). We continue by reparametrising the maps $f : Q/V \to V$ in (2.a) in terms of the kernel Σ_0^V of the group homomorphism

$$\Sigma^V \overset{\sim}{\to} \mathrm{Sym}(Q/V), \quad \pi \mapsto \tilde{\pi}, \quad \tilde{\pi}(qV) = \pi(q)V.$$

Claim 2.A. (1) For every $\sigma \in \mathrm{Sym}(Q/V)$ there is a unique $\sigma_\kappa \in \Sigma^V$ with $\sigma_\kappa\kappa = \kappa\sigma$. The map $\sigma \mapsto \sigma_\kappa : \mathrm{Sym}(Q/V) \to \Sigma^V$ is a group homomorphism splitting $\sim$.

(2) There is a bijection $\tau \leftrightarrow f$ between Σ_0^V and $\{f : Q/V \to V\}$ given by $\tau(\kappa(s)v) = \kappa(s)f(s)v$, $f(s) = \kappa(s)^{-1}\tau(\kappa(s))$.

(3) $\mathrm{sgn}(\sigma_\kappa) = \mathrm{sgn}(\sigma)^{|V|} = \mathrm{sgn}(\sigma)$ and $\mathrm{sgn}(\tau) = 1$ for all $\tau \in \Sigma_0^V$.

Proof. If $\sigma_\kappa \in \Sigma^V$ exists, then $\sigma_\kappa(q) = \sigma_\kappa(\kappa(qV)\kappa(qV)^{-1}q) = (\sigma_\kappa\kappa)(qV)\kappa(qV)^{-1}q$ $= \kappa(\sigma(qV))\kappa(qV)^{-1}q$ as $\kappa(qV)^{-1}q \in V$, and conversely. Finally, $\mathrm{sgn}(\sigma_\kappa) = \mathrm{sgn}(\sigma)^{|V|}$ since $\{s_i : i \mod b\}$ a cycle of σ implies $\{\kappa(s_i)v : i \mod b\}$ is a cycle of σ_κ for each $v \in V$; and $\mathrm{sgn}(\tau) = \prod_{s\in Q/V}\mathrm{sgn}(\tau \text{ on } \kappa(s)V) = \prod_{s\in Q/V}\mathrm{sgn}(v \mapsto f(s)v \text{ on } V) = \prod_{s\in Q/V}(-1)^{|V|-[V:\langle f(s)\rangle]} = 1$. □

Note that every $q \in Q$ is a unique product $q = \kappa(s)v$ with $s \in Q/V$, $v \in V$, and every $\pi \in \Sigma^V$ is a unique product $\pi = \sigma_\kappa\tau$ with $\sigma \in \mathrm{Sym}(Q/V)$, $\tau \in \Sigma_0^V$. Substituting (2.b) for the double product in (2.a) and using (2) and (3) of Claim 2.A,

$$\begin{aligned}\kappa(\sigma s)f(s)\kappa(s)^{-1} &= \sigma_\kappa(\kappa(s))f(s)\kappa(s)^{-1} = \sigma_\kappa(\kappa(s)f(s))\kappa(s)^{-1}\\ &= \sigma_\kappa(\tau(\kappa(s)))\kappa(s)^{-1} = (\sigma_\kappa\tau)(\kappa(s))\kappa(s)^{-1} = (\sigma_\kappa\tau)(\kappa(s)v)(\kappa(s)v)^{-1}\\ &= \pi(q)q^{-1},\end{aligned}$$

we obtain the assertion of Lemma 2.1. □

If, as before, $A \le U \le G$ has $V = U/A$, then, for $e \in \Lambda_\wedge A$, $\mathrm{vr}_V(e) \overset{\mathrm{def}}{=} \mathrm{ver}_U^A(e)$ defines a ring endomorphism of $\Lambda_\wedge A$ satisfying $\mathrm{vr}_V(a) = \prod_{v\in V} a^v$ for all $a \in A$. This condition determines vr_V uniquely, as A 'generates' $\Lambda_\wedge A$ additively: indeed, picking a central open $\Gamma \simeq \mathbb{Z}_l$ in A and writing $A = \dot{\bigcup}_a a\Gamma$, $\Lambda_\wedge A = \bigoplus_a a \cdot \Lambda_\wedge\Gamma$, the element e becomes $e = \sum_a ac_a$ with unique $c_a \in \Lambda_\wedge\Gamma$ and $\mathrm{vr}_V(e) = \sum_a \mathrm{vr}_V(a)\Psi_V(c_a)$ with $\Psi_V : \Lambda_\wedge\Gamma \to \Lambda_\wedge\Gamma$ the ring homomorphism induced by $\gamma \mapsto \gamma^{|V|}$ for $\gamma \in \Gamma$. In particular, we have

$$\text{(2.c)} \qquad \mathrm{vr}_V(e)^q = \mathrm{vr}_{V^q}(e^q) \quad \text{for } V \le Q,\ e \in \Lambda_\wedge A,\ q \in Q,$$

because $\mathrm{vr}_V(a)^q = \prod_{v\in V} a^{vq} = \prod_{v\in V} a^{qv^q} = \prod_{w\in V^q} a^{qw} = \mathrm{vr}_{V^q}(a^q)$ for all $a \in A$.

Lemma 2.2. *For all $Q' \le Q$ and all $e \in \Lambda_\wedge A$,* $\sum\limits_{V\le Q'} \mu_{Q'}(V) \prod\limits_{s\in Q'/V} \mathrm{vr}_V(e^s) \equiv 0$ mod $\mathrm{tr}_{Q'}(\Lambda_\wedge A)$, *where s runs over a set of left coset representatives of V in Q'; i.e., $Q' = \dot{\bigcup}_s sV$.*

Proof. We first observe that this holds for all $a \in A$ because $\mathrm{vr}_V(a^s) = \prod_{v \in V} a^{sv}$, so $\prod_{s \in Q'/V} \mathrm{vr}_V(a^s) = \prod_{q' \in Q'} a^{q'}$ is independent of $V \leq Q'$, and $\sum_{V \leq Q'} \mu_{Q'}(V) = 0$. It therefore suffices to prove additivity of the left side of the claimed congruence, i.e.,

$$\sum_{V \leq Q'} \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V((e_0 + e_1)^s)$$
$$\equiv \sum_{V \leq Q'} \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V(e_0^s) + \sum_{V \leq Q'} \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V(e_1^s) \bmod \mathrm{tr}_{Q'}(\Lambda_\wedge A).$$

We proceed by induction on $|Q'|$; the case $Q' = 1$ is trivial.

Let $F = F(Q')$ denote the set of maps f from Q' to $\mathbb{F}_2$, with Q'-action $(fq')(x) = f(x(q')^{-1})$ for all $x \in Q'$. Then

$$\prod_{s \in Q'/V} (e_0 + e_1)^s = \sum_{f \in F^V} \prod_{s \in Q'/V} e^s_{f(s)},$$

because the set of fixed points F^V of V on F is the set of all $f : Q'/V \to \mathbb{F}_2$.

Defining $\mathfrak{F} = \mathfrak{F}(Q') = \{(V, f) : V \leq Q',\ f \in F^V\}$, we have

$$\sum_{V \leq Q'} \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V((e_0 + e_1)^s)$$
$$= \sum_{V \leq Q'} \mu_{Q'}(V) \sum_{f \in F^V} \prod_{s \in Q'/V} \mathrm{vr}_V(e^s_{f(s)}) = \sum_{(V,f) \in \mathfrak{F}} \tilde{\mu}(V, f)$$

where $\tilde{\mu}(V, f) = \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V(e^s_{f(s)})$.

Since $f \in F^V$ implies $fq' \in F^{V^{q'}}$, $\mathfrak{F}$ becomes a Q'-set by $(V, f)^{q'} = (V^{q'}, fq')$, and we obtain that

$$\tilde{\mu}((V, f))^{q'} = \tilde{\mu}((V, f)^{q'})$$

since, by (2.c),

$$\begin{aligned}\tilde{\mu}(V, f)^{q'} &= \mu_{Q'}(V) \textstyle\prod_{s \in Q'/V} \mathrm{vr}_{V^{q'}}(e^{sq'}_{f(s)}) = \mu_{Q'}(V^{q'}) \prod_{s_1 \in Q'/V^{q'}} \mathrm{vr}_{V^{q'}}(e^{s_1}_{(fq')(s_1)}) \\ &= \tilde{\mu}(V^{q'}, fq'),\end{aligned}$$

as $s_1 = sq'$ has $(fq')(s_1) = f(s)$.

We have thus reduced the claimed congruence to

$$\sum_{(V,f) \in \mathfrak{F}} \tilde{\mu}(V, f) \equiv \tilde{\mu}(V, \mathbf{0}) + \tilde{\mu}(V, \mathbf{1}) \mod \mathrm{tr}_{Q'}(\Lambda_\wedge A)$$

where $\mathfrak{F} = \mathfrak{F}(Q')$ and $\mathbf{0}, \mathbf{1}$ denote the obvious constant functions.

It now suffices to analyze, for a fixed $f \in F$, the Q'-orbit sums over $(V, f) \in \mathfrak{F}$. Set $W = \mathrm{St}_{Q'}(f)$. Note that $W = Q'$ occurs only for $f \in F^{Q'} = \{\mathbf{0}, \mathbf{1}\}$, and then in the same way on both sides. Thus we may assume $W < Q'$ from now on.

Set $e_* = \prod_{x \in Q'/W} e^x_{f(x)} \in \Lambda_\wedge A$ for a fixed choice of coset representatives x of W in Q'. The $V \leq Q'$ for which (V, f) are in $\mathfrak{F}$ are those with $f \in F^V$, i.e., $V \leq W$. Observe that (V_1, f) and (V_2, f) are in the same Q'-orbit if and only if V_1 and V_2 are conjugate subgroups of W. So our sum of Q'-orbits involving (V, f) is[10]

$$\sum_{V \preceq W} \mathrm{tr}_{Q'/\mathrm{St}_{Q'}(V,f)}(\tilde{\mu}(V, f))$$

[10] $\mathrm{tr}_{Q'/V}(e) = \sum_{s \in Q'/V} e^{s^{-1}},\ V \leq Q',\ e \in (\Lambda_\wedge A)^V$.

where $\preceq$ means that V runs through a set of representatives of conjugacy classes of subgroups of W. Since $\mathrm{St}_{Q'}(V,f) = \mathrm{St}_{N_{Q'}(V)}(f) = W \cap N_{Q'}(V) = N_W(V)$, it follows that

$$\text{(2.d)} \qquad \sum_{V \preceq W} \mathrm{tr}_{Q'/\mathrm{St}_{Q'}(V,f)}(\tilde{\mu}(V,f)) = \mathrm{tr}_{Q'/W}(\sum_{V \preceq W} \mathrm{tr}_{W/N_W(V)}(\tilde{\mu}(V,f)).$$

We next analyze $\tilde{\mu}(V,f)$ for each $V \leq W$. Decomposing W as $W = \dot{\bigcup}_y yV$, hence $Q' = \dot{\bigcup}_{x,y} xyV$, we obtain

$$\prod_{s \in Q'/V} e^s_{f(s)} = \prod_{x,y} e^{xy}_{f(xy)} \doteq \prod_y (\prod_x e^x_{f(x)})^y = \prod_y e^y_*$$

with $\doteq$ due to $fy^{-1} = f$. Thus

$$\tilde{\mu}(V,f) = \mu_{Q'}(V) \prod_{s \in Q'/V} \mathrm{vr}_V(e^s_{f(s)}) = \mu_W(V) \prod_{s \in W/V} \mathrm{vr}_V(e^s_*);$$

hence

$$\begin{aligned} \mathrm{tr}_{W/N_W(V)}\tilde{\mu}(V,f) &= \sum_{t \in W/N_W(V)} \tilde{\mu}(V,f)^{t^{-1}} \\ &\doteq \sum_{t \in W/N_W(V)} \tilde{\mu}(V^{t^{-1}},f) = \sum_{t \in W/N_W(V)} \mu_W(V^{t^{-1}}) \prod_{s \in W/V^{t^{-1}}} \mathrm{vr}_{V^{t^{-1}}}(e^s_*) \end{aligned}$$

with $\doteq$ by $ft^{-1} = f$.

We substitute this into our sum (2.d) to get

$$\begin{aligned} &\mathrm{tr}_{Q'/W}(\sum_{V \preceq W} \mathrm{tr}_{W/N_W(V)}(\tilde{\mu}(V,f)) \\ &\quad = \mathrm{tr}_{Q'/W}(\sum_{V \leq W} \mu_W(V) \prod_{s \in W/V} \mathrm{vr}_V(e^s_*)) \\ &\quad \in \mathrm{tr}_{Q'/W}(\mathrm{tr}_W(\Lambda_\wedge A))d = \mathrm{tr}_{Q'}(\Lambda_\wedge A), \end{aligned}$$

by the induction hypothesis as $W \neq Q'$. Thus, Lemma 2.2 is verified. □

We now turn to the *proof* of Theorem 2.

Proof. In terms of the map $\mathrm{vr}_V : \Lambda_\wedge A \to \Lambda_\wedge A$ (where $V = U/A$ and $A \leq U \leq G$), Lemma 2.1 becomes

$$\mathrm{ver}^A_U \mathrm{res}^U_G \epsilon = \sum_{\pi \in \Sigma^V} \tilde{r}(\pi) \prod_{s \in Q/V} \mathrm{vr}_V(e^s_{\pi(s)s^{-1}})$$

with $\tilde{r}(\pi) = \mathrm{sgn}(\pi) \prod_{q \in Q} a_{\pi(q)q^{-1},q} \in \Lambda_\wedge A$. Multiplying this by $\mu_Q(V)$ and summing over $V \leq Q$, we obtain

$$\text{(2.e)} \quad \textstyle\sum_{V \leq Q} \mu_Q(V) \mathrm{ver}^A_U \mathrm{res}^U_G \epsilon = \sum_{\pi \in \Sigma} \tilde{r}(\pi) \sum_{V \leq \mathrm{St}_Q(\pi)} \mu_Q(V) \prod_{s \in Q/V} \mathrm{vr}_V(e^s_{\pi(s)s^{-1}})$$

because $[\, \pi \in \Sigma^V \iff V \leq \mathrm{St}_Q(\pi)\,]$. We consider the action of conjugation of Q on this sum, starting with

Claim 2.B. $\tilde{r}(\pi)^q = \tilde{r}(\pi^q)$ for all $q \in Q$.

Proof. First, $\mathrm{sgn}(\pi^q) = \mathrm{sgn}(\pi)$ holds since $\{x_i : i \mod b\}$ a cycle of π implies $\{x_i q : i \mod b\}$ is a cycle of π^q. Second,

$$(\prod_{q_1} a_{\pi(q_1)q_1^{-1},q_1})^q = \prod_{q_1} a^q_{\pi(q_1)q_1^{-1},q_1} \stackrel{1}{=} \prod_{q_1} a_{\pi(q_1)q_1^{-1},q_1 q} \prod_{q_1} a^{-1}_{\pi(q_1),q} \prod_{q_1} a_{q_1,q}$$
$$\stackrel{2}{=} \prod_{q_1} a_{\pi(q_1)q_1^{-1},q_1 q} \stackrel{3}{=} \prod_{q_1} a_{\pi(q_1 q^{-1})q q_1^{-1},q_1} = \prod_{q_1} a_{\pi^q(q_1)q_1^{-1},q_1}$$

with $\stackrel{1}{=}$ due to the cocycle relation, $\stackrel{2}{=}$ to π permuting the q_1, and $\stackrel{3}{=}$ to the substitution $q_1 \rightsquigarrow q_1 q^{-1}$. $\square$

Continuing with the proof of Theorem 2, the right-hand side of (2.e) is in $\mathrm{tr}_Q(\Lambda_\wedge A)$ if

$$\text{(2.f)} \qquad \textstyle\sum_{V \le \mathrm{St}_Q(\pi)} \mu_Q(V) \prod_{s \in Q/V} \mathrm{vr}_V(e^s_{\pi(s)s^{-1}}) \equiv 0 \mod \mathrm{tr}_{\mathrm{St}_Q(\pi)}(\Lambda_\wedge A)$$

holds for all $\pi \in \Sigma$. Namely, assuming (2.f), its left side can be written as $\mathrm{tr}_{\mathrm{St}_Q(\pi)}(\alpha)$ for some $\alpha \in \Lambda_\wedge A$. Since $\tilde{r}(\pi) \in (\Lambda_\wedge A)^{\mathrm{St}_Q(\pi)}$, it follows that

$$\mathrm{tr}_{Q/\mathrm{St}_Q(\pi)}(\tilde{r}(\pi)\mathrm{tr}_{\mathrm{St}_Q(\pi)}(\alpha)) = \mathrm{tr}_Q(\tilde{r}(\pi)\alpha)$$

is the orbit sum of π in (2.e), by Claim 2.B.

We next observe that (2.f) is a consequence of Lemma 2.2. To see this, let $\pi \in \Sigma^V$ be given and set $Q' = \mathrm{St}_Q(\pi)$ and $e = \prod_{x \in Q/Q'} e^x_{\pi(x)x^{-1}} \in \Lambda_\wedge A$, where $Q = \dot{\bigcup}_x xQ'$. Setting $Q' = \dot{\bigcup}_y yV$, then $Q = \dot{\bigcup}_{x,y} xyV$ and the V-term in (2.f) is

$$\mu_Q(V) \prod_{x,y} \mathrm{vr}_V(e^{xy}_{\pi(xy)(xy)^{-1}}) \doteq \mu_Q(V) \prod_{x,y} \mathrm{vr}_V(e^{xy}_{\pi(x)x^{-1}}) = \mu_{Q'}(V) \prod_{y \in Q'/V} \mathrm{vr}_V(e^y)$$

where $\doteq$ results from $\pi^y = \pi$.

Collecting everything so far, we see that Theorem 2 follows from (2.f) and that this holds because of Lemma 2.2. $\square$

3. Congruences between abelian pseudomeasures, II

This section is a sequel to [RW7]. As much as possible we continue with the notation used there (except that K there becomes $\mathcal{K}$ here).

Let p be a fixed prime number, $\mathcal{K}$ a totally real number field finite over $\mathbb{Q}$, L a totally real finite Galois extension of $\mathcal{K}$ of p-power degree with Galois group $\Sigma = G(L/\mathcal{K})$, and S a fixed finite set of *non-archimedean* primes of $\mathcal{K}$ containing all primes above p and those which ramify in L. Further, denote by $\mathcal{K}_S$ the maximal abelian extension of $\mathcal{K}$ which is unramified (at all non-archimedean primes) outside S and set $G_S = G(\mathcal{K}_S/\mathcal{K})$. Serre's pseudomeasure $\lambda_{\mathcal{K}} = \lambda_{\mathcal{K},S}$ has the property that $(1-g)\lambda_{\mathcal{K}}$ is in the completed group ring $\mathbb{Z}_p[[G_S]]$ for all $g \in G_S$ [Se].

Letting $\mathcal{F}$ run through the intermediate fields of $L/\mathcal{K}$, we denote by

- $\mathcal{F}_S$ the maximal abelian extension of $\mathcal{F}$ unramified outside the primes of $\mathcal{F}$ above S,
- $\mathcal{F}_S^+$ its maximal totally real subfield (hence $\mathcal{F} \subset \mathcal{F}_S^+$),
- $\lambda_{\mathcal{F},S}$ Serre's pseudomeasure with respect to $\mathcal{F}$ and S,
- $H_S = G(L_S/L)$, $H_S^+ = G(L_S^+/L)$.

