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SMOOTHNESS OF THE TRUNCATED DISPLAY FUNCTOR

EIKE LAU

Contents

Introduction

The notion of displays over p-adic rings arises naturally both in Cartier theory and in crystalline Dieudonné theory.

In Cartier theory, displays are a categorised form of structure equations of Cartier modules of formal Lie groups. This is the original perspective in [Zi1]. Passing from a structure equation to the module corresponds to Zink's functor BT from displays to formal Lie groups, which induces an equivalence between nilpotent displays and p-divisible formal groups by [Zi1, La1]. The theory includes at its basis a description of the Dieudonné crystal of a p-divisible group $\mathrm{BT}(\mathscr{P})$ in terms of the nilpotent display $\mathscr{P}$. We view this as a passage from Cartier theory to crystalline Dieudonné theory.

On the other hand, let G be a p-divisible group over a p-adic ring R, and let D be the covariant Dieudonné crystal of G. It is well known that the Frobenius of D restricted to the Hodge filtration is divisible by p. If the ring of Witt vectors $W(R)$ has no p-torsion, this gives a natural display structure on the value of D on $W(R)$. We show that this construction extends in a unique way to a functor from p-divisible groups to displays over an arbitrary p-adic ring R,

$$\Phi_R : (p\text{-div}/R) \to (\mathrm{disp}/R).$$

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The proof uses that the stacks of truncated p-divisible groups are smooth algebraic stacks with smooth transition morphisms by [Il2], which implies that in a universal case the ring of Witt vectors has no p-torsion.

There is a natural notion of truncated displays over rings of characteristic p. While a display is given by an invertible matrix over $W(R)$ if a suitable basis of the underlying module is fixed, a truncated display is given by an invertible matrix over the truncated Witt ring $W_n(R)$ for a similar choice of basis.[1] The functors Φ_R induce functors from truncated Barsotti-Tate (BT) groups to truncated displays of the same level

$$\Phi_{n,R} : (p\text{-div}_n/R) \to (\text{disp}_n/R).$$

For varying rings R of characteristic p they induce a morphism from the stack of truncated BT groups of level n to the stack of truncated displays of level n, which we denote by

$$\phi_n : \overline{\mathscr{BT}_n} \to \mathscr{D}isp_n.$$

The following is the central result of this article; see Theorem 4.5.

Theorem A. *The morphism ϕ_n is a smooth morphism of smooth algebraic stacks over $\mathbb{F}_p$, which is an equivalence on geometric points.*

Let us sketch the proof. The deformation theory of nilpotent displays together with the crystalline deformation theory of p-divisible groups implies that the restriction of the functor Φ to infinitesimal p-divisible groups is formally étale in the sense that it induces an equivalence of infinitesimal deformations. It follows that the smooth locus of ϕ_n contains all points of $\overline{\mathscr{BT}_n}$ that correspond to infinitesimal groups; since the smooth locus is open it must be all of $\overline{\mathscr{BT}_n}$.

For a truncated BT group G we denote by $\underline{\mathrm{Aut}}^o(G)$ the sheaf of automorphisms of G which become trivial on the associated truncated display.

Theorem B. *Let G_1 and G_2 be truncated BT groups over a ring R of characteristic p with associated truncated displays $\mathscr{P}_1$ and $\mathscr{P}_2$. The group scheme $\underline{\mathrm{Aut}}^o(G_i)$ is commutative, infinitesimal, and finite flat over R. The natural morphism $\underline{\mathrm{Isom}}(G_1, G_2) \to \underline{\mathrm{Isom}}(\mathscr{P}_1, \mathscr{P}_2)$ is a torsor under $\underline{\mathrm{Aut}}^o(G_i)$ for each i.*

This is more or less a formal consequence of Theorem A; see Theorem 4.7. If G is a truncated BT group of dimension r, codimension s, and level n, one can show that the degree of $\underline{\mathrm{Aut}}^o(G)$ is equal to p^{rsn}. In particular, the functors $\Phi_{n,R}$ are usually far from being an equivalence. The situation changes if one passes to the limit Φ_R. Namely, we have the following application of Theorems A and B; see Theorem 5.1.

Theorem C. *For a p-adic ring R, the functor Φ_R induces an equivalence between infinitesimal p-divisible groups and nilpotent displays over R.*

Let us sketch the argument. For a p-divisible group G over a ring R of characteristic p the situation is controlled by the projective limit of finite flat group schemes

$$\underline{\mathrm{Aut}}^o(G) = \varprojlim_n \underline{\mathrm{Aut}}^o(G[p^n]).$$

If the group G and its dual have a non-trivial étale part at some point of $\operatorname{Spec} R$, one can see directly that $\underline{\mathrm{Aut}}^o(G)$ is non-trivial, which explains the restriction to

[1] Truncated displays of a different kind are studied in [Bü] in a more general setting.

infinitesimal groups in Theorem C. One has to show that $\underline{\mathrm{Aut}}^o(G)$ is trivial if G is infinitesimal. If $\underline{\mathrm{Aut}}^o(G)$ were non-trivial, the first homology of its cotangent complex would be non-trivial, which would contradict the fact that Φ is formally étale for infinitesimal groups.

As a second application of Theorems A and B we obtain an alternative proof of the following result of Gabber; see Theorem 6.4.

Theorem D. *The category of p-divisible groups over a perfect ring R of characteristic p is equivalent to the category of Dieudonné modules over R.*

As in the case of perfect fields, a Dieudonné module over R is a projective $W(R)$-module M of finite type with a Frobenius-linear endomorphism F and a Frobenius^{-1}-linear endomorphism V such that $FV = p = VF$. One deduces formally an equivalence between commutative finite flat group schemes of p-power order over R and an appropriate category of finite Dieudonné modules. Over perfect valuation rings this equivalence is proved by Berthelot [Be], and in general it is proved by Gabber by a reduction to the case of valuation rings. Theorem D follows from Theorems A and B since they show that the morphism ϕ_n is represented by a morphism of groupoids of affine schemes which induces an isomorphism of the perfect hulls.

Finally, we study the relation between the functors Φ_R and BT_R. One can form the composition

$$(p\text{-div}/R) \xrightarrow{\Phi_R} (\mathrm{disp}/R) \xrightarrow{\mathrm{BT}_R} (\text{formal groups}/R).$$

Here both functors induce inverse equivalences when restricted to formal p-divisible groups and nilpotent displays.

Theorem E. *For each p-divisible group G over a p-adic ring R, the formal group $\mathrm{BT}_R(\Phi_R(G))$ is naturally isomorphic to the formal completion $\hat{G}$.*

See Theorem 8.3. In other words, we have obtained a passage from crystalline Dieudonné theory to Cartier theory: The natural display structure on the Dieudonné crystal of G, viewed as a structure equation of a Cartier module, gives the Cartier module of $\hat{G}$.

1. Preliminaries

1.1. **Properties of ring homomorphisms.** All rings are commutative with a unit. Let $f : A \to B$ be a ring homomorphism.

We call f *ind-étale* (resp. *ind-smooth*) if B can be written as a filtered direct limit of étale (resp. smooth) A-algebras. In the ind-étale case the transition maps in the filtered system are necessarily étale. We call f an ∞-*smooth covering* if there is a sequence of faithfully flat smooth ring homomorphisms $A = B_0 \to B_1 \to B_2 \to \cdots$ with $B \cong \varinjlim B_i$.

We call f *reduced* if f is flat and if the geometric fibres of f are reduced. This differs from EGA IV, 6.8.1, where in addition the fibres of f are assumed to be Noetherian. If f is reduced, then for each reduced A-algebra A' the ring $B \otimes_A A'$ is reduced. Every ind-smooth homomorphism is reduced.

Assume that A and B are Noetherian. By the Popescu desingularisation theorem, [Po, Thm. 2.5] and [Sw], f is ind-smooth if and only if f is regular; recall that f is regular if f is flat and if $B \otimes_A L$ is a regular ring when L is a finite extension of a residue field of a prime of A.

Without the Noetherian hypothesis again, we say that f is *quasi-étale* if the cotangent complex $L_{B/A}$ is acyclic, and that f is *quasi-smooth* if the augmentation $L_{B/A} \to \Omega_{B/A}$ is a quasi-isomorphism and if $\Omega_{B/A}$ is a projective B-module. Quasi-smooth implies formally smooth, and quasi-étale implies formally étale; see [Il1, III, Proposition 3.1.1] and its proof.

1.2. **Affine algebraic stacks.** Let Aff be the category of affine schemes. Let $\mathscr{X}$ be a category which is fibered in groupoids over Aff. For a topology τ on Aff, $\mathscr{X}$ is called a τ-stack if τ-descent is effective for $\mathscr{X}$. We call $\mathscr{X}$ an *affine algebraic stack* if $\mathscr{X}$ is an fpqc stack, if the diagonal morphism $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable affine, and if there is an affine scheme X with a faithfully flat morphism $X \to \mathscr{X}$, called a presentation of $\mathscr{X}$. Equivalently, $\mathscr{X}$ is the fpqc stack associated to a flat groupoid of affine schemes.

Let P be a property of ring homomorphisms which is stable under base change. A representable affine morphism of fpqc stacks is said to have the property P if its pull-back to affine schemes has the property P. In particular, one can demand that an affine algebraic stack has a presentation with the property P, called a P-presentation.

Assume that the property P is stable under composition and satisfies the following descent condition: If a composition of ring homomorphisms $v \circ u$ and v both have the property P and if v is faithfully flat, then u has the property P. One example is $P =$ reduced. Let $\mathscr{X}$ be an affine algebraic stack which has a P-presentation $X \to \mathscr{X}$. A morphism of affine algebraic stacks $\mathscr{X} \to \mathscr{Y}$ is said to have the property P if the composition $X \to \mathscr{X} \to \mathscr{Y}$ has the property P. This does not depend on the P-presentation of $\mathscr{X}$.

Let $\mathscr{X}$ be an affine algebraic stack with a reduced presentation $X \to \mathscr{X}$. We call $\mathscr{X}$ *reduced* if X is reduced; this does not depend on the reduced presentation. In general, there is a maximal reduced closed substack $\mathscr{X}_{\mathrm{red}}$ of $\mathscr{X}$. Indeed, the inverse images of X_{red} under the two projections $X \times_{\mathscr{X}} X \to X$ are equal because they coincide with $(X \times_{\mathscr{X}} X)_{\mathrm{red}}$; thus X_{red} descends to a substack of $\mathscr{X}$.

Assume that $\mathscr{X}$ is a locally Noetherian Artin algebraic stack and that Y is a locally Noetherian scheme. We call a morphism $Y \to \mathscr{X}$ *regular* if for a smooth presentation $X \to \mathscr{X}$ the projection $Y \times_{\mathscr{X}} X \to X$ is regular. This is independent of the smooth presentation of $\mathscr{X}$.

1.3. **The stack of p-divisible groups.** We fix a non-negative integer h. Let $\mathscr{BT} = \mathscr{BT}^h$ be the stack of p-divisible groups of height h, viewed as a fibered category over the category of affine schemes. Thus for an affine scheme X, $\mathscr{BT}(X)$ is the category with p-divisible groups of height h over X as objects and with isomorphisms of p-divisible groups as morphisms. Similarly, for each non-negative integer n, let $\mathscr{BT}_n = \mathscr{BT}_n^h$ be the stack of truncated Barsotti-Tate groups of height h and level n. This is an Artin algebraic stack of finite type over $\mathbb{Z}$ with affine diagonal; see [W, Prop. 1.8] and [La1, Sec. 2]. The truncation morphisms

$$\tau_n : \mathscr{BT}_{n+1} \to \mathscr{BT}_n$$

are smooth and surjective by [Il2, Thm. 4.4 and Prop. 1.8]. Note that $\mathscr{BT}_n$ has pure dimension zero since the dense open substack $\mathscr{BT}_n \times \operatorname{Spec} \mathbb{Q}$ is the classifying space of the finite group $\mathrm{GL}_h(\mathbb{Z}/p^n\mathbb{Z})$.

Lemma 1.1. *The fibered category $\mathscr{BT}$ is an affine algebraic stack in the sense of section 1.2. There is a presentation $\pi : X \to \mathscr{BT}$ such that π and the compositions $X \xrightarrow{\pi} \mathscr{BT} \xrightarrow{\tau} \mathscr{BT}_n$ for $n \geq 0$ are ∞-smooth coverings; in particular $X \to \operatorname{Spec} \mathbb{Z}$ is an ∞-smooth covering.*

Proof. This follows from the properties of $\mathscr{BT}_n$ and τ_n, using that $\mathscr{BT}$ is the projective limit of $\mathscr{BT}_n$ for $n \to \infty$. More precisely, the diagonal of $\mathscr{BT}$ is representable affine because a projective limit of affine schemes is affine. We choose smooth presentations $\psi_n : Y_n \to \mathscr{BT}_n$ with affine Y_n, and define recursively another sequence of smooth presentations $\pi_n : X_n \to \mathscr{BT}_n$ with affine X_n by $X_1 = Y_1$ and $X_{n+1} = Y_{n+1} \times_{\mathscr{BT}_n} X_n$. Let

$$X = \varprojlim_n X_n = \varprojlim_n (X_n \times_{\mathscr{BT}_n} \mathscr{BT}) \tag{1.1}$$

and let $\pi : X \to \mathscr{BT}$ be the limit of the morphisms π_n. The transition maps in the second system in (1.1) are smooth and surjective because all ψ_n are smooth and surjective. Thus π is presentation of $\mathscr{BT}$ and an ∞-smooth covering. The transition maps in the first system in (1.1) are smooth and surjective because the truncation morphisms τ_n are smooth and surjective too. Thus $X \to X_n \to \mathscr{BT}_n$ is an ∞-smooth covering. □

We refer to section 7 for presentations of $\mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ where the covering space is Noetherian, and closer to $\mathscr{BT}$ in some sense.

1.4. **Newton stratification.** In the following we write

$$\overline{\mathscr{BT}} = \mathscr{BT} \times \operatorname{Spec} \mathbb{F}_p; \qquad \overline{\mathscr{BT}_n} = \mathscr{BT}_n \times \operatorname{Spec} \mathbb{F}_p.$$

We call a Newton polygon of height h a polygon that appears as the Newton polygon of a p-divisible group of height h.

Lemma 1.2. *For each Newton polygon ν of height h there is a unique reduced closed substack $\mathscr{BT}_\nu$ of $\overline{\mathscr{BT}}$ such that the geometric points of $\mathscr{BT}_\nu$ are the p-divisible groups with Newton polygon $\preceq \nu$.*

Proof. We consider a reduced presentation $X \to \overline{\mathscr{BT}}$ with affine X, defined by the p-divisible group G over X. The points of X where G has Newton polygon $\preceq \nu$ form a closed subset of X; see [Ka, Thm. 2.3.1]. The corresponding reduced subscheme X_ν of X descends to a reduced substack of $\mathscr{BT}$ because the inverse images of X_ν under the two projections $X \times_{\mathscr{BT}} X \to X$ are reduced and coincide on geometric points, so they are equal. □

By a well-known boundedness principle, there is an integer N depending on h such that the Newton polygon of a p-divisible group G of height h is determined by its truncation $G[p^N]$.

Lemma 1.3. *For $n \geq N$ there is a unique reduced closed substack $\mathscr{BT}_{n,\nu}$ of $\overline{\mathscr{BT}_n}$ such that we have a Cartesian diagram*

$$\begin{tikzcd}
\mathscr{BT}_\nu \arrow[r] \arrow[d] & \mathscr{BT} \arrow[d, "\tau"] \\
\mathscr{BT}_{n,\nu} \arrow[r] & \mathscr{BT}_n,
\end{tikzcd}$$

where τ is the truncation. In particular, the closed immersion $\mathscr{BT}_\nu \to \mathscr{BT}$ is a morphism of finite presentation.

Proof. A reduced presentation $X \to \overline{\mathscr{BT}}$ composed with τ is a reduced presentation of $\overline{\mathscr{BT}_n}$. As in the proof of Lemma 1.2, the reduced subscheme X_ν of X descends to a reduced substack of $\overline{\mathscr{BT}_n}$. Since $\mathscr{BT}_n$ is of finite type, the immersion $\mathscr{BT}_{n,\nu} \to \mathscr{BT}_n$ is a morphism of finite presentation. $\square$

Along the same lines, one can consider the locus of infinitesimal groups:

Lemma 1.4. *There are unique reduced closed substacks $\mathscr{BT}^o \subseteq \overline{\mathscr{BT}}$ and $\mathscr{BT}_n^o \subseteq \overline{\mathscr{BT}_n}$ for $n \geq 1$ such that the geometric points of $\mathscr{BT}^o$ and $\mathscr{BT}_n^o$ are precisely the infinitesimal groups. There is a Cartesian diagram*

$$\begin{tikzcd}
\mathscr{BT}^o \arrow[r] \arrow[d] & \mathscr{BT}_{n+1}^o \arrow[r] \arrow[d] & \mathscr{BT}_n^o \arrow[d] \\
\mathscr{BT} \arrow[r, "\tau"] & \mathscr{BT}_{n+1} \arrow[r, "\tau"] & \mathscr{BT}_n .
\end{tikzcd}$$

In particular, the closed immersion $\mathscr{BT}^o \to \mathscr{BT}$ is of finite presentation.

Proof. Let G be a p-divisible group or truncated Barsotti-Tate group of positive level over an $\mathbb{F}_p$-scheme X. Since the points of X where the fibre of G is infinitesimal form a closed subset of X, the substacks $\mathscr{BT}^o$ and $\mathscr{BT}_n^o$ exist; see the proof of Lemma 1.2. The diagram is Cartesian since the truncation morphisms τ are reduced, and since G is infinitesimal if and only if $G[p]$ is infinitesimal. The vertical immersions are of finite presentation because $\mathscr{BT}_n$ is of finite type. $\square$

2. The display functor

2.1. **Frame formalism.** We recall some constructions from [La2] and [La3]. Let $\mathcal{F} = (S, I, R, \sigma, \sigma_1)$ be a frame in the sense of [La2] with $p\sigma_1 = \sigma$ on I.

