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LOOP GROUPS AND TWISTED K-THEORY II

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Elliptic operators appear in different guises in the representation theory of compact Lie groups. The Borel-Weil construction [BW], phrased in terms of holomorphic functions, has at its heart the $\bar{\partial}$ operator on Kähler manifolds. The $\bar{\partial}$ operator and differential geometric vanishing theorems figure prominently in the subsequent generalization by Bott [B]. An alternative approach using an algebraic Laplace operator was given by Kostant [K3]. The Atiyah-Bott proof [AB] of Weyl's character formula uses a fixed point theorem for the $\bar{\partial}$ complex. On a spin Kähler manifold $\bar{\partial}$ can be expressed in terms of the Dirac operator. This involves a shift, in this context the ρ-shift[1] whose analog for loop groups appears in our main theorem. Dirac operators may be used instead of $\bar{\partial}$ in these applications to representation theory, and indeed they often appear explicitly.

In this paper we introduce a new construction: the *Dirac family* attached to a representation of a Lie group $\mathcal{G}$ which is either compact or the loop group of a compact Lie group; in the latter case the representation is restricted to having positive energy. The Dirac family is a collection of Fredholm operators parametrized by an affine space, equivariant for an affine action of a central extension of $\mathcal{G}$ by the circle group $\mathbb{T}$. For an irreducible representation the support of the family is the coadjoint orbit given by the Kirillov correspondence, and the entire construction is reminiscent of the Fourier transform of the character [K2, Rule 6]. The Dirac family represents a class in twisted equivariant K-theory, so we obtain a map from representations to K-theory. For compact Lie groups it is a nonstandard realization of the twisted equivariant Thom homomorphism, which was proved long ago to be an isomorphism.[2] Our main result, Theorem 3.44, is that this map from representations to K-theory is an isomorphism when $\mathcal{G}$ is a loop group. The existence of a construction along these lines was first suggested by Graeme Segal; cf. [AS, §5].

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[1] 2ρ is the sum of the positive roots of a compact Lie group G; it may be identified with the first Chern class of the flag manifold associated to G.

[2] It is interesting to note that the proof of this purely topological result uses Dirac operators and index theory.

The basic Dirac operator (1.10) which appears in the Dirac family has been termed the *cubic Dirac operator*. Any Lie group G has a distinguished line segment in its affine space of linear connections: the endpoints are the flat connections ∇_L, ∇_R that are the infinitesimal versions of the global parallelisms via left and right translation. If G is compact, then the Levi-Civita connection of a bi-invariant metric is $\frac{1}{2}\nabla_L + \frac{1}{2}\nabla_R$. The cubic[3] Dirac operator is associated to the connection $\frac{2}{3}\nabla_L + \frac{1}{3}\nabla_R$. This particular Dirac operator was introduced by Slebarski [S2] and used by Kostant in [K4]. It also enjoys nice analytic properties, as in [G]. Alekseev and Meinrenken [AM] interpret this Dirac operator as a differential on a quantized version of the Weil algebra. It also appears in the generalized geometry of Hitchin [H1, §10]. The infinite dimensional version appears in Taubes' work [T] on loop spaces. It was subsequently used by Landweber [L] in a generalization of Kostant's paper to loop groups. We later learned that many relevant formulas were independently known in the physics literature, and indeed our Fredholm family, which is a gauge-coupled Dirac-Ramond operator, was first flagged in relation to twisted (nonequivariant) K-theory by Mickelsson [M].

The finite dimensional case is developed in detail in §1. Let G be any compact Lie group, V a finite dimensional unitary representation, and S the spin space of the adjoint representation. The Dirac family consists of endomorphisms of $V \otimes S$ parametrized by the dual $\mathfrak{g}^*$ of the Lie algebra of G. It is equivariant for the coadjoint action of G, possibly centrally extended to act on the spin space, and the endomorphisms are invertible outside a compact set. In fact, if V is irreducible we show in Proposition 1.19 that the the endomorphisms fail to be invertible only on a single coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. The interpretation in terms of the equivariant Thom isomorphism is Theorem 1.28.