Note that $G(L_S/\mathcal{F})^{\mathrm{ab}} = G(\mathcal{F}_S/\mathcal{F})$ and, for $\mathcal{F} \subseteq \mathcal{F}'$, that $G(L_S/\mathcal{F}')$ is an open subgroup of $G(L_S/\mathcal{F})$. This yields the transfer map $G(\mathcal{F}_S/\mathcal{F}) \to G(\mathcal{F}'_S/\mathcal{F}')$, and in particular,

$$\mathrm{ver}_{\mathcal{K}}^{\mathcal{F}} : G_S \to G(\mathcal{F}_S/\mathcal{F}),\ \mathrm{ver}_{\mathcal{F}}^{L} : G(\mathcal{F}_S/\mathcal{F}) \to H_S,\ \mathrm{ver}_{\mathcal{K}}^{L} = \mathrm{ver}_{\mathcal{F}}^{L} \circ \mathrm{ver}_{\mathcal{K}}^{\mathcal{F}} : G_S \to H_S.$$

We recall that the Möbius function $\mu = \mu_\Sigma$ of the poset of subgroups of the finite group Σ is defined by

$$\mu(1) = 1, \qquad \mu(\Sigma') = -\sum_{1 \le \Sigma'' < \Sigma'} \mu(\Sigma'') \quad \text{for } 1 \neq \Sigma' \le \Sigma.$$

For $\mathcal{K} \subseteq \mathcal{F} \subseteq L$ and $g \in G_S$ set $\tilde{\lambda}_{\mathcal{F}} = 2^{-[\mathcal{F}:\mathbb{Q}]}\lambda_{\mathcal{F},S}$ and $\tilde{\lambda}_{g_{\mathcal{F}}} = (1-g_{\mathcal{F}})\tilde{\lambda}_{\mathcal{F}}$, where $g_{\mathcal{F}} = \mathrm{ver}_{\mathcal{K}}^{\mathcal{F}} g$. Moreover, denote the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathcal{F}_\infty/\mathcal{F}$ by $\Gamma_{\mathcal{F}}$.

Theorem 5. *There exists $g \in G_S$ so that $g_{\mathcal{F}}$ has non-trivial image in $\Gamma_{\mathcal{F}}$ for $\mathcal{K} \subseteq \mathcal{F} \subseteq L$ and the image of*

$$\sum_{\mathcal{K} \subseteq \mathcal{F} \subseteq L} \mu_\Sigma(G(L/\mathcal{F}))\mathrm{ver}_{\mathcal{F}}^{L}(\tilde{\lambda}_{g_{\mathcal{F}}}),$$

under $\mathbb{Z}_p[[H_S]] \to \mathbb{Z}_p[[H_S^+]]$, is in the trace ideal $\mathrm{tr}_\Sigma(\mathbb{Z}_p[[H_S^+]])$ of $\mathbb{Z}_p[[H_S^+]]^\Sigma$.

The organization of the *proof* parallels [RW7]. The new ingredient is the identification of the congruence (3.4⋆) of Lemma 3.4 as the difference of constant terms of q-expansions at two cusps of a Hilbert modular form of Eisenstein type. This modular form is exhibited in §3.2 and then studied via the q-expansion principle of Deligne and Ribet; the hypothesis of Lemma 3.4 is deduced, in Lemma 3.6, from a property of Möbius coefficients in [HIO]. The proof, in §3.3, of the main result of this section, Theorem 5, is then a computation of constant terms of q-expansions at the “special cusps” of Lemma 3.5.

Our special cusps are a simple device to avoid comparing constant coefficients of q-expansions of F and $F_{|k}U_\beta$ at *arbitrary* cusps in [RW7, Lemma 6]. Having overlooked the need for this comparison in [RW7] implies that we now have its Theorem only for special $g = g_{\mathcal{K}} \in G_S$, i.e., those in Lemma 3.7: this can be deduced from Theorem 5 for odd p by [RW7, Lemma 5]. However, Theorem 5 is better suited for the application to equivariant Iwasawa theory; see Remark 4.1.

3.1. **A sufficient condition for a pseudomeasure congruence.** For a coset x of an open subgroup U of $G(\mathcal{F}_S/\mathcal{F})$ set $\delta^{(x)}(g) = 1$ or 0 according as $g \in x$ or not. Then, for even integers $k \ge 1$, define $\tilde{\zeta}_{\mathcal{F}}(1-k, \delta^{(x)}) = 2^{-[\mathcal{F}:\mathbb{Q}]}\zeta_{\mathcal{F},S}(1-k,\delta^{(x)}) \in \mathbb{Q}$ to be $2^{-[\mathcal{F}:\mathbb{Q}]}$ times the value at $1-k$ of the partial ζ-function for the set of integral ideals $\mathfrak{a}$ of $\mathcal{F}$ prime to S with Artin symbol $(\mathfrak{a}, \mathcal{F}_S/\mathcal{F})$ in x. Note that the definition of $\tilde{\zeta}_{\mathcal{F}}(1-k,\delta^{(x)})$ extends linearly to locally constant functions ε on $G(\mathcal{F}_S/\mathcal{F})$ with values in a $\mathbb{Q}$-vector space and gives values $\tilde{\zeta}_{\mathcal{F}}(1-k,\varepsilon)$ in that vector space, as usual.

Let $\mathcal{N} = \mathcal{N}_{\mathcal{F},p} : G(\mathcal{F}_S/\mathcal{F}) \to {\mathbb{Z}_p}^\times$ be that continuous character whose value on $(\mathfrak{a}, \mathcal{F}_S/\mathcal{F})$ for an integral ideal $\mathfrak{a}$ of $\mathcal{F}$ prime to S is its absolute norm $\mathcal{N}_{\mathcal{F}}\mathfrak{a}$. For $g \in G(\mathcal{F}_S/\mathcal{F})$, $k \ge 1$, and ε a locally constant $\mathbb{Q}_p$-valued function on $G(\mathcal{F}_S/\mathcal{F})$ we define, following [DR],

$$\tilde{\Delta}_g(1-k,\varepsilon) = \tilde{\zeta}_{\mathcal{F}}(1-k,\varepsilon) - \mathcal{N}(g)^k\tilde{\zeta}_{\mathcal{F}}(1-k,\varepsilon_g) \in \mathbb{Q}_p,$$

where $\varepsilon_g(g') = \varepsilon(gg')$ for $g' \in G(\mathcal{F}_S/\mathcal{F})$.

Theorem (0.4) **of** [DR]**.** *Let $\varepsilon_1, \varepsilon_2, \dots$ be a finite sequence of locally constant functions $G(\mathcal{F}_S/\mathcal{F}) \to \mathbb{Q}_p$ so that $\sum_{k\geq 1} \varepsilon_k(g')(\mathcal{N}g')^{k-1} \in \mathbb{Z}_p$ for all $g' \in G(\mathcal{F}_S/\mathcal{F})$. Then*

$$\sum_{k\geq 1} \tilde{\Delta}_g(1-k, \varepsilon_k) \in \mathbb{Z}_p \quad \text{for all } g \in G(\mathcal{F}_S/\mathcal{F}).^{11}$$

Call an open subgroup U of $G(\mathcal{F}_S/\mathcal{F})$ *admissible* if $\mathcal{N}(U) \subset 1 + p\mathbb{Z}_p$ and define $m_{\mathcal{F}}(U) \geq 1$ by $\mathcal{N}(U) = 1 + p^{m_{\mathcal{F}}(U)}\mathbb{Z}_p$.

Lemma 3.1. *If U runs through the cofinal system of admissible open subgroups of $G(\mathcal{F}_S/\mathcal{F})$, then $\mathbb{Z}_p[[G(\mathcal{F}_S/\mathcal{F})]] = \varprojlim_U \mathbb{Z}_p[G(\mathcal{F}_S/\mathcal{F})/U]/p^{m_{\mathcal{F}}(U)}\,\mathbb{Z}_p[G(\mathcal{F}_S/\mathcal{F})/U]$.*

Proof. See [RW7, Lemma 1]. □

Lemma 3.2. *For $h \in G(\mathcal{F}_S/\mathcal{F})$ there is a unique element $\tilde{\lambda}_h \in \mathbb{Z}_p[[G(\mathcal{F}_S/\mathcal{F})]]$, independent of k, whose image in $\mathbb{Z}_p[G(\mathcal{F}_S/\mathcal{F})/U]/p^{m_{\mathcal{F}}(U)}$ is*

$$\sum_{x\in G(\mathcal{F}_S/\mathcal{F})/U} \tilde{\Delta}_h(1-k, \delta^{(x)})\mathcal{N}(x)^{-k}x \mod p^{m_{\mathcal{F}}(U)}\mathbb{Z}_p[G(\mathcal{F}_S/\mathcal{F})/U]$$

for all admissible U, where $\mathcal{N}$ here also denotes the homomorphism $G(\mathcal{F}_S/\mathcal{F})/U \to (\mathbb{Z}_p/p^{m(U)})^\times$ induced by our previous $\mathcal{N}$. Moreover, if $\tilde{\lambda}_{\mathcal{F}}$ is $2^{-[\mathcal{F}:\mathbb{Q}]}$ times the pseudomeasure of [Se]*, then $(1-h)\tilde{\lambda}_{\mathcal{F}} = \tilde{\lambda}_h$.*

Proof. The lemma follows from the theorem above; see [RW7, Proposition 2]. □

Lemma 3.3. (1) *Let V be an admissible open subgroup of H_S. If $U \leq (\mathrm{ver}_{\mathcal{F}}^L)^{-1}(V)$, then $m_{\mathcal{F}}(U) \geq m_L(V) - e_{\mathcal{F}}$ where $[L:\mathcal{F}] = p^{e_{\mathcal{F}}}$.*

(2) *Let $s \in G(L_S/\mathcal{K})$ be an extension of $\sigma \in \Sigma$. Then $(\mathcal{F}^\sigma)_S = (\mathcal{F}_S)^s$ and, setting $g_{\mathcal{F}} = \mathrm{ver}_{\mathcal{K}}^{\mathcal{F}}(g)$ for $g \in G_S$, $g_{\mathcal{F}}^s = g_{\mathcal{F}^\sigma}$. Moreover, $\mathrm{ver}_{\mathcal{F}}^L(\tilde{\lambda}_{g_{\mathcal{F}}})^s = \mathrm{ver}_{\mathcal{F}^\sigma}^L(\tilde{\lambda}_{g_{\mathcal{F}^\sigma}})$.*

Proof. Statement (1) is due to $\mathcal{N}_L(\mathrm{ver}_{\mathcal{F}}^L g) = \mathcal{N}_{\mathcal{F}}(g)^{[L:\mathcal{F}]}$ for $g \in G(\mathcal{F}_S/\mathcal{F})$. For (2), the first claim follows from $(\mathcal{F}_S)^s \supseteq \mathcal{F}^s = \mathcal{F}^\sigma$ and $G((\mathcal{F}_S)^s/\mathcal{F}^s) = G(\mathcal{F}_S/\mathcal{F})^s$, the latter implying $(\mathcal{F}_S)^s \subseteq (\mathcal{F}^\sigma)_S$ and then $\mathcal{F}_S \subseteq (\mathcal{F}^\sigma)_S^{s^{-1}} \subseteq \mathcal{F}_S$, hence equality everywhere.

The second claim is a direct consequence of the definition of group transfer 'ver'. Namely $\mathrm{ver}_{\mathcal{K}}^{\mathcal{F}}(g)$ is a certain product built with respect to coset representatives of $G(L_S/\mathcal{F})$ in $G(L_S/\mathcal{K})$, which s takes to coset representatives of $G(L_S/\mathcal{F}^\sigma)$ in $G(L_S/\mathcal{K})$, whence the multiplicativity of s yields $\mathrm{ver}_{\mathcal{K}}^{\mathcal{F}}(g)^s = \mathrm{ver}_{\mathcal{K}}^{\mathcal{F}^\sigma}(g^s)$. But $g \in G_S = G(L_S/\mathcal{K})^{\mathrm{ab}}$ implies $g^s = g$.

Finally, concerning the last claimed equality, by Lemma 3.2 and $\mathcal{N}_{\mathcal{F}^\sigma}(x^s) = \mathcal{N}_{\mathcal{F}}(x)$ it suffices to show

$$\tilde{\Delta}_{g_{\mathcal{F}^\sigma}}(1-k, \delta_{\mathcal{F}^\sigma}^{(x^s)}) = \tilde{\Delta}_{g_{\mathcal{F}}}(1-k, \delta_{\mathcal{F}}^{(x)})$$

for x a coset of any admissible open $G(L/\mathcal{F})$-stable subgroup U of $G(\mathcal{F}_S/\mathcal{F})$. Thus it suffices to show

$$\tilde{\zeta}_{\mathcal{F}^\sigma}(1-k, \delta_{\mathcal{F}^\sigma}^{(x^s)}) = \tilde{\zeta}_{\mathcal{F}}(1-k, \delta_{\mathcal{F}}^{(x)})$$

for all such x, since[12] $(\delta_{\mathcal{F}^\sigma}^{(x^s)})_{g_{\mathcal{F}^\sigma}} = \delta_{\mathcal{F}^\sigma}^{(g_{\mathcal{F}^\sigma}^{-1}x^s)} = \delta_{\mathcal{F}^\sigma}^{((g_{\mathcal{F}}^{-1}x)^s)}$ and $(\delta_{\mathcal{F}}^{(x)})_{g_{\mathcal{F}}} = \delta_{\mathcal{F}}^{(g_{\mathcal{F}}^{-1}x)}$. Viewing $\delta_{\mathcal{F}}^{(x)}$ as a complex-valued function on $G(\mathcal{F}_S/\mathcal{F})/U$ and writing it as a $\mathbb{C}$-linear combination of the (abelian) characters χ of $G(\mathcal{F}_S/\mathcal{F})/U$, it now suffices

[11] For $p = 2$ compare [RW7, §5] and [RoW].

[12] Recall our convention $\varepsilon_g(g') = \varepsilon(gg')$.

to check that $\tilde\zeta_{\mathcal F}(1-k,\chi)=\tilde\zeta_{\mathcal F^\sigma}(1-k,\chi^s)$, and this follows from the compatibility of the Artin L-functions with inflation and induction. Indeed, inflating χ from $G(\mathcal F_S/\mathcal F)/U$ to $G(\mathcal F_S/\mathcal F)$ and further to $G(L_S/\mathcal F)$ and then inducing up to $G(L_S/\mathcal K)$, and analogously with $\chi,\mathcal F,U$ replaced by $\chi^s,\mathcal F^\sigma,U^s$ (note that U^s is well-defined), we have

$$\operatorname{ind}_{G(L_S/\mathcal F)}^{G(L_S/\mathcal K)}\operatorname{infl}_{G(\mathcal F_S/\mathcal F)/U}^{G(L_S/\mathcal F)}(\chi)=\operatorname{ind}_{G(L_S/\mathcal F^\sigma)}^{G(L_S/\mathcal K)}\operatorname{infl}_{G(\mathcal F_S^s/\mathcal F^\sigma)/U}^{G(L_S/\mathcal F^\sigma)}(\chi^s).$$

Lemma 3.3 is established. □

Set $\Sigma_{\mathcal F}=G(L/\mathcal F)$, and, for any set X carrying a natural Σ-action, denote the stabilizer subgroup of $x\in X$ in Σ by $\mathrm{St}_\Sigma(x)=\{\sigma\in\Sigma:\sigma(x)=x\}$. Also, $\mathbb Z_{(p)}\subset\mathbb Q$ is the localization of $\mathbb Z$ at its prime ideal $p\mathbb Z$.

Lemma 3.4. *For $g\in G_S$ set $g_{\mathcal F}=\mathrm{ver}_{\mathcal K}^{\mathcal F}g$ for all $\mathcal K\subseteq\mathcal F\subseteq L$. Then*

$$(3.4\star)\qquad \sum_{\mathcal K\subseteq\mathcal F\subseteq L}\mu_\Sigma(\Sigma_{\mathcal F})\tilde\Delta_{g_{\mathcal F}}(1-|\Sigma_{\mathcal F}|k,\varepsilon_L\mathrm{ver}_{\mathcal F}^L)\equiv 0\mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb Z_{(p)},$$

for all even locally constant $\mathbb Z_{(p)}$-valued functions ε_L on H_S, implies that

$$\mathfrak s_g\overset{\text{def}}{=}\sum_{\mathcal F}\mu_\Sigma(\Sigma_{\mathcal F})\mathrm{ver}_{\mathcal F}^L(\tilde\lambda_{g_{\mathcal F}})$$

has image, under the map $\mathbb Z_p[[H_S]]\to\mathbb Z_p[[H_S^+]]$, in $\mathrm{tr}_\Sigma(\mathbb Z_p[[H_S^+]])$.

Proof. For the *proof* of the lemma we first recall that a locally constant function ε_L on H_S is *even* if $\varepsilon_L(c_wh)=\varepsilon_L(h)$ for all $h\in H_S$ and all 'Frobenius elements' c_w at the archimedean primes w of L, i.e., at the restrictions $c_w\in H_S$ of complex conjugation with respect to the embeddings $L_S\hookrightarrow\mathbb C$ inducing w on L. We denote by C the group generated by the c_w's, so $H_S^+=H_S/C$.

We next observe that $\mathfrak s_g\in\mathbb Z_p[[H_S]]^\Sigma$. Namely, $\mathrm{ver}_{\mathcal F}^L(\tilde\lambda_{g_{\mathcal F}})^s=\mathrm{ver}_{\mathcal F^\sigma}^L(\tilde\lambda_{g_{\mathcal F}}^s)=\mathrm{ver}_{\mathcal F^\sigma}^L(\tilde\lambda_{g_{\mathcal F^\sigma}})$, by (2) of Lemma 3.3. Moreover, $\mu_\Sigma(\Sigma_{\mathcal F})=\mu_\Sigma(\Sigma_{\mathcal F^\sigma})$.

Turning finally to the image of $\mathfrak s_g$ under the map $\mathbb Z_p[[H_S]]\to\mathbb Z_p[[H_S^+]]$, we first replace the diagram in [RW7, p. 718] by the diagram below, in which $N=\ker\mathrm{ver}_{\mathcal F}^L$:

$$\begin{array}{ccc}\mathbb Z_p[[G(\mathcal F_S/\mathcal F)]] & \to & \varprojlim_{U\geq N}\mathbb Z_p[G(\mathcal F_S/\mathcal F)/U]/p^{m_{\mathcal F}(U)}\\ \mathrm{ver}_{\mathcal F}^L\downarrow & & \downarrow\\ \mathbb Z_p[[H_S]] & \overset{\simeq}{\to} & \varprojlim_{\Sigma-\text{stable }V}\mathbb Z_p[H_S/V]/p^{m_L(V)-e_{\mathcal F}}.\end{array}$$

Recall here that the right vertical map takes $(x_U)_U$ to $(y_V)_V$ by means of

$$\Xi:\quad \mathbb Z_p[G(\mathcal F_S/\mathcal F)/U]/p^{m_{\mathcal F}(U)}\xrightarrow{\mathrm{ver}}\mathbb Z_p[H_S/V]/p^{m_{\mathcal F}(U)}\to\mathbb Z_p[H_S/V]/p^{m_L(V)-e_{\mathcal F}},$$

whenever $U\leq(\mathrm{ver}_{\mathcal F}^L)^{-1}(V)$.[13]

Since the $m_L(V)$'s are unbounded, there are admissible open Σ-stable $V\leq H_S$ with $m_L(V)-e_{\mathcal F}\geq e_{\mathcal K}$ $(\forall\mathcal F)$. For any such V then $\mathbb Z_p[H_S/V]/p^{m_L(V)-e_{\mathcal F}}$ maps onto $\mathbb Z_p[H_S/V]/|\Sigma|$ and we write the image of $\mathfrak s_g$ here as $\sum_{y\in H_S/V}c_yy$. Because Σ fixes $\mathfrak s_g$, $c_{y^\sigma}=c_y$ for all σ. Since $\sum_{\sigma\in\Sigma\bmod \mathrm{St}_\Sigma(y)}c_{y^\sigma}y^\sigma=c_y\sum y^\sigma$, it follows that $\mathfrak s_g$ will be in $\mathrm{tr}_\Sigma(\mathbb Z_p[H_S/V])+|\Sigma|\mathbb Z_p[H_S/V]$ provided that

$$c_y\equiv 0\mod |\mathrm{St}_\Sigma(y)|.$$

[13]Note that 'ver' is the $\mathbb Z_p$-linear map induced by the group homomorphism obtained by factoring $G(\mathcal F_S/\mathcal F)\overset{\mathrm{ver}_{\mathcal F}^L}{\to}H_S\to H_S/V$ through $G(\mathcal F_S/\mathcal F)\to G(\mathcal F_S/\mathcal F)/U$.