In this article, the main example is the following. For a p-adic ring R, we denote by $W(R)$ the ring of p-typical Witt vectors and by f and v the Frobenius and Verschiebung of $W(R)$. Let $I_R = v(W(R))$ and let $f_1 : I_R \to W(R)$ be the inverse of v. Then

$$\mathscr{W}_R = (W(R), I_R, R, f, f_1)$$

is a frame with $pf_1 = f$. Windows over $\mathscr{W}_R$ in the sense of [La2] are (not necessarily nilpotent) displays over R in the sense of [Zi1] and [Me2].

For an S-module M let $M^{(1)}$ be its σ-twist, and for a σ-linear map of S-modules $\alpha : M \to N$ let $\alpha^\sharp : M^{(1)} \to N$ be its linearisation. A filtered F-V-module over $\mathcal{F}$ is a quadruple $(P, Q, F^\sharp, V^\sharp)$, where P is a projective S-module of finite type with a filtration $IP \subseteq Q \subseteq P$ such that P/Q is projective over R, and where $F^\sharp : P^{(1)} \to P$ and $V^\sharp : P \to P^{(1)}$ are homomorphisms of S-modules with $F^\sharp V^\sharp = p$ and $V^\sharp F^\sharp = p$. There is a functor

$$\Upsilon : (\mathcal{F}\text{-windows}) \to (\text{filtered } F\text{-}V\text{-modules over } \mathcal{F})$$
$$(P, Q, F, F_1) \mapsto (P, Q, F^\sharp, V^\sharp)$$

such that $F^\sharp$ is the linearisation of F, and $V^\sharp$ is determined by the relation $V^\sharp(F_1(x)) = 1 \otimes x$ for $x \in Q$; see [Zi1, Lemma 10] in the case of displays and [La3, Lemma 2.3]. If S has no p-torsion, Υ is fully faithful.

Assume that S and R are p-adic rings and that I is equipped with divided powers which are compatible with the canonical divided powers of p. For a p-divisible group G over R we denote by $\mathbb{D}(G)$ the *covariant* Dieudonné crystal of G. By a standard construction, it gives rise to a functor

$$\Theta : (p\text{-div}/R) \to (\text{filtered } F\text{-}V\text{-modules over } \mathcal{F})$$
$$G \mapsto (P, Q, F^\sharp, V^\sharp);$$

see [La3, Constr. 3.14]. Here $P = \mathbb{D}(G)_{S\to R}$, the submodule Q is the kernel of $P \to \mathrm{Lie}(G)$, and the Frobenius and Verschiebung of $G \otimes_R R/pR$ induce $V^\sharp$ and $F^\sharp$. Note that $F^\sharp$ is equivalent to a σ-linear map $F : P \to P$.

If S has no p-torsion, there is a unique σ-linear map $F_1 : Q \to P$ such that (P, Q, F, F_1) is an $\mathcal{F}$-window that gives back $(P, Q, F^\sharp, V^\sharp)$ when Υ is applied; see [Ki, Lemma A.2] and [La3, Prop. 3.15]. In other words, there is a unique functor

$$\Phi : (p\text{-div}/R) \to (\mathcal{F}\text{-windows})$$

together with an isomorphism $\Theta \cong \Upsilon \circ \Phi$.

2.2. **The display functor.** Let R be a p-adic ring. The ideal I_R carries natural divided powers which are compatible with the canonical divided powers of p. Moreover the ring $W(R)$ is p-adic; see [Zi1, Prop. 3]. Thus we have a functor Θ for the frame $\mathscr{W}_R$, which we denote by

$$\Theta_R : (p\text{-div}/R) \to (\text{filtered } F\text{-}V\text{-modules over } \mathscr{W}_R).$$

We also have a functor $\Upsilon_R : (\mathrm{disp}/R) \to (\text{filtered } F\text{-}V\text{-modules over } \mathscr{W}_R)$.

Proposition 2.1. *For each p-adic ring R there is a functor*

$$\Phi_R : (p\text{-div}/R) \to (\mathrm{disp}/R)$$

together with an isomorphism $\Theta_R \cong \Upsilon_R \circ \Phi_R$ compatible with base change in R. This determines Φ_R up to unique isomorphism.

In other words, for each p-divisible group G over a p-adic ring R with filtered F-V-module $\Theta_R(G) = (P, Q, F^\sharp, V^\sharp)$ there is a unique map $F_1 : Q \to P$ which is functorial in G and R such that (P, Q, F, F_1) is a display which induces $V^\sharp$; here F is defined by $F(x) = F^\sharp(1 \otimes x)$.

Proof of Proposition 2.1. Let $X = \operatorname{Spec} A \xrightarrow{\pi} \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ be a reduced presentation, given by a p-divisible group G over A; see Lemma 1.1. We write $X \times_{\mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p} X = \operatorname{Spec} B$. The rings A and B have no p-torsion; the rings A/pA and B/pB are reduced. Thus the p-adic completions $\hat{A}$ and $\hat{B}$ have no p-torsion and are reduced. In particular, the functors $\Phi_{\hat{A}}$ and $\Phi_{\hat{B}}$ exist and are unique, which implies that they commute with base change by arbitrary homomorphisms between $\hat{A}$ and $\hat{B}$.

Since displays over a p-adic ring R are equivalent to compatible systems of displays over R/p^nR for $n \geq 1$, to prove the proposition it suffices to show that there is a unique functor Φ_R if p is nilpotent in R. Let H be a p-divisible group over R. It defines a morphism $\alpha : \operatorname{Spec} R \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}/p^m\mathbb{Z}$ for some m. We define S

and T such that the following diagram has Cartesian squares, where π_1 and π_2 are the natural projections.

$$\begin{tikzcd}
\operatorname{Spec} T \arrow[r, shift left, "\psi_1"] \arrow[r, shift right, "\psi_2"'] \arrow[d, "\alpha''"] & \operatorname{Spec} S \arrow[r, "\psi"] \arrow[d, "\alpha'"] & \operatorname{Spec} R \arrow[d, "\alpha"] \\
\operatorname{Spec} B \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \operatorname{Spec} A \arrow[r, "\pi"] & \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p.
\end{tikzcd}$$

Then $T \cong S \otimes_R S$ such that ψ_1 and ψ_2 are the projections, and ψ is faithfully flat. Let $H_S = \psi^* H$. We have descent data $u : \pi_1^* G \cong \pi_2^* G$ and $v : \psi_1^* H_S \cong \psi_2^* H_S$ and an isomorphism $w : \alpha'^* G \cong H_S$ which preserves the descent data. Since p is nilpotent in R, the pair (α', α'') factors into

$$\begin{tikzcd}
\operatorname{Spec} T \arrow[r, "\hat\alpha''"] \arrow[d, shift right, "\psi_2"'] \arrow[d, shift left, "\psi_1"] & \operatorname{Spec} \hat B \arrow[r] \arrow[d, shift right, "\hat\pi_2"'] \arrow[d, shift left, "\hat\pi_1"] & \operatorname{Spec} B \arrow[d, shift right, "\pi_2"'] \arrow[d, shift left, "\pi_1"] \\
\operatorname{Spec} S \arrow[r, "\hat\alpha'"] & \operatorname{Spec} \hat A \arrow[r] & \operatorname{Spec} A.
\end{tikzcd}$$

The isomorphism u induces $\hat u : \hat\pi_1^* G_{\hat A} \cong \hat\pi_2^* G_{\hat A}$, and w induces an isomorphism $\hat w : \hat\alpha'^* G_{\hat A} \cong H_S$ which transforms $\hat u$ into v. The isomorphism $\hat w$ induces an isomorphism of filtered F-V-modules

$$\hat\alpha'^* \Theta_{\hat A}(G_{\hat A}) \cong \Theta_S(H_S).$$

Thus the operator F_1 on $\Theta_{\hat A}(G_{\hat A})$ given by $\Phi_{\hat A}(G_{\hat A})$ induces an operator F_1 on $\Theta_S(H_S)$ which makes a display $\Phi_S(H_S)$. The descent datum of filtered F-V-modules $\psi_1^* \Theta_S(H_S) \cong \psi_2^* \Theta_S(H_S)$ induced by v preserves F_1 since $\hat w$ transforms $\hat u$ into v and since the isomorphism $\pi_1^* \Theta_{\hat A}(G_{\hat A}) \cong \pi_2^* \Theta_{\hat A}(G_{\hat A})$ induced by $\hat u$ preserves F_1 by the uniqueness of $\Phi_{\hat B}$. By fpqc descent, cf. [Zi1, Thm. 37], the operator F_1 on $\Theta_S(H_S)$ descends to an operator F_1 on $\Theta_R(H)$ which makes a display $\Phi_R(H)$. This display is uniquely determined by the requirement that the functors $\Phi_{\hat B}, \Phi_S, \Phi_R$ are compatible with base change by the given ring homomorphisms $\hat A \to S \leftarrow R$.

The construction implies that F_1 is preserved under base change by homomorphisms $R \to R'$ of p-adic rings and under isomorphisms of p-divisible groups over R. Since a homomorphism of p-divisible groups $g : G \to G'$ can be encoded by the automorphism $\left(\begin{smallmatrix} 1 & g \\ 0 & 1 \end{smallmatrix}\right)$ of $G' \oplus G$, it follows that F_1 is also preserved under homomorphisms of p-divisible groups over R. $\square$

Proposition 2.2. *A p-divisible group G over a p-adic ring R is infinitesimal (unipotent) if and only if the display $\Phi_R(G)$ is nilpotent (F-nilpotent).*

Proof. The p-divisible group G is infinitesimal or unipotent if and only if the geometric fibres of G in points of characteristic p have this property; see [Me1, Chap. II, Prop. 4.4]. Similarly, a display $\mathscr{P} = (P, Q, F, F_1)$ over R is nilpotent or F-nilpotent if and only if the geometric fibres of $\mathscr{P}$ in points of characteristic p have this property. Indeed, let $\bar P$ be the projective R/pR-module $P/(I_R P + pP)$. The display $\mathscr{P}$ is nilpotent (resp. F-nilpotent) if and only if the homomorphism $\bar V^\sharp : \bar P \to \bar P^{(1)}$ (resp. $\bar F^\sharp : \bar P^{(1)} \to \bar P$) is nilpotent, which can be verified at the geometric points since $\bar{P}$ is finitely generated. Thus the proposition follows from the case of perfect fields, which is well known. $\square$

Remark 2.3. There is a natural duality isomorphism $\Phi_R(G^\vee) \cong \Phi_R(G)^t$, where $G^\vee$ is the Serre dual of G, and where ^{t} denotes the dual display as in [Zi1, Def. 19]. Indeed, the crystalline duality theorem [BBM, 5.3] implies that the functor Θ_R is compatible with duality, and the assertion follows from the uniqueness part of Proposition 2.1. See also [La3, Cor. 3.26].

2.3. **The extended display functor.** Assume that $B \to R$ is a surjective homomorphism of p-adic rings whose kernel $\mathfrak{b} \subset B$ is equipped with divided powers δ that are compatible with the canonical divided powers of p. Then one can define a frame

$$\mathscr{W}_{B/R} = (W(B), I_{B/R}, R, f, \tilde{f}_1)$$

with $p\tilde{f}_1 = f$; see [La3, Section 2.2]. Windows over $\mathscr{W}_{B/R}$ are called displays for B/R. The ideal $I_{B/R}$ carries natural divided powers, depending on δ, which are compatible with the canonical divided powers of p; see [La2, Section 2.7]. Thus there is a functor Θ for $\mathscr{W}_{B/R}$, which we denote by

$$\Theta_{B/R} : (p\text{-div}/R) \to (\text{filtered } F\text{-}V\text{-modules over } \mathscr{W}_{B/R}).$$

We also have $\Upsilon_{B/R}$: (displays for B/R) $\to$ (filtered F-V-mod. over $\mathscr{W}_{B/R}$).

Proposition 2.4. *For each divided power extension $B \to R$ of p-adic rings which is compatible with the canonical divided powers of p, there is a functor*

$$\Phi_{B/R} : (p\text{-div}/R) \to (\textit{displays for } B/R)$$

together with an isomorphism $\Theta_{B/R} \cong \Upsilon_{B/R} \circ \Phi_{B/R}$ compatible with base change in (B, R, δ). This determines $\Phi_{B/R}$ up to unique isomorphism.

Proof of Proposition 2.4. We may assume that p is nilpotent in B. For a given p-divisible group H over R we choose a p-divisible group H_1 over B which lifts H; this is possible since $B \to R$ is a nil-extension due to the divided powers. Necessarily we have to define $\Phi_{B/R}(H)$ as the base change of $\Phi_B(H_1)$ under the natural frame homomorphism $\mathscr{W}_B \to \mathscr{W}_{B/R}$. Here Φ_B is well-defined by Proposition 2.1. We have to show that the operator F_1 defined in this way on $\Theta_{B/R}(H)$ does not depend on the choice of H_1; then it follows easily that F_1 is compatible with base change in B/R and commutes with homomorphisms of p-divisible groups over R.

As in the proof of Proposition 2.1, the assertion is reduced to a universal situation. Let $X = \operatorname{Spec} A \xrightarrow{\pi} \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ be a presentation given by a p-divisible group G over X such that π and $X \to \operatorname{Spec} \mathbb{Z}_p$ are ∞-smooth coverings; see Lemma 1.1. Let $X' = \operatorname{Spec} A'$ be the p-adic completion of the divided power envelope of the diagonal $X \to X \times_{\operatorname{Spec} \mathbb{Z}_p} X$ and let G_1, G_2 be the inverse images of G under the two projections $X' \to X$. These are two lifts of G with respect to the diagonal morphism $X \to X'$. Since the divided power envelope of the diagonal of a smooth $\mathbb{Z}_p$-algebra has no p-torsion and since A is the direct limit of smooth $\mathbb{Z}_p$-algebras, A' has no p-torsion, and thus $W(A')$ has no p-torsion. Thus the operators F_1 on $\Theta_{A'/A}(G)$ defined by $\Phi_{A'}(G_1)$ and by $\Phi_{A'}(G_2)$ are equal.

The given p-divisible group H over R defines a morphism $\alpha : \operatorname{Spec} R \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$. Since π is an ∞-smooth covering and since a surjective smooth morphism of schemes has a section étale locally in the base, we can find a ring R' and a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r, "\alpha'"] \arrow[d, "\psi"'] & X \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r, "\alpha"] & \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p, \end{tikzcd}$$

where ψ is ind-étale and surjective. Since $\operatorname{Spec} R \to \operatorname{Spec} B$ is a nil-immersion, there is a unique ind-étale and surjective morphism $\operatorname{Spec} B' \to \operatorname{Spec} B$ which extends ψ. Since B' is flat over B, the given divided powers on the kernel of $B \to R$ extend to divided powers on the kernel of $B' \to R'$. Let H_1 and H_2 be two lifts of H to B and let $\beta_i : \operatorname{Spec} B' \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ be the morphism given by $H_i \otimes_B B'$ for $i = 1, 2$. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R' \arrow[r, "\alpha'"] \arrow[d] & X \arrow[d, "\pi"] \\ \operatorname{Spec} B' \arrow[r, "\beta_i"] & \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p. \end{tikzcd}$$

Since π is an ∞-smooth covering and since a smooth morphism satisfies the lifting criterion of formal smoothness for arbitrary nil-immersions of affine schemes, there are morphisms $\beta_i' : \operatorname{Spec} B' \to X$ for $i = 1, 2$ such that in the preceding diagram both triangles commute. They define a morphism $\beta' : \operatorname{Spec} B' \to X \times_{\operatorname{Spec} \mathbb{Z}_p} X$, which factors uniquely over $\beta'' : \operatorname{Spec} B' \to X'$, and we have isomorphisms $H_i \otimes_B B' \cong \beta''^* G_i$ that lift the given isomorphism $H \otimes_R R' \cong \alpha'^* G$. Thus the operators F_1 on $\Theta_{B'/R'}(H \otimes_R R')$ defined by $\Theta_{B'}(H_1 \otimes_B B')$ and by $\Theta_{B'}(H_2 \otimes_B B')$ are equal. Since $W(B) \to W(B')$ is injective, it follows that the operators F_1 on $\Theta_{B/R}(H)$ defined by $\Theta_B(H_1)$ and by $\Theta_B(H_2)$ are equal as well. $\square$

Remark 2.5. As in Remark 2.3 we have $\Phi_{B/R}(G^\vee) \cong \Phi_{B/R}(G)^t$.

2.4. **Consequences.** Let us recall the Dieudonné crystal of a nilpotent display. If $B \to R$ is a divided power extension of p-adic rings which can be written as the projective limit of divided power extensions $B_n \to R_n$ with $p^n B_n = 0$, for example if the divided powers are compatible with the canonical divided powers of p, the base change functor from nilpotent displays for B/R to nilpotent displays over R is an equivalence of categories by [Zi1, Thm. 44]. Using this, one defines the Dieudonné crystal $\mathbb{D}(\mathscr{P})$ of a nilpotent display $\mathscr{P}$ over R as follows: if $(\tilde{P}, \tilde{Q}, F, F_1)$ is the unique lift of $\mathscr{P}$ to a display for B/R, then

$$\mathbb{D}(\mathscr{P})_{B/R} = \tilde{P}/I_B\tilde{P}.$$

By duality, the same applies to F-nilpotent displays.

Remark 2.6. It is well known that the crystalline deformation theorem for p-divisible groups in [Me1] holds for not necessarily nilpotent divided powers if the groups are infinitesimal or unipotent; see [G1, Sec. 4] and [G2, p. 111].

