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GLOBAL WELL-POSEDNESS AND SCATTERING FOR THE DEFOCUSING, L^2-CRITICAL NONLINEAR SCHRÖDINGER EQUATION WHEN $d \geq 3$

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1. Introduction

The d-dimensional, L^2-critical nonlinear Schrödinger initial value problem is given by

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0,x) &= u_0 \in L^2(\mathbf{R}^d), \end{aligned} \tag{1.1}$$

where $F(u) = \mu |u|^{4/d} u$, $\mu = \pm 1$, $u(t) : \mathbf{R}^d \to \mathbf{C}$. (1.1) is said to be defocusing when $\mu = +1$ and focusing when $\mu = -1$. L^2-critical refers to scaling. If $u(t,x)$ solves (1.1) on $[0,T]$ with initial data $u(0,x) = u_0(x)$, then

$$\lambda^{d/2} u(\lambda^2 t, \lambda x) \tag{1.2}$$

solves (1.1) on $[0, \frac{T}{\lambda^2}]$ with initial data $\lambda^{d/2} u_0(\lambda x)$. (1.2) preserves the $L^2(\mathbf{R}^d)$ norm of a solution to (1.1):

$$\|\lambda^{d/2} u_0(\lambda x)\|_{L_x^2(\mathbf{R}^d)} = \|u_0(x)\|_{L_x^2(\mathbf{R}^d)}. \tag{1.3}$$

It was observed in [6] that the solution to (1.1) conserves the quantities mass

$$M(u(t)) = \int |u(t,x)|^2 dx = M(u(0)) \tag{1.4}$$

and energy

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t,x)|^2 dx + \frac{\mu d}{2(d+2)} \int |u(t,x)|^{\frac{2d+4}{d}} dx = E(u(0)). \tag{1.5}$$

Remark 1.1. $E(u(t))$ is positive definite when $\mu = +1$.

A solution to (1.1) satisfies Duhamel's formula.

Definition 1.2. $u : I \times \mathbf{R}^d \to \mathbf{C}$, $I \subset \mathbf{R}$ is a solution to (1.1) if for any compact $J \subset I$, $u \in C_t^0 L_x^2(J \times \mathbf{R}^d) \cap L_{t,x}^{\frac{2(d+2)}{d}}(J \times \mathbf{R}^d)$, and for all $t, t_0 \in I$,

$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^{t} e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \tag{1.6}$$

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The space $L_{t,x}^{\frac{2(d+2)}{d}}(J\times \mathbf{R}^d)$ arises naturally from Strichartz estimates. This norm is also invariant under (1.2).

Definition 1.3. A solution u to (1.1) defined on $I \subset \mathbf{R}$ blows up forward in time if there exists $t_0 \in I$ such that

$$\int_{t_0}^{\sup(I)} \int |u(t,x)|^{\frac{2(d+2)}{d}} dxdt = \infty. \tag{1.7}$$

u blows up backward in time if there exists $t_0 \in I$ such that

$$\int_{\inf(I)}^{t_0} \int |u(t,x)|^{\frac{2(d+2)}{d}} dxdt = \infty. \tag{1.8}$$

Definition 1.4. A solution $u(t,x)$ to (1.1) is said to scatter forward in time if there exists $u_+ \in L^2(\mathbf{R}^d)$ such that

$$\lim_{t\to\infty} \|u(t,x) - e^{it\Delta}u_+\|_{L^2(\mathbf{R}^d)} = 0. \tag{1.9}$$

A solution is said to scatter backward in time if there exists $u_- \in L^2(\mathbf{R}^d)$ such that

$$\lim_{t\to-\infty} \|u(t,x) - e^{it\Delta}u_-\|_{L^2(\mathbf{R}^d)} = 0. \tag{1.10}$$

(1.1) is locally well-posed for any $u_0 \in L^2(\mathbf{R}^d)$ with time of existence depending on the profile of the initial data.

Theorem 1.5. *Given $u_0 \in L^2(\mathbf{R}^d)$ and $t_0 \in \mathbf{R}$, there exists a maximal lifespan solution u to* (1.1) *defined on $I \subset \mathbf{R}$ with $u(t_0) = u_0$. Moreover,*

1. For any $d \geq 1$, there exists $\epsilon(d) > 0$ such that if $\|u_0\|_{L^2(\mathbf{R}^d)} < \epsilon(d)$, then (1.1) *is globally well-posed and scatters both forward and backward in time.*

2. I is an open neighborhood of t_0.

3. If $\sup(I)$ or $\inf(I)$ is finite, then u blows up in the corresponding time direction.

4. The solution map that takes initial data to the corresponding solution is uniformly continuous on compact time intervals for bounded sets of initial data.

5. If $\sup(I) = \infty$ and u does not blow up forward in time, then u scatters forward to a free solution. If $\inf(I) = -\infty$ and u does not blow up backward in time, then u scatters backward to a free solution.

Proof. See [6], [7]. □

In the defocusing case there are no known counterexamples to global well-posedness and scattering for (1.1), $u_0 \in L^2(\mathbf{R}^d)$ of arbitrary size. Therefore the following has been conjectured.

Conjecture 1.6. *For $d \geq 1$, the defocusing, mass critical nonlinear Schrödinger initial value problem* (1.1)*, $\mu = +1$ is globally well-posed for $u_0 \in L^2(\mathbf{R}^d)$ and all solutions scatter to a free solution as $t \to \pm\infty$.*

This conjecture has been affirmed in the radial case for $d \geq 2$.

Theorem 1.7. *When $d = 2$, $\mu = +1$,* (1.1) *is globally well-posed and solutions scatter for $u_0 \in L^2(\mathbf{R}^2)$ radial.*

Proof. See [26]. □

Theorem 1.8. *When $d \geq 3$, $\mu = +1$,* (1.1) *is globally well-posed and solutions scatter for $u_0 \in L^2(\mathbf{R}^d)$ radial.*

Proof. See [43], [30]. $\square$

In this paper we remove the radial condition for the case when $d \geq 3$ and prove

Theorem 1.9. (1.1) *is globally well-posed and solutions scatter for $u_0 \in L^2(\mathbf{R}^d)$, $d \geq 3$.*

Remark 1.10. [26] and [30] also proved global well-posedness and scattering for the focusing, mass-critical initial value problem

$$\begin{aligned} iu_t + \Delta u &= -|u|^{4/d} u, \\ u(0,x) &= u_0, \end{aligned} \tag{1.11}$$

with radial data and mass less than the mass of the ground state. There are known counterexamples to (1.1) being globally well-posed and scattering for all $u_0 \in L^2(\mathbf{R}^d)$ in the focusing case. These counterexamples all have mass greater than or equal to the mass of the ground state. It has therefore been conjectured that (1.11) is globally well-posed and scattering for $u_0 \in L^2$, u_0 has mass less than the mass of the ground state. Many of the tools used in this paper apply equally well to $\mu = -1$. So whenever possible we will prove theorems for both the focusing and defocusing case. The focusing problem will be fully addressed in an upcoming paper.

1.1. **Outline of proof.** To prove spacetime bounds for solutions to critical nonlinear Schrödinger problems with large data it suffices to consider a special class of solutions that are concentrated in both space and frequency. The seminal result, proved in [4], introduced the induction on energy method, proving global well-posedness and scattering for the defocusing energy-critical initial value problem in $\mathbf{R}^3$ and $\mathbf{R}^4$ for radial data. See [12], [35], [50], and [41] for more work on the defocusing, energy-critical initial value problem in dimensions $d \geq 3$.

A method closely related to induction on energy is the concentration compactness method, which has been used since the 1980s to study elliptic partial differential equations. (See for example [5].) More recently [21] applied this method to the focusing, radially symmetric, energy critical Schrödinger problem (see [29] for a treatment of the nonradial problem), and [22] applied this method to the focusing energy critical wave equation. [23] applied concentration compactness to the $\dot{H}^{1/2}$ critical initial value problem in three dimensions. See [24], [25] as well for more information on this method.

In the mass-critical case, [26] and [30] used concentration compactness to prove Theorems 1.7 and 1.8. Since (1.1) is globally well-posed for small $\|u_0\|_{L^2(\mathbf{R}^d)}$, if (1.1) is not globally well-posed for all $u_0 \in L^2(\mathbf{R}^d)$, then there must exist a minimum $\|u_0\|_{L^2(\mathbf{R}^d)} = m_0$ where global well-posedness fails. [45] showed that for Conjecture 1.6 to fail, there must exist a special minimal mass blow-up solution to (1.1) on some open interval $I \subset \mathbf{R}$,

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} = \infty, \tag{1.12}$$

$\|u(t)\|_{L_x^2(\mathbf{R}^d)} = m_0$ for all $t \in I$, u concentrated in space and frequency. Moreover, we can take a limit of special minimal mass blow-up solutions to obtain an even more special minimal mass blow-up solutions to (1.1). We prove Theorem 1.9

by contradiction, showing that one of these very special minimal mass blow-up solutions cannot occur.

Definition 1.11. A set $K \subset L^2(\mathbf{R}^d)$ is precompact in $L^2(\mathbf{R}^d)$ if it has compact closure in $L^2(\mathbf{R}^d)$. From the Arzelà-Ascoli theorem, a set $K \subset L^2(\mathbf{R}^d)$ is precompact if and only if there exists a compactness modulus function, $C(\eta) < \infty$ for all $\eta > 0$ such that

$$\int_{|x| \geq C(\eta)} |f(x)|^2 dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 d\xi < \eta. \tag{1.13}$$

Let

$$A_{\pm}(m) = \sup\{\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}((-\infty,\infty) \times \mathbf{R}^d)} : u \text{ solves (1.1)}, \mu = \pm 1, \|u(t)\|_{L^2(\mathbf{R}^d)} = m\}. \tag{1.14}$$

By Theorem 1.5, in order to prove Theorem 1.9 it suffices to prove $A_+(m) < \infty$ for any $m < \infty$. [45] proved a stability lemma, which implies that $A(m)$ is a continuous function of m. This in turn implies that $\{m : A_+(m) = \infty\}$ is a closed set. Therefore, if global well-posedness and scattering does not hold in the defocusing case for all $u_0 \in L^2(\mathbf{R}^d)$, then there must be a minimal m_0 with $A_+(m_0) = \infty$.

Remark 1.12. Likewise in the focusing case it would suffice to prove $A_-(m) < \infty$ when m is less than the mass of the ground state.

The set of blow-up solutions to (1.1) with $u(t) \in L^2(\mathbf{R}^d)$ has a natural GK structure, where $K \subset L^2(\mathbf{R}^d)$ and G is the $(2d+1)$-dimensional Lie group $G = \mathbf{R}^d \times (0,\infty) \times \mathbf{R}^d$. It is quite clear that shifting the origin generates a d-dimensional symmetry group for solutions to (1.1). (1.2) implies that we can change the scale by a fixed constant in the multiplicative group $(0,\infty)$. Finally the Galilean transformation generates the d-dimensional phase shift symmetry group. These symmetries do not alter $\|u(t)\|_{L_x^2(\mathbf{R}^d)}$.

Theorem 1.13. *Suppose $u(t,x)$ solves*

$$\begin{aligned} iu_t + \Delta u &= F(u), \\ u(0,x) &= u_0. \end{aligned} \tag{1.15}$$

Then $v(t,x) = e^{-it|\xi_0|^2} e^{ix\cdot\xi_0} u(t, x - 2\xi_0 t)$ solves the initial value problem

$$\begin{aligned} iv_t + \Delta v &= F(v), \\ v(0,x) &= e^{ix\cdot\xi_0} u(0,x). \end{aligned} \tag{1.16}$$

Proof. This follows by direct calculation. □

[45] proved that for Conjecture 1.6 to fail, there must exist a maximal interval $I \subset \mathbf{R}$ with $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} = \infty$, and u blows up both forward and backward in time. Moreover, for any $t \in I$, $\{u(t,x)\}/G$ is precompact. By the Arzelà-Ascoli theorem this implies that there exist functions $x(t), \xi(t) : I \to \mathbf{R}^d$, $N(t) : I \to (0,\infty)$, such that for any $\eta > 0$ there exists $C(\eta) < \infty$ with

$$\int_{|x - x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t,x)|^2 dx < \eta \tag{1.17}$$

and

$$\int_{|\xi-\xi(t)|\geq C(\eta)N(t)} |\hat{u}(t,\xi)|^2 d\xi < \eta. \tag{1.18}$$

Remark 1.14. If $u(t,x)$ obeys (1.17) and (1.18) and $v(t,x) = e^{-it|\xi_0|^2} e^{ix\cdot\xi_0} u(t, x - 2\xi_0 t)$, then

$$\int_{|\xi-\xi_0-\xi(t)|\geq C(\eta)N(t)} |\hat{v}(t,\xi)|^2 d\xi < \eta, \tag{1.19}$$

$$\int_{|x-2\xi_0 t - x(t)|\geq \frac{C(\eta)}{N(t)}} |v(t,x)|^2 dx < \eta. \tag{1.20}$$

This will be useful to us later because it shifts $\xi(t)$ by a fixed amount $\xi_0 \in \mathbf{R}^d$. For example, this allows us to set $\xi(0) = 0$.

Definition 1.15. A solution $u(t,x)$ is said to be almost periodic if there exists a group of symmetries G of the equation such that $\{u(t) : t \in I\}/G$ is a precompact set.

Since $\{u(t) : t \in I\}$ is precompact in L^2/G, if u is a minimal mass blow-up solution to (1.1), a sequence $u(t_n)$, $t_n \in I$ has a subsequence that converges in L^2/G. Thus

Theorem 1.16. *Suppose Conjecture* 1.6 *fails. Then there exists a maximal lifespan solution u on $I \subset \mathbf{R}$, u blows up forward and backward in time, and for all $t \in I$, u is almost periodic modulo the group $G = (0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$ which consists of scaling symmetries, translational symmetries, and Galilean symmetries. That is, for any $t \in I$,*

$$u(t,x) = \frac{1}{N(t)^{d/2}} e^{ix\cdot\xi(t)} Q_t(\frac{x - x(t)}{N(t)}), \tag{1.21}$$

where $Q_t(x) \in K \subset L^2(\mathbf{R}^d)$ and K is a precompact subset of $L^2(\mathbf{R}^d)$.