We compute the coefficient c_y of the image of $\mathfrak{s}_g$. By Lemma 3.2 with $U = (\mathrm{ver}^L_{\mathcal{F}})^{-1}(V)$ (compare [RW7, (ii) on p. 719] noting that any k is allowed), $\mathrm{ver}^L_{\mathcal{F}}(\tilde{\lambda}_{g_{\mathcal{F}}})$ has image

$$\sum_{x \in G(\mathcal{F}_S/\mathcal{F})/U} \tilde{\Delta}_{g_{\mathcal{F}}}(1 - |\Sigma_{\mathcal{F}}|k, \delta^{(x)}_{\mathcal{F}}) \mathcal{N}_{\mathcal{F}}(x)^{-|\Sigma_{\mathcal{F}}|k} \mathrm{ver}^L_{\mathcal{F}}(x),$$

where $\delta^{(x)}_{\mathcal{F}}$ is the characteristic function of the coset $x \subseteq G(\mathcal{F}_S/\mathcal{F})$.

Since $G(\mathcal{F}_S/\mathcal{F}) \overset{\mathrm{ver}^L_{\mathcal{F}}}{\to} H_S \to H_S/V$ has kernel $U = (\mathrm{ver}^L_{\mathcal{F}})^{-1}(V)$, either y is not in the image of $\mathrm{ver}^L_{\mathcal{F}}$ or $y = \mathrm{ver}^L_{\mathcal{F}}(x^{(\mathcal{F})})$ for a unique $x^{(\mathcal{F})} \in G(\mathcal{F}_S/\mathcal{F})/U$. Note that $\delta^{(y)}_L \mathrm{ver}^L_{\mathcal{F}}$ is $= 0$ in the first case and $= \delta^{(x^{(\mathcal{F})})}_{\mathcal{F}}$ in the second, when also $\mathcal{N}_{\mathcal{F}}(x^{(\mathcal{F})})^{-|\Sigma_{\mathcal{F}}|k} = \mathcal{N}_L(\mathrm{ver}^L_{\mathcal{F}}(x^{(\mathcal{F})}))^{-k}$. Thus, Möbius-summing over $\mathcal{F}$, we obtain

$$c_y = \Big(\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}}) \tilde{\Delta}_{g_{\mathcal{F}}}(1 - |\Sigma_{\mathcal{F}}|k, \delta^{(y)}_L \mathrm{ver}^L_{\mathcal{F}})\Big) \mathcal{N}_L(y)^{-k}.$$

Our hypothesis (3.4$\star$) now implies that $\mathfrak{s}_g$ is in $\mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V]) + |\Sigma|\mathbb{Z}_p[H_S/V]$ for all $V \geq C$ (recall that $\delta^{(y)}_L$ is even when $V \geq C$ and $y \in H_S/V$). Since $\mathfrak{s}_g$ is fixed by Σ and $|\Sigma|\mathbb{Z}_p[H_S/V]^\Sigma \subseteq \mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V])$, it follows that $\mathfrak{s}_g \in \mathrm{tr}_\Sigma(\mathbb{Z}_p[H_S/V])$.

This finishes the proof of Lemma 3.4. □

3.2. **Applying the q-expansion principle of** [DR]**.** Given an even integer k and an even locally constant $\mathbb{Z}_{(p)}$-valued function ε_L on H_S, choose an open subgroup V of H_S so that ε_L is constant on each coset H_S/V and let $\mathfrak{f} \subset |\Sigma|\mathfrak{o}_{\mathcal{K}}$ be an integral ideal, with all its prime factors contained in S, so that, for all $\mathcal{K} \subseteq \mathcal{F} \subseteq L$, $\mathfrak{f}\mathfrak{o}_{\mathcal{F}}$ is a multiple of the conductor of the field fixed by $(\mathrm{ver}^L_{\mathcal{F}})^{-1}(V)$ acting on $\mathcal{F}_S$.

As in [DR, p. 229] we write $\hat{\mathcal{K}}$ for the ring of 'finite' adèles of $\mathcal{K}$ and let $j : \hat{\mathcal{K}}^\times \to G_S, \psi : \hat{\mathcal{K}}^\times \to \varprojlim_{\mathfrak{f}'} G_{\mathfrak{f}'}$, with $\mathfrak{f}'$ running over the multiples of $\mathfrak{f}$ that have all their prime divisors in S, be the maps defined in [DR, p. 243].

Lemma 3.5. *There exist $\gamma \in \hat{\mathcal{K}}^\times$ so that*

(a) *γ and γ^{-1} are in $1 + \hat{\mathfrak{f}}$ and*

(b) *the image $g \in G_S$ of γ under $j : \hat{\mathcal{K}}^\times \to G_S$ has $\mathrm{ver}^{\mathcal{F}}_{\mathcal{K}}(g) \in G(\mathcal{F}_S/\mathcal{F})$ not in the kernel of $G(\mathcal{F}_S/\mathcal{F}) \to \Gamma_{\mathcal{F}}$, for all $\mathcal{K} \subseteq \mathcal{F} \subseteq L$.*

Proof. Choose $f > 0$ in $\mathbb{Z} \cap \mathfrak{f}$, and let $\alpha \in \hat{\mathbb{Q}}^\times$ be the 'finite' idèle with components $1 + f$ (respectively 1) at primes $q|f$ (respectively $q \nmid f$). Then α and α^{-1} belong to $1 + \widehat{\mathbb{Z}f}$. For every extension $\mathbb{Q} \hookrightarrow \mathcal{F}$ let $\alpha_{\mathcal{F}}$ be the image of α under the diagonal inclusion $\hat{\mathbb{Q}}^\times \hookrightarrow \hat{\mathcal{F}}^\times$. We use $\varprojlim_{\mathfrak{f}'} G_{\mathfrak{f}'} = G_S$, on identifying the inverse limit of ray class groups with the one of the corresponding Galois groups, as is conventional in [DR, p. 240]; so $j = \psi = (\psi_{\mathfrak{f}'})_{\mathfrak{f}'}$ by [DR, 2.33].

Now $\psi_{\mathfrak{f}'}(\alpha_{\mathcal{K}}) = \psi_{\mathfrak{f}'}((1+f)^{-1}\alpha_{\mathcal{K}})$, since $1 + f \in \mathcal{K}$ is totally positive, and $(1+f)^{-1}\alpha_{\mathcal{K}}$ has $\mathfrak{q}$-component 1 (respectively $(1+f)^{-1}$) when $\mathfrak{q}|\mathfrak{f}$ (respectively $\mathfrak{q} \nmid \mathfrak{f}$), so that the (fractional) ideal of $\mathfrak{o} = \mathfrak{o}_{\mathcal{K}}$ 'generated' by $(1+f)^{-1}\alpha_{\mathcal{K}}$ is $\mathfrak{a} = (1+f)^{-1}\mathfrak{o}$, which is coprime to $\mathfrak{f}$.

Recall that $\mathcal{K}_S$ contains the $\mathbb{Z}_p$-cyclotomic extension $\mathcal{K}_\infty$ of $\mathcal{K}$. By [Se, 2.2], $g = \psi(\alpha_{\mathcal{K}}) \in G_S$ acts on p-power roots of unity as $g = (\mathfrak{a}, \mathcal{K}_S/\mathcal{K})$, i.e., by raising them to the power $\mathcal{N}(\mathfrak{a}) = (1+f)^{-[\mathcal{K}:\mathbb{Q}]}$. It follows that the image of g under $G_S \to \Gamma_{\mathcal{K}}$ acts non-trivially, so (a) holds for $\alpha_{\mathcal{K}}$ and also (b) when $\mathcal{K} = \mathcal{F}$.

The argument for $\mathcal{F}$ is the same with $\mathcal{K}, \mathfrak{f}, S$ replaced by $\mathcal{F}, \mathfrak{f}\mathfrak{o}_{\mathcal{F}}, S_{\mathcal{F}}$ and shows that $\psi_{\mathcal{F}}(\alpha_{\mathcal{F}})$ acts on p-power roots of unity by $(1+f)^{-[\mathcal{F}:\mathbb{Q}]}$. Note also that $\psi_{\mathcal{F}}(\alpha_{\mathcal{F}}) = \mathrm{ver}_{\mathcal{K}}^{\mathcal{F}}(g)$ by the usual relation between inclusion and transfer. Setting $\gamma = \alpha_{\mathcal{K}}$, we then get (a) and (b).

This completes the proof of Lemma 3.5. $\square$

The next three results, Lemmas 6 and 7 and Proposition 8 of [RW7], concern Hilbert modular forms with emphasis on their q-expansions. [RW7, Lemma 6] constructs a Hecke operator U_β on $\mathcal{M}_k(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$, [RW7, Lemma 7] discusses restriction $\mathrm{res}_{\mathcal{F}}^{\mathcal{K}} : \mathcal{M}_k(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_{\mathcal{F}}), \mathbb{C}) \to \mathcal{M}_{[\mathcal{F}:\mathcal{K}]k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ for field extensions $\mathcal{F}/\mathcal{K}$, and [RW7, Proposition 8] is our bridge to [DR].

With k, ε_L, and $\mathfrak{f}$ as at the beginning of the section and any $g \in G_S$, we next exhibit a Hilbert modular form $\mathcal{E}$ in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ with the constant term of its standard q-expansion equal to $\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}}) \tilde{\Delta}_{g_{\mathcal{F}}}(1 - |\Sigma_{\mathcal{F}}|k, \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L)$ (compare (3.4$\star$)).

First, by [DR, (6.1)], in the form of [RW7, Proposition 8], there are modular forms

$$E_{\mathcal{F}} \stackrel{\mathrm{def}}{=} G_{|\Sigma_{\mathcal{F}}|k, \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L} \in \mathcal{M}_{|\Sigma_{\mathcal{F}}|k}(\Gamma_{00}(\mathfrak{f}\mathfrak{o}_{\mathcal{F}}), \mathbb{C})$$

of weight $|\Sigma_{\mathcal{F}}|k$ with standard q-expansion

$$\tilde{\zeta}_{\mathcal{F}}(1 - |\Sigma_{\mathcal{F}}|k, \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L) + \sum_{\substack{\mu \gg 0 \\ \mu \in \mathfrak{o}_{\mathcal{F}}}} \Big(\sum_{\substack{\mu \in \mathfrak{a} \subseteq \mathfrak{o}_{\mathcal{F}} \\ \mathfrak{a} \text{ prime to } S}} \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L(\mathfrak{a}) \mathcal{N}_{\mathcal{F}}(\mathfrak{a})^{|\Sigma_{\mathcal{F}}|k-1} \Big) q_{\mathcal{F}}^{\mu}.$$

Appealing to [RW7, Lemmas 7 and 6], we apply $\mathrm{res}_{\mathcal{F}}^{\mathcal{K}}$ and the Hecke operator $U_{[\mathcal{F}:\mathcal{K}]}$ to the modular form $E_{\mathcal{F}}$ displayed above and obtain, for each $\mathcal{F}$, the new modular form

$$\mathcal{E}_{\mathcal{F}} = (\mathrm{res}_{\mathcal{F}}^{\mathcal{K}} E_{\mathcal{F}})_{|_{[\mathcal{F}:\mathcal{K}]|\Sigma_{\mathcal{F}}|k}} U_{[\mathcal{F}:\mathcal{K}]}$$

of weight $|\Sigma|k$ in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ and with standard q-expansion

$$\tilde{\zeta}_{\mathcal{F}}(1 - |\Sigma_{\mathcal{F}}|k, \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L) + \sum_{\substack{\alpha \gg 0 \\ \alpha \in \mathfrak{o}_{\mathcal{K}}}} \Big(\sum_{[\alpha]_{\mathcal{F}}} \varepsilon_L(\mathfrak{a}_{\mathcal{F}} \mathfrak{o}_L) \mathcal{N}_{\mathcal{F}}(\mathfrak{a}_{\mathcal{F}})^{[L:\mathcal{F}]k-1} \Big) q_{\mathcal{K}}^{\alpha},$$

where, for $\mathcal{K} \subseteq \mathcal{F} \subseteq L$, $[\alpha]_{\mathcal{F}}$ denotes the set of all pairs $(\alpha_{\mathcal{F}}, \mathfrak{a}_{\mathcal{F}})$ satisfying

$$0 \ll \alpha_{\mathcal{F}} \in \mathfrak{a}_{\mathcal{F}} \subseteq \mathfrak{o}_{\mathcal{F}}, \qquad \mathfrak{a}_{\mathcal{F}} \text{ prime to } S, \qquad \mathrm{tr}_{\mathcal{F}/\mathcal{K}}(\alpha_{\mathcal{F}}) = [\mathcal{F} : \mathcal{K}]\alpha.$$

Here we have used $(\varepsilon_L \circ \mathrm{ver}_{\mathcal{F}}^L)(\mathfrak{a}_{\mathcal{F}}) = (\varepsilon_L \mathrm{ver}_{\mathcal{F}}^L)(\mathfrak{a}_{\mathcal{F}}, \mathcal{F}_S/\mathcal{F}) = \varepsilon_L(\mathfrak{a}_{\mathcal{F}} \mathfrak{o}_L)$.

We Möbius-sum all these $\mathcal{E}_{\mathcal{F}}$ and arrive at the modular form

$$\mathcal{E} \stackrel{\mathrm{def}}{=} \sum_{\mathcal{K} \subseteq \mathcal{F} \subseteq L} \mu_\Sigma(\Sigma_{\mathcal{F}}) \mathcal{E}_{\mathcal{F}}$$

in $\mathcal{M}_{|\Sigma|k}(\Gamma_{00}(\mathfrak{f}), \mathbb{C})$ whose standard q-expansion has constant coefficient

$$\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}}) \tilde{\zeta}_{\mathcal{F}}(1 - |\Sigma_{\mathcal{F}}|k, \varepsilon_L \mathrm{ver}_{\mathcal{F}}^L) \tag{3.0}$$

and higher coefficients

$$\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}}) \sum_{[\alpha]_{\mathcal{F}}} \varepsilon_L(\mathfrak{a}_{\mathcal{F}} \mathfrak{o}_L) \mathcal{N}_{\mathcal{F}}(\mathfrak{a}_{\mathcal{F}})^{[L:\mathcal{F}]k-1}, \tag{3.α}$$

at $0 \ll \alpha \in \mathfrak{o}_{\mathcal{K}}$.

Lemma 3.6. *Assume that k is an even positive integer and ε_L is an even locally constant $\mathbb{Z}_p$-valued function on H_S. Then*

$$\sum_{\mathcal{K}\subseteq\mathcal{F}\subseteq L}\mu_\Sigma(\Sigma_\mathcal{F})\sum_{[\alpha]_\mathcal{F}}\varepsilon_L(\mathfrak{a}_\mathcal{F}\mathfrak{o}_L)\mathcal{N}_\mathcal{F}(\mathfrak{a}_\mathcal{F})^{[L:\mathcal{F}]k-1}\equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}.$$

Proof. Utilizing the natural action of Σ on the set

$$[\alpha]_L=\{(\alpha_L,\mathfrak{a}_L):0\ll\alpha_L\in\mathfrak{a}_L\subseteq\mathfrak{o}_L,\ \mathfrak{a}_L \text{ prime to } S,\ \mathrm{tr}_{L/\mathcal{K}}(\alpha_L)=[L:\mathcal{K}]\alpha\},$$

given by $(\alpha_L,\mathfrak{a}_L)^\sigma=(\alpha_L^\sigma,\mathfrak{a}_L^\sigma)$, we identify the set $[\alpha]_\mathcal{F}$ with the subset $[\alpha]_L^{\Sigma_\mathcal{F}}$ of $\Sigma_\mathcal{F}$-fixed points in $[\alpha]_L$ by means of the map

$$\iota_\mathcal{F}:[\alpha]_\mathcal{F}\to[\alpha]_L,\quad(\alpha_\mathcal{F},\mathfrak{a}_\mathcal{F})\mapsto(\alpha_\mathcal{F},\mathfrak{a}_\mathcal{F}\mathfrak{o}_L).$$

Indeed, $\iota_\mathcal{F}$ is obviously injective and has image $[\alpha]_L^{\Sigma_\mathcal{F}}$ because

- $\alpha_L\in\mathcal{F}$ if $\alpha_L^\sigma=\alpha_L$ for all $\sigma\in\Sigma_\mathcal{F}$,
- $\mathfrak{a}_L=\mathfrak{a}_L^\sigma$ for all $\sigma\in\Sigma_\mathcal{F}$ implies $\mathfrak{a}_L=\mathfrak{a}_\mathcal{F}\mathfrak{o}_L$ for some integral ideal $\mathfrak{a}_\mathcal{F}$ of $\mathcal{F}$, since $\mathfrak{a}_L$ is prime to S and whence every prime divisor of $\mathfrak{a}_L$ is unramified in $L/\mathcal{K}$.

As a first consequence, formula (3.α) can be rewritten as

$$(3.\alpha')\qquad \sum_{\mathcal{K}\subseteq\mathcal{F}\subseteq L}\mu_\Sigma(\Sigma_\mathcal{F})\sum_{(\beta,\mathfrak{b})\in[\alpha]_L^{\Sigma_\mathcal{F}}}\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-\frac{1}{|\Sigma_\mathcal{F}|}},$$

as $\mathcal{N}_L(\mathfrak{a}_\mathcal{F}\mathfrak{o}_L)=\mathcal{N}_\mathcal{F}(\mathfrak{a}_\mathcal{F})^{[L:\mathcal{F}]}$.

We isolate the part of the sum (3.α') that belongs to a fixed $(\beta,\mathfrak{b})\in[\alpha]_L$. It is

$$\sum_{\mathcal{F}\text{ so }\Sigma_\mathcal{F}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})}\mu_\Sigma(\Sigma_\mathcal{F})\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^{k-\frac{1}{|\Sigma_\mathcal{F}|}},$$

and correspondingly, with $(\beta,\mathfrak{b})$ replaced by $(\beta,\mathfrak{b})^\sigma$,

$$\sum_{\mathcal{F}\text{ so }\Sigma_\mathcal{F}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma}\mu_\Sigma(\Sigma_\mathcal{F})\varepsilon_L(\mathfrak{b}^\sigma)\mathcal{N}_L(\mathfrak{b}^\sigma)^{k-\frac{1}{|\Sigma_\mathcal{F}|}}.$$

The group Σ acts on H_S and so on ε_L by ${\varepsilon_L}^\sigma(h)=\varepsilon_L(h^{\sigma^{-1}})$. We now consider the part of the sum (3.α') that belongs to the $\mathrm{St}_\Sigma(\varepsilon_L)$-orbit of $(\beta,\mathfrak{b})$:

$$\text{(i)}\qquad \sum_{\sigma\in[\mathrm{St}_\Sigma(\varepsilon_L)\cap\mathrm{St}_\Sigma(\beta,\mathfrak{b})\backslash\mathrm{St}_\Sigma(\varepsilon_L)]}\varepsilon_L(\mathfrak{b})\mathcal{N}_L(\mathfrak{b})^k\sum_{\mathcal{F}\text{ so }\Sigma_\mathcal{F}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma}\mu_\Sigma(\Sigma_\mathcal{F})\mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\Sigma_\mathcal{F}|}}.$$

Here, $[\mathrm{St}_\Sigma(\varepsilon_L)\cap\mathrm{St}_\Sigma(\beta,\mathfrak{b})\backslash\mathrm{St}_\Sigma(\varepsilon_L)]$ is a set of right coset representatives of $\mathrm{St}_\Sigma(\varepsilon_L)\cap\mathrm{St}_\Sigma(\beta,\mathfrak{b})$ in $\mathrm{St}_\Sigma(\varepsilon_L)$. Note that the sum (3.$\alpha'$) is the sum of all such orbit sums (i).