More precisely, assume that $B \to R = B/\mathfrak{b}$ is a divided power extension of rings in which p is nilpotent. We have a natural functor $G \mapsto (H, V)$ from p-divisible groups G over B to p-divisible groups H over R together with a lift of the Hodge filtration of H to a direct summand $V \subset \mathbb{D}(H)_{B/R}$. Here $\mathbb{D}(H)$ is the covariant Dieudonné crystal given by [BBM]. If the divided powers on $\mathfrak{b}$ are nilpotent, this functor is an equivalence of categories by the deformation theorem [Me1, V.1.6] together with the comparison of the crystals of [Me1] and of [BBM] in [BM, 3.2.11]. For general divided powers, the functor induces an equivalence between unipotent (infinitesimal) p-divisible groups G over B and pairs (H, V), where H is unipotent (infinitesimal).

Let us indicate a proof of this fact. First, by the crystalline duality theorem [BBM, 5.3] it suffices to consider unipotent p-divisible groups.

For each commutative formal Lie group $L = \operatorname{Spf} A$ over B there is a homomorphism $\log_L : \operatorname{Ker}[L(B) \to L(R)] \to \operatorname{Ker}[\operatorname{Lie}(L) \to \operatorname{Lie}(L_R)]$ such that $\log_{\hat{\mathbb{G}}_m}$ is given by the usual logarithm series. It can be described as follows. Let $A^* \subseteq \operatorname{Hom}_B(A, B)$ be the module of continuous homomorphisms, equipped with the multiplication and comultiplication induced by those of A (reversely). Then the Cartier dual $L^* = \underline{\operatorname{Hom}}(L, \mathbb{G}_m)$ is isomorphic to the affine group scheme $\operatorname{Spec} A^*$, and we have isomorphisms $L \cong \underline{\operatorname{Hom}}(L^*, \mathbb{G}_m)$ and $\underline{\operatorname{Lie}}(L) \cong \underline{\operatorname{Hom}}(L^*, \mathbb{G}_a)$. This is similar to the Cartier duality of commutative finite flat group schemes. Under these isomorphisms $\log_L$ is induced by $\log_{\hat{\mathbb{G}}_m}$. If the divided powers on $\mathfrak{b}$ are nilpotent, $\log_G$ is an isomorphism; its inverse is given by the usual exponential series for $\mathbb{G}_m$. Let us call L unipotent if the augmentation ideal $J \subset A^*$ is a nil-ideal. If L is unipotent, arbitrary divided powers on $\mathfrak{b}$ induce nilpotent divided powers on $\mathfrak{b}J$, which implies that $\log_L$ is an isomorphism again.

The construction of the Dieudonné crystal for nilpotent divided powers in [Me1] is based on the exponential of the formal completion of the universal vector extension EG of a p-divisible group G over B. If G is a unipotent p-divisible group, the formal completion of EG is a unipotent formal Lie group. Therefore, in the case of unipotent p-divisible groups, the construction of the Dieudonné crystal of [Me1] and the proof of the deformation theorem [Me1, V.1.6] are valid for not necessarily nilpotent divided powers. The comparison of crystals in [BM, 3.2.11] carries over to this case as well. □

Recall that we denote by $\mathbb{D}(G)$ the covariant Dieudonné crystal of a p-divisible group G.

Corollary 2.7. *Let $B \to R$ be a divided power extension of p-adic rings which is compatible with the canonical divided powers of p, and let G be a p-divisible group over R which is unipotent or infinitesimal. There is a natural isomorphism of projective B-modules*

$$\mathbb{D}(G)_{B/R} \cong \mathbb{D}(\Phi_R(G))_{B/R}.$$

Proof. If G is infinitesimal (unipotent), then $\Phi_R(G)$ is nilpotent (F-nilpotent), and the display $\Phi_{B/R}(G) = (\tilde{P}, \tilde{Q}, F, F_1)$ is the unique lift of $\Phi_R(G)$ to a nilpotent (F-nilpotent) display for B/R. Since $\tilde{P} = \mathbb{D}(G)_{W(B)\to R}$ and since the projection $W(B) \to B$ is a homomorphism of divided power extensions of R, we get an isomorphism $\mathbb{D}(G)_{B/R} \cong \tilde{P}/I_B\tilde{P} = \mathbb{D}(\Phi_R(G))_{B/R}$. □

Corollary 2.8. *The restriction of Φ_R to infinitesimal or unipotent p-divisible groups is formally étale; i.e., for a surjective homomorphism $B \to R$ of rings in which p is nilpotent with nilpotent kernel, the category of infinitesimal (unipotent)* *p-divisible groups over B is equivalent to the category of such groups G over R together with a lift of $\Phi_R(G)$ to a display over B.*

Proof. If the kernel of $B \to R$ carries divided powers compatible with the canonical divided powers of p, in view of Corollary 2.7 the assertion follows from the crystalline deformation theorem of p-divisible groups (see Remark 2.6) and its counterpart for nilpotent (unipotent) displays [Zi1, Prop. 45]. Thus the corollary holds for $B \to B/pB$ and for $R \to R/pR$. Hence we may assume that B is annihilated by p. Then $B \to R$ is a finite composition of divided power extensions with trivial divided powers, which are compatible with the canonical divided powers of p since pB is zero. □

The following is a special case of Theorem 5.1.

Corollary 2.9. *Let R be a complete local ring with perfect residue field of characteristic p. The functor Φ_R induces an equivalence between p-divisible groups over R with infinitesimal (unipotent) special fibre and displays over R with nilpotent (F-nilpotent) special fibre.*

Proof. Over perfect fields this is classical. The general case follows by Corollary 2.8 and by passing to the limit over $R/\mathfrak{m}_R^n$. □

3. Truncated displays

3.1. **Preliminaries.** For a p-adic ring R and a positive integer n let $W_n(R)$ be the ring of truncated Witt vectors of length n and let $I_{n,R} \subset W_n(R)$ be the kernel of the augmentation to R. The Frobenius of $W(R)$ induces a ring homomorphism

$$f : W_{n+1}(R) \to W_n(R).$$

The inverse of the Verschiebung of $W(R)$ induces a bijective f-linear map

$$f_1 : I_{n+1,R} \to W_n(R).$$

If R is an $\mathbb{F}_p$-algebra, the Frobenius of $W(R)$ induces a ring endomorphism f of $W_n(R)$, and the ideal $I_{n+1,R}$ of $W_{n+1}(R)$ is a $W_n(R)$-module.

Definition 3.1. A *pre-display* over an $\mathbb{F}_p$-algebra R is a sextuple

$$\mathscr{P} = (P, Q, \iota, \varepsilon, F, F_1),$$

where P and Q are $W(R)$-modules with homomorphisms

$$I_R \otimes_{W(R)} P \xrightarrow{\varepsilon} Q \xrightarrow{\iota} P$$

and where $F : P \to P$ and $F_1 : Q \to P$ are f-linear maps such that the following relations hold: The compositions $\iota\varepsilon$ and $\varepsilon(1 \otimes \iota)$ are the multiplication homomorphisms, and we have $F_1\varepsilon = f_1 \otimes F$.

If P and Q are $W_n(R)$-modules, we call $\mathscr{P}$ a pre-display of level n.

The axioms imply that $F\iota = pF_1$; cf. [Zi1, Eq. (2)].

Pre-displays over R form an abelian category (pre-disp$/R$) which contains the category of displays (disp$/R$) as a full subcategory. Pre-displays of level n over R form an abelian subcategory (pre-disp$_n/R$) of (pre-disp$/R$). For a homomorphism of p-adic rings $\alpha : R \to R'$, the restriction of scalars defines a functor

$$\alpha^* : (\text{pre-disp}/R') \to (\text{pre-disp}/R).$$

It has a left adjoint $\mathscr{P} \mapsto W(R') \otimes_{W(R)} \mathscr{P}$ given by the tensor product in each component. The restriction of α^* to pre-displays of level n is a functor $(\text{pre-disp}_n/R') \to (\text{pre-disp}_n/R)$ with left adjoint $\mathscr{P} \mapsto W_n(R') \otimes_{W_n(R)} \mathscr{P}$.

3.2. **Truncated displays.** Truncated displays of level n are pre-displays of level n with additional properties. We begin with the conditions imposed on the linear data of the pre-display. For an $\mathbb{F}_p$-algebra R let

$$J_{n+1,R} = \operatorname{Ker}(W_{n+1}(R) \to W_n(R)).$$

Definition 3.2. A *truncated pair* of level n over an $\mathbb{F}_p$-algebra R is a quadruple $\mathscr{B} = (P, Q, \iota, \varepsilon)$, where P and Q are $W_n(R)$-modules with homomorphisms

$$I_{n+1,R} \otimes_{W_n(R)} P \xrightarrow{\ \varepsilon\ } Q \xrightarrow{\ \iota\ } P$$

such that the following properties hold.

(1) The compositions $\iota\varepsilon$ and $\varepsilon(1 \otimes \iota)$ are the multiplication maps.
(2) The $W_n(R)$-module P is projective of finite type.
(3) The R-module $\operatorname{Coker}(\iota)$ is projective.
(4) We have an exact sequence, where $\bar{\varepsilon}$ is induced by ε:

$$0 \to J_{n+1,R} \otimes_R \operatorname{Coker}(\iota) \xrightarrow{\bar{\varepsilon}} Q \xrightarrow{\iota} P \to \operatorname{Coker}(\iota) \to 0. \tag{3.1}$$

A *normal decomposition* for a truncated pair consists of projective $W_n(R)$-modules $L \subseteq Q$ and $T \subseteq P$ such that we have bijective homomorphisms

$$L \oplus T \xrightarrow{\iota+1} P, \qquad L \oplus (I_{n+1,R} \otimes_{W_n(R)} T) \xrightarrow{1+\varepsilon} Q.$$

Each pair (L, T) of projective $W_n(R)$-modules of finite type defines a unique truncated pair for which (L, T) is a normal decomposition.

Lemma 3.3. *Every truncated pair $\mathscr{B}$ admits a normal decomposition.*

Proof. Let $\bar{L} = \operatorname{Coker}(\varepsilon)$, an R-module, and $\bar{T} = \operatorname{Coker}(\iota)$, a projective R-module. The 4-term exact sequence (3.1) induces a short exact sequence

$$0 \to \bar{L} \xrightarrow{\bar{\iota}} P/I_{n,R}P \to \bar{T} \to 0. \tag{3.2}$$

Thus $\bar{L}$ is a projective R-module. Let L and T be projective $W_n(R)$-modules which lift $\bar{L}$ and $\bar{T}$. Let $L \to Q$ and $T \to P$ be homomorphisms which commute with the obvious projections to $\bar{L}$ and $\bar{T}$, respectively. The exact sequence (3.2) implies that the homomorphism $\iota + 1 : L \oplus T \to P$ becomes an isomorphism over R, so it is an isomorphism as both sides are projective. Let $\mathscr{B}'$ be the truncated pair defined by (L, T). We have a homomorphism of truncated pairs $\mathscr{B}' \to \mathscr{B}$ such that the associated homomorphism of the 4-term sequences (3.1) is an isomorphism except possibly at Q. Hence it is an isomorphism by the 5-Lemma. $\square$

Definition 3.4. A truncated display of level n over an $\mathbb{F}_p$-algebra R is a pre-display $\mathscr{P} = (P, Q, \iota, \varepsilon, F, F_1)$ over R such $(P, Q, \iota, \varepsilon)$ is a truncated pair of level n and such that the image of F_1 generates P as a $W_n(R)$-module.

Let (disp_n/R) be the category of truncated displays of level n over R. This is a full subcategory of the abelian category $(\text{pre-disp}_n/R)$.

If $(P, Q, \iota, \varepsilon)$ is a truncated pair with given normal decomposition (L, T), the set of pairs (F, F_1) such that $(P, Q, \iota, \varepsilon, F, F_1)$ is a truncated display is bijective to the set of f-linear isomorphisms $\Psi : L \oplus T \to P$ such that $\Psi|_L = F_1|_L$ and $\Psi|_T = F|_T$.

If L and T are free $W_n(R)$-modules, Ψ is described by an invertible matrix over $W_n(R)$. This is analogous to the case of displays; see [Zi1, Lemma 9] and the subsequent discussion. The triple (L, T, Ψ) is called a normal representation of $(P, Q, \iota, \varepsilon, F, F_1)$.

Let k be a perfect field of characteristic p. A truncated Dieudonné module of level n over k is a triple (M, F, V), where M is a free $W(k)$-module of finite rank with an f-linear map $F : M \to M$ and an f^{-1}-linear map $V : M \to M$ such that $FV = p = VF$. If $n = 1$ we require that $\operatorname{Ker} F = \operatorname{Im} V$, which is equivalent to $\operatorname{Ker} V = \operatorname{Im} F$.

Lemma 3.5. *Truncated displays of level n over a perfect field k are equivalent to truncated Dieudonné modules of level n over k.*

Proof. Multiplication by p gives an isomorphism $W_n(k) \cong I_{n+1,k}$. Thus truncated displays of level n are equivalent to quintuples $\mathscr{P} = (P, Q, \iota, \epsilon, F_1)$, where P and Q are free $W_n(k)$-modules with homomorphisms $P \xrightarrow{\epsilon} Q \xrightarrow{\iota} P$ with $\epsilon\iota = p$ and $\iota\epsilon = p$ such that the sequence

$$Q \xrightarrow{\iota} P \xrightarrow{p^{n-1}\epsilon} Q \xrightarrow{\iota} P$$

is exact, and where $F_1 : Q \to P$ is a bijective f-linear map. The exactness is automatic if $n \geq 2$. The operator F of the truncated display is given by $F = F_1\epsilon$. Let $V = \iota F_1^{-1}$. The assignment $\mathscr{P} \mapsto (P, F, V)$ is an equivalence between truncated displays and truncated Dieudonné modules. □

Lemma 3.6. *For a homomorphism of $\mathbb{F}_p$-algebras $\alpha : R \to R'$ there is a unique base change functor*

$$\alpha_* : (\mathrm{disp}_n/R) \to (\mathrm{disp}_n/R')$$

with a natural isomorphism

$$\mathrm{Hom}_{(\text{pre-disp}_n/R)}(\mathscr{P}, \alpha^*\mathscr{P}') \cong \mathrm{Hom}_{(\mathrm{disp}_n/R)}(\alpha_*\mathscr{P}, \mathscr{P}')$$

for all truncated displays $\mathscr{P}$ of level n over R and $\mathscr{P}'$ of level n over R'.

Proof. This is straightforward. In terms of normal representations, α_* is given by $(L, T, \Psi) \mapsto (W_n(R') \otimes_{W_n(R)} L, W_n(R') \otimes_{W_n(R)} T, f \otimes \Psi)$. □

Remark 3.7. If α is ind-étale, then $W_n(R) \to W_n(R')$ is ind-étale, and the ideal $v^m(I_{n-m,R'})$ is equal to $W_n(R') \otimes_{W_n(R)} v^m(I_{n-m,R})$ for $0 \leq m \leq n$. This is proved in [LZ, Prop. A.8] if α is étale, and the functor W_n preserves filtered direct limits of rings. As a consequence we obtain:

Corollary 3.8. *For each truncated display $\mathscr{P}$ of level n over R there is a natural homomorphism of pre-displays over R',*

$$W_n(R') \otimes_{W_n(R)} \mathscr{P} \to \alpha_*\mathscr{P}.$$

If α is ind-étale, this homomorphism is an isomorphism.

Proof. In view of Remark 3.7, this follows from the proof of Lemma 3.6. □

Lemma 3.9. *Assume that $\alpha : R \to R'$ is a faithfully flat ind-étale homomorphism of $\mathbb{F}_p$-algebras. If $\mathscr{P}$ is a pre-display of level n over R such that the tensor product $\mathscr{P}' = W_n(R') \otimes_{W_n(R)} \mathscr{P}$ is a truncated display of level n over R', then $\mathscr{P}$ is a truncated display of level n over R.*

Proof. The pre-display $\mathscr{P}$ is a truncated display if and only if P is projective of finite type over $W_n(R)$, the homomorphism $F_1^\sharp : Q^{(1)} \to P$ is surjective, and the 4-term sequence (3.1) is exact. These properties descend from $\mathscr{P}'$ to $\mathscr{P}$ since in all components of $\mathscr{P}$ and of (3.1), the passage from $\mathscr{P}$ to $\mathscr{P}'$ is given by the tensor product with the faithfully flat homomorphism $W_n(\alpha)$; see Remark 3.7. □

Lemma 3.10. *For each $\mathbb{F}_p$-algebra R there are unique truncation functors*

$$\tau_n : (\mathrm{disp}/R) \to (\mathrm{disp}_n/R),$$
$$\tau_n : (\mathrm{disp}_{n+1}/R) \to (\mathrm{disp}_n/R),$$

together with a natural isomorphism

$$\mathrm{Hom}_{(\text{pre-disp}/R)}(\mathscr{P}, \mathscr{P}') \cong \mathrm{Hom}_{(\mathrm{disp}_n/R)}(\tau_n\mathscr{P}, \mathscr{P}')$$

if $\mathscr{P}$ is a display or truncated display of level $n+1$ over R and if $\mathscr{P}'$ is a truncated display of level n over R. The truncation functors are compatible with base change in R.

Proof. Again this is straightforward. In terms of normal representations, τ_n is given by $(L, T, \Psi) \mapsto (W_n(R) \otimes_{W(R)} L, W_n(R) \otimes_{W(R)} T, f \otimes \Psi)$. □

Lemma 3.11. *For an $\mathbb{F}_p$-algebra R, the category* (disp/R) *is the projective limit over n of the categories* (disp_n/R).

