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ALGEBRAIC K-THEORY VIA BINARY COMPLEXES

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This paper is dedicated to the memory of Daniel Quillen.

Introduction

In [11], Quillen defines the notion of *exact* category. Each such category $\mathcal{N}$ is an additive category together with a suitable collection of sequences called *short exact sequences*. Quillen constructs a space by gluing simplices to each other, using the exact sequences of $\mathcal{N}$ to determine the simplices and the gluing instructions. Then for each $n \in \mathbb{N}$ he defines $K_n\mathcal{N}$ as a homotopy group of the space. The construction is engineered so that $K_0\mathcal{N}$ is isomorphic to the Grothendieck group of $\mathcal{N}$. Because they are defined as homotopy groups, these K-groups are difficult to compute, with the case where $\mathcal{N}$ is the category of finite-dimensional vector spaces over a finite field being one of the few cases where they are all known. Until now, there has been no description of the K-groups not involving homotopy theory.

This paper[1] resulted from contemplation of a result of Nenashev [10], which depends on his earlier work [8, 9] and on work of Sherman [13, 14]. Nenashev provides generators and relations for $K_1\mathcal{N}$, where $\mathcal{N}$ is an exact category; the generators are *double exact sequences* in $\mathcal{N}$:

$$0 \longrightarrow N' \underset{j}{\overset{i}{\rightrightarrows}} N \underset{q}{\overset{p}{\rightrightarrows}} N'' \longrightarrow 0,$$

and the relations are those coming from short exact sequences of generators together with some other ones, including his six-term relation. By letting the double exact sequences be longer and multi-dimensional, we succeed, in Corollary 7.4, in giving explicit generators and relations for $K_n\mathcal{N}$, for any n; the generators are what we call *acyclic binary chain multicomplexes* of dimension n, and the relations are those coming from short exact sequences and those coming from acyclic binary chain multicomplexes that are trivial in one of the n directions. Said more succinctly, we realize $K_n\mathcal{N}$ as the Grothendieck group of a cube of exact categories; see Corollary 7.2. The basic tool used is an operation $\mathcal{N} \mapsto \Omega\mathcal{N}$ that transforms an exact category into a split pair of exact categories and that can be iterated to yield a cube. On the corresponding K-theory spectra, it amounts to taking the connective part of the loop space; see Corollary 6.5.

Whether our presentation of the higher algebraic K-groups results in any consequences for their computation remains to be seen.

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For background information on K-theory of exact categories with weak equivalences, the reader may refer to section A of the appendix. We speak of K-theory spectra rather than K-theory spaces throughout. That makes the proofs involving loop spaces simpler, because replacing a space X by its loop space discards $\pi_0 X$, which is just a set, but the corresponding operation on a connective spectrum X discards the Eilenberg-MacLane spectrum of the abelian group $\pi_0 X$, which fits into a handy fibration sequence with X. The extra structure retained allows us to avoid distracting digressions to deal with the behavior of low-dimensional homotopy groups and sets.

1. Long exact sequences in exact categories

Definition 1.1. A bounded chain complex N in an exact category $\mathcal{N}$ that arises by being spliced together from short exact sequences of $\mathcal{N}$ will be called a *long exact sequence* or an *acyclic chain complex* in $\mathcal{N}$. In other words, the differentials $d_i : N_i \to N_{i-1}$ can be factored as $N_i \to Z_{i-1} \to N_{i-1}$ so that $0 \to Z_i \to N_i \to Z_{i-1} \to 0$ is a short exact sequence of $\mathcal{N}$ for each i.

The component short exact sequences of an acyclic chain complex are unique up to isomorphism, because $Z_i \to N_i$ is a kernel of d_i, for all i. An exact functor $\mathcal{N}' \to \mathcal{N}$ sends long exact sequences of $\mathcal{N}'$ to long exact sequences of $\mathcal{N}$.

This definition of acyclicity is not expected to have good properties for an arbitrary exact category. For possibly better definitions, see Remark 6.7.

We have restricted our attention to bounded complexes in Definition 1.1, partly because it's the categories of bounded complexes whose K-theory can be understood, and partly because there are examples of unbounded acyclic complexes of modules over a ring where each module is projective, but the images of the differentials are not.

An exact category is conveniently constructed as a full subcategory of an abelian category by imposing various conditions on its collection of objects. The following definition allows us to discuss those conditions abstractly.

Definition 1.2. We will say that a property of objects in an abelian category $\mathcal{A}$ is *closed under extensions* if, for any exact sequence $0 \to M' \to M \to M'' \to 0$ of $\mathcal{A}$ where M' and M'' have the property, then so does M. If M' has the property whenever M and M'' do, then we say that the property is *closed under kernels of epimorphisms*, and if M'' has the property whenever M and M' do, then we say that the property is *closed under cokernels of monomorphisms.* A full subcategory $\mathcal{N} \subseteq \mathcal{A}$ is said to be *closed* in one of the preceding senses if the class of objects of $\mathcal{A}$ isomorphic to an object of $\mathcal{N}$ is closed in that sense.

Definition 1.3. An *admissible* embedding of an exact category $\mathcal{N}$ is a fully faithful exact functor $h : \mathcal{N} \hookrightarrow \mathcal{A}$, where: (1) $\mathcal{A}$ is an abelian category; (2) the class of objects of $\mathcal{A}$ isomorphic to an object of $h(\mathcal{N})$ is closed under extension; (3) h *reflects* exactness, in the sense that a sequence $E : 0 \to N' \to N \to N'' \to 0$ of $\mathcal{N}$ is an exact sequence of $\mathcal{N}$ if $h(E)$ is an exact sequence of $\mathcal{A}$.

Recall (from [11, §2] and [17, (A.7.1)]) that exact categories in the sense of Quillen are characterized by having admissible embeddings.

Definition 1.4. If $\mathcal{N}$ has an admissible embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ so that every bounded acyclic chain complex in $\mathcal{A}$ whose objects are in $\mathcal{N}$ also has the images of its differentials isomorphic to objects of $\mathcal{N}$, then we will say that $\mathcal{N}$ *supports long exact sequences.*

Since the image of a map is defined only up to isomorphism, if it is isomorphic to an object of $\mathcal{N}$, a representative for it may be chosen that is an object of $\mathcal{N}$.

When $\mathcal{N}$ supports long exact sequences, as demonstrated by an embedding $\mathcal{N} \hookrightarrow \mathcal{A}$, a bounded chain complex in $\mathcal{N}$ is acyclic in $\mathcal{N}$ if and only if it is acyclic in $\mathcal{A}$; moreover, all the usual properties of bounded acyclic chain complexes arising from diagram chasing arguments and homology arguments in $\mathcal{A}$ apply to bounded acyclic chain complexes in $\mathcal{N}$: for example, in a short exact sequence of bounded chain complexes in $\mathcal{N}$, if two of the terms are acyclic, then so is the third; a homotopy equivalence of bounded chain complexes is a quasi-isomorphism (for the definition, see Definition 2.6 below); a retract of a bounded acyclic chain complex is acyclic; a retract of a quasi-isomorphism between bounded chain complexes is a quasi-isomorphism.

Lemma 1.5. *Let $\mathcal{N}$ be an exact category. If $\mathcal{N}$ supports long exact sequences, then any retraction in $\mathcal{N}$ can be completed to a direct sum diagram.*

Proof. Given arrows p and i in $\mathcal{N}$ with $pi = 1$, we see that $0 \leftarrow \bullet \overset{p}{\leftarrow} \bullet \xleftarrow{1-ip} \bullet \overset{i}{\leftarrow} \bullet \leftarrow 0$ is an acyclic chain complex in $\mathcal{A}$ whose objects are in $\mathcal{N}$, and thus the image of the idempotent endomorphism $1 - ip$ lies in $\mathcal{N}$. $\square$

Thomason has shown that the converse of Lemma 1.5 is true, providing an intrinsic and simple characterization of the exact categories that support long exact sequences: see the proof of Corollary 6.5.

Any conjunction of properties of objects in an abelian category $\mathcal{A}$, each of which is closed under extensions and closed under kernels of epimorphisms (or closed under extensions and closed under cokernels of monomorphisms), specifies an exact subcategory of $\mathcal{A}$ that supports long exact sequences. Examples of such properties of R-modules include: being finitely generated, being finitely presented, being torsion free, flatness, projectivity, injectivity, being of finite projective dimension, being of projective dimension $\leq n$, being of flat dimension $\leq n$, and being stably free, but not being free (because there are stably free modules that are not free). The analogous local properties of quasicoherent sheaves on a scheme also provide examples.