Additionally, $N(0) = 1$, $N(t) \leq 1$ on $[0,\infty)$, $[0,\infty) \subset I$, and $\xi(0) = x(0) = 0$.

Proof. See [45] and section four of [43]. □

Remark 1.17. This is also true of a minimal mass blow-up solution to the focusing problem (1.11).

Remark 1.18. To prove Theorem 1.9 it suffices to prove that a solution to (1.1) of the form of Theorem 1.16 does not occur. To that end, for the remainder of the paper let $A \lesssim B$ denote $A \lesssim_{u,d} B$, where u is a minimal mass blow-up solution to (1.1) and d is the dimenson. Likewise $\lesssim_s$ refers to $\lesssim_{u,d,s}$.

Lemma 1.19. *Let u be a minimal mass blow-up solution to* (1.1) *on I that is almost periodic modulo G. Then there exists $\delta(u)$ such that for all $t_0 \in I$,*

$$[t_0 - \delta N(t_0)^2, t_0 + \delta N(t_0)^2] \subset I, \tag{1.22}$$

and

$$N(t) \sim N(t_0), \qquad |\xi(t) - \xi(t_0)| \lesssim N(t_0). \tag{1.23}$$

Proof. See lemma 5.18 of [28]. □

Remark 1.20. Additionally since u is almost periodic there exists $\eta(u) > 0$ such that

$$\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}([t_0-\delta N(t_0)^2, t_0+\delta N(t_0)^2]\times\mathbf{R}^d)} \geq \eta(u). \tag{1.24}$$

Definition 1.21. Divide $[0,\infty)$ into consecutive intervals J_k such that $\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)} = 1$. We call these intervals the intervals of local constancy. If $J \subset [0,\infty)$ is a union of consecutive intervals of local constancy, then

$$\sum_{J_k} N(J_k) \sim \int_J N(t)^3 dt. \tag{1.25}$$

For convenience let $J_k(t)$ denote the interval J_k to which t belongs.

Therefore, possibly after modifying the $C(\eta)$ in (1.17), (1.18) by a fixed constant, we can choose $N(t) : I \to (0,\infty)$, $\xi(t) : I \to \mathbf{R}^d$ such that

$$|\frac{d}{dt}N(t)| \lesssim N(t)^3, \tag{1.26}$$

$$|\frac{d}{dt}\xi(t)| \lesssim N(t)^3. \tag{1.27}$$

Furthermore,

Lemma 1.22. *If $u(t,x)$ is a minimal mass blow-up solution on an interval J, then*

$$\int_J N(t)^2 dt \lesssim \|u\|^{\frac{2(d+2)}{d}}_{L_{t,x}^{\frac{2(d+2)}{d}}(J\times\mathbf{R}^d)} \lesssim 1 + \int_J N(t)^2 dt. \tag{1.28}$$

Proof. See [30]. □

Combining (1.26), (1.27), and Theorem 1.16, we have proved

Theorem 1.23. *Suppose* (1.1) *is not globally well-posed and scattering for all $u_0 \in L^2(\mathbf{R}^d)$. Then there exists a minimal mass blow-up solution to* (1.1) *which satisfies* (1.17), (1.18), *$\xi(0) = x(0) = 0$, $N(0) = 1$, u blows up forward and backward in time, $N(t) \leq 1$ on $[0,\infty)$, and*

$$|\frac{d}{dt}N(t)|, |\frac{d}{dt}\xi(t)| \lesssim N(t)^3. \tag{1.29}$$

To prove Theorem 1.9 it suffices to prove that we cannot have a minimal mass blow-up solution to (1.1) that satisfies the properties of Theorem 1.23. The rest of the paper will work toward proving such a solution does not exist. We will say a solution to (1.1) satisfying these properties is a solution in the form of Theorem 1.23. It therefore suffices to exclude two scenarios separately:

$$\int_0^\infty N(t)^3 dt < \infty \tag{1.30}$$

and

$$\int_0^\infty N(t)^3 dt = \infty. \tag{1.31}$$

Borrowing terminology from [27], [49] we call (1.30) a rapid frequency cascade and (1.31) a quasi-soliton.

The main new ingredient of this paper is a long-time Strichartz estimate. This estimate will be heavily utilized to exclude scenarios (1.30) and (1.31). This estimate is proved by combining an induction on frequency estimate with the bilinear Strichartz estimate.

Theorem 1.24. *Suppose $J \subset [0,\infty)$ is compact, $d \geq 3$, u is a minimal mass blow-up solution to* (1.1) *in the form of Theorem* 1.23 *with $\mu = \pm 1$, $\int_J N(t)^3 dt = K$, and $u \in L_t^\infty \dot{H}_x^{s_0 - 1/2}(J \times \mathbf{R}^d)$. Then*

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}), \tag{1.32}$$

where $\sigma_J(N)$ is a frequency envelope that majorizes

$$\inf_{t \in J} \|P_{|\xi - \xi(t)| > N} u(t)\|_{L_x^2(\mathbf{R}^d)}. \tag{1.33}$$

More generally suppose $\frac{1}{2} \leq s_0 < 1 + \frac{4}{d}$. Then

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \lesssim \frac{K^{1/2}}{N^{s_0}} \|u\|_{L_t^\infty \dot{H}_x^{s_0 - 1/2}(J \times \mathbf{R}^d)} + \sigma_J(\frac{N}{2}). \tag{1.34}$$

When $\int_0^\infty N(t)^3 dt < \infty$, we use a method similar to the method used in [26], [30], and [44]. We use (1.32) to prove that a rapid frequency cascade must possess additional regularity, in particular $u(t) \in L_t^\infty \dot{H}_x^s([0,\infty) \times \mathbf{R}^d)$ for $0 < s < 1 + 4/d$. Since $\int N(t)^3 dt < \infty$, $N(t) \searrow 0$ as $t \to \infty$, this contradicts conservation of energy.

To preclude the quasi-soliton we rely on a frequency localized interaction Morawetz estimate. (See [12] for such an estimate in the energy-critical case. [12] dealt with the energy-critical equation, $u(t) \in \dot{H}^1$, and thus truncated to high frequencies.) The interaction Morawetz estimates scale like $\int_J N(t)^3 dt$, and in fact are bounded below by some constant times $\int_J N(t)^3 dt$. Using this fact we prove a contradiction for $\int_0^T N(t)^3 dt = K(T)$ sufficiently large. Since we are truncating to low frequencies, this method is very similar to the almost Morawetz estimates that are often used in conjunction with the I-method. (See [2], [8], [10], [11], [13], [14], [15], [16], [18], and [17] for more information on the I-method.) (1.32) allows us to control the errors that arise from frequency truncation and prove

Theorem 1.25. *For any $\eta > 0$ there exists $K_0(\eta)$ such that if $\int_J N(t)^3 dt = K$, $K \geq K_0$, then*

$$K \lesssim \int_J \int_{\mathbf{R}^d \times \mathbf{R}^d} (-\Delta\Delta |x-y|) |P_{\leq CK} u(t,x)|^2 |P_{\leq CK} u(t,y)|^2 dx dy dt \lesssim \eta K. \tag{1.35}$$

This leads to a contradiction in the case when $\int_0^\infty N(t)^3 dt = \infty$, since K can be made arbitrarily large.

Outline of the paper: In §2, we describe some harmonic analysis and properties of the linear Schrödinger equation that will be needed later in the paper. In particular we discuss Strichartz estimates. We also discuss the bilinear Strichartz estimates and the fractional chain rule.

In §3 we prove Theorem 1.24 and use this result to exclude the rapid frequency cascade. Finally in §4 we obtain the frequency localized interaction Morawetz estimate, excluding the quasi-soliton, which completes the proof of Theorem 1.9.

2. The linear Schrödinger equation

In this section we will introduce some tools from harmonic analysis that will be needed later in the paper.

2.1. **Linear Strichartz estimates.**

Definition 2.1. A pair (p, q) will be called an admissible pair for $d \geq 3$ if $p \geq 2$ and

$$\frac{2}{p} = d(\frac{1}{2} - \frac{1}{q}). \tag{2.1}$$

Let

$$S^0(I \times \mathbf{R}^d) = \{u \in C_t^0(I, L^2(\mathbf{R}^d)) : \|u\|_{S^0(I \times \mathbf{R}^d)} < \infty\}, \tag{2.2}$$

with norm

$$\|u\|_{S^0(I \times \mathbf{R}^d)} \equiv \sup_{(p,q) \text{ admissible}} \|u\|_{L_t^p L_x^q(I \times \mathbf{R}^d)}. \tag{2.3}$$

Let $N^0(I \times \mathbf{R}^d)$ be the space dual to $S^0(I \times \mathbf{R}^d)$ with appropriate norm.

Theorem 2.2. *If $u(t, x)$ solves the initial value problem*

$$\begin{gathered} iu_t + \Delta u = f(t), \\ u(0, x) = u_0, \end{gathered} \tag{2.4}$$

on an interval I, then

$$\|u\|_{S^0(I \times \mathbf{R}^d)} \lesssim \|u_0\|_{L^2(\mathbf{R}^d)} + \|f\|_{N^0(I \times \mathbf{R}^d)}. \tag{2.5}$$

Proof. See [42] for the case when $p > 2$, $\tilde{p} > 2$, and [20] for the proof when $p = 2$, $\tilde{p} = 2$, or both. We will rely very heavily on the double endpoint case, or when both $p = 2$ and $\tilde{p} = 2$. □

Remark 2.3. The lack of an endpoint Strichartz estimate when $d = 2$ causes a technical difficulty for (2.2) in the case $d = 2$ since one could well have $\|e^{it\Delta} u_0\|_{S^0(I \times \mathbf{R}^2)} = \infty$ for some $I \subset \mathbf{R}$ even if $u_0 \in L^2(\mathbf{R}^2)$. Because this paper only considers $d \geq 3$ we do not address that issue here.

We will also make heavy use of the bilinear Strichartz estimates throughout the paper.

Lemma 2.4. *Suppose $\hat{v}(t, \xi)$ is supported on $|\xi - \xi_0| \leq M$ and $\hat{u}(t, \xi)$ is supported on $|\xi - \xi_0| > N$, for some $\xi_0 \in \mathbf{R}^d$, $M \ll N$. Then for the interval $I = [a, b]$, $d \geq 1$,*

$$\|uv\|_{L_{t,x}^2(I \times \mathbf{R}^d)} \lesssim \frac{M^{(d-1)/2}}{N^{1/2}} \|u\|_{S_*^0(I \times \mathbf{R}^d)} \|v\|_{S_*^0(I \times \mathbf{R}^d)}, \tag{2.6}$$

$$\|u\|_{S_*^0(I \times \mathbf{R}^d)} \equiv \|u(a)\|_{L^2(\mathbf{R}^d)} + \|(i\partial_t + \Delta)u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)}. \tag{2.7}$$

Proof. See [50]. □

We will also need the Littlewood-Paley partition of unity. Let $\phi \in C_0^\infty(\mathbf{R}^d)$ be a radial function, $0 \leq \phi \leq 1$,

$$\phi(x) = \begin{cases} 1, & |x| \leq 1; \\ 0, & |x| > 2. \end{cases} \tag{2.8}$$

Define the frequency truncation

$$\mathcal{F}(P_{\leq N}u) = \phi(\frac{\xi}{N})\hat{u}(\xi). \tag{2.9}$$

Let $P_{>N}u = u - P_{\leq N}u$ and $P_N u = P_{\leq 2N}u - P_{\leq N}u$. For convenience of notation let $u_N = P_N u$, $u_{\leq N} = P_{\leq N}u$, and $u_{>N} = P_{>N}u$.

Lemma 2.5. *For any $1 \leq p \leq \infty$, N,*

$$\|P_N f\|_{L^p(\mathbf{R}^d)}, \|P_{\leq N} f\|_{L^p(\mathbf{R}^d)}, \|P_{>N} f\|_{L^p(\mathbf{R}^d)} \lesssim \|f\|_{L^p(\mathbf{R}^d)}. \tag{2.10}$$

Proof. See [46]. □

2.2. **Fractional chain rule.** A function F is called algebraic if $F(x)$ can be written as a polynomial in x and $\bar{x}$. When $d \geq 3$ the nonlinearity $F(u) = \mu|u|^{4/d}u$ is no longer algebraic, which introduces complications to the analysis of (1.1). Indeed, the Fourier transform of $F(u)$ is not the convolution of Fourier transforms of u, and thus $F(u_{<N})$ need not be truncated in frequency. Instead, we will use the fractional chain rule.

Lemma 2.6. *Let G be a Hölder continuous function of order $0 < \alpha < 1$. Then for every $0 < s < \alpha$, $1 < p < \infty$, $\frac{s}{\alpha} < \sigma < 1$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \in (1, \infty]$, $p_2 \in (1, \infty)$,*

$$\||\nabla|^s G(u)\|_{L_x^p(\mathbf{R}^d)} \lesssim \||u|^{\alpha - \frac{s}{\sigma}}\|_{L_x^{p_1}(\mathbf{R}^d)} \||\nabla|^\sigma u\|_{L_x^{\frac{s}{\sigma}p_2}(\mathbf{R}^d)}^{s/\sigma}. \tag{2.11}$$

Proof. See [50]. □

Corollary 2.7. *Let $0 \leq s < 1 + 4/d$. On any spacetime slab $I \times \mathbf{R}^d$,*

$$\||\nabla|^s F(u)\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(I \times \mathbf{R}^d)} \lesssim \||\nabla|^s u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)} \|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(I \times \mathbf{R}^d)}^{4/d}. \tag{2.12}$$

Proof. See [30]. □

Corollary 2.8. *For $0 \leq s < 1 + \frac{4}{d}$,*

$$\||\nabla|^s F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J \times \mathbf{R}^d)} \lesssim \|u\|_{L_t^\infty L_x^2(J \times \mathbf{R}^d)}^{4/d} \||\nabla|^s u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}. \tag{2.13}$$

Proof. We use an argument similar to the argument found in [30] to prove Corollary 2.7. □

Case 1. $0 \leq s \leq 1$. The case $s = 0$ is Hölder's inequality and case $s = 1$ is the chain rule. For $0 < s < 1$ use the fractional chain rule.

