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GLOBAL RIGIDITY OF HIGHER RANK ANOSOV ACTIONS ON TORI AND NILMANIFOLDS

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WITH AN APPENDIX BY JAMES F. DAVIS

1. Introduction

An Anosov diffeomorphism f on a torus $\mathbb{T}^n$ is *affine* if f lifts to an affine map on $\mathbb{R}^n$. By a classical result of Franks and Manning, any Anosov diffeomorphism g on $\mathbb{T}^n$ is topologically conjugate to an affine Anosov diffeomorphism. More precisely, there is a homeomorphism $\phi : \mathbb{T}^n \mapsto \mathbb{T}^n$ such that $f = \phi \circ g \circ \phi^{-1}$ is an affine Anosov diffeomorphism. We call ϕ the Franks-Manning conjugacy. The linear part of f is the map induced by g on $H_1(\mathbb{T}^n)$.

Anosov diffeomorphisms are rarely C^1-conjugate to affine ones. For example, one can perturb a linear Anosov diffeomorphism locally around a fixed point p to change the conjugacy class of the derivative at p. The resulting diffeomorphism will still be Anosov but cannot be C^1-conjugate to its linearization. The situation is radically different for $\mathbb{Z}^k$-actions with many Anosov diffeomorphisms. In other words, Anosov diffeomorphisms rarely commute with other Anosov diffeomorphisms.

It follows easily from the result for a single Anosov diffeomorphism that an Anosov $\mathbb{Z}^k$-action α on $\mathbb{T}^n$ is topologically conjugate to a $\mathbb{Z}^k$-action by affine Anosov diffeomorphisms. We call this action the *linearization* of α and denote it by ρ. Again, for any $a \in \mathbb{Z}^k$ the linear part of $\rho(a)$ is the map induced by $\alpha(a)$ on $H_1(\mathbb{T}^n)$. The logarithms of the moduli of the eigenvalues of these linear parts define additive maps $\lambda_i : \mathbb{Z}^k \mapsto \mathbb{R}$, which extend to linear functionals on $\mathbb{R}^k$. A *Weyl chamber* of ρ is a connected component of $\mathbb{R}^k - \bigcup_i \ker\lambda_i$.

Theorem 1.1. *Let α be a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$, on a torus $\mathbb{T}^n$ and let ρ be its linearization. Suppose that there is a $\mathbb{Z}^2$ subgroup of $\mathbb{Z}^k$ such that $\rho(a)$ is ergodic for every nonzero $a \in \mathbb{Z}^2$. Further assume that there is an Anosov element for α in each Weyl chamber of ρ. Then α is C^∞-conjugate to ρ.*

Furthermore, for a linear $\mathbb{Z}^k$-action on $\mathbb{T}^n$ having a $\mathbb{Z}^2$ subgroup acting by ergodic elements is equivalent to several other properties, in particular to being genuinely higher rank [37]. A linear $\mathbb{Z}^k$-action is called *genuinely higher rank* if for all finite index subgroups Z of $\mathbb{Z}^k$, no quotient of the Z-action factors through a finite extension of $\mathbb{Z}$. Hence we obtain the following corollary.

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Corollary 1.2. *Let α be a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$, on a torus $\mathbb{T}^n$. Suppose that the linearization ρ of α is genuinely higher rank. Further assume that there is an Anosov element for α in each Weyl chamber of ρ. Then α is C^∞-conjugate to ρ.*

We can define Weyl chambers for the action α itself. In fact these Weyl chambers will turn out to be the same for α and ρ. Hence existence of Anosov elements for α in every Weyl chamber of ρ is equivalent to existence of Anosov elements for α in every Weyl chamber of α.

We refer to our paper [11] for a brief survey of other results and methods in the classification of higher rank Anosov actions. Our global rigidity results above are optimal except that we require an Anosov element in every Weyl chamber. Rodriguez Hertz in [34] classifies higher rank actions on tori assuming only one Anosov element. However, his work requires multiple additional hypotheses such as bunching conditions and low dimensionality of coarse Lyapunov spaces. In particular, the hypotheses in [34] require that the rank of the acting group has to grow linearly with the dimension of the torus. It is a conjecture due to Katok and the third author that global rigidity holds assuming α has one Anosov element. We discuss this conjecture in more detail at the end of this introduction.

Let us briefly describe our proof, which crucially uses the Franks-Manning conjugacy ϕ for some Anosov element of the action. As we noted, ϕ also conjugates any commuting diffeomorphism to an affine map. Consequently, each element of the action gives a functional equation for ϕ. This yields explicit series representations for its projection ϕ_V to any generalized joint eigenspace V of ρ. The existence of Anosov elements of α in every Weyl chamber allows us to define coarse Lyapunov foliations as finest nontrivial intersections of stable and unstable foliations of Anosov elements. Since the latter are continuous, so are the coarse Lyapunov foliations. It is precisely here that the existence of an Anosov element in each Weyl chamber is used. We then employ the continuity of the coarse Lyapunov foliations to obtain uniform estimates for contraction and expansion. Thus elements close to a Weyl chamber wall act almost isometrically along suitable coarse Lyapunov foliations, or more precisely, we can make their exponents in these estimates as close to 0 as we wish, and in particular smaller than the size of the exponent in the exponential decay we get from exponential mixing. We use such elements to study the regularity of ϕ_V along each coarse Lyapunov foliation $\mathcal{W}$. Using exponential mixing for Hölder functions we show that the partial derivatives along $\mathcal{W}$ exist as distributions dual to spaces of Hölder functions. Then we adapt ideas from a paper by Rauch and Taylor to show that ϕ is smooth. We emphasize that the rigidity of $\mathbb{Z}^k$-actions for $k \geq 2$ is due to the co-existence of (almost) isometric and hyperbolic behavior in the actions. This utterly fails for $\mathbb{Z}$-actions.

The paper is organized as follows. We first explain general definitions, constructions and results for higher rank Anosov actions in Section 2. In Section 3 we turn to actions on tori and nilmanifolds and use the Franks-Manning conjugacy to derive special properties of such actions. Most importantly, we will develop uniform growth estimates for elements near the Weyl chamber walls of the action in Section 3.2. We then turn to the case of the torus as it is substantially more elementary than the nilmanifold case. In Section 4, we establish exponential mixing for $\mathbb{Z}^k$-actions by ergodic affine automorphisms on a torus. For smooth actions on tori with the standard smooth structure, we prove in Section 5 the existence of partial derivatives in all directions as distributions dual to Hölder functions. This concludes the proof for the case of standard tori using the general regularity result that we establish in Section 8. For exotic tori, i.e., manifolds that are homeomorphic to but not diffeomorphic to tori, in dimensions at least 5 we can pass to a finite cover with the standard smooth structure. For dimension 4 we give a special argument in Section 6.

Finally, we adapt our arguments to nilmanifolds: Let N be a simply connected nilpotent Lie group. We call a diffeomorphism of N *affine* if it is a composition of an automorphism of N with a left translation by an element of N. If $\Gamma \subset N$ is a discrete subgroup, we call the quotient N/Γ a *nilmanifold*. An *infra-nilmanifold* M is a manifold finitely covered by a nilmanifold. Diffeomorphisms of M covered by affine diffeomorphisms of N are again called *affine*. The Franks-Manning conjugacy theorem generalizes to infra-nilmanifolds: Suppose M' is a smooth manifold homeomorphic with an infra-nilmanifold. Then every Anosov diffeomorphism of M' is conjugate to an affine diffeomorphism of M by a homeomorphism ϕ. We call ϕ the Franks-Manning conjugacy. Given an action α of $\mathbb{Z}^k$ on M' which contains an Anosov diffeomorphism, then its Franks-Manning conjugacy jointly conjugates all $\alpha(a), a \in \mathbb{Z}^k$, to affine diffeomorphisms $\rho(a)$. We call ρ the linearization of α. Now we can state our main result for nilmanifolds:

Theorem 1.3. *Let α be a C^∞-action of $\mathbb{Z}^k$, $k \geq 2$, on a compact infra-nilmanifold N/Γ and let ρ be its linearization. Suppose that there is a $\mathbb{Z}^2$ subgroup of $\mathbb{Z}^k$ such that $\rho(a)$ is ergodic for every nonzero $a \in \mathbb{Z}^2$. Further assume that there is an Anosov element for α in each Weyl chamber of ρ. Then α is C^∞-conjugate to ρ.*

Our main result reduces to the case of standard nilmanifolds, i.e. nilmanifolds with the differentiable structure coming from the ambient Lie group. Indeed, there are no Anosov diffeomorphisms on nontoral nilmanifolds in dimensions 4 or less, and the result by J. Davis in the appendix shows that any nilmanifold of dimension at least 5 is finitely covered by a standard nilmanifold.

For standard nilmanifolds we proceed similarly to the toral case. We adapt arguments of Margulis and Qian [32] to reduce regularity of the conjugacy to regularity of the solution of a cohomology equation. The relevant cocycle however takes values in a nilpotent group and is not directly amenable to our approach. Instead, we consider suitable factors of the cocycle in various abelian quotients of the derived series of N. Again we prove regularity of coboundaries for the resulting cocycles by exponential mixing of the $\mathbb{Z}^k$ action, uniform expansion and contraction of elements close to Weyl chamber walls, and showing existence of derivatives via distributions dual to Hölder functions. Unlike in the toral case, exponential mixing of actions by affine automorphisms does not follow from elementary Fourier analysis. Rather this was established by Gorodnik and the third author in [14]. We remark that this approach yields the first rigidity results for higher rank actions on general nilmanifolds. Earlier cocycle and local rigidity results, by A. Katok and the third author, were only proved for actions which were higher rank both on the toral factor as well as the fibers (e.g. [27]). There, cocycles were straightened out separately on the base and the fibers. Exponential mixing of these actions thus allows for a much simpler and direct approach and is also used in [15] to prove cocycle rigidity results.

Epilogue: We conclude this paper with some remarks about the conjecture by Katok and Spatzier that genuinely higher rank abelian Anosov actions are smoothly conjugate to affine actions. Using the arguments of our earlier paper [11], we can show that the conjugacy is always smooth along almost every leaf of each coarse Lyapunov foliation. However, we have no further evidence in support of this conjecture and in fact have some doubts about its truth. In [13], Gogolev constructed a diffeomorphism of a torus which is Hölder conjugate to an Anosov diffeomorphism but itself is not Anosov. Thus having one Anosov element may not imply that most elements are Anosov. In [7], Farrell and Jones constructed Anosov diffeomorphisms on exotic tori. In light of this construction it seems obvious to ask:

Question 1.4. *Are there genuinely higher rank Anosov $\mathbb{Z}^k$ actions on exotic tori?*

As exotic tori are finitely covered by standard tori, such actions would lift to actions on standard tori. The latter could not be smoothly equivalent to their linearizations since a smoothness result for conjugacy would descend to the C^0 conjugacy between the exotic and standard tori. Thus such examples would also give counterexamples to the conjecture by Katok and Spatzier even when the underlying smooth structure on the torus is standard.

We remark here that the construction in [7], further explained and simplified in [8], does not adapt easily to the case of actions of higher rank abelian groups. Indeed because of the delicate cutting and pasting arguments used in their constructions, it would be hard to guarantee that different elements continue to commute. As a consequence of Theorem 1.1, a positive answer to Question 1.4 can only occur for an action where relatively few elements are Anosov. Furthermore, by the results in [34], a positive answer to Question 1.4 seems unlikely if the dynamically defined foliations for the action have dimensions 1 or 2. The Farrell-Jones construction proceeds by cutting and pasting exotic spheres into the torus. This suggests, in order to construct examples for Question 1.4, that one would want to glue in the exotic sphere in a manner somehow subordinate to the dynamical foliations using their high dimension.

2. Preliminaries

Throughout the paper, the smoothness of diffeomorphisms, actions, and manifolds is assumed to be C^∞, even though all definitions and some of the results can be formulated in lower regularity.

2.1. Anosov actions of $\mathbb{Z}^k$.

Let a be a diffeomorphism of a compact manifold M. We recall that a is *Anosov* if there exist a continuous a-invariant decomposition of the tangent bundle $TM = E^s_a \oplus E^u_a$ and constants $K > 0$, $\lambda > 0$ such that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|Da^n(v)\| &\le Ke^{-\lambda n}\|v\| \text{ for all } v \in E^s_a, \\ \|Da^{-n}(v)\| &\le Ke^{-\lambda n}\|v\| \text{ for all } v \in E^u_a. \end{aligned} \tag{1}$$

The distributions E^s_a and E^u_a are called the *stable* and *unstable* distributions of a.

Now we consider a $\mathbb{Z}^k$ action α on a compact manifold M via diffeomorphisms. The action is called *Anosov* if there is an element which acts as an Anosov diffeomorphism. For an element a of the acting group we denote the corresponding diffeomorphisms by $\alpha(a)$ or simply by a if the action is fixed.

The distributions E^s_a and E^u_a are Hölder continuous and tangent to the stable and unstable foliations $\mathcal{W}^s_a$ and $\mathcal{W}^u_a$ respectively [18]. The leaves of these foliations are C^∞ injectively immersed Euclidean spaces. Locally, the immersions vary continuously in the C^∞ topology. In general, the distributions E^s and E^u are only Hölder continuous transversally to the corresponding foliations.

2.2. **Lyapunov exponents and coarse Lyapunov distributions.** First we recall some basic facts from the theory of nonuniform hyperbolicity for a single diffeomorphism; see for example [2]. Let a be a diffeomorphism of a compact manifold M preserving an ergodic probability measure μ. By Oseledec's Multiplicative Ergodic Theorem, there exist finitely many numbers χ_i and an invariant measurable splitting of the tangent bundle $TM = \bigoplus E_i$ on a set of full measure such that the forward and backward Lyapunov exponents of $v \in E_i$ are χ_i. This splitting is called *Lyapunov decomposition*. We define the stable distribution of a with respect to μ as $E_a^- = \bigoplus_{\chi_i<0} E_i$. The subspace $E_a^-(x)$ is tangent μ-a.e. to the stable manifold $W_a^-(x)$. More generally, given any $\theta < 0$ we can define the strong stable distribution by $E_a^\theta = \bigoplus_{\chi_i \le \theta} E_i$ which is tangent μ-a.e. to the strong stable manifold $W_a^\theta(x)$. $W_a^\theta(x)$ is a smoothly immersed Euclidean space. For a sufficiently small ball $B(x)$, the connected component of $W_a^\theta(x) \cap B(x)$, called a *local* manifold, can be characterized by the exponential contraction property: for any sufficiently small $\varepsilon > 0$ there exists $C = C(x)$ such that

$$W_a^{\theta,loc}(x) = \{y \in B(x) \mid \operatorname{dist}(a^n x, a^n y) \le C e^{(\theta+\varepsilon)n} \quad \forall n \in \mathbb{N}\}. \tag{2}$$

The unstable distributions and manifolds are defined similarly. In general, E_a^- is only measurable and depends on the measure μ. However, if a is an Anosov diffeomorphism, then E_a^- for any measure always agrees with the continuous stable distribution E_a^s. Indeed, E_a^s cannot contain a vector with a nontrivial component in some E_j with $\chi_j \ge 0$ since such a vector does not satisfy (1). Hence $E_a^s \subset \bigoplus_{\chi_i<0} E_i$. Similarly, the unstable distribution $E_a^u \subset \bigoplus_{\chi_i>0} E_i$. Since $TM = E_a^s \oplus E_a^u$, both inclusions have to be equalities.