Until further notice, we assume that the exact categories under consideration support long exact sequences. A separate argument will be given later to remove the assumption from the main results.

2. Chain complexes

In this section we review standard material about chain complexes.

Definition 2.1. For an exact category $\mathcal{N}$, let $C\mathcal{N}$ denote the category of bounded chain complexes in $\mathcal{N}$; the differential $d = d_N$ of a chain complex N is a map of degree -1 on the underlying graded object $gr\, N = (k \mapsto N_k)$ of N. We identify a chain complex N with the pair $(gr\, N, d)$. The short exact sequences of $C\mathcal{N}$ are defined to be those sequences $0 \to N' \to N \to N'' \to 0$ such that $0 \to N'_k \to N_k \to N''_k \to 0$ is a short exact sequence of $\mathcal{N}$ for each $k \in \mathbb{Z}$. The category $\mathcal{N}$ is embedded in $C\mathcal{N}$ as the subcategory of complexes concentrated in degree 0. For a graded object N of $\mathcal{N}$ and $i \in \mathbb{Z}$, we define the shifted graded object $N[i]$ to be

$k \mapsto N_{i+k}$. For $N \in C\mathcal{N}$ and $i \in \mathbb{Z}$, we define the shifted chain complex $N[i] \in C\mathcal{N}$ by shifting the underlying graded object and taking $(-1)^i d$ as the differential. For integers j and k, we let $C_{[j,k]}\mathcal{N}$ denote the full exact subcategory of $C\mathcal{N}$ whose objects are those objects N with $N_i = 0$ unless $j \leq i \leq k$; we say that such an N is *supported* on the interval $[j, k]$.

Remark 2.2. One sees that $C\mathcal{N}$ is an exact category. If $\mathcal{N} \hookrightarrow \mathcal{A}$ is an admissible embedding that shows that $\mathcal{N}$ supports long exact sequences, then $C\mathcal{N} \hookrightarrow C\mathcal{A}$ is an admissible embedding that shows that $C\mathcal{N}$ supports long exact sequences, too. The same statement holds for $C_{[j,k]}\mathcal{N}$.

Definition 2.3. Given a diagram $M \xleftarrow{f} N \xrightarrow{g} P$ in $C\mathcal{N}$ we define the *double mapping cylinder* $\mathbb{I}(M \xleftarrow{f} N \xrightarrow{g} P)$ by taking $gr\, M \oplus gr\, N[-1] \oplus gr\, P$ as the underlying graded object and equipping it with the differential

$$\begin{pmatrix} d_M & -f & 0 \\ 0 & -d_N & 0 \\ 0 & g & d_P \end{pmatrix}.$$

Definition 2.4. We introduce the *left* inclusion $\lambda : M \to \mathbb{I}(M \leftarrow N \to P)$ and the *right* inclusion $\rho : P \to \mathbb{I}(M \leftarrow N \to P)$. Moreover, given a commutative square

$$\begin{tikzcd} N \arrow[r, "g"] \arrow[d, "f"] & P \arrow[d, "s"] \\ M \arrow[r, "r"] & Q \end{tikzcd}$$

the matrix $\begin{pmatrix} r & 0 & s \end{pmatrix}$ defines a *projection* map $\pi_{r,s} : \mathbb{I}(M \xleftarrow{f} N \xrightarrow{g} P) \to Q$ satisfying the equations $\pi_{r,s}\lambda = r$ and $\pi_{r,s}\rho = s$.

Definition 2.5. Given a map $N \xrightarrow{g} P$ in $C\mathcal{N}$ we define the *mapping cone* $\mathbb{V}(g) := \mathbb{I}(0 \leftarrow N \xrightarrow{g} P)$.

Definition 2.6. We say that a map in $C\mathcal{N}$ is a *quasi-isomorphism* if its mapping cone is acyclic.

Definition 2.7. Given a map $N \xrightarrow{g} P$ in $C\mathcal{N}$ we define the *mapping cylinder* $\mathbb{T}(g) := \mathbb{I}(N \xleftarrow{1_N} N \xrightarrow{g} P)$. The left inclusion $\lambda : N \to \mathbb{T}(g)$ will be called the *front inclusion*, the right inclusion $\rho : P \to \mathbb{T}(g)$ will be called the *back inclusion*, and the projection $\pi_{g,1} : \mathbb{T}(g) \to P$ will be called the *projection*. The maps satisfy the equations $\pi_{g,1}\lambda = g$ and $\pi_{g,1}\rho = 1_P$.

We see that the projection and the back inclusion of a mapping cylinder are quasi-isomorphisms, because the complementary summand they split off from the mapping cylinder is the mapping cone of an identity map.

With $\mathbb{T}$ serving as the *cylinder functor*, we see that the category q of quasi-isomorphisms in $C\mathcal{N}$ is a *category of weak equivalences* in $C\mathcal{N}$ with the extension, saturation and cylinder axioms [18, (1.2) and (1.6)]. In particular, $qC\mathcal{N}$ is an exact category with weak equivalences (see Definition A.1).

We recall the following notation of [18]. If $w\mathcal{N}$ is an exact category with weak equivalences (Definition A.1), then let $\mathcal{N}^w$ denote the full subcategory of $\mathcal{N}$ consisting of the objects N such that $0 \rightarrowtail N$ is in w. Since w satisfies the extension axiom, $\mathcal{N}^w$ is closed under extensions in $\mathcal{N}$, and thus is an exact category; that is why we included the extension axiom in Definition A.1.

We introduce the alternative notation $C^q\mathcal{N}$ for the exact category $(C\mathcal{N})^q$ of acyclic bounded chain complexes in $\mathcal{N}$. If $\mathcal{N}$ supports long exact sequences, then so does $C^q\mathcal{N}$, because a short exact sequence of $C\mathcal{N}$ with two of its objects in $C^q\mathcal{N}$ has all three of its objects in $C^q\mathcal{N}$.

3. Binary chain complexes

Definition 3.1. A *binary chain complex* in $\mathcal{N}$ is a chain complex in $\mathcal{N}$ with an extra differential. In other words, it is a triple (N, d, d'), where N is a $\mathbb{Z}$-graded object of $\mathcal{N}$ with maps $d : N \to N[-1]$ (the *top* differential) and $d' : N \to N[-1]$ (the *bottom* differential) satisfying $d^2 = d'^2 = 0$. A *map* between two binary chain complexes is a map between the underlying graded objects that commutes with both differentials. Let $B\mathcal{N}$ denote the category of bounded binary chain complexes in $\mathcal{N}$. For integers j and k, let $B_{[j,k]}\mathcal{N}$ denote the full exact subcategory of $B\mathcal{N}$ whose objects are those objects N with $N_i = 0$ unless $j \leq i \leq k$.

We can regard a binary chain complex also as a pair of chain complexes with the same underlying graded object. In other words, it is a pair of pairs $((N, d), (N', d'))$ such that $N = N'$.

Definition 3.2. The diagonal map $\Delta : C\mathcal{N} \to B\mathcal{N}$, defined by $\Delta(N, d) := (N, d, d)$, is split by the *top* and *bottom* forgetful functors $\top, \bot : B\mathcal{N} \to C\mathcal{N}$, defined by $\top(N, d, d') := (N, d)$ and $\bot(N, d, d') := (N, d')$. We define $gr(N, d, d') := N$. We say that an object of $B\mathcal{N}$ is *acyclic*, or that an arrow of $B\mathcal{N}$ is a *quasi-isomorphism*, or that a sequence of objects of $B\mathcal{N}$ is *exact*, if and only if its image under the functor $(\top, \bot) : B\mathcal{N} \to C\mathcal{N} \times C\mathcal{N}$ has the same property. Let q denote the category of quasi-isomorphisms in $B\mathcal{N}$. There are unique notions of *double mapping cylinder* (2.3), *mapping cylinder* (2.7), and *mapping cone* (2.5) in $B\mathcal{N}$ preserved by $\top$ and $\bot$, for which we use the same notation as above ($\mathbb{I}$, $\mathbb{T}$ and $\mathbb{V}$).

The category $B\mathcal{N}$ is an exact category, and the category q of quasi-isomorphisms in $B\mathcal{N}$ is a category of weak equivalences with the extension, saturation and cylinder properties. We introduce the notation $B^q\mathcal{N} := (B\mathcal{N})^q$. If $\mathcal{N}$ supports long exact sequences, then so do $B\mathcal{N}$ and $B^q\mathcal{N}$, because a short exact sequence of $B\mathcal{N}$ with two of its objects in $B^q\mathcal{N}$ has all three of its objects in $B^q\mathcal{N}$.