Theorem 2.9. *Assume $F : \mathbf{C} \to \mathbf{C}$ is a C^1 map, $F(0) = 0$ and for $\mu \in L^1([0, 1])$,*

$$|F'(\tau v + (1 - \tau)w)| \leq \mu(\tau)[G(v) + G(w)]. \tag{2.14}$$

Then for $0 < s < 1$, $p \in (1, \infty)$, $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2}$, $q_1 \in (1, \infty]$, $q_2 \in (1, \infty)$,

$$\|F(u)\|_{H^{s,p}(\mathbf{R}^d)} \leq C\|G(u)\|_{L^{q_1}(\mathbf{R}^d)} \|u\|_{H^{s,q_2}(\mathbf{R}^d)}. \tag{2.15}$$

Proof. See chapter 2, proposition 5.1 of [48]. □

Case 2. $d \geq 3$, $1 < s < \min\{1 + \frac{4}{d}, 2\}$. Use the chain rule and fractional product rule (see [46] for more details).

(2.16)
$$\begin{aligned} \| |\nabla|^s F(u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} &\lesssim \| F_z(u) + F_{\bar{z}}(u) \|_{L_t^\infty L_x^{d/2}(J \times \mathbf{R}^d)} \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} \\ &+ \| |\nabla|^{s-1} [F_z(u) + F_{\bar{z}}(u)] \|_{L_t^{\frac{2s}{s-1}} L_x^q (J \times \mathbf{R}^d)} \| \nabla u \|_{L_t^{2s} L_x^p (J \times \mathbf{R}^d)}, \end{aligned}$$

with

$$\frac{1}{p} = \frac{(d-2)}{2ds} + \frac{s-1}{2s}, \tag{2.17}$$

$$\frac{1}{q} = \frac{2}{d} + \frac{(s-1)(d-2)}{2ds} - \frac{s-1}{2s}. \tag{2.18}$$

By interpolation,

$$\| \nabla u \|_{L_t^{2s} L_x^p (J \times \mathbf{R}^d)} \lesssim \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)}^{1/s} \| u \|_{L_t^\infty L_x^2 (J \times \mathbf{R}^d)}^{(s-1)/s}. \tag{2.19}$$

Now use Lemma 2.6. Choose σ with $\frac{s-1}{4/d} < \sigma < 1$. Let $\frac{1}{p_1} = \frac{2}{d} - \frac{s-1}{2\sigma}$ and $\frac{1}{p_2} = \frac{(s-1)(d-2)}{2ds} + \frac{(s-\sigma)(s-1)}{2s\sigma}$. Both $F_z(z)$ and $F_{\bar{z}}(z)$ are Hölder continuous functions of order $\frac{4}{d}$. Without loss of generality consider $F_z(u)$:

$$\| |\nabla|^{s-1} F_z(u(t)) \|_{L_x^q (\mathbf{R}^d)} \lesssim \| |u(t)|^{4/d - \frac{s-1}{\sigma}} \|_{L_x^{p_1}(\mathbf{R}^d)} \| |\nabla|^\sigma u(t) \|_{L_x^{(\frac{s-1}{\sigma}) p_2} (\mathbf{R}^d)}^{\frac{s-1}{\sigma}}. \tag{2.20}$$

By interpolation

$$\| |\nabla|^\sigma u(t) \|_{L_t^{\frac{2s}{\sigma}} L_x^{(\frac{s-1}{s}) p_2} (J \times \mathbf{R}^d)}^{\frac{s-1}{\sigma}} \lesssim \| |\nabla|^s u \|_{L_t^2 L_x^{\frac{2d}{d-2}} (J \times \mathbf{R}^d)}^{\frac{s-1}{s}} \| u \|_{L_t^\infty L_x^2 (J \times \mathbf{R}^d)}^{(\frac{s-1}{\sigma})(\frac{s-\sigma}{s})}. \tag{2.21}$$

Finally,

$$\| |u|^{4/d - \frac{s-1}{\sigma}} \|_{L_t^\infty L_x^{p_1} (J \times \mathbf{R}^d)} \lesssim \| u \|_{L_t^\infty L_x^2 (J \times \mathbf{R}^d)}^{2/p_1}. \tag{2.22}$$

Summing up our terms, the corollary is proved in this case also.

Case 3. $d = 3$. Take $2 \leq s < 7/3$. Then

$$\begin{aligned} &\| |\nabla|^s F(u) \|_{L_t^2 L_x^{6/5} (J \times \mathbf{R}^3)} \\ &= \| |\nabla|^{s-2} [F_z(u) \Delta u + F_{\bar{z}}(u) \Delta \bar{u} + 2 F_{z\bar{z}}(u) |\nabla u|^2 + F_{zz}(u) (\nabla u)^2 \\ &\qquad + F_{\bar{z}\bar{z}}(u) (\nabla \bar{u})^2] \|_{L_t^2 L_x^{6/5} (J \times \mathbf{R}^3)}. \end{aligned} \tag{2.23}$$

F_{zz}, $F_{z\bar{z}}$, $F_{\bar{z}\bar{z}}$ are Hölder continuous of order $1/3$, while F_z and $F_{\bar{z}}$ are in fact differentiable, so use Lemma 2.6 and interpolate as in the previous case. □

At various points in the proof of Theorem 1.9 we will also rely on the Sobolev embedding lemma.

Lemma 2.10. *If* $\frac{1}{p} = \frac{1}{2} - \frac{\rho}{d}$ *and* $\rho < \frac{d}{2}$, *then*

$$\dot{H}^\rho(\mathbf{R}^d) \subset L^p(\mathbf{R}^d)$$

and

$$\| u \|_{L^p(\mathbf{R}^d)} \lesssim_p \| u \|_{\dot{H}^\rho(\mathbf{R}^d)}.$$

In §4 we will also rely heavily on the Hardy-Littlewood-Sobolev inequality.

Lemma 2.11. *Suppose* $\frac{r}{d} = 1 - (\frac{1}{p} - \frac{1}{q})$, $1 < p < \infty$, $1 < q < \infty$, *and* $0 < r < d$. *Then let*

$$g(x) = \int \frac{1}{|x-y|^r} f(y) dy. \tag{2.24}$$

$$\|g\|_{L^q(\mathbf{R}^d)} \lesssim_{p,q} \|f\|_{L^p(\mathbf{R}^d)}. \tag{2.25}$$

Corollary 2.12. *Suppose* $K(x)$ *is a kernel,*

$$|K(x)| \lesssim 1, \tag{2.26}$$

and

$$|\nabla K(x)| \lesssim \frac{1}{|x|}. \tag{2.27}$$

Let

$$w(x) = \int K(x-y) \cdot (\nabla f(y)) g(y) dy. \tag{2.28}$$

Then $w(x) = G(x) + H(x)$, *where for* $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|G\|_{L_x^\infty(\mathbf{R}^d)} \lesssim \|\nabla g\|_{L_x^p(\mathbf{R}^d)} \|f\|_{L_x^{p'}(\mathbf{R}^d)}, \tag{2.29}$$

$$\|H\|_{L_x^{\frac{6d}{5}}(\mathbf{R}^d)} \lesssim \||\nabla|^{2/3} g\|_{L_x^{\frac{2d}{d-1}}(\mathbf{R}^d)} \|f\|_{L_x^{\frac{2d}{d+2}}(\mathbf{R}^d)}, \tag{2.30}$$

and

$$\|H\|_{L_x^{3d}(\mathbf{R}^d)} \lesssim \||\nabla|^{2/3} g\|_{L_x^{\frac{2d}{d+2}}(\mathbf{R}^d)} \|f\|_{L_x^{\frac{2d}{d-2}}(\mathbf{R}^d)}. \tag{2.31}$$

Proof. This is proved by integration by parts and the Hardy-Littlewood-Sobolev inequality:

$$\begin{aligned} \int K(x-y) \cdot (\nabla f(y)) g(y) dy = &- \int K(x-y) \cdot (\nabla g(y)) f(y) dy \\ &+ \int (\nabla \cdot K(x-y)) g(y) f(y) dy. \end{aligned}$$

Let

$$G(x) = - \int K(x-y) \cdot (\nabla g(y)) f(y) dy$$

and

$$H(x) = \int (\nabla \cdot K(x-y)) g(y) f(y) dy.$$

Apply Hölder's inequality and $|K(x-y)| \lesssim 1$ to $G(x)$, Lemma 2.11, $|\nabla K(x-y)| \lesssim \frac{1}{|x-y|}$, and the Sobolev embedding theorem to $H(x)$. □

3. Rapid cascade

In this section we exclude the first of two blow-up solutions, the rapid frequency cascade.

Theorem 3.1. *There does not exist a minimal mass blow-up solution to* (1.1), $\mu = +1$, *in the form of Theorem* 1.23 *when*

$$\int_0^\infty N(t)^3 dt < \infty. \tag{3.1}$$

As in [26], [30], [43] we defeat this scenario by proving that such a minimal mass blow-up solution must possess additional regularity. (3.1) implies $\liminf_{t\to\infty} N(t) = 0$, which contradicts conservation of energy.

Remark 3.2. In fact (3.1) and $|N'(t)| \lesssim N(t)^3$ imply

$$\lim_{t\to\infty} N(t) = 0.$$

In this section we prove Theorem 1.24, from which we can obtain additional regularity in the case of the rapid frequency cascade. These estimates will also be used in §4 to preclude scenario (1.31) from occurring, although we will not prove any additional regularity in that case.

First define a frequency envelope that majorizes $\inf_{t\in J} \|P_{|\xi-\xi(t)|>N} u(t)\|_{L_x^2(\mathbf{R}^d)}$.

Definition 3.3. Suppose $N = 2^k$ for some $k \in \mathbf{Z}$, $J = [0, T]$ a compact interval. Let

$$\sigma_J(N) = \sum_{j=-\infty}^{\infty} 2^{-|j-k|/3} \inf_{t\in J} \|P_{|\xi-\xi(t)|\geq 2^j} u(t)\|_{L_x^2(\mathbf{R}^d)}. \tag{3.2}$$

Then if $M \leq N$,

$$\sigma_J(N) \leq \sigma_J(M) \leq (\frac{M}{N})^{1/3} \sigma_J(N). \tag{3.3}$$

See [39], [40] for more information on frequency envelopes.

We now prove (3.3). We have

$$\sum_{j=-\infty}^{\infty} 2^{-|j-k|/3} a_j = \sum_{j=-\infty}^{\infty} 2^{-|j|/3} a_{j+k}$$

and

$$\sum_{j=-\infty}^{\infty} 2^{-|j-l|/3} a_j = \sum_{j=-\infty}^{\infty} 2^{-|j|/3} a_{j+l}.$$

If a_j is monotone decreasing, then for all j, $a_{j+k} \leq a_{j+l}$. This proves the first inequality in (3.3). The second inequality follows from the fact that $2^{|j-l|/3} a_j \leq 2^{|l-k|/3} 2^{|j-k|/3} a_j$.

Theorem 1.24 is proved by an induction argument on N. Fix $J \subset [0, \infty)$ to be a union of a number of consecutive intervals of local constancy. Unless stated otherwise, $L_t^p L_x^q$ refers to $L_t^p L_x^q(J \times \mathbf{R}^d)$. Start with the base case.

Lemma 3.4. *Theorem* 1.24 *holds for* $N \leq N_{\min} = \inf_{t\in J} N(t)$.

Proof. It is a simple consequence of Lemma 1.22, conservation of mass, and the Strichartz inequality that

$$\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}}^2 \lesssim 1 + \int_J N(t)^2 dt \lesssim 1 + \frac{1}{N_{\min}} \int_J N(t)^3 dt \lesssim 1 + \frac{K}{N_{\min}}. \tag{3.4}$$

This implies (1.32). Next, $u \in L_t^\infty \dot{H}_x^{s_0-1/2}(J \times \mathbf{R}^d)$, $N(0) = 1$, the Sobolev embedding theorem, (1.17), and Hölder's inequality imply that

$$\|u\|_{L_t^\infty \dot{H}^{s_0-1/2}(J\times\mathbf{R}^d)} \gtrsim_{s_0} 1. \tag{3.5}$$

$N(t) \leq 1$ and (3.5) then imply $\dfrac{\|u(t)\|_{L_t^\infty \dot{H}_x^{s_0-1/2}}}{N_{\min}^{s_0-1/2}} \gtrsim 1$, so (1.34) follows as well. $\square$

Next we make the inductive step. In the interest of first exposing the main idea we will obtain an estimate conducive to induction when $\xi(t) \equiv 0$. This case is already fairly interesting on its own, generalizing the radial case to the case when $u(0,x)$ is symmetric across the $x_1 = 0$, ..., $x_d = 0$ hyperplanes. Indeed u symmetric across any d linearly independent hyperplanes is enough to imply $\xi(t) \equiv 0$. After this, we will treat the case when $\xi(t)$ is time dependent, which necessarily introduces a few additional complications.

Lemma 3.5. *Fix $\eta > 0$, let $C_0(\eta) = C(\eta^{\frac{3(d+1)}{4}})$ defined in (1.17), (1.18). If $\xi(t) \equiv 0$, then when $d = 3$, for any $t_0 \in J$,*

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^6} \lesssim_s \|P_{>N} u(t_0)\|_{L_x^2} + \sum_{M \leq \eta N} (\frac{M}{N})^s \|u_{>M}\|_{L_t^2 L_x^6} \\ + \eta^3 \|u_{>\eta N}\|_{L_t^2 L_x^6(J \times \mathbf{R}^3)} + \frac{C_0^{3/2} K^{1/2}}{(\eta N)^{1/2}} (\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k \times \mathbf{R}^3)}). \end{aligned} \tag{3.6}$$

When $d \geq 4$,

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim_s \|P_{>N} u(t_0)\|_{L_x^2} + \sum_{M \leq \eta N} (\frac{M}{N})^s \|u_{>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \\ + \eta^3 \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} + \frac{C_0^{4-6/d} K^{2/d}}{(\eta N)^{2/d}} (\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k \times \mathbf{R}^d)})^{4/d} \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}}^{1-4/d}. \end{aligned} \tag{3.7}$$

Proof. Define a cutoff $\chi(t) \in C_0^\infty(\mathbf{R}^d)$ in physical space,

$$\chi(t,x) = \begin{cases} 1, & |x - x(t)| \leq \frac{C_0}{N(t)}; \\ 0, & |x - x(t)| > \frac{2C_0}{N(t)}. \end{cases} \tag{3.8}$$

Let J_k be an interval of local constancy. Recall that $N(J_k) = \sup_{t \in J_k} N(t)$. Then

$$\begin{aligned} \|P_{>N} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \lesssim \|P_{>N} F(u_{\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \\ + \|(u_{>\eta N}) |u_{>C_0 N(J_k)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \\ + \|(u_{>\eta N})(1 - \chi(t)) |u_{\leq C_0 N(J_k)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \\ + \|(u_{>\eta N}) \chi(t) |u_{\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)}. \end{aligned} \tag{3.9}$$

By Bernstein's inequality and (2.13), for any $0 \leq s < 1 + 4/d$,

$$\|P_{>N} F(u_{\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \lesssim_s \frac{1}{N^s} \||\nabla|^s u_{\leq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbf{R}^d)} \|u\|_{L_t^\infty L_x^2(J_k \times \mathbf{R}^d)}^{4/d} \tag{3.10}$$

$$\lesssim_s \sum_{M \leq \eta N} (\frac{M}{N})^s \|u_{>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbf{R}^d)}. \tag{3.11}$$

For the next two terms we use (1.17) and (1.18). Since mass is concentrated in both frequency and space, we can deal with the mass outside these balls perturbatively.