Now we consider the case of $\mathbb{Z}^k$ actions. Let μ be an ergodic probability measure for a $\mathbb{Z}^k$ action α on a compact manifold M. By commutativity, the Lyapunov decompositions for individual elements of $\mathbb{Z}^k$ can be refined to a joint invariant splitting for the action. The following proposition from [22] describes the Multiplicative Ergodic Theorem for this case. See [20] for more details on the Multiplicative Ergodic Theorem and related notions for higher rank abelian actions.

Proposition 2.1. *There are finitely many linear functionals χ on $\mathbb{Z}^k$, a set of full measure $\mathcal{P}$, and an α-invariant measurable splitting of the tangent bundle $TM = \bigoplus E_\chi$ over $\mathcal{P}$ such that for all $a \in \mathbb{Z}^k$ and $v \in E_\chi$, the Lyapunov exponent of v is $\chi(a)$, i.e.*

$$\lim_{n\to\pm\infty} n^{-1} \log \|Da^n(v)\| = \chi(a),$$

where $\|\cdot\|$ is a continuous norm on TM.

The splitting $\bigoplus E_\chi$ is called the *Lyapunov decomposition*, and the linear functionals χ, extended to linear functionals on $\mathbb{R}^k$, are called the *Lyapunov exponents* of α. The hyperplanes $\ker\chi \subset \mathbb{R}^k$ are called the *Lyapunov hyperplanes* or *Weyl chamber walls*, and the connected components of $\mathbb{R}^k - \bigcup_\chi \ker\chi$ are called the *Weyl chambers* of α. The elements in the union of the Lyapunov hyperplanes are called *singular*, and the elements in the union of the Weyl chambers are called *regular*.

Consider a $\mathbb{Z}^k$ action by *automorphisms* of a torus $M = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ or, more generally, a nilmanifold $M = N/\Gamma$, where N is a simply connected nilpotent Lie group and $\Gamma \subset G$ is a (cocompact) lattice. In this case, the Lyapunov decomposition is determined by the eigenspaces of the $d \times d$ matrix that defines the toral automorphism or by the eigenspaces of the induced automorphism on the Lie algebra of N. In particular, every Lyapunov distribution is smooth and in the toral case integrates to a linear foliation. The Lyapunov exponents are given by the logarithms of the moduli of the eigenvalues. Hence they are independent of the invariant measure and give uniform estimates of expansion and contraction rates.

In the nonalgebraic case, the individual Lyapunov distributions are in general only measurable and depend on the given measure. This can already be seen for a single diffeomorphism, even if Anosov. However, as we observed above, the full stable distribution E^s_a of an Anosov element a always agrees with $\bigoplus_{\chi(a)<0} E_\chi$ on a set of full measure for any measure.

For higher rank actions, *coarse Lyapunov distributions* play a similar role to the stable and unstable distributions for an Anosov diffeomorphism. For any Lyapunov functional χ the coarse Lyapunov distribution is the direct sum of all Lyapunov spaces with Lyapunov exponents, as functionals, positively proportional to χ:

$$E^\chi = \bigoplus E_{\chi'}, \quad \chi' = c\,\chi \text{ with } c > 0.$$

For an algebraic action such a distribution is a finest nontrivial intersection of the stable distributions of certain Anosov elements of the action. For nonalgebraic actions, however, this is not a priori clear. It was shown in [24, Proposition 2.4] that, in the presence of sufficiently many Anosov elements, the coarse Lyapunov distributions are well-defined, continuous, and tangent to foliations with smooth leaves. We quote the discrete time version [23, Proposition 2.2]. We denote the set of all Anosov elements in $\mathbb{Z}^k$ by $\mathcal{A}$.

Proposition 2.2. *Let α be an Anosov action of $\mathbb{Z}^k$ and let μ be an ergodic probability measure for α with full support. Suppose that there exists an Anosov element in every Weyl chamber defined by μ. Then for each Lyapunov exponent χ the coarse Lyapunov distribution can be defined as*

$$E^\chi(p) = \bigcap_{\{a\in\mathcal{A}\,\mid\,\chi(a)<0\}} E^s_a(p) \;=\; \bigoplus_{\{\chi'=c\,\chi\,\mid\,c>0\}} E_{\chi'}(p)$$

on the set $\mathcal{P}$ of full measure where the Lyapunov splitting exists. Moreover, E^χ is Hölder continuous, and thus it can be extended to a Hölder distribution tangent to the foliation $\mathcal{W}^\chi = \bigcap_{\{a\in\mathcal{A}\,\mid\,\chi(a)<0\}} \mathcal{W}^s_a$ with uniformly C^∞ leaves.

Note that ergodic measures with full support always exist if a $\mathbb{Z}^k$ action contains a transitive Anosov element. A natural example is given by the measure μ of maximal entropy for such an element, which is unique [25, Corollary 20.1.4] and hence is invariant under the whole action. We emphasize that it is precisely here where we use the assumption that every Weyl chamber contains an Anosov element. We will use Proposition 2.2 in the next section to get uniform estimates for elements close to Weyl chamber walls.

Since a coarse Lyapunov distribution is defined by a collection of positively proportional Lyapunov exponents, it can be uniquely identified with the subset of $\mathbb{R}^k$ where these functionals are positive (resp. negative). This subset is called the *positive (resp. negative) Lyapunov half-space*. Similarly, a coarse Lyapunov distribution can be defined with the oriented Lyapunov hyperplane that separates the corresponding positive and negative Lyapunov half-spaces.

3. $\mathbb{Z}^k$ ACTIONS ON TORI AND NILMANIFOLDS AND UNIFORM ESTIMATES

From now on we consider Anosov $\mathbb{Z}^k$ actions on tori and nilmanifolds. In this section, we explore the special features we obtain thanks to the Franks-Manning conjugacy. This allows us to control invariant measures, Lyapunov exponents, and even upper bounds of expansion for elements close to a Weyl chamber wall (cf. Section 3.2).

3.1. **Invariant measures and Lyapunov exponents.** Let f be an Anosov diffeomorphism of a torus $M = \mathbb{T}^d$ or, more generally, of a nilmanifold $M = N/\Gamma$. By the results of Franks and Manning in [12, 31], f is topologically conjugate to an Anosov automorphism $A : M \to M$; i.e., there exists a homeomorphism $\phi : M \to M$ such that $A \circ \phi = \phi \circ f$. The conjugacy ϕ is bi-Hölder; i.e., both ϕ and ϕ^{-1} are Hölder continuous with some Hölder exponent γ.

Now we consider an Anosov $\mathbb{Z}^k$ action α on a nilmanifold M. Fix an Anosov element a for α. Then we have ϕ which conjugates $\alpha(a)$ to an automorphism A. By [39, Corollary 1], any homeomorphism of M commuting with A is an affine automorphism. Hence we conclude that ϕ conjugates α to an action ρ by affine automorphisms. We will call ρ an *algebraic action* and refer to it as the *linearization* of α.

Now we describe the preferred invariant measure for α (cf. [21, Remark 1]). We denote by λ the normalized Haar measure on the nilmanifold M. Note that λ is invariant under any affine automorphism of M and is the unique measure of maximal entropy for any affine Anosov automorphism.

Proposition 3.1 ([11, Proposition 2.4]). *The action α preserves an absolutely continuous measure μ with smooth positive density. Moreover, $\mu = \phi_*^{-1}(\lambda)$ and for any Anosov element $a \in \mathbb{Z}^k$, μ is the unique measure of maximal entropy for $\alpha(a)$.*

In the next proposition we show that the Lyapunov exponents of (α, μ) and (ρ, λ) are positively proportional and that the corresponding coarse Lyapunov foliations are mapped into each other by the conjugacy ϕ. From now on, instead of indexing a coarse Lyapunov foliation by a representative of the class of positively proportional Lyapunov functionals, we index them numerically; i.e., we write $\mathcal{W}^i$ instead of $\mathcal{W}^\chi$, implicitly identifying the finite collection of equivalence classes of Lyapunov exponents with a finite set of integers.

Proposition 3.2. *Assume there is an Anosov element in every Weyl chamber. Then:*

1. *The Lyapunov exponents of (α, μ) and (ρ, λ) are positively proportional, and thus the Lyapunov hyperplanes and Weyl chambers are the same.*
2. *For any coarse Lyapunov foliation $\mathcal{W}^i_\alpha$ of α,*
$$\phi(\mathcal{W}^i_\alpha) = \mathcal{W}^i_\rho,$$
where $\mathcal{W}^i_\alpha$ is the corresponding coarse Lyapunov foliation for ρ.

Remark. In fact, one can show that (1) holds for Lyapunov exponents and coarse Lyapunov foliations of (α, ν) for any α-invariant measure ν, so, in particular, the Lyapunov exponents of all α-invariant measures are positively proportional and the coarse Lyapunov splittings are consistent with the continuous one defined in Proposition 2.2.

Remark. We do not claim at this point that the Lyapunov exponents of (α, μ) and (ρ, λ) (or of different invariant measures for α) are equal. Of course, if α is shown to be smoothly conjugate to ρ, then this is true a posteriori.

Proof. The proposition is the discrete time analogue of [11, Proposition 2.5]. We include the proof for the sake of completeness. First we observe that the conjugacy ϕ maps the stable manifolds of α to those of ρ. More precisely, for any $a \in \mathbb{Z}^k$ and any for μ-a.e. $x \in M$ we have

$$\phi(W^-_{\alpha(a)}(x)) = W^-_{\rho(a)}(\phi(x)). \tag{3}$$

Indeed, it suffices to establish this for local manifolds, which are characterized by the exponential contraction as in (2). Since ϕ is bi-Hölder, it preserves the property that $\mathrm{dist}(x_n, y_n)$ decays exponentially, which implies (3). In particular, for any Anosov $a \in \mathbb{Z}^k$ and any $x \in M$ we have $\phi(W^s_{\alpha(a)}(x)) = W^s_{\rho(a)}(\phi(x))$. Hence the formula for $\mathcal{W}^i_\alpha$ given in Proposition 2.2 implies (2) once we establish (1).

To establish (1) it suffices to show that the oriented Lyapunov hyperplanes of (α, μ) and (ρ, λ) are the same. Suppose that an oriented Lyapunov hyperplane L of one action, say α, is not an oriented Lyapunov hyperplane of the other action ρ. Then we can take $\mathbb{Z}^k$ elements $a \in L^+$ and $b \in L^-$ which are not separated by any Lyapunov hyperplane of either action other than L. Then, $E^-_{\alpha(b)} = E^-_{\alpha(a)} \oplus E$, where E is the coarse Lyapunov distribution of α corresponding to L. Similarly, since we assumed that L^+ is not a positive Lyapunov half-space for ρ, we have $E^-_{\rho(b)} \subseteq E^-_{\rho(a)}$. We conclude that

$$W^-_{\alpha(a)} \subsetneq W^-_{\alpha(b)} \quad \text{but} \quad W^-_{\rho(a)} \supseteq W^-_{\rho(b)},$$

which contradicts (3) since ϕ is a homeomorphism. □

3.2. **Uniform estimates for elements near a Lyapunov hyperplane.** The uniform estimates proved in this section will play a crucial role in the proof of the main theorem. They give us upper bounds with small exponents for the expansion in certain directions for elements close to the Weyl chamber walls. This almost isometric behavior together with strong hyperbolic behavior in other directions and exponential mixing will force the convergence of suitable series as distributions.

We first address estimates for the first derivatives of these elements. We fix a positive Lyapunov half-space $L^+ \subset \mathbb{R}^k$ and the corresponding Lyapunov hyperplane L. We denote the corresponding coarse Lyapunov distributions for α and ρ by E and $\bar{E}$ respectively. Recall that $\gamma > 0$ denotes a Hölder exponent of ϕ and ϕ^{-1}.

Lemma 3.3. *For a given coarse Lyapunov distribution E of α there exist linear functionals χ_m and χ_M on $\mathbb{R}^k$ positive on the Lyapunov half-space L^+ corresponding to E such that for any invariant ergodic measure ν of $\alpha(b)$ we have*

$$\chi_m(b) \le \chi_\nu(b) \le \chi_M(b) \qquad \forall\, b \in L^+ \cap \mathbb{Z}^k,$$

where $\chi_\nu(b)$ is any Lyapunov exponent of $(\alpha(b), \nu)$ corresponding to the distribution E. Equivalently, we have $\chi_M(c) \le \chi_\nu(c) \le \chi_m(c)$ for all $c \in L^- \cap \mathbb{Z}^k$.

Proof. The Lyapunov exponents of ρ corresponding to $\bar{E}$ are functionals positive on L^+. Let $\bar{\chi}_m$ and $\bar{\chi}_M$ be the smallest and the largest ones on L^+. We will show that $\chi_m = \gamma\bar{\chi}_m$ and $\chi_M = \gamma^{-1}\bar{\chi}_M$ satisfy the conclusion of the lemma.

First we will prove the second inequality, which is slightly easier. Suppose that $\chi_\nu(b) > \chi_M(b)$ for some Lyapunov exponent of $(\alpha(b), \nu)$ corresponding to the distribution E. Let E' be the distribution spanned by the Lyapunov subspaces of $(\alpha(b), \nu)$ corresponding to Lyapunov exponents greater than $\chi_M(b) + \varepsilon$. Then, for some $\varepsilon > 0$, E' has nonzero intersection with the distribution E. The strong unstable distribution $E'(x)$ is tangent for ν-a.e. x to the corresponding local strong unstable manifold $W'(x)$. Hence the intersection $F(x)$ of $W'(x)$ with the leaf $W(x)$ of the coarse Lyapunov foliation corresponding to E is a submanifold of positive dimension. We take $y \in F(x)$ and denote $y_n = \alpha(-nb)(y)$ and $x_n = \alpha(-nb)(x)$. Then x_n and y_n converge exponentially with the rate at least $\chi_M(b) + \varepsilon$. Since the conjugacy ϕ is γ bi-Hölder it is easy to see that

$$\operatorname{dist}(\phi(x_n), \phi(y_n)) = \operatorname{dist}(\rho(-nb)(x), \rho(-nb)(y))$$

decreases at a rate faster than $\gamma\,\chi_M(b)$. But this is impossible since ϕ maps $W(x)$ to $\bar{W}(\phi(x))$, the leaf of corresponding Lyapunov foliation of ρ, which is contracted by $\rho(-b)$ at a rate at most $\bar{\chi}_M(b) = \gamma\,\chi_M(b)$.