4. The K-theory of binary chain complexes

For an exact category $\mathcal{N}$, we regard $K\mathcal{N}$ as a spectrum and similarly for an exact category with weak equivalences. See Definitions A.2 and A.3.

The map $K\mathcal{N} \to KqC\mathcal{N}$ arising from the embedding $\mathcal{N} \hookrightarrow C\mathcal{N}$ is a homotopy equivalence of spectra. The proof in [17, (1.11.7)] depends on an assumption, (1.11.3.1), that is slightly stronger than our assumption that $\mathcal{N}$ supports long exact sequences, but the proof goes through almost verbatim, so we don't repeat it here. The proof in [2] works for any exact category.

Definition 4.1. Let $\mathcal{C}$ be a category. Let $Ar\,\mathcal{C}$ denote the category of arrows in $\mathcal{C}$. If f is an arrow of $\mathcal{C}$, let $[f]$ denote the corresponding object of $Ar\,\mathcal{C}$. We may refer to it as a *pair* of objects of $\mathcal{C}$.

Definition 4.2. We regard the exact functor $iC^q\mathcal{N} \xrightarrow{\Delta} iB^q\mathcal{N}$ as an arrow in the category of exact categories with weak equivalences. Thus we may use Definition 4.1 to define $\Omega\mathcal{N} := [iC^q\mathcal{N} \xrightarrow{\Delta} iB^q\mathcal{N}]$ and $\Omega_{[j,k]}\mathcal{N} := [iC^q_{[j,k]}\mathcal{N} \xrightarrow{\Delta} iB^q_{[j,k]}\mathcal{N}]$. They are objects in the category of arrows in the category of exact categories with weak equivalences, or, more briefly, they are pairs of exact categories.

The relative K-theory spectrum $K\Omega\mathcal{N}$ (see Definition A.4) is the factor of $KB^q\mathcal{N}$ complementary to the "image" of $K\Delta : KC^q\mathcal{N} \to KB^q\mathcal{N}$. It is our main object of study, which begins with the following theorem.

Theorem 4.3. *If $\mathcal{N}$ is an exact category that supports long exact sequences, then there is a natural homotopy equivalence*

$$K\Omega\mathcal{N} \sim \Omega K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}].$$

Proof. Here "natural" means functorial in $\mathcal{N}$, and "homotopy equivalence" means a sequence of homotopy equivalences in various directions (a "zigzag"). We use ΩX, where X is a spectrum, as notation for the shifted spectrum $X[-1]$. Waldhausen's fibration theorem (see [18, (1.6.4)] or [17, (1.8.2)]), generalized from spaces to spectra in the usual way, yields the homotopy Cartesian squares in the following diagram (the spectra in their lower left-hand corners are contractible):

$$\begin{tikzcd} KiC^q\mathcal{N} \arrow[r] \arrow[d] & KiC\mathcal{N} \arrow[d] \\ KqC^q\mathcal{N} \arrow[r] & KqC\mathcal{N}, \end{tikzcd} \qquad \begin{tikzcd} KiB^q\mathcal{N} \arrow[r] \arrow[d] & KiB\mathcal{N} \arrow[d] \\ KqB^q\mathcal{N} \arrow[r] & KqB\mathcal{N}. \end{tikzcd}$$

The functor Δ provides a map from the left square to the right square, yielding the following homotopy cartesian square on relative K-theory, with contractible lower left-hand corner:

$$\begin{tikzcd} K[iC^q\mathcal{N} \xrightarrow{\Delta} iB^q\mathcal{N}] \arrow[r] \arrow[d] & K[iC\mathcal{N} \xrightarrow{\Delta} iB\mathcal{N}] \arrow[d] \\ K[qC^q\mathcal{N} \xrightarrow{\Delta} qB^q\mathcal{N}] \arrow[r] & K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}]. \end{tikzcd}$$

The map $KiC\mathcal{N} \xrightarrow{K\Delta} KiB\mathcal{N}$ is a homotopy equivalence, because the additivity theorem and compatibility with inductive limits apply to show, using the naive admissible filtration that any chain complex or binary chain complex has, that its source and target are compatibly homotopy equivalent to $Ki\coprod_{\mathbb{Z}}\mathcal{N}$. Thus the upper right corner of the square above is also contractible, providing the desired homotopy equivalence. Introducing the abbreviations

$$\begin{aligned} i\Delta\mathcal{N} &:= [iC\mathcal{N} \xrightarrow{\Delta} iB\mathcal{N}], \\ i\Delta^q\mathcal{N} &:= [iC^q\mathcal{N} \xrightarrow{\Delta} iB^q\mathcal{N}] = \Omega\mathcal{N}, \\ q\Delta\mathcal{N} &:= [qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}], \\ q\Delta^q\mathcal{N} &:= [qC^q\mathcal{N} \xrightarrow{\Delta} qB^q\mathcal{N}], \end{aligned}$$

we may express the homotopy equivalence as the following sequence of natural homotopy equivalences: $\Omega K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}] = \Omega K q\Delta\mathcal{N} \xrightarrow{\sim} \Omega K[i\Delta\mathcal{N} \to q\Delta\mathcal{N}] \xleftarrow{\sim} \Omega K[i\Delta^q\mathcal{N} \to q\Delta^q\mathcal{N}] \xrightarrow{\sim} \Omega K[i\Delta^q\mathcal{N} \to 0] \cong K i\Delta^q\mathcal{N} = K\Omega\mathcal{N}$. Alternatively, a similar natural zigzag of homotopy equivalences could be constructed using any functorial construction of homotopy fibers or homotopy cofibers of maps of spectra, thereby avoiding the use of multi-relative K-theory. ☐

Remark 4.4. The naive filtration of a chain complex does not preserve quasi-isomorphisms; thus it cannot be used to show that $KqC\mathcal{N} \xrightarrow{K\Delta} KqB\mathcal{N}$ is a homotopy equivalence. But it can be used to show that $K_0qC\mathcal{N} \xrightarrow{K_0\Delta} K_0qB\mathcal{N}$ is an isomorphism, a fact that follows from the theorem above and the fact that Δ has a left inverse (e.g., $\perp$). The isomorphism is compatible with $\pi_{-1}K\Omega\mathcal{N} = 0$.

Corollary 4.5. *There is a natural homotopy equivalence*

$$K\Omega\mathcal{N} \sim \Omega^2 K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}].$$

Proof. We have a commutative diagram

$$\begin{tikzcd}
KqC\mathcal{N} \arrow[r, "K\Delta"] \arrow[dr, "1"'] & KqB\mathcal{N} \arrow[r] \arrow[d, "K\top"] & K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}] \\
& KqC\mathcal{N} \arrow[d] & \\
& K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}] &
\end{tikzcd}$$

involving two fibration sequences up to homotopy, which by a diagram chase, provides a homotopy equivalence $\Omega K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}] \xrightarrow{\sim} K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}]$, yielding the result.

To make the naturality clear, we realize the homotopy equivalence explicitly as the composition of the equivalences and isomorphisms in the following sequence, which involves K-theory of 2-dimensional cubes of exact categories, mentioned in the appendix:

$$\Omega^2 K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}] \cong \Omega^2 K \left[\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ qB\mathcal{N} \arrow[r, "\top"] & qC\mathcal{N} \end{tikzcd} \right]$$

$$\xrightarrow{\sim} \Omega^2 K \left[\begin{tikzcd} qC\mathcal{N} \arrow[r, "1"] \arrow[d, "\Delta"] & qC\mathcal{N} \arrow[d, "1"] \\ qB\mathcal{N} \arrow[r, "\top"] & qC\mathcal{N} \end{tikzcd} \right] \xrightarrow{\sim} \Omega^2 K \left[\begin{tikzcd} qC\mathcal{N} \arrow[r] \arrow[d, "\Delta"] & 0 \arrow[d] \\ qB\mathcal{N} \arrow[r] & 0 \end{tikzcd} \right]$$

$$\cong \Omega K[qC\mathcal{N} \xrightarrow{\Delta} qB\mathcal{N}] \sim K\Omega\mathcal{N}.$$

☐

Let t denote the category of maps f in $B\mathcal{N}$ such that $\top f$ is in q, and let b denote the category of maps f in $B\mathcal{N}$ such that $\perp f$ is in q. They are categories of weak equivalences satisfying the extension, saturation, and cylinder axioms. If $\mathcal{N}$ supports long exact sequences, then so do $B^t\mathcal{N}$ (and $B^b\mathcal{N}$), because a short exact sequence of $B\mathcal{N}$ with two of its objects in $B^t\mathcal{N}$ has all three of its objects in $B^t\mathcal{N}$.