Our main application is to positive energy representations of loop groups. In §2 we review and develop the concept of energy in the theory of loop groups. From the beginning we work with twisted loop groups, that is, the group of gauge transformations L_PG of a principal G-bundle $P \to S^1$. We introduce the slightly larger group of gauge transformations which cover a rigid rotation of S^1. Then a central extension $(L_PG)^\tau$ of L_PG by the circle group $\mathbb{T}$ is said to be *admissible* if it extends over this larger group—so is rotation-invariant—and if there is an invariant bilinear form which pairs the Lie algebras of the center $\mathbb{T}$ and the rotations $\mathbb{T}_{\text{rot}}$; see Definition 2.10 for the precise conditions.[4] This bilinear form, central in Kac's algebraic theory [K1, §6], plays a crucial role here as it sets up a correspondence between connections on $P \to S^1$ and linear splittings of the central extension of loop algebras (Lemma 2.18). The space of connections enters also in Definition 2.8, which associates to the central extension $(L_PG)^\tau$ a *twisting* of the equivariant K-theory of G acting on itself by conjugation. One novelty here is the inclusion of a *grading*, a homomorphism $L_PG \to \mathbb{Z}/2\mathbb{Z}$, in τ; it affects a component of the associated twisting. Infinitesimal rotations measure energy, but the precise definition depends

[3] "Cubic" refers to the tangent vector to this line of connections, which is the invariant 3-form on G. The Dirac operator attached to *any* connection on this line has a cubic term in its local formula. The particular connection and Dirac operator of interest is distinguished by the coefficient in front of that term. The apparent asymmetry between left and right is explained by our use of *left* translation to trivialize the spin bundle on G.

[4] We prove in an appendix that if the Lie algebra of G is semisimple, then any central extension is admissible. On the other hand, if G is a torus of dimension at least two, then there exist non-admissible central extensions.

on a choice of connection. Thus we obtain a family of energies parametrized by the space $\mathcal{A}_P$ of connections. Following [PS] in §2.5 we introduce positive energy representations, extending the standard definition to allow for gradings. For a fixed admissible graded central extension $(L_PG)^\tau$ there is a finite set of isomorphism classes of irreducibles. They generate a free abelian group we denote $R^\tau(L_PG)$.

The Dirac family construction is taken up again in §3, now in the infinite dimensional setting of loop groups. The adjoint spin representation S of the loop group determines a distinguished central extension $(L_PG)^\sigma$. We form a family of Dirac operators by tensoring a positive energy representation of $(L_PG)^{\tau-\sigma}$ with spinors. This gives a family of Fredholm operators parametrized by $\mathcal{A}_P$, equivariant for the central extension $(L_PG)^\tau$. Here we encounter the *adjoint shift*[5] by σ. This Fredholm family represents an element of twisted K-theory in the model developed in Part I, and so the Dirac construction induces a homomorphism

$$\Phi : R^{\tau-\sigma}(L_PG) \longrightarrow K_G^{\tau+\dim G}(G[P]). \tag{0.1}$$

Here $G[P]$ is the union of components of G consisting of all holonomies of connections on $P \to S^1$, and G acts on it by conjugation. Our main result, Theorem 3.44, asserts that Φ is an isomorphism.

The proof, presented in §4 for the case when G is connected with a torsion-free fundamental group, is computational: we compute both sides of (0.1) and check that Φ induces an isomorphism. We deduce the result for this class of groups from the special cases of tori and simply connected groups. The positive energy representations of the loop groups in these cases—the left-hand side of (0.1)—are enumerated in [PS]. The twisted equivariant K-theory—the right-hand side of (0.1)—is computed in [FHT1, §4]. The primary work here is the analysis of the kernel of the Dirac family. This is parallel to the finite dimensional case in §1, and in fact reduces to it. One of the main points is a Weitzenböck-type formula (3.36) which relates the square of Dirac to energy. Specific examples are written out in §4.4.

In Part III [FHT3] we complete the proof of Theorem 3.44 for any compact Lie group. We also generalize to a wider class of central extensions of loop groups which do not rely on energy. The positive energy condition on representations is replaced by an integrability condition at the Lie algebra level [K1], and the whole treatment there relies much more on Lie algebraic methods. We also give a variation which incorporates energy more directly into (0.1), and several other complements. In many cases the twisted equivariant K-theory $K_G^{\tau+\dim G}(G)$ is a ring, in fact, a Frobenius ring. In those cases the twisting τ is derived from a more primitive datum—a "consistent orientation"—as we explain in [FHT4].