Proof. It is easy to see that the truncation functor from displays over R to compatible systems of truncated displays of level n over R is fully faithful. For a given compatible system of truncated displays $(\mathscr{P}_n)_{n\geq 1}$ we define $\mathscr{P} = \varprojlim_n \mathscr{P}_n$ componentwise. The proof of Lemma 3.3 shows that each normal decomposition of $\mathscr{P}_n$ can be lifted to a normal decomposition of $\mathscr{P}_{n+1}$. It follows easily that $\mathscr{P}$ is a display. □

3.3. **Descent.** We recall the descent of projective modules over truncated Witt rings. Let $R \to R'$ be a faithfully flat homomorphism of rings in which p is nilpotent and let $R'' = R' \otimes_R R'$. We denote by $\mathscr{V}_n(R)$ the category of projective $W_n(R)$-modules of finite type and by $\mathscr{V}_n(R'/R)$ the category of modules in $\mathscr{V}_n(R')$ together with a descent datum relative to $R \to R'$.

Lemma 3.12. *The obvious functor $\gamma : \mathscr{V}_n(R) \to \mathscr{V}_n(R'/R)$ is an equivalence.*

Proof. First we note that for each flat $W_n(R)$-module M the complex

$$C_n(M) = [0 \to M \to M \otimes_{W_n(R)} W_n(R') \to M \otimes_{W_n(R)} W_n(R'') \to \cdots]$$

is exact. Indeed, this is clear if $n = 1$, and in general $C_n(M)$ is an extension of $C_1(M \otimes_{W_n(R), f^{n-1}} R)$ and $C_{n-1}(M \otimes_{W_n(R)} W_{n-1}(R))$.

It follows that the functor γ is fully faithful. We have to show that γ is essentially surjective. If R' is a finite product of localisations of R, then $W_n(R) \to W_n(R')$ has the same property, and thus γ is an equivalence. Hence we may always pass to an open cover of $\operatorname{Spec} R$ by spectra of localisations of R. For M' in $\mathscr{V}_n(R'/R)$ the descent datum induces a descent datum for the projective R'-module $M'/I_{R'}M'$, which is effective. By passing to a localisation of R we may assume that the descended R-module is free.

For fixed r let $\mathscr{V}_n^o(R)$ be the category of modules M in $\mathscr{V}_n(R)$ together with an isomorphism of R-modules $\beta : R^r \cong M/I_RM$; homomorphisms in $\mathscr{V}_n^o(R)$ preserve the β's. In view of the preceding remarks it suffices to show that the obvious functor $\mathscr{V}_n^o(R) \to \mathscr{V}_n^o(R'/R)$ is essentially surjective. Since every object of $\mathscr{V}_n^o(R)$ is isomorphic to the standard object $(W_n(R)^r, \mathrm{id})$, we have to show that all objects in $\mathscr{V}_n^o(R'/R)$ are isomorphic. This means that for the sheaf of groups $\underline{A}$ on the category of affine R-schemes defined by $\underline{A}(\operatorname{Spec} S) = \operatorname{Aut}(W_n(S)^r, \mathrm{id})$, the Čech cohomology group $\check{H}^1(R'/R, \underline{A})$ is trivial. This is true because $\underline{A}$ has a finite filtration with quotients isomorphic to quasi-coherent modules. $\square$

We turn to descent of truncated pairs. Let $R \to R'$ be a faithfully flat homomorphism of $\mathbb{F}_p$-algebras. We denote by $\mathscr{C}_n(R)$ the category of truncated pairs of level n over R and by $\mathscr{C}_n(R'/R)$ the category of truncated pairs of level n over R' together with a descent datum relative to $R \to R'$.

Lemma 3.13. *The obvious functor $\gamma : \mathscr{C}_n(R) \to \mathscr{C}_n(R'/R)$ is an equivalence.*

Proof. For a truncated pair $\mathscr{B}$ over R we denote by $\mathscr{B}', \mathscr{B}''$ the base change to R', R'', etc. We have an exact sequence $0 \to P \to P' \to P''$ by the proof of Lemma 3.12, and thus $0 \to Q \to Q' \to Q''$ by the 4-term exact sequence (3.1). It follows easily that the functor γ is fully faithful. We show that γ is essentially surjective by a variant of the proof of Lemma 3.12. Again, the assertion holds if R' is a finite product of localisations of R, and thus we may pass to an open cover of $\operatorname{Spec} R$ defined by localisations. For $\mathscr{B}'$ in $\mathscr{C}_n(R'/R)$ the given descent datum induces a descent datum for the projective R-modules $\operatorname{Coker}(\iota)$ and $\operatorname{Coker}(\varepsilon)$. By passing to a localisation of R we may assume that the descended R-modules are free.

For fixed r, s let $\mathscr{C}_n^o(R)$ be the category of truncated pairs $\mathscr{B}$ in $\mathscr{C}_n(R)$ together with isomorphisms $\beta_1 : R^r \cong \operatorname{Coker}(\iota)$ and $\beta_2 : R^s \cong \operatorname{Coker}(\varepsilon)$ of R-modules; homomorphisms in $\mathscr{C}_n^o(R)$ preserve the β_i. It suffices to show that $\mathscr{C}_n^o(R) \to \mathscr{C}_n^o(R'/R)$ is essentially surjective. By Lemma 3.3 and its proof, all objects of $\mathscr{C}_n^o(R)$ are isomorphic. For fixed $(\mathscr{B}, \beta_1, \beta_2)$ in $\mathscr{C}_n^o(R)$ with normal decomposition (L, T) the group $\operatorname{Aut}(\mathscr{B}, \beta_1, \beta_2)$ can be identified with the group of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A \in \operatorname{Aut}(L)$, $B \in \operatorname{Hom}(T, L)$, $C \in \operatorname{Hom}(L, I_{n+1,R} \otimes_{W_n(R)} T)$, and $T \in \operatorname{Aut}(T)$ such that $A \equiv \mathrm{id}$ and $D \equiv \mathrm{id}$ modulo I_R. The sheaf of groups $\underline{A} = \underline{\operatorname{Aut}}(\mathscr{B}, \beta_1, \beta_2)$ has a finite filtration with quotients isomorphic to quasi-coherent modules. Thus $\check{H}^1(R'/R, \underline{A})$ is trivial, which implies that all objects of $\mathscr{C}_n^o(R'/R)$ are isomorphic. $\square$

Proposition 3.14. *Faithfully flat descent is effective for truncated displays over $\mathbb{F}_p$-algebras.*

Proof. By Lemmas 3.13 and 3.3 it suffices to show that for a given truncated pair $\mathscr{B}$ over an $\mathbb{F}_p$-algebra R with a normal decomposition (L, T), the truncated display structures on $\mathscr{B}$ form an fpqc sheaf on the category of affine schemes over $\operatorname{Spec} R$. This is true because these structures correspond to f-linear isomorphisms $L \oplus T \to P$. $\square$

3.4. **Smoothness.** As in section 1.3 we fix a positive integer h. We denote by $\mathscr{D}isp_n \to \operatorname{Spec} \mathbb{F}_p$ the stack of truncated displays of level n and rank h. Thus $\mathscr{D}isp_n(\operatorname{Spec} R)$ is the groupoid of truncated displays of level n and rank h over R. The truncation functors induce morphisms

$$\tau_n : \mathscr{D}isp_{n+1} \to \mathscr{D}isp_n.$$

Proposition 3.15. *The fibered category $\mathscr{D}isp_n$ is a smooth Artin algebraic stack of dimension zero over $\mathbb{F}_p$ with affine diagonal. The morphism τ_n is smooth and surjective of relative dimension zero.*

Proof. By Proposition 3.14, $\mathscr{D}isp_n$ is an fpqc stack. In order to see that its diagonal is affine we have to show that for truncated displays $\mathscr{P}_1$ and $\mathscr{P}_2$ over an $\mathbb{F}_p$-algebra R the sheaf $\underline{\mathrm{Isom}}(\mathscr{P}_1, \mathscr{P}_2)$ is represented by an affine scheme. By passing to an open cover of $\operatorname{Spec} R$ we may assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ have normal decompositions by free modules. Then homomorphisms of the underlying truncated pairs are represented by an affine space. To commute with F and F_1 is a closed condition, and a homomorphism of truncated pairs is an isomorphism if and only if it induces isomorphisms on $\operatorname{Coker}(\iota)$ and $\operatorname{Coker}(\varepsilon)$, which means that two determinants are invertible. Thus $\underline{\mathrm{Isom}}(\mathscr{P}_1, \mathscr{P}_2)$ is an affine scheme.

For each integer d with $0 \leq d \leq h$, let $\mathscr{D}isp_{n,d}$ be the substack of $\mathscr{D}isp_n$ where the projective module $\operatorname{Coker}(\iota)$ has rank d. Let X_n be the functor on affine $\mathbb{F}_p$-schemes such that $X_n(\operatorname{Spec} R)$ is the set of invertible $W_n(R)$-matrices of rank h. Then X_n is an affine open subscheme of the affine space of dimension nh^2 over $\mathbb{F}_p$. We define a morphism $\pi_{n,d} : X_n \to \mathscr{D}isp_{n,d}$ such that the truncated display $\pi_{n,d}(M)$ is given by the normal representation (L, T, Ψ), where $L = W_n(R)^{h-d}$ and $T = W_n(R)^d$, and M is the matrix representation of Ψ. Let $G_{n,d}$ be the sheaf of groups such that $G_{n,d}(R)$ is the group of invertible matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with $A \in \operatorname{Aut}(L)$, $B \in \operatorname{Hom}(T, L)$, $C \in \operatorname{Hom}(L, I_{n+1,R} \otimes_{W_n(R)} T)$, and $T \in \operatorname{Aut}(T)$ for L and T as above. Then $G_{n,d}$ is an affine open subscheme of the affine space of dimension nh^2 over $\mathbb{F}_p$. The morphism $\pi_{n,d}$ is a $G_{n,d}$-torsor. Thus $\mathscr{D}isp_{n,d}$ and $\mathscr{D}isp_n$ are smooth algebraic stacks of dimension zero over $\mathbb{F}_p$.

The truncation morphism τ_n is smooth and surjective because it commutes with the obvious projection $X_{n+1} \to X_n$, which is smooth and surjective. The relative dimension of τ is the difference of the dimensions of its source and target, which are both zero. $\square$

Let $\mathscr{D}isp \to \operatorname{Spec} \mathbb{F}_p$ be the stack of displays over $\mathbb{F}_p$-algebras.

Corollary 3.16. *The fibered category $\mathscr{D}isp$ is an affine algebraic stack over $\mathbb{F}_p$, which has a presentation $\pi : X \to \mathscr{D}isp$ such that π and the compositions $X \to \mathscr{D}isp \xrightarrow{\tau} \mathscr{D}isp_n$ for $n \geq 0$ are ∞-smooth coverings.*

Proof. By Lemma 3.11, $\mathscr{D}isp$ is the projective limit over n of $\mathscr{D}isp_n$. Thus the corollary follows from Proposition 3.15 by the proof of Lemma 1.1. $\square$

3.5. **Nilpotent truncated displays.** Let R be an $\mathbb{F}_p$-algebra. For each truncated display $\mathscr{P} = (P, Q, \iota, \varepsilon, F, F_1)$ of positive level n over R there is a unique homomorphism

$$V^\sharp : P \to P^{(1)} = W_n(R) \otimes_{f, W_n(R)} P$$

such that $V^\sharp(F_1(x)) = 1 \otimes x$ for $x \in Q$. If $F^\sharp : P^{(1)} \to P$ denotes the linearisation of F, we have $F^\sharp V^\sharp = p$ and $V^\sharp F^\sharp = p$. This is analogous to the case of displays; see [Zi1, Lemma 10]. The construction of $V^\sharp$ is compatible with truncation. We call $\mathscr{P}$ nilpotent if for some n the n-th iterate of $V^\sharp$

$$P \to P^{(1)} \to \cdots \to P^{(n)}$$

is zero. Since the ideal $I_{n,R}$ is nilpotent, $\mathscr{P}$ is nilpotent if and only if the truncation $\tau_1(\mathscr{P})$ of level 1 is nilpotent. A display over R is nilpotent if and only if all its truncations are nilpotent.

Lemma 3.17. *There are unique reduced closed substacks $\mathscr{D}isp^o \subseteq \mathscr{D}isp$ and $\mathscr{D}isp^o_n \subseteq \mathscr{D}isp_n$ for $n \geq 1$ such that the geometric points of $\mathscr{D}isp^o$ and $\mathscr{D}isp^o_n$*

are precisely the nilpotent (truncated) displays. There is a Cartesian diagram

$$\begin{tikzcd}
\mathscr{D}isp^o \arrow[r] \arrow[d] & \mathscr{D}isp^o_{n+1} \arrow[r] \arrow[d] & \mathscr{D}isp^o_n \arrow[d] \\
\mathscr{D}isp \arrow[r, "\tau"] & \mathscr{D}isp_{n+1} \arrow[r, "\tau"] & \mathscr{D}isp_n.
\end{tikzcd}$$

In particular, the closed immersion $\mathscr{D}isp^o \to \mathscr{D}isp$ is of finite presentation.

Proof. Over a field, a truncated display of level 1 and rank h is nilpotent if and only if the h-th iterate of $V^\sharp$ vanishes. Thus for a display or truncated display of positive level $\mathscr{P}$ over an $\mathbb{F}_p$-algebra R the points of $\operatorname{Spec} R$ where $\mathscr{P}$ is nilpotent form a closed subset. Since $\mathscr{D}isp$ and $\mathscr{D}isp_n$ have reduced presentations and since the truncation morphisms τ are reduced, the existence of the reduced closed substacks $\mathscr{D}isp^o$ and $\mathscr{D}isp^o_n$ and the Cartesian diagram follow; cf. Lemma 1.4. □

4. Smoothness of the truncated display functor

4.1. **The truncated display functor.** We begin with the observation that the display functors Φ_R induce truncated display functors on each level. Recall that $(p\text{-div}_n/R)$ is the category of truncated Barsotti-Tate groups of level n over R, and (disp_n/R) is the category of truncated displays of level n over R, which is defined if R is an $\mathbb{F}_p$-algebra.

Proposition 4.1. *For each $\mathbb{F}_p$-algebra R and each positive integer n there is a unique functor*

$$\Phi_{n,R} : (p\text{-div}_n/R) \to (\text{disp}_n/R)$$

which is compatible with base change in R and with the truncation functors from $n+1$ to n on both sides such that the system $(\Phi_{n,R})_{n\geq 1}$ induces Φ_R in the projective limit.

Proof. Let $(p\text{-div}_n/R)'$ be the category of all G in $(p\text{-div}_n/R)$ which can be written as the kernel of an isogeny of p-divisible groups $H_0 \to H_1$ over R. First we define a functor

$$\Phi'_{n,R} : (p\text{-div}_n/R)' \to (\text{pre-disp}_n/R).$$

For G in $(p\text{-div}_n/R)'$ we choose an isogeny of p-divisible groups $H_0 \to H_1$ with kernel G and define

$$\Phi'_{n,R}(G) = \operatorname{Coker}(\tau_n\Phi_R(H_0) \to \tau_n\Phi_R(H_1)),$$

where Φ_R is given by Proposition 2.1, and where τ_n is the truncation from displays to truncated displays of level n. If $g : G \to G'$ is a homomorphism in $(p\text{-div}_n/R)'$ such that G is the kernel of $H_0 \to H_1$ and G' is the kernel of $H'_0 \to H'_1$, we define $\Phi'_{n,R}(g)$ as follows. Let $H''_0 = H_0 \times H'_0$, let $G \to H''_0$ be given by $(1, g)$, and let $H''_1 = H''_0/G$. The projections $H_0 \leftarrow H''_0 \to H'_0$ extend uniquely to homomorphisms of complexes $H_* \leftarrow H''_* \to H'_*$, where the first arrow is a quasi-isomorphism. This means that its cone is exact, which is preserved by $\tau_n \circ \Phi_R$. Thus the homomorphisms of complexes

$$\tau_n\Phi_R(H_*) \leftarrow \tau_n\Phi_R(H''_*) \to \tau_n\Phi_R(H'_*)$$

induce a homomorphism of pre-displays $\Phi'_{n,R}(g) : \Phi'_{n,R}(G) \to \Phi'_{n,R}(G')$ on the cokernels. It is easy to verify that $\Phi'_{n,R}$ is a well-defined functor, which is independent of the chosen isogenies; see also [La2, 8.5] and [La3, 4.1].

Since Φ_R and τ_n are compatible with base change in R, for a ring homomorphism $\alpha : R \to R'$ and for G in $(p\text{-div}_n/R)'$ we get a natural homomorphism of pre-displays over R',

$$u' : W_n(R') \otimes_{W_n(R)} \Phi'_{n,R}(G) \to \Phi'_{n,R'}(G \otimes_R R').$$

If α is ind-étale, Corollary 3.8 implies that u' is an isomorphism. Since W_n preserves ind-étale coverings of rings by [LZ, Prop. A.8], cf. Remark 3.7, ind-étale descent is effective for pre-displays of level n.

Assume that G is the p^n-torsion of a p-divisible group H over R. Then we can use the isogeny $p^n : H \to H$ in the construction of $\Phi'_{n,R}(G)$. Since p^n annihilates $W_n(R)$, it follows that $\Phi'_{n,R}(G) = \tau_n\Phi_R(H)$. In particular, in this case the pre-display $\Phi'_{n,R}(G)$ is a truncated display of level n.

For each $G \in (p\text{-div}_n/R)$ there is a sequence of faithfully flat smooth ring homomorphisms $R = R_0 \to R_1 \to R_2 \cdots$ such that, if we write $R' = \varinjlim R_i$, the group $G \otimes_R R'$ is the p^n-torsion of a p-divisible group over R'; see Lemma 1.1. Since a surjective smooth morphism of schemes has a section étale locally in the base, we can assume that $R \to R'$ is ind-étale. By Lemma 3.9 it follows that the image of $\Phi'_{n,R}$ lies in (disp_n/R), and by ind-étale descent we get a unique extension of $\Phi'_{n,R}$ to a functor $\Phi_{n,R}$ as in the proposition which is compatible with ind-étale base change in R.