Definition 4.6. Suppose $\mathcal{N}$ is an exact category. Suppose we are given objects N and P of $B\mathcal{N}$, and a pair of maps $r : \top N \to \top P$ and $r' : \bot N \to \bot P$ in $C\mathcal{N}$. Then we define the *binary mapping cylinder* $\mathbb{T}(r, r') := (\mathbb{T}(r), \mathbb{T}(r')) \in B\mathcal{N}$, by observing that the two mapping cylinders $\mathbb{T}(r)$ and $\mathbb{T}(r')$ have the same underlying graded object, namely $gr\, N \oplus gr\, N[-1] \oplus gr\, P$. The two front inclusions $\top N \rightarrowtail \mathbb{T}(r)$ and $\bot N \rightarrowtail \mathbb{T}(r')$ agree on underlying graded objects, because they are both given by the inclusion $gr\, N \rightarrowtail gr\, N \oplus gr\, N[-1] \oplus gr\, P$ of the first factor, and thus they combine to give a map $\lambda : N \rightarrowtail \mathbb{T}(r, r')$, which we call the *front inclusion*; $\top\lambda$ is a quasi-isomorphism if r is, and $\bot\lambda$ is a quasi-isomorphism if r' is. Similarly, we have the *back inclusion* $\rho : P \rightarrowtail \mathbb{T}(r, r')$, which is a quasi-isomorphism. If $r = r'$, then they give rise to a unique map $\tilde{r} = (r, r') : N \to P$, and the binary mapping cylinder reduces to the mapping cylinder: $\mathbb{T}(r, r') = \mathbb{T}(\tilde{r})$.

Remark 4.7. In the context of the definition above, the projections $\pi_{r,1} : \mathbb{T}(r) \to \top P$ and $\pi_{r',1} : \mathbb{T}(r') \to \bot P$ for the two mapping cylinders do not agree on the underlying graded objects, because their matrices are $\begin{pmatrix} r & 0 & 1_P \end{pmatrix}$ and $\begin{pmatrix} r' & 0 & 1_P \end{pmatrix}$. Thus generally there is no projection $\mathbb{T}(r, r') \to P$ in the category $B\mathcal{N}$.

Theorem 4.8. *If $\mathcal{N}$ is an exact category that supports long exact sequences, then the map $K\top : KtB\mathcal{N} \to KqC\mathcal{N}$ is a homotopy equivalence.*

Proof. We consider the exact functor $F : qC\mathcal{N} \to tB\mathcal{N}$ defined by $F(N, d) := (N, d, 0)$. The composite $\top \circ F$ is the identity functor. The composite $F \circ \top$ is the exact endofunctor of $tB\mathcal{N}$ given by $(N, d, d') \mapsto (N, d, 0)$. Use the binary mapping cylinder of Definition 4.6 to define another exact endofunctor G of $tB\mathcal{N}$ by $G(N, d, d') := \mathbb{T}((N, d) \xrightarrow{1} (N, d), (N, d') \xrightarrow{0} (N, 0))$. Observe that $\top G(N, d, d')$ is the mapping cylinder of an identity map, so its front inclusion is a quasi-isomorphism. Thus, the front and back inclusions $1 \xrightarrow[\sim]{\lambda} G \xleftarrow[\sim]{\rho} F \circ \top$ provide weak equivalences of $tB\mathcal{N}$, i.e., arrows of t, and thus, according to [18, Prop. 1.3.1], induce homotopies on K-theory, rendering $KF \circ K\top$ homotopic to the identity and showing that KF is a homotopy inverse for $K\top$. $\square$

Corollary 4.9. *If $\mathcal{N}$ is an exact category that supports long exact sequences, then there is a natural homotopy equivalence $K\Omega\mathcal{N} \sim \Omega KqB^t\mathcal{N}$.*

Proof. We apply Waldhausen's fibration theorem (see [18, (1.6.4)] or [17, (1.8.2)]) to obtain the fibration sequence in the bottom row of the following diagram:

$$\begin{tikzcd}
 & KqB\mathcal{N} \arrow[r, "K\top"] & KqC\mathcal{N} \arrow[r] & K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}] \\
KqB^t\mathcal{N} \arrow[r] & KqB\mathcal{N} \arrow[r] \arrow[u, "1"'] & KtB\mathcal{N}. \arrow[u, "K\top", "\sim"']
\end{tikzcd}$$

The top row is also a fibration sequence. The right-hand vertical map is a homotopy equivalence (by the theorem), and thus the vertical maps induce a homotopy equivalence on the homotopy fibers, yielding a homotopy equivalence $KqB^t\mathcal{N} \sim \Omega K[qB\mathcal{N} \xrightarrow{\top} qC\mathcal{N}]$, which can be expressed as a natural zigzag. Applying Corollary 4.5 yields the result. $\square$

5. Stably potentially acyclic graded objects

The following relation was used in the proof of the cofinality theorem [3, (1.1)].

Definition 5.1. An additive functor $F : \mathcal{M} \to \mathcal{N}$ between additive categories is called *cofinal* if for any object $N \in \mathcal{N}$, there exists an object $N' \in \mathcal{N}$ and an object $M \in \mathcal{M}$, such that $N \oplus N' \cong FM$. We say that objects N and N' of $\mathcal{N}$ are *equivalent modulo the image of* F if and only if there exist objects M and M' of $\mathcal{M}$ such that $N \oplus FM \cong N' \oplus FM'$.

Equivalence modulo the image of F is an equivalence relation, coarser than isomorphism.

Lemma 5.2. *Suppose an additive functor $F : \mathcal{M} \to \mathcal{N}$ between additive categories is cofinal. The set G_F of equivalence classes $\langle N\rangle$ modulo the image of F, associated to objects $N \in \mathcal{N}$, together with the binary operation defined by $\langle N\rangle + \langle N'\rangle := \langle N \oplus N'\rangle$, is an abelian group.*

Proof. Write $N \sim N'$ for the relation. One checks that the binary operation $\langle N\rangle + \langle N'\rangle$ is well defined, with $0 := \langle 0\rangle$ serving as the identity element. The operation is commutative and associative, because direct sum in $\mathcal{N}$ is commutative and associative up to isomorphism. Cofinality shows that additive inverses exist, and thus G_F is an abelian group. $\square$

Definition 5.3. For an exact category $\mathcal{N}$, we let $Gr\,\mathcal{N}$ denote the exact category of bounded $\mathbb{Z}$-graded objects of $\mathcal{N}$. An object $N \in Gr\,\mathcal{N}$ is called *potentially* acyclic if there is a differential d on N that makes the resulting chain complex (N, d) acyclic. It is called *stably* potentially acyclic if there is a potentially acyclic graded object N' such that $N \oplus N'$ is potentially acyclic. Let $\chi(N)$ denote the class $\sum_i(-1)^i[N_i]$ in $K_0\mathcal{N}$.

Lemma 5.4. *A graded object N of an exact category $\mathcal{N}$ is stably potentially acyclic if and only if $\chi(N) = 0$ in $K_0\mathcal{N}$.*

Proof. We consider equivalence modulo the image of the forgetful functor $F : C^q\mathcal{N} \to Gr\,\mathcal{N}$. In other words, $N \sim N'$ if there exist potentially acyclic objects X and X' such that $N \oplus X \cong N' \oplus X'$. Since $N \oplus N[-1]$ is always potentially acyclic, F is cofinal, and Lemma 5.2 applies to show that the group G_F of equivalence classes $\langle N\rangle$ is an abelian group, in which $\langle N[-1]\rangle = -\langle N\rangle$. Observe that $\langle N\rangle = 0$ if and only if N is stably potentially acyclic.