The first inequality can be established similarly. Suppose that $\chi_\nu(b) < \chi_m(b)$ for some Lyapunov exponent of $(\alpha(b), \nu)$ corresponding to the distribution E. Let $E'' \subset E$ be the Lyapunov distribution corresponding to this exponent. We cannot assert that E'' is tangent to an invariant foliation, so we consider a curve l tangent to a vector $0 \neq v \in E''(x)$ for some ν-typical x. Then the exponent of v with respect to $\alpha(-b)$ is $-\chi_\nu(b)$. However, since $\phi(l) \subset \bar{W}(\phi(x))$, we can obtain as above that l is contracted by $\alpha(-b)$ at the rate at least $\chi_m(b)$. It is easy to see that this is impossible. □

Proposition 3.4. *Let E be a coarse Lyapunov distribution and $L^+ \subset \mathbb{R}^k$ be the corresponding Lyapunov half-space for α. Then for any element $b \in L^+$ any $\varepsilon > 0$ there exists $C = C(b, \varepsilon)$ such that*

$$C^{-1} e^{(\chi_m - \varepsilon)n}\|v\| \leq \|D(\alpha(nb))v\| \leq C e^{(\chi_M + \varepsilon)n}\|v\| \quad \textit{for all } v \in E, n \in \mathbb{N}, \tag{4}$$

where χ_m and χ_M are as in Lemma 3.3.

Proof. In the proof we will abbreviate $\alpha(b)$ to b. Consider functions $a_n(x) = \log\|Db^n|_E(x)\|$, $n \in \mathbb{N}$. Since the distribution E is continuous, so are the functions a_n. The sequence a_n is subadditive, i.e. $a_{n+k}(x) \leq a_n(b^k(x)) + a_k(x)$. The Subadditive and Multiplicative Ergodic Theorems imply that for every b-invariant ergodic measure ν the limit $\lim_{n\to\infty} a_n(x)/n$ exists for ν-a.e. x and equals the largest Lyapunov exponent of (b, ν) on the distribution E. The latter is at most $\chi_M(b)$ by Lemma 3.3. Thus the exponential growth rate of $\|Db^n|_E(x)\|$ is at most $\chi_M(b)$ for all b-invariant ergodic measures. Since $\|Db^n|_E(x)\|$ is continuous, this implies the uniform exponential growth estimate, as in the second inequality in (4) (see [36, Theorem 1] or [34, Proposition 3.4]). The first inequality in (4) follows similarly by observing that the exponential growth rate of $\|Db^{-n}|_E(x)\|$ is at most $-\chi_m(b)$. □

Lemma 3.5. *Assume that there is an Anosov element in every Weyl chamber. Then for any $a \in \mathbb{Z}^k$, $\alpha(a)$ is Anosov if and only if its linearization $\rho(a)$ is Anosov.*

Proof. It is classical that if a is Anosov, so is its linearization. So assume that $\rho(a)$ is Anosov. Then a does not belong to any Lyapunov hyperplane of ρ and hence of α. Then Proposition 3.4 applied to a or $-a$ implies that any coarse Lyapunov distribution of α is either uniformly contracted or uniformly expanded by $\alpha(a)$. This implies that $\alpha(a)$ is Anosov since the coarse Lyapunov distributions span TM. $\square$

3.3. **Higher derivatives and estimates on compositions.** In this subsection, we recall a basic estimate on higher derivatives of compositions of diffeomorphisms. The main point is that the exponential growth rate is entirely controlled by the first derivative.

Let ψ be a diffeomorphism of a compact manifold M. Given a function f on M, in local coordinates we have a vector-valued function f^k consisting of f and its partial derivatives up to order k. Using a finite collection of charts and a subordinate partition of unity, one can define the C^k norm of f as $\sup_x \|f^k(x)\|$. It is easy to check that different choices of charts and/or partition of unity give rise to equivalent C^k norms. We will also write $\|f(x)\|_k = \|f^k(x)\|$ for the corresponding norm at x. More generally, let $\mathcal{F}$ be a foliation of M by smooth manifolds. Given a function f which is continuous and differentiable along $\mathcal{F}$ we can again locally define a vector-valued function $f^{k,\mathcal{F}}(x)$ consisting of f and its partial derivative to order k along $\mathcal{F}$ and let $\|f(x)\|_{k,\mathcal{F}} = \|f^{k,\mathcal{F}}(x)\|$. Fixing a finite collection of foliation charts and a subordinate partition of unity, this allows us to define C^k norms corresponding to only taking derivatives along $\mathcal{F}$, by $\|f\|_{k,\mathcal{F}} = \sup_{x\in M} \|f(x)\|_{k,\mathcal{F}}$. Once again it is easy to check that different choices of charts and/or partitions of unity give rise to equivalent norms. In this setting, for a homeomorphism ψ of M that is smooth along $\mathcal{F}$ with all derivatives continuous transversely, we define $\|\psi(x)\|_{k,\mathcal{F}} = \sup \|f \circ \psi(x)\|_{k,\mathcal{F}}$, where the supremum is over functions f such that $\|f(\psi(x))\|_{k,\mathcal{F}} = 1$. We then define $\|\psi\|_{k,\mathcal{F}} = \sup_{x\in M} \|\psi(x)\|_{k,\mathcal{F}}$.

Lemma 3.6. *Let ψ be a diffeomorphism of a manifold M preserving a foliation $\mathcal{F}$ by smooth leaves. Let $N_k = \|\psi\|_{k,\mathcal{F}}$. Then there exists a polynomial P depending only on k and the dimension of the leaves of $\mathcal{F}$ such that for every $m \in \mathbb{N}$,*

$$\|\psi^m\|_{k,\mathcal{F}} \leq N_1^{mk} P(mN_k). \tag{5}$$

This type of estimate is used frequently in the dynamics literature, particularly in KAM theory, and is usually referred to as an *estimate on compositions*. This lemma is essentially [9, Lemma 6.4] and a proof is contained in Appendix B of that paper. There are many other proofs of (5) in the literature, though mostly only in the case where the foliation $\mathcal{F}$ is trivial, i.e. when the only leaf of $\mathcal{F}$ is the manifold M. Most proofs should adapt easily to the foliated setting.

4. Exponential mixing for $\mathbb{Z}^k$ actions on tori

Consider a diffeomorphism a on a manifold preserving a probability measure μ. Given two Hölder functions f, g, we consider the matrix coefficients $\langle a^k f, g\rangle$, where the bracket refers to the standard inner product on $L^2(\mu)$. For an Anosov diffeomorphism a, the matrix coefficients of Hölder functions decay exponentially fast in k for either an invariant volume or the measure of maximal entropy, as follows easily from symbolic dynamics. D. Lind established exponential decay for Hölder functions for ergodic toral automorphisms in [30]. This is considerably harder, as there is no suitable symbolic dynamics. Instead he shows that dual orbits of

Fourier coefficients diverge fast as one has good lower bounds on the distances from integer points to neutral subspaces along stable and unstable subspaces. This precisely is Katznelson's lemma on rational approximation of invariant subspaces. We adapt Lind's argument to prove exponential decay of matrix coefficients of Hölder functions for $\mathbb{Z}^k$ actions by ergodic automorphisms with a bound depending on the norm of the element in $\mathbb{Z}^k$. Even if the $\mathbb{Z}^k$ action contains only Anosov elements this is not trivial since we seek a bound in terms of the norm of $a \in \mathbb{Z}^k$. In addition, some elements in $\mathbb{Z}^k$ will be arbitrarily close to the Lyapunov hyperplanes and thus have little, if any, expansion in certain directions. Thus one essentially has to deal with the partially hyperbolic case. We remark that Damjanović and Katok obtained estimates of exponential divergence of Fourier coefficients for the dual action induced by a $\mathbb{Z}^k$ action by ergodic toral automorphisms [4].

Finally, Gorodnik and the third author generalized exponential decay of matrix coefficients to ergodic automorphisms and $\mathbb{Z}^k$ actions of such on nilmanifolds [14]. We will report on this development in more detail in Section 7 when we prove the nilmanifold version of our main result. The arguments required for the nilmanifold case are substantially more complicated and rely on work by Green and Tao on equidistribution of polynomial sequences [16]. For this reason, and to keep our exposition for the case of toral automorphisms self-contained and elementary, we present our adaptation of Lind's arguments.

Let τ be a $\mathbb{Z}^k$ action by ergodic automorphisms of $\mathbb{T}^n$. We begin by recalling Katznelson's Lemma. For a proof, see [4, Lemma 4.1].

Lemma 4.1. *Let A be an $N \times N$ matrix with integer coefficients. Suppose that $\mathbb{R}^N$ splits as $\mathbb{R}^N = V \oplus V'$ with V and V' invariant under A and such that $A\mid_V$ and $A\mid_{V'}$ do not have common eigenvalues. If $V \cap \mathbb{Z}^N = \{0\}$, then there exists a constant C such that*

$$d(z, V) \geq C\|z\|^{-N}$$

for all $z \in \mathbb{Z}^N$. Here $\|z\|$ denotes the Euclidean norm and d the Euclidean distance.

Consider the finest decomposition into $\tau(\mathbb{Z}^k)$-invariant subspaces E_i of $\mathbb{R}^n = \bigoplus_i E_i$. All E_i are subspaces of generalized eigenspaces of the elements of $\tau(\mathbb{Z}^k)$. Let λ_i denote the Lyapunov exponent defined by the vectors in E_i. Then $e^{\lambda_i(a)}$ is the absolute value of the eigenvalue of $\tau(a)$ on E_i. It is well known that the $\lambda_i(a)$ are the Lyapunov exponents of $\tau(a)$. Pick an inner product with respect to which the E_j are mutually orthogonal. Let $|||v|||$ denote its norm. Since all norms on $\mathbb{R}^n$ are equivalent, we can pick $D > 0$ such that $\frac{1}{D}\|v\| \leq |||v||| \leq D\|v\|$. Finally note that for any $a \in \mathbb{Z}^k$, $\tau(a)$ expands $v \in E_j$ by at least $e^{\lambda_j(a)}$.

Lemma 4.2. *If $\tau(a)$ is an ergodic toral automorphism, then for some i, $\lambda_i(a) \neq 0$.*

This follows immediately from Kronecker's theorem that the eigenvalues of an integer matrix are roots of unity if they all lie on the unit circle. However, let us give a simple direct proof.

Proof. Consider the Jordan decomposition $\tau(a) = bc$ of a, where b is semisimple, c unipotent and $\tau(a)$ and b commute. Then for all i, $\lambda_i(a) = \lambda_i(b)$. If all $\lambda_i(a) = 0$, then b lies in a compact subgroup. Since $\tau(a)$ is ergodic, no eigenvalue of $\tau(a)$ is a root of unity, and hence no power of b is 1. Hence powers of b approximate 1 arbitrarily closely. Hence $tr\, \tau(a)^l = tr\, b^l$ is arbitrarily close to n for suitable l. Since $\tau(a)^l \in SL(n, \mathbb{Z})$, $tr\ \tau(a)^l$ is an integer, and thus $tr\ \tau(a)^l = n$. On the other hand, however, $tr\ \tau(a)^l < n$ since the eigenvalues of b^l cannot be real. This is the final contradiction. $\square$

We will need a slightly stronger variant of this lemma. For $a \in \mathbb{Z}^k$, set $S(a) = \max_i \lambda_i(\tau(a))$. Then $S(a) \neq 0$ for $\tau(a)$ ergodic.

Lemma 4.3. *Suppose that for all $0 \neq a \in \mathbb{Z}^k$, $\tau(a)$ acts ergodically. Then $\inf\{S(a) \mid 0 \neq a \in \mathbb{Z}^k\} > 0$.*

Explicit lower bounds can be found in the literature, e.g. in [3]. We give an easy soft argument for a positive lower bound.

Proof. First suppose that all elements in $\mathbb{Z}^k$ are semisimple. If $\tau(a)$ is semisimple, then $\tau(a)$ expands each E_j precisely by $e^{\lambda_j(a)}$ with respect to $|||v|||$. Suppose $S(a_l) \to 0$ for a sequence of mutually distinct $1 \neq a_l \in \mathbb{Z}^k$. Then there are infinitely many $\tau(a_l)$ which expand distances w.r.t. $|||\ldots|||$ by at most $\frac{2}{D}$. Hence distances w.r.t. $\|\ldots\|$ get expanded by at most 2. Pick any integer vector $z \in \mathbb{Z}^n$. As the images $a_l(e_1)$ are integer vectors of norm at most $2\|z\|$, for some $a_l \neq a_j$, $a_l(z) = a_j(z)$. Hence $a_j^{-1} a_l$ cannot be ergodic.

Next consider the general case. Consider a generating set $a_1, \ldots, a_k$ of $\mathbb{Z}^k$. Suppose $a_1 \in \mathbb{Z}^k$ has a Jordan decomposition $\tau(a_1) = b_1\, c_1$ with b_1 semisimple and c_1 unipotent. Since $\tau(a_1) \in SL(n, \mathbb{Z})$ both b_1 and c_1 are in $SL(n, \mathbb{Q})$. Since c_1 is unipotent, the subspace $W_1 = \{v \mid c_1 v = v\}$ of eigenvectors with eigenvalue 1 is nontrivial and is defined over $\mathbb{Q}$. Also, W_1 is $\tau(\mathbb{Z}^k)$-invariant, and $\tau(\mathbb{Z}^k)$ acts faithfully on W_1 since otherwise some element $\tau(a)$ for $a \in \mathbb{Z}^k$ has eigenvalue 1 and is not ergodic. Also $\tau(a)\mid_{W_1}$ is semisimple. Inductively, we define a descending sequence of rational $\tau(\mathbb{Z}^k)$-invariant subspaces $W_1 \supset W_2 \supset \ldots \supset W_k$ on which $\mathbb{Z}^k$ acts faithfully. In addition, $\tau(a_i)\mid_{W_i}$ is semisimple. Hence $\mathbb{Z}^k$ acts faithfully on W_k and every element acts semisimply. By the special case above, $\inf\{S(a\mid_{W_K}) \mid 1 \neq a \in \mathbb{Z}^k\} > 0$. Since $\inf\{S(a) \mid 0 \neq a \in \mathbb{Z}^k\} \geq \inf\{S(a\mid_{W_K}) \mid 0 \neq a \in \mathbb{Z}^k\}$, the claim follows. $\square$

Note that the λ_i and hence S extend to continuous functions on $\mathbb{R}^k$.