For an arbitrary homomorphism $\alpha : R \to R'$ of $\mathbb{F}_p$-algebras and for G in $(p\text{-div}_n/R)$, by ind-étale descent, the above homomorphisms u' induce a homomorphism of pre-displays over R',

$$u : W_n(R') \otimes_{W_n(R)} \Phi_{n,R}(G) \to \Phi_{n,R'}(G \otimes_R R').$$

Since $\Phi_{n,R}(G)$ and $\Phi_{n,R'}(G \otimes R')$ are truncated displays, u induces a base change homomorphism of truncated displays over R',

$$\tilde{u} : \alpha_*\Phi_{n,R}(G) \to \Phi_{n,R'}(G \otimes_R R').$$

We claim that $\tilde{u}$ is an isomorphism. If G is the p^n-torsion of a p-divisible group H over R, this is true because then $\Phi'_{n,R}(G) = \tau_n\Phi_R(H)$. The general case follows by passing to an ind-étale covering of R. $\square$

Remark 4.2. For the construction of $\Phi_{n,R}$ as a functor from $(p\text{-div}_n/R)$ to (pre-disp$_n/R$) one can also use the theorem of Raynaud [BBM, 3.1.1] that a commutative finite flat group scheme can be embedded into an Abelian variety locally in the base. However, an additional argument is needed to ensure that the image of $\Phi_{n,R}$ consists of truncated displays.

Remark 4.3. If k is a perfect field of characteristic p, in view of Lemma 3.5, the functor $\Phi_{n,k} : (p\text{-div}_n/k) \to (\text{disp}_n/k)$ is an equivalence of categories by classical Dieudonné theory.

Remark 4.4. The definition of the dual display carries over to truncated displays, and the functor $\Phi_{n,R}$ preserves duality because this holds for Φ_R; see Remark 2.3. We leave out the details.

4.2. **Smoothness.** The functors $\Phi_{n,R}$ for variable $\mathbb{F}_p$-algebras R induce a morphism of algebraic stacks over $\mathbb{F}_p$,

$$\phi_n : \overline{\mathscr{BT}_n} \to \mathscr{D}isp_n.$$

The source and target of ϕ_n are smooth over $\mathbb{F}_p$ of pure dimension zero. For a perfect field k of characteristic p the functor

$$\phi_n(k) : \overline{\mathscr{BT}_n}(k) \to \mathscr{D}isp_n(k)$$

is an equivalence; see Remark 4.3.

Theorem 4.5. *The morphism ϕ_n is smooth and surjective.*

Proof. Let $\mathscr{U} \subset \overline{\mathscr{BT}_n}$ be the open substack where ϕ_n is smooth. We consider a geometric point in $\overline{\mathscr{BT}_n}(k)$ for an algebraically closed field k, given by a truncated Barsotti-Tate group G over k. The tangent space $t_G(\overline{\mathscr{BT}_n})$ is the set of isomorphism classes of deformations of G over $k[\varepsilon]$. Let $\mathscr{P} = \phi_n(G)$. Since $\overline{\mathscr{BT}_n}$ and $\mathscr{D}isp_n$ are smooth, G lies in $\mathscr{U}(k)$ if and only if ϕ_n induces a surjective map on tangent spaces

$$t_G(\phi_n) : t_G(\overline{\mathscr{BT}_n}) \to t_{\mathscr{P}}(\mathscr{D}isp_n).$$

There is a p-divisible group H over k such that $G \cong H[p^n]$. Let $\tilde{\mathscr{P}} = \Phi_k(H)$ be the associated display; thus $\mathscr{P} = \tau_n\tilde{\mathscr{P}}$. We have a commutative square of tangent spaces

$$\begin{tikzcd} t_H(\overline{\mathscr{BT}}) \arrow[r, "t_H(\phi)"] \arrow[d, "t_H(\tau)"'] & t_{\tilde{\mathscr{P}}}(\mathscr{D}isp) \arrow[d, "t_{\tilde{\mathscr{P}}}(\tau)"] \\ t_G(\overline{\mathscr{BT}_n}) \arrow[r, "t_G(\phi_n)"] & t_{\mathscr{P}}(\mathscr{D}isp_n), \end{tikzcd}$$

where τ denotes the truncation morphisms and where $\phi : \overline{\mathscr{BT}} \to \mathscr{D}isp$ is induced by the functors Φ_R for $\mathbb{F}_p$-algebras R. Here $t_{\tilde{\mathscr{P}}}(\tau)$ is surjective because the truncation morphisms $\mathscr{D}isp_{m+1} \to \mathscr{D}isp_m$ for $m \geq n$ are smooth.

If G is infinitesimal or unipotent, H is infinitesimal or unipotent as well, and the map $t_H(\phi)$ is bijective by Corollary 2.8. Thus $\mathscr{U}$ contains all infinitesimal and unipotent groups, and $\mathscr{U} = \overline{\mathscr{BT}_n}$ by Lemma 4.6 below. □

Lemma 4.6. *Let $\mathscr{U}$ be an open substack of $\overline{\mathscr{BT}_n}$ that contains all points which correspond to infinitesimal or unipotent groups. Then $\mathscr{U} = \overline{\mathscr{BT}_n}$.*

Proof. For an algebraically closed field k and $G \in \overline{\mathscr{BT}_n}(k)$ we have to show that G lies in $\mathscr{U}(k)$. We write $G = H[p^n]$ for a p-divisible group H over k. Let K be an algebraic closure of $k((t))$ and let R be the ring of integers of K. Let ν be the Newton polygon of H and let β be the unique linear Newton polygon with $\beta \preceq \nu$. By [O1, Thm. 3.2] there is a p-divisible group H'' over R with generic Newton polygon ν and special Newton polygon β. Since K is algebraically closed, there is an isogeny $H''_K \to H \otimes_k K$. Let C be its kernel, let $C_R \subset H''$ be the schematic closure of C, and let $H' = H''/C_R$. Then $H'_K \cong H \otimes_k K$, and the special fibre H'_k is isoclinic. We obtain a commutative diagram where g is given by G, and g' is given by $H'[p^n]$:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "g"] \\ \operatorname{Spec} R \arrow[r, "g'"] & \overline{\mathscr{BT}_n}. \end{tikzcd}$$

Here $g'^{-1}(\mathscr{U})$ is an open subset of $\operatorname{Spec} R$, which contains the closed point since the special fibre of $H'[p^n]$ is unipotent or infinitesimal. Thus $g'^{-1}(\mathscr{U})$ is all of $\operatorname{Spec} R$, which implies that G lies in $\mathscr{U}(k)$. $\square$

We consider the diagonal morphism

$$\Delta : \overline{\mathscr{BT}_n} \to \overline{\mathscr{BT}_n} \times_{\mathscr{D}isp_n} \overline{\mathscr{BT}_n}$$

and view it as a morphism over $\overline{\mathscr{BT}_n} \times \overline{\mathscr{BT}_n}$. Let X be an affine $\mathbb{F}_p$-scheme. For $g : X \to \overline{\mathscr{BT}_n} \times \overline{\mathscr{BT}_n}$, corresponding to two truncated Barsotti-Tate groups G_1 and G_2 over X, with associated truncated displays $\mathscr{P}_1$ and $\mathscr{P}_2$, the inverse image of Δ under g is the morphism of affine X-schemes

$$\underline{\operatorname{Isom}}(G_1, G_2) \to \underline{\operatorname{Isom}}(\mathscr{P}_1, \mathscr{P}_2).$$

For $G \in \overline{\mathscr{BT}}_n(X)$, with associated truncated display $\mathscr{P}$, let

$$\underline{\operatorname{Aut}}^o(G) = \operatorname{Ker}(\underline{\operatorname{Aut}}\, G \to \underline{\operatorname{Aut}}\, \mathscr{P}).$$

This is an affine group scheme over X. For varying X and G we obtain a relative affine group scheme $\underline{\operatorname{Aut}}^o(G^{\mathrm{univ}})$ over $\overline{\mathscr{BT}_n}$. Let

$$\pi_1, \pi_2 : \overline{\mathscr{BT}_n} \times_{\mathscr{D}isp_n} \overline{\mathscr{BT}_n} \to \overline{\mathscr{BT}_n}$$

be the two projections.

Theorem 4.7. *The representable affine morphism Δ is finite, flat, radicial, and surjective. The group scheme $\underline{\operatorname{Aut}}^o(G^{\mathrm{univ}}) \to \overline{\mathscr{BT}_n}$ is commutative and finite flat, and Δ is a torsor under $\pi_i^* \underline{\operatorname{Aut}}^o(G^{\mathrm{univ}})$ for $i = 1, 2$.*

Proof. We write $\mathscr{X} = \overline{\mathscr{BT}_n}$ and $\mathscr{Y} = \mathscr{D}isp_n$. Let $\pi : X \to \mathscr{X}$ be a smooth presentation with affine X. We can assume that X has pure dimension m, which implies that π has pure dimension m. By Theorem 4.5, the composition $\psi = \phi_n \circ \pi : X \to \mathscr{X} \to \mathscr{Y}$ is a smooth presentation of pure dimension m as well. It follows that $X' = X \times_{\mathscr{X}} X$ and $Y' = X \times_{\mathscr{Y}} X$ are smooth $\mathbb{F}_p$-schemes of pure dimension $2m$. The natural morphism $\phi' : X' \to Y'$ can be identified with the inverse image of $\Delta : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ under the smooth presentation $X \times X \to \mathscr{X} \times \mathscr{X}$.

Since $\phi_n : \mathscr{X} \to \mathscr{Y}$ is an equivalence on geometric points, ϕ' is bijective on geometric points. Since X' and Y' are equidimensional, the irreducible components of X' are in bijection to the irreducible components of Y'. Thus ϕ' is flat; see [Ma, Thm. 23.1]. Let Z be the normalisation of Y' in the purely inseparable extension of function fields defined by $X' \to Y'$. Then $Z \to Y'$ is bijective on geometric points, so $X' \to Z$ is bijective on geometric points, and $X' = Z$ by Zariski's main theorem. Thus ϕ' is finite, flat, radicial, and surjective, which implies that Δ is finite, flat, radicial, and surjective.

Recall that a morphism $T \to S$ with an action of an S-group A on T is called a quasi-torsor if for each $S' \to S$ the fibre $T(S')$ is either empty or isomorphic to $A(S')$ as an $A(S')$-set. Clearly Δ is a quasi-torsor under the obvious right action of $\pi_1^* \underline{\operatorname{Aut}}^o(G^{\mathrm{univ}})$ and under the obvious left action of $\pi_2^* \underline{\operatorname{Aut}}^o(G^{\mathrm{univ}})$. Since Δ is finite, flat, and surjective, it follows that the quasi-torsor Δ is a torsor, and that $\pi_i^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ is finite and flat over $\mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$. Since ϕ_n is smooth and surjective, the same holds for the projections π_i, and it follows that $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ is finite and flat over $\mathscr{X}$.

It remains to show that $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ is commutative. It suffices to show this on a dense open substack of $\mathscr{X}$, and thus is suffices to show that for the finite group scheme $G = (\mathbb{Z}/p^n\mathbb{Z})^r \times (\mu_{p^n})^s$ over a field k the k-group scheme $\underline{\mathrm{Aut}}^o(G)$ is commutative. Now $\underline{\mathrm{Hom}}(\mu_{p^n}, \mathbb{Z}/p^n\mathbb{Z})$ is zero, and the group schemes $\underline{\mathrm{Aut}}((\mathbb{Z}/p^n\mathbb{Z})^r)$ and $\underline{\mathrm{Aut}}((\mu_{p^n})^s)$ are étale. Since ϕ_n is an equivalence on geometric points, it follows that $\underline{\mathrm{Aut}}^o(G)$ is contained in the group scheme $\{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \mid a \in \mu_{p^n}^{rs}\}$, which is commutative. □

Remark 4.8. For $G = (\mathbb{Z}/p^n\mathbb{Z})^r \times (\mu_{p^n})^s$ as above, $\underline{\mathrm{Aut}}^o(G)$ is in fact equal to $\{\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right) \mid a \in \mu_{p^n}^{rs}\}$. Since ordinary groups are dense in $\overline{\mathscr{BT}_n}$, it follows that on the open and closed substack of $\overline{\mathscr{BT}_n}$ where the universal group has dimension s and codimension r, the degree of the finite flat group scheme $\underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ is equal to p^{rsn}.

To prove the first equality, it suffices to show that the truncated displays $\mathscr{P}_1 = \Phi_{n,\mathbb{F}_p}(\mathbb{Z}/p^n\mathbb{Z})$ and $\mathscr{P}_2 = \Phi_{n,\mathbb{F}_p}(\mu_{p^n})$ satisfy $\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2) = 0$. For an $\mathbb{F}_p$-algebra R, if $i : I_{n+1,R} \to W_n(R)$ denotes the natural homomorphism, we have

$$\mathscr{P}_1 = (W_n(R), W_n(R), \mathrm{id}, i, f, pf),$$

$$\mathscr{P}_2 = (W_n(R), I_{n+1,R}, i, \mathrm{id}, f_1, f).$$

Thus $\underline{\mathrm{Hom}}(\mathscr{P}_1, \mathscr{P}_2)$ can be identified with the set of $a \in I_{n+1,R}$ such that $f_1(a) = i(a)$, or equivalently $a = v(i(a))$, which implies that $a = 0$.

By passing to the limit, Theorems 4.5 and 4.7 give the following information on the morphism $\phi : \overline{\mathscr{BT}} \to \mathscr{D}isp$. For a p-divisible group G over an $\mathbb{F}_p$-algebra R with associated display $\mathscr{P}$ let $\underline{\mathrm{Aut}}^o(G)$ be the kernel of $\underline{\mathrm{Aut}}(G) \to \underline{\mathrm{Aut}}(\mathscr{P})$. This is an affine group scheme over R, which is the projective limit over n of the finite flat group schemes $\underline{\mathrm{Aut}}^o(G[p^n])$; thus $\underline{\mathrm{Aut}}^o(G)$ is commutative and flat. Let G^{univ} be the universal p-divisible group and let $\pi_1, \pi_2 : \mathscr{D}isp \times_{\overline{\mathscr{BT}}} \mathscr{D}isp \to \mathscr{D}isp$ be the two projections.

Corollary 4.9. *The morphism ϕ is faithfully flat, and its diagonal is a torsor under the flat affine group scheme $\pi_i^* \underline{\mathrm{Aut}}^o(G^{\mathrm{univ}})$ for $i = 1, 2$.* □

The limit $\underline{\mathrm{Aut}}^o(G) = \varprojlim_n \underline{\mathrm{Aut}}^o(G[p^n])$ can show quite different behaviour depending on G; see Corollary 5.6 in the next section.

5. Classification of formal p-divisible groups

As an application of Theorems 4.5 and 4.7 together with Corollary 2.8 we will prove the following.

Theorem 5.1. *For each p-adic ring R, the functor Φ_R from p-divisible groups over R to displays over R induces an equivalence*

$$\Phi_R^1 : (\textit{formal } p\textit{-divisible groups}/R) \to (\textit{nilpotent displays}/R).$$

It is known by [Zi1] and [La1] that the functor BT_R defined in [Zi1] from displays over R to formal Lie groups over R induces an equivalence between nilpotent displays over R and formal p-divisible groups over R. The relation between Φ_R and BT_R is discussed in section 8.

Lemma 5.2. *For $n \geq 1$ there is a Cartesian diagram of Artin algebraic stacks*

$$\begin{CD} \mathscr{BT}_n^o @>{\phi_n^o}>> \mathscr{D}isp_n^o \\ @VVV @VVV \\ \overline{\mathscr{BT}_n} @>{\phi_n}>> \mathscr{D}isp_n, \end{CD}$$

where the vertical arrows are the immersions given by Lemmas 1.4 *and* 3.17*, and where ϕ_n is given by the functors $\Phi_{n,R}$. The projective limit over n is a Cartesian diagram of affine algebraic stacks*

$$\begin{CD} \mathscr{BT}^o @>{\phi^o}>> \mathscr{D}isp^o \\ @VVV @VVV \\ \overline{\mathscr{BT}} @>{\phi}>> \mathscr{D}isp, \end{CD}$$

where ϕ is given by the functors Φ_R.

Proof. Since ϕ_n is smooth, the inverse image of $\mathscr{D}isp_n^o$ under ϕ_n is a reduced closed substack $\mathscr{BT}_n'$ of $\overline{\mathscr{BT}_n}$. By classical Dieudonné theory, the geometric points of $\mathscr{BT}_n'$ and of $\mathscr{BT}_n^o$ coincide; thus $\mathscr{BT}_n^o = \mathscr{BT}_n'$, and we get the first Cartesian square. The projective limit of $\mathscr{BT}_n^o$ is $\mathscr{BT}^o$ by Lemma 1.4, and the projective limit of $\mathscr{D}isp_n^o$ is $\mathscr{D}isp^o$ by Lemma 3.17. Hence the second Cartesian square follows from the first one. $\square$

The essential part of Theorem 5.1 is the following result.