An object of $\mathcal{N}$ is regarded as a graded object by concentrating it in degree 0. If $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of graded objects, then $N' \oplus N[1] \oplus N''[2]$ is potentially acyclic, allowing us to conclude that $\langle N'\rangle + \langle N''\rangle = \langle N\rangle$ and providing a well-defined map $j : K_0\mathcal{N} \to G_F$. Observe that $\langle N[i]\rangle = (-1)^i\langle N\rangle$ and thus $\langle N\rangle = \langle \bigoplus_i N_i[-i]\rangle = \sum_i\langle N_i[-i]\rangle = \sum_i(-1)^i\langle N_i\rangle$. Thus the map $\chi : G_F \to K_0\mathcal{N}$ defined by $\chi\langle N\rangle := \chi(N)$ is a well-defined isomorphism inverse to j. We see that $\langle N\rangle = 0$ if and only if $\chi\langle N\rangle = 0$, yielding the result. $\square$

Remark 5.5. As an aside, we mention that the proof of Lemma 5.4 can be made into an effective computational tool that converts a proof of an equation $\chi(N) = 0$ in $K_0\mathcal{N}$ into a proof that N is stably potentially acyclic. In the case where $N = 0$, the result is a graded object N', as in Definition 5.3, that is made acyclic in two ways; i.e., the result is an acyclic binary complex.

Definition 5.6. Suppose that $\mathcal{N}$ is an exact category that supports long exact sequences. Define x to be the subcategory of $C\mathcal{N}$ whose arrows $N \to N'$ are those satisfying $\chi(N) = \chi(N')$.

Definition 5.7. For $t \in \mathbb{Z}$ write V^tX for the stage of the Postnikov filtration of a spectrum X obtained from X by killing $\pi_s X$ for all $s < t$. In particular, V^0X is the connective part of X.

Remark 5.8. The category x is a category of weak equivalences satisfying the extension, saturation, and cylinder axioms. We introduce the notation $C^x\mathcal{N} := (C\mathcal{N})^x$. Its objects are the chain complexes N satisfying $\chi(N) = 0$. It is an exact category that supports long exact sequences, because a short exact sequence of $C\mathcal{N}$ with two of its objects in $C^x\mathcal{N}$ has all three of its objects in $C^x\mathcal{N}$. Thomason's cofinality theorem [17, (1.10.1)] provides a fibration sequence of spectra up to homotopy:

$$KqC^x\mathcal{N} \to KqC\mathcal{N} \to \text{“}K_0\mathcal{N}\text{”},$$

where “G” denotes the Eilenberg-MacLane spectrum whose only nonvanishing homotopy group is a G in dimension 0. (Presumably Thomason was working with the stable model structure of [1, Theorem 2.3].)

Thus $K_0qC^x\mathcal{N} = 0$ and $K_iqC^x\mathcal{N} \cong K_iqC\mathcal{N} \cong K_i\mathcal{N}$ for $i > 0$. We see that Thomason's theorem says that $KqC^x\mathcal{N} \sim V^1KqC\mathcal{N}$.

Observe that if $(N, d, d') \in B^b\mathcal{N}$, then $(N, d) \in C^x\mathcal{N}$, because acyclicity of (N, d') shows that $\chi(N) = 0$. Hence $\top$ induces an exact functor $\top : tB^b\mathcal{N} \to qC^x\mathcal{N}$.

Theorem 5.9. *If $\mathcal{N}$ is an exact category that supports long exact sequences, then the map $K\top : KtB^b\mathcal{N} \to KqC^x\mathcal{N}$ is a homotopy equivalence.*

Proof. We apply Waldhausen's approximation theorem (see [18, (1.6.7)] or [17, (1.9.1)]) to the map $\top : tB^b\mathcal{N} \to qC^x\mathcal{N}$; its conclusion is what we want for the spaces at level 1 of the Ω-spectra involved here. As required, $tB^b\mathcal{N}$ satisfies the saturation axiom and the cylinder axiom, and $qC^x\mathcal{N}$ satisfies the saturation axiom. The required *approximation property* consists of two parts. The first part, that a map f in $B^b\mathcal{N}$ is in t if and only if $\top f$ is in q, is the definition of t.

The second part of the approximation property states that given an object $N \in B^b\mathcal{N}$ and a map $r : \top N \to P$ in $C^x\mathcal{N}$, there exists a cofibration (admissible monomorphism) $s : N \rightarrowtail T$ in $B^b\mathcal{N}$ and a quasi-isomorphism $u : \top T \xrightarrow{\sim} P$ so that $u \circ \top s = r$.

Since $\chi(P) = 0$, by Lemma 5.4 we can find a potentially acyclic graded object Y of $\mathcal{N}$ such that $gr\,P \oplus Y$ is potentially acyclic. Let y be an acyclic differential on Y, and let z be an acyclic differential on $gr\,P \oplus Y$. Observe that the inclusion $in_1 : P \rightarrowtail P \oplus (Y, y)$ and the projection $pr_1 : P \oplus (Y, y) \twoheadrightarrow P$ are quasi-isomorphisms. Define $r' := in_1 \circ r$. We use the binary mapping cylinder of Definition 4.6 to define $T := \mathbb{T}(\top N \xrightarrow{r'} P \oplus (Y, y), \perp N \xrightarrow{0} (gr\,P \oplus Y, z))$; the underlying graded objects of the sources and targets of the two maps are the same, as required. We take $s : N \rightarrowtail T$ to be its front inclusion, which is an admissible monomorphism, and we take $u' := \pi_{r',1} : \top T = \mathbb{T}(r') \to P \oplus (Y, y)$ to be the projection, which is a quasi-isomorphism. We compute $u' \circ \top s = \pi_{r',1}\lambda = r'$, using an identity given in Definition 2.7. The map $u := pr_1 \circ u'$ is a quasi-isomorphism, because pr_1 and u' are.

Since $\perp T$ is the mapping cylinder of a map whose target, $(gr\, P \oplus Y, z)$, is acyclic, it is acyclic, and hence T is an object of $B^b\mathcal{N}$, as required. We compute $u \circ \top s = pr_1 \circ u' \circ \top s = pr_1 \circ r' = pr_1 \circ in_1 \circ r = r$, thereby satisfying the second part of the approximation property and completing the proof. ☐

Corollary 5.10. *If $\mathcal{N}$ is an exact category that supports long exact sequences, then there is a natural homotopy equivalence of spectra $K\Omega\mathcal{N} \sim V^0\Omega K\mathcal{N}$, as well as a natural homotopy equivalence of spaces $K\Omega\mathcal{N}\langle 1\rangle \sim V^1\Omega(K\mathcal{N}\langle 1\rangle)$, where V^1 is the functor that takes the connected component of the base point of a space.*

Proof. Thomason's cofinality theorem (see Remark 5.8) shows that $\Omega KqC^x\mathcal{N} \sim \Omega V^1 K\mathcal{N} \sim V^0\Omega K\mathcal{N}$. Interchanging top and bottom differentials gives an isomorphism $KqB^b\mathcal{N} \cong KqB^t\mathcal{N}$. Observe that $qB^b\mathcal{N} = tB^b\mathcal{N}$, because any map between acyclic chain complexes is a quasi-isomorphism. Now we combine Corollary 4.9 with Theorem 5.9.

In other words, we assemble the following sequence of natural homotopy equivalences, equalities, and isomorphisms into a natural zigzag: $K\Omega\mathcal{N} \sim \Omega KqB^t\mathcal{N} \cong \Omega KqB^b\mathcal{N} = \Omega KtB^b\mathcal{N} \sim \Omega KqC^x\mathcal{N} \sim \Omega V^1 K\mathcal{N} \sim V^0\Omega K\mathcal{N}$.

The remark about K-theory spaces follows from the observation that our model for the K-theory spectrum is an Ω-spectrum from level 1 onward. ☐

6. Passing to arbitrary exact categories

Now we drop the standing assumption that the exact categories under consideration support long exact sequences and begin working to generalize Corollary 5.10 to arbitrary exact categories. In that generality, the notion of quasi-isomorphism is not useful, for it may not be true that the composite of two quasi-isomorphisms is a quasi-isomorphism.

The full subcategory $C^q\mathcal{N} \subseteq C\mathcal{N}$ of acyclic complexes is closed under extensions. To see that, consider a short exact sequence $0 \to (N', d') \to (N, d) \to (N'', d'') \to 0$ of $C\mathcal{N}$ with (N', d') and (N'', d'') in $C^q\mathcal{N}$. Let $\mathcal{N} \hookrightarrow \mathcal{A}$ be an admissible embedding into an abelian category (see Definition 1.3). By a diagram chase, (N, d) is acyclic in $\mathcal{A}$, because (N', d') and (N'', d'') are. By induction on n and the 3×3 lemma one sees that $0 \to \operatorname{im} d'_n \to \operatorname{im} d_n \to \operatorname{im} d''_n \to 0$ is exact in $\mathcal{A}$ for each n. By assumption, $\operatorname{im} d'_n$ and $\operatorname{im} d''_n$ are in $\mathcal{N}$ (up to isomorphism), so by admissibility, $\operatorname{im} d_n$ is also in $\mathcal{N}$ (up to isomorphism). Hence $(N, d) \in C^q\mathcal{N}$.

