

## EVEN GALOIS REPRESENTATIONS AND THE FONTAINE–MAZUR CONJECTURE. II

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### 1. INTRODUCTION

Let  $G_{\mathbf{Q}}$  denote the absolute Galois group of  $\mathbf{Q}$ , and let

$$\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

be a continuous irreducible representation unramified away from finitely many primes. In [FM95], Fontaine and Mazur conjecture that if  $\rho$  is semi-stable at  $p$ , then either  $\rho$  is the Tate twist of an even representation with finite image or  $\rho$  is modular. In [Kis09], Kisin establishes this conjecture in almost all cases under the additional hypotheses that  $\rho|_{D_p}$  has distinct Hodge–Tate weights and  $\rho$  is *odd* (see also [Eme]). The oddness condition in Kisin’s work is required in order to invoke the work of Khare and Wintenberger [KW09a, KW09b] on Serre’s conjecture. If  $\rho$  is even and  $p > 2$ , however, then  $\bar{\rho}$  will never be modular. Indeed, when  $\rho$  is even and  $\rho|_{D_p}$  has distinct Hodge–Tate weights, the conjecture of Fontaine and Mazur predicts that  $\rho$  does not exist. In [Cal11], some progress was made towards proving this claim under the additional assumption that  $\rho$  was *ordinary* at  $p$ . The main result of this paper is to remove this condition. Up to conjugation, the image of  $\rho$  lands in  $\mathrm{GL}_2(\mathcal{O})$ , where  $\mathcal{O}$  is the ring of integers of some finite extension  $L/\mathbf{Q}_p$  (see Lemme 2.2.1.1 of [BM02]). Let  $\mathbf{F}$  denote the residue field, and let  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\mathbf{F})$  denote the corresponding residual representation. We prove:

**Theorem 1.1.** *Let  $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$  be a continuous Galois representation which is unramified except at a finite number of primes. Suppose that  $p > 7$ , and, furthermore, that*

- (1)  $\rho|_{D_p}$  is potentially semi-stable with distinct Hodge–Tate weights.
- (2) The representation  $\bar{\rho}$  is absolutely irreducible and not of dihedral type.
- (3)  $\bar{\rho}|_{D_p}$  is not a twist of a representation of the form  $\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$  where  $\omega$  is the mod- $p$  cyclotomic character.

Then  $\rho$  is modular.

Taking into account the work of Colmez [Col10] and Emerton [Eme], this follows directly from the main result of Kisin [Kis09] when  $\rho$  is odd. Thus, it suffices to assume that  $\rho$  is even and derive a contradiction. As in [Cal11], the main idea is to

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use *potential automorphy* to construct from  $\rho$  a RAESDC automorphic representation  $\pi$  for  $\mathrm{GL}(n)$  over some totally real field  $F$  whose existence is incompatible with the evenness of  $\rho$ . It was noted in [Cal11] that improved automorphy lifting theorems would lead to an improvement in the main results of that paper. Using the recent work of Barnet–Lamb, Gee, Geraghty, and Taylor [BLGGT10], it is a simple matter to deduce the main theorem of this paper if  $\rho$  is a twist of a crystalline representation sufficiently deep in the Fontaine–Laffaille range (explicitly, if *twice* the difference of the Hodge–Tate weights is at most  $p-2$ ). However, if one wants to apply the main automorphy lifting theorem (Theorem 4.2.1) of [BLGGT10] more generally, then (at the very least) one has to assume that  $\rho|_{D_p}$  is potentially crystalline. Even under this assumption, one runs into the difficulty of showing that  $\rho|_{D_p}$  is *potentially diagonalizable* (in the notation of that paper), which seems out of reach at present. Instead, we use an idea we learned from Gee (which is also crucially used in [BLGGT10, BLGG09, BLGG10]) of tensoring together certain “shadow” representations in order to maneuver ourselves into a situation in which we can show that a certain representation (which we would like to prove is automorphic) lies on the same component (of a particular local deformation ring) as an automorphic representation. In [BLGGT10, BLGG09], it is important that one restricts, following the idea of M. Harris, to tensoring with representations induced from characters, since then one is still able to prove the modularity of the original representation. In contrast, we shall need to tensor together representations with large image. Ultimately, we construct (from  $\rho$ ) a regular algebraic self-dual automorphic representation for  $\mathrm{GL}(9)$  over a totally real field  $E^+$  with a corresponding  $p$ -adic Galois representation  $\varrho : G_{E^+} \rightarrow \mathrm{GL}_9(\overline{\mathbf{Q}}_p)$ . If  $\rho$  is even, then (by construction) it will be the case that  $\mathrm{Trace}(\varrho(c)) = +3$  for any complex conjugation  $c$ . This contradicts the main theorem of [Tay10], and thus  $\rho$  must be odd. In order to understand the local deformation rings that arise, and in order to construct an appropriate shadow representation, we shall have to use the full strength of the results of Kisin [Kis09] for totally real fields in which  $p$  splits completely. This is the reason why condition (3) of Theorem 1.1 is required, even when  $\rho$  is even.

It will be convenient to prove the following, which, in light of the main theorem of [Kis09], implies Theorem 1.1. Recall that  $\omega$  denotes the mod- $p$  cyclotomic character.

**Theorem 1.2.** *Let  $F^+$  be a totally real field in which  $p$  splits completely. Let  $\rho : G_{F^+} \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$  be a continuous Galois representation unramified except at a finite number of primes. Suppose that  $p > 7$ , and, furthermore, that*

- (1)  $\rho|_{D_v}$  is potentially semi-stable with distinct Hodge–Tate weights for all  $v|p$ .
- (2) The representation  $\mathrm{Sym}^2 \bar{\rho}|_{G_{F^+(\zeta_p)}}$  is irreducible.
- (3) If  $v|p$ , then  $\bar{\rho}|_{D_v}$  is not a twist of a representation of the form  $\begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$ .

Then, for every real place of  $F^+$ ,  $\rho$  is odd.

*Remark 1.3.* Under the conditions of Theorem 1.2, it follows that  $\rho$  is potentially modular over an extension in which  $p$  splits completely (see Remark 3.6).

In section 5, we give some applications of our theorem to universal deformation rings. In particular, we construct (unrestricted) universal deformation rings of large dimension such that none of the corresponding Galois representations is geometric

(= de Rham). For an even more concrete application, we prove the following (see Corollary 5.5):

**Theorem 1.4.** *There exists a surjective even representation  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{SL}_2(\mathbf{F}_{11})$  with no geometric deformations.*

Contrast this result with Corollary 1(a) of Ramakrishna [Ram02], which implies that  $\bar{\rho}$  does admit a deformation to a surjective representation  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{SL}_2(\mathbf{Z}_{11})$  unramified outside finitely many primes.

*Remark 1.5.* A word on notation. There are only finitely many letters that can plausibly be used to denote a global field, and thus, throughout the text, we have resorted to using subscripts. In order to prepare the reader, we note now the existence in the text of a sequence of inclusions of totally real fields:

$$F^+ \subseteq F_1^+ \subseteq F_2^+ \subseteq F_3^+ \subseteq F_4^+ \subseteq F_5^+ \subseteq F_6^+ \subseteq F_7^+,$$

and corresponding degree two CM extensions  $F_3 \subseteq \dots \subseteq F_7$ . The subscript implicitly records (except for one instance) the number of times a theorem of Moret-Bailly (Theorem 3.1) is applied. (This is not literally true, since many of the references we invoke also appeal to variations of this theorem.)

As usual, the abbreviations RAESDC and RACSDC for an automorphic representation  $\pi$  for  $\mathrm{GL}(n)$  stand for regular, algebraic, essentially self-dual, and cuspidal and for regular, algebraic, conjugate self-dual, and cuspidal, respectively.

## 2. LOCAL DEFORMATION RINGS

Let  $E$  be a finite extension of  $\mathbf{Q}_p$  (the coefficient field), and let  $V$  be a  $d$ -dimensional vector space over  $E$  with a continuous action of  $G_K$ , where  $K/\mathbf{Q}_p$  is a finite extension. Let us suppose that  $V$  is potentially semi-stable [Fon94]. Let

$$\tau : I_K \rightarrow \mathrm{GL}_d(\overline{\mathbf{Q}}_p)$$

be a continuous representation of the inertia subgroup of  $K$ . Fix an embedding  $K \hookrightarrow \overline{\mathbf{Q}}_p$ . Attached to  $V$  is a  $d$ -dimensional representation of the Weil–Deligne group of  $K$ . If the restriction of this representation to the inertia subgroup is equivalent to  $\tau$ , we say that  $V$  is of *type*  $\tau$ . Also associated to  $V$  is a  $p$ -adic Hodge type  $\mathbf{v}$ , which records the breaks in the Hodge filtration associated to  $V$  considered as a de Rham representation (cf. [Kis08], §2.6). Let  $\mathbf{F}$  be a finite field of characteristic  $p$ , and let us now fix a representation

$$\bar{\rho} : G_K \rightarrow \mathrm{GL}_n(\mathbf{F}).$$

Let  $R_{\bar{\rho}}^{\square}$  be the universal framed deformation ring of  $\bar{\rho}$  considered as a  $W(\mathbf{F})$ -algebra. The following theorem is a result of Kisin (see [Kis08], Theorem 2.7.6).