Because of

- $\Sigma_\mathcal{F}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma \iff \Sigma_{\mathcal{F}^{\sigma^{-1}}}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})$ (as $\mathrm{St}_\Sigma(\beta,\mathfrak{b})^\sigma=(\mathrm{St}_\Sigma(\beta,\mathfrak{b}))^\sigma$ and $\Sigma_\mathcal{F}^{\sigma^{-1}}=\Sigma_{\mathcal{F}^{\sigma^{-1}}}$),
- $\mu_\Sigma(\Sigma_\mathcal{F})=\mu_\Sigma(\Sigma_{\mathcal{F}^{\sigma^{-1}}})$ (a direct consequence of the definition of the Möbius function), and
- $|\Sigma_\mathcal{F}|=|\Sigma_{\mathcal{F}^{\sigma^{-1}}}|$,

the inner sums of (i) are independent of σ. Hence, if we can show

$$\text{(ii)}\qquad \sum_{\mathcal{F}\text{ so }\Sigma_\mathcal{F}\leq\mathrm{St}_\Sigma(\beta,\mathfrak{b})}\mu_\Sigma(\Sigma_\mathcal{F})\mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\Sigma_\mathcal{F}|}}\equiv 0\mod |\mathrm{St}_\Sigma(\varepsilon_L)\cap\mathrm{St}_\Sigma(\beta,\mathfrak{b})|\mathbb{Z}_{(p)},$$

then sum (i) is $\equiv 0 \mod |\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$ and the proof of the proposition will be complete.

For (ii), we first shorten the notation by setting $P = \mathrm{St}_\Sigma(\beta, \mathfrak{b}) \le \Sigma$ and $r = \mathcal{N}_L(\mathfrak{b})^{-\frac{1}{|\mathrm{St}_\Sigma(\beta,\mathfrak{b})|}}$ which is a unit in $\mathbb{Z}_{(p)}$ because $\mathfrak{b} = \mathfrak{a}_M \mathfrak{o}_L$ if $\Sigma_M = P$ (compare with the proof of (3.α') above). This turns the left-hand side of (ii) into $\sum_{1\le P'\le P} \mu_P(P') r^{[P:P']}$, as obviously $\mu_\Sigma(P') = \mu_P(P')$. Applying now Claim 3.A below we obtain

$$\sum_{1\le P'\le P} \mu_P(P') r^{[P:P']} \equiv 0 \mod |P|\mathbb{Z}_{(p)},$$

which is even stronger than (ii).

Claim 3.A. Let P be a finite p-group and r a unit in $\mathbb{Z}_{(p)}$. Then

$$\sum_{1\le P'\le P} \mu_P(P') r^{[P:P']} \equiv 0 \mod |P|\mathbb{Z}_{(p)} \ .$$

Proof. Let $z \in \mathbb{Z}_p$ satisfy $z^{p-1} = 1$ and $r \equiv z \mod p$; hence

$$r^{p^n} \equiv z^{p^n} = z \mod p^{n+1}, \quad \text{for } n \ge 0.$$

From [HIO, Corollary 3.5] we obtain $|P'| \,\Big|\, p\mu_{P'}(P')$, for $P' \le P$. Therefore, and as $\mu_P(P') = \mu_{P'}(P')$, $\mu_P(P') r^{[P:P']} \equiv \mu_P(P') z \mod |P|\mathbb{Z}_p$. Consequently,

$$\sum_{1\le P'\le P} \mu_P(P') r^{[P:P']} \equiv z \cdot \sum_{1\le P'\le P} \mu_P(P') = 0 \mod |P|\mathbb{Z}_p \ ,$$

as $\sum_{1\le P'\le P} \mu_P(P') = 0$, by the definition of the Möbius function.

Since $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)}$, this proves Claim 3.A and also ends the proof of Lemma 3.6. $\square$

Choosing $\gamma \in \hat{\mathcal{K}}^\times$ with $j(\gamma) = g$ and denoting by $\mathcal{E}_\gamma$ the q-expansion of $\mathcal{E}$ at the cusp determined by γ, set $\mathcal{E}(\gamma) \overset{\mathrm{def}}{=} \mathcal{N}_K(\gamma_p)^{-|\Sigma|k}\mathcal{E}_\gamma$ (so $\mathcal{E}(1)$ is the standard q-expansion of $\mathcal{E}$). Then [DR, (0.3)] implies that

> *the constant term of $\mathcal{E}(1) - \mathcal{E}(\gamma)$ is contained in $p^n\mathbb{Z}_{(p)}$, provided that $\mathcal{E}(1)$ has all non-constant coefficients contained in $p^n\mathbb{Z}_{(p)}$,*

and, by Lemma 3.6, this applies with $p^n = |\mathrm{St}_\Sigma(\varepsilon_L)|$, so the constant term of $\mathcal{E}(1) - \mathcal{E}(\gamma)$ is in $|\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$.

3.3. **Conclusion of the proof: Special cusps.** By what has been said at the end of the previous subsection we need to compute the constant coefficients of $\mathcal{E}(\gamma)$ and $\mathcal{E}(1) - \mathcal{E}(\gamma)$. We do this for the γ's of Lemma 3.5 and then prove Theorem 5.

Lemma 3.7. *Setting $g = j(\gamma)$, with γ as in Lemma 3.5, the constant term of $\mathcal{E}(\gamma)$ is*

$$\mathcal{N}_\mathcal{K}(g)^{|\Sigma|k} \cdot \big(\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_\mathcal{F}) \tilde{\zeta}_\mathcal{F}(1 - |\Sigma_\mathcal{F}|k, (\varepsilon_L \mathrm{ver}^L_\mathcal{F})_{g_\mathcal{F}})\,\big).$$

Proof. We first show that $\mathrm{res}^\mathcal{K}_\mathcal{F} E_\mathcal{F}$ has constant term

$$\mathcal{N}_\mathcal{K}((\gamma))^{|\Sigma|k} \tilde{\zeta}_\mathcal{F}(1 - |\Sigma_\mathcal{F}|k, (\varepsilon_L \mathrm{ver}^L_\mathcal{F})_{g_\mathcal{F}})$$

at the cusp determined by $\gamma \in \hat{\mathcal{K}}^\times$. By (2) of [RW7, Lemma 7], this constant term of $\operatorname{res}^{\mathcal{K}}_{\mathcal{F}} E_{\mathcal{F}}$ is equal to the one of $E_{\mathcal{F}}$ at the cusp determined by $\gamma \in \hat{\mathcal{F}}^\times$, whence, by (2) of [RW7, Proposition 8], equals

$$\mathcal{N}_{\mathcal{F}}((\gamma)_{\mathcal{F}})^{|\Sigma_{\mathcal{F}}|k}\tilde{\zeta}_{\mathcal{F}}(1-|\Sigma_{\mathcal{F}}|k,(\varepsilon_L \operatorname{ver}^L_{\mathcal{F}})_{g_{\mathcal{F}}}) = \mathcal{N}_{\mathcal{K}}((\gamma))^{|\Sigma|k}\tilde{\zeta}_{\mathcal{F}}(1-|\Sigma_{\mathcal{F}}|k,(\varepsilon_L \operatorname{ver}^L_{\mathcal{F}})_{g_{\mathcal{F}}}),$$

because $(\gamma)_{\mathcal{F}} = (\gamma)_{\mathcal{K}}\mathfrak{o}_{\mathcal{F}}$.

We next check that $(\operatorname{res}^{\mathcal{K}}_{\mathcal{F}} E_{\mathcal{F}})_{||\Sigma|k} U_{[\mathcal{F}:\mathcal{K}]}$ and $\operatorname{res}^{\mathcal{K}}_{\mathcal{F}} E_{\mathcal{F}}$ have the same constant term at the cusp determined by $\gamma \in \hat{\mathcal{K}}^\times$. By (a) of Lemma 3.5, $M \stackrel{\text{def}}{=} \left(\begin{smallmatrix}\gamma & 0\\ 0 & \gamma^{-1}\end{smallmatrix}\right)$ is in $\widehat{\Gamma_{00}(\mathfrak{f})}$, so $M = M_1M_2$ with $M_2 = I$ in the notation of [DR, p. 262]; hence, by [DR, 5.8], the constant terms referred to above are the constant terms of the standard q-expansions of $((\operatorname{res}^{\mathcal{K}}_{\mathcal{F}})_{||\Sigma|k} U_{[\mathcal{F}:\mathcal{K}]})|M_2 = (\operatorname{res}^{\mathcal{K}}_{\mathcal{F}})_{||\Sigma|k} U_{[\mathcal{F}:\mathcal{K}]}$ and $(\operatorname{res}^{\mathcal{K}}_{\mathcal{F}} E_{\mathcal{F}})|M_2 = \operatorname{res}^{\mathcal{K}}_{\mathcal{F}} E_{\mathcal{F}}$, which agree by [RW7, Lemma 6].

Möbius-summing, we have shown that $\mathcal{E}_\gamma$ has constant term

$$\mathcal{N}_{\mathcal{K}}((\gamma))^{|\Sigma|k}\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}})\tilde{\zeta}_{\mathcal{F}}(1-|\Sigma_{\mathcal{F}}|k,(\varepsilon_L \operatorname{ver}^L_{\mathcal{F}})_{g_{\mathcal{F}}});$$

hence $\mathcal{E}(\gamma)$ has the required constant term, because $\mathcal{N}_{\mathcal{K}}(\gamma_p)^{-1}\mathcal{N}_K((\gamma)) = \mathcal{N}_{\mathcal{K},p}(g)$, by (3) of [RW7, Proposition 8].

This completes the proof of Lemma 3.7. □

We can now complete the *proof* of Theorem 5. It follows from equation (3.0) and Lemma 3.7 that for such γ the constant term of $\mathcal{E}(1) - \mathcal{E}(\gamma)$ is

$$\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}})[\tilde{\zeta}_{\mathcal{F}}(1-|\Sigma_{\mathcal{F}}|k, \varepsilon_L\operatorname{ver}^L_{\mathcal{F}}) - \mathcal{N}_{\mathcal{K}}(g)^{|\Sigma|k}\tilde{\zeta}_{\mathcal{F}}(1-|\Sigma_{\mathcal{F}}|k,(\varepsilon_L\operatorname{ver}^L_{\mathcal{F}})_{g_{\mathcal{F}}})]$$
$$= \sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}})\tilde{\Delta}_{g_{\mathcal{F}}}(1-|\Sigma_{\mathcal{F}}|k,\varepsilon_L\operatorname{ver}^L_{\mathcal{F}}),$$

the latter since $\mathcal{N}_{\mathcal{F}}(\operatorname{ver}^{\mathcal{F}}_K g) = \mathcal{N}_{\mathcal{K}}(g)^{[\mathcal{F}:\mathcal{K}]}$.

Note at this stage that it is this sum, $\sum_{\mathcal{F}} \mu_\Sigma(\Sigma_{\mathcal{F}})\tilde{\Delta}_{g_{\mathcal{F}}}(1-|\Sigma_{\mathcal{F}}|k,\varepsilon_L\operatorname{ver}^L_{\mathcal{F}})$, which is referred to in (3.4$\star$). The last sentence of §3.2 now implies that this sum is $\equiv 0$ mod $|\mathrm{St}_\Sigma(\varepsilon_L)|\mathbb{Z}_{(p)}$, for every even locally constant ε_L, thus verifying the hypothesis of Lemma 3.4.

Theorem 5 is now the conclusion of Lemma 3.4. □

4. Deriving Theorem 3 from §3

For Theorem 3 we identify the (Deligne-Ribet) prime number p occurring in the preceding section with the given (Iwasawa) prime number l, the fields $\mathcal{K}$, L of §3 with our field k and the fixed field K^A of the given abelian normal open subgroup A in $G = G(K/k)$, respectively; so L is a Galois extension of k with group Q (the Σ in §3) and the intermediate fields f in L/k (the $\mathcal{F}$ in §3) correspond to the subgroups U/A of Q. Moreover, the set S of §3 gets replaced by the set S' of all non-archimedean primes of the set S fixed in the introduction[14]. Set $G_{S'} = G(k_{S'}/k)$, $H_{S'} = G(L_{S'}/L)$.[15] Note that the cyclotomic $\mathbb{Z}_l$-extension L_∞ of L is contained in $L^+_{S'}$ and abbreviate $G(L_\infty/L)$ by Γ_L.

[14]Hence our K here is contained in $L_{S'}$.

[15]If Leopoldt's conjecture fails, then these groups are not of our type, though abelian.

Let λ be a pseudomeasure on $H_{S'}$ in the language of [Se], i.e., an element of the total ring of fractions of the commutative ring $\mathbb{Z}_l[[H_{S'}]]$ so that $(1-h)\lambda$ is in $\mathbb{Z}_l[[H_{S'}]]$ for all $h \in H_{S'}$.

Lemma 4.1. *There is a unique pseudomeasure λ^A on A so that $(1-a)\lambda^A$ is the image, under $H_{S'} \twoheadrightarrow A$, of $(1-h)\lambda$ for every $h \in H_{S'}$ with image $a \in A$. Moreover, λ^A is in $\Lambda_\bullet A$.*

We call λ^A the deflation $\mathrm{defl}^A_{H_{S'}}(\lambda)$ of the pseudomeasure λ. Note that if λ is Serre's pseudomeasure λ_L, then $\lambda_L^A = \lambda_{K/L}$.

Proof. Pick an element $h \in H_{S'}$ with non-trivial image under $H_{S'} \to \Gamma_L$. Let Γ_h be the subgroup of $H_{S'}$ topologically generated by h and let $M_h^{-1}\mathbb{Z}_l[[H_{S'}]]$ be the localization of $\mathbb{Z}_l[[H_{S'}]]$ by inverting the multiplicative set $M_h = \mathbb{Z}_l[[\Gamma_h]] \setminus l\mathbb{Z}_l[\Gamma_h]]$. Then $\lambda \in M_h^{-1}\mathbb{Z}_l[[H_{S'}]]$ because $(1-h)\lambda \in \mathbb{Z}_l[[H_{S'}]]$. The natural map $\mathrm{defl}^A_{H_{S'}}$: $\mathbb{Z}_l[[H_{S'}]] \twoheadrightarrow \mathbb{Z}_l[[A]] = \Lambda A$ induces $\mathrm{defl}^A_{H_{S'}} : M_h^{-1}\mathbb{Z}_l[[H_{S'}]] \xrightarrow{\eta_h} \Lambda_\bullet A$. Note that $\Lambda_\bullet A$ is independent, as a subring of $\mathcal{Q}A$, of the choice of h, by Remark 1.1.

Then $\eta_h(\lambda) \in \Lambda_\bullet A$ is independent of the choice of h as above, for if h' is another, then, with a the image of h under $H_{S'} \twoheadrightarrow A$,

$$(1-a)\eta_h(\lambda) = \eta_h((1-h)\lambda) \doteq \eta_{h'}((1-h)\lambda) = (1-a)\eta_{h'}(\lambda),$$

with $\doteq$ due to $(1-a)\lambda \in \mathbb{Z}_l[[H_{S'}]]$. Thus $\eta_h(\lambda) = \eta_{h'}(\lambda)$ because $1-a \in (\Lambda_\bullet A)^\times$. This common image is λ^A.

Finally, λ^A is a pseudomeasure on A for, if $a \in A$, then, choosing a preimage $b \in H_{S'}$ for a, we have $(1-b)\lambda \in \mathbb{Z}_l[[H_{S'}]]$ mapping to $\eta_h((1-b)\lambda) = (1-a)\lambda^A$ in ΛA for any h as above. Thus, Lemma 4.1 is shown. □

Theorem 5 says that there exists a $g \in G_{S'}$, with g_f having image $\neq 1$ in Γ_f for $k \subseteq f \subseteq L$, so that

$$\sum_{k\subseteq f\subseteq L} \mu_Q(G(L/f))\mathrm{ver}_f^L(\tilde{\lambda}_{g_f}) \tag{4.a}$$

has image in $\mathrm{tr}_Q(\mathbb{Z}_l[[H^+_{S'}]])$, under $\mathrm{defl}^{H^+_{S'}}_{H_{S'}} : \mathbb{Z}_l[[H_{S'}]] \to \mathbb{Z}_l[[H^+_{S'}]]$.

We first remove the $\tilde{}$ on λ_{g_f} in (4.a). Recall that $\tilde{\lambda}_f = 2^{-[f:\mathbb{Q}]}\lambda_f$ implies $\tilde{\lambda}_{g_f} = 2^{-[f:\mathbb{Q}]}\lambda_{g_f}$. So it suffices to show that

$$\mu_Q(G(L/f))2^{-[f:\mathbb{Q}]} \equiv \mu_Q(G(L/f))2^{-[L:\mathbb{Q}]} \mod |Q|,$$

and this congruence in turn is a consequence of $|Q|$ being a power of $l \neq 2$ and

$$\begin{aligned}&\mu_Q(G(L/f))(2^{-[f:\mathbb{Q}]} - 2^{-[L:\mathbb{Q}]}) = 2^{-[L:\mathbb{Q}]}\mu_Q(G(L/f))\Big((2^{[L:f]-1})^{[f:\mathbb{Q}]} - 1\Big),\\ &2^{[L:f]-1} \equiv 1 \bmod l \text{ by } 2^l \equiv 2 \bmod l, \quad \text{so} \quad (2^{[L:f]-1})^{[f:\mathbb{Q}]} \equiv 1 \bmod l[f:\mathbb{Q}],\end{aligned}$$

so it now suffices to show that $\mu_Q(G(L/f))l[f:\mathbb{Q}]$ is divisible by $|Q| = [L:k]$. Due to [HIO, Corollary 3.5], $\mu_Q(G(L/f))l$ is divisible by $[L:f]$, whence $\mu_Q(G(L/f))l[f:\mathbb{Q}]$ by $[L:f][f:\mathbb{Q}] = [L:\mathbb{Q}]$.

Next, setting $h = \mathrm{ver}_k^L g = \mathrm{ver}_f^L \mathrm{ver}_k^f g = \mathrm{ver}_f^L g_f$, we have $(1-g_f)\lambda_f = \lambda_{g_f} \in \mathbb{Z}_l[[G(f_{S'}/f)]]$ implying that

$$\lambda_f \in M_{g_f}^{-1}\mathbb{Z}_l[[G(f_{S'}/f)]]$$

(in the notation of the proof of Lemma 4.1); hence

$$\mathrm{ver}_f^L(\lambda_f) \in M_h^{-1}\mathbb{Z}_l[[H_{S'}]]$$

because $\mathrm{ver}_f^L = \mathrm{ver}_{G(L_{S'}/f)^{\mathrm{ab}}}^{H_{S'}} = \mathrm{ver}_{G(f_{S'}/f)}^{H_{S'}}$. Thus

$$\sum_{k\subseteq f\subseteq L} \mu_Q(G(L/f))\mathrm{ver}_f^L((1-g_f)\lambda_f) = (1-h)\sum_{k\subseteq f\subseteq L}\mu_Q(G(L/f))\mathrm{ver}_f^L(\lambda_f)$$

implies that every term of the second sum is in $M_h^{-1}\mathbb{Z}_l[[H_{S'}]]$. Applying $\mathrm{defl}_{H_{S'}}^A = \mathrm{defl}_{H_{S'}^+}^A \mathrm{defl}_{H_{S'}}^{H_{S'}^+}$ to (4.a) then yields that

$$\text{(4.b)} \qquad \textstyle\sum_{k\subseteq f\subseteq L}\mu_Q(G(L/f))\mathrm{defl}_{H_{S'}}^A \mathrm{ver}_f^L(\lambda_f)$$

is in $\mathrm{defl}_{H_{S'}^+}^A(\mathrm{tr}_Q(M_{h^+}^{-1}\mathbb{Z}_l[[H_{S'}]])) = \mathrm{tr}_Q(\Lambda_\bullet A)$, where $h^+ = \mathrm{defl}_{H_{S'}}^{H_{S'}^+} h$.