It follows that $C^q\mathcal{N}$ is an exact category. By the same reasoning, considering the top and bottom differentials separately, $B^q\mathcal{N}$ is an exact category.

Recall from [3, §1] that a full subcategory $\mathcal{M} \subseteq \mathcal{N}$ closed under extensions in $\mathcal{N}$ (see Definition 1.2) is called *cofinal* in $\mathcal{N}$ if for any $N \in \mathcal{N}$ there exists $N' \in \mathcal{N}$ such that $N \oplus N' \in \mathcal{M}$. We may harmlessly modify the definition to require only that $N \oplus N'$ is isomorphic to an object of $\mathcal{M}$; direct sums are defined only up to isomorphism, so the modification is an improvement. (See also Definition 5.1.) For such a functor, the cofinality theorem [3, (1.1)] (or [4]) states that the map $K_i\mathcal{M} \to K_i\mathcal{N}$ is an isomorphism for $i > 0$ and an injection for $i = 0$. The following lemma documents part of the proof there.

Lemma 6.1. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ is a cofinal full subcategory closed under extensions. Then for any objects $N_1, \ldots, N_k$ with the same class in* $\operatorname{coker}(K_0\mathcal{M} \to K_0\mathcal{N})$*, there is an object $N' \in \mathcal{N}$ such that $N_i \oplus N'$ is isomorphic to an object of $\mathcal{M}$, for all i.*

We use it in the next lemma.

Lemma 6.2. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ is a cofinal full subcategory closed under extensions. Consider $\mathcal{M}$ to be an exact category by equipping it with the sequences of $\mathcal{M}$ that are short exact sequences of $\mathcal{N}$. Then for any $(N, d, d') \in B^q\mathcal{N}$ there is an object $(L, e) \in C^q\mathcal{N}$ such that $(N, d, d') \oplus (L, e, e) \in B^q\mathcal{M}$ (up to isomorphism).*

Proof. For any k, the identity $[\operatorname{im} d_k] = \sum_{i=k}^{\infty}(-1)^{i-k}[N_i] = [\operatorname{im} d'_k]$ in $K_0\mathcal{N}$ allows us to apply Lemma 6.1 to find $N'_k \in \mathcal{N}$ such that $\operatorname{im} d_k \oplus N'_k \in \mathcal{M}$ (up to isomorphism) and $\operatorname{im} d'_k \oplus N'_k \in \mathcal{M}$ (up to isomorphism). Let

$$L = \bigoplus_k (N'_k[-k] \oplus N'_k[-(k-1)]),$$

equipped with the differential e derived from the identity maps $N_k \to N_k$, making (L, e) acyclic. We see that $\operatorname{im}(d \oplus e)_k \cong \operatorname{im} d_k \oplus N'_k \in \mathcal{M}$ and $\operatorname{im}(d' \oplus e)_k \cong \operatorname{im} d'_k \oplus N'_k \in \mathcal{M}$. Since the images of the differentials of $(N \oplus L, d \oplus e, d' \oplus e)$ are in $\mathcal{M}$, and $\mathcal{M}$ is closed under extensions in $\mathcal{N}$, it follows that the terms of $N \oplus L$ are in $\mathcal{M}$ (up to isomorphism), too, since each term is an extension of two images, yielding the result. ☐

Corollary 6.3. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ is a cofinal full subcategory closed under extensions, regarded as an exact category as in Lemma* 6.2*. Then the same is true for $C^q\mathcal{M} \subseteq C^q\mathcal{N}$ and $B^q\mathcal{M} \subseteq B^q\mathcal{N}$.*

Corollary 6.4. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ is a cofinal full subcategory closed under extensions, regarded as an exact category as in Lemma* 6.2*. Then the map $K\Omega\mathcal{M} \to K\Omega\mathcal{N}$ is a homotopy equivalence.*

Proof. It is enough to show that $K_i\Omega\mathcal{M} \to K_i\Omega\mathcal{N}$ is an isomorphism for each i. It is a direct summand of the map $K_iB^q\mathcal{M} \to K_iB^q\mathcal{N}$ (because Δ is split by $\perp$), so is an isomorphism for $i > 0$ and is injective for $i = 0$, by the cofinality theorem. It is also surjective for $i = 0$, because Lemma 6.2 implies that $K_0B^q\mathcal{M} \oplus K_0C^q\mathcal{N} \to K_0B^q\mathcal{N}$ is surjective. ☐

Corollary 6.5. *If $\mathcal{M}$ is an exact category, then there is a natural homotopy equivalence of spectra $K\Omega\mathcal{M} \sim V^0\Omega K\mathcal{M}$.*

Proof. According to Thomason [17, (A.9.2)], there is a natural procedure, which adds images for the idempotent maps, that embeds $\mathcal{M}$ as a full subcategory, closed under extensions, in an exact category $\mathcal{N}$ that supports long exact sequences. (The appropriate admissible embedding $\mathcal{N} \hookrightarrow \mathcal{A}$ to use is the Gabriel-Quillen embedding, which satisfies an even stronger property [17, (1.11.3.1)]: that a map of $\mathcal{N}$ sent to an epimorphism of $\mathcal{A}$ is an admissible epimorphism of $\mathcal{N}$; for the proof, see [17, (A.7.16(b))].) Since $\mathcal{M}$ is cofinal in $\mathcal{N}$, the map $K\Omega\mathcal{M} \to K\Omega\mathcal{N}$ is a natural homotopy equivalence, by Corollary 6.4, and $V^0\Omega K\mathcal{M} \to V^0\Omega K\mathcal{N}$ is a natural homotopy equivalence, by the cofinality theorem, yielding the result from Corollary 5.10. ☐

Remark 6.6. It would be good to have a construction of the map $K\Omega\mathcal{M} \to \Omega K\mathcal{M}$, for any exact category $\mathcal{M}$ (which might involve formally inverting some homotopy equivalences), without first adding images for idempotent maps.

Remark 6.7. Let $\mathcal{M}$ and $\mathcal{N}$ be as in the proof of Corollary 6.5, and let M be a bounded chain complex of $\mathcal{M}$ that is acyclic in $\mathcal{N}$. The kernels $Z_i \in \mathcal{N}$ of the differentials $d_i : M_i \to M_{i-1}$ all lie in $\operatorname{coker}(K_0\mathcal{M} \to K_0\mathcal{N})$, and all but finitely many of them are zero, so by Lemma 6.1 there is an object X such that $Z_i \oplus X \in \mathcal{M}$ (up to isomorphism), for all i. Since $Z_i = 0$ for some i, it follows that X is isomorphic to an object of $\mathcal{M}$, and thus we may assume $X \in \mathcal{M}$. Hence there is a bounded acyclic chain complex M' of $\mathcal{M}$, obtained as a direct sum of shifts of the chain complex $[\cdots \to 0 \to X \xrightarrow{1} X \to 0 \to \cdots]$, such that $M \oplus M'$ is isomorphic to an acyclic chain complex of $\mathcal{M}$.

Thus it seems that Definition 1.1 could be improved, for an arbitrary exact category $\mathcal{M}$, by saying that M is *acyclic* if it is a direct summand of a chain complex obtainable by splicing short exact sequences together. It might also be a good idea to require the complementary summand to arise by splicing short exact sequences. Rewriting this paper from that point of view might be possible, but the absence of the kernels of the differentials of M would prevent some arguments in this paper and in [17] from going through verbatim. See [2] for an idea about how to deal with that.

An alternative approach would be to try to use Schlichting's definition of an acyclic chain complex [12, Section 3.1.3]. A chain complex called "acyclic" here is called "strictly acyclic" there, and an "acyclic" chain complex is defined there to be one homotopy equivalent to a strictly acyclic chain complex. Schlichting's definition is based on the observation of Neeman and of Keller that a bounded complex becomes zero in the derived category of $\mathcal{M}$ if and only if it is acyclic in Schlichting's sense.

7. Iteration of the construction

Because $\Omega\mathcal{N}$ is a pair of exact categories, we may iterate the construction of $\Omega\mathcal{N}$ from $\mathcal{N}$. Iterating n times yields an n-dimensional cubical diagram $\Omega^n\mathcal{N}$ of exact categories; its multi-relative K-theory spectrum $K\Omega^n\mathcal{N}$ is the n-fold iterated homotopy cofiber of the corresponding cube of K-theory spectra and is considered in the following corollary.