**Theorem 2.1** (Kisin). *There exists a quotient  $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}$  of  $R_{\bar{\rho}}^{\square}$  such that the  $\overline{\mathbf{Q}}_p$  points of the scheme  $\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$  are exactly the  $\overline{\mathbf{Q}}_p$  points of  $\mathrm{Spec}(R_{\bar{\rho}}^{\square})$  that are potentially semi-stable of type  $\tau$  and Hodge type  $\mathbf{v}$ . It is unique if it is assumed to be reduced and  $p$ -torsion free.*

Note that restricting  $\bar{\rho}$  to some finite index subgroup  $G_L$  induces a functorial map of corresponding local deformation rings:

$$\mathrm{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p]) \rightarrow \mathrm{Spec}(R_{\bar{\rho}|_{G_L}}^{\square, \tau, \mathbf{v}}[1/p]),$$

where, by abuse of notation,  $\tau$  in the second ring denotes the restriction of  $\tau$  to  $I_L$  (and correspondingly with  $\mathbf{v}$ ). We use  $\mathbb{1}$  to denote the trivial type.

**Definition 2.2.** A point of  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$  is *very smooth* if it defines a smooth point on  $\text{Spec}(R_{\bar{\rho}|_{G_L}}^{\square, \tau, \mathbf{v}}[1/p])$  for every finite extension  $L/K$ .

In §1.3 and §1.4 of [BLGGT10], various notions of equivalence are defined between representations. We would like to define a mild (obvious) extension of these definitions when  $v|p$ . Suppose that  $\rho_1$  and  $\rho_2$  are two continuous  $d$ -dimensional representations of  $G_K$  with coefficients in some finite extension  $E$  over  $\mathbf{Q}_p$ . Let  $\mathcal{O}$  denote the ring of integers of  $E$ . Let us assume that  $\rho_1$  and  $\rho_2$  come with a specific integral structure, i.e., a given  $G_K$ -invariant  $\mathcal{O}$ -lattice. Equivalently, we may suppose that  $\rho_1$  and  $\rho_2$  are representations  $G_K \rightarrow \text{GL}_d(\mathcal{O})$ . In particular, the mod- $p$  reductions  $\bar{\rho}_1$  and  $\bar{\rho}_2$  are well defined. Such extra structure arises, for example, if the representations  $\rho_i$  are the local representations attached to global representations whose mod- $p$  reductions are absolutely irreducible.

**Definition 2.3.** Suppose  $\rho_1$  and  $\rho_2$  are two continuous  $G_K$ -representations with given integral structure. If  $\rho_1$  and  $\rho_2$  are potentially semi-stable, we say that  $\rho_1 \Downarrow \rho_2$  (respectively,  $\rho_1 \rightsquigarrow \rho_2$ ) if  $\bar{\rho} := \bar{\rho}_1 \simeq \bar{\rho}_2$ , the representations  $\rho_1$  and  $\rho_2$  have the same type  $\tau$ , the same Hodge type  $\mathbf{v}$ , and lie on the same irreducible component of  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$ , and, furthermore, that  $\rho_1$  corresponds to a very smooth point of  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$  (respectively, smooth point of  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$ ).

*Remark 2.4.* If  $\rho_1 \Downarrow \rho_2$ , we say (following [BLGGT10], §1.3, §1.4) that  $\rho_1$  *very strongly connects* to  $\rho_2$  (or  $\rho_1$  “zap”  $\rho_2$ ). (If  $\rho_1 \rightsquigarrow \rho_2$ , then  $\rho_1$  *strongly connects* to  $\rho_2$ , or  $\rho_1$  “squig”  $\rho_2$ .) If  $\rho_1 \Downarrow \rho_2$ , then clearly  $\rho_1 \rightsquigarrow \rho_2$ , and moreover  $\rho_1|_{G_L} \Downarrow \rho_2|_{G_L}$  (and hence  $\rho_1|_{G_L} \rightsquigarrow \rho_2|_{G_L}$ ) for any finite extension  $L/K$ .

*Remark 2.5.* If  $\rho_1$  and  $\rho_2$  are both potentially crystalline representations, then one may also consider the ring  $R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, cr}$  parametrizing representations which are potentially crystalline (cf. [Kis08]). One may subsequently define the notions  $\Downarrow$  and  $\rightsquigarrow$  relative to this ring. Since  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, cr}[1/p])$  is smooth (*ibid.*), the relationships  $\rho_1 \Downarrow \rho_2$  and  $\rho_1 \rightsquigarrow \rho_2$  are symmetric, and one may simply write  $\rho_1 \sim \rho_2$  (“ $\rho_1$  connects to  $\rho_2$ ”; cf. [BLGGT10]). The scheme  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$  is not in general smooth, so we must impose strong connectedness in the potentially semi-stable case in order for the arguments of [BLGGT10] to apply. In this paper, whenever we have  $p$ -adic representations  $\rho_1$  and  $\rho_2$ , we shall write  $\rho_1 \sim \rho_2$  only when  $\rho_1$  and  $\rho_2$  are potentially crystalline, and by writing this we mean that they are connected relative to  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}, cr}[1/p])$ .

In order to deduce in any particular circumstance that  $\rho_1 \Downarrow \rho_2$ , it will be useful to have some sort of criteria to determine when  $\rho_1$  corresponds to a very smooth point. If  $V$  is a potentially semi-stable representation of  $G_K$ , let  $D = D_{\text{pst}}(V)$  denote the corresponding weakly admissible  $(\varphi, N, \text{Gal}(\bar{K}/K))$ -module. Let

$$D(k) \subset \bar{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p^{nr}} D$$

denote the subspace generated by the (generalized) eigenvectors of Frobenius of slope  $k$ . Since  $N\varphi = p\varphi N$ , there is a natural map  $N : D(k+1) \rightarrow D(k)$ .

**Lemma 2.6.** *Let  $V$  be a potentially semi-stable representation of  $G_K$  of type  $\tau$  and Hodge type  $\mathbf{v}$ . Suppose that  $N : D(k+1) \rightarrow D(k)$  is an isomorphism whenever the target and source are nonzero. Then  $V$  is a very smooth point on  $\text{Spec}(R_{\bar{\rho}}^{\square, \tau, \mathbf{v}}[1/p])$ .*

*Proof.* The explicit condition follows from the proof of Lemma 3.1.5 of [Kis08].  $\square$

The condition of Lemma 2.6 is a (somewhat brutal) way of insisting that the monodromy operator  $N$  is as nontrivial as possible, given the action of Frobenius. We note in passing that by Theorem 3.3.4 of [Kis08],  $\text{Spec}(R_p^{\square, \tau, \nu}[1/p])$  admits a formally smooth dense open subscheme.

**Example 2.7.** If  $V$  is a 2-dimensional representation that is potentially semi-stable but not potentially crystalline, then  $\text{Sym}^{n-1}(V)$  satisfies the conditions of Lemma 2.6 and so is very smooth for all  $n$ .

*Remark 2.8.* One expects that a local  $p$ -adic representation associated to an RACDSC automorphic representation  $\pi$  is very smooth on the corresponding local deformation ring. In fact, this would follow by the proof of Lemma 1.3.2 of [BLGGT10] if one had local-global compatibility at all primes (cf. Conjecture 1.1 and Theorem 1.2 of [TY07]). Since local-global compatibility is still unknown, however, we must take more care in ensuring that the local representations associated to automorphic representations we construct are (very) smooth.

### 3. REALIZING LOCAL REPRESENTATIONS

It will be useful in the sequel to quote the following extension of a theorem of Moret–Bailly.

**Theorem 3.1.** *Let  $E$  be a number field and let  $S$  be a finite set of places of  $E$ . Let  $F/E$  be an auxiliary finite extension of number fields. Suppose that  $X/E$  is a smooth geometrically connected variety. Suppose that: For  $v \in S$ ,  $\Omega_v \subset X(E_v)$  is a nonempty open (for the  $v$ -topology) subset. Then there is a finite Galois extension  $H/E$  and a point  $P \in X(H)$  such that*

- (1)  $H/E$  is linearly disjoint from  $F/E$ .
- (2) Every place  $v$  of  $S$  splits completely in  $H$ , and if  $w$  is a prime of  $H$  above  $v$ , there is an inclusion  $P \in \Omega_v \subset X(H_w)$ .
- (3) Suppose that for any place  $u$  of  $\mathbf{Q}$ ,  $S$  contains either every place  $v|u$  of  $E$  or no such places. Then one can choose  $H$  to be a compositum  $EM$  where:
  - (a)  $M/\mathbf{Q}$  is a totally real Galois extension.
  - (b) If there exists a  $v \in S$  and a prime  $p$  such that  $v|p$ , then  $p$  splits completely in  $M$ .

*Proof.* Omitting part (3), this is (a special case of) Proposition 2.1 of [HSBT10]. To prove the additional statement, it suffices to apply Proposition 2.1 of [HSBT10] to the restriction of scalars  $Y = \text{Res}_{E/\mathbf{Q}}(X)$ . (Note that being smooth and geometrically connected is preserved under taking the restriction of scalars of a separable extension; see, for example, Theorem A.5.9 of [CGP10].)  $\square$

As a first application of this theorem, we prove the following result, which shows that the inverse Galois problem can be solved “potentially”, even with the imposition of local conditions at a finite number of primes.