Lemma 4.4. *Suppose for all $0 \neq a \in \mathbb{Z}^k$, $\tau(a)$ acts ergodically. Then for all $0 \neq a \in \mathbb{R}^k$, $S(a) > 0$. Thus $0 < \sigma := \frac{1}{2} \inf\{S(a) \mid a \in \mathbb{R}^k, \| a \| = 1\}$.*

Proof. Suppose $S(a) = 0$ for some $0 \neq a \in \mathbb{R}^k$. Since the line $ta, t \in \mathbb{R}$ comes arbitrarily close to integer points in $\mathbb{Z}^k$, we can find $t_l \in \mathbb{R}$ and $a_l \in \mathbb{Z}^k$ with $a_l - t_l a \to 0$ as $l \to \infty$. As $S(t_l a) = 0$ for all l, it follows readily that $S(a_l) \to 0$ in contradiction to the last lemma. The last claim follows as S is continuous. $\square$

Let $B(d)$ denote the ball of radius d in $\mathbb{Z}^k$.

Lemma 4.5. *Let $1 < r < e^{\frac{\sigma}{n+2}}$. Set $H_l = \{z \in \mathbb{Z} \mid -r^l \leq z \leq r^l\}^n$. Then we have for all sufficiently large l and $a \in \mathbb{Z}^k$ with $\|a\| \geq l$,*

$$\tau(a)(H_l) \cap H_l = \{0\}.$$

Proof. Fix a constant $b > 0$ such that for all $r > 0$, $[-r, r]^n$ is contained in the ball $B_{br}(0)$ of radius br about 0.

Suppose that there is a sequence $l_m \to \infty$ and $a_{l_m} \in \mathbb{Z}^k$ with $\alpha_{l_m} := \|a_{l_m}\| \geq l_m$ such that $\tau(a_{l_m})(H_{l_m}) \cap H_{l_m} \neq \{0\}$. Passing to a subsequence we may assume that $\frac{a_{l_m}}{\alpha_{l_m}} \to a$ converges to $a \in \mathbb{R}^k$. Since $S(a) \geq 2\sigma$, $\lambda_i(a) \geq \sigma$ for some i. Hence we get for all large m that $\lambda_i(a_{l_m}) \geq l_m \sigma$.

Let $E = \bigoplus_{j \neq i} E_j$. By Katznelson's Lemma applied to E, there is a constant $C > 0$ such that for $0 \neq z \in \mathbb{Z}^n$, the distance $d(z, E) > C\|z\|^{-n}$. Suppose $z_{l_m} \in H_{l_m}$ with $\tau(a_{l_m})z_{l_m} \in H_{l_m}$. Then we get

$$\|z_{l_m}\| < br^{l_m} \text{ and } \|\tau(a_{l_m})z_{l_m}\| < br^{l_m}.$$

Denote by π_i the projection to E_i along E. Then $\|\pi_i(z_{l_m})\| = d(z_{l_m}, E) \geq C\|z_{l_m}\|^{-n} > Cb^{-n}r^{-nl_m}$.

As E and E_i are transversal and have constant angle, there is a constant M such that for all $v \in \mathbb{R}^n$, $\pi_i(v) \leq M\|v\|$. Hence $\|\tau(a_{l_m})(\pi_i(z_{l_m}))\| = \|\pi_i(\tau(a_{l_m})z_{l_m})\| < Mbr^{l_m}$. On the other hand, we will show below that

$$\|\tau(a_{l_m})(\pi_i(z_{l_m}))\| \geq \frac{1}{D}e^{\sigma l_m}b^{-n}r^{-nl_m}.$$

Indeed, this estimate is clear when $\tau(a)$ is semisimple but needs more care when $\tau(a)$ has nontrivial Jordan form. This estimate will yield a contradiction to the Lyapunov exponent $\lambda_i(a)$ of a to be at least σ. Here are the details.

Set $v_{l_m} := \frac{\pi_i(z_{l_m})}{\|\pi_i(z_{l_m})\|}$. By the estimates above we get

$$\|\tau(a_{l_m})(v_{l_m})\| \leq \frac{Mbr^{l_m}}{\|\pi_i z_{l_m}\|} \leq MC^{-1}b^{n+1}r^{(n+1)l_m}.$$

Set $b_{l_m} := a - \frac{a_{l_m}}{\alpha_{l_m}}$. Then $b_{l_m} \to 0$. For all large m, we may assume that b_{l_m} expands vectors by a factor of at most r. Since $l_m \leq \alpha_{l_m}$ this implies that

$$\begin{aligned}\|\tau(\alpha_{l_m}a)(v_{l_m})\| = \|\tau(\alpha_{l_m}b_{l_m})\tau(a_{l_m})(v_{l_m})\| &\leq MC^{-1}b^{n+1}r^{(n+1)l_m}r^{\alpha_{l_m}}\\ &\leq MC^{-1}b^{n+1}r^{(n+2)\alpha_{l_m}}.\end{aligned}$$

Find a basis $w_1, \ldots, w_s$ of E_i which brings a to Jordan form. Write $v_{l_m} = x^1_{l_m}w_1 + \cdots + x^s_{l_m}w_s$. Passing to a subsequence the v_{l_m} converge. Suppose j is the last coordinate such that $x^j_{l_m} \to x^j \neq 0$. Then $\tau(\alpha_{l_m}a)(v_{l_m})$ has j-coordinate of absolute value $x^j e^{\alpha_{l_m}\,\lambda_i(a)}$. Since the sup norm determined by the basis $w_1, \ldots, w_r$ is equivalent to the standard Euclidean norm, there is a constant M' such that $\|\tau(\alpha_{l_m}a)(v_{l_m})\| > M'x^j e^{\alpha_{l_m}\,\lambda_i(a)}$. Hence

$$M'x^j e^{\alpha_{l_m}\,\sigma} < M'x^j e^{\alpha_{l_m}\,\lambda_i(a)} < MC^{-1}b^{n+1}r^{(n+2)\alpha_{l_m}}.$$

This is impossible for large l_m by choice of r and σ. □

We will use the approximation by Fejér kernel functions $K_l(t) = \sum_{j=-l}^{l}\big(1 - \frac{|j|}{l+1}\big)e^{2\pi ijt}$, and we refer to [28, chapter I] for details.

Set $F_l(t_1, \ldots, t_n) = K_l(t_1)\ldots K_l(t_n)$. For continuous $f : \mathbb{T}^n \mapsto \mathbb{R}$, $K_l \star f$ is supported on H_l. Endow the space

$$H_\theta = \{f : \mathbb{T}^n \to \mathbb{R} \mid f \text{ is Hölder with Hölder exponent } \theta\}$$

for $0 < \theta < 1$ with the norm

$$\|f\|_\theta = \|f\|_\infty + \sup_{t,h\neq 0}\frac{\|f(t+h) - f(t)\|}{\|h\|^\theta}.$$

As in [28, p. 21, Exercise 1], we get

Lemma 4.6. *There is a constant $C = C(\theta)$ such that the map $H_\theta \mapsto L_\infty(\mathbb{T}^n)$ given by $f \mapsto F_m \star f$ satisfies the estimate*

$$\|F_m \star f - f\|_\infty \leq C(\theta)\|f\|_\theta m^{-\theta}.$$

Theorem 4.7. *Suppose $\mathbb{Z}^k$ acts affinely on $\mathbb{T}^n$ such that for all $0 \neq a \in \mathbb{Z}^k$, $\tau(a)$ acts ergodically. Let f and g be two Hölder functions on $\mathbb{T}^n$ with Hölder exponents θ. Then there exists $r > 1$ such that for any $a_l \in \mathbb{Z}^k$ with $\|a_l\| \geq l$ we can bound the matrix coefficients*

$$\left|\langle a_l f, g\rangle - \int_{\mathbb{T}^n} f \int_{\mathbb{T}^n} g\right| < C(\theta)\ \ (4\|f\|_\theta\|g\|_2 + 2\|g\|_\theta\|f\|_2)\ \ r^{-\theta l}.$$

In particular, the matrix coefficients decay exponentially fast.

Proof. We can assume that $\int_{\mathbb{T}^n} f = \int_{\mathbb{T}^n} g = 0$ are both 0 by subtracting the constants $\int_{\mathbb{T}^n} f$ and $\int_{\mathbb{T}^n} g$ from f and g respectively.

We pick $1 < r < e^{\frac{\sigma}{n+2}}$ as in Lemma 4.5, where σ is as in Lemma 4.4. Let $m = [r^l]$, the largest integer smaller than r^l. Set $f_l = K_m \star f$ and $g_l = K_m \star g$ with frequencies in H_l. Then $\int_{\mathbb{T}^n} f_l = \int_{\mathbb{T}^n} g_l = 0$ and $\|f - f_l\|_\infty \leq 2C(\theta)\|f\|_\theta(r^l)^{-\theta}$ and $\|g - g_l\|_\infty < 2C(\theta)\|g\|_\theta(r^l)^{-\theta}$, where the 2 accounts for the discrepancy coming from m versus r^l. By the last lemma, we get

$$\langle a_l(f), g\rangle = \langle a_l f, (g - g_l)\rangle + \langle a_l(f - f_l), g_l\rangle + \langle a_l(f_l), g_l\rangle.$$

The last term is eventually 0 since the constant term is 0 and a_l moves H_l off itself. The first term is bounded by

$$\|f\|_2\|g - g_l\|_\infty \leq 2C(\theta)\|g\|_\theta\|f\|_2\ r^{-\theta l}.$$

Take l large enough so that $\|g - g_l\|_\infty < 2C(\theta)\|g\|_\theta(r^l)^{-\theta} < 2$. Then the second term is bounded by

$$\|g_l\|_2\|f - f_l\|_\infty \leq 2C(\theta)\|f\|_\theta\|g_l\|_2\ r^{-\theta l} \leq 4C(\theta)\|f\|_\theta\|g\|_2\ r^{-\theta l}.$$

This yields the desired estimate □

Corollary 4.8. *The same statement as above holds for any Anosov $\mathbb{Z}^k$ action with $k > 1$ where every element acts ergodically.*

Proof. This combines Theorem 4.7, the existence of a Hölder conjugacy, and the fact that we define matrix coefficients with respect to the push-forward measure, which is the unique smooth invariant measure by Proposition 3.1. □

5. Regularity and the proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 by showing that the Franks-Manning conjugacy ϕ between the $\mathbb{Z}^k$ actions α and ρ is smooth. We will use ϕ and the uniform exponential estimates along the coarse Lyapunov foliations of α from Section 3.2, but we will not use Anosov elements explicitly in this section. Instead, we will use the subgroup $\mathbb{Z}^2$ consisting of ergodic elements that we postulated in Theorem 1.1. Theorem 4.7 gives exponential mixing with uniform estimates along this $\mathbb{Z}^2$. This allows us to define distributions on Hölder functions which correspond to the components of the conjugacy and their derivatives. First, however, we will make some reductions to the general case.

By passing to a finite index subgroup of $\mathbb{Z}^k$ we can assume that the action α has a common fixed point. First we reduce the problem to the case when α acts on the torus with the standard differentiable structure. Note that a construction due to Farrell and Jones shows that there exist Anosov diffeomorphisms of exotic tori [7]. However, every exotic torus of dimension at least 5 has a finite cover which is diffeomorphic to the standard torus [38, Chapter 15 A, last unitalicized paragraph]. In this case we can consider the lifts of the actions and the conjugacy. Clearly, the smoothness of ϕ follows from the smoothness of its lift. We will give an independent argument in Section 6 for the case of 4-dimensional tori. Hence, without loss of generality, we can assume that α acts on the same standard torus as ρ. In dimensions 2 and 3, by Remark A.5 in the Appendix, there are no exotic differentiable structures, though this fact is not strictly needed here. In dimension 3, Theorem 1.1 follows from the main result of [34]. As explained in Section 6, there are no higher rank Anosov actions on tori in dimension 2.

By changing coordinates we can also assume that 0 is a common fixed point for both α and ρ. Then there exists a unique conjugacy ϕ in the homotopy class of the identity satisfying $\phi(0) = 0$. We can lift ϕ to the map $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\tilde{\phi}(0) = 0$ and write it as $\tilde{\phi} = I + h$, where $h : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathbb{Z}^n$ periodic.

Consider an element a in $\mathbb{Z}^2$ and abbreviate $\alpha(a)$ to a and $\rho(a)$ to $\bar{A}$. We denote their lifts to $\mathbb{R}^n$ that fix 0 by $\tilde{a}$ and A respectively and note that A is linear. Since ϕ is a conjugacy and the lifts fix 0, they satisfy $\tilde{\phi} \circ \tilde{a} = A \circ \tilde{\phi}$. Hence we obtain

$$(I + h)(\tilde{a}(x)) = A\,(I + h)(x),$$

which is equivalent to

$$h(x) = A^{-1}(\tilde{a}(x) - A(x)) + A^{-1}(h(\tilde{a}(x))) = Q(x) + A^{-1}(h(\tilde{a}(x))),$$

where $Q(x) = A^{-1}(\tilde{a}(x) - A(x))$. Note that $Q(x)$ is smooth since a is smooth with respect to the standard differentiable structure (this will be crucial later). Since h is $\mathbb{Z}^n$ periodic it is easy to see that $A^{-1}(h(\tilde{a}(x)))$ and hence $Q(x)$ are also $\mathbb{Z}^n$ periodic. For the remainder of this section we will view h and Q as functions from $\mathbb{T}^n$ to $\mathbb{R}^n$. The functional equation on $\mathbb{T}^n$ becomes

$$h(x) = Q(x) + A^{-1}(h(ax)). \tag{6}$$

Fix a coarse Lyapunov foliation $\mathcal{V}$ of α and the corresponding linear coarse Lyapunov foliation $\bar{\mathcal{V}}$ of ρ. Let V be the subspace of $\mathbb{R}^n$ parallel to $\bar{\mathcal{V}}$ and W be the complementary A invariant subspace, which is parallel to the sum of all coarse Lyapunov foliations of ρ different from $\bar{\mathcal{V}}$. Denote by $h_V : \mathbb{R}^n \to V$ the projection of h to V along W. Since V is A-invariant, projecting equation (6) and letting A_V denote the restriction of A to V we obtain

$$h_V(x) = Q_V(x) + A_V^{-1}(h_V(ax)) =: F_V(h_V)(x), \tag{7}$$

where Q_V denotes the projection of Q to V along W.

We will use the functional equation (7) with well-chosen elements a to study the derivatives of h_V along the coarse Lyapunov foliations of α. These derivatives exist, a priori, only in the sense of distribution on smooth functions. The crucial element of the proof is Lemma 5.1 below, which shows that these distributional derivatives extend to functionals on the spaces of Hölder functions. We emphasize that this lemma is quite general and may be useful in other situations. The main ingredients are the uniform exponential estimates with arbitrarily small exponents along coarse Lyapunov foliations, and exponential mixing for Hölder functions. The key idea is that in our estimates for derivatives, the exponential decay coming from exponential mixing overcomes small exponential growth coming from derivatives.