The twistings of equivariant K-theory we encounter in this paper have a special form relative to the general theory of Part I. Suppose a compact Lie group G acts on a space X. Let P be a space with the action of a topological group $\mathcal{G}$ that the normal subgroup $\mathcal{N} \subset \mathcal{G}$ acts freely, and there are isomorphisms $\mathcal{P}/\mathcal{N} \cong X$ and $\mathcal{G}/\mathcal{N} \cong G$ compatible with the group actions. In the language of Part I, the quotient maps define a *local equivalence of groupoids* $\mathcal{P}//\mathcal{G} \to X//G$. In this paper

[5] For $G = \mathrm{SU}(2)$ it is the famous $k \to k+2$ shift which occurs in low dimensional physics: conformal field theory, Chern-Simons theory, etc. It is a loop group analog of the ρ-shift for compact Lie groups. For an interpretation in terms of the intrinsic geometry of the loop group see [F].

we use twistings of $K_G(X)$ of the form

$$\tau = (\mathcal{P} \xrightarrow{\mathcal{N}} X, \mathcal{G}^\tau, \epsilon^\tau). \tag{0.2}$$

These consist of $\mathcal{P} \to X$ as above, a central extension $\mathcal{G}^\tau$ of $\mathcal{G}$ by the circle group $\mathbb{T}$, and a homomorphism $\epsilon^\tau : \mathcal{G} \to \mathbb{Z}/2\mathbb{Z}$, termed a *grading*. Together they define a graded central extension of $\mathcal{P}//\mathcal{G}$; compare [FHT1, §2.2].

§1. The finite dimensional case

§1.1. The spin representation (finite dimensional case). The basic reference is [ABS]. Let H be a finite dimensional real vector space with an inner product $\langle\cdot,\cdot\rangle$. Then there are central extensions $\mathrm{Pin}^\pm(H)$ and $\mathrm{Pin}^c(H)$ of the orthogonal group $\mathrm{O}(H)$ which fit into the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathrm{Pin}^\pm(H) & \longrightarrow & \mathrm{O}(H) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & \mathrm{Pin}^c(H) & \longrightarrow & \mathrm{O}(H) & \longrightarrow & 1 \end{array} \tag{1.1}$$

There is a split extension of Lie algebras

$$0 \longrightarrow i\mathbb{R} \longrightarrow \mathfrak{pin}^c(H) \leftrightarrows \mathfrak{o}(H) \longrightarrow 0, \tag{1.2}$$

the splitting induced from the first line of (1.1). (Throughout, $i = \sqrt{-1}$.) The Clifford algebra $\mathrm{Cliff}^\pm(H^*)$ of H^* is the universal unital associative algebra with a linear map $\gamma : H^* \to \mathrm{Cliff}^\pm(H^*)$ such that

$$\gamma(\mu)\gamma(\mu') + \gamma(\mu')\gamma(\mu) = \pm 2\langle\mu,\mu'\rangle, \qquad \mu,\mu' \in H^*. \tag{1.3}$$

Then $\mathrm{Cliff}^\pm(H^*)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, and the left-hand side of (1.3) is the graded commutator $\big[\gamma(\mu),\gamma(\mu')\big]$. Also, $\mathrm{Pin}^\pm(H) \subset \mathrm{Cliff}^\pm(H^*)$ and $\mathrm{Pin}^c(H) \subset \mathrm{Cliff}^\pm(H^*)\otimes\mathbb{C}$, and these groups inherit a grading ϵ from that of the Clifford algebra. (A *grading* of a Lie group G is a continuous homomorphism $\epsilon : G \to \mathbb{Z}/2\mathbb{Z}$.) Let $e^1,\dots,e^n$ be an orthonormal basis of H^*. Then $\gamma(e^1)\cdots\gamma(e^n) \in \mathrm{Cliff}(H^*)$ is a *volume form*, and is determined up to sign. If H is oriented we use oriented orthonormal bases and so fix the sign. In this paper we consider *complex* modules for $\mathrm{Cliff}^\pm(H^*)$, and we denote $\mathrm{Cliff}^-(H^*)\otimes\mathbb{C} \cong \mathrm{Cliff}^+(H^*)\otimes\mathbb{C}$ by $\mathrm{Cliff}^c(H^*)$. Also, if $H = \mathbb{R}^n$ with the standard metric, then we use the notation $C_n^\pm$ for the real Clifford algebras, C_n^c for the complex Clifford algebra, and $\mathrm{Pin}_n^\pm$ for the pin groups.