**Proposition 3.2.** *Let  $G$  be a finite group, let  $E/\mathbf{Q}$  be a finite extension, and  $S$  a finite set of places of  $E$ . Let  $F/E$  be an auxiliary finite extension of number fields. For each finite place  $v \in S$ , let  $H_v/E_v$  be a finite Galois extension together with a fixed inclusion  $\phi_v : \text{Gal}(H_v/E_v) \rightarrow G$  with image  $D_v$ . For each real infinite place  $v \in S$ , let  $c_v \in G$  be an element of order dividing 2. There exists a number*

field  $K/E$  and a finite Galois extension of number fields  $L/K$  with the following properties:

- (1) There is an isomorphism  $\text{Gal}(L/K) = G$ .
- (2)  $L/E$  is linearly disjoint from  $F/E$ .
- (3) All places in  $S$  split completely in  $K$ .
- (4) For all finite places  $w$  of  $K$  above  $v \in S$ , the local extension  $L_w/K_w$  is equal to  $H_v/E_v$ . Moreover, there is a commutative diagram:

$$\begin{array}{ccc}
 \text{Gal}(L_w/K_w) & \longrightarrow & D_w \subset G \\
 \parallel & & \parallel \\
 \text{Gal}(H_v/E_v) & \xrightarrow{\phi_v} & D_v \subset G.
 \end{array}$$

- (5) For all real places  $w|\infty$  of  $K$  above  $v \in S$ , complex conjugation  $c_w \in G$  is conjugate to  $c_v$ .

*Proof.* Suppose that  $G$  acts faithfully on  $n$  letters, and let  $G \hookrightarrow \Sigma$  denote the corresponding map from  $G$  to the symmetric group. (Any group admits such a faithful action, e.g., the regular representation.) There is an induced action of  $G$  on  $\mathbf{Q}[x_1, x_2, \dots, x_n]$ , and we may let  $X_G = \text{Spec}(\mathbf{Q}[x_1, x_2, \dots, x_n]^G)$ . There are corresponding morphisms

$$\mathbf{A}^n \rightarrow X_G \rightarrow X_\Sigma.$$

The scheme  $X_G$  is affine, irreducible, geometrically connected, and contains a Zariski dense smooth open subscheme. The variety  $X_\Sigma$  is canonically isomorphic to the affine space  $\mathbf{A}^n$  over  $\text{Spec}(\mathbf{Q})$  via the symmetric polynomials. Under the projection to  $X_\Sigma$ , a  $K$ -point of  $X_G$  (for any perfect field  $K$ ) gives a polynomial over  $K$  such that the Galois group of its splitting field  $L$  is a (not necessarily transitive) subgroup of  $G$ . Without loss of generality, we may enlarge  $S$  in the following way: For each conjugacy class  $\langle g \rangle \in G$ , we add to  $S$  an auxiliary finite place  $v$  and impose a local condition that the decomposition group at  $v$  is unramified and is the subgroup generated by  $g$ . For all  $v \in S$ , let  $\Omega_v \subset X_G(E_v)$  denote the set of points of  $X_G$  defined as follows. If  $v$  is a finite place, we suppose that the extension  $L_v/E_v$  is isomorphic to  $H_v/E_v$ , and moreover, the corresponding action of  $D_v$  on  $G$  is the one given by  $\phi_v$ . To construct such a point, we need to show that, given a Galois group  $D_v = \text{Gal}(H_v/E_v)$ , any permutation representation on  $\Sigma$  can be realized by an  $n$ -tuple of elements of  $H_v$ . By induction, it suffices to consider the case of transitive representations. For a point  $x \in \Sigma$  whose stabilizer is supposed to be  $C_v = \text{Gal}(H_v/A_v)$ , choose  $x$  to be a primitive element of  $A_v = H_v^{C_v}$ , and then extend this choice to the orbit of  $x$  under the abstract group  $D_v$  via the Galois action arising from  $\phi_v$ . If  $v$  is an infinite place, we simply require that  $L_v/E_v$  has Galois group  $\langle c_v \rangle$ . By assumption, these sets are nonzero, and by Krasner's Lemma they are open. We deduce by Theorem 3.1 (applied to the smooth open subscheme of  $X_G$ ) that there exists a Galois extension  $L/K$  with Galois group  $H \subset G$  with the required local decomposition groups at each place  $w$  above  $v$ . By construction, for every  $g \in G$  there exists a finite unramified place  $w$  in  $K$  such that the conjugacy class of Frobenius at  $w$  in  $\text{Gal}(L/K)$  is the conjugacy class of  $g$ . It follows that the intersection of  $H$  with every conjugacy class of  $G$  is nontrivial, and hence  $H = G$

by a well-known theorem of Jordan (see Theorem 4' of [Ser03]). Thus the theorem is established.  $\square$

*Remark 3.3.* A weaker version of Proposition 3.2, namely, that every finite group  $G$  occurs as the Galois group  $\text{Gal}(L/K)$  for some extension of number fields, is a trivial consequence of the fact that  $\Sigma = S_n$  occurs as the Galois group of some extension of  $\mathbf{Q}$ , since we may take  $L$  to be any such extension and  $K = L^G$ . If we insist that some place  $v$  splits completely in  $K$ , however, this will typically force  $L$  to also split completely at  $v$ .

Let  $\rho : G_{F^+} \rightarrow \text{GL}_2(\overline{\mathbf{Q}}_p)$  be as in Theorem 1.2. After increasing  $F^+$  (if necessary), we may assume that  $\rho$  is semi-stable at all primes of residue characteristic different from  $p$ . Attached to  $\rho$  is a residual representation  $\bar{\rho} : G_{F^+} \rightarrow \text{GL}_2(\mathbf{F})$  for some finite field  $\mathbf{F}$  of characteristic  $p$ .

**Proposition 3.4.** *There exists a totally real field  $F_1^+/F^+$  and a residual Galois representation  $\bar{\tau}_{\text{res}} : G_{F_1^+} \rightarrow \text{GL}_2(\mathbf{F})$  with the following properties:*

- (1) *All primes above  $p$  split completely in  $F_1^+$ .*
- (2) *The residual representation  $\bar{\tau}_{\text{res}} : G_{F_1^+} \rightarrow \text{GL}_2(\mathbf{F})$  has image containing  $\text{SL}_2(\mathbf{F}_p)$ .*
- (3) *For each  $v|p$  in  $F^+$ , and for each  $w$  above  $v$  in  $F_1^+$ , there is an isomorphism  $\bar{\tau}_{\text{res}}|_{D_w} \simeq \bar{\rho}|_{D_v}$ .*
- (4) *If  $v \nmid p$ , then  $\bar{\tau}_{\text{res}}|_{D_v}$  is unramified.*
- (5)  *$\bar{\tau}_{\text{res}}$  is totally odd at every real place of  $F_1^+$ .*
- (6)  *$F_1^+ \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$  and  $F_1^+ \cap \mathbf{Q}(\ker(\bar{\rho})) = \mathbf{Q}$ .*

*Proof.* Proposition 3.2 immediately guarantees a residual representation satisfying all the conditions with the possible exception of (4), which can be achieved by a further base extension.  $\square$

**Proposition 3.5.** *There exists a totally real field  $F_2^+/F_1^+$  and a Hilbert modular form  $f$  for  $F_2^+$  with a corresponding residual representation  $\bar{\rho}_f : G_{F_2^+} \rightarrow \text{GL}_2(\mathbf{F})$  with the following properties:*

- (1) *There is an isomorphism  $\bar{\rho}_f \simeq \bar{\tau}_{\text{res}}|_{G_{F_2^+}}$ .*
- (2) *All primes above  $p$  split completely in  $F_2^+$ .*
- (3)  *$F_2^+ \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$ ,  $F_2^+ \cap \mathbf{Q}(\ker(\bar{\rho})) = \mathbf{Q}$ , and  $F_2^+ \cap \mathbf{Q}(\ker(\bar{\tau}_{\text{res}})) = \mathbf{Q}$ .*

*Proof.* This follows immediately from Theorem 5.1.1 of [Sno], taking into account Proposition 8.2.1 of *ibid.* It suffices to note (in the notation of *ibid.*) that there always exists at least one definite type  $t(v)$  compatible with  $\bar{\tau}_{\text{res}}|_{D_v}$  for each  $v|p$ , but this is exactly Proposition 7.8.1 of [Sno].  $\square$

*Remark 3.6.* If  $\rho$  is odd for all infinite places of  $F_1^+$ , then we may take  $\bar{\tau}_{\text{res}}$  to be  $\bar{\rho}$ , and Proposition 3.5 implies that  $\bar{\rho}|_{G_{F_2^+}}$  is modular. By Theorem 2.2.18 of [Kis07], it follows in this case that  $\rho$  is modular over  $F_2^+$ . (This is not literally correct, because  $\bar{\rho}$  may not have image containing  $\text{SL}_2(\mathbf{F}_p)$  and so not virtually satisfy Condition (2) of Proposition 3.4. However, one may check that the only fact used about the image of  $\bar{\tau}$  so far is that it is irreducible.)

Having realized the representations  $\bar{\rho}|_{D_v}$  for  $v|p$  inside the mod- $p$  reduction of some Hilbert modular form  $f$ , we now realize the representations  $\rho|_{D_v}$  in characteristic zero as coming from Hilbert modular forms (to the extent that it is possible).