Lemma 5.1. *For any coarse Lyapunov foliation $\mathcal{V}'$ of α, possibly equal to $\mathcal{V}$, and for any $\theta > 0$ the derivatives of h_V of any order along $\mathcal{V}'$ exist as distributions on the space of θ-Hölder functions.*

Proof. Let $L, L^+, L^- \subset \mathbb{R}^k$ be the Lyapunov hyperplane and the positive and negative Lyapunov half-spaces corresponding to $\mathcal{V}$. Let L' be the Lyapunov hyperplane corresponding to $\mathcal{V}'$. In this proof we will choose a in the $\mathbb{Z}^2$ subgroup consisting of ergodic elements. We note that $\mathcal{V}$ and $\mathcal{V}'$ are coarse Lyapunov foliations for α-action of the full $\mathbb{Z}^k$ and that we make no assumptions on the relative positions of $\mathbb{Z}^2$, L, and L' in $\mathbb{R}^k$. We will choose a in a narrow cone in $\mathbb{Z}^2$ around $L' \cap \mathbb{Z}^2$, so that a will expand $\mathcal{V}'$ at most slowly. In case $\mathbb{Z}^2 \subset L'$, this automatically holds for all a in $\mathbb{Z}^2$. Since any such cone cannot be contained entirely in L^-, we can always choose such an $a \in \mathbb{Z}^2$ in L^+ or L.

If $a \in L^+$, then A_V^{-1} is a contraction. Then the operator F_V in (7) is a contraction on the space $C^0(\mathbb{T}^n, V)$. Hence it has a unique fixed point $\lim F_V^m(0)$, which therefore has to coincide with h_V. Thus we obtain

$$h_V(x) = \sum_{m=0}^{\infty} A_V^{-m}\, Q_V(a^m x). \tag{8}$$

If $a \in L$ the series in (8) does not converge in the space of continuous functions. However, it converges in the space $\mathcal{D}_0$ of distributions on smooth functions with zero average, and the equality in (8) holds in $\mathcal{D}_0$. To see this we iterate (7) to get

$$h_V(x) = \sum_{m=0}^{N-1} A_V^{-m}\, Q_V(a^m x) + A_V^{-N} h_V(a^N x). \tag{9}$$

Since $\|A_V^{-m}\|$ grows at most polynomially in m for $a \in L$, and since h_V is Hölder, Corollary 4.8 implies that the pairing $\langle A_V^{-N} h_V(a^N x), f\rangle \to 0$ for any Hölder function f with $\int_{\mathbb{T}^n} f = 0$. This establishes convergence and equality in (8) when both sides are considered as elements in $\mathcal{D}_0$.

We will use the notation of Section 3.3 for derivatives. Given a smooth function $g : \mathbb{T}^n \to \mathbb{R}^l$, we write $g^{k,\mathcal{V}'}$ for the vector consisting of the derivatives of g up to order k along the foliation $\mathcal{V}'$. If g is a vector-valued function on $\mathbb{T}^n$ and f is a scalar-valued function, we write gf for the vector function obtained by component-wise multiplication of g by f. We then write $\langle g, f\rangle$ for the vector obtained by integrating gf over $\mathbb{T}^n$. We will use the same notation $h_V^{k,\mathcal{V}'}$ for the vector of distributional derivatives of h_V along $\mathcal{V}'$ (see Section 8 for a detailed description of distributional derivatives in the context of foliations). Differentiating (8) term-wise we obtain the formula for $h_V^{k,\mathcal{V}'}$,

$$\langle h_V^{k,\mathcal{V}'}, f\rangle = \sum_{m=0}^{\infty} \langle A_V^{-m} (Q_V \circ a^m)^{k,\mathcal{V}'}, f\rangle. \tag{10}$$

Note that the derivative of a distribution is defined by its values on derivatives of test functions (16), and those have zero average. Thus convergence and equality in (10) hold in the space $\mathcal{D}$ of distributions on smooth functions, even if equality in (8) holds only in $\mathcal{D}_0$. Since Q_V is smooth, the pairings in the series in (10) are simply given by integration. To show that $h_V^{k,\mathcal{V}'}$ extends to a functional on the space of Hölder functions we will now estimate these pairings in terms of the Hölder norm of f.

We will use smooth approximations of f by convolutions $f_\varepsilon = f \star \phi_\varepsilon$, where the kernel is given by rescaling $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ of a fixed bump function ϕ and thus is supported on the ball of radius ε and satisfies

$$\phi_\varepsilon \geq 0, \quad \int_{\mathbb{T}^n} \phi_\varepsilon = 1, \quad \|\phi_\varepsilon\|_{C^k} = \varepsilon^{-(n+k)}\|\phi\|_{C^k}.$$

Then it is easy to check the following estimates, where $\|\cdot\|_k$ denotes the C^k norm for $k \geq 0$,

(11)
$$\|f_\varepsilon - f\|_0 \leq \varepsilon^\theta \|f\|_\theta \quad \text{for } 0 < \theta \leq 1 \quad \text{and} \qquad \|f_\varepsilon\|_{C^k} \leq c_k\, \varepsilon^{-n-k}\|f\|_0 \quad \text{for } k \in \mathbb{N},$$

where f is a θ-Hölder function and c_k is a constant depending only on k. First we estimate the pairings in (10) with f_ε. Note that $\|\cdot\|_l \leq \|\cdot\|_k$ if $l \leq k$. We have

$$\begin{aligned}\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, f_\varepsilon\rangle\| &\leq \|A_V^{-m}\| \cdot \|\langle (Q_V \circ a^m)^{k,\mathcal{V}'}, f_\varepsilon\rangle\| \\ &= \|A_V^{-m}\| \cdot \|\langle Q_V \circ a^m, (f_\varepsilon)^{k,\mathcal{V}'}\rangle\|\,.\end{aligned}$$

Since $\|(f_\varepsilon)^{k,\mathcal{V}'}\|_\theta \leq \|(f_\varepsilon)^{k,\mathcal{V}'}\|_1 \leq \|f_\varepsilon\|_{k+1}$, using Corollary 4.8 and (11) we can estimate

$$\begin{aligned}\|\langle Q_V \circ a^m, (f_\varepsilon)^{k,\mathcal{V}'}\rangle\| &\leq K_1\, r^{-m\|a\|\theta}\, \|Q_V\|_\theta \|(f_\varepsilon)^{k,\mathcal{V}'}\|_\theta \\ &\leq K_2\, r^{-m\|a\|\theta} \varepsilon^{-(n+k+1)} \|Q_V\|_\theta\, \|f\|_0\,.\end{aligned}$$

Since a is chosen in $L^+ \cup L$, $\|A_V^{-1}\|$ grows at most polynomially in $\|a\|$ and thus, for any $\eta > 0$, we can ensure that $\|A_V^{-1}\| < (1+\eta)^{\|a\|}$ for all a with sufficiently large norm. Thus we conclude from the two equations above that

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, f_\varepsilon\rangle\| \leq K_2\,(1+\eta)^{m\|a\|}\, r^{-m\|a\|\theta}\, \varepsilon^{-(n+k+1)}\|Q_V\|_\theta\, \|f\|_0\,. \tag{12}$$

Now we estimate the pairings in (10) with $f - f_\varepsilon$ using the supremum norm and estimating $\|A_V^{-m}\|$ as above:

$$\begin{aligned}&\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, (f - f_\varepsilon)\rangle\| \leq \|A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}\|_0 \cdot \|(f - f_\varepsilon)\|_0 \\ &\quad \leq \|A_V^{-m}\| \cdot \|Q_V \circ a^m\|_{k,\mathcal{V}'} \cdot \varepsilon^\theta \|f\|_\theta \leq (1+\eta)^{m\|a\|} \cdot \|a^m\|_{k,\mathcal{V}'} \cdot \|Q_V\|_{k,\mathcal{V}'} \cdot \varepsilon^\theta \|f\|_\theta.\end{aligned}$$

Here we used the notation of Section 3.3. Denoting $N_k = \|a\|_{k,\mathcal{V}'}$, and using equation (5) from Lemma 3.6 we conclude that

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, (f - f_\varepsilon)\rangle\| \leq (1+\eta)^{m\|a\|} \cdot N_1^{mk}\, P(mN_k) \cdot \varepsilon^\theta \cdot \|Q_V\|_k \cdot \|f\|_\theta\,.$$

Recall that we choose a in a cone around $L' \cap \mathbb{Z}^2$. For any $\eta > 0$, by taking the cone sufficiently narrow and using Proposition 3.4, we can ensure that $N_1 = \|a\|_{1,\mathcal{V}'} < (1+\eta)^{\|a\|}$ for any such a with sufficiently large norm. Then from the last equation we obtain that

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, (f - f_\varepsilon)\rangle\| \leq (1+\eta)^{m(k+1)\|a\|} \cdot P(mN_k) \cdot \varepsilon^\theta \cdot \|Q_V\|_k \cdot \|f\|_\theta\,. \tag{13}$$

For any fixed θ, we have a fixed rate of exponential decay with respect to m in (12), but the rate of exponential growth in (13) can be made arbitrarily slow. This allows us to choose an ε that gives exponentially decaying estimates for both (12) and (13). More precisely, we take

$$\varepsilon = r^{\frac{-m\|a\|\theta}{\theta+n+k+1}} \qquad \text{and denote} \quad \zeta = r^{\frac{\theta^2}{\theta+n+k+1}} > 1.$$

Then we obtain from (12) and (13) that

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, f_\varepsilon\rangle\| \le K_2\,(1+\eta)^{m\|a\|}\,\zeta^{-m\|a\|} \cdot \|Q_V\|_\theta\,\|f\|_0 \quad \text{and}$$

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, (f-f_\varepsilon)\rangle\| \le (1+\eta)^{(k+1)m\|a\|}\,P(mN_k)\,\zeta^{-m\|a\|} \cdot \|Q_V\|_k\,\|f\|_\theta.$$

For any k we can now choose a, and hence η, so that $\xi = \zeta \cdot (1+\eta)^{-(k+2)} > 1$. Since the polynomial P and constant N_k depend only on k and a, we can then estimate $P(mN_k) \le K_3(1+\eta)^{m\|a\|}$. Finally, we obtain from the last two equations that

$$\|\langle A_V^{-m}(Q_V \circ a^m)^{k,\mathcal{V}'}, f\rangle\| \le K_4\,\xi^{-m\|a\|} \cdot \|Q_V\|_k\,\|f\|_\theta.$$

Thus for any θ and k we obtain exponentially decreasing estimates for the terms in (10). We conclude that $\|\langle h_V^{k,\mathcal{V}'}, f\rangle\| \le C\|f\|_\theta$ and hence $h_V^{k,\mathcal{V}'}$ extends to a functional on the space of θ-Hölder functions.

Proof of Theorem 1.1. We discussed actions on two- and three-dimensional tori above, and we will prove Theorem 1.1 for four-dimensional tori with an exotic smooth structure in the next section. When the dimension is greater than four, as explained above, we can pass to a finite cover by smoothing theory and assume that the smooth structure is standard. Passing to a subgroup of finite index, we can also assume that α has a common fixed point. By Lemma 5.1, for any coarse Lyapunov foliation $\mathcal{V}'$ of α and for any $\theta > 0$ the derivatives of h_V of any order along $\mathcal{V}'$ exist as distributions on the space of θ-Hölder functions. Hence by Corollary 8.5, all h_V are C^∞. Since the subspaces V span, h is determined by the projections h_V. It follows that h is C^∞ and hence so is ϕ. It remains to show that ϕ is a diffeomorphism. Since ϕ is a homeomorphism, it suffices to show that the differential of ϕ is everywhere nondegenerate. This follows from Proposition 3.1 since we have $\lambda = \phi_*(\mu)$ and μ has smooth positive density.

□

6. Four-dimensional exotic tori

Now consider a higher rank Anosov action on a 4-dimensional torus with an exotic differentiable structure. Due to low dimension we are able adapt arguments from [11] to obtain the result in this case.

By passing to a finite index subgroup of $\mathbb{Z}^k$ we can assume that the linear part ρ acts by linear automorphisms from $SL(4,\mathbb{Z})$. We begin by analyzing possibilities for such actions on $\mathbb{T}^4$. Let $A \in SL(4,\mathbb{Z})$ be an Anosov element for ρ. First we claim that the characteristic polynomial of A is irreducible over $\mathbb{Q}$. Indeed, the only possible splitting would be into a product of quadratic terms and would imply existence of a rational invariant subspace of dimension two. Such a subspace would be invariant with respect to a finite index subgroup of $\mathbb{Z}^k$. The restriction of ρ to the corresponding torus would still be Anosov and contain a $\mathbb{Z}^2$ subgroup of ergodic elements, as ergodicity is equivalent to having no root of unity as an eigenvalue. The latter however is impossible since Anosov actions on $\mathbb{T}^2$ can only have rank one. More precisely, by the Dirichlet Unit Theorem the centralizer of an irreducible Anosov matrix in $SL(n,\mathbb{Z})$ is a finite extension of $\mathbb{Z}^d$, where d is $n-1$ minus the number of pairs of complex eigenvalues. Moreover, all nontrivial elements of this $\mathbb{Z}^d$ are semisimple. We conclude that $\rho(\mathbb{Z}^k)$ is a subgroup of such $\mathbb{Z}^d \subset SL(4, \mathbb{Z})$.

We note that ρ has four Lyapunov exponents (counted with multiplicity) and $\chi_1+\chi_2+\chi_3+\chi_4 = 0$ by volume preservation. If no two are negatively proportional, then ρ, and hence α, is so-called TNS (totally nonsymplectic) and smoothness of the conjugacy follows from [11, Theorem 1.1]. Now suppose that there are negatively proportional Lyapunov exponents. This case does not follow from any previous theorem but can still be handled using techniques from [11] and [22]. Note that in this case there are no positively proportional Lyapunov exponents, as otherwise for elements near the kernel of the negatively proportional ones all Lyapunov exponents will be close to zero by volume preservation, contradicting Lemma 4.3. This implies that $\rho(\mathbb{Z}^k)$ contains matrices with pure real spectrum and the coarse Lyapunov spaces for ρ are one-dimensional and totally irrational, so in particular the corresponding linear foliations of $\mathbb{T}^4$ are ergodic.

For the nonlinear action α the coarse Lyapunov foliations are also one-dimensional and any pair W_i, W_j is jointly integrable in the topological sense by the conjugacy to the linear action. By [22, Lemma 4.1], the joint foliation W_{ij} has smooth leaves. For each W_i consider a W_j which does not correspond to negatively proportional exponents. Then one can see as in [11, Proposition 5.2] that there is an element that contracts W_i faster than W_j and conclude that W_i and W_j are C^∞ along the leaves of W_{ij}. In place of measurable normal forms in [11], for one-dimensional foliations we can use the nonstationary linearization [26, Proposition A.1] which is continuous on M in the C^∞ topology. Hence a simple version of the holonomy argument [11, Proposition 8.1] works for any W_i using the holonomy along such W_j. The argument shows that the conjugacy ϕ is C^∞ along any $W_i(x)$ with the derivatives continuous on M. Then the smoothness of ϕ follows easily as in [11].