Corollary 7.1. *If $\mathcal{N}$ is an exact category, then there is a natural homotopy equivalence of spectra $K\Omega^n\mathcal{N} \sim V^0\Omega^n K\mathcal{N}$, for all $n \geq 0$.*

Proof. The case where $n = 0$ amounts to the statement that $K\mathcal{N}$ is connective, which is true. Arguing by induction for $n \geq 1$, we may assume that there is a natural homotopy equivalence $K\Omega^{n-1}\mathcal{N} \sim V^0\Omega^{n-1}K\mathcal{N}$. It is implemented by a natural zigzag of homotopy equivalences in various directions, where naturality means that the spectra and maps in the zigzag are functorial in the exact category $\mathcal{N}$. We will apply the assumption to the source and target of the arrow $\Omega\mathcal{N} = [C^q\mathcal{N} \xrightarrow{\Delta} B^q\mathcal{N}]$. The map $K\Delta$ in the homotopy fibration sequence

$$KC^q\mathcal{N} \xrightarrow{K\Delta} KB^q\mathcal{N} \to K\Omega\mathcal{N}$$

is split (by $K\top$), so

$$V^0\Omega^{n-1}KC^q\mathcal{N} \xrightarrow{K\Delta} V^0\Omega^{n-1}KB^q\mathcal{N} \to V^0\Omega^{n-1}K\Omega\mathcal{N}$$

is also a homotopy fibration sequence; the splitting is needed to ensure that the long exact sequence of homotopy groups remains exact when all the negative homotopy groups are replaced by 0, and the exactness can be used to prove that the sequence remains a homotopy fibration sequence. We combine it with the homotopy fibration sequence that characterizes the multi-relative K-theory spectrum $K\Omega^n\mathcal{N}$ in the following diagram:

$$\begin{tikzcd}
K\Omega^{n-1}C^q\mathcal{N} \arrow[r, "K\Delta"] \arrow[d, leftrightsquigarrow] & K\Omega^{n-1}B^q\mathcal{N} \arrow[r] \arrow[d, leftrightsquigarrow] & K\Omega^n\mathcal{N} \\
V^0\Omega^{n-1}KC^q\mathcal{N} \arrow[r, "K\Delta"] & V^0\Omega^{n-1}KB^q\mathcal{N} \arrow[r] & V^0\Omega^{n-1}K\Omega\mathcal{N}.
\end{tikzcd}$$

The naturality in the inductive assumption, applied to the exact functor $C^q\mathcal{N} \xrightarrow{\Delta} B^q\mathcal{N}$ of exact categories, gives a zigzag of commutative squares connecting the two vertical zigzags of homotopy equivalences in the diagram, yielding a natural zigzag of homotopy equivalences $K\Omega^n\mathcal{N} \sim V^0\Omega^{n-1}K\Omega\mathcal{N}$ between the two cofibers. Corollary 6.5 gives a natural homotopy equivalence $V^0\Omega^{n-1}K\Omega\mathcal{N} \sim V^0\Omega^{n-1}V^0\Omega K\mathcal{N}$. Finally, the homotopy equivalence $V^0\Omega^{n-1}V^0\Omega K\mathcal{N} \xrightarrow{\simeq} V^0\Omega^n K\mathcal{N}$ yields the result. $\square$

Corollary 7.2. *If $\mathcal{N}$ is an exact category, then there is a natural isomorphism of groups $K_n\mathcal{N} \cong K_0\Omega^n\mathcal{N}$.*

In the next corollary, we unwind the right-hand side of the isomorphism of Corollary 7.2 in order to make the presentation of the abelian group $K_n\mathcal{N}$ it provides explicit.

Definition 7.3. An *acyclic binary multicomplex of dimension n* in $\mathcal{N}$ is an object of the exact category $(B^q)^n\mathcal{N}$; i.e., it is a bounded $\mathbb{Z}^n$-graded object of $\mathcal{N}$ equipped with acyclic differentials d_i and d'_i in direction i, for each i with $1 \le i \le n$, where the maps d_i and d'_i commute with d_j and d'_j provided $i \neq j$. (Four pairs of maps have just been asserted to commute: d_i commutes with d_j, d_i commutes with d'_j, d'_i commutes with d_j, and d'_i commutes with d'_j.) A map between acyclic binary multicomplexes is a map of underlying graded objects that commutes with each of the differentials, and a short exact sequence is a short sequence consisting of short exact sequences in each component.

Corollary 7.4. *If $\mathcal{N}$ is an exact category, then the abelian group $K_n\mathcal{N}$ has the following presentation by generators and relations. There is one generator for each acyclic binary multicomplex of dimension n in $\mathcal{N}$. The relations are of two types: (1) those coming from short exact sequences, as in the definition of the Grothendieck group; (2) any generator that is* trivial, *in the sense that $d_i = d'_i$ for some i, is made to vanish.*

Proof. Let X be a cube of connective spectra, let X_v be the spectrum at the terminal vertex v of the cube, and let $\bar{X}$ denote the iterated homotopy cofiber of X. One sees that the abelian group $\pi_0\bar{X}$ is isomorphic to the quotient of π_0X_v by the subgroup generated by the images of the maps $\pi_0X_w \to \pi_0X_v$ arising from the edges $w \to v$ of the cube arriving at v. The proof goes by induction using the long exact sequence of homotopy groups associated to a homotopy fiber sequence of spectra.

We get the result by applying that to the situation at hand, in which $\bar{X} = K\Omega^n\mathcal{N}$, $X_v = K(B^q)^n\mathcal{N}$, the edges $w \to v$ correspond to the various diagonal maps $K\Delta : K(B^q)^iC^q(B^q)^{n-i-1}\mathcal{N} \to K(B^q)^n\mathcal{N}$, and $\pi_0\bar{X} \cong K_n\mathcal{N}$. □

8. Remarks

Remark 8.1. The *double short exact sequences* used by Nenashev as generators for $K_1\mathcal{N}$ in [10] and mentioned in the introduction of this paper are derived from the generators used by Sherman in [14] and concern the case where $n = 1$ and the acyclic binary complexes are supported on $[0, 2]$; i.e., they are the objects of the category $B^q_{[0,2]}\mathcal{N}$. Nenashev's six-term relation follows from our relations by taking the total complex of his 3×3-diagram and filtering it vertically and horizontally. If our map $K_0B^q_{[0,2]}\mathcal{N} \to K_1\mathcal{N}$ agrees with Nenashev's, then it is surjective, too. We have not checked that, but it can be shown directly that $K_0\Omega_{[0,2]}\mathcal{N} \to K_0\Omega\mathcal{N}$ is surjective, for any exact category $\mathcal{N}$, so acyclic binary complexes supported on $[0, 2]$ suffice to generate $K_1\mathcal{N}$; we omit the proof, because it is complicated. It follows that acyclic binary multicomplexes supported on $[0, 2]^n$ suffice to generate $K_n\mathcal{N}$. To see that, we show for each i that the map $\eta_i : K_0\Omega^{i+1}_{[0,2]}\Omega^{n-i-1}\mathcal{N} \to K_0\Omega^i_{[0,2]}\Omega^{n-i}\mathcal{N}$ is surjective, and then we compose those maps. The functors $\Omega_{[0,2]}$ and Ω commute up to natural isomorphism, so η_i is a direct summand of $K_0\Omega_{[0,2]}(B^q_{[0,2]})^i(B^q)^{n-i-1}\mathcal{N} \to K_0\Omega(B^q_{[0,2]})^i(B^q)^{n-i-1}\mathcal{N}$, which is surjective by our assertion with $\mathcal{N}$ replaced by the exact category $(B^q_{[0,2]})^i(B^q)^{n-i-1}\mathcal{N}$, yielding surjectivity of η_i.

Remark 8.2. Here is a heuristic argument for the generation of $K_1\mathcal{N}$ by acyclic binary complexes of length 2, as asserted in Remark 8.1. Any element of $\pi_{n+1}K\mathcal{N}$ can be viewed as a proof that $0 = 0$ in $\pi_nK\mathcal{N}$. A proof that $0 = 0$ in $K_0\mathcal{N}$ amounts to writing the element 0 in the free abelian group generated by the objects of $\mathcal{N}$ as a linear combination of expressions $\langle P\rangle - \langle P'\rangle - \langle P''\rangle$ arising from short exact sequences $0 \to P' \to P \to P'' \to 0$. Hence it amounts to an equation $\sum_i(\langle N_i\rangle - \langle N'_i\rangle - \langle N''_i\rangle) = \sum_j(\langle M_j\rangle - \langle M'_j\rangle - \langle M''_j\rangle)$. Writing $N := \bigoplus_i N_i$, etc., yields an isomorphism $N \oplus M' \oplus M'' \cong N' \oplus N'' \oplus M$. The exact sequences $0 \to N'\oplus M' \to N\oplus M'\oplus M'' \to N''\oplus M'' \to 0$ and $0 \to N'\oplus M' \to N'\oplus N''\oplus M \to N'' \oplus M'' \to 0$ have isomorphic objects and thus yield an acyclic binary complex of length 2 that seems to capture the essence of the proof, and thus every element of $K_1\mathcal{N}$ ought to arise this way. See also Remark 5.5.