**Proposition 3.7.** *There exists a Hilbert modular form  $g$  over  $F_2^+$  with the following properties:*

- (1) *The residual representation  $\bar{\rho}_g : G_{F_2^+} \rightarrow \mathrm{GL}_2(\mathbf{F})$  is equal to  $\bar{\rho}_f$ .*
- (2) *For each place  $w|p$  of  $F^+$  and  $v|w$  of  $F_2^+$ ,  $\rho_g|D_v \sim \rho|D_w$  if  $\rho|D_w$  is potentially crystalline, and  $\rho_g|D_v \not\sim \rho|D_w$  otherwise.*
- (3) *For each finite place  $v$  of  $F_2^+$  away from  $p$ ,  $\rho_g|D_v$  is unramified.*

*Proof.* Consider the modular representation  $\bar{\rho}_f = \bar{r}_{\mathrm{res}}|G_{F_2^+}$  constructed in Proposition 3.5. By construction, it is modular of minimal level and is unramified outside  $p$ . It follows from Theorem 2.2.18 and Corollary 2.2.17 of [Kis09] that (in the notation of *ibid.*)  $M_\infty$  is faithful as an  $\bar{R}_\infty$ -module. To orient the reader,  $M_\infty$  is a module which is built (by Taylor–Wiles patching) from a sequence of faithful Hecke modules of finite level, and  $\bar{R}_\infty$  is a (patched) decorated universal deformation ring which encodes deformations of a prescribed type at all  $v|p$ . In particular, the ring  $\bar{R}_\infty$  detects *all* the components of the local deformation rings at  $v|p$  of the given type. Hence, at least morally (since at this point we are working with patched modules) the faithfulness of  $M_\infty$  says that “every component” of the corresponding local deformation rings is realized globally. To show this formally, we need to pass from the patched level back down to the finite level by setting all the auxiliary Taylor–Wiles variables  $\Delta_\infty$  equal to zero. Explicitly, the Taylor–Wiles–Kisin method yields an isomorphism  $\bar{R}_\infty = \bar{R}_{\Sigma_p}^{\square, \psi}[[x_1, \dots, x_g]]$  of  $\bar{R}_\infty$  as a power series ring over a tensor product of local deformation rings. Here  $\Sigma_p$  denotes the set of places dividing  $p$ . Consider a component  $Z$  of  $\mathrm{Spec}(\bar{R}_\infty)$  such that the characteristic zero points of  $Z$  lie on the same local component as  $\rho|D_v$  for  $v|p$  (if  $\rho|D_v$  lies on multiple components, choose any component). Since all the (equivalent) conditions of Lemma 2.2.11 of *ibid.* hold, we know (as in the proof of and notation of that lemma) that  $\bar{R}_\infty$  is a finite torsion free  $\mathcal{O}[[\Delta_\infty]]$ -module. In particular,  $Z$  surjects onto  $\mathrm{Spec}(\mathcal{O}[[\Delta_\infty]])$ . In particular, there is a nontrivial fibre at 0. Since  $M_0 = M_\infty \otimes_{\mathcal{O}[[\Delta_\infty]]} \mathcal{O}$  is a space of *classical* modular forms (of minimal level), we deduce the existence of  $g$ . Note that if  $\rho|D_w$  is potentially semi-stable but not potentially crystalline, then  $\rho_g$  is very smooth by Example 2.7, and hence  $\rho_g|D_v \not\sim \rho|D_w$  for  $v|p$ . □

*Remark 3.8.* The faithfulness of  $M_\infty$  as an  $\bar{R}_\infty$ -module is not a clear consequence of the Fontaine–Mazur conjecture. That is, *a priori*, the collection of all global representations may surreptitiously conspire to avoid a given local component. Thus, without any further ideas, the new cases of the Fontaine–Mazur conjecture proved by Emerton [Eme] do not allow us to realize all local representations globally when

$$\bar{\rho}|D_p \sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}.$$

We have now constructed a Hilbert modular form  $g$  whose  $p$ -adic representation is a “shadow” of  $\rho$ , that is, lies on the same component as  $\rho$  of every local deformation space at a place dividing  $p$ . However, the global mod- $p$  representations  $\bar{\rho}$  and  $\bar{\rho}_g$  are unrelated. In order to prove a modularity statement, we will need to construct a second pair of shadow representations with the same residual representation as  $\bar{\rho}$  and  $\bar{\rho}_g$ . It will never be the case, however, that  $\bar{\rho}$  will be automorphic over a totally real field unless  $\bar{\rho}$  is totally odd. The main idea of [Cal11] was to consider the representation  $\mathrm{Sym}^2(\bar{\rho})$  as a conjugate self-dual representation over some CM field.



In the sequel, we shall construct a pair of RACSDC ordinary crystalline shadow representations which realize the mod- $p$  representations  $\text{Sym}^2(\bar{\rho})$  and  $\text{Sym}^2(\bar{\rho}_g)$ . By abuse of notation, let  $\mathbf{v}$  denote the Hodge type of  $\rho$  for any prime dividing  $p$ , so  $\mathbf{v}$  is literally a collection of Hodge types for each  $v|p$  in  $F^+$ . Similarly, suppose that  $\rho$  is potentially semi-stable of type  $\tau$ , where  $\tau$  refers to a collection of types for all  $v|p$  (adding subscripts would add nothing to the readability of the following argument).

**Proposition 3.9.** *Let  $p > 7$  be prime. There exists a totally real field  $F_5^+/F_2^+$ , a CM extension  $F_5/F_2^+$ , and a RACDSC automorphic representation  $\pi$  over  $F_5$  with a corresponding Galois representation  $\rho_\pi : G_{F_5} \rightarrow \text{GL}_3(\overline{\mathbf{Q}}_p)$  such that:*

- (1)  $\rho_\pi$  is unramified at all places not dividing  $p \cdot \infty$ .
- (2) For every  $v|p$ ,  $\rho_\pi|_{G_v}$  is ordinary and crystalline with Hodge type  $\text{Sym}^2 \mathbf{w}$ , where  $\mathbf{w}$  is a Hodge type of some 2-dimensional de Rham representation.
- (3) For all  $v|p$  and for all  $i$ , there is an inequality  $\dim \text{gr}^i(\text{Sym}^2 \mathbf{v} \otimes \text{Sym}^2 \mathbf{w}) \leq 1$ .
- (4) The image of the restriction of  $\bar{\rho}$  to  $G_{F_5^+}$  is the image of  $\bar{\rho}$  on  $G_{\mathbf{Q}}$ .
- (5) The restriction of  $\bar{\rho}_g$  to  $G_{F_5^+}$  has image containing  $\text{SL}_2(\mathbf{F}_p)$ .
- (6) The residual Galois representation  $\bar{\rho}_\pi : G_{F_5} \rightarrow \text{GL}_3(\mathbf{F})$  is isomorphic to the restriction of  $\text{Sym}^2(\bar{\rho})$  to  $G_{F_5}$ .
- (7) The Hilbert modular form  $g$  remains modular over  $F_5^+$ .
- (8) The compatible family of Galois representations associated to  $\pi$  is irreducible after restriction to any finite index subgroup of  $G_{F_5}$ .
- (9) If  $\rho_g|_{D_p}$  is not potentially crystalline, the representation  $\rho_\pi \otimes \text{Sym}^2(\rho_g)|_{D_p}$  is a very smooth point of

$$\text{Spec}(R^{\square, \text{Sym}^2(\tau) \otimes \text{Sym}^2(1), \text{Sym}^2 \mathbf{v} \otimes \text{Sym}^2 \mathbf{w}}[1/p]).$$

- (10)  $F_5 \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$ .
- (11)  $\bar{\rho}_\pi|_{D_v}$  is trivial for every  $v|p$ .

*Proof.* Let  $F_3^+/F_2^+$  be a totally real field for which  $\text{Sym}^2(\bar{\rho})$  becomes completely trivial at all  $v|p$ . (In general, the field  $F_3^+$  will be highly ramified at  $p$ .) Increasing  $F_3^+$  if necessary, assume that the restriction of  $\text{Sym}^2(\bar{\rho})$  is unramified outside  $v|p$ , and that there exists a CM extension  $F_3/F_3^+$  which is totally unramified at all finite places. By Proposition 3.3.1 of [BLGGT10],  $\text{Sym}^2(\bar{\rho})$  admits minimal ordinary automorphic lifts over some CM extension  $F_4 = F_4^+ \cdot F_3$ . (Here by minimal we simply mean unramified outside  $p$  and ordinary and crystalline at  $v|p$ .) (We use the fact that  $\bar{\rho}$  is not of dihedral type, so  $\text{Sym}^2(\bar{\rho})$  is irreducible, and that  $p \geq 2(3+1)$ .) The resulting automorphic representation  $\pi$  satisfies condition (1). Note that  $F_4$  may be chosen to be disjoint from any auxiliary field. This implies that we may construct  $F_4$  so that conditions (1), (4), (5), (6), (10), and (11) hold. To preserve the modularity of  $g$ , let  $l$  denote an auxiliary prime which is totally split in  $F_4$ , and which admits a prime  $\lambda|l$  in  $F_2^+$  such that:

- (1)  $\rho_{g,\lambda}$  is crystalline in the Fontaine–Laffaille range, and the difference of any two weights is at most  $l - 2$ .
- (2)  $\bar{\rho}_{g,\lambda}$  is absolutely irreducible.