In order to derive Theorem 3 from this, we set $U = G(K/f)$ and show that

$$\text{(4.c)} \qquad \mathrm{defl}_{H_{S'}}^A \mathrm{ver}_f^L = \mathrm{defl}_{H_{S'}}^A \mathrm{ver}_{G(f_{S'}/f)}^{H_{S'}} = \mathrm{ver}_{U^{\mathrm{ab}}}^A \mathrm{defl}_{G(f_{S'}/f)}^{U^{\mathrm{ab}}}.$$

Here, the second equality follows from the commutative diagram

$$\begin{array}{ccccc} G(L_{S'}/K) & \rightarrowtail & G(L_{S'}/f) & \twoheadrightarrow & G(K/f) \\ \| & & \cup & & \cup \\ G(L_{S'}/K) & \rightarrowtail & G(L_{S'}/L) & \twoheadrightarrow & G(K/L) \end{array}$$

allowing transfers to be computed by using corresponding coset representatives, giving

$$\mathrm{defl}_{G(L_{S'}/L)}^{G(K/L)} \mathrm{ver}_{G(L_{S'}/f)}^{G(L_{S'}/L)} = \mathrm{ver}_{G(K/f)}^{G(K/L)} \mathrm{defl}_{G(L_{S'}/f)}^{G(K/f)}.$$

Factoring the transfers through the abelianisations

$$\begin{aligned} \mathrm{defl}_{G(L_{S'}/L)}^{G(K/L)} \mathrm{ver}_{G(L_{S'}/f)^{\mathrm{ab}}}^{G(L_{S'}/L)} \mathrm{defl}_{G(L_{S'}/f)}^{G(L_{S'}/f)^{\mathrm{ab}}} &= \mathrm{ver}_{G(K/f)^{\mathrm{ab}}}^{G(K/L)} \mathrm{defl}_{G(K/f)}^{G(K/f)^{\mathrm{ab}}} \mathrm{defl}_{G(L_{S'}/f)}^{G(K/f)} \\ &= \mathrm{ver}_{G(K/f)^{\mathrm{ab}}}^{G(K/L)} \mathrm{defl}_{G(f_{S'}/f)}^{G(K/f)^{\mathrm{ab}}} \mathrm{defl}_{G(L_{S'}/f)}^{G(L_{S'}/f)^{\mathrm{ab}}} \end{aligned}$$

and cancelling the (surjective) maps on the right gives (4.c).

Combining (4.b) with the commutative square

$$\begin{array}{ccc} M_{g_f}^{-1}\mathbb{Z}_l[[G(f_{S'}/f)]] & \xrightarrow{\mathrm{ver}_{G(f_{S'}/f)}^{H_{S'}}} & M_h^{-1}\mathbb{Z}_l[[H_{S'}]] \\ \mathrm{defl}_{G(f_{S'}/f)}^{U^{\mathrm{ab}}} \downarrow & & \mathrm{defl}_{H_{S'}}^A \downarrow \\ \Lambda_\bullet U^{\mathrm{ab}} & \xrightarrow{\mathrm{ver}_{U^{\mathrm{ab}}}^A} & \Lambda_\bullet A, \end{array}$$

which follows from (4.c), we get the statement of Theorem 3, by applying Lemma 4.1 and $U^{\mathrm{ab}} = G(K^{[U,U]}/K^U)$. □

Remark 4.1. Consider the localization $\mathbb{Z}_l[[H_{S'}^+]]_\bullet$ of [RW7, bottom of p. 715], which results from inverting the multiplicative set of elements of $\mathbb{Z}_l[[H_{S'}^+]]$ whose image in $\mathbb{Z}_l[[\Gamma_L]]$ is not in $l\mathbb{Z}_l[[\Gamma_L]]$. Observe that if $A \geq \langle z\rangle$ and $z \neq 1$, then $\hat{z} = \sum_{i=0}^{|z|-1} z^i$ has image $|z|$ in $\mathbb{Z}_l[[\Gamma_L]]$ and so is not in $l\mathbb{Z}_l[[\Gamma_L]]$; hence $\hat{z} \in \mathbb{Z}_l[[H_{S'}^+]]_\bullet^\times$ and $\hat{z}(z-1) = 0$ imply $z = 1$ in $\mathbb{Z}_l[[H_{S'}]]_\bullet$. It follows that specializing $H_{S'}^+$ to A induces a map from $\mathbb{Z}_l[[H_{S'}^+]]_\bullet$ to the $\Lambda_\bullet A$ of the current paper only when A is a pro-l group. In particular, the application of Theorem 5 in §3 to equivariant Iwasawa theory is different from that proposed in [RW7, pp. 715–716].

5. Proof of Theorem 4

We fix some notation. A is an abelian normal open subgroup of $G = G(K/k) = \langle z\rangle \times G[l]$ containing $\langle z\rangle$ with factor group $Q = G/A$, and Γ, C are central subgroups of G contained in A, with $\Gamma \simeq \mathbb{Z}_l$ open and C of exponent l. Further, $U \leq G$ is open and contains C and $\langle z\rangle$.

Recall that Res_G^U satisfies diagram (1.D2) and is discussed in §7 and [RW8, Proposition A]. In particular, we know that 'Res' is additive and transitive and that it preserves integrality. Also, if $g \in G$, then conjugation by g canonically induces maps $\Lambda_\wedge U \to \Lambda_\wedge U^g$ and $T(\Lambda_\wedge U) \to T(\Lambda_\wedge U^g)$, the latter by $\tau_U(u)^g = \tau_{U^g}(u^g)$ for $u \in U$.[16]

Lemma 5.1. (1) *If* $[G:U] = l$ *and* $\alpha \in \Lambda_\wedge\Gamma$, $g \in G$, *we have*

$$\mathrm{Res}_G^U(\alpha\tau_G(g)) = \begin{cases} \alpha\sum_{x\in G/U}\tau_U(g^x) & \text{if } g \in U, \\ \Psi(\alpha)\tau_U(g^l) & \text{if } g \notin U, \end{cases}$$

where $\Psi : \Lambda_\wedge\Gamma \to \Lambda_\wedge\Gamma$ *is the continuous* $\mathbb{Z}_l$*-linear ring homomorphism induced by* $\gamma \mapsto \gamma^l$ *for* $\gamma \in \Gamma$.

(2) *For* $g \in G$, $c \in C$, *and* $\alpha \in \Lambda_\wedge\Gamma$,

$$\mathrm{Res}_G^U(\alpha\tau_G(g(c-1))) = \alpha\sum_{x\in G/U}\dot\tau_U(g^x(c-1))$$

with $x \in G/U$ *meaning* $G = \dot\bigcup_x xU$ *and* $\dot\tau_U : \Lambda_\wedge G \to T(\Lambda_\wedge U)$ *defined by extending* τ_U *to take the value* 0 *outside of* $\Lambda_\wedge U$.

(3) *Let* $V \leq U$ *be open subgroups of* G *containing* $\langle z\rangle$ *and let* $g \in G$, $w \in T(\Lambda_\wedge G)$. *Then* $(\mathrm{Res}_U^V w)^g = \mathrm{Res}_{U^g}^{V^g}(w^g)$.

Proof. For $\alpha \in \Gamma$ the claimed formula (1) is already given at the bottom of [RW8, p. 127]; additivity and continuity then imply it in general.

For (2), we first remark that $\sum_{x\in G/U}\dot\tau(g^x(c-1))$ is independent of a special choice of coset representatives x of U in G and on replacing g by g^s for $s \in G$. We proceed by induction on $[G:U]$. If $U < G$, choose $U \leq G' < G$, $[G:G'] = l$, so $G' \lhd G$. Then

$$\begin{aligned}\mathrm{Res}_G^U(\alpha\tau_G(g(c-1))) &= \mathrm{Res}_{G'}^U\mathrm{Res}_G^{G'}(\alpha\tau_G(g(c-1)))\\ &= \mathrm{Res}_{G'}^U\left(\begin{cases} \alpha\sum_{x''\in G/G'}\tau_{G'}(g^{x''}(c-1)) & \text{if } g \in G', \\ \Psi(\alpha)\tau_{G'}((g^{x''}c)^l - (g^{x''})^l) & \text{if} g \notin G' \end{cases}\right)\\ &= \begin{cases} \sum_{x''\in G/G'}\mathrm{Res}_{G'}^U\alpha\tau_{G'}(g^{x''}(c-1)) & \text{if } g \in G', \\ 0 & \text{if } g \notin G'. \end{cases}\end{aligned}$$

In the first case, the induction hypothesis turns this into

$$\alpha\sum_{x''\in G/G'}\sum_{x'\in G'/U}\dot\tau_U(g^{x''x'}(c-1)) = \alpha\sum_{x\in G/U}\dot\tau_U(g^x(c-1)),$$

as required. In the second case, when $g \notin G'$, we must show that the right-hand side of the assertion is zero, which however is clear since $g^x \in U$ implies $g^x \in G'$; hence $g \in G'$. This finishes the proof of (2).

[16] $\tau_U : \Lambda_\wedge U \to T(\Lambda_\wedge U)$ is the natural map.

For (3), choose a sequence of subgroups $V = V_0 < V_1 < \cdots < V_n = U$ with $[V_{i+1} : V_i] = l$, $0 \le i \le n-1$, and then combine transitivity of 'Res' with induction on n to arrive at

$$(\mathrm{Res}_U^V w)^g = (\mathrm{Res}_{V_1}^V(\mathrm{Res}_U^{V_1} w))^g \stackrel{!}{=} \mathrm{Res}_{V_1^g}^{V^g}((\mathrm{Res}_U^{V_1} w)^g) = \mathrm{Res}_{V_1^g}^{V^g}(\mathrm{Res}_{U^g}^{V_1^g} w^g) = \mathrm{Res}_{U^g}^{V^g}(w^g),$$

in which the equality $\stackrel{!}{=}$ still needs to be verified. However, here we are in the index l case, so $V^g \lhd V_1^g$, and we can apply (1).

Lemma 5.1 is established. □

Lemma 5.2. *Denote the map $G \to G/C$ by $^-$ and define $\mathfrak{a} = \ker(\Lambda_\wedge G \to \Lambda_\wedge \overline{G})$, $\mathfrak{b} = \ker(\Lambda_\wedge A \to \Lambda_\wedge \overline{A})$, so $\mathfrak{a} \rightarrowtail \Lambda_\wedge G \twoheadrightarrow \Lambda_\wedge \overline{G}$ and $\mathfrak{b} \rightarrowtail \Lambda_\wedge A \twoheadrightarrow \Lambda_\wedge \overline{A}$ are exact. Then*

(i) *$\mathfrak{b}^Q/\mathrm{tr}_Q \mathfrak{b} \to (\Lambda_\wedge A)^Q/\mathrm{tr}_Q(\Lambda_\wedge A)$ is injective,*
(ii) *$\tau_G(\mathfrak{a}) \rightarrowtail T(\Lambda_\wedge G) \twoheadrightarrow T(\Lambda_\wedge \overline{G})$ is exact and $\mathrm{Res}_G^A \tau_G(\mathfrak{a}) = \mathrm{tr}_Q \mathfrak{b}$.*

Proof. Applying Tate cohomology to the sequence defining $\mathfrak{b}$, we see that the claimed injectivity (i) is a consequence of $H^{-1}(Q, \Lambda_\wedge \overline{A}) = 0$. In order to see this vanishing, choose a central open $\Gamma \simeq \mathbb{Z}_l$ of $\overline{G}$ contained in $\overline{A}$ and pick Q-orbit representatives $a_i\Gamma$ of $\overline{A}/\Gamma$. Set $Q_i = \mathrm{stab}_Q(a_i\Gamma)$. For $q_i \in Q_i$, $a_i^{q_i - 1} \in \Gamma$ has finite order; hence $a_i^{q_i} = a_i$, i.e., $Q_i = \mathrm{stab}_Q(a_i)$. Hence we have a set of representatives of Γ in $\overline{A}$ consisting of Q-orbits $Q_i\backslash Q$ for some $Q_i \le Q$ and consequently, [RW3, Lemma 5],

$$H^{-1}(Q, \Lambda_\wedge \overline{A}) = \bigoplus_i H^{-1}(Q, \mathrm{ind}\,_{Q_i}^{Q}(\Lambda_\wedge \Gamma)) = \bigoplus_i H^{-1}(Q_i, \Lambda_\wedge \Gamma) = 0,$$

as $\Lambda_\wedge \Gamma$ has $\mathbb{Z}_l$-torsion $= 0$.

For the first claim in (ii) we only need to check exactness at the middle, or, more precisely, that $\mathrm{defl}_G^{\overline{G}} \tau_G(v) = 0$ implies $\tau_G(v) \in \tau_G(\mathfrak{a})$. Now, $\mathrm{defl}_G^{\overline{G}}(v) = \sum_i [\overline{w}_i, \overline{x}_i]$ with $w_i, x_i \in \Lambda_\wedge G$ implies that $v - \sum_i [w_i, x_i] \in \ker \mathrm{defl}_G^{\overline{G}} = \mathfrak{a}$, so apply τ_G and arrive at $\tau_G(v) \in \tau_G(\mathfrak{a})$.

Regarding the second claim of (ii), the elements of $\mathfrak{a}$ are $\Lambda_\wedge \Gamma$-linear combinations of $\tau_G(g(c-1))$. By (2) of Lemma 5.1, Res_G^A of this equals $\sum_{x \in G/A} \dot{\tau}_A(g^x(c-1)) = \mathrm{tr}_Q(g(c-1))$ if $g \in A$ and 0 if $g \notin A$; note that the $g(c-1)$, $g \in A$, $c \in C$ generate $\mathfrak{b}$ as $\Lambda_\wedge \Gamma$-module.

This finishes the proof of the lemma. □

Lemma 5.3. *Using notation as in Lemma 5.2, we have $\mathbb{L}(1 + \mathrm{tr}_Q \mathfrak{b}) \subset \mathrm{tr}_Q(\Lambda_\wedge A)$.*

Proof. Let $\beta = \sum_{a \in A, c \in C} \beta_{a,c} a(c-1)$ be an element in $\mathfrak{b}$, where $\beta_{a,c} \in \Lambda_\wedge \Gamma$ for some central open $\Gamma \simeq \mathbb{Z}_l$ contained in A. Now

$$\mathbb{L}(1 + \mathrm{tr}_Q \beta) = \frac{1}{l} \log \frac{(1 + \mathrm{tr}_Q \beta)^l}{\Psi(1 + \mathrm{tr}_Q \beta)},$$

by the argument given in [RW3, pp. 39–40], which also works in the situation when G is l-elementary and the unit u to which $\mathbb{L}$ is applied (see [RW3, p. 39, $(*)$]) is in $\Lambda_\wedge A$ rather than in $\Lambda\Gamma$ (the ring $\mathfrak{O}$ there is $\mathbb{Z}_l$ here, so the Frobenius automorphism Fr is trivial). The point is that [RW3, p. 40, $(**)$] applies to an abelian situation and so we still need only consider degree 1 characters.

Thus $\mathbb{L}(1+\mathrm{tr}_Q\beta) \in \mathrm{tr}_Q(\Lambda_\wedge A)$ if $\frac{(1+\mathrm{tr}_Q\beta)^l}{\Psi(1+\mathrm{tr}_Q\beta)} \equiv 1 \mod l\mathrm{tr}_Q(\Lambda_\wedge A)$. Since $(1+\mathrm{tr}_Q\beta)^l \equiv 1+(\mathrm{tr}_Q\beta)^l \mod l\mathrm{tr}_Q(\Lambda_\wedge A)$, it suffices to show that $(\mathrm{tr}_Q\beta)^l \equiv \Psi(\mathrm{tr}_Q\beta) \mod l\mathrm{tr}_Q(\Lambda_\wedge A)$.

Now, as $\Psi(a) = a^l$ for $a \in A$ [RW3, p. 33], $\Psi(\mathrm{tr}_Q\beta) = \sum_{a,c}\Psi(\beta_{a,c})(\mathrm{tr}_Q((ac)^l) - \mathrm{tr}_Q(a^l)) = 0$, since $(ac)^l = a^l c^l = a^l$, and we are left to check that $(\mathrm{tr}_Q\beta)^l \equiv 0 \mod l\mathrm{tr}_Q(\Lambda_\wedge A)$. But $\mathrm{tr}_Q\beta = \sum_c \kappa_c(c-1)$ with $\kappa_c = \mathrm{tr}_Q(\sum_a \beta_{a,c}a) \in \mathrm{tr}_Q(\Lambda_\wedge A)$, so

$$(\mathrm{tr}_Q\beta)^l \equiv \sum_c \kappa_c^l (c-1)^l \mod l\mathrm{tr}_Q(\Lambda_\wedge A)$$

as $\mathrm{tr}_Q(\Lambda_\wedge A)$ is an ideal in $(\Lambda_\wedge A)^Q$ and $c \in (\Lambda_\wedge A)^Q$. So $(c-1)^l \equiv 0 \mod l(\Lambda_\wedge A)^Q$ establishes Lemma 5.3. □

We are now in a position to prove Theorem 4.

If U is an open subgroup of G containing $\langle z\rangle$ and N is a finite normal subgroup of U, we write $t_{U/N}$ for t_{K^N/K^U}; similarly, we write λ_A for $\lambda_{K/K^A} \in (\Lambda_\wedge A)^\times = K_1(\Lambda_\wedge A)$. We recall from [RW8, Lemma 2] and [RW5, Lemma 2.1] that $\mathrm{Res}_G^U t_G = t_U$, $\mathrm{defl}_U^{U/N} t_U = t_{U/N}$, in this notation.

As above, set $\overline{G} = G/C$ and denote the canonical surjection $G \twoheadrightarrow \overline{G}$ by $\overline{}$. Also recall the short exact sequences

$$\mathfrak{a} \rightarrowtail \Lambda_\wedge G \twoheadrightarrow \Lambda_\wedge \overline{G}, \qquad \mathfrak{b} \rightarrowtail \Lambda_\wedge A \twoheadrightarrow \Lambda_\wedge \overline{A}.$$

For the definitions to follow we use the commutative square

$$\begin{array}{ccc} (\Lambda_\wedge G)^\times & \twoheadrightarrow & (\Lambda_\wedge \overline{G})^\times \\ \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow \\ K_1(\Lambda_\wedge G) & \rightarrow & K_1(\Lambda_\wedge \overline{G}) \end{array}$$

in which the (natural) vertical maps are surjective, since $\Lambda_\wedge G$, $\Lambda_\wedge \overline{G}$ are semi-local rings. Moreover, the top horizontal map is surjective as well, because $\ker(\Lambda_\wedge G \to \Lambda_\wedge \overline{G}) \subset \mathrm{rad}(\Lambda_\wedge G)$.

By Proposition 1.3, $t_{\overline{G}} \in T(\Lambda_\wedge \overline{G})$ implies $L_{K^C/k} = \mathrm{Det}\,\theta$ with $\theta \in K_1(\Lambda_\wedge \overline{G})$. Observe that $\mathrm{res}_{\overline{G}}^{\overline{A}}\theta = \lambda_{\overline{A}}$, because

$$\mathrm{Det}(\mathrm{res}_{\overline{G}}^{\overline{A}}\theta) = \mathrm{res}_{\overline{G}}^{\overline{A}}(\mathrm{Det}\theta) = \mathrm{res}_{\overline{G}}^{\overline{A}}(L_{K^C/k}) = L_{K^C/K^A} = \mathrm{Det}\lambda_{\overline{A}}$$

and $SK_1(\Lambda_\wedge \overline{A}) = 1$, as $\Lambda_\wedge \overline{A}$ is commutative semilocal [CR, (45.12)].