The isomorphism classes of $\mathbb{Z}/2\mathbb{Z}$-graded complex $\mathrm{Cliff}^c(H^*)$-modules form a semigroup whose group completion is isomorphic to $\mathbb{Z}$ if $\dim H$ is odd and $\mathbb{Z}\oplus\mathbb{Z}$ if $\dim H$ is even. If $\dim H$ is even, then the two positive generators are distinguished by the action of a volume form on the even component of an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded module. If $\dim H$ is odd, then $\mathrm{Cliff}^c(H^*)$ has a nontrivial center (as an ungraded algebra) which is isomorphic to C_1^c; an element in the center has the form $a + b\omega$ where $a,b \in \mathbb{C}$ and ω is a volume form. In the odd case we always consider $\mathrm{Cliff}^c(H^*)$-modules with a commuting C_1^c-action which is not necessarily that of the center, even though we do not explicitly mention that action. The group completion of isomorphism classes is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$. The positive generators are distinguished by the action of the product of a volume form in $\mathrm{Cliff}^c(H^*)$ and a volume form in C_1^c on the even component. For any H the equivalence relation of K-theory (K^0 if $\dim H$ is even, K^1 if $\dim H$ is odd) identifies the positive generator of one summand $\mathbb{Z}$ with the negative generator of the other.

Let S be an irreducible complex $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}^c(H^*)$-module. Then S restricts to a graded representation of $\mathrm{Pin}^c(H)$, so to a projective representation of $\mathrm{O}(H)$. At the Lie algebra level we use the splitting (1.2) to obtain a representation of $\mathfrak{o}(H)$. There exist *compatible* metrics on S—metrics such that $\mathrm{Pin}^c(H)$ acts by unitary transformations and such that $\gamma(\mu)$ is a skew-Hermitian transformation for each $\mu \in H^*$.

§1.2. The cubic Dirac operator. Let G be a compact Lie group and $\langle\cdot,\cdot\rangle$ a G-invariant inner product on the Lie algebra $\mathfrak{g}$. We apply §1.1 to $V = \mathfrak{g}$ using the adjoint representation $G \to \mathrm{O}(\mathfrak{g})$. Thus by pullback from (1.1) with $H = \mathfrak{g}$ we obtain a graded central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow G^\sigma \longrightarrow G \longrightarrow 1 \tag{1.4}$$

and a canonically split extension of Lie algebras

$$0 \longrightarrow i\mathbb{R} \longrightarrow \mathfrak{g}^\sigma \leftrightarrows \mathfrak{g} \longrightarrow 0. \tag{1.5}$$

If G is connected and simply connected, then (1.4) is canonically split as well; in general it is induced from an extension by $\mathbb{Z}/2\mathbb{Z}$. Choose an *irreducible* $\mathbb{Z}/2\mathbb{Z}$-graded complex $\mathrm{Cliff}^c(\mathfrak{g}^*)$-module S with a compatible metric. Then S carries a unitary representation of G^σ.