There exists $C_0(\eta)$ such that

$$\begin{aligned}
&\|(u_{>\eta N})|u_{>C_0(\eta)N(J_k)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k\times\mathbf{R}^d)} \\
&\quad + \|(u_{>\eta N})(1-\chi(t))|u_{\leq C_0(\eta)N(J_k)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k\times\mathbf{R}^d)} \\
&\lesssim \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k\times\mathbf{R}^d)} [\|(1-\chi(t))u\|_{L_t^\infty L_x^2}^{4/d} + \|u_{>C_0(\eta)N(t)}\|_{L_t^\infty L_x^2}^{4/d}] \\
&\leq \eta^3 \|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k\times\mathbf{R}^d)}.
\end{aligned}$$

Finally take

$$\|(P_{>\eta N}u)\chi(t)|u_{\leq C_0(\eta)N(J_k)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k\times\mathbf{R}^d)}. \tag{3.12}$$

We will use (2.6) to estimate (3.12) and then sum over all the subintervals.

When $d = 3$: Applying the bilinear estimates, mass conservation, and Hölder's inequality,

$$\begin{aligned}
&\|(P_{>\eta N}u)\chi(t)|u_{\leq C_0 N(J_k)}|^{4/3}\|_{L_t^2 L_x^{6/5}(J_k\times\mathbf{R}^3)} \\
&\quad\leq \|(P_{>\eta N}u)(u_{\leq C_0 N(J_k)})\|_{L_{t,x}^2(J_k\times\mathbf{R}^3)} \|\chi(t)\|_{L_t^\infty L_x^6(J_k\times\mathbf{R}^d)} \|u\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)}^{1/3} \\
&\quad\lesssim \frac{C_0 N(J_k)}{(\eta N)^{1/2}} (\frac{C_0}{N(J_k)})^{1/2} \|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)} \|u\|_{S_*^0(J_k\times\mathbf{R}^d)}.
\end{aligned}$$

By (3.4),

$$\|u\|_{S_*^0(J_k\times\mathbf{R}^d)} = \|u_0\|_{L^2(\mathbf{R}^d)} + \||u|^{4/d}u\|_{L_{t,x}^{\frac{2(d+2)}{d+4}}(J_k\times\mathbf{R}^d)} \lesssim 1. \tag{3.13}$$

Summing over the subintervals J_k,

(3.14)

$$\|(P_{>\eta N}u)\chi(t)|u_{\leq C_0 N(J_k(t))}|^{4/3}\|_{L_t^2 L_x^{6/5}} \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} \frac{K^{1/2}}{N^{1/2}} (\sup_{J_k} \|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)}).$$

When $d \geq 4$: To simplify notation let $\frac{1}{q} = \frac{2(d-2)}{d^2}$ and $\frac{1}{p} = \frac{1}{q} + \frac{2}{d}$. Then

$$\begin{aligned}
&\|(P_{>\eta N}u)\chi(t)|u_{\leq C_0 N(J_k(t))}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\quad\leq \||(P_{>\eta N}u)(u_{\leq C_0 N(J_k(t))})|^{4/d}\chi(t)\|_{L_t^{d/2} L_x^p} \|(P_{>\eta N}u)^{1-4/d}\|_{L_t^{2d/(d-4)} L_x^{\frac{2d^2}{(d-2)(d-4)}}}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\|[(P_{>\eta N}u)(u_{\leq C_0 N(J_k)})]^{4/d}\chi(t)\|_{L_t^{d/2} L_x^p(J_k\times\mathbf{R}^d)} \\
&\quad\leq \|(P_{>\eta N}u)(u_{\leq C_0 N(J_k)})\|_{L_{t,x}^2(J_k\times\mathbf{R}^d)}^{4/d} \|\chi(t)\|_{L_t^\infty L_x^q(J_k\times\mathbf{R}^d)} \\
&\quad\lesssim \frac{(C_0 N(J_k))^{\frac{2(d-1)}{d}}}{(\eta N)^{2/d}} \|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)}^{4/d} \|u\|_{S_*^0(J_k\times\mathbf{R}^d)}^{4/d} (\frac{C_0}{N(J_k)})^{\frac{2(d-2)}{d}} \\
&\quad\lesssim C_0^{4-6/d} (\frac{N(J_k)}{\eta N})^{2/d} \|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)}^{4/d}.
\end{aligned}$$

Again summing over all subintervals,

$$\begin{aligned}
&\|[(P_{>\eta N}u)(u_{\leq C_0 N(J_k(t))})]^{4/d}\chi(t)\|_{L_t^{d/2}L_x^p} \\
&\lesssim (\sum N(J_k))^{2/d}\frac{C_0^{4-6/d}}{(\eta N)^{2/d}}(\sup_{J_k}\|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)})^{4/d} \\
&\lesssim \frac{K^{2/d}}{N^{2/d}}\frac{C_0^{4-6/d}}{\eta^{2/d}}(\sup_{J_k}\|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)})^{4/d}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(u_{>\eta N})\chi(t)|u_{\leq C_0 N(J_k(t))})|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\
&\lesssim \frac{C_0^{2-4/d}}{\eta^{2/d}}\frac{K^{2/d}}{N^{2/d}}(\sup_{J_k}\|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)})^{4/d}\|u_{>\eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}}^{1-4/d}.
\end{aligned}$$

By Strichartz estimates, when $d=3$,

$$\begin{aligned}
\|u_{>N}\|_{L_t^2L_x^6} &\lesssim_s \|P_{>N}u(t_0)\|_{L_x^2} + \sum_{M\leq \eta N}(\frac{M}{N})^s\|u_{>M}\|_{L_t^2L_x^6} \\
&+ \eta^3\|u_{>\eta N}\|_{L_t^2L_x^6} + \frac{C_0^{3/2}K^{1/2}}{(\eta N)^{1/2}}(\sup_{J_k}\|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^3)}).
\end{aligned}\tag{3.15}$$

This proves Lemma 3.5 when $d=3$. When $d\geq 4$,

$$\begin{aligned}
&\|u_{>N}\|_{L_t^2L_x^{\frac{2d}{d-2}}} \lesssim_s \|P_{>N}u(t_0)\|_{L_x^2} \\
&+ \sum_{M\leq \eta N}(\frac{M}{N})^s\|u_{>M}\|_{L_t^2L_x^{\frac{2d}{d-2}}} \\
&+ \eta^3\|u_{>\eta N}\|_{L_t^2L_x^{\frac{2d}{d-2}}} + \frac{C_0^{4-6/d}K^{2/d}}{(\eta N)^{2/d}}(\sup_{J_k}\|u_{>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)})^{4/d}\|u_{>\eta N}\|_{L_t^2L_x^{\frac{2d}{d-2}}}^{1-4/d}. \quad \square
\end{aligned}\tag{3.16}$$

3.1. $\xi(t)$ **time dependent.** When $\xi(t)$ is time dependent we run into a bit of difficulty with the projection of the Duhamel term. Consider the case when $J=[0,T]$, $d=3$, $N(t)\equiv 1$ and $\xi(t)=(t,0,0)$ to illustrate this idea. The low frequencies at time $t=0$ will be the high frequencies at some later time. Indeed, at time $t>N$, $\xi=0$ will belong to the set

$$\{|\xi-\xi(t)|>N\}.$$

Therefore we partition J into subintervals J^l where $|\xi(t_1)-\xi(t_2)|\ll N$ for $t_1,t_2\in J^l$ and use the Duhamel formula on each J^l separately.

Remark 3.6. The subintervals J^l should not be confused with the previously mentioned intervals of local constancy. Rather, the partition into subintervals J^l depends on N and each J^l will contain many intervals of local constancy. For $N\gg K$, J will be partitioned into a single J^l.

By (1.27),

$$\int_J |\frac{d}{dt}\xi(t)|dt\lesssim K.\tag{3.17}$$

Therefore, partition J into $\leq C(u)\frac{K}{N} + 1$ subintervals J^l such that

$$\int_{J_l} |\frac{d}{dt}\xi(t)|dt \leq \frac{N}{100} \tag{3.18}$$

on each subinterval. For any $t_0^l \in J^l$,

$$\|P_{|\xi-\xi(t)|>N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J^l\times\mathbf{R}^d)} \lesssim \|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J^l\times\mathbf{R}^d)} \tag{3.19}$$

$$\lesssim \|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|_{L_x^2(\mathbf{R}^d)} + \|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J^l\times\mathbf{R}^d)} \tag{3.20}$$

$$\lesssim \|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|_{L_x^2(\mathbf{R}^d)} + \|P_{|\xi-\xi(t)|>\frac{N}{4}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J^l\times\mathbf{R}^d)}. \tag{3.21}$$

The last inequality follows from (3.18). Summing in L_t^2,

$$\begin{aligned}\|P_{|\xi-\xi(t)|>N} u\|^2_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} \lesssim \sum_{J^l\subset J} &\|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u\|^2_{L_x^2(R^d)} \\ &+ \|P_{|\xi-\xi(t)|>\frac{N}{4}}(|u|^{4/d}u)\|^2_{L_t^2 L_x^{\frac{2d}{d+2}}(J^l\times\mathbf{R}^d)}.\end{aligned} \tag{3.22}$$

Choose $t_0^l \in J^l$ such that

$$\|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|_{L_x^2(\mathbf{R}^d)} = \inf_{t\in J_l} \|P_{|\xi-\xi(t)|>\frac{N}{2}} u(t)\|_{L_x^2(\mathbf{R}^d)}.$$

On one J^l, $\|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|_{L_x^2(\mathbf{R}^d)} = \inf_{t\in J} \|P_{|\xi-\xi(t)|>\frac{N}{2}} u(t)\|_{L_x^2(\mathbf{R}^d)}$. On all the other J^l subintervals we are satisfied with the crude estimate

$$\|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|_{L_x^2(\mathbf{R}^d)} \leq \|P_{|\xi-\xi(t)|>\frac{N}{2}} u(t)\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)}.$$

Therefore

$$\Big(\sum_{J^l\subset J} \|P_{|\xi-\xi(t_0^l)|>\frac{N}{2}} u(t_0^l)\|^2_{L_x^2(\mathbf{R}^d)}\Big)^{\frac{1}{2}} \lesssim \Big(\frac{K}{N}\Big)^{\frac{1}{2}} \|P_{|\xi-\xi(t)|>\frac{N}{2}} u(t)\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)} + \sigma_J\Big(\frac{N}{2}\Big). \tag{3.23}$$

Combining (3.23) with

$$\sum_{J^l\subset J} \|P_{|\xi-\xi(t)|>\frac{N}{4}}(|u|^{4/d}u)\|^2_{L_t^2 L_x^{\frac{2d}{d+2}}(J^l\times\mathbf{R}^d)} = \|P_{|\xi-\xi(t)|>\frac{N}{4}}(|u|^{4/d}u)\|^2_{L_t^2 L_x^{\frac{2d}{d+2}}(J\times\mathbf{R}^d)},$$

we have

$$\begin{aligned}\|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J\times\mathbf{R}^d)} \lesssim \Big(\frac{K}{N}\Big)^{1/2} &\|P_{|\xi-\xi(t)|>\frac{N}{2}} u(t)\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)} + \sigma_J\Big(\frac{N}{2}\Big) \\ &+ \|P_{|\xi-\xi(t)|>\frac{N}{4}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J\times\mathbf{R}^d)}.\end{aligned} \tag{3.24}$$

Remark 3.7. Now that (3.24) has been obtained, the subintervals J^l satisfying (3.18) will not be needed for the rest of the paper. The only subintervals of J that will appear from this point forward are the intervals of local constancy.

Now we are ready to prove an induction lemma.

Lemma 3.8. *Suppose $\xi(t)$ is time dependent, and u is in the form of Theorem 1.24. When $d = 3$, for $\frac{1}{2} \leq s_0 \leq 1 + \frac{4}{d}$, $0 \leq s \leq 1 + \frac{4}{d}$,*

$$\|u_{|\xi-\xi(t)|\geq N}\|_{L_t^2 L_x^6} \lesssim_{s,s_0} (\frac{K}{N})^{1/2} \|u_{|\xi-\xi(t)|\geq \frac{N}{2}}\|_{L_t^\infty L_x^2} + \sigma_J(\frac{N}{2}) \tag{3.25}$$

$$+ \sum_{M \leq \eta N} (\frac{M}{N})^s \|u_{|\xi-\xi(t)|\geq M}\|_{L_t^2 L_x^6} + \eta^3 \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^2 L_x^6} \tag{3.26}$$

$$+ C_0(\eta)^{3/2} (\frac{K}{\eta N})^{1/2} \frac{1}{N^{s_0 - 1/2}} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}}. \tag{3.27}$$

When $d \geq 4$,

$$\|u_{|\xi-\xi(t)|\geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \lesssim_{s,s_0} (\frac{K}{N})^{1/2} \|u_{|\xi-\xi(t)|\geq \frac{N}{2}}\|_{L_t^\infty L_x^2} + \sigma_J(\frac{N}{2}) \tag{3.28}$$

$$+ \sum_{M \leq \eta N} (\frac{M}{N})^s \|u_{|\xi-\xi(t)|\geq M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} + \eta^3 \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \tag{3.29}$$

$$+ C_0^{4-6/d} (\frac{K}{\eta N})^{2/d} \frac{1}{N^{(4s_0-2)/d}} \|u_{|\xi-\xi(t)|\geq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}}^{1-4/d} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}}^{4/d}. \tag{3.30}$$

Remark 3.9. (3.4) implies $\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} < \infty$. Because $u \in L_t^\infty \dot{H}_x^{s_0-1/2}$, the quantities (3.27) and (3.30) make sense.