The above square gives a $\vartheta \in K_1(\Lambda_\wedge G)$ with $\mathrm{defl}_G^{\overline{G}}\vartheta = \theta$. Define $\vartheta' \in K_1(\Lambda_\wedge A)$ by $\mathrm{res}_G^A\vartheta = \lambda_A\vartheta'$; hence $\mathrm{defl}_A^{\overline{A}}\vartheta' = 1$.

Further define $\xi = \mathbb{L}(\vartheta)$, $\xi' = \mathbb{L}(\vartheta')$. Then, using diagrams (1.D1) and (1.D2), $\xi \in T(\Lambda_\wedge G)$ has $\mathrm{defl}_G^{\overline{G}}\xi = t_{\overline{G}}$, $\mathrm{Res}_G^A\xi = t_A + \xi'$, and $\mathrm{defl}_A^{\overline{A}}\xi' = 0$.

The exact sequences displayed above give rise to the commutative diagram

$$\begin{array}{ccccc} \tau_G(\mathfrak{a}) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & T(\Lambda_\wedge \overline{G}) \\ \downarrow & & \mathrm{Res}_G^A \downarrow & & \mathrm{Res}_{\overline{G}}^{\overline{A}} \downarrow \\ \mathfrak{b} & \rightarrowtail & \Lambda_\wedge A & \twoheadrightarrow & \Lambda_\wedge \overline{A} \end{array}$$

with top sequence exact by Lemma 5.1. We need to modify ξ by adding an element $\alpha \in \tau_G(\mathfrak{a})$ (so without changing $\mathrm{defl}_G^{\overline{G}}\xi$) to arrange that $\mathrm{Res}_G^A(\xi+\alpha) = t_A$; i.e., we need to prove that $\xi' \in \mathrm{Res}_G^A(\tau_G(\mathfrak{a}))$.

Now, λ_A is Q-invariant, by the proof of [RW5, Lemma 3.1], and $\mathrm{res}\,_G^A$ takes Q-invariant values, whence $\vartheta' \in 1+\mathfrak{b}^Q$. We claim that

$$\vartheta' \in 1 + \mathrm{tr}_Q \mathfrak{b}. \tag{5.a}$$

Write $\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{res}\,_G^U\vartheta = \lambda_{U^{\mathrm{ab}}}\vartheta'_U$ for $A \le U \le G$. Then $\vartheta'_U \in K_1(\Lambda_\wedge U^{\mathrm{ab}}) = (\Lambda_\wedge U^{\mathrm{ab}})^\times$ has $\mathrm{defl}_{U^{\mathrm{ab}}}^{U/[U,U]\cdot C}\vartheta'_U = 1$ because

$$\begin{aligned}\lambda_{U/[U,U]C}\mathrm{defl}_{U^{\mathrm{ab}}}^{U/[U,U]C}(\vartheta'_U) &= \mathrm{defl}_{U^{\mathrm{ab}}}^{U/[U,U]C}\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{res}\,_G^U\vartheta = \mathrm{defl}_{\overline{U}}^{U/[U,U]C}\mathrm{defl}_U^{\overline{U}}\mathrm{res}\,_G^U\vartheta\\ &= \mathrm{defl}_{\overline{U}}^{U/[U,U]C}\mathrm{res}\,_{\overline{G}}^{\overline{U}}\mathrm{defl}_G^{\overline{G}}\vartheta = \mathrm{defl}_{\overline{U}}^{U/[U,U]C}\mathrm{res}\,_{\overline{G}}^{\overline{U}}\theta = \lambda_{U/[U,U]C},\end{aligned}$$

with the last equality by $SK_1(\Lambda_\wedge U/[U,U]C) = 1$, as before.

Summing up, $\vartheta'_U \in \ker(\,(\Lambda_\wedge U^{\mathrm{ab}})^\times \to (\Lambda_\wedge U/[U,U]\cdot C)^\times\,)$ so $\vartheta'_U = 1+\alpha$ where α is a $\Lambda_\wedge\Gamma$-linear combination of elements $\tilde{u}(\tilde{c}-1)$ with $u\in U$, $c \in C$, and $\tilde{~} : U \to U^{\mathrm{ab}}$ the canonical map (with $\Gamma \simeq \mathbb{Z}_l$ some central open subgroup of G contained in A). Then if $A < U$, $\mathrm{ver}_{U^{\mathrm{ab}}}^A$ takes c to $c^{[U:A]} = 1$ and thus $\tilde{u}(\tilde{c}-1)$ to zero, i.e., $\mathrm{ver}_{U^{\mathrm{ab}}}^A\vartheta'_U = 1$. As a first result we therefore have

$$U \neq A \implies \mathrm{ver}_U^A\mathrm{res}\,_G^U\vartheta = \mathrm{ver}_{U^{\mathrm{ab}}}^A\lambda_{U^{\mathrm{ab}}}.$$

Now insert ϑ into the "Möbius-Wall" congruence of Theorem 2 and obtain

$$\lambda_A\vartheta' + \sum_{A<U\le G}\mu_Q(U/A)\mathrm{ver}_{U^{\mathrm{ab}}}^A\lambda_{U^{\mathrm{ab}}} \equiv 0 \mod \mathrm{tr}_Q(\Lambda_\wedge A).$$

Comparing this[17] with the abelian pseudomeasure congruence

$$\sum_{A\le U\le G}\mu_Q(U/A)\mathrm{ver}_{U^{\mathrm{ab}}}^A(\lambda_{U^{\mathrm{ab}}}) \equiv 0 \mod \mathrm{tr}_Q(\Lambda_\wedge A)$$

of Theorem 3 gives $\vartheta' \equiv 1 \mod \mathrm{tr}_Q(\Lambda_\wedge A)$, as λ_A is a unit in $(\Lambda_\wedge A)^Q$ and $\mathrm{tr}_Q(\Lambda_\wedge A)$ an ideal. Therefore

$$\vartheta' \in (1+\mathfrak{b}^Q)\cap(1+\mathrm{tr}_Q(\Lambda_\wedge A)) = 1 + (\mathfrak{b}^Q \cap \mathrm{tr}_Q(\Lambda_\wedge A)) = 1+\mathrm{tr}_Q\mathfrak{b},$$

with the last equality due to Lemma 5.2. This proves (5.a).

Turning back to the proof of Theorem 4, we know that $\xi' = \mathbb{L}(\vartheta')$ is in $\mathfrak{b}^Q$. By (5.a) and Lemma 5.3, we also have $\xi' \in \mathrm{tr}_Q(\Lambda_\wedge A)$, hence $\xi' \in \mathfrak{b}^Q \cap \mathrm{tr}_Q(\Lambda_\wedge A) = \mathrm{tr}_Q(\mathfrak{b}) = \mathrm{Res}_G^A(\tau_G(\mathfrak{a}))$, by Lemma 5.2.

Thus the proof of Theorem 4 is complete. □

6. Proof of the THEOREM

Recall the notation of the beginning of §5, so $G = G(K/k) = \langle z\rangle \times G[l]$ is l-elementary, with $\langle z\rangle$ a finite cyclic group of order prime to l and $G[l]$ a pro-l group, and A is an abelian normal open subgroup of G containing $\langle z\rangle$.

We define $\mathfrak{c}_U^{\mathrm{ab}}$ by the exact sequence $0 \to \mathfrak{c}_U^{\mathrm{ab}} \to \Lambda_\wedge(U^{\mathrm{ab}}) \to \Lambda_\wedge(U/C[U,U]) \to 0$.

Lemma 6.1. *Let C have order l and let $U \ge A$ satisfy $C \cap [U,U] = 1$. Denote the normalizer of U in G by $N = N_G(U)$ and let Y be a set of representatives of N/U-orbits in $U/\Gamma C[U,U]$. Then $\mathrm{tr}_{N/U}(\mathfrak{c}_U^{\mathrm{ab}})$ has $\Lambda_\wedge\Gamma$-basis*

$$\{\mathrm{tr}_{N/U}(\tilde{y}(\tilde{c}-1)) : y \in Y_1,\ 1 \neq c \in C\}$$

[17]It is only here that Theorems 2 and 3 make their appearance.

where Y_1 is the set of $y \in Y$ that have preimage $\tilde{y}$ in $U/[U,U]$ which is fixed by $\mathrm{stab}_{N/U}(y)$, *and $\tilde{c}$ is the image of $c \in C$ in $U/[U,U]$.*

Proof. For the *proof*, we use $C \cap [U,U] = 1 = \Gamma \cap [U,U]$ to identify C, Γ with their images in U^{ab} (hence c with $\tilde{c}$). We investigate the N/U-structure of $0 \to \mathfrak{c}_U^{\mathrm{ab}} \to \Lambda_\wedge U^{\mathrm{ab}} \to \Lambda_\wedge(U^{\mathrm{ab}}/C) \to 0$ via the $\Lambda_\wedge\Gamma$-bases coming from the N/U-action on $C \rightarrowtail U^{\mathrm{ab}}/\Gamma \twoheadrightarrow U^{\mathrm{ab}}/\Gamma C$ by [RW3, Lemma 5].

Now Y is a set of representatives of N/U-orbits on $U^{\mathrm{ab}}/\Gamma C$. If $\hat{y}$ is a preimage of $y \in Y$ under $U^{\mathrm{ab}}/\Gamma \to U^{\mathrm{ab}}/\Gamma C$, then $\mathrm{stab}_{N/U}(y)$ either fixes $\hat{y}$ (which will be referred to as Case 1) or moves $\hat{y}$ (which will be referred to as Case 2); moreover this case distinction is independent of the choice of $\hat{y}$. This permits us to analyze the map $\Lambda_\wedge U^{\mathrm{ab}} \to \Lambda_\wedge(U^{\mathrm{ab}}/C)$ one $y \in Y$ at a time in terms of the map of N/U-sets from the preimage of the N/U-orbit of y to the N/U-orbit of y it induces. This is because of the permutation $\Lambda_\wedge\Gamma$-basis given by choosing preimages $\tilde{y}$ of $\hat{y}$ under $U^{\mathrm{ab}} \to U^{\mathrm{ab}}/\Gamma$ with $\mathrm{stab}_{N/U}(\tilde{y}) = \mathrm{stab}_{N/U}(\hat{y})$, as in the proof of Lemma 5.2.

Thus, in Case 1, the preimage of the N/U-orbit of y is $\dot{\bigcup}_{c \in C}(N/U\text{-orbit of } \hat{y}c)$, so l copies of $\frac{N/U}{\mathrm{stab}_{N/U}(y)}$ as N/U-sets, and the map is $\hat{y}^n c \mapsto y^n$ for $n \in N, c \in C$. So the kernel on $\Lambda_\wedge\Gamma$-permutation modules has $\Lambda_\wedge\Gamma$-basis $\{\tilde{y}^n(c-1) : n \in \frac{N/U}{\mathrm{stab}_{N/U}(y)}, 1 \neq c \in C\}$.

Similarly, in Case 2, the preimage of the N/U-orbit of y is the N/U-orbit of $\hat{y}$: here $\hat{y}^z = \hat{y}\gamma_y(z)$, with γ_y a homomorphism $\mathrm{stab}_{N/U}(y) \twoheadrightarrow C$, has $\mathrm{stab}_{N/U}(\hat{y})$ as its kernel. Now the kernel on $\Lambda_\wedge\Gamma$-permutation modules has $\Lambda_\wedge\Gamma$-basis $\{\tilde{y}^n - \tilde{y} : n \in \frac{N/U}{\mathrm{stab}_{N/U}(\hat{y})}\}$.

Hence $\mathfrak{c}_U^{\mathrm{ab}}$ has $\Lambda_\wedge\Gamma$-basis the union of these over $y \in Y$, and $\mathrm{tr}_{N/U}(\mathfrak{c}_U^{\mathrm{ab}})$ has the claimed $\Lambda_\wedge\Gamma$-basis since Y_1 consists of the $y \in Y$ in Case 1 and $\mathrm{tr}_{N/U}(\tilde{y}^n - \tilde{y}) = 0$ for all $y \in Y$ in Case 2. This proves the lemma. $\square$

Lemma 6.2. *If $v \in T(\Lambda_\wedge G)$ has $\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{Res}_G^U v = 0$ for all subgroups U of G containing A, then $v = 0$.*

Proof. The *proof* is by induction on $[G:A]$. Fix a central open $\Gamma \simeq \mathbb{Z}_l$ in A and an n so that $l^n \equiv 1 \mod |z|$ and l^n is a multiple of the exponent of $G[l]/\Gamma$. By the diagram of (ii) of Lemma 7.1 and $\mathrm{defl}_G^{G^{\mathrm{ab}}} v = 0$, there is an $\omega \in K_1(\Lambda_\wedge G)$ so that $\mathbb{L}(\omega) = v$ and $\mathrm{defl}_G^{G^{\mathrm{ab}}} \omega$ is a torsion element of $K_1(\Lambda_\wedge G^{\mathrm{ab}})$.

Consider $\mathrm{Res}_G^U v$ with $U \geq A$ having index l in G. Then $\mathrm{defl}_V^{V^{\mathrm{ab}}}\mathrm{Res}_U^V(\mathrm{Res}_G^U v) = \mathrm{defl}_V^{V^{\mathrm{ab}}}\mathrm{Res}_G^V v = 0$, for all $A \leq V \leq U$, so the induction hypothesis yields $\mathrm{Res}_G^U v = 0$. Thus $\mathbb{L}(\mathrm{res}_G^U\omega) = \mathrm{Res}_G^U(\mathbb{L}\omega) = 0$ implies $\mathbf{L}(\mathrm{Det}(\mathrm{res}_G^U\omega)) = \mathrm{Tr}(\mathbb{L}(\mathrm{res}_G^U\omega)) = 0$; hence $\mathrm{Det}(\mathrm{res}_G^U\omega)(\chi_1)^l = \Psi((\mathrm{Det}(\mathrm{res}_G^U\omega))(\psi_l\chi_1)$ for all characters χ_1 of U, by the definition of $\mathbf{L}$.

If χ_1 is an irreducible character of U with kernel containing Γ, then $\chi_1 = \beta \otimes \varpi$ with β, ϖ irreducible characters of U with kernels containing $U[l], \langle z\rangle\Gamma$, respectively. Note that $\psi_l^n\chi_1 = \psi_l^n\beta \otimes \psi_l^n\varpi = \beta \otimes 1 = \beta$; hence

$$\begin{aligned}\mathrm{Det}(\mathrm{res}_G^U\omega)(\chi_1)^{l^n} &= \Psi^n(\mathrm{Det}(\mathrm{res}_G^U\omega)(\psi_l^n\chi_1)) = \Psi^n((\mathrm{res}_G^U(\mathrm{Det}\omega))(\beta)) \\ &= \Psi^n((\mathrm{Det}\omega)(\mathrm{ind}_U^G\beta))\end{aligned}$$

is torsion. This holds because $U \lhd G$ l-elementary implies that $\beta = \mathrm{res}\,^U_G\beta'$ where $\beta' = \mathrm{infl}^G_{G^{\mathrm{ab}}}\beta''$, hence $\mathrm{ind}\,^G_U\beta = \beta'\mathrm{ind}\,^G_U 1$ and $\mathrm{ind}\,^G_U 1 = \sum_{i=1}^l \alpha_i$ with α_i the irreducible characters of G having $\mathrm{res}\,^U_G\alpha_i = 1$; now $\alpha_i = \mathrm{infl}^G_{G^{\mathrm{ab}}}\alpha'_i$ so

$$\begin{aligned}(\mathrm{Det}\omega)(\mathrm{ind}\,^G_U\beta) &= (\mathrm{Det}\omega)(\textstyle\sum_{i=1}^l \beta'\alpha_i) = \prod_{i=1}^l (\mathrm{Det}\omega)(\mathrm{infl}^G_{G^{\mathrm{ab}}}(\beta''\alpha'_i))\\ &= \textstyle\prod_{i=1}^l \mathrm{Det}(\mathrm{defl}^{G^{\mathrm{ab}}}_G\omega)(\beta''\alpha'_i)\end{aligned}$$

is torsion. Thus $(\mathrm{Det}(\mathrm{res}\,^U_G\omega))(\chi_1)$ is torsion for all such χ_1 and $U \geq A$ of index l in G.

Now if χ is one of the finitely many irreducible characters of G with $\Gamma \subseteq \ker(\chi)$, then (by Clifford theory) either $\chi = \mathrm{ind}\,^G_U\chi_1$ with $U \geq A$ of index l when $(\mathrm{Det}\omega)(\chi) = (\mathrm{res}\,^U_G(\mathrm{Det}\omega))(\chi_1)$ is torsion or $\chi = \mathrm{infl}^G_{G^{\mathrm{ab}}}\alpha$ when $(\mathrm{Det}\omega)(\chi) = \mathrm{Det}(\mathrm{defl}^{G^{\mathrm{ab}}}_G\omega)(\alpha)$ is again torsion. Every irreducible character of G has the form $\chi \otimes \rho$ with such a χ and ρ of type W [18]; hence $(\mathrm{Det}\omega)(\chi \otimes \rho) = \rho^\sharp((\mathrm{Det}\omega)(\chi))$ is torsion of order at most that of $(\mathrm{Det}\omega)(\chi)$. Thus $\mathrm{Det}\omega$ is a torsion element in $\mathrm{HOM}(R_l(G), (\Lambda_\wedge\Gamma_k)^\times)$ and so $\mathrm{Tr}(v) = \mathrm{Tr}(\mathbb{L}\omega) = \mathbf{L}(\mathrm{Det}\omega) = 0$ implies $v = 0$. □

Remark 6.1. When G is abelian pro-l, Proposition 5.1 of [RW6] gives a description of the kernel of $\mathbb{L}$ on $(\Lambda_\wedge G)^\times$. This can be (and originally was) used to prove the lemma for pro-l groups. The present proof for l-elementary groups is shorter than extending that proposition from $\Lambda_\wedge$ to $\Lambda_\wedge^\beta$ (for the notation compare the proof of Lemma 7.1).

We next state

Lemma 6.3 (Uniqueness Principle). *If $\xi \in T(\Lambda_\wedge G)$ has $\mathrm{defl}^{U^{\mathrm{ab}}}_U\mathrm{Res}^U_G\xi = t_{U^{\mathrm{ab}}}$ for all subgroups U of G containing a fixed abelian normal open subgroup A of G, then $\xi = t_G$. In particular, $t_G \in T(\Lambda_\wedge G)$.*

Proof. This holds because $l^n t_G$ is integral for large enough natural n (see [RW4, Proposition 9]). Setting $v = l^n t_G - l^n\xi$, we see that $\mathrm{defl}^{U^{\mathrm{ab}}}_U\mathrm{Res}^U_G v = l^n t_{U^{\mathrm{ab}}} - l^n t_{U^{\mathrm{ab}}} = 0$, so $v = l^n(t_G - \xi) = 0$ by Lemma 6.2. However, $T(\Lambda_\wedge G)$ is torsionfree as $\mathrm{Tr} : T(\mathcal{Q}_\wedge G) \to \mathrm{Hom}^*(R_l(G), \mathcal{Q}^{\mathrm{c}}_\wedge\Gamma_k)$ is injective. □

Proof. Now we start the *proof* of the THEOREM. If it were false, there would exist a counterexample K/k, with $t_{K/k}$ not in $T(\Lambda_\wedge G)$, for which the Galois group G would have commutator subgroup $[G,G]$ of *minimal* order; among these groups G there would be one with centre $Z(G)$ of *minimal* index $[G : Z(G)]$.