We may now apply Theorem 4.5.1 of [BLGGT10] to deduce the existence of an extension  $F_5 = F_5^+ \cdot F_4$  of  $F_4$  so that  $\pi$  and  $g$  *simultaneously* remain automorphic over  $F_5$  (by base change, if  $g$  is automorphic over  $F_5$ , it is also so over  $F_5^+$ ). Note that the

key condition to check in order to apply Theorem 4.5.1 of [BLGGT10] is potential diagonalizability. This is easy by construction: the  $p$ -adic Galois representations associated to  $\pi$  are ordinary for all  $v|p$ , and the  $\lambda$ -adic Galois representations associated to  $g$  are crystalline in the Fontaine–Laffaille range, and  $\lambda|l$  is unramified in  $F_4$ . Thus we can ensure that condition (7) holds. It is also easy to see that  $F_5$  can be chosen to preserve the conditions that we have already established above. Note that the amount of freedom allowed in choosing the Hodge type is equivalent to the freedom to choose  $\mu$  in Proposition 3.2.1 of [BLGGT10], and it is easy to find a choice of weight such that  $\pi$  satisfies conditions (2) and (3). In particular, one could let  $\mathbf{w}$  be any regular Hodge type which is independent of the choice of embedding  $F_5 \rightarrow \overline{\mathbf{Q}}_p$  and for which there are sufficiently large gaps between nonzero terms of  $\text{gr}^i(\mathbf{w})$ . To verify that the compatible system associated to  $\pi$  is irreducible over any finite index subgroup of  $G_{F_5}$  (and so satisfies condition (8)), we invoke Theorem 2.2.1 of [BR92]. It suffices to note that  $\overline{\rho}_\pi$  has a nonsolvable image, and thus  $\pi$  is not induced from an algebraic Hecke character over a solvable extension. Finally, we must show that  $\pi$  can be chosen to satisfy (9). The slopes of Frobenius of an ordinary crystalline representation are given by the breaks in the Hodge filtration. In particular, we may choose a  $\mathbf{w}$  so that the integers  $i$  such that  $\text{gr}^i \text{Sym}^2(\mathbf{w}) \neq 0$  each differ by  $\geq 4$ . If  $\text{Sym}^2(\rho_g)$  is potentially semi-stable but not potentially crystalline, it follows from Lemma 2.6 that for such a  $\mathbf{w}$  that any such tensor product  $\rho_\pi \otimes \text{Sym}^2(\rho_g)$  will be very smooth. Thus, choosing  $\mathbf{w}$  appropriately, very smoothness is automatically satisfied.  $\square$

We may now construct a Hilbert modular form  $h$  as follows.

**Proposition 3.10.** *Let  $p > 5$ . There exists a totally real field  $F_6^+/F_5^+$  and a Hilbert modular form  $h$  over  $F_6^+$  with a corresponding Galois representation  $\rho_h : G_{F_6^+} \rightarrow \text{GL}_2(\overline{\mathbf{Q}}_p)$  such that:*

- (1) *For every  $v|p$ ,  $\rho_h|D_v$  is ordinary and crystalline with Hodge type  $\mathbf{w}$ .*
- (2) *The residual representation  $\overline{\rho}_h : G_{F_6^+} \rightarrow \text{GL}_2(\mathbf{F})$  is isomorphic to the restriction of  $\overline{\rho}_g$ .*
- (3) *The images of  $\overline{\rho}$  and  $\overline{\rho}_g$  remain unchanged upon restriction to  $G_{F_6^+}$ .*
- (4) *The Hilbert modular form  $g$  remains modular over  $F_6^+$ , and the RACDSC representation  $\pi$  remains modular over  $F_6 = F_6^+ \cdot F_5$ .*
- (5) *For all places  $v$  not dividing  $p$ ,  $\rho_h|D_v \sim \rho|D_v$ .*
- (6)  *$F_6 \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}$ .*

*Proof.* We may prove this by modifying the proof of Proposition 3.9 as follows. Modify the field  $F_3^+$  so that  $\overline{\rho}_g$  is ordinary at all  $v|p$  (but still has large image), and such that  $\overline{\rho}_g|D_v$  admits a modular lift which is a ramified and semi-stable lift for those primes  $v \nmid p$  which ramify in  $\rho$ , ordinary for  $v|p$ , and unramified everywhere else. The existence of such a lift follows from Theorem 6.1.9 of [BLGG10], at least in the weight 0 case; the existence of a lift in weight  $\mathbf{w}$  then follows from Hida theory. Then let  $F_6^+$  denote a field for which (using Theorem 4.5.1 of [BLGGT10] again) we can simultaneously establish the modularity of this lift (which will correspond to  $h$ ), as well as the modularity of the  $p$ -adic and  $\lambda$ -adic Galois representations associated to  $\pi$  and  $g$  respectively. (It may have been more consistent to have combined Propositions 3.9 and 3.10 into a single proposition, but it would have been more unwieldy.)  $\square$

4. THE PROOF OF THEOREM 1.2

Let  $L/\mathbf{Q}$  denote a field which contains the coefficient field of  $g$  and  $\pi$ . Let  $(L, \rho_{g,\lambda})$  denote the compatible family of Galois representations associated to  $g$ . There exists a prime of  $\mathcal{O}_F$  dividing  $p$  such that the corresponding mod- $p$  representation is  $\bar{\rho}_g$ , which, by construction, has a nonsolvable image. It follows that the form  $g$  does not have complex multiplication, and hence the images of  $\rho_{g,\lambda}$  for all  $\lambda$  contain an open subgroup of  $\mathrm{SL}_2(\mathbf{Z}_l)$  where  $\lambda|l$ .

**Proposition 4.1.** *There exists a totally real field  $F_7^+/F_6^+$ , a quadratic CM extension  $F_7/F_7^+$  and a RACDSC automorphic representation  $\Pi$  of  $\mathrm{GL}(9)/F_7$  such that:*

- (1) *The compatible family of Galois representations associated to  $\Pi$  is the restriction to  $G_{F_7}$  of the family*

$$(L, \mathrm{Sym}^2(\rho_{g,\lambda}) \otimes \rho_{\pi,\lambda}).$$

- (2) *The images of  $\bar{\rho}$  and  $\bar{\rho}_h$  remain unchanged upon restriction to  $G_{F_7}$ .*

*Proof.* The compatible system is essentially self-dual, orthogonal (automatically since  $n = 9$  is odd), and has distinct Hodge–Tate weights (by assumption (3) of Proposition 3.9). Let us verify that it is irreducible. Since  $g$  does not have CM (as  $\bar{\rho}_g$  is not dihedral), the Zariski closure of  $\mathrm{Sym}^2(\rho_{g,\lambda})$  is  $\mathrm{PGL}(2)$ . Since  $\rho_{\pi,\lambda}$  is irreducible, it follows that the tensor product will be irreducible unless  $\rho_{\pi,\lambda}$  is a twist of  $\mathrm{Sym}^2(\rho_{g,\lambda})$ . By multiplicity one [JS81] for  $\mathrm{GL}(3)$ , we deduce that  $\mathrm{Sym}^2(g)$  is a twist of  $\pi$ . This contradicts the fact that the mod- $p$  residual representations  $\mathrm{Sym}^2(\bar{\rho}_g)$  and  $\bar{\rho}_\pi$  are not twists of each other, since one extends to a totally odd representation ( $\mathrm{Sym}^2\bar{\tau}_{\mathrm{res}}$ ) of  $G_{F_1^+}$  and the other to a representation ( $\mathrm{Sym}^2(\bar{\rho})$ ) of  $G_{F_1^+}$  which is even at some infinite place. The potential automorphy follows from Theorem A of [BLGGT10] (as follows from the proof of that theorem, the field  $F_7^+$  can be chosen to be disjoint from any finite auxiliary field, establishing condition (2)). □

Let us now write  $E^+ = F_7^+$  and  $E = F_7$ , and consider the representations  $\rho$ ,  $\rho_g$ ,  $\rho_\pi$ , and  $\rho_h$  as representations of  $G_E$ . Without loss of generality, we may assume that  $\rho$  is even for at least one real place of  $F^+$  (and hence also of  $E^+$ ). Let us consider the representation

$$\varrho : \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h) : G_E \rightarrow \mathrm{GL}_9(\overline{\mathbf{Q}}_p).$$

By construction, we observe that  $\bar{\varrho} = \mathrm{Sym}^2(\bar{\rho}) \otimes \mathrm{Sym}^2(\bar{\rho}_h) = \bar{\rho}_\pi \otimes \mathrm{Sym}^2(\bar{\rho}_g) = \bar{\rho}(\Pi)$  is residually modular. Moreover, we find that

$$\rho(\Pi)|_{D_v} = \rho_\pi \otimes \mathrm{Sym}^2(\rho_g)|_{D_v} \rightsquigarrow \mathrm{Sym}^2(\rho) \otimes \mathrm{Sym}^2(\rho_h)|_{D_v} = \varrho|_{D_v}$$

for all  $v$ , with the possible exception of  $v|p$ . By Lemma 3.4.3 of Geraghty [Ger09], the ordinary deformation rings are smooth and connected (by Proposition 3.9 (11),  $\bar{\rho}_\pi|_{D_v}$  is trivial for  $v|p$ ), and hence, for  $v|p$ ,

$$\rho_\pi|_{D_v} \sim \mathrm{Sym}^2(\rho_h)|_{D_v}.$$

On the other hand, by construction (Proposition 3.7 (2)), we also have (for  $v|p$ ) that

$$\mathrm{Sym}^2(\rho_g)|_{D_v} \not\sim \mathrm{Sym}^2(\rho)|_{D_v}.$$

If  $\rho|_{D_v}$  is potentially crystalline, then all four representations are potentially crystalline at  $v$ , and we deduce that

$$\rho_\pi \otimes \text{Sym}^2(\rho_g)|_{D_v} \sim \text{Sym}^2(\rho) \otimes \text{Sym}^2(\rho_h)|_{D_v}.$$

On the other hand, if  $\rho|_{D_v}$  is not potentially crystalline, then neither is  $\rho_g|_{D_v}$ , and we deduce from condition (9) of Lemma 2.6 that the left-hand side corresponds to a very smooth point of the corresponding local deformation ring

$$\text{Spec}(R^{\square, \text{Sym}^2(\tau) \otimes \text{Sym}^2(1), \text{Sym}^2 \mathbf{v} \otimes \text{Sym}^2 \mathbf{w}}[1/p]).$$