Proof. If $J_k = [a_k, b_k]$ is an interval of local constancy,

$$\|P_{|\xi-\xi(t)|> \frac{N}{4}} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \tag{3.31}$$

$$\lesssim \|P_{|\xi-\xi(t)|> \frac{N}{4}} F(u_{|\xi-\xi(a_k)|\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \tag{3.32}$$

$$+ \|(u_{|\xi-\xi(a_k)|>\eta N}) |u_{|\xi-\xi(a_k)|>C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \tag{3.33}$$

$$+ \|(u_{|\xi-\xi(a_k)|>\eta N})(1-\chi(t))|u|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \tag{3.34}$$

$$+ \|(u_{|\xi-\xi(a_k)|>\eta N}) \chi(t) |u_{|\xi-\xi(a_k)|\leq C_0 N(t)}|^{4/d}\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)}. \tag{3.35}$$

Choose $\eta_1(u) > 0$ sufficiently small so that if $t \in J_k$, $|\xi(t) - \xi(a_k)| \leq \frac{N(J_k)}{100\eta_1}$. We take for granted that $C_0(\eta) \gg \frac{1}{\eta_1}$.

First suppose $N(J_k) \leq \eta\eta_1 N$. By Bernstein's inequality and $e^{-ix\cdot\xi(t)} F(u) = F(e^{-ix\cdot\xi(t)} u)$,

$$\begin{aligned} &\|P_{|\xi-\xi(t)|> \frac{N}{4}} F(u_{|\xi-\xi(a_k)|\leq \eta N})\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \\ &\lesssim \frac{1}{N^s} \||\nabla - i\xi(t)|^s u_{|\xi-\xi(a_k)|\leq \eta N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbf{R}^d)} \end{aligned} \tag{3.36}$$

$$\lesssim \sum_{M \leq 2\eta N} (\frac{M}{N})^s \|u_{|\xi-\xi(t)|>M}\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J_k \times \mathbf{R}^d)}. \tag{3.37}$$

Next, by (1.17) and (1.18),

$$\begin{aligned}(3.38)\qquad &\|(u_{|\xi-\xi(a_k)|\geq\eta N})(1-\chi(t))|u_{|\xi-\xi(a_k)|\leq C_0N(J_k)}|^{4/d}\|_{L_t^2L_x^{\frac{2d}{d+2}}(J_k\times\mathbf{R}^d)}\\ &+\|(u_{|\xi-\xi(a_k)|\geq\eta N})|u_{|\xi-\xi(a_k)|>C_0N(t)}|^{4/d}\|_{L_t^2L_x^{\frac{2d}{d+2}}(J_k\times\mathbf{R}^d)}\\ &\lesssim\eta^3\|u_{|\xi-\xi(a_k)|\geq\eta N}\|_{L_t^2L_x^{\frac{2d}{d-2}}(J_k\times\mathbf{R}^d)}.\end{aligned}$$

This takes care of (3.32), (3.33) and (3.34).

As in the case when $\xi(t)\equiv 0$, making a bilinear estimate when $d=3$,

$$(3.39)\qquad \|\chi(t)(u_{|\xi-\xi(a_k)|>\eta N})|u_{|\xi-\xi(a_k)|\leq C_0N(J_k)}|^{4/3}\|_{L_t^2L_x^{6/5}(J_k\times\mathbf{R}^3)}$$

$$\begin{aligned}(3.40)\qquad &\lesssim\|\chi(t)\|_{L_t^\infty L_x^6(J_k\times\mathbf{R}^3)}\|(u_{|\xi-\xi(a_k)|>\eta N})(u_{|\xi-\xi(a_k)|\leq C_0N(J_k)})\|_{L_{t,x}^2(J_k\times\mathbf{R}^3)}\\ &\quad\times\|u_{|\xi-\xi(a_k)|\leq C_0N(J_k)}\|_{L_t^\infty L_x^2(J_k\times\mathbf{R}^3)}^{1/3}\end{aligned}$$

$$(3.41)\qquad \lesssim C_0^{3/2}\frac{N(J_k)^{1/2}}{\eta^{1/2}N^{1/2}}\|u_{|\xi-\xi(a_k)|>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^3)}.$$

Likewise, for $d\geq 4$,

$$(3.42)\qquad \|\chi(t)|u_{|\xi-\xi(a_k)|>\eta N}||u_{|\xi-\xi(a_k)|\leq C_0N(J_k)}|^{4/d}\|_{L_t^2L_x^{\frac{2d}{d+2}}(J_k\times\mathbf{R}^d)}$$

$$(3.43)\qquad \lesssim C_0^{4-6/d}\left(\frac{N(J_k)}{\eta N}\right)^{2/d}\|u_{|\xi-\xi(a_k)|>\eta N}\|_{S_*^0(J_k\times\mathbf{R}^d)}^{4/d}\|u_{|\xi-\xi(t)|>\frac{\eta N}{2}}\|_{L_t^2L_x^{\frac{2d}{d-2}}(J_k\times\mathbf{R}^d)}^{1-4/d}.$$

(3.5), (1.18), and $N(t)\leq 1$ imply

$$(3.44)\qquad |\xi(t)|^s\lesssim\frac{\|u\|_{L_t^\infty\dot H_x^s(J\times\mathbf{R}^d)}}{\|u\|_{L_t^\infty L_x^2(J\times\mathbf{R}^d)}}.$$

Therefore,

$$(3.45)\qquad \|e^{-ix\cdot\xi(t)}u(t,x)\|_{\dot H^{s_0}(\mathbf{R}^d)}\lesssim_{s_0}\|u\|_{L_t^\infty\dot H_x^{s_0}(J\times\mathbf{R}^d)}.$$

Next, Theorem 2.2 and Duhamel's principle imply

$$\begin{aligned}(3.46)\qquad \||\nabla|^{s-1/2}u\|_{S^0(J_k\times\mathbf{R}^d)}&\lesssim\||\nabla|^{s-1/2}u(a_k)\|_{L^2(\mathbf{R}^d)}\\ &\quad+\||\nabla|^{s-1/2}u\|_{S^0(J_k\times\mathbf{R}^d)}\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)}^{4/d}.\end{aligned}$$

By the continuity method,

$$(3.47)\qquad \||\nabla|^{s-1/2}u\|_{S^0(J_k\times\mathbf{R}^d)}\lesssim\|u\|_{L_t^\infty\dot H_x^{s-1/2}(J_k\times\mathbf{R}^d)}.$$

This also holds for $e^{-ix\cdot\xi(a_k)}u$. Therefore,

$$\begin{aligned}(3.48)\qquad &\||\nabla|^{s-1/2}(e^{-ix\cdot\xi(a_k)}u)\|_{S_*^0(J_k\times\mathbf{R}^d)}\\ &\lesssim\||\nabla|^{s-1/2}(e^{-ix\cdot\xi(a_k)}u(a_k))\|_{L_x^2(\mathbf{R}^d)}\\ &\quad+\||\nabla|^{s-1/2}(e^{-ix\cdot\xi(a_k)}u)\|_{S^0(J_k\times\mathbf{R}^d)}\|u\|_{L_{t,x}^{\frac{2(d+2)}{d}}(J_k\times\mathbf{R}^d)}^{4/d}\\ &\lesssim\|u\|_{L_t^\infty\dot H^{s-1/2}(J_k\times\mathbf{R}^d)}.\end{aligned}$$

When $N(J_k) \geq \eta\eta_1 N$,
(3.49)
$$\begin{aligned}\||\nabla - i\xi(t)|^{s-1/2} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} &\lesssim \||\nabla - i\xi(a_k)|^{s-1/2} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} \\ + \||\xi(t) - \xi(a_k)|^{s-1/2} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}(J_k \times \mathbf{R}^d)} &\lesssim \|u\|_{L_t^\infty \dot{H}_x^{s-1/2}(J_k \times \mathbf{R}^d)}.\end{aligned}$$

By (1.22), $\sum_{J_k \subset J} N(J_k) \sim \int_J N(t)^3 dt = K$. Therefore there are $\lesssim \frac{K}{N\eta_1\eta}$ intervals J_k with $N(J_k) \geq \eta\eta_1 N$. The proof of Lemma 3.8 is complete. □

Remark 3.10. By (3.44) we also have

$$\|P_{|\xi - \xi(t)| > \frac{N}{2}} u\|_{L_t^\infty L_x^2} \lesssim_s \frac{1}{N^{s-1/2}} \|u\|_{L_t^\infty \dot{H}_x^{s-1/2}}. \tag{3.50}$$

Proof of Theorem 1.24. Fix J. Let

$$f(N) = \|u_{|\xi - \xi(t)| \geq N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}}. \tag{3.51}$$

First take $d = 3$. By Lemma 3.8,

$$f(N) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + \sum_{M \leq \eta N} (\frac{M}{N}) f(M). \tag{3.52}$$

Let

$$c(J) = \sup_N \frac{1}{(\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2})} \cdot f(N), \tag{3.53}$$

taking the supremum over all dyadic integers N. Then

$$f(N) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + c(J) \sum_{M \leq \eta N} (\frac{M}{N})((\frac{K}{M})^{1/2} + \sigma_J(\frac{M}{2})). \tag{3.54}$$

Since $\sigma_J(N) \leq \sigma_J(M) \leq (\frac{N}{M})^{1/3} \sigma_J(N)$,

$$f(N) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + c(J)\eta^{1/2} (\frac{K}{N})^{1/2} + c(J)\eta^{2/3} \sigma_J(\frac{N}{2}). \tag{3.55}$$

Fixing $\eta(u) > 0$ sufficiently small,

$$c(J) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}}. \tag{3.56}$$

□

Remark 3.11. We have thus far ignored the possibility that (3.53) is infinite. However, by (3.4), $\|u\|_{L_t^2 L_x^{\frac{2d}{d-2}}(J \times \mathbf{R}^d)} < \infty$. Let

$$c_\epsilon(J) = \sup_N \frac{1}{(\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + \epsilon} f(N). \tag{3.57}$$

Plugging this into (3.53),

$$f(N) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + c_\epsilon(J)\eta^{1/2} (\frac{K}{N})^{1/2} + c_\epsilon(J)\eta^{2/3} + \epsilon\eta. \tag{3.58}$$

This implies

$$c_\epsilon(J) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} + 1 + \eta + \eta^{2/3} c_\epsilon(J). \tag{3.59}$$

Again fixing $\eta(u) > 0$ sufficiently small, $c_\epsilon(J) \lesssim \frac{C_0^{3/2}}{\eta^{1/2}} + 2$. This bound is independent of $\epsilon > 0$. Thus (1.32) holds when $d = 3$.

Next choose $\frac{1}{2} \leq s_0 < 1 + \frac{4}{d}$ and let $s = \frac{s_0}{2} + \frac{1}{2} + \frac{2}{d}$. Let

$$c(J, s_0) = \sup_N \frac{1}{(\frac{K^{1/2}}{N^{s_0}})\|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}} + \sigma_J(\frac{N}{2})} f(N). \tag{3.60}$$

Then

$$\begin{aligned} f(N) \lesssim_{s_0} & \frac{C_0^{3/2}}{\eta^{1/2}} \frac{K^{1/2}}{N^{s_0}} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}} + \sigma_J(\frac{N}{2}) \\ & + c(J, s_0) \sum_{M \leq \eta N} (\frac{M}{N})^s ((\frac{K}{M})^{s_0} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}} + \sigma_J(\frac{M}{2})). \end{aligned} \tag{3.61}$$

Therefore,

$$c(J, s_0) \lesssim_{s_0} \frac{C_0^{3/2}}{\eta^{1/2}} + \eta^{s - s_0} c(J, s_0). \tag{3.62}$$

Again fixing $\eta(s_0) > 0$ sufficiently small,

$$c(J, s_0) \lesssim_{s_0} \frac{C_0^{3/2}}{\eta(s_0)^{1/2}}. \tag{3.63}$$

This proves (1.34) when $d = 3$.

Next, for $d \geq 4$,

$$f(N) \lesssim (\frac{K}{N})^{1/2} + \sigma_J(\frac{N}{2}) + \sum_{M \leq \eta N} (\frac{M}{N}) f(M) + C_0^{4-6/d} (\frac{K}{\eta N})^{2/d} f(\eta N)^{1-4/d} \tag{3.64}$$

$$\lesssim \frac{C_0^{d-3/2}}{\eta^{d/4-1}} (\frac{K}{\eta N})^{1/2} + \sigma_J(\frac{N}{2}) + \sum_{M \leq \eta N} (\frac{M}{N}) f(M). \tag{3.65}$$

Then proceed as in the case when $d = 3$ to prove (1.32). Likewise, for $\frac{1}{2} \leq s_0 < 1 + \frac{4}{d}$, let $s = \frac{s_0}{2} + \frac{1}{2} + \frac{2}{d}$. Then

$$f(N) \lesssim_{s_0} \frac{C_0^{d-3/2}}{\eta^{(\frac{4}{d}-1)s}} (\frac{K}{\eta N})^{1/2} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}} + \sigma_J(\frac{N}{2}) + \sum_{M \leq \eta N} (\frac{M}{N})^s f(M). \tag{3.66}$$

Proceed as in the $d = 3$ case to prove (1.34). □

Remark 3.12. Notice that we have also proved

$$\|P_{|\xi - \xi(t)| > N} F(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim_{s_0} (\frac{K^{1/2}}{N^{s_0}} \|u\|_{L_t^\infty \dot{H}_x^{s_0-1/2}} + \sigma_J(\frac{N}{2})). \tag{3.67}$$

Theorem 1.24 and (3.67) are enough to give additional regularity.