Fix a basis $\{e_a\}$ of $\mathfrak{g}$ and let $\{e^a\}$ be the dual basis of $\mathfrak{g}^*$. Define tensors g and f by

$$\begin{aligned} \langle e_a, e_b\rangle &= g_{ab}, & \langle e^a, e^b\rangle &= g^{ab}, \\ [e_a, e_b] &= f^c_{ab} e_c, & \langle [e_a, e_b], e_c\rangle &= f_{abc}. \end{aligned}$$

The invariance of the inner product implies that f_{abc} is skew in the indices. Define the invariant 3-form

$$\Omega = \frac{1}{6} f_{abc} e^a \wedge e^b \wedge e^c \tag{1.6}$$

on $\mathfrak{g}$. Let γ^a denote[6] Clifford multiplication by e^a, and σ_a the action of e_a on spinors (via the splitting (1.5)); both are skew-Hermitian transformations of S, the former odd and the latter even. Then we have

$$\begin{aligned} \sigma_a &= \frac{1}{4} f_{abc}\gamma^b\gamma^c, \\ [\gamma^a, \gamma^b] &= -2g^{ab}, \\ [\sigma_a, \sigma_b] &= f^c_{ab}\sigma_c, \\ [\sigma_a, \gamma^b] &= -f^b_{ac}\gamma^c. \end{aligned} \tag{1.7}$$

Now any element of $\mathfrak{g}$ may be identified with a left-invariant vector field on G, and so acts on smooth functions by differentiation. Let R_a denote the differentiation corresponding to e_a. It satisfies

$$[R_a, R_b] = f^c_{ab} R_c. \tag{1.8}$$

We use left translation to identify the tensor product $C^\infty(G) \otimes S$ with spinor fields on G, and let R_a, σ_a, γ^a operate on the tensor product. Then

$$[R_a, \sigma_b] = [R_a, \gamma^b] = 0. \tag{1.9}$$

[6]We view Cliff^c as the complexification of Cliff^- as reflected by the sign in the second formula of (1.7).

Introduce the Dirac operator

$$D_0 = i\gamma^a R_a + \frac{i}{3}\gamma^a \sigma_a \;=\; i\gamma^a R_a + \frac{i}{12} f_{abc}\gamma^a\gamma^b\gamma^c \;=\; i\gamma^a R_a + \frac{i}{2}\gamma(\Omega). \tag{1.10}$$

The last term is, up to a factor, Clifford multiplication by the 3-form Ω, accomplished using the canonical vector space isomorphism between the exterior algebra and Clifford algebra. The Dirac operator D_0 is an odd formally skew-adjoint operator on smooth spinor fields. Observe that D_0 is G^σ-invariant. It is not the Riemannian Dirac operator, nor is it the Dirac operator associated to the natural parallel transport on G using either left or right translation.[7] Define the *Dirac family* of skew-adjoint Dirac operators parametrized by $\mu \in \mathfrak{g}^*$:

$$D_\mu = \gamma(\mu) + D_0 = \mu_a\gamma^a + D_0, \qquad \mu = \mu_a e^a \in \mathfrak{g}^*. \tag{1.12}$$

The map $\mu \mapsto D_\mu$ is G^σ-equivariant.[8]

Recall that the Peter-Weyl theorem gives an orthogonal direct sum decomposition of the L^2 spinor fields, where we trivialize the spin bundle using left translation:

$$L^2(G)\otimes S \;\;\cong \bigoplus_{V \text{ irreducible}} V^*\otimes V\otimes S.$$

Here V runs over (representatives of) equivalence classes of irreducible representations of G. In each summand the group G operates via the projective spin representation on S, via right translation on V, and via left translation on V^*. Each summand is a finite dimensional space of smooth spinor fields, and D_μ preserves the decomposition. Since D_μ only involves (infinitesimal) right translation, it operates trivially on V^*. We study each summand separately and drop the factor V^* on which D_μ acts trivially. In other words, for each finite dimensional irreducible unitary representation V of G we consider the Dirac family of *finite dimensional* odd skew-adjoint operators

$$\begin{aligned} D(V) : \mathfrak{g}^* &\longrightarrow \operatorname{End}(V\otimes S)\\ \mu &\longmapsto D_\mu \end{aligned} \tag{1.13}$$

given by formulas (1.12) and (1.10), where now R is the infinitesimal action of $\mathfrak{g}$ on the representation V. Note that V need not be irreducible, though, to analyze the operator it is convenient to assume so.