Hence

$$\rho(\Pi)|_{D_v} = \rho_\pi \otimes \text{Sym}^2(\rho_g)|_{D_v} \not\sim \text{Sym}^2(\rho) \otimes \text{Sym}^2(\rho_h)|_{D_v} = \varrho|_{D_v},$$

and thus  $\rho(\Pi)|_{D_v} \rightsquigarrow \varrho|_{D_v}$ . We now prove that  $\bar{\varrho}$  has adequate image (in the sense of [Tho10]). Recall that the fixed fields corresponding to  $\text{Sym}^2(\bar{\rho})$  and  $\text{Sym}^2(\bar{\rho}_h)$  are disjoint by construction. The images of  $\text{Sym}^2(\bar{\rho}_h)$  are either  $\text{PSL}_2(\mathbf{F})$  or  $\text{PGL}_2(\mathbf{F})$ , by construction. Since  $\text{PSL}_2(\mathbf{F})$  is simple, by Goursat’s Lemma, the image of  $\bar{\varrho}$  contains (with index at most two) the direct product of the images of  $\text{Sym}^2(\bar{\rho})$  and  $\text{PSL}_2(\mathbf{F})$ . Since both  $\text{Sym}^2(\bar{\rho})$  and  $\text{PSL}_2(\mathbf{F})$  have adequate image, their direct product satisfies the cohomological condition of adequateness, since  $H^1(\Gamma \times \Gamma', V \otimes V') = 0$  whenever  $H^1(\Gamma, W) = H^1(\Gamma', W') = 0$  by inflation restriction (the irreducible constituents of  $\text{Sym}^2(V \otimes V')$  are of the form  $W \otimes W'$ , where  $W$  and  $W'$  are irreducible constituents of  $\text{Sym}^2(W)$  and  $\text{Sym}^2(W')$  respectively). Since the image of  $\bar{\varrho}$  contains this product with index dividing two, its image still satisfies the cohomological condition of adequateness, by inflation-restriction (and the fact that  $p \neq 2$ ). We deduce that  $\bar{\varrho}$  is adequate by Lemma 2(ii) of [GHTT10]. It follows from Theorem 7.1 below (cf. Theorem 2.2.1 of [BLGGT10]) that  $\varrho$  is modular over  $E$ . (Since  $n = 9$  is odd, the slightly regular condition is vacuous.) Since the Galois representations  $\rho$  and  $\rho_h$  extend to the totally real subfield  $E^+$ , so does the representation  $\varrho$ , and hence (by [AC89])  $\varrho$  comes from a RAESDC representation for  $\text{GL}(9)/E^+$ . By assumption, however, there exists a real place of  $E^+$  and a corresponding complex conjugation  $c_v \in G_{E^+}$  such that  $\rho(c_v)$  is a scalar. It follows that the image  $\varrho(c_v)$  of  $c_v$  is conjugate to

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix},$$

which has trace  $+3$ . Since the representation  $\varrho$  is irreducible, automorphic, and 9 is odd, this contradicts the main theorem (Proposition A) of [Tay10], which says that such representations must be “odd” in the sense that the trace of complex conjugation must be  $\pm 1$ . This completes the proof of Theorem 1.2.

*Remark 4.2.* In [Cal11], we proved (Theorem 1.3) that for an imaginary quadratic field  $K$ , ordinary representations  $\rho : G_K \rightarrow \text{GL}_2(\overline{\mathbf{Q}}_p)$  (satisfying certain supplementary hypotheses) had parallel weight. One may ask whether the methods of this paper can be used to generalize that result (presuming that  $p$  splits in  $K$ ). Starting with the tensor representation  $\psi = \rho \otimes \rho^c$ , one is led to a 16-dimensional representation  $\varrho = \psi \otimes \rho_f \otimes \rho_g$  for non-CM Hilbert modular forms  $f$  and  $g$ , which one may show is modular for  $\text{GL}(16)$  over some totally real field  $E^+$ . It is not apparent, however, how this might lead to a contradiction, since  $\varrho(c)$  has trace 0 for any complex conjugation  $c$ , and known cases of functoriality do not yet allow



to being unramified over a finite extension. Thus, up to twist,  $\rho|_{I_v}$  has finite image, and, in particular, the projective image of  $\rho|_{I_v}$  is finite. The only finite subgroups of  $\mathrm{PGL}_2(\overline{\mathbf{Q}}_p) \simeq \mathrm{PGL}_2(\mathbf{C})$  are either cyclic, dihedral,  $A_4$ ,  $S_4$ , or  $A_5$ . Thus, the projective image of  $\bar{\rho}|_{I_v}$  must be one of these groups. By assumption, the projective image of  $\bar{\rho}|_{I_v}$  is a nondihedral group of order divisible by  $p > 7$ ; hence  $\rho|_{I_v}$  cannot be finite up to twist either, and  $\rho$  does not exist.  $\square$

We have the following corollary:

**Corollary 5.2.** *There exist absolutely irreducible representations  $\bar{\rho}$  such that the subset  $\mathfrak{X}^{\mathrm{geom}} \subset \mathfrak{X}$  is not Zariski dense. In fact, there exist representations such that  $\mathfrak{X}^{\mathrm{geom}}$  is empty, but  $\mathfrak{X}$  has arbitrary large dimension.*

*Proof.* Let  $F^+/\mathbf{Q}$  be a totally real field, and let  $\bar{\rho} : G_{F^+} \rightarrow \mathrm{GL}_2(\mathbf{F})$  be a continuous irreducible representation satisfying the conditions of Theorem 5.1. Suppose, furthermore, that  $\psi_v \neq \omega^{-1}$  for any  $v|p$ . The existence of such representations is guaranteed by Proposition 3.2. More precisely, start with an auxiliary totally real field  $E^+$  of degree  $> 1$  and fix an infinite place  $v$  of  $E$ . Then use Proposition 3.2 to construct  $\bar{\rho}$  which are odd for all  $w|\infty$  in  $F^+$  with  $w \nmid v$ , and even for all  $w|v$ . By Theorem 5.1,  $\mathfrak{X}^{\mathrm{geom}} = \emptyset$  for any finite set of auxiliary primes  $S$ . Using the strategy of the proof of Theorem 1(a) of [Ram02], one deduces the existence of sets  $S$  for which there exists a family of Galois representations

$$\rho : G_{F^+,S} \rightarrow \mathrm{GL}_2(W(\mathbf{F})[[T_1, \dots, T_n]])$$

of (relative over  $W(\mathbf{F})$ ) dimension  $n = \delta + 2r$ , where  $\delta$  is the Leopoldt defect and  $r$  is the number of infinite places of  $F^+$  at which  $\bar{\rho}$  is odd, and such that every specialization of  $\rho$  is *reducible* at  $D_v$  for all  $v|p$ . (More generally, one may replace  $F^+$  by any number field  $F$  and obtain a family of dimension  $\delta + 2s + 2r$ , where  $r$  is the number of real places at which  $\bar{\rho}$  is odd, and  $s$  is the number of complex places.) (If one fixes the determinant, this decreases to  $s + 2r$ .) The special case when  $F$  is an imaginary quadratic field is Lemma 7.6 of [CM09], but the proof for an arbitrary number field is essentially the same. By construction,  $2r = [F^+ : \mathbf{Q}] - [F^+ : E^+]$ . In particular, we can make  $r$  (and thus  $\mathfrak{X}$ ) arbitrarily large.  $\square$

Similarly, we note the following cases of the Fontaine–Mazur conjecture which do not require any assumption on the Hodge–Tate weights or the parity of  $\bar{\rho}$ :

**Corollary 5.3.** *Let  $\rho : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$  be a continuous Galois representation which is unramified except at a finite number of primes. Suppose that  $p > 7$ , and, furthermore, that:*

- (1)  $\rho|_{D_p}$  is potentially semi-stable.
- (2) The residual representation  $\bar{\rho}$  is absolutely irreducible and is not of dihedral type.
- (3)  $\bar{\rho}|_{D_p}$  is of the form  $\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}$ , where:
  - (a)  $*$  is ramified.
  - (b)  $\psi_1/\psi_2 \neq \omega$ , and  $\psi_1/\psi_2|_{I_p}$  has order  $> 2$ .

Then  $\rho$  is modular.

Although Proposition 3.2 guarantees the existence of infinitely many even Galois representations over totally real fields with image containing  $\mathrm{SL}_2(\mathbf{F}_p)$ , it may also

be of interest to construct at least one example over  $\mathbf{Q}$  (with  $p \geq 11$ ). We shall do this now.

**Lemma 5.4.** *Let  $K/\mathbf{Q}$  be a degree 11 extension with splitting field  $L/\mathbf{Q}$  such that:*

- (1) 11 is totally ramified in  $K$ .
- (2)  $G = \text{Gal}(L/\mathbf{Q}) = \text{PSL}_2(\mathbf{F}_{11})$ .
- (3)  $\text{ord}_{11}(\Delta_{K/\mathbf{Q}}) \not\equiv 0 \pmod{10}$ .

Let  $\mathfrak{p}$  denote a prime above  $p := 11$  in  $L$ , and let  $I \subseteq D \subset G$  denote the corresponding inertia and decomposition groups. Then  $I = D$  has order 55 and is the full Borel subgroup of  $G$ .