Theorem 3.13. *Suppose u is a minimal mass blow-up solution to* (1.1), $\mu = \pm 1$, *u is in the form of Theorem* 1.23, *and* $\int_0^\infty N(t)^3 dt = K < \infty$. *Then for* $0 \leq s < 1 + \frac{4}{d}$,

$$\|u(t, x)\|_{L_t^\infty \dot{H}_x^s((0,\infty) \times \mathbf{R}^d)} \lesssim_s K^s. \tag{3.68}$$

Proof. By (1.27),

$$\int_0^\infty |\frac{d}{dt}\xi(t)|dt \lesssim K. \tag{3.69}$$

Choose $C(d,m_0)$ sufficiently large so that

$$\int_0^\infty |\frac{d}{dt}\xi(t)|dt \leq \frac{C(d,m_0)}{100}K. \tag{3.70}$$

By conservation of mass, for any $s \geq 0$,

$$\|P_{\leq C(d,m_0)K} u(t,x)\|_{L_t^\infty \dot{H}_x^s((0,\infty)\times\mathbf{R}^d)} \lesssim_s K^s. \tag{3.71}$$

Let $J(T) = [0,T]$. Then

$$\liminf_{t\to\infty} N(t) = 0 \tag{3.72}$$

and (1.18) imply that for any $N > 0$,

$$\lim_{T\to\infty} \sigma_{J(T)}(\frac{N}{2}) = 0. \tag{3.73}$$

Theorem 1.24 implies that for $\frac{1}{2} \leq s < 1 + \frac{4}{d}$,

$$\|u_{|\xi-\xi(t)|>N}\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,\infty)\times\mathbf{R}^d)} \lesssim_s \frac{K^{1/2}}{N^s}\|u\|_{L_t^\infty \dot{H}_x^{s-1/2}([0,\infty)\times\mathbf{R}^d)}, \tag{3.74}$$

and (3.67) implies that

$$\|P_{|\xi-\xi(t)|>N}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,\infty)\times\mathbf{R}^d)} \lesssim_s \frac{K^{1/2}}{N^s}\|u\|_{L_t^\infty \dot{H}_x^{s-1/2}([0,\infty)\times\mathbf{R}^d)}. \tag{3.75}$$

Also, for any $N \geq CK$, by (3.70) and $\xi(0) = 0$,

$$\liminf_{t\to+\infty} \|P_{>N}u(t)\|_{L_x^2(\mathbf{R}^d)} = 0. \tag{3.76}$$

So for $N \geq CK$,

$$\|P_{>N}u(t_0)\|_{L_x^2(\mathbf{R}^d)}^2 \leq \int_{t_0}^\infty |\frac{d}{dt}\langle P_{>N}u(t), P_{>N}u(t)\rangle|dt, \tag{3.77}$$

$$\frac{d}{dt}\langle P_{>N}u(t), P_{>N}u(t)\rangle = 2\mu Re\langle iP_{>N}(|u|^{4/d}u)(t), P_{>N}u(t)\rangle. \tag{3.78}$$

Therefore, for any $N \geq C(d,m_0)K$, $0 \leq t_0 < \infty$,

$$\begin{aligned} &\|P_{>N}u(t_0)\|_{L_x^2(\mathbf{R}^d)}^2 \\ &\quad\lesssim \|P_{|\xi-\xi(t)|>\frac{N}{2}}(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}((0,\infty)\times\mathbf{R}^d)} \|u_{|\xi-\xi(t)|>\frac{N}{2}}\|_{L_t^2 L_x^{\frac{2d}{d-2}}((0,\infty)\times\mathbf{R}^d)} \\ &\quad\lesssim_s \frac{K}{N^{2s}}\|u\|_{L_t^\infty \dot{H}_x^{s-1/2}((0,\infty)\times\mathbf{R}^d)}^2. \end{aligned} \tag{3.79}$$

We make the elementary estimate

$$\begin{aligned} \|u\|_{L_t^\infty \dot{H}_x^s((0,\infty)\times\mathbf{R}^d)} \leq{}& \|P_{\leq C(d,m_0)K}u\|_{L_t^\infty \dot{H}_x^s((0,\infty)\times\mathbf{R}^d)} \\ &+ \|P_{>C(d,m_0)K}u\|_{L_t^\infty \dot{H}_x^s((0,\infty)\times\mathbf{R}^d)}. \end{aligned} \tag{3.80}$$

Using (3.71) for the first term in (3.80) and (3.79) for the second, for any $\gamma \leq s < 1 + \frac{4}{d}$, $0 \leq \gamma < \frac{1}{2}$,

$$\|u\|_{L_t^\infty \dot{H}_x^s((0,\infty)\times\mathbf{R}^d)} \lesssim_{s,\gamma} K^s + K^\gamma\|u\|_{L_t^\infty \dot{H}_x^{s-\gamma}((0,\infty)\times\mathbf{R}^d)}. \tag{3.81}$$

Iterating (3.81) starting with $s = \gamma$ and conservation of mass completes the proof of Theorem 3.13. $\square$

Proof of Theorem 3.1. For $T_1 > T_2$,

$$|\xi(T_1) - \xi(T_2)| \lesssim \int_{T_2}^{T_1} N(t)^3 dt \to 0 \tag{3.82}$$

as $T_2 \to \infty$. Therefore, the limit

$$\xi_\infty = \lim_{t\to\infty} \xi(t) \tag{3.83}$$

exists. By performing a Galilean transformation we may set $\xi_\infty = 0$. For $t > 0$, $1 < s < 1 + 4/d$,

$$\|u(t)\|_{\dot{H}^1_x} \lesssim \|u_{|\xi-\xi(t)|\geq C(\eta)N(t)}\|_{\dot{H}^1} + \|u_{|\xi-\xi(t)|\leq C(\eta)N(t)}\|_{\dot{H}^1} \tag{3.84}$$

$$\lesssim \|u_{|\xi-\xi(t)|\geq C(\eta)N(t)}\|_{L^2_x}^{1-1/s} \|u\|_{\dot{H}^s}^{1/s} + \|u_{|\xi-\xi(t)|\leq C(\eta)N(t)}\|_{\dot{H}^1} \tag{3.85}$$

$$\lesssim_u \eta^{1-1/s} + |\xi(t)| + C(\eta)N(t). \tag{3.86}$$

Since $\xi(t) \to 0$, $N(t) \to 0$, the right-hand side may be made arbitrarily small. By the Gagliardo-Nirenberg inequality,

$$\|u(t)\|_{L_x^{\frac{2(d+2)}{d}}}^{\frac{2(d+2)}{d}} \lesssim \|u(t)\|_{\dot{H}^1}^2 \|u(t)\|_{L_x^2}^{4/d} \lesssim \|u(t)\|_{\dot{H}^1}^2. \tag{3.87}$$

This implies $E(u(t)) \to 0$, which implies $u \equiv 0$, which is a contradiction. $\square$

Remark 3.14. Theorem 3.13 is true for the focusing case as well; however, E is not positive definite. Nevertheless, when the mass lies below the ground state the Gagliardo-Nirenberg inequality allows us to make an estimate similar to (3.86). We postpone addressing this issue to a later paper.

4. Quasi-soliton

In this section we defeat the second minimal mass blow-up scenario, the quasi-soliton. Exclusion of this scenario will complete the proof of Theorem 1.9. To exclude this scenario it suffices to prove Theorem 1.25.

Theorem 1.25 utilizes the interaction Morawetz estimates of [11] for $d = 3$ and of [44] for $d \geq 4$. See also [34] and [9]. Take the tensor product of two solutions to (1.1). Let x refer to the first d variables in $\mathbf{R}^d \times \mathbf{R}^d$ and y refer to the second d variables. Let

$$M(t) = Im \int\int_{\mathbf{R}^d\times\mathbf{R}^d} |u(t,y)|^2 \overline{u(t,x)} \frac{(x-y)}{|x-y|} \cdot \nabla u(t,x) dx dy. \tag{4.1}$$

[11] proved

$$\|u\|_{L^4_{t,x}(I\times\mathbf{R}^3)}^4 \lesssim \int_I \partial_t M(t) dt, \tag{4.2}$$

and [44] proved

$$\int_I \int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{1}{|x-y|^3} |u(t,x)|^2 |u(t,y)|^2 dx dy dt \lesssim \int_I \partial_t M(t) dt. \tag{4.3}$$

The interaction Morawetz estimate is well suited to defeat the quasi-soliton for two reasons. First, $M(t)$ is Galilean invariant, a fact well exploited by [34]. Indeed,

$$M(t) = \partial_t \int |x-y||u(t,x)|^2|u(t,y)|^2 dxdy. \tag{4.4}$$

The quantity $|x-y||u(t,x)|^2|u(t,y)|^2$ is invariant under the transformation $u \mapsto e^{ix\cdot\xi(t)}u$, as is $M(t)$. Let $\tilde{M}(t)$ be the new $M(t)$ after making the transformation $u \mapsto e^{ix\cdot\xi(t)}u$. Then

$$\tilde{M}(t) = Im \int\int_{\mathbf{R}^d\times\mathbf{R}^d} |u(t,y)|^2\overline{u(t,x)}\frac{(x-y)}{|x-y|}\cdot(\nabla - i\xi(t))u(t,x)dxdy, \tag{4.5}$$

$$\begin{aligned} &Im \int\int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{(x-y)}{|x-y|}\cdot(i\xi(t))\overline{u}(t,x)\overline{u}(t,y)u(t,x)u(t,y)]dxdy \\ &= \int\int_{\mathbf{R}^d\times\mathbf{R}^d} \xi(t)\cdot\frac{(x-y)}{|x-y|}|u(t,x)|^2|u(t,y)|^2dxdy. \end{aligned} \tag{4.6}$$

Because $|u(t,x)|^2|u(t,y)|^2$ is even in $x-y$ and $\frac{(x-y)}{|x-y|}$ is odd in $x-y$, (4.6) $\equiv 0$ and $\tilde{M}(t) = M(t)$.

The second reason these estimates are quite useful is that they scale like $\int_I N(t)^3 dt$ for a minimal mass blow-up solution to (1.1). When $d=3$, Hölder's inequality and (1.17) imply

$$\frac{99m_0^2}{100} \leq \int_{|x-x(t)|\leq \frac{C(\frac{m_0^2}{100})}{N(t)}} |u(t,x)|^2dx \lesssim \frac{1}{N(t)^{3/2}}\|u(t,x)\|_{L_x^4(\mathbf{R}^3)}^2. \tag{4.7}$$

Therefore,

$$\int_I N(t)^3dt \lesssim \|u\|_{L_{t,x}^4(I\times\mathbf{R}^3)}^4. \tag{4.8}$$

Remark 4.1. Compare to [12], where in the energy critical case $\|u\|_{L_{t,x}^4}$ scales like

$$\int_I N(t)^{-1}dt. \tag{4.9}$$

In higher dimensions,

$$(\frac{99m_0^2}{100})^2 \leq \int_{|x-x(t)|\leq \frac{C(\frac{m_0^2}{100})}{N(t)}} \int_{|y-x(t)|\leq \frac{C(\frac{m_0^2}{100})}{N(t)}} |u(t,x)|^2|u(t,y)|^2dxdy. \tag{4.10}$$

This implies that

$$N(t)^3 \lesssim \int \frac{1}{|x-y|^3}|u(t,x)|^2|u(t,y)|^2dxdy \tag{4.11}$$

and

$$\int_I N(t)^3dt \lesssim \int_I \partial_t M(t)dt. \tag{4.12}$$

If $\int_0^\infty N(t)^3dt = \infty$, then this forces

$$\int_0^\infty \partial_t M(t)dt = \infty. \tag{4.13}$$

To exclude the quasi-soliton it would suffice to prove $\sup_{t \in (0,\infty)} |M(t)| < \infty$. By the fundamental theorem of calculus this would imply that

$$\int_0^T \partial_t M(t) dt \leq C < \infty \tag{4.14}$$

for any T, which contradicts (4.13). This is precisely the method used in [43] to treat radial data. However, neither the methods of [43], which rely very heavily on radial symmetry, nor the methods of the previous section will readily yield additional regularity for the nonradial quasi-soliton. First, we could have $N(t) \equiv 1$, and thus $\sigma_J(\frac{N}{2})$ would not converge to zero when $J = [0,T]$, $T \nearrow \infty$. Secondly, $\int_0^\infty N(t)^3 dt$ is not uniformly bounded, so $K \nearrow \infty$ as $T \to \infty$.

Instead we rely on a frequency truncated interaction Morawetz estimate. [12] introduced a frequency localized version of (4.2) for the energy critical nonlinear Schrödinger equation on $\mathbf{R}^3$ to prove global well-posedness and scattering. In that case $u(t) \in \dot{H}^1(\mathbf{R}^3)$, so the Morawetz estimates were localized to high frequencies. Here $u(t) \in L^2(\mathbf{R}^3)$, so we localize to low frequencies.

This method also has a great deal in common with the almost Morawetz estimates frequently used in conjunction with the I-method. (See [8], [14], and [18] for the two-dimensional case, and [17] for the three-dimensional case.)

By (1.27), $\xi(0) = 0$, there exists a constant $C(u)$ such that for $J = [0,T]$, $\int_0^T N(t)^3 dt = K$, $|\xi(t)| \leq \frac{CK}{100}$ for all $t \in [0,T]$. By (1.18), $N(t) \leq 1$, (4.8), and $|\xi(t)| \leq \frac{CK}{100}$ on $[0,T]$, when $d = 3$,

$$\int_J N(t)^3 dt \lesssim \|P_{\leq CK} u(t,x)\|_{L^4_{t,x}(J \times \mathbf{R}^3)}^4. \tag{4.15}$$

Using (4.12) in place of (4.8) when $d \geq 4$,

$$\int_J N(t)^3 dt \lesssim \int_J \int \int \frac{1}{|x-y|^3} |P_{\leq CK} u(t,x)|^2 |P_{\leq CK} u(t,y)|^2 dx dy dt. \tag{4.16}$$

As in §3 let $L_t^p L_x^q$ denote $L_t^p L_x^q(J \times \mathbf{R}^d)$.