Theorem 5.3. *The morphism $\phi^o : \mathscr{BT}^o \to \mathscr{D}isp^o$ is an isomorphism.*

Proof. We use the following notation:

$$\mathscr{X} = \mathscr{BT}^o, \qquad \mathscr{X}_n = \mathscr{BT}_n^o,$$

$$\mathscr{Y} = \mathscr{D}isp^o, \qquad \mathscr{Y}_n = \mathscr{D}isp_n^o.$$

As in the proof of Lemma 1.1 we choose smooth presentations $X_n \to \mathscr{X}_n$ with affine X_n such that the truncation morphisms $\mathscr{X}_{n+1} \to \mathscr{X}_n$ lift to morphisms $X_{n+1} \to X_n$, where $X_{n+1} \to X_n \times_{\mathscr{X}_n} \mathscr{X}_{n+1}$ is smooth and surjective. Since $\mathscr{X}_n \to \mathscr{Y}_n$ is smooth and surjective by Theorem 4.5 and Lemma 5.2, the composition $X_n \to \mathscr{X}_n \to \mathscr{Y}_n$ is a smooth presentation. Let $X = \varprojlim_n X_n$ and

$$X_n' = X_n \times_{\mathscr{X}_n} X_n, \qquad X' = X \times_{\mathscr{X}} X = \varprojlim_n X_n',$$

$$Y_n' = X_n \times_{\mathscr{Y}_n} X_n, \qquad : Y' = X \times_{\mathscr{Y}} X = \varprojlim_n Y_n'.$$

Here $X \to \mathscr{X}$ is a faithfully flat presentation because for $Z \to \mathscr{X}$ with affine Z we have $Z \times_{\mathscr{X}} X = \varprojlim_n (Z \times_{\mathscr{X}_n} X_n)$, and a projective limit of faithfully flat affine Z-schemes is a faithfully flat affine Z-scheme. Similarly, the composition

$X \to \mathscr{X} \to \mathscr{Y}$ is a faithfully flat presentation. We have an infinite commutative diagram:

$$\begin{tikzcd} X_1' \arrow[d] & X_2' \arrow[l] \arrow[d] & X_3' \arrow[l] \arrow[d] & \cdots \arrow[l] \\ Y_1' & Y_2' \arrow[l] & Y_3' \arrow[l] & \cdots \arrow[l] . \end{tikzcd}$$

The theorem means that the limit $X' \to Y'$ is an isomorphism.

Let G_n be the infinitesimal truncated Barsotti-Tate group over X_n which defines the presentation $X_n \to \mathscr{X}_n$ and let $\pi_n : Y_n' \to X_n$ be the first projection. By Theorem 4.7, $X_n' \to Y_n'$ is a torsor under the commutative infinitesimal finite flat group scheme $A_n = \underline{\mathrm{Aut}}^o(\pi_n^* G_n)$. The truncation induces a homomorphism of finite flat group schemes over Y_{n+1}'

$$\psi_n : A_{n+1} \to A_n \times_{Y_n'} Y_{n+1}'$$

and a morphism

$$X_{n+1}' \to X_n' \times_{Y_n'} Y_{n+1}'$$

which is equivariant with respect to ψ_n.

Lemma 5.4. *For each m there is an $n \geq m$ such that the transition homomorphism*

$$\psi_{m,n} : A_n \to A_m \times_{Y_m'} Y_n'$$

is zero.

If Lemma 5.4 is proved, it follows that there is a unique diagonal morphism which makes the following diagram commute:

$$\begin{tikzcd} X_m' \arrow[d] & X_n' \arrow[l] \arrow[d] \\ Y_m' & Y_n' . \arrow[l] \arrow[ul] \end{tikzcd}$$

Thus $\varprojlim_n X_n' \to \varprojlim_n Y_n'$ is an isomorphism, and Theorem 5.3 follows.

The essential case of Lemma 5.4 is the following. Consider a geometric point $y : \operatorname{Spec} k \to Y'$ for an algebraically closed field k. Let $A_{n,k} = A_n \times_{Y_n'} \operatorname{Spec} k$ and $Z_n = X_n' \times_{Y_n'} \operatorname{Spec} k$. Thus Z_n is an $A_{n,k}$-torsor. We have homomorphisms of finite k-group schemes $A_{n+1,k} \to A_{n,k}$ and equivariant morphisms $Z_{n+1} \to Z_n$. Since k is algebraically closed and since $A_{n,k}$ is infinitesimal, $Z_n(k)$ has precisely one element, and there are compatible isomorphisms $Z_n \cong A_{n,k}$. For each m the images of $A_{n,k} \to A_{m,k}$ for $n \geq m$ stabilise to a subgroup scheme A_m' of $A_{m,k}$, and $A_{m+1}' \to A_m'$ is an epimorphism. Let $A_n' = \operatorname{Spec} B_n$ and $B = \varinjlim_n B_n$.

The geometric point $y : \operatorname{Spec} k \to Y'$ corresponds to two formal p-divisible groups G_1 and G_2 over k together with an isomorphism of the associated displays $\alpha : \Phi_k(G_1) \cong \Phi_k(G_2)$. The k-scheme $X' \times_{Y'} \operatorname{Spec} k$ can be identified with $\varprojlim_n Z_n = \varprojlim_n A_n' = \operatorname{Spec} B$ and classifies lifts of α to an isomorphism $G_1 \cong G_2$. Thus Corollary 2.8 implies that B is a formally étale k-algebra. It follows that the cotangent complex $L_{B/k}$ has trivial homology in degrees 0 and 1; see Lemma 5.5 below. The epimorphism $A_{n+1}' \to A_n'$ induces an injective homomorphism $H_1(L_{B_n/k}) \otimes_{B_n} B_{n+1} \to H_1(L_{B_{n+1}/k})$; see [Me1, Chap. I, Prop. 3.3.4]. Since

$L_{B/k} = \varinjlim_n L_{B_n/k}$ it follows that $H_1(L_{B_n/k})$ is zero, which implies that the finite infinitesimal group scheme A'_n is zero for each n. Thus $X' \times_{Y'} \operatorname{Spec} k \cong \operatorname{Spec} k$.

Let us now prove Lemma 5.4. First we note that $X'_{n+1} \to X'_n$ is smooth and surjective because this morphism can be factored as follows:

$$X_{n+1} \times_{\mathscr{X}_{n+1}} X_{n+1} \to (X_n \times_{\mathscr{X}_n} \mathscr{X}_{n+1}) \times_{\mathscr{X}_{n+1}} (X_n \times_{\mathscr{X}_n} \mathscr{X}_{n+1}) \to X_n \times_{\mathscr{X}_n} X_n.$$

The first arrow is smooth and surjective by our assumptions on X_n, and the second arrow is smooth and surjective because this holds for $\mathscr{X}_{n+1} \to \mathscr{X}_n$ by Lemma 1.4. Since $X'_n \to Y'_n$ is faithfully flat, it follows that $Y'_{n+1} \to Y'_n$ is faithfully flat as well.

Let $U_{m,n} \subset Y'_n$ be the open set of all points y such that the fibre $(\psi_{m,n})_y$ is non-zero. Since Y'_n is reduced, it suffices to show that for each m there is an n such that $U_{m,n}$ is empty. Assume that for some m the set $U_{m,n}$ is non-empty for all $n \geq m$. Let $V_{m,n}$ be the set of generic points of $U_{m,n}$. Since $Y'_{n+1} \to Y'_n$ is flat, we have $V_{m,n+1} \to V_{m,n}$. Since $V_{m,n}$ is finite and non-empty, the projective limit over n of $V_{m,n}$ is non-empty. Hence there is a geometric point $y : \operatorname{Spec} k \to Y'$ such that $\operatorname{Spec} k \to Y'_n$ lies in $V_{m,n}$ for each $n \geq m$. If we perform the above construction for y, the group A'_m is non-zero, which is impossible. Thus Lemma 5.4 and Theorem 5.3 are proved. □

Lemma 5.5. *A ring homomorphism $\alpha : A \to B$ is formally étale if and only if the cotangent complex $L_{B/A}$ has trivial homology in degrees* 0 *and* 1.

Proof. Clearly α is formally unramified if and only if $\Omega_{B/A} = H_0(L_{B/A})$ is zero. Let us assume that this holds. Since the obstructions to formal smoothness lie in $\operatorname{Ext}^1_B(L_{B/A}, M)$ for varying B-modules M, the implication $\Leftarrow$ of the lemma is clear. So assume that α is formally smooth. We write $B = R/I$ for a polynomial ring R over A; let $C = R/I^2$. Since $L_{R/A}$ is a free R-module in degree zero, the natural homomorphisms $H_1(L_{C/A}) \to H_1(L_{C/R})$ and $H_1(L_{B/A}) \to H_1(L_{B/R})$ are injective. The homomorphism $H_1(L_{C/R}) \to H_1(L_{B/R})$ can be identified with $I^2/I^4 \to I/I^2$, which is zero. Thus $u : H_1(L_{C/A}) \to H_1(L_{B/A})$ is zero as well. Since α is formally smooth, id_B factors into A-algebra homomorphisms $B \to C \xrightarrow{\pi} B$, where π is the projection. Thus u is surjective, and $H_1(L_{B/A})$ is zero. □

Proof of Theorem 5.1. We may assume that p is nilpotent in R. Since Φ_R is an additive functor, in order to show that Φ^1_R is fully faithful it suffices to show that for two formal p-divisible groups G_1 and G_2 over R with associated nilpotent displays $\mathscr{P}_1$ and $\mathscr{P}_2$, the map $\gamma : \operatorname{Isom}(G_1, G_2) \to \operatorname{Isom}(\mathscr{P}_1, \mathscr{P}_2)$ is bijective. We define an ideal $I \subset R$ by the Cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} R/I & \longrightarrow & \mathscr{BT}^o \times \mathscr{BT}^o \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{(G_1, G_2)} & \mathscr{BT} \times \mathscr{BT}. \end{array}$$

If we write $G'_1 = G_1 \otimes_R R/I$, etc., $\operatorname{Isom}(G'_1, G'_2) \to \operatorname{Isom}(\mathscr{P}'_1, \mathscr{P}'_2)$ is bijective by Theorem 5.3. Since $\mathscr{BT}^o \to \mathscr{BT}$ is of finite presentation by Lemma 1.4 and since I is a nil-ideal, I is nilpotent. By Corollary 2.8 it follows that γ is bijective. We show that Φ^1_R is essentially surjective. Since $\mathscr{D}isp^o \to \mathscr{D}isp$ is of finite presentation by Lemma 3.17, for a given nilpotent display $\mathscr{P}$ over R we find a nilpotent ideal $I \subset R$ such that the associated morphism $\operatorname{Spec} R/I \to \operatorname{Spec} R \to \mathscr{D}isp$ factors over

$\mathscr{D}isp^o$. By Theorem 5.3, $\mathscr{P}_{R/I}$ lies in the image of $\Phi^1_{R/I}$. Thus $\mathscr{P}$ lies in the image of Φ^1_R by Corollary 2.8. $\square$

Corollary 5.6. *Let G be a p-divisible group over an $\mathbb{F}_p$-algebra R. The affine group scheme $\underline{\mathrm{Aut}}^o(G)$ is trivial if and only if for all $x \in \operatorname{Spec} R$ the fibre G_x is connected or unipotent.*

Proof. Assume that G is a p-divisible group over an algebraically closed field k such that $H = \mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$ is a direct summand of G. By Remark 4.8, the group $\underline{\mathrm{Aut}}^o(H[p^n])$ is isomorphic to μ_{p^n}, with transition maps $\mu_{p^{n+1}} \to \mu_{p^n}$ given by $\zeta \mapsto \zeta^p$. Thus $\underline{\mathrm{Aut}}^o(H)$ is non-trivial, which implies that $\underline{\mathrm{Aut}}^o(G)$ is non-trivial as well. This proves the implication $\Rightarrow$.

Assume now that all fibres of G over R are connected or unipotent. Let $\operatorname{Spec} R/I$ and $\operatorname{Spec} R/J$ be the inverse images of $\mathscr{BT}^o$ under the morphisms $\operatorname{Spec} R \to \mathscr{BT}$ defined by G and by $G^\vee$. Then I and J are finitely generated by Lemma 1.4, and IJ is a nil-ideal; thus IJ is nilpotent. In order to show that $\underline{\mathrm{Aut}}^o(G)$ is trivial we may replace R by R/IJ. Then $R \to R/I \times R/J$ is injective. Since $\underline{\mathrm{Aut}}^o(G)$ is flat over R by Corollary 4.9, we may further replace R by R/I and by R/J. Since $\underline{\mathrm{Aut}}^o(G) \cong \underline{\mathrm{Aut}}^o(G^\vee)$ by Remark 2.3, in both cases the assertion follows from Theorem 5.3. $\square$

6. Dieudonné theory over perfect rings

The results of this section were obtained earlier by Gabber by a different method. For a perfect ring R of characteristic p we consider the Dieudonné ring

$$D(R) = W(R)\{F, V\}/J,$$

where $W(R)\{F, V\}$ is the non-commutative polynomial ring in two variables over $W(R)$ and where J is the ideal generated by the relations $Fa = f(a)F$ and $aV = Vf(a)$ for $a \in W(R)$, and $FV = VF = p$.

Definition 6.1. A *projective Dieudonné module over R* is a $D(R)$-module which is a projective $W(R)$-module of finite type. A *truncated Dieudonné module of level n over R* is a $D(R)$-module M which is a projective $W_n(R)$-module of finite type; if $n = 1$ we require also that $\operatorname{Ker}(F) = \operatorname{Im}(V)$ and $\operatorname{Ker}(V) = \operatorname{Im}(F)$ and that M/VM is a projective R-module. An *admissible torsion $W(R)$-module* is a finitely presented $W(R)$-module of projective dimension ≤ 1 which is annihilated by a power of p. A *finite Dieudonné module over R* is a $D(R)$-module which is an admissible torsion $W(R)$-module.

We denote by (Dieud/R), by (Dieud_n/R), and by (Dieud^f/R) the categories of projective, truncated level n, and finite Dieudonné modules over R, respectively. For a homomorphism of perfect rings $R \to R'$ the scalar extension by $W(R) \to W(R')$ induces functors from projective, truncated, or finite Dieudonné modules over R to such modules over R'.

Lemma 6.2. *If M is a projective or truncated Dieudonné module of level ≥ 2, then M/pM is a truncated Dieudonné module of level 1.*

Proof. We can replace M by M/p^2M. The operators F and V of M induce operators $\bar{F}$ and $\bar{V}$ of $\bar{M} = M/pM$. Assume that $\bar{F}(\bar{x}) = 0$ for an element $x \in M$. Then $F(x) \in pM$ and thus $F(x) = F(V(y))$ for some $y \in M$. Hence $px = pV(y)$ and thus $\bar{x} = \bar{V}(\bar{y})$. This shows that $\operatorname{Ker}(\bar{F}) = \operatorname{Im}(\bar{F})$, and similarly $\operatorname{Ker}(\bar{V}) = \operatorname{Im}(\bar{V})$.

Since these relations remain true after base change, for each point of $\operatorname{Spec} R$ the dimensions of the fibres of M/VM and of M/FM add up to the dimension of the fibre of $\bar{M}$. Thus the dimension of the fibre of M/VM is an upper and lower semicontinuous function on $\operatorname{Spec} R$, and M/VM is a projective R-module since R is reduced. □

Lemma 6.3. *Truncated Dieudonné modules of level $n \geq 1$ over R are equivalent to truncated displays of level n over R, and projective Dieudonné modules over R are equivalent to displays over R.*

Proof. This extends Lemma 3.5. Since multiplication by p is a isomorphism $W_n(R) \cong I_{n+1,R}$ and since truncated displays have normal decompositions by Lemma 3.3, truncated displays of level n over R are equivalent to quintuples $\mathscr{P} = (P, Q, \iota, \epsilon, F_1)$ where P and Q are projective $W_n(R)$-modules of finite type with homomorphisms $P \xrightarrow{\epsilon} Q \xrightarrow{\iota} P$ such that $\epsilon\iota = p$ and $\iota\epsilon = p$, and where $F_1 : Q \to P$ is a bijective f-linear map, such that

($\star$) $\operatorname{Coker}(\iota)$ is projective over R, and $Q \xrightarrow{\iota} P \xrightarrow{p^{n-1}\epsilon} Q \xrightarrow{\iota} P$ is exact.

By the proof of Lemma 6.2, condition ($\star$) is automatic if $n \geq 2$. It follows that truncated displays of level $n \geq 1$ are equivalent to truncated Dieudonné modules of level n by $\mathscr{P} \mapsto (P, F, V)$ with $F = F_1\epsilon$ and $V = \iota F_1^{-1}$. The equivalence between displays and projective Dieudonné modules follows easily; see also [La3, Lemma 2.4]. □

Theorem 6.4. *For a perfect ring R of characteristic p, the functors*

$$\Phi_{n,R} : (p\text{-div}_n/R) \to (\mathrm{Dieud}_n/R),$$

$$\Phi_R : (p\text{-div}/R) \to (\mathrm{Dieud}/R)$$

are equivalences; these functors are well-defined by Lemma 6.3.

Here Φ_R and $\Phi_{n,R}$ are defined by $G \mapsto (\mathbb{D}(G)_{W(R)/R}, F, V)$, where $\mathbb{D}(G)$ is the covariant Dieudonné crystal of G, and where F and V are induced by the Verschiebung $G^{(1)} \to G$ and Frobenius $G \to G^{(1)}$.

Proof. For an $\mathbb{F}_p$-algebra A, the perfection A^{per} is the direct limit of Frobenius $A \to A \to \cdots$. For an $\mathbb{F}_p$-scheme X, the perfection X^{per} is the projective limit of Frobenius $X \leftarrow X \leftarrow \cdots$. This is a local construction, which coincides with the perfection of rings in the case of affine schemes.

Since the functor $\Phi_{n,R}$ is additive, it is fully faithful if for two groups G_1 and G_2 in $(p\text{-div}_n/R)$ with associated truncated displays $\mathscr{P}_1$ and $\mathscr{P}_2$, the map $\gamma : \operatorname{Isom}(G_1, G_2) \to \operatorname{Isom}(\mathscr{P}_1, \mathscr{P}_2)$ induced by $\Phi_{n,R}$ is bijective. The morphism of affine R-schemes $\underline{\operatorname{Isom}}(G_1, G_2) \to \underline{\operatorname{Isom}}(\mathscr{P}_1, \mathscr{P}_2)$ is a torsor under an infinitesimal finite flat group scheme by Theorem 4.7. Thus $\underline{\operatorname{Isom}}(G_1, G_2)^{\mathrm{per}} \to \underline{\operatorname{Isom}}(\mathscr{P}_1, \mathscr{P}_2)^{\mathrm{per}}$ is an isomorphism, which implies that γ is bijective.