*Proof.* Since 11 is totally ramified in  $K/\mathbf{Q}$ , the inertia group  $I$  has order divisible by  $[K : \mathbf{Q}] = 11$ . Since  $I \subset D$  is solvable, it follows that  $D$  is contained inside a Borel subgroup of  $G$ . Let  $F$  and  $E$  denote the images of  $L$  and  $K$  under their embedding into  $\overline{\mathbf{Q}}_p$  corresponding to  $\mathfrak{p}$ . We note that  $D = \text{Gal}(F/\mathbf{Q}_p)$ , and we have the following diagram:

$$\begin{array}{ccccc}
 L & \xrightarrow{\quad} & K & \xrightarrow{\quad} & \mathbf{Q} \\
 \downarrow & & \downarrow & & \downarrow \\
 F & \xrightarrow[\text{ef} = 1, 5]{} & E & \xrightarrow[e = 11]{} & \mathbf{Q}_p.
 \end{array}$$

It suffices to assume that  $|I| = 11$  and deduce a contradiction. Suppose that  $|D| = 11$ , so  $F = E$  is abelian over  $\mathbf{Q}_p$ . By local class field theory,  $F/\mathbf{Q}_p$  is (up to an unramified twist) given by the degree  $p = 11$  subfield of the  $p^2$ -roots of unity. Thus

$$\Delta_{E/\mathbf{Q}_p} = \Delta_{F/\mathbf{Q}_p} = 11^{20},$$

and thus  $\text{ord}_{11}(\Delta_{E/\mathbf{Q}_p}) \equiv 0 \pmod{10}$ , a contradiction.

Suppose that  $|D| = 55$  and  $|I| = 11$ . Let  $I_n \subseteq I$  denote the lower ramification groups. The  $p$ -adic valuation of the discriminant of  $F/\mathbf{Q}_p$  is given by the following formula:

$$\text{ord}_p(\Delta_{F/\mathbf{Q}_p}) = \frac{|D|}{|I|} \cdot \sum_{n=0}^{\infty} (|I_n| - 1).$$

By assumption,  $|D|/|I| = 5$  and  $|I_n| = 11$  or 1 for all  $n$ . We deduce that  $\text{ord}_p(\Delta_{F/\mathbf{Q}_p}) \equiv 0 \pmod{50}$ . On the other hand,

$$\Delta_{F/\mathbf{Q}_p} = N_{F/E}(\Delta_{F/E}) \cdot (\Delta_{E/\mathbf{Q}})^5.$$

Since  $F/E$  is unramified, we deduce that

$$\text{ord}_{11}(\Delta_{E/\mathbf{Q}_p}) = \frac{1}{5} \text{ord}_{11}(\Delta_{F/\mathbf{Q}_p}) \equiv 0 \pmod{10}.$$

Since  $\text{ord}_{11}(\Delta_{K/\mathbf{Q}}) = \text{ord}_{11}(\Delta_{E/\mathbf{Q}_p})$ , the lemma follows. □

**Corollary 5.5.** *There exists a surjective even representation  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \text{SL}_2(\mathbf{F}_{11})$  with no geometric deformations.*

*Proof.* Let  $K$  be the field obtained by adjoining to  $\mathbf{Q}$  a root of the irreducible polynomial

$$x^{11} + 154 \cdot x^{10} + 8591 \cdot x^9 + 207724 \cdot x^8 + 1846031 \cdot x^7 - 2270598 \cdot x^6 - 63850600 \cdot x^5 + 73646034 \cdot x^4 + 582246423 \cdot x^3 - 1610954576 \cdot x^2 + 1500989952 \cdot x - 481890304.$$

This polynomial was obtained by specializing the parameters  $a$  and  $t$  of a polynomial found by Malle (Theorem 9.1 of [Mal00]) to  $a = 14$  and  $t = -419$  respectively. One may verify that 11 is totally ramified in  $K/\mathbf{Q}$ , that the splitting field  $L/\mathbf{Q}$  is totally real with Galois group  $G = \mathrm{PSL}_2(\mathbf{F}_{11})$ , and that the discriminant has a prime factorization as follows:

$$\Delta_{K/\mathbf{Q}} = 11^{12} \cdot 133462088669841218191^4.$$

Since  $\mathrm{ord}_{11}(\Delta_{K/\mathbf{Q}}) = 12$ , it follows from Lemma 5.4 that the inertia group  $D$  at 11 is the full Borel subgroup of  $G$ . We note also the factorization

$$133462088669841218191 \cdot \mathcal{O}_K = \mathfrak{p}_1 \mathfrak{p}_2^2 \mathfrak{p}_3^2 \mathfrak{q}_1^2 \mathfrak{q}_2,$$

where  $\mathfrak{p}_i$  and  $\mathfrak{q}_i$  have residue degrees 1 and 2 respectively. It follows that the residue degree and the ramification index of every prime above 133462088669841218191 in  $L$  is even (in fact, 2). Since  $133462088669841218191 \equiv 3 \pmod{4}$ , it follows from a theorem of Böge (Theorem 1.1 of [Klü00]) that  $L$  embeds in a  $\mathrm{SL}_2(\mathbf{F}_{11})$ -extension  $N/\mathbf{Q}$ . The action of complex conjugation on this extension is, by construction, either trivial or by  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ . In the former case,  $N$  is totally real. In the latter case, we may twist by some (any) quadratic character of an imaginary quadratic field, and the corresponding field extension (which we still call  $N$ ) will be totally real. Let  $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{Gal}(N/\mathbf{Q}) = \mathrm{SL}_2(\mathbf{F}_{11})$  denote the corresponding representation. We now show that  $\bar{\rho}|_{D_{11}}$  satisfies the conditions of Theorem 5.1. Since the decomposition group at 11 maps surjectively onto the Borel subgroup of  $\mathrm{PSL}_2(\mathbf{F}_{11})$ , it is contained in the Borel subgroup of  $\mathrm{SL}_2(\mathbf{F}_{11})$ . Any such representation may be twisted (in  $\mathrm{GL}_2(\mathbf{F}_{11})$ ) to be of the form

$$\begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$$

for some character  $\psi$ . If  $\psi$  has order two, then the image of  $D_{11}$  will not surject onto the Borel subgroup of  $\mathrm{PSL}_2(\mathbf{F}_{11})$  (twisting does not affect this projection). If  $\psi = \omega$ , however, then  $\det(\bar{\rho}) = \omega \cdot \chi^2$  for some character  $\chi$ . Yet  $\bar{\rho}$  has image in  $\mathrm{SL}_2(\mathbf{F}_{11})$  and thus has trivial determinant, whilst  $\omega$  is not the square of any character. Thus  $\psi \neq \omega$ , and Theorem 5.1 applies.  $\square$

### 6. SOME REMARKS ON THE CONDITION $p > 7$

One may wonder if the condition that  $p > 7$  is used in an essential way in this argument. At the very least, one will require that the representation

$$\mathrm{Sym}^2 \bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_3(\mathbf{F}_p)$$

be adequate. This fails to have adequate image if the image of  $\bar{\rho}$  is  $\mathrm{SL}_2(\mathbf{F}_p)$  and  $p \leq 7$ . The author expects that for  $p = 7$  it should be sufficient to assume that the projective image of  $\bar{\rho}$  is either  $A_4, S_4, A_5$  or contains  $\mathrm{PSL}_2(\mathbf{F}_{49})$ , that for  $p = 5$  that projective image is either  $A_4, S_4$ , or contains  $\mathrm{PSL}_2(\mathbf{F}_{25})$ , and that for  $p = 3$  the image contains  $\mathrm{PSL}_2(\mathbf{F}_{27})$ . The main technical issue to address is exactly what form of adequateness is required in Proposition 3.2.1 of [BLGGT10], although another issue is that many of the references we cite include assumptions on  $p$  which would



also need to be modified (using [Tho10]). The methods (in principle) also apply with  $p = 2$ , although many more technical ingredients would need to be generalized in this case, in particular, the work of [Kis09].

7. A REMARK ON POTENTIAL MODULARITY THEOREMS

In recent modularity lifting results [BLGHT09, BLGGT10, BLGG09, BLGG10, Ger09] for  $l$ -adic representations, a weak form of local-global compatibility at primes  $v|l$  is invoked (in this section only, we work with  $l$ -adic representations rather than  $p$ -adic representations in order to be most compatible with [BLGHT09]), namely, that automorphic forms of level co-prime to  $l$  give rise to crystalline Galois representations (of the correct weight). In general, local-global compatibility for RACDSC cuspidal forms for  $GL(n)$  is only known in the crystalline case (as follows from [TY07]), although partial results are known in the semi-stable case. In this section, we show how to prove modularity results similar to Theorem 2.2.1 of [BLGHT09] without local-global compatibility, allowing for a modularity lifting theorem in the potentially semi-stable case. We claim no great originality, as the proof is essentially the same as the proof of Theorem 2.2.1 of [BLGGT10] (or Theorem 7.1 of [Tho10]) with the addition of one simple ingredient (Lemma 7.3 below). (The authors of [BLGGT10] inform me that they have a different method for dealing with the potentially semi-stable case, which was not included in [BLGGT10] for space reasons.) (One should also compare the statement of this theorem to Theorem 7.1 of [Tho10].)