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[7]See the introduction for further discussion. The choice of coefficient in the second term of (1.10) is made so that the square of the Dirac operator has no first order term $g^{ab}R_a\sigma_b$:

$${D_0}^2 = \frac{1}{2}[D_0, D_0] = g^{ab}(R_aR_b + \frac{1}{3}\sigma_a\sigma_b). \tag{1.11}$$

The first term in (1.11) is the nonpositive Laplace operator on spinor fields, formed using right translation as parallel transport. Of course, the entire operator ${D_0}^2$ is nonpositive as it is the square of a skew-adjoint operator. Formula (1.11) follows from a straightforward computation using (1.7), (1.8), and (1.9), and it has an easy extension to the computation of the square of (1.12). We could use these formulas to prove Proposition 1.19, but instead give an argument which more directly extends to the infinite dimensional case; see the proof of Proposition 3.33.

[8]There is a real version of the Dirac family, which we have not completely investigated. Roughly speaking, we replace the spin space S by the vector space $\mathrm{Cliff}^-(\mathfrak{g}^*)$: the group G^σ acts on the left and there is a commuting action of $C^-_{\dim G}$ on the right. Then in the Dirac family $D(V) : \mathfrak{g}^* \to \operatorname{End}\bigl(V\otimes \mathrm{Cliff}^-(\mathfrak{g}^*)\bigr)$ we rotate both copies of $\mathfrak{g}^*$ to $\sqrt{-1}\mathfrak{g}^*$ thus obtaining a Real family of operators under the involution of complex conjugation. It represents an element of twisted KR-theory.

§1.3. The kernel of D_μ and the Kirillov correspondence. Fix a maximal torus T in the identity component $G_1 \subset G$, and let $\mathfrak{t} \subset \mathfrak{g}$ be its Lie algebra. The lattice of characters of T, or *weights* of G, is $\Lambda = \mathrm{Hom}(T, \mathbb{T})$. Write a character of T as a homomorphism $e^{i\lambda} : T \to \mathbb{T}$ for some $\lambda \in \mathfrak{t}^*$, and so embed $\Lambda \subset \mathfrak{t}^*$. Under the adjoint action of T the complexification of $\mathfrak{g}$ splits into a sum of the complexification of $\mathfrak{t}$ and one-dimensional *root* spaces $\mathfrak{g}_{\pm\alpha}$:

$$\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}). \tag{1.14}$$

The roots form a finite set $\Delta \subset \Lambda$ and come in pairs $\pm\alpha$ whose kernels form the infinitesimal diagram, a set of hyperplanes in $\mathfrak{t}$. The components of the complement of the infinitesimal diagram are simply permuted by the *Weyl group* $W = N(T)/T$. A *Weyl chamber* is a choice of component, and it determines the set Δ^+ of positive roots, that is, roots which have positive values on the Weyl chamber. The bracket $i[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is a line in $\mathfrak{t}$, and the *coroot* H_α is the element in that line so that $\mathrm{ad}(H_\alpha)$ acts as multiplication by 2 on $\mathfrak{g}_\alpha$. The coroot H_α is positive if α is a positive root. The kernels of the coroots from a set of hyperplanes in $\mathfrak{t}^*$, and the *dual Weyl chamber* is the component of the complement which takes positive values on the positive coroots. A weight is *dominant* if it lies in the closure of the dual Weyl chamber. A weight $\mu \in \Lambda$ is *regular* if the Weyl group W acts on μ with trivial stabilizer.

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

It is an element of $\mathfrak{t}^*$ which is not necessarily in the weight lattice Λ. We follow [PS, §2.7] and label an irreducible representation by its *lowest* weight, which is *antidominant*.

Lemma 1.15. *As an ungraded representation S is a sum of irreducible representations of G_1^σ of lowest weight $-\rho$.*