7. The nilmanifold case

In this section we will describe the adaptations of our arguments needed for the case of an Anosov action on an infranilmanifold M. Passing to finite covers, we can assume that N/Γ is a nilmanifold. Next we reduce to the case when the differentiable structure on N/Γ is standard, i.e., given by the ambient Lie group structure. First we note that there are no nilmanifolds of dimension at most 4 supporting an Anosov automorphism besides the torus. Hence we can employ the theorem of J. Davis, proved in the appendix, that every exotic nilmanifold in dimension at least 5 has a finite cover with standard differentiable structure. This allows us to lift the actions to ones smooth with respect to a standard differentiable structure, as in the beginning of Section 5. Thus the main theorem follows for nilmanifolds of dimension at least 5 provided it holds for actions on standard nilmanifolds. We will now give a proof of the main theorem in this setup.

First note that the arguments from Section 3 allowing uniform control of exponents work verbatim. That certain distributions are dual to the space of Hölder functions will again be key to our arguments. This requires exponential mixing of the action which does not follow easily from Fourier analysis or more generally representation theory anymore. Instead we evoke a recent result by Gorodnik and the third author [15]. This is far less elementary than the results in Section 4 and uses recent results of Green and Tao [16] on the equidistribution of polynomial sequences.

Theorem 7.1 (Gorodnik-Spatzier). *Consider a $\mathbb{Z}^k$ action α by ergodic affine diffeomorphisms on an infra-nilmanifold. Then for any $0 < \theta < 1$ there is $0 < \lambda < 1$ such that for any two θ-Hölder functions $f, g : X \mapsto \mathbb{R}$ we get*

$$\left| \langle f \circ \alpha(z), g \rangle - \int_{\mathbb{T}^n} f \int_{\mathbb{T}^n} g \right| \leq O_\theta(\lambda^{\|z\|}) \|f\|_\theta \|g\|_\theta, \tag{14}$$

where $\|z\|$ denotes some fixed norm on $\mathbb{Z}^k$.

We need to establish regularity of the solutions to the cocycle equations employed in Section 5. We are inspired by the approach of Margulis and Qian in [32, Lemma 6.5]. However, while they write their equations in exponential coordinates and directly study the solutions in these coordinates, we will reduce the cocoycle equation to a series of equations, one for each term of the derived series of N. This yields abelian-valued cocycle equations to which we can apply the arguments from the toral case. Here are the details.

As in Section 5 we consider the lift $\tilde{\phi} : N \to N$ of the Franks-Manning conjugacy $\phi : N/\Gamma \to N/\Gamma$. We can write it as a product $\tilde{\phi} = h \cdot I$, where $h : N \to N$ satisfies

$$(h \cdot I)(a(x)) = A\big((h \cdot I)(x)\big) \tag{15}$$

on N and projects to the map from N/Γ to N.

Let N' be the commutator subgroup of N. Pick a splitting of the Lie algebra $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}_0$ of N, where $\mathcal{N}'$ is the Lie algebra of N'. Note that $\mathcal{N}_0$ is not a Lie algebra. Let $N_0 = \exp \mathcal{N}_0$, where exp is the exponential map. Now we decompose h as a product $h = h_1 \cdot h_0$, where h_0 takes values in N_0 and h_1 takes values in N', in the following way. We take h_0 to be the exponential of the $\mathcal{N}_0$ component of $\exp^{-1} h$ and define $h_1 = h \cdot (h_0)^{-1}$. One can see that $h_1 \in N'$ from the Campbell-Hausdorff formula since all brackets are in $\mathcal{N}'$. Note that h_0 and h_1 project to maps from N/Γ to N.

Step 1. We first show that h_0 is smooth. Let $\bar{h} : N \to N' \backslash N$ be the composition of h with the projection $N \to N' \backslash N$. Note that h_0 is smooth precisely when $\bar{h}$ is smooth, since by construction $\exp^{-1} h_0$ and $\exp^{-1} \bar{h}$ are just related by the identification of $\mathcal{N}_0$ with the Lie algebra of $N' \backslash N$. Write the group operation in $N' \backslash N$ additively. Denote by $\bar{A}$ the induced automorphism of $N' \backslash N$. Then we get

$$(I + \bar{h})(a(x)) = \bar{A}\,(I + \bar{h})(x).$$

Now we can use exactly the same arguments as in Section 5 and in particular exponential mixing to show that $\bar{h}$ is smooth.

Step 2. We write (15) in terms of the decomposition $h = h_1 \cdot h_0$:

$$h_1(a(x)) h_0(a(x)) a(x) = A(h_1(x)) A(h_0(x)) A(x).$$

This gives the formula

$$h_1(x) = A^{-1}(h_1(a(x))) A^{-1}(h_0(a(x))) A^{-1}(a(x)) x^{-1} h_0(x)^{-1}.$$

Since the automorphism A leaves N' invariant it follows that both $h_1(x)$ and $A^{-1}(h_1(a(x)))$ belong to N'. Hence the function

$$Q_1(x) := A^{-1}(h_0(a(x))) A^{-1}(a(x)) x^{-1} h_0(x)^{-1}$$

also takes values in N'. In addition, $Q_1(x)$ is smooth by construction and satisfies the functional equation

$$h_1(x) = A^{-1}(h_1(a(x)))Q_1(x).$$

Since h_1 projects to a map from N/Γ, then so does $A^{-1}(h_1(a(x)))$ and, from the equation, $Q_1(x)$. Thus the equation holds in $C^0(N/\Gamma, N')$.

Now mod out by the second derived group N'' and denote the projected maps by bars. Again we write multiplication in $N''\backslash N'$ additively to get

$$\bar{h}_1(x) = (\overline{A\mid_{N'}})^{-1}(\bar{h}_1(a(x))) + \overline{Q}_1(x).$$

We can analyze the solution to this equation once again using the methods from the basic toral case, and in particular exponential mixing and uniqueness of solutions. We conclude that $\bar{h}_1$ is a smooth function. Continue this analysis by decomposing N' in terms of N'' and a complement N_1 to N'' inside N'. Since the derived series terminates for a nilpotent Lie group, we see that h is a smooth function.

8. Wavefront sets

We establish regularity properties of a distribution whose derivatives along a foliation $\mathcal{F}$ are dual to Hölder functions in a suitable fashion. While the definitions and concepts will be developed for foliations, the proof will be entirely local on an open subset of $R^{n_1} \times R^{n_2}$ and only uses partial derivatives along the second factor. However, it will be important to develop the appropriate notions for foliations for our application to the conjugacy problem in the main part of the paper.

The main theorem is a variation of results of Rauch and Taylor in [35] who assume that derivatives of the distribution along a foliation belong to various function spaces. The novelty here is that the derivatives are allowed to be distributions, of a precise order less than 0. While we only deal with the particular case of distributions dual to certain Hölder functions, we expect this to be true much more generally.

We first lay out our assumptions on the foliation. Let x and y denote the coordinates of the first and second factor of a point in $R^{n_1} \times R^{n_2}$. Suppose $z = \Gamma(x, y)$ is a bi-Hölder homeomorphism of an open subset $O \subset R_x^{n_1} \times R_y^{n_2}$ into $\mathbb{R}^{n_1+n_2}$ with the property that Γ has y-derivatives of all orders and these derivatives are Hölder in (x, y). We further assume that for a fixed x, $\Gamma(x, \_\,)$ is an immersion on each $\{x\} \times R_y^{n_2}$. Then we call Γ a *foliation chart*, or more precisely, a *Hölder foliation chart with smooth leaves.* On a manifold, Hölder foliations $\mathcal{F}$ with smooth leaves are defined by patching foliation charts. If $\mathcal{F}$ can be defined by using smooth foliation charts Γ, we call $\mathcal{F}$ *smooth.* Note that the $x \times R^{n_2}$ for $x \in \mathbb{R}^{n_1}$ define a smooth foliation $\mathcal{Y}$ of $\mathbb{R}^{n_1+n_2}$.

We will further assume $\mathcal{F}$ is *strongly absolutely continuous*, i.e. there is a continuous function $J(x, y) > 0$ such that all y-derivatives of J exist and are Hölder in x and y and such that for any compactly supported continuous function u on $\Gamma(O)$,

$$\int u(z)dz = \int u(\Gamma(x, y))J(x, y)dxdy.$$

Note that if a function $u(z)$ has partial derivatives along the foliation $\mathcal{F}$, then $u \circ \Gamma(x, y)$ has partial y-derivatives. In addition, the dependence of these latter derivatives on x is continuous or Hölder if the partial derivatives of u along $\mathcal{F}$

are continuous or Hölder. Thus the partials $\partial_y^\beta(u(\Gamma(x,y)))$ are well-defined, and it makes sense to discuss their regularity.

We will now define derivatives along the foliation $\mathcal{F}$ on a manifold M defined by foliation charts Γ. Fix a standard basis for $R_y^{n_2}$, parallel translate it over $\mathbb{R}^{n_1+n_2}$ and consider the push-forward under Γ. This defines vector fields V_j tangent to $\mathcal{F}$ which are smooth along the leaves of $\mathcal{F}$ and whose derivatives along $\mathcal{F}$ of any order depend Hölder transversely to $\mathcal{F}$. We say that a function f has derivatives of order up to k along $\mathcal{F}$ if for any sequence $V_{j_1},\dots,V_{j_k}$ the derivatives $V_{j_1}\dots V_{j_k}(f)$ exist. If M is endowed with a Riemannian metric, equivalently we can require the following: consider any smooth vector fields $X_1,\dots,X_k$ on M, and denote their orthogonal projections to the tangent spaces of $\mathcal{F}$ by $Z_1,\dots,Z_k$. Then f has derivatives up to order k along $\mathcal{F}$ if the derivatives $Z_1\dots Z_k(f)$ exist.

Lemma 8.1. *Under the above assumptions, the derivatives of Γ^{-1} along $\mathcal{F}$ are also Hölder.*

Proof. This follows from the standard formulas for differentiating the inverse of immersions, and the assumptions on Hölderness of Γ and its derivatives along $\mathcal{Y}$. Note that the correspondence of the Hölder coefficients, while complicated, is explicit. $\square$

In our main theorem below, we will allow the Hölder exponents of the higher order derivatives of both Γ and J to get worse with the order. In the following we will use a fixed nonincreasing sequence α_k such that all $\mathcal{Y}$ or $\mathcal{F}$ derivatives of both Γ, Γ^{-1} and J of order at most k are Hölder with Hölder exponent α_k. This is possible by the last lemma. Note that the vector fields V_j defined above and their derivatives along $\mathcal{F}$ up to order k depend α_k-Hölder transversely to $\mathcal{F}$.

Fix a Riemannian metric on M. Next, we introduce the space $C_{\mathcal{F}}^{\alpha,k}$ of compactly supported α-Hölder functions on M which in addition have derivatives along $\mathcal{F}$ of all orders $\leq k$ and all such derivatives are α-Hölder as functions on M. Then $C_{\mathcal{F}}^{\alpha,k}$ is a Banach space with the norm given by the finite sequence of α-Hölder norms of the derivatives along $\mathcal{F}$ of order $\leq k$. If M is compact, the norm is independent of the Riemannian metric chosen up to bi-Lipschitz equivalence. Note that $C_{\mathcal{F}}^{\alpha,k}$ is closed under multiplication. We let $(C_{\mathcal{F}}^{\alpha,k})^*$ be the dual space to $C_{\mathcal{F}}^{\alpha,k}$. Note that any compactly supported smooth function on M naturally belongs to any $C_{\mathcal{F}}^{\alpha,k}$. Hence any element in $(C_{\mathcal{F}}^{\alpha,k})^*$ defines a distribution on smooth functions on M. Alternatively, $(C_{\mathcal{F}}^{\alpha,k})^*$ is the space of distributions (dual to smooth functions) which extend to continuous linear functionals on $C_{\mathcal{F}}^{\alpha,k}$. As for notation, we will also write the pairing $D(\phi)=\langle D,\phi\rangle$ for $D\in(C_{\mathcal{F}}^{\alpha,k})^*$ and $\phi\in C_{\mathcal{F}}^{\alpha,k}$. All of these notions apply to the special case of $\mathcal{F}=\mathcal{Y}$.

We will work with a foliation chart Γ and use the above notation for the case $M=\Gamma(O)$.

Lemma 8.2. *Under composition with Γ, functions in $C_{\mathcal{F}}^{\alpha,k}$ pull back to functions in $C_{\mathcal{Y}}^{\alpha\alpha_k,k}$. Conversely, functions in $C_{\mathcal{Y}}^{\beta,k}$ pull back to functions in $C_{\mathcal{F}}^{\beta\alpha_k,k}$ under composition with Γ^{-1}. Consequently, we can also pull back distributions in $(C_{\mathcal{F}}^{\beta\alpha_k,k})^*$ by Γ to get distributions in $(C_{\mathcal{Y}}^{\beta,k})^*$.*

Proof. Both assertions are standard and follow simply from the fact that Hölder exponents multiply under composition and don't change under addition and multiplication. The last statement is obtained by taking duals. The pull-back for distributions means push-forward by Γ^{-1}. $\square$

Now we define distributional derivatives. Let us first consider partial derivatives along y-directions for the $\mathcal{Y}$ foliation. These are the derivatives we will use in the proof of the main theorem below. Fix a standard basis for $R_y^{n_2}$ and parallel translate it over $\mathbb{R}^{n_1+n_2}$. Then the $\frac{\partial}{\partial y_i}$ derivative of a distribution $D \in (C_{\mathcal{Y}}^{\alpha,k})^*$ is defined by evaluating on $h \in C_{\mathcal{F}}^{\alpha,k+1}$ via

$$\langle \frac{\partial}{\partial y_i}(D), h\rangle = -\langle D, \frac{\partial}{\partial y_i}(h)\rangle. \tag{16}$$

Note that $\frac{\partial}{\partial y_i}(D)$ is only defined on $C_{\mathcal{Y}}^{\alpha,k+1}$, and hence, $\frac{\partial}{\partial y_i}(D) \in (C_{\mathcal{Y}}^{\alpha,k+1})^*$.

Similarly, we define distributional derivatives along $\mathcal{F}$. Fix a standard basis for $R_y^{n_2}$, parallel translate it over $\mathbb{R}^{n_1+n_2}$ and consider the push-forward under Γ. This defines vector fields V_j tangent to $\mathcal{F}$ which are smooth along the leaves of $\mathcal{F}$ and whose derivatives along $\mathcal{F}$ of order up to k depend α_k-Hölder transversely for α_k as above. Assume in the following that $\alpha \leq \alpha_k$. Indeed the $V_i(h)$ involve the coefficients of Γ, and this assumption will insure that taking derivatives along the V_j does not affect Hölder exponents. More precisely we have $V_i(h) \in C_{\mathcal{F}}^{\alpha,k}$ for $h \in C_{\mathcal{F}}^{\alpha,k+1}$ as the V_i are α-Hölder by assumption on α. Hence we can define the derivative of a distribution $D \in (C_{\mathcal{F}}^{\alpha,k})^*$ by evaluating on $h \in C_{\mathcal{F}}^{\alpha,k+1}$ via

$$\langle V_i(D), h\rangle = -\langle D, V_i(h)\rangle. \tag{17}$$

Note that $V_i(D)$ is only defined on $C_{\mathcal{F}}^{\alpha,k+1}$, and hence, $V_i(D) \in (C_{\mathcal{F}}^{\alpha,k+1})^*$.