Since $[G,G] \neq 1$, by [RW3, Theorem 9], and $[G,G]$ is an l-group, as G is l-elementary, we may choose a central subgroup $C \leq [G,G]$ of order l in G and then a *maximal* abelian normal subgroup A of G, necessarily containing C and $\langle z\rangle$. We also fix a central open $\Gamma \simeq \mathbb{Z}_l$ inside A. Since $|[\overline{G},\overline{G}]| < |[G,G]|$, Theorem 4 guarantees the existence of

$$\xi \in T(\Lambda_\wedge G) \quad \text{with } \mathrm{defl}^{\overline{G}}_G\xi = t_{\overline{G}} \text{ and } \mathrm{Res}^A_G\xi = t_A, \tag{6.1}$$

where, as before, $^-$ denotes the canonical surjection $G \twoheadrightarrow G/C$. To defeat the counterexample G it suffices, by the Uniqueness Principle (Lemma 6.3), to find such a

[18] I.e., $\mathrm{res}\,^H_G\rho = 1$.

ξ so that $\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{Res}_G^U\xi = t_{U^{\mathrm{ab}}}$ for all subgroups $U \geq A$ of G. Observe that this already holds for U with $[U,U] \geq C$: for then $\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{Res}_G^U\xi = \mathrm{defl}_{U/C}^{U^{\mathrm{ab}}}\mathrm{defl}_U^{U/C}\mathrm{Res}_G^U\xi = \mathrm{defl}_{\overline{U}}^{U^{\mathrm{ab}}}\mathrm{Res}_{\overline{G}}^{\overline{U}}\mathrm{defl}_G^{\overline{G}}\xi = \mathrm{defl}_{\overline{U}}^{U^{\mathrm{ab}}}\mathrm{Res}_{\overline{G}}^{\overline{U}}t_{\overline{G}} = t_{U^{\mathrm{ab}}}$.

On the other hand, for $U \geq A$ with $[U,U] \ngeq C$, hence $C \cap [U,U] = 1$, then $|[U,U]| < |[G,G]|$ implies[19] $t_U \in T(\Lambda_\wedge U)$, by our hypothesis on G, permitting us to define

$$\xi_U = \mathrm{Res}_G^U\xi - t_U \in T(\Lambda_\wedge U)$$

and to define the *support* of ξ by

$$\mathrm{supp}(\xi) = \{U \geq A : C \cap [U,U] = 1 \text{ and } \xi_U \neq 0\}.$$

Note that $\mathrm{supp}(\xi) \neq \emptyset$ for our counterexample G by the Uniqueness Principle (Lemma 6.3).

To investigate $U \in \mathrm{supp}(\xi)$, we state

Claim 6.A. (a) If $A \leq V \leq U$ and $C \cap [U,U] = 1$, then $\mathrm{Res}_U^V\xi_U = \xi_V$ and $\xi_V^g = \xi_{V^g}$ for $g \in G$.

(b) G acts on $\mathrm{supp}(\xi)$ by conjugation.

(c) $A \notin \mathrm{supp}(\xi)$.

Proof. Recall that $t_U = \mathrm{Res}_G^U t_G$. Now (a) results from

$$\mathrm{Res}_U^V\xi_U = \mathrm{Res}_U^V(\mathrm{Res}_G^U\xi - \mathrm{Res}_G^U t_G) = \mathrm{Res}_G^V\xi - \mathrm{Res}_G^V t_G = \xi_V$$

and

$$\xi_V^g = \mathrm{Res}_G^V(\xi - t_G)^g = \mathrm{Res}_{G^g}^{V^g}(\xi^g - t_G^g) = \mathrm{Res}_G^{V^g}(\xi - t_G) = \xi_{V^g},$$

by (3) of Lemma 5.1, which at the same time implies (b); (c) follows from (6.1) and the definition of 'supp'. □

Moreover, we let, as in Lemma 6.1, $N = N_G(U)$ be the normalizer of U in G. Define $\mathfrak{c}_U$ and $\mathfrak{c}_U^{\mathrm{ab}}$ by the exact sequences

$$0 \to \mathfrak{c}_U \to \Lambda_\wedge(N/[U,U]) \to \Lambda_\wedge(U/C[U,U]) \to 0,$$
$$0 \to \mathfrak{c}_U^{\mathrm{ab}} \to \Lambda_\wedge(U^{\mathrm{ab}}) \to \Lambda_\wedge(U/C[U,U]) \to 0.$$

Claim 6.B. If $U \in \mathrm{supp}(\xi)$, then $\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U \in \mathrm{tr}_{N/U}(\mathfrak{c}_U^{\mathrm{ab}})$.

Proof. We first note that $t_{N/[U,U]}$ is in $T(\Lambda_\wedge(N/[U,U]))$: for the commutator subgroup $[N,N]/[U,U]$ of $N/[U,U]$ has smaller order than $[G,G]$ unless $[U,U] = 1$ and $[N,N] = [G,G]$, in which case $A \notin \mathrm{supp}(\xi)$ implies $N < G$, because A is *maximal* abelian normal in G; but then $Z(G) \leq N$ implies $[N/[U,U] : Z(N/[U,U])] \leq [N : Z(N)] \leq [N : Z(G)] < [G : Z(G)]$, contrary to the minimality hypothesis on G.

Writing $\mathrm{defl}_N^{N/[U,U]}\mathrm{Res}_G^N\xi = t_{N/[U,U]} + z_U$, with $z_U \in T(\Lambda_\wedge(N/[U,U]))$, and $\tau = \tau_{N/[U,U]}$, we consider the commutative diagram

$$\begin{array}{ccccc} & & T(\Lambda_\wedge N) & \twoheadrightarrow & T(\Lambda_\wedge \overline{N}) \\ & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ \tau(\mathfrak{c}_U) & \rightarrowtail & T(\Lambda_\wedge(N/[U,U])) & \twoheadrightarrow & T(\Lambda_\wedge(N/C[U,U])), \end{array}$$

with all surjective maps deflations and exact bottom row by (ii) of Lemma 5.2 applied to $N/[U,U] \geq U/[U,U] \geq C[U,U]/[U,U]$ in place of $G \geq A \geq C$; moreover, we also obtain $\mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}\tau(\mathfrak{c}_U) = \mathrm{tr}_{N/U}(\mathfrak{c}_U^{\mathrm{ab}})$.

[19] Recall that $t_U = t_{K/K^U}$ by the Galois correspondence.

Since $\mathrm{Res}_G^N\xi \in T(\Lambda_\wedge N)$ has $\mathrm{defl}_N^{\overline{N}}\mathrm{Res}_G^N\xi = \mathrm{Res}_{\overline{G}}^{\overline{N}}\mathrm{defl}_G^{\overline{G}}\xi = \mathrm{Res}_{\overline{G}}^{\overline{N}}t_{\overline{G}} = t_{\overline{N}}$, the diagram implies $z_U \in \tau(\mathfrak{c}_U)$. Thus

$$\begin{aligned}\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U + t_{U^{\mathrm{ab}}} &= \mathrm{defl}_U^{U^{\mathrm{ab}}}(\mathrm{Res}_G^U\xi - t_U) + t_{U^{\mathrm{ab}}}\\ &= \mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{Res}_N^U(\mathrm{Res}_G^N\xi)\\ &= \mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}\mathrm{defl}_N^{N/[U,U]}(\mathrm{Res}_G^N\xi)\\ &= \mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}(z_U + t_{N/[U,U]}) = \mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}z_U + t_{U^{\mathrm{ab}}}\end{aligned}$$

implies $\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U = \mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}z_U \in \mathrm{Res}_{N/[U,U]}^{U^{\mathrm{ab}}}\tau(\mathfrak{c}_U)$. Combining with the previous paragraph yields the claim. $\square$

Now continuing with the *proof*, it follows that the THEOREM holds if we can modify ξ, subject to (6.1) holding, so that $\mathrm{supp}(\xi)$ is empty. Since this is not possible for our G, by hypothesis, there must exist a ξ for which $\mathrm{supp}(\xi)$ has *minimal cardinality* $\neq 0$.

Since $\mathrm{supp}(\xi)$ is non-empty, it contains a U with *minimal* $[U : A]$. By Lemma 6.1 we may write

$$\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U = \textstyle\sum_{y\in Y_1,\, 1\neq c\in C}\alpha(y,c)\mathrm{tr}_{N/U}(\tilde{y}(\tilde{c}-1)) \tag{6.2}$$

with unique $\alpha(y,c)$ in $\Lambda_\wedge\Gamma$. As $A \notin \mathrm{supp}(\xi)$, every V with $A \leq V < U$ has $C \cap [V,V] = 1$ and $\xi_V = 0$; equivalently, we may restrict our attention to maximal such V and hence may assume that V has index l in U (so, in particular, $[U,U] \lhd V$). Now

$$\mathrm{Res}_{U^{\mathrm{ab}}}^{V/[U,U]}(\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U) = \mathrm{defl}_V^{V/[U,U]}\mathrm{Res}_U^V\xi_U = \mathrm{defl}_V^{V/[U,U]}\xi_V = 0,$$

by (a) of Claim 6.A. We apply $\mathrm{Res}_{U^{\mathrm{ab}}}^{V/[U,U]}$ to (6.2), observing (by (2) of Lemma 5.1) that, for $u \in U^{\mathrm{ab}}$, $\mathrm{Res}_{U^{\mathrm{ab}}}^{V/[U,U]}(u(\tilde{c}-1)) = lu(\tilde{c}-1)$ if $u \in V/[U,U]$, and $= 0$, if not, because U^{ab} is abelian. It follows that

$$0 = \sum_{y\in Y_1\cap(V/\Gamma C[U,U]),\, c\neq 1} l\alpha(y,c)\mathrm{tr}_{N/U}(\tilde{y}(\tilde{c}-1))$$

whenever $A \leq V \leq U$, $[U : V] = l$, and, therefore, that $\alpha(y,c) = 0$ unless $y \in Y_1$ is not in $V/\Gamma C[U,U]$.

In particular, if U/A is non-cyclic, then every element of U/A is contained in a maximal V/A for some V; hence $\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U = 0$. But then the Uniqueness Principle (Lemma 6.3), applied to $A \lhd U$ instead of $A \lhd G$, implies that $\xi_U = 0$. Thus $U \notin \mathrm{supp}(\xi)$, contrary to assumption.

It follows that our U with minimal $[U : A]$ in $\mathrm{supp}(\xi)$ has U/A cyclic. Then $[U,U] \leq A$, $U/A \simeq \frac{U/\Gamma C[U,U]}{A/\Gamma C[U,U]}$, and

$$\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U = \sum_{y\in Y_1,\langle yA\rangle = U/A,\, 1\neq c\in C}\alpha(y,c)\mathrm{tr}_{N/U}(\tilde{y}(\tilde{c}-1)), \tag{6.3}$$

because U/A now has a unique maximal subgroup V/A and so $y \in Y_1$ is not in $V/\Gamma C[U,U]$ precisely when yA generates U/A.

Now our THEOREM essentially follows from the next result.

Claim 6.C. Assume that U/A is cyclic. Set, in the notation of (6.3),

$$\xi'' = \sum_{y\in Y_1,\,\langle yA\rangle=U/A,\,1\neq c\in C} \alpha(y,c)\tau_G(y'(c-1)) \quad in\ T(\Lambda_\wedge G),$$

with preimages $y' \in U$ of $\tilde{y}$ under $\mathrm{defl}_U^{U^{\mathrm{ab}}}$. Then

(i) $\mathrm{defl}_U^{U^{\mathrm{ab}}}\mathrm{Res}_G^U\xi'' = \mathrm{defl}_U^{U^{\mathrm{ab}}}\xi_U$, and
(ii) if $A \le U_1 \le G$, then $\mathrm{Res}_G^{U_1}\xi'' \neq 0$ implies $\exists\, g\in G : U^g \le U_1$.

Proof. Recall that the y in the ξ''-sum have $\langle yA\rangle = U/A$. Applying (2) of Lemma 5.1 gives

$$\mathrm{Res}_G^U\xi'' = \sum_{y\in Y_1,\,\langle yA\rangle=U/A,\,1\neq c} \alpha(y,c) \sum_{x\in G/U} \dot{\tau}_U((y')^x(c-1)).$$

Note that $(y')^x \in U$ implies $(yA)^x \in U/A$; hence $(U/A)^x = U/A$, i.e., $x\in N$. Now we have

$$\mathrm{Res}_G^U\xi'' = \sum_{y,c}\alpha(y,c)\sum_{x\in N/U}\tau_U((y')^x(c-1));$$

hence applying $\mathrm{defl}_U^{U^{\mathrm{ab}}}$ gives (i).

For (ii) note that y still has $\langle yA\rangle = U/A$, but we now apply (2) of Lemma 5.1 with U replaced by U_1. Some term $\mathrm{Res}_G^{U_1}\tau_G(y'(c-1))$ in this sum must be $\neq 0$, by hypothesis; but this term is $\sum_{x_1\in G/U}\dot{\tau}_{U_1}((y')^{x_1}(c-1))$, so we must have a non-zero term here, i.e., $(y')^{x_1}\in U_1$ for some x_1. Now $(U/A)^{x_1}\subseteq U_1/A$ implies $U^{x_1}\subseteq U_1$. □

We apply Claim 6.C and set $\xi' \stackrel{\mathrm{def}}{=} \xi-\xi''$. Then $\mathrm{Res}_G^A\xi' = t_A$, by (ii) with $U_1 = A$; moreover, due to the appearance of the elements $1\neq c\in C$ in ξ'', $\mathrm{defl}_G^{G/C}\xi' = t_{G/C}$; thus ξ' satisfies (6.1). Further, $\mathrm{supp}(\xi')\subseteq\mathrm{supp}(\xi)$: for if $U_1\in\mathrm{supp}(\xi')$, then $C\cap[U_1,U_1] = 1$ and $\xi'_{U_1}\neq 0$; hence $\xi_{U_1} = \xi'_{U_1} + \mathrm{Res}_G^{U_1}\xi''$ is non-zero unless $\mathrm{Res}_G^{U_1}\xi''\neq 0$. But in that case $U_1\supseteq U^g$ for some $g\in G$ by (ii); hence Claim 6.A implies $\xi_{U_1}\neq 0$, as $\mathrm{Res}_{U_1}^{U^g}\xi_{U_1} = \xi_{U^g}\neq 0$ by $U\in\mathrm{supp}(\xi)$. But now (i) and (ii) of Claim 6.C imply $\mathrm{defl}_U^{U^{\mathrm{ab}}}\xi'_U = 0$ and $\xi'_{U_1} = 0$ for $A\le U_1 < U$; hence $\xi'_U = 0$ by the Uniqueness Principle (Lemma 6.3). Thus $U\notin\mathrm{supp}(\xi')$, which contradicts the minimal cardinality of $\mathrm{supp}(\xi)$ and therefore finishes the proof of the THEOREM. □

7. l-ELEMENTARY GROUPS

Recall that $G = G(K/k) = \langle z\rangle\times G[l]$ is l-elementary.

Lemma 7.1. (i) *The logarithm* $\mathbb{L} : K_1(\Lambda_\wedge G)\to T(\mathcal{Q}_\wedge G)$ *of diagram* (1.D1) *has image in* $T(\Lambda_\wedge G)$.

(ii) *Let* $\mathfrak{a}$ *be the kernel of* $\mathrm{defl}_G^{G^{\mathrm{ab}}} : \Lambda_\wedge G\twoheadrightarrow\Lambda_\wedge G^{\mathrm{ab}}$. *Then the commutative diagram*

$$\begin{array}{ccccc} 1+\mathfrak{a} & \rightarrowtail & (\Lambda_\wedge G)^\times & \twoheadrightarrow & (\Lambda_\wedge G^{\mathrm{ab}})^\times \\ \downarrow\!\!\!\downarrow & & \mathbb{L}\downarrow & & \mathbb{L}^{\mathrm{ab}}\downarrow \\ \tau_G(\mathfrak{a}) & \rightarrowtail & T(\Lambda_\wedge G) & \twoheadrightarrow & \Lambda_\wedge G^{\mathrm{ab}} \end{array}$$

has exact rows and surjective left vertical map.

Proof. For (i), abbreviate $G[l]$ as U. Each ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible character β of $\langle z\rangle$ induces a $\mathbb{Z}_l$-algebra homomorphism $\mathbb{Z}_l[\langle z\rangle] \to {\mathbb{Z}_l}^{\mathrm{c}}$ with image $\mathbb{Z}_l[\beta]$, hence surjective ring homomorphisms

$$\mathbb{Q}_l[\langle z\rangle] \twoheadrightarrow \mathbb{Q}_l(\beta), \quad \Lambda_\wedge G \twoheadrightarrow \mathbb{Z}_l[\beta] \otimes_{\mathbb{Z}_l} \Lambda_\wedge U \overset{\mathrm{def}}{=} \Lambda^\beta_\wedge U, \qquad \mathcal{Q}_\wedge G \twoheadrightarrow \mathcal{Q}^\beta_\wedge U.$$

Applying the functors K_1 and T gives the southeast and southwest arrows of the diagram (with Γ, RG, RU short for $\Gamma_k, R_l(G), R_l(U)$, respectively):

$$\begin{tikzcd}
K_1(\Lambda_\wedge G) \arrow[rrr] \arrow[dr] \arrow[ddd] & & & T(\mathcal{Q}_\wedge G) \arrow[dl] \arrow[ddd] \\
& K_1(\Lambda^\beta_\wedge U) \arrow[r] \arrow[d] & T(\mathcal{Q}^\beta_\wedge U) \arrow[d] & \\
& \mathrm{HOM}^{(\beta)}(RU,(\Lambda^{\mathrm{c}}_\wedge\Gamma)^\times) \arrow[r] & \mathrm{Hom}^{(\beta)}(RU,\mathcal{Q}^{\mathrm{c}}_\wedge\Gamma) & \\
\mathrm{HOM}(RG,(\Lambda^{\mathrm{c}}_\wedge\Gamma)^\times) \arrow[ur] \arrow[rrr] & & & \mathrm{Hom}^*(RG,\mathcal{Q}^{\mathrm{c}}_\wedge\Gamma) \arrow[ul]
\end{tikzcd}$$

with large square from diagram (1.D1) of §1 and small square [RW3, 2. of Proposition 11] with unramified coefficients $\mathbb{Z}_l[\beta]$, which are abbreviated by the superscript β. The northwest and northeast arrows $f \mapsto f^\beta$ are defined by $f^\beta(\varpi) = f(\beta \otimes \varpi)$.

To see that the left quadrilateral commutes[20] let $H' = H \cap U$ (recalling that H is the kernel of $G \to \Gamma_k$), hence $H = \langle z\rangle \times H'$, and let $\beta(x)$ denote the image of $x \in \Lambda_\wedge G$ in $\Lambda^\beta_\wedge U$. We must check that $(\mathrm{Det}x)^\beta(\varpi) = (\mathrm{Det}\beta(x))(\varpi)$, i.e.,[21] $\det_{\mathcal{Q}^{\mathrm{c}}_\wedge\Gamma_k}(x \mid \mathfrak{V}_{\beta\otimes\varpi}) = \det(\beta(x) \mid \mathfrak{V}_\varpi)$. Here,

$$\mathfrak{V}_\varpi = \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H']}(V_\varpi, {\mathbb{Q}_l}^{\mathrm{c}} \otimes_{\mathbb{Q}_l(\beta)} \mathcal{Q}^\beta_\wedge U) = \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H']}(V_\varpi, \mathcal{Q}^{\mathrm{c}}_\wedge U)$$

and

$$\begin{aligned}
\mathfrak{V}_{\beta\otimes\varpi} &= \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H]}(V_{\beta\otimes\varpi}, \mathcal{Q}^{\mathrm{c}}_\wedge G) \\
&= \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[\langle z\rangle]\otimes_{{\mathbb{Q}_l}^{\mathrm{c}}}{\mathbb{Q}_l}^{\mathrm{c}}[H']}(V_\beta \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} V_\varpi, {\mathbb{Q}_l}^{\mathrm{c}}[\langle z\rangle] \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} \mathcal{Q}^{\mathrm{c}}_\wedge U) \\
&= \mathrm{Hom}_{{\mathbb{Q}_l}^{\mathrm{c}}[H']}(V_\beta \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} V_\varpi, V_\beta \otimes_{{\mathbb{Q}_l}^{\mathrm{c}}} \mathcal{Q}^{\mathrm{c}}_\wedge U).
\end{aligned}$$

Then $h \mapsto 1 \otimes h$ is an isomorphism $\mathfrak{V}_\varpi \to \mathfrak{V}_{\beta\otimes\varpi}$ of vector spaces over $\mathcal{Q}^{\mathrm{c}}_\wedge\Gamma_k$ and one checks that $(1 \otimes h)x = 1 \otimes h \cdot \beta(x)$.