Proof. Let $\mathfrak{p} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ be the sum of the positive root spaces, so that (1.14) reads $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{p} \oplus \bar{\mathfrak{p}}$. Then as a $\mathbb{Z}/2\mathbb{Z}$-graded representation of the central extension T^σ of T we have the $\mathbb{Z}/2\mathbb{Z}$-graded tensor product decomposition

$$\begin{aligned} S = S(\mathfrak{g}^*) &\cong S(\mathfrak{t}^*) \otimes \bigwedge \bar{\mathfrak{p}}^* \otimes (\mathrm{Det}\,\mathfrak{p}^*)^{1/2} \\ &\cong S(\mathfrak{t}^*) \otimes \bigwedge \mathfrak{p} \ \ \otimes (\mathrm{Det}\,\bar{\mathfrak{p}} \ \ )^{1/2}, \end{aligned} \tag{1.16}$$

where $S(\mathfrak{t}^*)$ is a fixed irreducible complex $\mathbb{Z}/2\mathbb{Z}$-graded spin module of $\mathfrak{t}^*$. We claim that the weights of T^σ which occur in $S' = \bigwedge \mathfrak{p} \otimes (\mathrm{Det}\,\bar{\mathfrak{p}})^{1/2}$ are those of the irreducible representation of G_1^σ of lowest weight $-\rho$. The lemma follows directly from the claim as T^σ acts trivially on $S(\mathfrak{t}^*)$. To verify the claim, note that the character of $\bigwedge \mathfrak{p}$ as an ungraded representation of T is $\prod_{\alpha \in \Delta^+} (1 + e^{i\alpha})$, so the character[9] of S'

[9]Characters of representations of T^σ on which the central $\mathbb{T}$ acts as scalar multiplication form a torsor for the character group of T and may be identified with a subset of $\mathfrak{t}^*$ via the splitting (1.5). The character (1.17) may be expressed as a sum of exponentials of elements of $\mathfrak{t}^*$ which are necessarily in this subset.

is

$$\prod_{\alpha\in\Delta^+}(e^{i\alpha/2}+e^{-i\alpha/2})=\prod_{\alpha\in\Delta^+}\frac{e^{i\alpha}-e^{-i\alpha}}{e^{i\alpha/2}-e^{-i\alpha/2}}. \tag{1.17}$$

That this is the character of the irreducible representation of lowest weight $-\rho$ follows from the Weyl character formula; or, we observe that (1.17) is Weyl invariant, has lowest weight $-\rho$, and that $\dim S' = 2^{\#\Delta^+}$ has the correct dimension by the Weyl dimension formula. □

We now analyze the operator (1.13). Fix $\mu \in \mathfrak{g}^*$. Choose a maximal torus T_μ in the identity component of the stabilizer subgroup $Z_\mu \subset G$. If μ is regular, the identity component of Z_μ *is* T_μ. Then μ lies in ${\mathfrak{t}_\mu}^* \subset \mathfrak{g}^*$, i.e., μ annihilates the root spaces. Furthermore, choose a Weyl chamber so that μ is *antidominant*.[10] Let $\rho = \rho(\mu) \in \mathfrak{t}^*_\mu$ be half the sum of positive roots, defined relative to T_μ. Under $G_1 \subset G$ there is a decomposition

$$V \cong \bigoplus V_{,}{-\lambda} \tag{1.18}$$

with $V_{-\lambda}$ the $(-\lambda)$-isotypical component of V, where λ runs over the dominant weights of G_1.

Proposition 1.19. *Suppose V is irreducible. The operator $D_\mu(V)$ is nonsingular unless μ is regular and $-\lambda = \mu + \rho$ is a lowest weight in the decomposition* (1.18) *of V under G_1. In that case,*

$$\operatorname{Ker} D_\mu = K_{-\lambda} \otimes S_{-\rho} \subset V \otimes S, \tag{1.20}$$

where $K_{-\lambda}$ is the $(-\lambda)$-weight space of V in the decomposition under $T_\mu \subset G_1$ and $S_{-\rho}$ is the $\mathbb{Z}/2\mathbb{Z}$-graded $(-\rho)$-weight space of S in the decomposition under $T^\sigma_\mu \subset G^\sigma_1$. The stabilizer $Z_\mu \subset G$ of μ, a group with identity component T_μ, acts irreducibly on $K_{-\lambda}$. The Clifford algebra $\mathrm{Cliff}^c(\mathfrak{t}^*_\mu)$ *acts irreducibly on $S_{-\rho}$.*

If $G = G_1$ is connected, then $Z_\mu = T_\mu$ and $\dim K_{-\lambda} = 1$. The latter is a standard fact in the representation theory of connected compact Lie groups. The corresponding fact in the disconnected case—that Z_μ acts irreducibly on $K_{-\lambda}$—is less standard; it may be found in [DK, §4.13] for example.