Let us show that $\Phi_{n,R}$ is essentially surjective. If $X \to \mathscr{BT}_n$ is a smooth presentation with affine X, the composition $X \to \mathscr{BT}_n \to \mathscr{D}isp_n$ is smooth and surjective by Theorem 4.5. For a given truncated display $\mathscr{P}$ of level n over R let $\operatorname{Spec} S = X \times_{\mathscr{D}isp_n} \operatorname{Spec} R$ and let $R' = S^{\mathrm{per}}$. Then R' is a perfect faithfully flat R-algebra such that $\mathscr{P}_{R'}$ lies in the image of $\Phi_{n,R'}$. Since $R'' = R' \otimes_R R'$ is perfect, the functors $\Phi_{n,R'}$ and $\Phi_{n,R''}$ are fully faithful. Thus $\mathscr{P}$ lies in the image of $\Phi_{n,R}$ by faithfully flat descent.

By passing to the projective limit it follows that Φ_R is an equivalence. $\square$

We denote by (p-grp$/R$) the category of commutative finite flat group schemes of p-power order over R.

Corollary 6.5. *The covariant Dieudonné crystal defines a functor*

$$\Phi_R^f : (p\text{-grp}/R) \to (\mathrm{Dieud}^f/R)$$

which is an equivalence of categories.

Over a perfect field this is classical, over perfect valuation rings the result is proved in [Be], and the general case was first proved by Gabber by a reduction to the case of valuation rings. Theorem 6.4 is an immediate consequence of Corollary 6.5.

The functor Φ_R^f can be defined by $G \mapsto (\mathbb{D}(G)_{W(R)}, F, V)$ as above.

Proof of Corollary 6.5. By a standard construction, the functor Φ_R and its inverse Φ_R^{-1} induce formally a functor Φ_R^f as in Corollary 6.5 and a functor Ψ_R^f in the opposite direction, which are mutually inverse; this new definition of Φ_R^f coincides with the previous one by the construction of Φ_R.

Let us explain this in more detail. Since Zariski descent is effective for finite flat group schemes and for finite Dieudonné modules, in order to define Φ_R^f and Ψ_R^f we may always pass to an open cover of $\operatorname{Spec} R$. For each group G in (p-grp$/R$), by a theorem of Raynaud [BBM, Thm. 3.1.1] there is an open cover of $\operatorname{Spec} R$ where G can be written as the kernel of an isogeny of p-divisible groups $H_0 \to H_1$. We define $\Phi_R^f(G) = \mathrm{Coker}[\Phi_R(H_0) \to \Phi_R(H_1)]$. This is independent of the chosen isogeny, functorial in G, and compatible with localisations in R; see the proof of Proposition 4.1.

A homomorphism of projective Dieudonné modules $u : N_0 \to N_1$ over R is called an isogeny if u becomes bijective when p is inverted. Then u is injective, and its cokernel is a finite Dieudonné module M. In this case we define $\Psi_R^f(M) = \mathrm{Ker}[\Phi_R^{-1}(N_0) \to \Phi_R^{-1}(N_1)]$. This depends only on M, the construction is functorial in M, compatible with localisations in R, and inverse to Φ_R^f when it is defined.

It remains to show that each finite Dieudonné module M over R can be written as the cokernel of an isogeny of projective Dieudonné modules locally in $\operatorname{Spec} R$. It is easy to find a commutative diagram

$$\begin{tikzcd}
N_1 \arrow[r, "\epsilon"] \arrow[d, "\pi"] & Q \arrow[r, "\iota"] \arrow[d, "\psi"] & N_1 \arrow[d, "\pi"] \\
M \arrow[r, "F"] & M \arrow[r, "V"] & M,
\end{tikzcd}$$

where Q and N are free $W(R)$-modules of the same finite rank, where ι, ϵ, π are $W(R)$-linear maps, and where ψ is an f-linear map, such that $\iota\epsilon = p$ and $\epsilon\iota = p$ and π, ψ are surjective. The kernel of π is a projective $W(R)$-module N_0 of finite type. If we find a bijective f-linear map $F_1 : Q \to N_1$ with $\pi F_1 = \psi$, we can define $F = F_1\epsilon$ and $V = \iota F_1^{-1}$, and M is the cokernel of the isogeny of Dieudonné modules $N_0 \to N_1$. Thus it suffices to show that F_1 exists locally in $\operatorname{Spec} R$, which is an easy application of Nakayama's lemma. $\square$

7. Small presentations

In addition to the infinite-dimensional presentation of the stack $\mathscr{BT}$ constructed in Lemma 1.1 one can also find presentations where the covering space is Noetherian, or even of finite type. This will be used in section 8. Assume that G is a p-divisible group of height h over a $\mathbb{Z}_p$-algebra A. It defines a morphism

$$\pi : \operatorname{Spec} A \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p.$$

For each positive integer m we also consider the restriction

$$\pi^{(m)} : \operatorname{Spec} A/p^m A \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}/p^m\mathbb{Z}.$$

We recall that the points of $\overline{\mathscr{BT}}$ are pairs (k, H), where H is a p-divisible group of height h over a field k of characteristic p, modulo the equivalence relation with $(k, H) \sim (k', H')$ if and only if there is a common extension k'' of k and of k' such that $H_{k''} \cong H'_{k''}$.

Proposition 7.1. *There is a pair (A, G) with the following properties:*

(i) *The ring A is p-adic and excellent.*
(ii) *For each maximal ideal $\mathfrak{m}$ of A, the residue field $A/\mathfrak{m}$ is perfect, and the group $G \otimes \hat{A}_{\mathfrak{m}}$ is a universal deformation of its special fibre.*
(iii) *All points of $\overline{\mathscr{BT}}$ which correspond to isoclinic p-divisible groups lie in the image of π.*

Proposition 7.2. *Assume that (A, G) satisfies (i) and (ii). Then π is ind-smooth, and $\pi^{(m)}$ is ind-smooth and quasi-étale. If (iii) holds as well, the morphisms π and $\pi^{(m)}$ are also faithfully flat.*

We note the following consequence of Propositions 7.1 and 7.2.

Corollary 7.3. *There is a presentation $\pi : \operatorname{Spec} A \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ such that A is an excellent p-adic ring, π is ind-smooth, and the residue fields of the maximal ideals of A are perfect.* $\square$

Let us prove Proposition 7.1. As explained in [NVW, Sec. 2], pairs (A, G) that satisfy (i)–(iii) can be constructed using integral models of suitable PEL-Shimura varieties. In that case, $A/p^n A$ is smooth over $\mathbb{Z}/p^n\mathbb{Z}$. For completeness we give another construction using the Rapoport-Zink isogeny spaces of p-divisible groups.

Proof of Proposition 7.1. Let $\mathbb{G}$ be a p-divisible group over $\mathbb{F}_p$ which is descent in the sense of [RZ, Def. 2.13]. By [RZ, Thm. 2.16] there is a formal scheme M over $\operatorname{Spf} \mathbb{Z}_p$ which is locally formally of finite type and which represents the following functor on the category of rings R in which p is nilpotent: $M(R)$ is the set of isomorphism classes of pairs (G, ρ), where G is a p-divisible group over R and where $\rho : \mathbb{G} \otimes R/pR \to G \otimes_R R/pR$ is a quasi-isogeny. Let G^{univ} be the universal group over M.

If $U = \operatorname{Spf} A$ is an affine open subscheme of M, then A is an I-adic Noetherian $\mathbb{Z}_p$-algebra such that A/I is of finite type over $\mathbb{F}_p$. Thus A is p-adic and excellent; see [Va, Thm. 9]. The restriction of G^{univ} to U defines a p-divisible group G over $\operatorname{Spec} A$ because p-divisible groups over $\operatorname{Spf} A$ and over $\operatorname{Spec} A$ are equivalent; see [Me1, Lemma 4.16]. For each maximal ideal $\mathfrak{m}$ of A the residue field $A/\mathfrak{m}$ is finite, and the definition of M implies that $G \otimes \hat{A}_{\mathfrak{m}}$ is a universal deformation. Thus (i) and (ii) hold.

For $0 \leq d \leq h$ let $\mathbb{G}_d$ be a decent isoclinic p-divisible group of height h and dimension d over $\mathbb{F}_p$, and let M_d be the associated isogeny space over $\operatorname{Spf} \mathbb{Z}_p$. It is well known that there is a positive integer δ depending on h such that if two p-divisible groups of height h over an algebraically closed field k are isogeneous, then there is an isogeny between them of degree at most δ. Thus there is a finite set of affine open subschemes $\operatorname{Spf} A_{d,i}$ of M_d such that each isoclinic p-divisible group of height h over k appears in the universal group over $\operatorname{Spec} A_{d,i,\mathrm{red}}$ for some (d,i). Let A be the product of all $A_{d,i}$ and let G be the p-divisible group over A defined by the universal groups over $A_{d,i}$. Then (A,G) satisfies (i)–(iii). $\square$

Let us prove the first part of Proposition 7.2.

Lemma 7.4. *If (A,G) satisfies (i) and (ii), then π is ind-smooth.*

Proof. Let $X = \operatorname{Spec} A$ and $\mathscr{X} = \mathscr{B}\mathscr{T} \times \operatorname{Spec} \mathbb{Z}_p$ and $\mathscr{X}_n = \mathscr{B}\mathscr{T}_n \times \operatorname{Spec} \mathbb{Z}_p$. For $T \to \mathscr{X}$ with affine T, the affine scheme $X \times_{\mathscr{X}} T$ is the projective limit of the affine schemes $X \times_{\mathscr{X}_n} T$. Since ind-smooth homomorphisms are stable under direct limits of rings, to prove the lemma it suffices to show that the composition

$$\pi_n : X \xrightarrow{\pi} \mathscr{X} \to \mathscr{X}_n$$

is ind-smooth for each $n \geq 1$. By Popescu's theorem this means that π_n is regular. By EGA IV, 6.8.3 this holds if and only if for each closed point $x \in X$ the composition

$$\hat{X} = \operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X \to \mathscr{X}_n$$

is regular. Let k be the perfect residue field of x. Since $\mathbb{Z}_p \to W(k)$ is regular it suffices to show that the resulting map

$$\hat{X} \to \mathscr{X}_{n,k} = \mathscr{X}_n \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} W(k)$$

is regular. Let $Y \to \mathscr{X}_{n,k}$ be a smooth presentation such that $\pi_n(x)$ lifts to a k-valued point y of Y. One can find a p-divisible group H over $\hat{Y} = \operatorname{Spec} \hat{\mathcal{O}}_{Y,y}$ such that the special fibre H_y is equal to G_x and such that the truncation $H[p^n]$ is the inverse image of the universal group over $\mathscr{X}_n$; see [Il2, Thm. 4.4]. Since $G_{\hat{X}}$ is assumed to be universal, we get a unique $\psi : \hat{Y} \to \hat{X}$ such that $\psi^* G_{\hat{X}}$ is equal to H as deformations of G_x. Thus we have the following commutative diagram with $\mathscr{X}_k = \mathscr{X} \times_{\operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} W(k)$:

$$\begin{tikzcd}
\hat{X} \arrow[r] & X \arrow[r, "G"] & \mathscr{X}_k \arrow[r] & \mathscr{X}_{n,k} \\
& \hat{Y} \arrow[ul, "\psi"] \arrow[ur, "H"'] \arrow[rr] & & Y. \arrow[u]
\end{tikzcd}$$

Since $\mathscr{X}_n$ is smooth over $\mathbb{Z}_p$ by [Il2, Thm. 4.4], $\hat{\mathcal{O}}_{Y,y}$ is a power series ring over $W(k)$. The same holds for $\hat{\mathcal{O}}_{X,x}$ since this is a universal deformation ring. The diagram induces a diagram of the tangent spaces at the images of the closed point of $\hat{Y}$. Here $\hat{X} \to \mathscr{X}_k$ is bijective on the tangent spaces since $G_{\hat{X}}$ is a universal deformation, $\mathscr{X}_k \to \mathscr{X}_{n,k}$ is bijective on the tangent spaces by [Il2, Thm. 4.4], and $\hat{Y} \to \mathscr{X}_{n,k}$ is surjective on the tangent spaces since $Y \to \mathscr{X}_{n,k}$ is smooth. Thus ψ is surjective on the tangent spaces, which implies that $\hat{\mathcal{O}}_{Y,y}$ is a power series ring over

$\hat{\mathcal{O}}_{X,x}$; in particular ψ is faithfully flat. Since $\hat{Y} \to Y \to \mathscr{X}_{n,k}$ is regular it follows that $\hat{X} \to \mathscr{X}_{n,k}$ is regular. □

Now we prove the second part of Proposition 7.2. This is not used later.

Lemma 7.5. *If (A, G) satisfies (i) and (ii), then $\pi^{(m)}$ is quasi-étale.*

Proof. Let $X^{(m)} = \operatorname{Spec} A/p^m A$ and $\mathscr{X}^{(m)} = \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}/p^m\mathbb{Z}$. For a morphism $Y \to \mathscr{X}^{(m)}$ with affine Y let $Z = X \times_{\mathscr{X}} Y$ and let $\pi' : Z \to Y$ be the second projection. We have to show that $L_{Z/Y}$ is acyclic. Here π' is ind-smooth by Lemma 7.4. Thus $L_{Z/Y}$ is isomorphic to $\Omega_{Z/Y}$, and it suffices to show that $\pi^{(m)}$ is formally unramified.

For an arbitrary morphism $g : \operatorname{Spec} B \to \mathscr{X}^{(m)}$, given by a p-divisible group H over B, we write $\Lambda_H = \operatorname{Lie}(H) \otimes_B \operatorname{Lie}(H^\vee)$. There is a Kodaira-Spencer homomorphism $\kappa_H : \Lambda_H \to \Omega_B$, which is surjective if and only if g is formally unramified. For a closed point $x \in X^{(m)}$ let A_x be the complete local ring at x and let $\hat{X}^{(m)} = \operatorname{Spec} A_x$. We have $i : \hat{X}^{(m)} \to X^{(m)}$. There is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & i^*\Omega_{X^{(m)}} \arrow[r] \arrow[d, "i^*\kappa_G"] & \Omega_{\hat{X}^{(m)}} \arrow[r] \arrow[d, "\kappa_{i^*G}"] & \Omega_{\hat{X}^{(m)}/X^{(m)}} \arrow[r] & 0 \\ & i^*\Lambda_G \arrow[r, equal] & \Lambda_{i^*G}. & & \end{tikzcd}$$

The upper line is exact because $\hat{X}^{(m)} \to X^{(m)}$ is regular, thus ind-smooth, which implies that $L_{\hat{X}^{(m)}/X^{(m)}}$ is concentrated in degree zero. Since A_x is isomorphic to a power series ring $W_m(k)[[t_1, \ldots, t_r]]$ for a perfect field k, the A_x-module $\Omega_{\hat{X}^{(m)}}$ is free with basis $dt_1, \ldots, dt_r$. These elements appear in $i^*\Omega_{X^{(m)}}$, and thus $\Omega_{\hat{X}^{(m)}/X^{(m)}}$ is zero. The homomorphism κ_{i^*G} is an isomorphism because i^*G is assumed to be universal. Thus $i^*\kappa_G$ is an isomorphism for all x, which implies that κ_G is an isomorphism, and $\pi^{(m)}$ is formally unramified, as desired. □

Finally we prove the last part of Proposition 7.2.

Lemma 7.6. *Assume that (A, G) satisfies (i) and (ii). Then π is surjective if and only if (iii) holds.*

Proof. It suffices to prove the implication $\Leftarrow$. We write $X = \operatorname{Spec} A$ and $\mathscr{X} = \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$. Let k be an algebraically closed field of characteristic p and let $H : \operatorname{Spec} k \to \mathscr{X}$ be a geometric point, i.e. a p-divisible group H over k. It suffices to show that $X_H = X \times_{\mathscr{X}} \operatorname{Spec} k$ is non-empty. Let K be an algebraic closure of $k((t))$ and let R be the ring of integers in K. As in the proof of Lemma 4.6 we find a p-divisible group H' over R with generic fibre $H \otimes_k K$ and with isoclinic special fibre. We consider the morphism $\operatorname{Spec} R \to \mathscr{X}$ defined by H'. The projection $X \times_{\mathscr{X}} \operatorname{Spec} R \to \operatorname{Spec} R$ is flat because it is ind-smooth by Lemma 7.4, and its image contains the closed point of $\operatorname{Spec} R$ by (iii). Thus the projection is surjective, which implies that $X \times_{\mathscr{X}} \operatorname{Spec} K = X_H \otimes_k K$ is non-empty; thus X_H is non-empty. □

Proof of Proposition 7.2. Use Lemmas 7.4, 7.5, and 7.6. □

8. Relation with the functor BT

For a p-adic ring R we consider the following commutative diagram of categories, where f. = formal, g. = groups, n. = nilpotent. The vertical arrows are the inclusions. The functor BT_R is defined in [Zi1, Thm. 81]. Its restriction to nilpotent displays gives formal p-divisible groups by [Zi1, Cor. 89]:

$$\begin{tikzcd} (p\text{-div}/R) \arrow[r, "\Phi_R"] & (\mathrm{disp}/R) \arrow[r, "\mathrm{BT}_R"] & (\mathrm{f.\,g.}/R) & \\ (\mathrm{f.}\ p\text{-div}/R) \arrow[u] \arrow[r, "\Phi^1_R"] & (\mathrm{n.\,disp}/R) \arrow[u] \arrow[r, "\mathrm{BT}^1_R"] & (\mathrm{f.}\ p\text{-div}/R) \arrow[u] \arrow[r, "\Phi^1_R"] & (\mathrm{n.\,disp}/R). \end{tikzcd}$$

Here BT^1_R is an equivalence by [Zi1] if R is excellent and by [La1] in general, and Φ^1_R is an equivalence by Theorem 5.1.