Theorem 4.2. *Let $M(t)$ be the interaction Morawetz quantity*

$$M(t) = Im \iint_{\mathbf{R}^d \times \mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 \overline{P_{\leq CK} u(t,x)} \frac{(x-y)}{|x-y|} \cdot (\nabla - i\xi(t)) P_{\leq CK} u(t,x) dx dy. \tag{4.17}$$

Then for any $\eta > 0$ there exists $K_0(\eta)$ such that for $K \geq K_0$,

$$\int_0^T \partial_t M(t) dt \lesssim \eta K. \tag{4.18}$$

Proof. By the fundamental theorem of calculus, (4.18) $= M(T) - M(0)$. Then

$$Im \int \int_{\mathbf{R}^d \times \mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 \overline{P_{\leq CK} u(t,x)} \frac{(x-y)}{|x-y|} \cdot (\nabla - i\xi(t)) P_{\leq CK} u(t,x) dx dy \tag{4.19}$$

$$\leq \|P_{\leq CK} u\|_{L_t^\infty L_x^2}^3 \|(\nabla - i\xi(t)) P_{\leq CK} u\|_{L_t^\infty L_x^2}. \tag{4.20}$$

We estimate $\|P_{\leq CK} u\|_{L_t^\infty L_x^2}$ by conservation of mass. Also, by almost periodicity and $N(t) \leq 1$, for any $\eta > 0$ there exists $C(\eta) > 0$ such that

$$\|P_{|\xi - \xi(t)| > C(\eta)} u(t,x)\|_{L_x^2(\mathbf{R}^d)} \leq \eta. \tag{4.21}$$

If $K > \frac{C(\eta)}{\eta}$, then

$$\begin{aligned}
&\|(\nabla - i\xi(t))P_{\leq CK}u\|_{L_t^\infty L_x^2} \\
&\lesssim \|(\nabla - i\xi(t))P_{|\xi-\xi(t)|\leq \eta K}u\|_{L_t^\infty L_x^2} + K\|P_{|\xi-\xi(t)|>\eta K}u\|_{L_t^\infty L_x^2} \\
&\lesssim \eta K.
\end{aligned} \tag{4.22}$$

Let $K_0(\eta) = \frac{C(\eta)}{\eta}$. □

Next, since $P_{\leq CK}$ is a Fourier multiplier,

$$\partial_t(P_{\leq CK}u) = i\Delta P_{\leq CK}u - iF(P_{\leq CK}u) + iF(P_{\leq CK}u) - iP_{\leq CK}F(u). \tag{4.23}$$

If we had only

$$\partial_t(P_{\leq CK}u) = i\Delta(P_{\leq CK}u) - iF(P_{\leq CK}u), \tag{4.24}$$

then the proof of Theorem 1.25 would be complete. We could copy the arguments from [11] and [44] exactly, replacing u with $P_{\leq CK}u$ and using (4.15) and (4.16). However, we also need to deal with the error terms that arise from the fact that

$$F(P_{\leq CK}u) - P_{\leq CK}F(u) \neq 0. \tag{4.25}$$

Borrowing a convention from [14] let

$$\begin{aligned}
\mathcal{G} &= i\Delta(P_{\leq CK}u) - iF(P_{\leq CK}u), \\
\mathcal{N} &= iF(P_{\leq CK}u) - iP_{\leq CK}F(u),
\end{aligned} \tag{4.26}$$

$$\partial_t P_{\leq CK}u = \mathcal{G} + \mathcal{N}, \tag{4.27}$$

$$\begin{aligned}
\frac{d}{dt}M(t) = {} & M_{\mathcal{G}}(t) + M_{\mathcal{N}}(t) \\
& - Re\iint |P_{\leq CK}u(t,y)|^2|P_{\leq CK}u(t,x)|^2\frac{(x-y)}{|x-y|}\cdot\xi'(t)dxdy,
\end{aligned} \tag{4.28}$$

where $M_{\mathcal{G}}(t)$ are the terms in $\frac{d}{dt}M(t)$ with $\partial_t P_{\leq CK}u$ replaced with $\mathcal{G}$ and $M_{\mathcal{N}}(t)$ are the terms in $\frac{d}{dt}M(t)$ with $\partial_t P_{\leq CK}u$ replaced with $\mathcal{N}$. Once again, $\frac{(x-y)}{|x-y|}$ odd gives the Galilean invariance

$$Re\int\int |P_{\leq CK}u(t,y)|^2|P_{\leq CK}u(t,x)|^2\frac{(x-y)}{|x-y|}\cdot\xi'(t)dxdy \equiv 0. \tag{4.29}$$

Next, since $\overline{P_{\leq CK}u}F(P_{\leq CK}u) - P_{\leq CK}uF(\overline{P_{\leq CK}u}) = 0$,

$$M_{\mathcal{G}}(t) = Re \int\int [(\overline{P_{\leq CK}u})\Delta(P_{\leq CK}u) - (P_{\leq CK}u)\Delta(\overline{P_{\leq CK}u})](t,y) \times \overline{P_{\leq CK}u}(t,x)\frac{(x-y)}{|x-y|}\cdot \nabla P_{\leq CK}u(t,x)dxdy \tag{4.30}$$

$$-Re \int\int |P_{\leq CK}u(t,y)|^2 \Delta\overline{P_{\leq CK}u}(t,x)\frac{(x-y)}{|x-y|}\cdot \nabla P_{\leq CK}u(t,x)dxdy \tag{4.31}$$

$$+Re \int\int |P_{\leq CK}u(t,y)|^2 \overline{P_{\leq CK}u}(t,x)\frac{(x-y)}{|x-y|}\cdot \nabla\Delta P_{\leq CK}u(t,x)dxdy \tag{4.32}$$

$$+Re \int\int |P_{\leq CK}u(t,y)|^2 F(\overline{P_{\leq CK}u})(t,x)\frac{(x-y)}{|x-y|}\cdot \nabla P_{\leq CK}u(t,x)dxdy \tag{4.33}$$

$$-Re \int\int |P_{\leq CK}u(t,y)|^2 \overline{P_{\leq CK}u}(t,x)\frac{(x-y)}{|x-y|}\cdot \nabla F(P_{\leq CK}u)(t,x)dxdy \tag{4.34}$$

$$\begin{aligned}&+Im\iint[\overline{P_{\leq CK}u}\Delta P_{\leq CK}u - P_{\leq CK}u\Delta\overline{P_{\leq CK}u}](t,y)|P_{\leq CK}u(t,x)|^2\frac{(x-y)}{|x-y|}\cdot\xi(t)dxdy\\ &+Im\iint|P_{\leq CK}u(t,y)|^2[\overline{P_{\leq CK}u}\Delta P_{\leq CK}u - P_{\leq CK}u\Delta\overline{P_{\leq CK}u}](t,x)\frac{(x-y)}{|x-y|}\cdot\xi(t)dxdy.\end{aligned} \tag{4.35}$$

(4.35) $\equiv 0$, so it is possible to replace the $\nabla - i\xi(t)$ in $M_{\mathcal{G}}(t)$ with ∇ and perform the same integration by parts computations that are found in [11] and [44]. Therefore

$$M_{\mathcal{G}}(t) \gtrsim \int\int (-\Delta\Delta|x-y|)|P_{\leq CK}u(t,x)|^2|P_{\leq CK}u(t,y)|^2 dxdy. \tag{4.36}$$

It remains to estimate the error arising from $M_{\mathcal{N}}(t)$. Let $\mathcal{E} = \int_0^T M_{\mathcal{N}}(t)dt$. It follows by direct calculation that

Lemma 4.3.

$$\mathcal{E} = Re\int_0^T \int\int_{\mathbf{R}^d\times\mathbf{R}^d} [P_{\leq CK}F(\bar{u}) - F(\overline{P_{\leq CK}u})](t,x)|P_{\leq CK}u(t,y)|^2 \times\frac{(x-y)}{|x-y|}\cdot(\nabla - i\xi(t))P_{\leq CK}u(t,x)dxdydt \tag{4.37}$$

$$+Re\int_0^T \int\int_{\mathbf{R}^d\times\mathbf{R}^d} |P_{\leq CK}u(t,y)|^2\overline{P_{\leq CK}u(t,x)} \times\frac{(x-y)}{|x-y|}\cdot(\nabla - i\xi(t))[F(P_{\leq CK}u) - P_{\leq CK}F(u)](t,x)dxdydt \tag{4.38}$$

$$\begin{aligned}&+ Re\int_0^T \int\int_{\mathbf{R}^d\times\mathbf{R}^d} \frac{(x-y)}{|x-y|}\cdot[\overline{P_{\leq CK}u}(t,x)(\nabla - i\xi(t))P_{\leq CK}u(t,x)]\\ &\times [P_{\leq CK}F(\bar{u})(t,y)P_{\leq CK}u(t,y) - P_{\leq CK}F(u)(t,y)\overline{P_{\leq CK}u}(t,y)]dxdydt.\end{aligned} \tag{4.39}$$

At this point it is beneficial to say a few words about the focusing mass critical problem. The interaction Morawetz estimates of [11] and [44] rely very heavily on $\mu = +1$. When $\mu = -1$ the computations are no longer positive definite and the estimates used in the defocusing case will not work for the focusing problem, even for mass below the mass of the ground state. Therefore, it will be necessary to introduce a new interaction Morawetz estimate adapted to the focusing problem in an upcoming paper. This Morawetz estimate will rely on a quantity $M(t)$ of the form (4.17), but with $\frac{(x-y)}{|x-y|}$ replaced by a real-valued function $a(t, x-y)$. It is easy to see that if $a(t, x-y)$ is an odd function of $x-y$ for all t, then we have the Galilean invariance of (4.6). Secondly, if

$$|a(t, x-y)| \lesssim 1, \tag{4.40}$$

then (4.20) also holds. Therefore we prove,

Theorem 4.4. *Suppose u is a minimal mass blow-up solution to* (1.1)*, in the form of Theorem* 1.23*, $\mu = \pm 1$. Suppose $\int_0^T N(t)^3 dt = K$. Let*

$$\tilde{M}(t) = Im \int\int_{\mathbf{R}^d \times \mathbf{R}^d} |Iu(t,y)|^2 \overline{Iu(t,x)} a(t, x-y) \cdot \nabla P_{\leq CK} u(t,x) dx dy, \tag{4.41}$$

where

$$a(t,x) = -a(t,-x), \tag{4.42}$$

$$|a(t,x)| \lesssim 1, \tag{4.43}$$

and

$$|\nabla_x a(t,x)| \lesssim \frac{1}{|x|}. \tag{4.44}$$

Then $\tilde{M}(t)$ is Galilean invariant,

$$\sup_{t \in (0,\infty)} |\tilde{M}(t)| \lesssim 1, \tag{4.45}$$

and for any $\eta > 0$ there exists $K_0(\eta)$ such that for $K > K_0$,

$$\tilde{\mathcal{E}} \lesssim \eta K, \tag{4.46}$$

where

$$\tilde{\mathcal{E}} = Re \int_0^T \int\int_{\mathbf{R}^d \times \mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 [P_{\leq CK} F(\bar{u}) - F(\overline{P_{\leq CK} u})](t,x) \times a(t, x-y) \cdot (\nabla - i\xi(t)) P_{\leq CK} u(t,x)\} dx dy dt \tag{4.47}$$

$$+ Re \int_0^T \int\int_{\mathbf{R}^d \times \mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 \overline{P_{\leq CK} u}(t,x) \times a(t, x-y) \cdot (\nabla - i\xi(t)) [F(P_{\leq CK} u) - P_{\leq CK} F(u)](t,x) dx dy dt \tag{4.48}$$

$$+ Re \int_0^T \int\int_{\mathbf{R}^d \times \mathbf{R}^d} a(t, x-y) \cdot [\overline{P_{\leq CK} u}(t,x) (\nabla - i\xi(t)) P_{\leq CK} u(t,x)] \times [P_{\leq CK} F(\bar{u})(t,y) P_{\leq CK} u(t,y) - P_{\leq CK} F(u)(t,y) \overline{P_{\leq CK} u}(t,y)] dx dy dt. \tag{4.49}$$

Theorem 4.4 implies Theorem 1.25 since $\frac{(x-y)}{|x-y|}$ certainly satisfies (4.42)–(4.44). To prove Theorem 4.4 we need some intermediate lemmas. First notice that by Theorem 1.24 and conservation of mass,

$$\|P_{|\xi - \xi(t)| > N} u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T] \times \mathbf{R}^d)} \lesssim ((\frac{K}{N})^{1/2} + 1). \tag{4.50}$$

Lemma 4.5. *Under the hypotheses of Theorem* 1.16*, for any* $\frac{1}{2} < s \leq 1$*,*

$$\| |\nabla|^s [e^{-ix\cdot\xi(t)} P_{\leq CK} u] \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \lesssim_s K^s. \tag{4.51}$$

More generally, for any $p > 2$*,* (p,q) *is d-admissible, and* $\frac{1}{p} < s \leq 1$*,*

$$\| |\nabla|^s (e^{-ix\cdot\xi(t)} P_{\leq CK} u) \|_{L_t^p L_x^q (J\times\mathbf{R}^d)} \lesssim_{s,p} K^s. \tag{4.52}$$

Proof.

$$\begin{aligned} &\| |\nabla|^s (e^{-ix\cdot\xi(t)} P_{\leq CK} u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim_s \sum_{N \leq 2CK} N^s \| P_N (e^{-ix\cdot\xi(t)} P_{\leq CK} u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim_s \sum_{N \leq 2CK} N^s \| P_N (e^{-ix\cdot\xi(t)} u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \lesssim_s \sum_{N\leq 2CK} N^s (\frac{K}{N})^{1/2} \lesssim_s K^s. \end{aligned}$$

(4.52) follows by interpolating (4.51) with (1.18) and summing. □

Lemma 4.6. *Under the hypotheses of Theorem* 1.23*, for* $1/2 < s \leq 1$*,*

$$\| |\nabla|^s (e^{-ix\cdot\xi(t)} P_{\leq CK} F(u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim_s K^s. \tag{4.53}$$

Proof.

$$\begin{aligned} &\| |\nabla|^s (e^{-ix\cdot\xi(t)} P_{\leq CK} F(u)) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim_{d,s} \| |\nabla|^s P_{\leq 2CK} F(e^{-ix\cdot\xi(t)} u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim \sum_{N \leq 2CK} N^s \| P_N F(e^{-ix\cdot\xi(t)} u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)}. \end{aligned} \tag{4.54}$$