**Theorem 7.1.** *Let  $F$  be an imaginary CM field with maximal totally real subfield  $F^+$ . Suppose  $l$  is odd and let  $n$  be a positive integer. Let*

$$r : G_F \rightarrow GL_n(\overline{\mathbf{Q}}_l)$$

*be a continuous representation and let  $\bar{r}$  denote the corresponding residual representation. Also, let*

$$\mu : G_{F^+} \rightarrow \overline{\mathbf{Q}}^\times$$

*be a continuous homomorphism. Suppose that  $(r, \mu)$  enjoys the following properties:*

- (1)  $r^c \simeq r^\vee \epsilon_l^{1-n} \mu|_{G_F}$ .
- (2)  $\mu(c_v)$  is independent of  $v|\infty$ .
- (3) The reduction  $\bar{r}$  is absolutely irreducible and  $\bar{r}(G_{F(\zeta_l)}) \subset GL_n(\overline{\mathbf{F}}_l)$  is adequate.
- (4) There is a RAECDSC automorphic representation  $(\pi, \chi)$  of  $GL_n(\mathbf{A}_F)$  with the following properties:
  - (a)  $(\bar{r}, \bar{\mu}) \simeq (\bar{r}_{l,\iota}(\pi), \bar{r}_{l,\iota}(\chi))$ .
  - (b) For all places  $v \nmid l$  of  $F$  at which  $\pi$  or  $r$  is ramified, we have

$$r_{l,\iota}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}.$$

- (c) For all places  $v|l$  of  $F$ ,  $r|_{G_{F_v}}$  is potentially semi-stable and we have

$$r_{l,\iota}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}.$$

- (d) If  $n$  is even,  $\pi$  has slightly regular weight.

*Then  $(r, \mu)$  is automorphic.*

*Remark 7.2.* The only difference between this theorem and Theorem 2.2.1 of [BLGGT10] is that:

- (1) We do *not* assume that  $\pi$  is potentially unramified above  $l$ .
- (2) We require for  $v|l$  that  $r_{l,\iota}(\pi)|_{G_{F_v}} \rightsquigarrow r|_{G_{F_v}}$  rather than  $r_{l,\iota}(\pi)|_{G_{F_v}} \sim r|_{G_{F_v}}$ .
- (3) We impose that  $\pi$  has slightly regular weight (this is only a condition when  $n$  is even). This is because we require that the Galois representation associated to  $\pi$  can be realized geometrically. Perhaps using the methods of [Che09] this assumption can be eliminated. Alternatively, one could try to work with the representation  $r_{l,\iota}(\pi)^{\otimes 2}$ , which can be realized geometrically (see [Car10]).

*Proof.* We make the following minor adjustment to the proof of Theorem 7.1 of [Tho10] (cf. Theorem 2.2.1 of [BLGGT10]). The character  $\chi$  may be untwisted after some solvable ramified extension. We now modify the deformation problem considered in the proof of Theorem 3.6.1 of [BLGG09] as follows. At  $v|l$ , we consider deformations (with a fixed finite collection of Hodge types  $\mathbf{v}$ ) that become potentially semi-stable over some fixed extension  $L/K$ , where  $L$  will be determined below. The argument proceeds in the same manner *providing* there exists a map from  $R \rightarrow \mathbf{T}$ , and (in light of the fact that the deformation rings at  $l$  might have nonsmooth points in characteristic zero) utilizing the hypothesis that  $r_{l,\iota}(\pi)|_{G_{F_v}}$  corresponds to a smooth point in the local deformation ring. For any fixed level structure, the Galois representations arising from quotients of  $\mathbf{T}_{Q_n}$  are potentially semi-stable over some extension  $L/K$  by a theorem of Tsuji [Tsu98]. However, to define the appropriate ring  $R$ , we must ensure that the corresponding types of these Galois representations lie in some finite set *independent* of the set of auxiliary primes  $Q_n$ . In particular, we are required to show that we may find a *fixed*  $L/K$  such that the Galois representations obtained by adding any set of auxiliary Taylor–Wiles primes are semi-stable over the same field  $L$ .

**Lemma 7.3.** *Let  $K/\mathbf{Q}_l$  be a finite extension, and let  $X$  be a proper flat scheme over  $\text{Spec}(\mathcal{O}_K)$  with smooth generic fibre. Then there exists a finite extension  $L/K$  with the following property: For every finite étale map  $\pi : Y \rightarrow X$ , the étale cohomology groups  $H^i(Y_{\overline{K}}, \overline{\mathbf{Q}}_l)$  become semi-stable as representations of  $G_L$ .*

*Proof.* After making a finite extension  $L/K$ , there exists (via the theory of alterations [dJ96]) a proper hypercovering  $X_\bullet$  of  $X$  such that for all  $n \leq 2 \dim(X)$ :

- (1)  $X_n$  is proper and flat over  $\text{Spec}(\mathcal{O}_L)$ .
- (2)  $X_n$  has smooth generic fibre and semi-stable special fibre.

By cohomological descent, there is a spectral sequence

$$H^m(X_{n,\overline{L}}, \overline{\mathbf{Q}}_l) \Rightarrow H^{m+n}(X_{\overline{L}}, \overline{\mathbf{Q}}_l).$$

The cohomology groups on the left are semi-stable by Tsuji’s proof of  $C_{\text{st}}$ . Since the property of being semi-stable is preserved by taking subquotients, it follows that the  $G_L$ -representation  $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$  (for  $i \leq 2 \dim(X)$ ) has an exhaustive filtration by semi-stable  $G_L$ -modules. Hence the semi-simplification of  $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$  is semi-stable. Since  $H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$  is also de Rham [Fal88], it follows that  $H^i(X_{\overline{K}}, \overline{\mathbf{Q}}_l) = H^i(X_{\overline{L}}, \overline{\mathbf{Q}}_l)$  is itself semi-stable as a  $G_L$ -representation for  $i \leq 2 \dim(X)$ . The cohomology of  $X_{\overline{K}}$  vanishes outside this range, so the claim follows for all  $i$ . This recovers Tsuji’s Theorem  $C_{\text{pst}}$  (indeed, this is essentially Tsuji’s argument). Let us now consider a finite étale morphism  $Y \rightarrow X$ . We may form a hypercovering

$Y_\bullet = X_\bullet \times_X Y$  of  $Y$ . The properties (1) and (2) of the hypercovering  $X_\bullet$  are preserved under base change by a finite étale map, and thus the cohomology of  $Y$  is also semi-stable over  $L$ .  $\square$

*Remark 7.4.* For an expositional account of the theory of hypercoverings and cohomological descent in the étale topology, see [Con].

Consider the (compact) Shimura variety  $\mathrm{Sh}$  (over  $\mathrm{Spec}(\mathcal{O}_K)$ ) associated to the unitary similitude group  $G$  as in [CHL, HT01, Shi10], where  $\mathcal{L} = \mathcal{L}_\xi$  is a lisse sheaf corresponding to the automorphic vector bundle for an irreducible algebraic representation  $\xi$  of  $G$ . Let  $\mathcal{A}^m$  denote the  $m$ th self-product of the universal abelian variety over  $\mathrm{Sh}$ , and let  $\pi : \mathcal{A}^m \rightarrow \mathrm{Sh}$  denote the (smooth, proper) projection. For a suitable  $m$ , one can write  $\mathcal{L} = eR^m\pi_*\overline{\mathbf{Q}}_l(r)$  for some  $m = m_\xi$  and  $r = r_\xi$ , and  $e$  is some idempotent (cf. [HT01], p.98). Finally, let  $\mathrm{Sh}(N)$  denote the finite étale cover of  $\mathrm{Sh}$  corresponding to the addition of an auxiliary level  $N$ -structure for some  $N$  co-prime to  $p$ . Let  $\mathcal{A}^m(N)$  denote the base change of  $\mathcal{A}^m$  to  $\mathrm{Sh}(N)$ ; it is finite étale over  $\mathcal{A}^m$ . The Leray spectral sequence gives a map

$$H^p(\mathrm{Sh}(N)_{\overline{K}}, R^q\pi_*\overline{\mathbf{Q}}_l(r)) \Rightarrow H^{p+q}(\mathcal{A}^m(N)_{\overline{K}}, \overline{\mathbf{Q}}_l(r)).$$

Multiplication by  $n$  on  $\mathcal{A}$  induces the map  $n^j$  on  $R^j\pi_*\overline{\mathbf{Q}}_l$ . The formation of the spectral sequence is compatible with this map, and hence it commutes with the differentials in the spectral sequence, which correspondingly degenerates (cf. the argument of Deligne, p.169 of [Del73]). Thus

$$H^n(\mathrm{Sh}(N)_{\overline{K}}, \mathcal{L}) = eH^n(\mathrm{Sh}(N)_{\overline{K}}, R^m\pi_*\overline{\mathbf{Q}}_l(r))$$

occurs as a subquotient of  $H^i(\mathcal{A}(N)_{\overline{K}}, \overline{\mathbf{Q}}_l(r))$  for some  $i$ . Let  $X = \mathcal{A}^m$  and  $Y = \mathcal{A}^m(N)$ . By Lemma 7.3, we deduce that

$$H^n(\mathrm{Sh}(N)_{\overline{K}}, \mathcal{L})$$

is semi-stable over a fixed extension  $L/K$  for all  $N$  depending only on  $m$  and  $\xi$ . If  $\pi$  has slightly regular weight, the Galois representation associated to  $\pi$  in [Shi10] can be realized geometrically in the étale cohomology of an automorphic sheaf on  $\mathrm{Sh}$  as considered above. Moreover, the Galois representations corresponding to automorphic forms arising in the Taylor–Wiles constructions at auxiliary primes arise in the étale cohomology of the same sheaf on  $\mathrm{Sh}(N)$  for some auxiliary level  $N$ . It follows that the local Galois representations associated to the Hecke rings  $\mathbf{T}_{Q_n}$  are all quotients of a local deformation ring involving a fixed finite set of types, which is the necessary local input for the modularity lifting theorem (Theorem 7.1) of [Tho10].  $\square$

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