Remark 8.3. The computation of the tame symbol in [3, (7.7)] was probably the first place that acyclic binary complexes were encountered in K-theory. Actually, the differentials there go in opposite directions, but there is a conjectural fix for that. Namely, given exact sequences $E = [A \rightarrowtail B \twoheadrightarrow C]$ and $F = [C \rightarrowtail B \twoheadrightarrow A]$, the two exact sequences $[0 \rightarrowtail A \twoheadrightarrow A] \oplus E \oplus [C \rightarrowtail C \twoheadrightarrow 0]$ and $[A \rightarrowtail A \twoheadrightarrow 0] \oplus F \oplus [0 \rightarrowtail C \twoheadrightarrow C]$ have the same underlying graded object, and thus determine an acyclic binary complex. The corresponding classes in K_1 are related; see [15, (*) above Theorem 10].

Remark 8.4. The acyclic binary multicomplexes supported on $[0, 1]^n$ correspond to the commuting n-tuples of automorphisms discussed in [6]. They include products of units, such as the Steinberg symbols in K_2 of a ring, as well as the operation $A \star B$ of Milnor [7], which produces an element of $K_2 R$ from two commuting elementary automorphisms.

Definition 8.5. For a ring R, let $\mathcal{P}R$ denote the exact category of finitely generated projective left R-modules.

Remark 8.6. The Dennis-Stein symbols [16] in K_2 of a commutative ring possibly correspond to certain acyclic binary multicomplexes supported on $[0,2] \times [0,1]$. Given elements a and b of a ring R such that $1 + ab$ is a unit, Dennis and Stein define

$$\langle a, b\rangle := x_{21}(-b/(1+ab))x_{12}(a)x_{21}(b)x_{12}(-a/(1+ab))h_{12}(1+ab)^{-1} \in K_2 R,$$

where $h_{ij}(u) := w_{ij}(u)w_{ij}(-1)$ and $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$. Given an invertible n by n matrix θ over R, let $I(\theta) \in B^q_{[0,1]}\mathcal{P}R$ be the acyclic binary complex whose top differential is $\theta : R^n \to R^n$ and whose bottom differential is $id_n : R^n \to R^n$. For a unit u of R, let $I(u) := I(u \cdot id_1)$. Given elements a and b of a ring R such that $1 - ab$ is a unit, the identity

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1-ab \end{pmatrix} = \begin{pmatrix} 1 & b \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1-ab & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

shows that

$$\alpha := \begin{pmatrix} 1 & a \\ 0 & 1-ab \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} 1-ab & a \\ 0 & 1 \end{pmatrix}$$

are conjugate matrices, and hence that $I(\alpha) \cong I(\beta)$. The triangular form of α and β shows that we have exact sequences $0 \to I(1) \to I(\alpha) \to I(1-ab) \to 0$ and $0 \to I(1-ab) \to I(\beta) \to I(1) \to 0$; these are two short exact sequences in $B^q\mathcal{P}R$ whose objects, in reverse order, are isomorphic. Using Remark 8.3, we get an object of $B^qB^q\mathcal{P}R$, hence a class in $K_0\Omega^2\mathcal{P}R$, which is isomorphic to K_2R. Perhaps the corresponding element of K_2R is $\langle -a, b\rangle$.

Appendix A. Exact categories with weak equivalences

In this section, we introduce K-theory of exact categories with weak equivalences, modeled after Waldhausen's notion of category with cofibrations and weak equivalences [18, (1.1)] and motivated by Thomason's treatment in [17].

Let $\mathcal{N}$ be an *exact category*. It is an additive category together with a collection of *short exact* sequences that has an admissible embedding (see Definition 1.3).

We will assume that the exact categories under consideration in this paper are small, so their K-theory spaces are defined. We also assume each exact category comes with a chosen zero object called 0.

As Waldhausen does, we regard an exact category $\mathcal{N}$ as a *category with cofibrations* (in the sense of [18, (1.1)]) by letting the cofibrations be the admissible monomorphisms. The cofibration sequences are thereby identified with the short exact sequences, the coproduct $N \vee N'$ of objects N and N' of $\mathcal{N}$ is identified with the direct sum $N \oplus N'$, and the zero object $*$ is identified with the zero object 0.

Definition A.1. An *exact category with weak equivalences* $w\mathcal{N}$ is an exact category $\mathcal{N}$ with a category of weak equivalences $w \subseteq \mathcal{N}$ satisfying the extension axiom, in the sense of [18, (1.2)].

Definition A.2. If $w\mathcal{N}$ is an exact category with weak equivalences, we let $Kw\mathcal{N}$ denote its K-theory spectrum; see, for example, [17, Definition 1.5.3]. Its space at level n, for $n > 0$, is realized concretely by the construction $S.$ of Waldhausen, iterated n times [18, (1.3) and (1.5.3)]:

$$Kw\mathcal{N}\langle n\rangle := |wS.^n\mathcal{N}|.$$

Defined that way, it is an Ω-spectrum starting at $n = 1$; i.e., the map $Kw\mathcal{N}\langle n\rangle \to \Omega Kw\mathcal{N}\langle n+1\rangle$ is a homotopy equivalence for $n \geq 1$. For an integer m we let $K_m w\mathcal{N}$ denote the homotopy group $\pi_m Kw\mathcal{N}$. It is the m-th *K-group* of $w\mathcal{N}$.

Definition A.3. If $\mathcal{N}$ is an exact category, let i denote the subcategory of it whose arrows are the isomorphisms, regarded as a category of weak equivalences. We define the K-theory spectrum of $\mathcal{N}$ by $K\mathcal{N} := Ki\mathcal{N}$, and we define $K_m\mathcal{N} := K_m i\mathcal{N}$.

Suppose $f : v\mathcal{M} \to w\mathcal{N}$ is an exact functor between exact categories equipped with categories of weak equivalences; this means that f is an additive functor $\mathcal{M} \to \mathcal{N}$ that sends short exact sequences of $\mathcal{M}$ to short exact sequences of $\mathcal{N}$ and sends arrows in v to arrows in w. As in Definition 4.1, we let $[f]$ refer to f as an object in its category of arrows.

Definition A.4. Let $K[f]$ denote the relative K-theory spectrum, whose defining attribute is that it is the spectrum that fits into the following fibration sequence up to homotopy:

$$Kv\mathcal{M} \xrightarrow{f} Kw\mathcal{N} \to K[f].$$

The spectrum $K[f]$ is realized concretely by the construction $S.S.[f]$ of Waldhausen [18, (1.5.5) and (1.5.7)], developed using [18, (1.5.3)] by repeating the $S.$-construction n times, with $n \geq 1$:

$$K[f]\langle n\rangle := |S.^n S.[f]|.$$

Here $S.[f]$ is a certain simplicial exact category with weak equivalences defined in [18, (1.5.4)].

To speak of the sequence above as a homotopy fibration sequence, one needs to specify a preferred null-homotopy for the composite map. It is provided by the commutative square

$$\begin{tikzcd} Kv\mathcal{M} \arrow[r, "f"] \arrow[d] & Kw\mathcal{N} \arrow[d] \\ K[1_{vM}] \arrow[r] & K[f] \end{tikzcd}$$

and the contractibility of $K[1_{vM}]$; see [18, 1.5.7]. In the body of the paper we will leave the presence of such preferred null-homotopies and auxiliary contractible spaces implicit.

For an integer m we let $K_m[f]$ denote the homotopy group $\pi_m K[f]$. It is the m-th *K-group* of $[f]$.

The notion of a relative K-theory spectrum can be generalized to provide a combinatorial construction of the multi-relative K-theory space $K\mathcal{N}$ of an n-dimensional cube $\mathcal{N}$ of exact categories, as discussed in [5, §4] for spaces. Extending the construction to yield a spectrum or to handle exact categories with weak equivalences presents no difficulties.

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