Now write $F(e^{-ix\cdot\xi(t)}u) = F(P_{\leq N}(e^{-ix\cdot\xi(t)}u)) + O(|P_{>N}(e^{-ix\cdot\xi(t)}u)||u|^{4/d})$ and use Theorem 1.24 to estimate

$$\begin{aligned} \| P_N F(P_{\leq N}(e^{-ix\cdot\xi(t)}u)) \|_{L_t^2 L_x^{\frac{2d}{d+2}}} &\lesssim N^{-1} \| \nabla P_{\leq N}(e^{-ix\cdot\xi(t)}u) \|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|u\|_{L_t^\infty L_x^2}^{4/d} \\ &\lesssim N^{-1} \sum_{M \leq N} M(1 + (\frac{K}{M})^{1/2}) \lesssim 1 + (\frac{K}{N})^{1/2} \end{aligned} \tag{4.55}$$

and

$$\| |P_{>N}(e^{-ix\cdot\xi(t)}u)||u|^{4/d} \|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim 1 + (\frac{K}{N})^{1/2}. \tag{4.56}$$

Therefore,

$$\| |\nabla|^s [e^{-ix\cdot\xi(t)} P_{\leq CK} F(u)] \|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \sum_{N \leq 2CK} N^s (1 + (\frac{K}{N})^{1/2}) \lesssim K^s. \tag{4.57}$$ □

Lemma 4.7. *Under the hypotheses of Theorem* 1.16*,*

$$\| P_{\leq CK} F(u) - F(P_{\leq CK} u) \|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim o_K(1), \tag{4.58}$$

where $o_K(1) \to 0$ *as* $K \to \infty$*.*

Proof. Let $\rho(K)$ be some quantity to be specified later, $\rho(K) \leq \frac{1}{2}$, $\rho(K) \to 0$ as $K \to \infty$, and $\rho(K) \geq (1+K)^{-1/10}$. Let $u_l = P_{\leq \rho(K)CK} u$. Then

$$\begin{aligned}
\|P_{\leq CK}F(u_l) - F(u_l)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} &= \|P_{\geq CK}F(u_l)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\
&\lesssim \frac{1}{K}\|\nabla u_l\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \|u_l\|^{4/d}_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \lesssim \rho(K).
\end{aligned} \tag{4.59}$$

Now let $v = u - u_l$. By Taylor's theorem,

$$F(u) - F(u_l) = v\int_0^1 F'(u_l + \tau v)d\tau = vG(u_l, v), \tag{4.60}$$

(4.61)

$$F(P_{\leq CK}u) - F(u_l) = (P_{\leq CK}v)\int_0^1 F'(u_l + \tau P_{\leq CK}v)d\tau = (P_{\leq CK}v)G(u_l, P_{\leq CK}v).$$

Using the fundamental theorem of calculus when $|\xi_1| \gg |\xi_2|$, $|\phi(\xi)| \lesssim 1$ for $|\xi_1| \lesssim |\xi_2|$, ϕ is the multiplier used in (2.8) to define the frequency projection,

$$|\phi(\frac{\xi_1+\xi_2}{CK}) - \phi(\frac{\xi_1}{CK})| \lesssim \frac{|\xi_2|}{|\xi_1|}. \tag{4.62}$$

Therefore,

$$\|P_{\leq CK}(vG(u_l, P_{\leq CK}v)) - (P_{\leq CK}v)G(u_l, P_{\leq CK}v)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \tag{4.63}$$

$$\lesssim \frac{1}{K^{3/d}}\|P_{\leq CK}v\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \||\nabla|^{3/d}G(u_l, P_{\leq CK}v)\|_{L_t^\infty L_x^{d/2}([0,T]\times\mathbf{R}^d)}, \tag{4.64}$$

$F' \in C^{0,\min\{4/d,1\}}$, $|F'(x)| \lesssim_d |x|^{4/d}$, and (1.18) imply that there exists $\eta(K) \leq 1$, $\eta(K) \to 0$ as $K \to \infty$ such that

$$\||\nabla|^{3/d}G(u_l, P_{\leq CK}v)\|_{L_t^\infty L_x^{d/2}([0,T]\times\mathbf{R}^d)} \lesssim \eta(K)K^{3/d}. \tag{4.65}$$

Therefore,

$$(4.63) \lesssim \frac{\eta(K)}{\rho(K)^{1/2}}. \tag{4.66}$$

Finally,

$$\begin{aligned}
&\|P_{\leq CK}(v[G(u_l, v) - G(u_l, P_{\leq CK}v)])\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\
&\lesssim \|v\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \|G(u_l, v) - G(u_l, P_{\leq CK}v)\|_{L_t^\infty L_x^{d/2}([0,T]\times\mathbf{R}^d)},
\end{aligned} \tag{4.67}$$

(4.68)

$$\|G(u_l, v) - G(u_l, P_{\leq CK}v)\|_{L_t^\infty L_x^{d/2}([0,T]\times\mathbf{R}^d)} \lesssim \|P_{\geq CK^{9/10}}u\|^{\min(4/d,1)}_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \lesssim \eta(K).$$

Combining (4.59), (4.66), and (4.68),

$$\|P_{\leq CK}F(u) - F(P_{\leq CK}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim \rho(K) + \frac{\eta(K)}{\rho(K)^{1/2}}. \tag{4.69}$$

Choosing $\rho(K) \to 0$ sufficiently slowly completes the proof of the lemma. □

We are now ready to estimate the first term in $\tilde{\mathcal{E}}$.

Corollary 4.8. *For any $\eta > 0$ there exists $K_0(\eta)$ such that for $K \geq K_0$,*

$$(4.47) \lesssim \eta K. \tag{4.70}$$

Proof. By $|a(t,x-y)| \lesssim 1$, Lemmas 4.5 and 4.7,

$$(4.47) \lesssim \|P_{\leq CK} u\|^2_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \|e^{ix\cdot\xi(t)}\nabla(e^{-ix\cdot\xi(t)} P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \times \|P_{\leq CK} F(u) - F(P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim \eta K. \qquad \square$$

In order to estimate (4.48) and (4.49) we need one additional lemma.

Corollary 4.9. *For any $\eta > 0$ there exists $K_0(\eta)$ such that for $K > K_0$,*

$$(4.48) \lesssim \eta K.$$

Proof. Let

$$w(t,y) = \int_{\mathbf{R}^d} (e^{ix\cdot\xi(t)} \overline{P_{\leq CK} u}(t,x)) a(t,x-y) \nabla\cdot (e^{-ix\cdot\xi(t)}[P_{\leq CK} F(u(t,x)) - F(P_{\leq CK} u(t,x))]) dx.$$

Then by Corollary 2.12,

$$w(t,y) = G(t,y) + H(t,y), \tag{4.71}$$

with, for $K > K_0$,

$$\|G(t,y)\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \lesssim \|P_{\leq CK} F(u) - F(P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \times \|\nabla(e^{-ix\cdot\xi(t)} P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \lesssim \eta K \tag{4.72}$$

and

$$\|H(t,y)\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0,T]\times\mathbf{R}^d)} \lesssim \|P_{\leq CK} F(u) - F(P_{\leq CK} u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \times \||\nabla|^{2/3}(e^{-ix\cdot\xi(t)} P_{\leq CK} u)\|_{L_t^4 L_x^{\frac{2d}{d-1}}([0,T]\times\mathbf{R}^d)} \lesssim \eta K^{2/3}. \tag{4.73}$$

By Hölder's inequality and conservation of mass,

$$\int_0^T \int_{\mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 |G(t,y)| dy dt \leq \|G(t,y)\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \|P_{\leq CK} u(t,y)\|^2_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \lesssim \eta K.$$

By Sobolev embedding and Lemma 4.5,

$$\int_0^T \int_{\mathbf{R}^d} |P_{\leq CK} u(t,y)|^2 |H(t,y)| dy dt \leq \|H(t,y)\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0,T]\times\mathbf{R}^d)} \|P_{\leq CK} u(t,y)\|^2_{L_t^8 L_x^{\frac{12d}{6d-5}}([0,T]\times\mathbf{R}^d)} \lesssim \eta K.$$

This implies that $(4.48) \lesssim \eta K$. $\square$

Corollary 4.10. *For any $\eta > 0$ there exists $K_0(\eta)$ such that for $K > K_0$,*

$$(4.49) \lesssim \eta K. \tag{4.74}$$

Proof.

$$P_{\leq CK}(|u|^{4/d} u)\overline{P_{\leq CK} u} = |u|^{2+4/d} + (1 - P_{\leq CK})(|u|^{4/d} u)\overline{(P_{\leq CK} - 1) u} + (P_{\leq CK} - 1)(|u|^{4/d} u)(\overline{P_{\leq CK} u}) + P_{\leq CK}(|u|^{4/d} u)\overline{(P_{\leq CK} - 1) u},$$

$$Im[|u|^{2+4/d}] \equiv 0. \tag{4.75}$$

Next, let

$$w_1(t,x) = \int_{\mathbf{R}^d} a(t,x-y) Im[(1-P_{\leq CK})(|u|^{4/d}u)(t,y)\overline{(P_{\leq CK}-1)u(t,y)}]dy, \tag{4.76}$$

$$w_2(t,x) = \int_{\mathbf{R}^d} a(t,x-y) Im[(P_{\leq CK}-1)(|u|^{4/d}u)(t,y)\overline{(P_{\leq CK}u)(t,y)}]dy, \tag{4.77}$$

and

$$w_3(t,x) = \int_{\mathbf{R}^d} a(t,x-y) Im[P_{\leq CK}(|u|^{4/d}u)(t,y)\overline{(P_{\leq CK}-1)u(t,y)}]dy. \tag{4.78}$$

By Hölder's inequality, Lemma 4.5, Lemma 4.6, and $|\xi - \xi(t)| \sim |\xi|$ for $|\xi| \geq CK$,

$$\begin{aligned}&\|w_1\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim \|(1-P_{\leq CK})u\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \|(1-P_{\leq CK})(|u|^{4/d}u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim 1.\end{aligned}$$

Next consider (4.77):

$$e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)(|u|^{4/d}u) = \nabla\cdot\nabla\Delta^{-1}(e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)(|u|^{4/d}u)). \tag{4.79}$$

By Corollary 2.12 we have $w_2 = G_2 + H_2$, where

$$\begin{aligned}&\|G_2\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim \|\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \\ &\qquad \times \|\nabla\Delta^{-1}(e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim K(\frac{1}{K}) = 1.\end{aligned}$$

We used Lemmas 4.5, 4.6 and the fact that $|\xi - \xi(t)| \gtrsim K$ on the support of $(1-P_{\leq CK})$. Then

$$\begin{aligned}&\|H_2\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim \||\nabla|^{2/3}(e^{-ix\cdot\xi(t)}P_{\leq CK}u)\|_{L_t^4 L_x^{\frac{2d}{d-1}}([0,T]\times\mathbf{R}^d)} \\ &\qquad \times \|\nabla\Delta^{-1}(e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\ &\quad \lesssim \eta K^{-1/3}.\end{aligned}$$

Similarly, $w_3 = G_3 + H_3$, with

$$\begin{aligned}
&\|G_3\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \\
&\quad \lesssim \|\nabla\Delta^{-1}(e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \\
&\qquad \times \|\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \lesssim 1, \\
&\|H_3\|_{L_t^1 L_x^{3d}([0,T]\times\mathbf{R}^d)} \\
&\quad \lesssim \||\nabla|^{2/3}(e^{-ix\cdot\xi(t)}P_{\leq CK}(|u|^{4/d}u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,T]\times\mathbf{R}^d)} \\
&\qquad \times \|\nabla\Delta^{-1}(e^{-ix\cdot\xi(t)}(P_{\leq CK}-1)u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}([0,T]\times\mathbf{R}^d)} \\
&\quad \lesssim K^{-1/3}.
\end{aligned}$$

By Hölder's inequality, for $K > \frac{C(\eta)}{\eta}$,

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^d} |F_1(t,x) + G_2(t,x) + G_3(t,x)||(\nabla - i\xi(t))P_{\leq CK}u(t,x)||P_{\leq CK}u(t,x)|dxdt \\
&\qquad\qquad \leq \|w_1(t,x) + G_2(t,x) + G_3(t,x)\|_{L_t^1 L_x^\infty([0,T]\times\mathbf{R}^d)} \\
&\quad \times\|\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}u)\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)}\|P_{\leq CK}u(t,x)\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \lesssim \eta K.
\end{aligned}$$

Next, by Hölder's inequality and Sobolev embedding,

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^d} |H_3(t,x)||\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}u(t,x))||P_{\leq CK}u(t,x)|dxdt \\
&\leq \|H_3\|_{L_t^1 L_x^{3d}([0,T]\times\mathbf{R}^d)}\|\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}u(t,x))\|_{L_t^\infty L_x^2([0,T]\times\mathbf{R}^d)} \\
&\quad \times \|e^{-ix\cdot\xi(t)}P_{\leq CK}u\|_{L_t^\infty L_x^{\frac{6d}{3d-2}}([0,T]\times\mathbf{R}^d)} \\
&\lesssim K^{-1/3}\eta K K^{1/3} = \eta K.
\end{aligned}$$

Finally by the Sobolev embedding theorem and Lemma 4.5,

$$\begin{aligned}
&\int_0^T \int_{\mathbf{R}^d} |H_2(t,x)||\nabla e^{-ix\cdot\xi(t)}P_{\leq CK}u(t,x)||P_{\leq CK}u(t,x)|dxdt \\
&\lesssim \|H_2\|_{L_t^{4/3} L_x^{\frac{6d}{5}}([0,T]\times\mathbf{R}^d)}\|\nabla(e^{-ix\cdot\xi(t)}P_{\leq CK}u)\|_{L_t^4 L_x^{\frac{2d}{d-1}}([0,T]\times\mathbf{R}^d)} \\
&\quad \times \|P_{\leq CK}u\|_{L_t^\infty L_x^{\frac{6d}{3d-2}}([0,T]\times\mathbf{R}^d)} \\
&\lesssim K^{-1/3}\eta K K^{1/3} = \eta K.
\end{aligned}$$

This completes the proof of Theorem 4.4. □

Theorem 4.4 implies Theorem 1.24, which excludes the quasi-soliton blow-up solution. This rules out scenario (1.31) and completes the proof of Theorem 1.9.

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