The same argument, with T, Tr rather than K_1, Det, yields the commutativity of the right quadrilateral, and the commutativity of the bottom quadrilateral follows from the formula [RW3, p. 37] for $\mathbf{L}$ by $\psi_l(\beta) = \beta^{\mathrm{Fr}}$. The diagram now implies that the top quadrilateral commutes.

Recall that, [RW3, Proposition 11], the logarithm $\mathbb{L}_\beta : K_1(\Lambda^\beta_\wedge U) \to T(\mathcal{Q}^\beta_\wedge U)$ is integral for all β. It thus suffices to show that if $x \in T(\mathcal{Q}_\wedge G)$ has image in $T(\Lambda^\beta_\wedge U)$ under the southwest arrow for every β, then $x \in T(\Lambda_\wedge G)$.

Letting β run through a set of representatives of the $G({\mathbb{Q}_l}^{\mathrm{c}}/\mathbb{Q}_l)$-action on the ${\mathbb{Q}_l}^{\mathrm{c}}$-irreducible characters of $\langle z\rangle$, we get an isomorphism $\mathbb{Z}_l[\langle z\rangle] \to \prod_\beta \mathbb{Z}_l[\beta]$. This induces isomorphisms $T(\Lambda_\wedge G) \to \bigoplus_\beta T(\Lambda^\beta_\wedge U)$ and $T(\mathcal{Q}_\wedge G) \to \bigoplus_\beta T(\mathcal{Q}^\beta_\wedge U)$. The first of these provides an $x' \in T(\Lambda_\wedge G)$ with the same images as x for all β, and the second gives $x = x' \in T(\Lambda_\wedge G)$.

We now prove (ii). The exact sequence defining $\mathfrak{a}$ gives the top row since $\mathfrak{a} \subseteq \mathrm{rad}(\Lambda_\wedge G)$, as $[G,G]$ is an l-group. The bottom row is exact by (ii) of Lemma 5.2. To

[20] This can also be obtained from [RW3, Theorem 1].

[21] Compare also [RW2, Proposition 6] and [RW3, Lemma 2].

see the vertical surjectivity, write $\mathfrak{u}$ for the kernel of $\mathrm{defl}_U^{U^{\mathrm{ab}}} : \Lambda_\wedge U \twoheadrightarrow \Lambda_\wedge U^{\mathrm{ab}}$ (with $U = G[l]$); also write $\mathfrak{u}^\beta$ for the kernel of $\Lambda_\wedge^\beta U \twoheadrightarrow \Lambda_\wedge^\beta U^{\mathrm{ab}}$ (with β as before). Then the map $1 + \mathfrak{u}^\beta \to \tau_U(\mathfrak{u}^\beta)$ induced by $\mathbb{L}_\beta : (\Lambda_\wedge^\beta U)^\times \to T(\Lambda_\wedge^\beta U)$ is surjective [RW3, 2b. of Proposition 11]. Identifying $\Lambda_\wedge G$ and $\prod_\beta \Lambda_\wedge^\beta U$ as in the last paragraph of the proof of (i) (also for the abelianizations) and assembling our asserted diagram in terms of the β-decomposition, noting that the commutativity of the square below follows from that of the top quadrangle in (i), we deduce that $1 + \mathfrak{a} \to \tau_G(\mathfrak{a})$ is also surjective:

$$\begin{array}{ccc} (\Lambda_\wedge G)^\times & \stackrel{\beta}{\to} & (\Lambda_\wedge^\beta U)^\times \\ \mathbb{L}\downarrow & & \mathbb{L}_\beta\downarrow \\ T(\Lambda_\wedge G) & \stackrel{\beta}{\to} & T(\Lambda_\wedge^\beta U). \end{array} \qquad \square$$

Recall that, for a pro-l group $G = G(K/k)$ and an open subgroup $G' \leq G$,

$$\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l(G), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k) \to \mathrm{Hom}^*(R_l(G'), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_{k'})$$

is defined in [RW8, §1]. We partially extend this definition to l-elementary G.

Lemma 7.2. *Let $G = \langle z\rangle \times G[l]$ be l-elementary. If G' is an open subgroup of G containing $\langle z\rangle$, define for $f \in \mathrm{Hom}^*(R_l(G), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k)$*

$$\mathrm{Res}_G^{G'} f = [\,\chi' \mapsto f(\mathrm{ind}\,_{G'}^G\chi') + \sum_{r\geq 1} \frac{\Psi^r}{l^r}\,(f(\psi_l^{r-1}\chi))\,] \in \mathrm{Hom}^*(R_l(G'), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_{k'}),$$

where $\chi' \in R_l(G')$, ψ_l denotes the l-th Adams operation, $\chi \stackrel{\mathrm{def}}{=} \psi_l(\mathrm{ind}\,_{G'}^G\chi') - \mathrm{ind}\,_{G'}^G(\psi_l\chi')$, and $k' = K^{G'}$. The diagram below, and so diagram (1.D2), commutes:

$$\begin{array}{ccccccc} K_1(\Lambda_\wedge G) & \stackrel{\mathrm{Det}}{\to} & \mathrm{HOM}(R_l(G),(\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times) & \stackrel{\mathbf{L}}{\to} & \mathrm{Hom}^*(R_l(G),\mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k) & \stackrel{\mathrm{Tr}}{\leftarrow} & T(\mathcal{Q}_\wedge G) \\ \mathrm{res}\,_G^{G'}\downarrow & & \mathrm{res}\,_G^{G'}\downarrow & & \mathrm{Res}_G^{G'}\downarrow & & \mathrm{Res}_G^{G'}\downarrow \\ K_1(\Lambda_\wedge G') & \stackrel{\mathrm{Det}}{\to} & \mathrm{HOM}(R_l(G'),(\Lambda_\wedge^{\mathrm{c}}\Gamma_{k'})^\times) & \stackrel{\mathbf{L}}{\to} & \mathrm{Hom}^*(R_l(G'),\mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_{k'}) & \stackrel{\mathrm{Tr}}{\leftarrow} & T(\mathcal{Q}_\wedge G'). \end{array}$$

Moreover, $\mathrm{Res}_G^{G'} : T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$ takes $T(\Lambda_\wedge G)$ to $T(\Lambda_\wedge G')$ and is transitive; i.e., $\mathrm{Res}_G^{G''} = \mathrm{Res}_{G'}^{G''}\mathrm{Res}_G^{G'}$ whenever $\langle z\rangle \leq G'' \leq G' \leq G$.

Proof. We first observe that $G^{l^{r_0}} \subset G'$ for a suitable power l^{r_0} of l; thus

$$\psi_l^{r_0-1}\chi = \psi_l^{r_0-1}(\psi_l(\mathrm{ind}\,_{G'}^G\chi') - \mathrm{ind}\,_{G'}^G(\psi_l\chi')) = 0,$$

for all $\chi' \in R_l(G')$ (compare [RW8, p. 119]): this again follows from

$$\chi(g) = \textstyle\sum_{t \text{ with } m(g^t)=1} \chi'(g^{lt}), \tag{7.1}$$

with $m(g) = \min\{r \geq 0 : g^{l^r} \in G'\}$ and $G = \dot{\bigcup}_t tG'$. From this, $\mathrm{Res}_G^{G'}$ is well-defined and the middle square commutes just as in [RW8, p. 120].

The map $\mathrm{Res}_G^{G'}$ on the very right is defined by transporting

$$\mathrm{Res}_G^{G'} : \mathrm{Hom}^*(R_l(G), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_k) \to \mathrm{Hom}^*(R_l(G'), \mathcal{Q}_\wedge^{\mathrm{c}}\Gamma_{k'})$$

to $T(\mathcal{Q}_\wedge G) \to T(\mathcal{Q}_\wedge G')$ by means of the isomorphism 'Tr'; in particular, the right square commutes. Also, commutativity of the left square is [RW2, Lemma 9]. Observing that $\mathbb{L} = \mathrm{Tr}^{-1}\mathbf{L}\mathrm{Det}$, diagram (1.D2) is obtained by arranging columns 1, 4, and 3 with $\mathbb{L}$ and Tr as horizontal maps.

Moreover, the rest follows as in [RW8, proof of Proposition A] using (7.1). $\square$

We close this section with adjusting the arguments in §1 for the proof of Proposition 1.3 for pro-l groups to l-elementary groups G.

Lemma 7.3. (i) *If* $t_{K/k} \in T(\Lambda_\wedge G)$*, then there is a unique torsion* $w \in \mathrm{HOM}(R_l(G), (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times)$ *with* $wL_{K/k} \in \mathrm{Det}K_1(\Lambda_\wedge G)$ *and* $\mathrm{defl}_G^{G^{\mathrm{ab}}} w = 1$.

(ii) *Moreover, if* G *has an abelian subgroup* A *of index* l*, then*

$$w = 1 \iff \mathrm{ver}^A_{G^{\mathrm{ab}}} \lambda_{K^{[G,G]}/k} \equiv \lambda_{K/k'} \mod \mathrm{tr}_Q(\Lambda_\wedge G')$$

where $k' = K^A$ *and* $Q = G/A$.

Proof. For (i), using the diagram in (ii) of Lemma 7.1 to replace the one in the proof of [RW5, Proposition 2.2], there exists a $y \in (\Lambda_\wedge G)^\times$ so $\mathbb{L}(y) = t_{K/k}$ and $\mathrm{defl}_G^{G^{\mathrm{ab}}} w = 1$. Following the proof of [RW5, Proposition 2.4], one defines w by $wL_{K/k} = \mathrm{Det}y$ and checks that $\mathrm{defl}_G^{G^{\mathrm{ab}}} w = 1$ and $\mathbf{L}(w) = 0$. This implies w is torsion by the indicated argument from [RW3, p. 46] by observing that, while $\psi_l^n \chi$ is only a character β of $G/G[l]$ for large n, $\beta = \mathrm{infl}_{G^{\mathrm{ab}}}^G \beta'$ still implies that $w(\beta) = (\mathrm{defl}_G^{G^{\mathrm{ab}}} w)(\beta') = 1$. The argument for the uniqueness of w still works because [RW3, Lemma 12] is already proved for $\Lambda_\wedge^\beta(G[l])$, in the notation of Lemma 7.1.

More precisely, in the notation of the proof of (ii) of Lemma 7.1, let $x \in 1 + \mathfrak{a}$ have $\mathrm{Det}x$ torsion. By the commutativity of the left quadrangle in the proof of (i) of Lemma 7.1, $\beta(x) \in 1 + \mathfrak{u}^\beta$ has $\mathrm{Det}\beta(x)$ torsion, hence we have $\mathrm{Det}\beta(x) = 1$, and so it suffices to observe that $\mathrm{HOM}(R_l(G), (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times) \to \prod_\beta \mathrm{HOM}^\beta(R_l(U), (\Lambda_\wedge^{\mathrm{c}}\Gamma_k)^\times)$ is injective.

To verify (ii), we can follow the proof of [RW5, equivalence of (1) and (2) in Proposition 3.2], except that we still need to show that the only torsion unit $e \in \Lambda_\wedge A$ congruent to $1 \mod \mathrm{tr}_Q(\Lambda_\wedge A)$ is 1, even when G is l-elementary. Decomposing the torsion subgroup H of G as $H = \langle z \rangle \times H'$, we have $A = \Gamma \times H'$ with $\Gamma \simeq \Gamma_{k'}$ and $\Lambda_\wedge A = \prod_\beta (\Lambda_\wedge^\beta \Gamma)[H']$. Now $\beta(e) \equiv 1 \mod \mathrm{tr}_Q((\Lambda_\wedge^\beta\Gamma)[H'])$. Hence $\beta(e) = 1$ by Higman's theorem for $(\Lambda_\wedge^\beta\Gamma)[H']$; see [RW3, p. 47]. This holds for all β; hence $e = 1$. □

Appendix: Deflating $\mho$

The proof of [RW2, Proposition 12(a)] refers to [RWt, Proposition 4.8] which, however, requires Leopoldt's conjecture (see [RWt, Lemma 3.4]). We recall the statement made in [RW2] (suppressing the index ∞ on K and G as well as the $\tilde{}$ on $\mho_S$):

> *If* N *is a finite normal subgroup of* G *with factor group* $\overline{G}$ *and fixed field* $\overline{K} = K^N$*, then* $\mathrm{defl}_G^{\overline{G}} : K_0T(\Lambda G) \to K_0T(\Lambda\overline{G})$ *takes* $\mho_S = \mho_{K/k,S}$ *to* $\overline{\mho}_S = \mho_{\overline{K}/k,S}$.

The refinement $\mho_{K/k,S}$ of the Iwasawa module $X = X_{K/k,S} \stackrel{\mathrm{def}}{=} G(M/K)$, with M the maximal abelian S-ramified l-extension of K, is described in [RW1, §1].

Here is a direct argument for the above claim.

Let $\overline{M}$ be the maximal abelian S-ramified l-extension of $\overline{K}$; hence $G(\overline{M}/\overline{K})$ is the biggest abelian pro-l quotient of $G(M/\overline{K})$. Consider the diagram below, where $\tilde{X}$ is the pushout of i and ver and where the transfer $G(M/\overline{K}) \to X^N$ factors through

$G(\overline{M}/\overline{K})$ since X^N is abelian pro-l. The bottom row is called the deflation of the top one in [RWt, §3.2].

$$\begin{tikzcd}
X = G(M/K) \arrow[r, tail] \arrow[d, phantom, "\cap"] & G(M/k) \arrow[r, two heads] \arrow[d, equal] & G(K/k) = G \arrow[d, two heads] \\
G(M/\overline{K}) \arrow[r, tail] \arrow[d, two heads] & G(M/k) \arrow[r, two heads] \arrow[d, two heads] & G(\overline{K}/k) = \overline{G} \arrow[d, equal] \\
\overline{X} = G(\overline{M}/\overline{K}) \arrow[r, tail, "i"] \arrow[d, "\mathrm{ver}"'] & G(\overline{M}/\overline{K}) \arrow[r, two heads] \arrow[d] & \overline{G} \arrow[d, equal] \\
X^N \arrow[r, tail] & \tilde{X} \arrow[r, two heads] & \overline{G}.
\end{tikzcd}$$

By the Appendix 4.A analogue of Lemma 3.2 in [RWt] the translation functor turns the bottom two rows into[22]

$$\begin{tikzcd}
\overline{X} \arrow[r, tail] \arrow[d] & \overline{Y} \arrow[r, two heads] \arrow[d] & \Delta\overline{G} \arrow[d, equal] \\
X^N \arrow[r, tail] & Y_N \arrow[r, two heads] & (\Delta G)_N
\end{tikzcd} \tag{$\star$}$$

by replacing the bottom one by the equivalent extension given by the ΛG-analogue of Lemma 3.3 [RWt]. Here we should note that Y has projective dimension ≤ 1 over ΛG [RW1, Theorem 1]; hence Y has projective dimension ≤ 1 over $\mathbb{Z}_l[N]$ [RW1, 2. of proof of Proposition 4], which does not need M to be finite.

Suppose that we know that $\mathrm{ver} : G(\overline{M}/\overline{K}) \to X^N$ is an isomorphism, hence that the extensions in ($\star$) are equivalent. If we use, [RW1, §1], $X \rightarrowtail Y \to \Lambda G \twoheadrightarrow \mathbb{Z}_l$ to compute $\mho$, then taking N-coinvariants computes the analogous $\mathrm{defl}_G^{\overline{G}}(\mho)$ for the bottom row of ($\star$) (cf. the analogy with [RWt, Lemma 4B.1, p. 41]). By diagram ($\star$), the same procedure for the top row computes $\overline{\mho}$. Thus $\mathrm{defl}_G^{\overline{G}}(\mho) = \overline{\mho}$.

Concerning $G(\overline{M}/\overline{K}) \xrightarrow{\mathrm{ver}} X^N$, let L be the maximal S-ramified Galois extension of k or, equivalently, of k_∞. Denote the corresponding Galois groups by $\mathfrak{G}$ and $\mathfrak{H}$, respectively; so $\mathfrak{H} \rightarrowtail \mathfrak{G} \twoheadrightarrow \Gamma_k$ is exact. Moreover, set $V = G(L/K)$ and $U = G(L/\overline{K})$, whence $V \rightarrowtail U \twoheadrightarrow N$.

Assume that we already know $\mathrm{scd}_l(\mathfrak{H}) = 2$. Then we proceed as follows. As U is open in $\mathfrak{H}$, it follows that also $\mathrm{scd}_l(U) = 2$ by [NSW, pp. 139–140]. The proof of [(i) $\Longrightarrow$ (ii)] of [NSW, Theorem 3.6.4, p. 160] gives the isomorphism $H^2(V,\mathbb{Z})(l)_N \xrightarrow{\mathrm{cor}} H^2(U,\mathbb{Z})(l)$ (see [NSW, 3.3.8, p. 142]) and so $(U^{\mathrm{ab}})_l \xrightarrow{\mathrm{ver}} (V^{\mathrm{ab}})_l^N$. Since $(U^{\mathrm{ab}})_l = G(\overline{M}/\overline{K}) = \overline{X}$, $(V^{\mathrm{ab}})_l = G(M/K) = X$ finishes the proof.

Hence it remains to show $\mathrm{scd}_l(\mathfrak{H}) = 2$. Now, $\mathrm{scd}_l(\mathfrak{H}) \leq 2$ is a consequence of the weak Leopoldt conjecture (see [NSW, 10.3.26, p. 549]) and then $\mathrm{scd}_l(\mathfrak{H}) = 2$ results from the remark following it, of which we add a proof: Assume $\mathrm{scd}_l(\mathfrak{H}) \leq 1$. Then $\mathrm{cd}_l(\mathfrak{H}) \leq 1$ and $\mathrm{cd}_l(\Gamma_k) = 1$; hence $2 = \mathrm{cd}_l(\mathfrak{G}) \leq \mathrm{cd}_l(\Gamma_k) + \mathrm{cd}_l(\mathfrak{H})$ implies $\mathrm{cd}_l(\mathfrak{H}) = 1$. Note that $\mathrm{cd}_l(\mathfrak{G}) = 2$ by [NSW, 10.9.3, p. 587]. Denoting a Sylow-l subgroup of $\mathfrak{H}$ by $\mathfrak{H}_l$, we have $\mathrm{cd}_l(\mathfrak{H}_l) = 1 = \mathrm{scd}_l(\mathfrak{H}_l)$ [NSW, 3.3.6, p. 141] and thus $H^2(\mathfrak{H}_l, \mathbb{Z})(l) = 0$ [NSW, 3.3.4, p. 139]. Hence $H^1(\mathfrak{H}_l, \mathbb{Q}_l/\mathbb{Z}_l) = 0$ and so $\mathfrak{H}_l = 1$, contradicting $\mathrm{cd}_l(\mathfrak{H}_l) = 1$.

[22] Here ΔG denotes the augmentation ideal of ΛG.

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