Lemma 8.1. *There is a natural isomorphism of functors* $\Phi^1_R \circ \mathrm{BT}^1_R \cong \mathrm{id}$.

We have a functor $\Upsilon_R : (\mathrm{disp}/R) \to$ (filtered F-V-modules over $\mathscr{W}_R$).

Let $\Upsilon^1_R : (\mathrm{n.\,disp}/R) \to$ (filtered F-V-modules over $\mathscr{W}_R$) be its restriction.

Proof of Lemma 8.1. By [Zi1, Thm. 94 and Cor. 97], for each nilpotent display $\mathscr{P}$ over R there is an isomorphism, functorial in $\mathscr{P}$ and in R,

$$u_R(\mathscr{P}) : \Upsilon^1_R(\Phi^1_R(\mathrm{BT}^1_R(\mathscr{P}))) \cong \Upsilon^1_R(\mathscr{P}).$$

We have to show that $u_R(\mathscr{P})$ commutes with F_1. This is automatic if R has no p-torsion because then $W(R)$ has no p-torsion, and $pF_1 = F$. Since Φ^1_R is an equivalence, we may assume that $\mathscr{P} = \Phi^1_R(G)$ for a formal p-divisible group G over R. We may also assume that p is nilpotent in R.

Let $\operatorname{Spec} A \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ be a presentation given by a p-divisible group H over A such that A is Noetherian; see Corollary 7.3. Let $J \subset A$ be the ideal such that $X \times_{\mathscr{BT}} \mathscr{BT}^o = \operatorname{Spec} A/J$ and let $\hat{A}$ be the J-adic completion of A. If A is constructed using isogeny spaces as in the proof of Proposition 7.1, then A is I-adic for an ideal I which contains J, and thus A is already J-adic. In any case, since $\mathbb{Z}_p \to A \to \hat{A}$ is flat, $\hat{A}$ has no p-torsion. Thus the compatible system $[u_{A/J^n}(\Phi^1_{A/J^n}(H))]_{n \geq 1}$ necessarily preserves F_1.

For G over R as above we consider $\operatorname{Spec} R \times_{\mathscr{BT}} \operatorname{Spec} A = \operatorname{Spec} S$. Since $\operatorname{Spec} R_{\mathrm{red}} \to \operatorname{Spec} R \to \mathscr{BT}$ factors over $\mathscr{BT}^o$, the ideal JS is a nil-ideal. Thus JS is nilpotent as J is finitely generated by Lemma 1.4; hence for sufficiently large n we have $\operatorname{Spec} S = \operatorname{Spec} R \times_{\mathscr{BT}} \operatorname{Spec} A/J^n$. By construction, $R \to S$ is faithfully flat, thus injective, and $G \otimes_R S \cong H \otimes_A S$. Thus $u_R(\Phi^1_R(G))$ preserves F_1 since this holds for $u_{A/J^n}(\Phi^1_{A/J^n}(H))$. □

Remark 8.2. The above proof of Lemma 8.1 uses that Φ^1_R is an equivalence, but this could be avoided by a finer (elementary) analysis of the stack of displays over rings in which p is nilpotent. Thus the facts that Φ^1_R and BT^1_R are equivalences can be derived from each other.

Since Φ^1_R is an equivalence, the isomorphism $\Phi^1_R \circ \mathrm{BT}^1_R \cong \mathrm{id}$ of Lemma 8.1 induces for each formal p-divisible group G over R an isomorphism

$$\rho_R(G) : G \cong \mathrm{BT}^1_R(\Phi^1_R(G)).$$

Theorem 8.3. *For each p-divisible group G over a p-adic ring R there is a unique isomorphism which is functorial in G and R,*

$$\tilde{\rho}_R(G) : \hat{G} \cong \mathrm{BT}_R(\Phi_R(G)),$$

which coincides with $\rho_R(G)$ if G is infinitesimal.

If G is an extension of an étale p-divisible group by an infinitesimal p-divisible group, Theorem 8.3 follows from Lemma 8.4 below because both sides of $\tilde{\rho}_R(G)$ preserve short exact sequences.

Lemma 8.4. *If G is étale, then $\mathrm{BT}_R(\Phi_R(G))$ is zero.*

Proof. Let $\Phi_R(G) = \mathscr{P} = (P, Q, F, F_1)$. We have $P = Q$, and $F_1 : P \to P$ is an f-linear isomorphism. Let N be a nilpotent R/p^nR-algebra for some n. By the definition of BT_R, the group $\mathrm{BT}_R(\mathscr{P})(N)$ is the cokernel of the endomorphism $F_1 - 1$ of $\hat{W}(N) \otimes_{W(R)} P$. This endomorphism is bijective because here F_1 is nilpotent since f is nilpotent on $\hat{W}(N)$. □

Proof of Theorem 8.3. Assume that G is a p-divisible group over an I-adic ring A such that $G_{A/I}$ is infinitesimal. Then $G_n = G_{A/I^n}$ is infinitesimal as well. Since formal Lie groups over A are equivalent to compatible systems of formal Lie groups over A/I^n for $n \geq 1$, the isomorphisms $\rho_{A/I^n}(G_n)$ define the desired isomorphism $\tilde{\rho}_A(G)$, which is clearly unique. The construction is functorial in the triple (A, I, G).

Assume that in addition a ring homomorphism $u : A \to B$ is given such that B is J-adic and such that $G_{B/J}$ is ordinary; we do not assume that $u(I) \subseteq J$. There is a unique exact sequence of p-divisible groups over B,

$$0 \to H \xrightarrow{\alpha} G_B \to H' \to 0,$$

such that H is of multiplicative type and H' is étale. Consider the following diagram of isomorphisms; cf. Lemma 8.4:

$$\begin{tikzcd} \hat{H} \arrow[r, "\tilde{\rho}_B(H)"] \arrow[d, "\alpha"'] & \mathrm{BT}_B(\Phi_B(H)) \arrow[d, "\alpha"] \\ \hat{G}_B \arrow[r, "(\tilde{\rho}_A(G))_B"] & (\mathrm{BT}_A(\Phi_A(G)))_B. \end{tikzcd} \tag{8.1}$$

Since the construction of $\tilde{\rho}_A(G)$ is functorial in (A, I), the diagram commutes if $u(I) \subseteq J$. If Theorem 8.3 holds, (8.1) commutes always. We show directly that (8.1) commutes in a special case that allows us to define $\tilde{\rho}_R$ in general by descent. As in the proof of Proposition 7.1 we consider a decent p-divisible group $\mathbb{G}$ over a perfect field k and an affine open subscheme $U = \operatorname{Spf} A$ of the isogeny space of $\mathbb{G}$ over $\operatorname{Spf} W(k)$. Let $U_{\mathrm{red}} = \operatorname{Spec} A/I$ and let G be the univeral p-divisible group over A. By passing to a connected component of U we may assume that A is integral. Since A has no p-torsion and since A/pA is regular, pA is a prime ideal. Let $B = \hat{A}_{pA}$ be the complete local ring of A at this prime and let $J = pB$.

Lemma 8.5. *For this choice of* (A, I, B, J, G) *the diagram* (8.1) *is defined and commutes.*

Proof. In order that (8.1) is defined we need that $G_{B/J}$ is ordinary. Then the defect of commutativity of (8.1) is an automorphism $\xi = \xi(A, I, B, J, G)$ of $\hat{H}$, which is functorial with respect to (A, I, B, J, G).

Step 1. For an arbitrary maximal ideal $\mathfrak{m}$ of A let $A_1 = \hat{A}_{\mathfrak{m}}$, let I_1 be the maximal ideal of A_1, let B_1 be the complete local ring of A_1 at the prime ideal pA_1, and let $J_1 = pB_1$. We have compatible injective homomorphisms $A \to A_1$ with $I \to I_1$ and $B \to B_1$ with $J \to J_1$. Moreover $A_1 \cong W(k_1)[[t_1, \dots, t_r]]$ for a finite extension k_1 of k, and G_{A_1} is a universal deformation. Thus G_{B_1/J_1} is ordinary, which implies that $G_{B/J}$ is ordinary, and it suffices to show that $\xi(A_1, I_1, B_1, J_1, G_{A_1}) = \mathrm{id}$.

Step 2. Next we achieve $r = 1$ by a blowing-up construction. Let k_2 be an algebraic closure of the function field $k_1(\{t_i/t_j\}_{1 \le i,j \le r})$, let $A_2 = W(k_2)[[t]]$, let $I_2 = tA_2$, let B_2 be the completion of A_2 at the prime ideal pA_2, and let $J_2 = pB_2$. Thus $B_2/J_2 = k_2((t))$. There is a natural local homomorphism $A_1 \to A_2$ with $t_i \mapsto [t_i/t_1]t$, which induces an injective homomorphism $B_1 \to B_2$. Thus it suffices to show that $\xi(A_2, I_2, B_2, J_2, G_{A_2}) = \mathrm{id}$.

Step 3. Let L be the completion of an algebraic closure of $k_2((t))$ and let $\mathcal{O} \subset L$ be its ring of integers. For a fixed $n \ge 1$ let $A' = W_n(\mathcal{O})$ and let I' be the kernel of $A' \to \mathcal{O}/t\mathcal{O}$; thus I' is generated by $([t], p)$. It is easy to see that A' is I'-adic, using that $\mathcal{O}$ is t-adic and that $p^r[t] = v^r[t^{p^r}]$. Let $B' = W_n(L)$ and $J' = pB'$. The homomorphism $A_2 \to A'$ defined by the inclusion $W(k_2) \to W(L)$ and by $t \mapsto [t]$ induces a homomorphism $B_2 \to B'$ such that the kernels of $B_2 \to B'$ for increasing n have zero intersection. Thus it suffices to show that $\xi(A', I', B', J', G_{A'}) = \mathrm{id}$.

Step 4. Since L is algebraically closed, H_L is isomorphic to $\mu_{p^\infty}^d$. Since G_L is ordinary, the inclusion $H_L \to G_L$ splits uniquely; i.e., we have a homomorphism $\psi_L : G_L \to \mu_{p^\infty}^d$ that induces an isomorphism of the formal completions. Since $\mathcal{O}$ is normal, the Serre dual of ψ_L extends to a homomorphism over $\mathcal{O}$; thus ψ_L extends to a homomorphism $\psi_{\mathcal{O}} : G_{\mathcal{O}} \to \mu_{p^\infty}^d$. The homomorphism $p^n\psi_{\mathcal{O}}$ extends to $\psi' : G \to \mu_{p^\infty}^d$ over A', and its restriction $\psi'_B : G_{B'} \to \mu_{p^\infty}^d$ over B induces an isogeny of the multiplicative parts, which commutes with the associated ξ's by functoriality. Thus it suffices to show that $\xi' = \xi(A', I', B', J', \mu_{p^\infty}) = \mathrm{id}$.

Since A' is p-adic and since μ_{p^∞} is infinitesimal over A'/pA', the element $\xi'' = \xi(A', pA', B', J', \mu_{p^\infty})$ is well-defined, and it induces ξ' by functoriality. But we have $\xi'' = \mathrm{id}$ because $A' \to B'$ maps pA' into $J' = pB'$. This proves Lemma 8.5. $\square$

We continue the proof of Theorem 8.3 and write $\mathrm{BT}_R(\Phi_R(G)) = G^+$. Let $\operatorname{Spec} A \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ be the presentation constructed in the proof of Proposition 7.1 and let G be the universal group over A. The ring A is I-adic such that $G_{A/I}$ is isoclinic. Thus the above construction applies and gives $\tilde{\rho}_A(G) : \hat{G} \cong G^+$; here the components of $\operatorname{Spec} A$ where G is étale do not matter in view of Lemma 8.4. Let $\operatorname{Spec} A \times_{\mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p} \operatorname{Spec} A = \operatorname{Spec} C$ and let $\hat{C}$ be the p-adic completion of C.

For an arbitrary ring R in which p is nilpotent we want to define $\tilde{\rho}_R$ by descent, starting from $\tilde{\rho}_A(G)$. This is possible if and only if the inverse images of $\tilde{\rho}_A(G)$ under the two projections $p_i : \operatorname{Spec} \hat{C} \to \operatorname{Spec} A$ coincide, i.e. if the following diagram of formal Lie groups over $\hat{C}$ commutes, where $u : p_1^* G \cong p_2^* G$ is the given descent isomorphism:

$$\begin{tikzcd} p_1^*\hat{G} \arrow[r, "\hat{u}"] \arrow[d, "p_1^*(\tilde{\rho})"'] & p_2^*\hat{G} \arrow[d, "p_2^*(\tilde{\rho})"] \\ p_1^*G^+ \arrow[r, "u^+"] & p_2^*G^+. \end{tikzcd} \tag{8.2}$$

Let $A = \prod A_i$ be a maximal decomposition so that each A_i is a domain, let B_i be the complete local ring of A_i at the prime pA_i, and let $B = \prod B_i$. Since the two homomorphisms $A \to C$ are flat and since $A \to B$ is flat and induces an injective map $A/p^nA \to B/p^nB$, the natural homomorphism $C \to B \otimes_A C \otimes_A B = C'$ induces an injective homomorphism of the p-adic completions $\hat{C} \to \hat{C}'$. Thus the commutativity of (8.2) can be verified over $\hat{C}'$. Let H be the multiplicative part of the ordinary p-divisible group G_B.

Since the construction of $\tilde{\rho}$ is functorial with respect to the projections of p-adic rings $q_1, q_2 : \operatorname{Spec} \hat{C}' \to \operatorname{Spec} B$, the following diagram of formal Lie groups over $\hat{C}'$ commutes:

$$\begin{tikzcd} q_1^*\hat{H} \arrow[r, "\hat{u}"] \arrow[d, "q_1^*(\tilde{\rho})"'] & q_2^*\hat{H} \arrow[d, "q_2^*(\tilde{\rho})"] \\ q_1^*H^+ \arrow[r, "u^+"] & q_2^*H^+. \end{tikzcd} \tag{8.3}$$

Lemma 8.5 implies that the inclusion $H \to G_B$ induces an isomorphism of diagrams (8.3) $\cong$ (8.2) $\otimes_{\hat{C}} \hat{C}'$. Thus (8.2) commutes as well. $\square$

8.1. **Complement to** [La1]**.** The proof that BT^1_R is an equivalence in [La1] proceeds along the following lines. First, by [Zi1] the functor is always faithful, and fully faithful if R is reduced over $\mathbb{F}_p$. Second, by using an ∞-smooth presentation of $\mathscr{BT}^o$ as in Lemma 1.1, one deduces that BT^1_R is essentially surjective if R is reduced over $\mathbb{F}_p$. Using this, one shows that BT^1_R is fully faithful in general, and the general equivalence follows.

The second step is based on the following consequence of the first step. A faithfully flat homomorphism of reduced rings $R \to S$ is called an *admissible covering* if $S \otimes_R S$ is reduced. In this case, a formal p-divisible group G over R lies in the image of BT^1_R if G_S lies in the image of BT^1_S. In [La1, Sec. 3], some effort is needed to find a sufficient supply of admissible coverings.

The proof can be simplified as follows if one starts with a reduced presentation $\pi : \operatorname{Spec} A' \to \mathscr{BT} \times \operatorname{Spec} \mathbb{Z}_p$ with excellent A' such that $A'/\mathfrak{m}$ is perfect for all maximal ideals $\mathfrak{m}$ of A'; see Corollary 7.3. Let $\operatorname{Spec} A \to \mathscr{BT}^o$ be the restriction of π. As in [La1] it suffices to show that the universal group G over A lies in the image of BT^1_A. Since A is excellent, this follows from [Zi1]. A direct argument goes as follows: The homomorphisms $A \to \prod_{\mathfrak{m}} A_{\mathfrak{m}}$ and $A_{\mathfrak{m}} \to \hat{A}_{\mathfrak{m}}$ are admissible coverings, which reduces the surjectivity of BT^1_A to the surjectivity of $\mathrm{BT}^1_{A/\mathfrak{m}^n}$. By deformation theory this is reduced to the case of the perfect fields $A/\mathfrak{m}$, which is classical.

8.2. **Erratum to** [La1]**.** [La1, Lemma 3.3] asserts that if R is a Noetherian ring and if $R \to S$ and $S \to T$ are admissible coverings, then $R \to T$ is an admissible covering too. This is false; see Example 8.7 below. The proof assumes incorrectly that a field extension L/K such that $L \otimes_K L$ is reduced must be separable. The following part loc. cit. is proved correctly.

Lemma 8.6. *Let $R \to S$ be a faithfully flat homomorphism of reduced rings where R is Noetherian such that for all minimal prime ideals $\xi \subset S$ and $\eta = \xi \cap R$ the field extension $R_\eta \to S_\xi$ is separable. Then $S \otimes_R S$ is reduced.*

The incorrect [La1, Lemma 3.3] is only used in the proof of [La1, Prop. 3.4], where it can be avoided as follows. For certain rings $A \to \hat{A} \to \hat{B}$ one needs that $\hat{B} \otimes_A \hat{B}$ is reduced. The proof shows that $A \to \hat{A}$ and $\hat{A} \to \hat{B}$ satisfy the hypotheses of Lemma 8.6. Thus $A \to \hat{B}$ satisfies these hypotheses as well, and the assertion follows.

Example 8.7. Let K be a field of characteristic p and let a, b, c be part of a p-basis of $K^{1/p}$ over K. Let $L = K(X, a + bX)$ and $M = L(Y, a + cY)$, where X and Y are algebraically independent over K. Then $L \otimes_K L$ and $M \otimes_L M$ are reduced, but $M \otimes_K M$ is not reduced. In particular, L is not separable over K.

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Added after posting

In section 2.1 we have to assume that the endomorphism σ of S preserves the natural extension to $I + pS$ of the given divided powers on I. This is satisfied for $S = W(R)$ in section 2.2 and for $S = W(B)$ in section 2.3.

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Institut für Mathematik, Universität Paderborn, D-33098 Paderborn, Germany

E-mail address: `elau@math.upb.de`