Note that pulling back derivatives $V_j(D)$ gives us $\frac{\partial}{\partial y_j}$ derivates of the pull-back of D on the appropriate function spaces.

Further define gD for $g \in C_{\mathcal{F}}^{\alpha,k}$ and $D \in (C_{\mathcal{F}}^{\alpha,k})^*$ by evaluating on a test function $\phi \in C_{\mathcal{F}}^{\alpha,k}$ by

$$(gD)(\phi) = \langle gD, \phi\rangle = \langle D, g\phi\rangle. \tag{18}$$

We conclude that $gD \in (C_{\mathcal{F}}^{\alpha,k})^*$. If D is given by integration against a compactly supported L^1- function u, then gD is given by integrating against gu.

Lemma 8.3. *Let $\alpha \leq \alpha_k$ and suppose that $g \in C_{\mathcal{F}}^{\alpha,k+1}$ and $D \in (C_{\mathcal{F}}^{\alpha,k})^*$. Then $V_i(g\,D) = V_i(g)\,D + gV_i(D)$ holds true in $(C_{\mathcal{F}}^{\alpha,k+1})^*$, i.e. as functionals on $C_{\mathcal{F}}^{\alpha,k+1}$.*

Proof. We check this by evaluating both sides on $\phi \in C_{\mathcal{F}}^{\alpha,k+1}$:

$$\langle V_i(g\,D), \phi\rangle = -\langle g\,D, V_i\phi\rangle = -\langle D, g\,(V_i\phi)\rangle = -\langle D, V_i(g\phi) - (V_i g)\phi\rangle =$$
$$\langle D, (V_i g)\phi\rangle - \langle D, V_i(g\phi)\rangle = \langle (V_i g)D, \phi\rangle + \langle V_i D, g\,\phi\rangle = \langle (V_i g)D, \phi\rangle + \langle g(V_i D), \phi\rangle.$$

$\square$

Note. The inner product $\langle D, (V_i g)\phi\rangle$ is not defined unless $g \in C_{\mathcal{F}}^{\alpha,k+1}$. Thus we need the higher regularity on g in the hypothesis of the previous lemma. This simple problem caused the introduction of the spaces of test functions $C_{\mathcal{F}}^{\alpha,k}$.

Let u be an L^1 function defined on a neighborhood of a point z_0. A vector ζ_0 is called *not singular* for u at z_0 if there exist an open set $\mathcal{U} \ni z_0$ and an open cone $\mathcal{Z} \subset \mathbb{R}^n \setminus \{0\}$ around ζ_0 such that for any positive integer N and any C^∞ function χ with support in $\mathcal{U}$ there exists a constant $C = C(N, \chi)$ so that

(19)
$$|\widehat{\chi u}\,(\zeta)| = \Big| \int u(z)\chi(z) \exp(-iz \cdot \zeta) dz \Big| \le C|\zeta|^{-N} \quad \text{for all } \zeta \in \mathcal{Z} \text{ with } |\zeta| > 1.$$

Otherwise, ζ_0 is called *singular* for u at z_0. The *wave front set* $WF(u)$ is defined as the set of all (z_0, ζ_0) such that ζ_0 is singular for u at z_0.

Theorem 8.4. *Suppose that $u(z)$ is an L^1 function. Let $\mathcal{F}$ be a Hölder foliation with smooth leaves which is also strongly absolutely continuous. Consider the distribution D defined by integration against $u(z)$. Assume that any derivative of D along $\mathcal{F}$ of any order belongs to $(C^\alpha_{\mathcal{F}})^*$ for all positive α. If $(z_0, \zeta_0) \in T^*(\mathbb{R}^n)\backslash 0$ is not conormal to $\mathcal{F}$, then*
$$(z_0, \zeta_0) \notin WF(u).$$

As an immediate corollary, we obtain the result needed in Section 5.

Corollary 8.5. *Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be Hölder foliations with smooth leaves on a manifold M which are also strongly absolutely continuous. Assume in addition that the tangent spaces to these foliations span the tangent spaces to M at all points.*

Now suppose that $u(z)$ is an L^1 function. Consider the distribution D defined by integration against $u(z)$. Assume that any derivative of D of any order along any $\mathcal{F}_i, i = 1, \dots, r$ belongs to $(C^\alpha_{\mathcal{F}_i})^*$ for all $1 \le i \le r$ and all positive α. Then u is C^∞.

Proof. Since the tangent spaces to the foliations span the tangent bundle everywhere, no vector $\zeta \neq 0$ can be conormal to all $\mathcal{F}_i$. Now it follows from Theorem 1.1 that $WF(u)$ is empty and hence u is smooth by e.g. [19, Section 8.1]. $\square$

The main idea in the proof of Theorem 8.4 is a simple generalization of an argument of Rauch and Taylor in [35]. However much more care has to be taken to make sure that various operations undertaken are well defined and allowed. In particular, we use integration by parts for derivatives along the foliation. This requires that the test functions in question are differentiable along $\mathcal{F}$ up to a suitable order. This led to the definition of the function spaces above.

Remark. The proof of Theorem 1.1 becomes easier if the foliation $\mathcal{F}$ has derivatives of all orders of a fixed Hölder class and the distribution in question together with its derivatives along $\mathcal{F}$ is dual to a fixed Hölder class.

Proof. We fix (z_0, ζ_0) which in not conormal to $\mathcal{F}$. By the definition of the wave front set it suffices to show that there exist an open set $\mathcal{U} \ni z_0$ and an open cone $\mathcal{Z} \subset \mathbb{R}^n \setminus \{0\}$ around ζ_0 such that for any $N > 0$ and any $\chi \in C_0^\infty(\mathcal{U})$ there exists a constant C so that

(20)
$$|\widehat{\chi u}\,(\zeta)| = \Big| \int u(z)\chi(z) \exp(-iz \cdot \zeta) dz \Big| \le C|\zeta|^{-N} \quad \text{for all } \zeta \in \mathcal{Z} \text{ with } |\zeta| > 1.$$

We define
$$\phi(x, y, \zeta) = -\Gamma(x, y) \cdot \zeta,$$

and note that, for a fixed ζ, the function ϕ is in $\mathcal{C}^{\alpha_k,k}$ for all k by the choice of α_k. Using a foliation chart and the strong absolute continuity of $\mathcal{F}$ we can write

$$\widehat{\chi u}\,(\zeta) = \int u(\Gamma(x,y))\,\chi(\Gamma(x,y))J(x,y)\exp(i\phi(x,y,\zeta))dxdy.$$

The hypothesis that (z_0,ζ_0) is not conormal to $\mathcal{F}$ implies that

$$d_y\phi(x,y,\zeta_0) \neq 0, \text{ where } \Gamma(x,y) = z.$$

Relabeling the y coordinates it follows that there exist a neighborhood $\mathcal{U}$ of z_0, an open cone $\mathcal{Z} \subset \mathbb{R}^n \setminus \{0\}$ around ζ_0, and $\delta > 0$ so that

$$\left|\frac{\partial\phi(x,y,\zeta)}{\partial y_1}\right| > \delta|\zeta|, \text{ when } (\Gamma(x,y),\zeta) \in \mathcal{U}\times\mathcal{Z}. \tag{21}$$

To obtain the desired decay in ζ we use the identity

$$\left(\frac{1}{i\partial\phi(x,y,\zeta)/\partial y_1}\frac{\partial}{\partial y_1}\right)\exp(i\phi(x,y,\zeta)) = \exp(i\phi(x,y,\zeta))$$

to deduce that

(22)

$$\widehat{\chi u}(\zeta) = \int u(\Gamma(x,y))\chi(\Gamma(x,y))J(x,y)\left(\frac{1}{i\partial\phi(x,y,\zeta)/\partial y_1}\frac{\partial}{\partial y_1}\right)^N\exp(i\phi(x,y,\zeta))dxdy.$$

We can expand

$$\left(\frac{1}{i\partial\phi(x,y,\zeta)/\partial y_1}\frac{\partial}{\partial y_1}\right)^N = \sum_{m=1}^N \psi_{m,N}(x,y,\zeta)\left(\frac{\partial}{\partial y_1}\right)^m. \tag{23}$$

To describe functions $\psi_{m,N}(x,y,\zeta)$ we note that $(g\frac{\partial}{\partial y_1})^N$ is a sum of terms of the form $P_m(\frac{\partial}{\partial y_1})^m$, where P_m is a polynomial in g and its first $N-m$ derivatives. Applying this to $g = \frac{1}{i\partial\phi(x,y,\zeta)/\partial y_1}$, we see that each function $\psi_{m,N}(x,y,\zeta)$ is a quotient of a polynomial in $\Gamma(x,y)\cdot\zeta$ and its first $N-m+1$ derivatives divided by a power of $i\partial\phi(x,y,\zeta)/\partial y_1$. Taking k derivatives of $\psi_{m,N}$ yields, by the product and quotient rules, a similar expression which involves derivatives of $\Gamma(x,y)$ of order $N-m+1+k$ and hence is Hölder with exponent $\alpha_{(k+N-m+1)}$. It follows that, for a fixed ζ and any $m = 1,\ldots,N$, the function $\psi_{m,N}(x,y,\zeta)$ is in $C_{\mathcal{Y}}^{\alpha_{(N+1)},m}$. Moreover, there exists a constant C such that

$$\|\psi_{m,N}\|_{\alpha_{(N+1)},m} \leq C\,|\zeta|^{-N} \quad \text{for all } \zeta \in \mathcal{Z} \text{ with } |\zeta| > 1. \tag{24}$$

Indeed, since $\phi(x,y,\zeta)$ is linear in ζ, both sides of (23) are homogeneous of degree $-N$ in ζ, and hence so are the functions $\psi_{m,N}$ and their derivatives. We conclude that the functions in (24) are rational functions in ζ of homogeneous degree $-N$ whose coefficients, as functions of (x,y), are Hölder on $\Gamma^{-1}(\mathcal{U})$. The Hölder norms of these coefficients are continuous in ζ and hence are uniformly bounded on $\mathcal{Z}\cap\{|\zeta| = 1\}$. Finally, using equation (21) we can bound the denominators away from zero and obtain (24).

Using (22) and (23) we can write $\widehat{\chi u}$ as a finite sum

$$\widehat{\chi u}(\zeta) = \sum_{m=1}^N \int u(\Gamma(x,y))\chi(\Gamma(x,y))J(x,y)\psi_{m,N}(x,y,\zeta)\left(\frac{\partial}{\partial y_1}\right)^m\exp(i\phi(x,y,\zeta))dxdy.$$

In the remainder of the proof we estimate each term of this sum. For this we denote

$$A = u(\Gamma(x,y))\,\chi(\Gamma(x,y)) \quad \text{and} \quad A^{\zeta}_{m,N} = u(\Gamma(x,y))\,\chi(\Gamma(x,y))\,J(x,y)\,\psi_{m,N}(x,y,\zeta)$$

and view A and $A^{\zeta}_{m,N}$ as the distributions given by integration, with a fixed ζ, against the corresponding functions. Since the functions $u\circ\Gamma, \chi\circ\Gamma, J$ and $\psi_{M,n}$ are in L^1, A and $A^{\zeta}_{m,N}$ lie in $(\mathcal{C}^{\alpha}_{\mathcal{Y}})^* = (\mathcal{C}^{\alpha,0}_{\mathcal{Y}})^*$ for all positive α, and $A^{\zeta}_{m,N} = J\psi_{m,N}A$ as elements of $(\mathcal{C}^{\alpha}_{\mathcal{Y}})^*$ with multiplication of distributions defined as in (18). Recall that $\phi(x,y,\zeta)$ is in $(C^{\alpha_m,m}_{\mathcal{Y}})$, so by the definition of derivatives of distributions for each term in $\widehat{\chi u}(\zeta)$ we obtain

$$\begin{aligned}
&\int u(\Gamma(x,y))\,\chi(\Gamma(x,y))\,J(x,y)\,\psi_{m,N}(x,y,\zeta)\left(\frac{\partial}{\partial y_1}\right)^m \exp(i\phi(x,y,\zeta))\,dxdy \\
&\quad = \langle A^{\zeta}_{m,N}, \Big(\left(\frac{\partial}{\partial y_1}\right)^m \exp(i\phi(x,y,\zeta)\Big)\rangle \\
&\quad = (-1)^m\langle\Big(\left(\frac{\partial}{\partial y_1}\right)^m (A^{\zeta}_{m,N})\Big), \exp(i\phi(x,y,\zeta))\rangle \\
&\quad = (-1)^m\langle\Big(\left(\frac{\partial}{\partial y_1}\right)^m (J\psi_{m,N}A)\Big), \exp(i\phi(x,y,\zeta))\rangle\,,
\end{aligned}$$

where the pairing is in the sense of $(C^{\alpha_m,m}_{\mathcal{Y}})^*$ for $1 \le m \le N$. Now we apply the Leibniz rule, Lemma 8.3, m times to write $\left(\frac{\partial}{\partial y_1}\right)^m (J\psi_{m,N}A)$ as

$$\begin{aligned}
&\left(\frac{\partial}{\partial y_1}\right)^m (J\psi_{m,N}A) \\
&\qquad = \sum_{a+b+c=m} K_{a,b,c}\Big[\left(\frac{\partial}{\partial y_1}\right)^a J(x,y)\Big]\Big[\left(\frac{\partial}{\partial y_1}\right)^b (\psi_{m,N})\Big]\Big[\left(\frac{\partial}{\partial y_1}\right)^c A\Big].
\end{aligned}$$

The equation holds in $(C^{\alpha_{N+1},m}_{\mathcal{Y}})^*$ since A is in $(\mathcal{C}^{\alpha_{N+1}}_{\mathcal{Y}})^*$ and $\psi_{m,N}$ as well as J are in $C^{\alpha_{N+1},m}_{\mathcal{Y}}$. Finally, we can rewrite the pairing in $(C^{\alpha_{N+1},m}_{\mathcal{Y}})^*$ of each term in this sum with $\exp(i\phi(x,y,\zeta))$ as

$$\langle\Big(\Big[\left(\frac{\partial}{\partial y_1}\right)^a J(x,y)\Big]\cdot\Big[\left(\frac{\partial}{\partial y_1}\right)^b (\psi_{m,N})\Big]\cdot\Big[\left(\frac{\partial}{\partial y_1}\right)^c A\Big]\Big), \exp(i\phi(x,y,\zeta))\rangle$$

$$= \langle\left(\frac{\partial}{\partial y_1}\right)^c A\,, \Big(\Big[\left(\frac{\partial}{\partial y_1}\right)^a J(x,y)\Big]\cdot\Big[\left(\frac{\partial}{\partial y_1}\right)^b (\psi_{m,N})\Big]\cdot\exp(i\phi(x,y,\zeta))\rangle. \tag{25}$$

Now we use the assumption that derivatives of u and hence of the localization $u\chi$ along $\mathcal{F}$ exist as elements in $(\mathcal{C}^{\alpha}_{\mathcal{F}})^*$ for all positive α. Therefore, by Lemma 8.2, y-derivatives of the pull-back $A = (u\chi)\circ\Gamma$ also exist as elements in $(\mathcal{C}^{\alpha}_{\mathcal{Y}})^*$ for all positive α. Hence the pairing in (25) can be estimated by the (α_{N+1})-Hölder norm of the product $\big[\big(\frac{\partial}{\partial y_1}\big)^a J(x,y)\big]\big[\big(\frac{\partial}{\partial y_1}\big)^b (\psi_{m,N})\big]\exp(i\phi(x,y,\zeta))$. As $b, c \le m$, all three functions are (α_{N+1})-Hölder. Moreover, for all $\zeta \in \mathcal{Z}$ with $|\zeta| > 1$, $\|\big(\frac{\partial}{\partial y_1}\big)^a J(x,y)\|_{\alpha_{N+1}}$ is bounded by a fixed constant, $\|\big(\frac{\partial}{\partial y_1}\big)^b (\psi_{m,N})\|_{\alpha_{N+1}} \le C\,|\zeta|^{-N}$ by (24), and the norm $\|\exp(i\phi(x,y,\zeta))\|_{\alpha_{N+1}}$ can be estimated by $C'\,|\zeta|$. We conclude that each pairing in (25) can be estimated by $C''\,|\zeta|^{-N+1}$, and hence the same estimate holds for $|\widehat{\chi u}(\zeta)|$. Since N is arbitrary, the desired estimate (20) now follows and shows that any (z_0, ζ_0) which is not conormal to $\mathcal{F}$ is not the wave front set of u. $\square$

Appendix. A finite cover of an exotic nilmanifold is standard

A *nilmanifold* is the quotient G/L of a simply connected nilpotent Lie group G by a discrete cocompact subgroup L. Two homeomorphisms $f, g : X \to Y$ are *isotopic* if they are homotopic through homeomorphisms.

Theorem A.1. *Let $h : M \to G/L$ be a homeomorphism from a smooth manifold to a nilmanifold of dimension greater than four. Then there is a finite cover $\widehat{G/L} \to G/L$ so that the induced pull-back homeomorphism $\widehat{M} \to \widehat{G/L}$ is isotopic to a diffeomorphism.*

Theorem A.1 is a consequence of Lemma A.3 and Lemma A.4 stated below.

Definition A.2. A space N satisfies condition (*) if for any $i > 0$, for any finite abelian group T, for any finite cover $\hat{p} : \hat{N} \to N$, and for any $x \in H^i(\hat{N}; T)$, then there exists a finite cover $\tilde{p} : \tilde{N} \to \hat{N}$ so that $\tilde{p}^* x = 0$.

Lemma A.3. *Let $h : M \to N$ be a homeomorphism of smooth manifolds of dimension greater than four. Suppose N satisfies (*). Then there is a finite cover $\widehat{N} \to N$ so that the induced pull-back homeomorphism $\widehat{M} \to \widehat{N}$ is isotopic to a diffeomorphism.*

In particular any two smooth structures on N become diffeomorphic after passing to a finite cover. An existence result can be proved using similar techniques: any topological manifold of dimension greater than four which satisfies (*) has a finite cover which admits a smooth structure.

Lemma A.4. *Any nilmanifold satisfies condition (*).*

Proof. Since a finite cover of a nilmanifold is a nilmanifold, it will be notationally simpler to show that any nilmanifold satisfies condition (**) defined below.

A space N satisfies condition (**) if for any $i > 0$, for any finite abelian group T, and for any $x \in H^i(N; T)$, then there exists a finite cover $\widehat{p} : \widehat{N} \to N$ so that $\widehat{p}^* x = 0$.

We first verify condition (**) when $i = 1$. Indeed, the Universal Coefficient Theorem gives an isomorphism $H^1(N; T) \to \mathrm{Hom}(H_1(N); T)$ for all spaces N, and the Hurewicz Theorem gives an isomorphism $\pi_1(N, n_0)^{\mathrm{ab}} \to H_1(N)$ for a path-connected space N. Thus there is a natural isomorphism of contravariant functors from path-connected based spaces to abelian groups

$$\Phi(N, n_0) : H^1(N; T) \xrightarrow{\cong} \mathrm{Hom}(\pi_1(N, n_0), T).$$

Given $x \in H^1(N; T)$, there is a connected cover $\widehat{p} : \widehat{N} \to N$ and a base point $\widehat{n}_0 \in \widehat{N}$ so that

$$\widehat{p}_*(\pi_1(\widehat{N}, \widehat{n}_0)) = \ker(\Phi(N, n_0)(x) : \pi_1(N, n_0) \to T).$$

Since T is a finite group, $\widehat{p}$ is a finite cover. The commutative square

$$\begin{tikzcd} H^1(\widehat{N};T) \arrow[r, "\cong"] & \mathrm{Hom}(\pi_1(\widehat{N},\widehat{n}_0),T) \\ H^1(N;T) \arrow[r, "\cong"] \arrow[u, "\widehat{p}^*"] & \mathrm{Hom}(\pi_1(N,n_0),T) \arrow[u, "-\,\circ\,\widehat{p}_*"'] \end{tikzcd}$$

shows that $\widehat{p}^* x = 0$.

We now turn to the proof that any nilmanifold satisfies condition (**) when $i > 1$. The proof will be by induction on the dimension of the nilmanifold $N = G/L$, using the Gysin sequence of a principal S^1-bundle

$$S^1 \to N \xrightarrow{\pi} N/S^1,$$

where N/S^1 is a nilmanifold. To obtain this principal bundle note that the center $Z(G)$ is nontrivial since G is nilpotent. Furthermore, it can by shown that $Z(L) = L \cap Z(G)$ is a discrete cocompact subgroup of the real vector space $Z(G)$ (see [33, Proposition 2.17]). Choose a primitive element $l \in L \cap Z(G)$. Then $S^1 = \mathbb{R}\cdot l/\mathbb{Z}\cdot l$ acts freely on N and the quotient N/S^1 is the nilmanifold $(G/\mathbb{R}\cdot l)/(L/\mathbb{Z}\cdot l)$.

Let N be a nilmanifold. Assume by induction that condition (**) holds for all nilmanifolds of strictly smaller dimension. The Gysin sequence (see [5])

$$\cdots \to H^{i-2}(N/S^1;T) \xrightarrow{\cup e} H^i(N/S^1;T) \xrightarrow{\pi^*} H^i(N;T) \xrightarrow{\pi_!} H^{i-1}(N/S^1;T) \to \cdots$$

is an exact sequence associated to a principal S^1-fibration. By the inductive hypothesis, there exists a finite cover $\widetilde{p/S^1} : \widetilde{N/S^1} \to N/S^1$ so that $\widetilde{p/S^1}^*(\pi_! x) = 0$. (Note, here is where we use that $i > 1$.) Define $\widetilde{N}$ as the pull-back

$$\begin{tikzcd} \widetilde{N} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{p}"] & \widetilde{N/S^1} \arrow[d, "\widetilde{p/S^1}"] \\ N \arrow[r, "\pi"] & N/S^1. \end{tikzcd}$$

We have a map of principal S^1 bundles, hence a map of Gysin sequences (see the bottom two rows of the diagram below). By commutativity of the lower right square below and the exactness of the middle row, there is an $x' \in H^i(\widetilde{N/S^1};T)$ so that $\widetilde{\pi}^* x' = \widetilde{p}^* x$. By the inductive hypothesis again, there is a finite cover $\widehat{p/S^1} : \widehat{N/S^1} \to \widetilde{N/S^1}$ so that $\widehat{p/S^1}^*(x') = 0$. Defining $\widehat{N}$ as a pull-back, we have the diagram below:

$$\begin{tikzcd} H^i(\widehat{N/S^1};T) \arrow[r, "\widehat{\pi}^*"] & H^i(\widehat{N};T) \arrow[r, "\widehat{\pi}_!"] & H^{i-1}(\widehat{N/S^1};T) \\ H^i(\widetilde{N/S^1};T) \arrow[r, "\widetilde{\pi}^*"] \arrow[u, "\widehat{p/S^1}^*"] & H^i(\widetilde{N};T) \arrow[r, "\widetilde{\pi}_!"] \arrow[u, "\widehat{p}^*"] & H^{i-1}(\widetilde{N/S^1};T) \arrow[u, "\widehat{p/S^1}^*"] \\ H^i(N/S^1;T) \arrow[r, "\pi^*"] \arrow[u, "\widetilde{p/S^1}^*"] & H^i(N;T) \arrow[r, "\pi_!"] \arrow[u, "\widetilde{p}^*"] & H^{i-1}(N/S^1;T). \arrow[u, "\widetilde{p/S^1}^*"] \end{tikzcd}$$

Hence our desired finite cover is $\widetilde{p} \circ \widehat{p} : \widehat{N} \to N$. This completes the proof of the lemma. $\square$

In preparation for the proof of Lemma A.3 we review a bit of smoothing theory. The two definitive treatments are the books [29] and [17]; see also the recent survey [6]. A *smooth structure on a topological manifold* Σ is a pair (M, h), where M is a smooth manifold and $h : M \to \Sigma$ is a homeomorphism. Two smooth structures (M_1, h_1) and (M_2, h_2) are *isotopic* if there is a diffeomorphism $f : M_1 \to M_2$ so that h_1 is isotopic to $h_2 \circ f$. Let $\mathcal{T}_O(\Sigma)$ be the set of isotopy classes of smooth structures on Σ.

The fundamental theorem of smoothing theory says that a topological manifold of dimension greater than four admits a smooth structure if and only if its topological tangent bundle admits the structure of a vector bundle. Furthermore, isotopy classes of smooth structures are in bijective correspondence with bundle reductions. It will be easier (and slicker) to express this in terms of maps to classifying spaces, as in Part 2 of [17].

Let $Top(n)$ be the group of homeomorphisms of $\mathbb{R}^n$ fixing the origin. Give $Top(n)$ the compact open topology. Let $O(n)$ be the orthogonal group. Let $Top = \operatorname{colim} Top(n)$ and $O = \operatorname{colim} O(n)$. The quotient space Top/O admits the structure of an abelian H-space satisfying the following property: if Σ is a topological manifold of dimension greater than four, then the abelian group of homotopy classes $[\Sigma, Top/O]$ acts freely and transitively on the set of isotopy classes of smooth structures $\mathcal{T}_O(\Sigma)$. For smooth structures (M_1, h_1) and (M_2, h_2), let $d(h_1, h_2)$ be the unique element of $[\Sigma, Top]$ so that $d(h_1, h_2)[M_1, h_1] = [M_2, h_2] \in \mathcal{T}_O(\Sigma)$. Thus $d(h_1, h_2) = 0$ if and only if the homeomorphism $h_2^{-1} \circ h_1 : M_1 \to M_2$ is isotopic to a diffeomorphism.

The homotopy groups of Top/O are reasonably well-understood. Indeed,

$$\pi_i(Top/O) = 0, 0, 0, \mathbb{Z}/2, 0, 0, 0, \mathbb{Z}/28 \quad \text{for } i = 0, 1, 2, 3, 4, 5, 6, 7$$

and for $i \geq 5$, $\pi_i(Top/O) \cong \Theta_i$, the group of exotic smooth structures on the i-sphere. In particular, Top/O is simply connected and the homotopy groups $\pi_i(Top/O)$ are all finite.

Proof of Lemma A.3. Let Σ be a topological manifold of dimension greater than 4 which admits a smooth structure. Here are three observations. First, if $\widehat{p} : \widehat{\Sigma} \to \Sigma$ is a covering map, then the map $\widehat{p}^* : \mathcal{T}_O(\Sigma) \to \mathcal{T}_O(\widehat{\Sigma})$ is equivariant with respect to the group homomorphism $\widehat{p}^* : [\Sigma, Top/O] \to [\widehat{\Sigma}, Top/O]$. In other words, $\widehat{p}^*([\alpha] \cdot [M, h]) = \widehat{p}^*[\alpha] \cdot \widehat{p}^*[M, h]$. The geometric fact underlying this is that the pull-back of the tangent bundle of the base space under a covering map is the tangent bundle of the total space. Second, note that Σ admits the structure of a CW-complex, for example, by triangulating the smooth structure. Finally, note that if $f, g : X \to Y$ are maps from a CW-complex to a simply connected space, and $H(i-1) : X^{i-1} \times I \to Y$ is a homotopy from $f|_{X^{i-1}}$ to $g|_{X^{i-1}}$, there is a well-defined obstruction class $\mathcal{O} = \mathcal{O}^i(f, g, H(i-1)) \in H^i(X; \pi_i Y)$ (see [5, Theorem 7.12]. This class vanishes if and only if there is a homotopy $H(i) : X^i \times I \to Y$ from $f|_{X^i}$ to $g|_{X^i}$ which restricts to $H(i-1)|_{X^{i-2} \times I}$.

Let (M_1, h_1) and (M_2, h_2) be two smooth structures on a topological manifold Σ which satisfies condition (*). Assume $n = \dim \Sigma \geq 5$. Give Σ the structure of an n-dimensional CW-complex. Assume, by induction, that there exists a finite cover $\widehat{p}_{i-1} : \widehat{\Sigma}_{i-1} \to \Sigma$ so that $d(\widehat{p}_{i-1}h_1, \widehat{p}_{i-1}^* h_2)$ is represented by a map $\widehat{\Sigma}_{i-1} \to Top/O$ which is null-homotopic restricted to the $(i-1)$-skeleton. Let $\mathcal{O} \in H^i(\widehat{\Sigma}_{i-1}; \pi_i(Top/O))$ be the obstruction to extending to null-homotopy. By condition (*), there is a finite cover $\widehat{p}(i) : \widehat{\Sigma}_i \to \widehat{\Sigma}_{i-1}$ so that $\widehat{p}(i)^*\mathcal{O} = 0$. Then the finite cover $\widehat{p}_i := \widehat{p}_{i-1} \circ \widehat{p}(i) : \widehat{\Sigma}_i \to \Sigma$ satisfies the inductive hypothesis. Thus $\widehat{p}_n$ is a finite cover so that the smooth structures $\widehat{p}_n^* h_1$ and $\widehat{p}_n^* h_2$ are isotopic. $\square$

Remark A.5. Suppose Σ is a manifold of dimension 3 or less. Using the work of many mathematicians, most notably Rado and Moise, one can show (see [6]) that Σ admits a smooth structure and that any two smooth structures are isotopic.

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