Proof. Introduce a shifted version $\dot\pi_\mu : \mathfrak{g} \to \operatorname{End}(V \otimes S)$ of the infinitesimal Lie algebra action by the formula

$$\dot\pi_\mu(\xi) = \xi^a(R_a + \sigma_a - i\mu_a), \qquad \xi \in \mathfrak{g}.$$

Notice that $\dot\pi_\mu(\xi)$ is skew-adjoint. The inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ gives isomorphisms $\xi \mapsto \xi^*$ and $\mu \mapsto \mu_*$ between $\mathfrak{g}$ and its dual. Introduce also the self-adjoint operator

$$E_\mu = i\dot\pi_\mu(\mu_*) - \frac{|\mu|^2}{2}. \tag{1.21}$$

Then since $\mu \in \mathfrak{t}^*$ it follows that E_μ is constant on weight spaces of $V \otimes S$; its value on the space with weight ω is

$$\langle \frac{\mu}{2} - \omega, \mu\rangle. \tag{1.22}$$

[10] If μ is regular, it is contained in a unique (negative) dual Weyl chamber and so determines a notion of anti-dominance.

Since μ is antidominant, E_μ has its minimum on the lowest weight space. Straightforward computations with (1.7)–(1.10) give for $\xi \in \mathfrak{g}$ the second of the commutation relations

$$[D_\mu, \dot{\pi}_\mu(\xi)] = [-iE_\mu, \gamma(\xi^*)],$$
$$[D_\mu, \gamma(\xi^*)] = -2i\dot{\pi}_\mu(\xi);$$

the first is the infinitesimal form of the statement that D_μ is equivariant for the action of G^σ. Iterating we find

$$[D_\mu^2, \dot{\pi}_\mu(\xi)] = [-2E_\mu, \dot{\pi}_\mu(\xi)],$$
$$[D_\mu^2, \gamma(\xi^*)] = [-2E_\mu, \gamma(\xi^*)].$$

Now $V \otimes S$ is generated from the lowest weight space by applying operators $\dot{\pi}_\mu(\xi)$ and $\gamma(\xi^*)$. Hence $D_\mu^2 + 2E_\mu$ is constant on $V \otimes S$, and it is easy to check from (1.12) and (1.21) that this constant is independent of μ. Therefore, the self-adjoint nonpositive operator D_μ^2 has its maximum when E_μ achieves its minimum. By (1.22) and the remark which follows this happens at $\mu = -(\lambda + \rho)$ on the minimum weight space $K_{-\lambda} \otimes S_{-\rho}$. It remains to check that D_μ^2 does vanish on this space.

We use a complex basis compatible with (1.14). Fix nonzero elements $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$, and choose a basis e_t of $\mathfrak{t}$. Extend the inner products on $\mathfrak{g}, \mathfrak{g}^*$ to bilinear forms on the complexifications. Write

$$D_\mu = \gamma^a(iR_a + \mu_a) + \frac{i}{3}\gamma^a\sigma_a.$$

Then for $\alpha \in \Delta^+$ we have $R_{-\alpha} = \gamma^\alpha = 0$ on the lowest weight space $K_{-\lambda} \otimes S_{-\rho}$, so the only contribution from the first term is from the e_t, and these terms contribute $\gamma(\lambda + \mu)$. For the last term

$$\frac{i}{3}\gamma^a\sigma_a = \frac{i}{3}\sigma_a\gamma^a + \frac{i}{3}f^a_{ac}\gamma^c.$$

On the lowest weight space $\sigma_{-\alpha} = \gamma^\alpha = 0$ for $\alpha \in \Delta^+$. So the root indices contribute

$$\sum_{\alpha\in\Delta^+} \frac{i}{3} f^\alpha_{\alpha c}\gamma^c = \sum_{\alpha\in\Delta^+} \frac{i}{3} f^\alpha_{\alpha t}\gamma^t = \sum_{\alpha\in\Delta^+} \frac{i}{3}(-i)\alpha(e_t)\gamma^t = \frac{1}{3}\gamma(2\rho) = \frac{2}{3}\gamma(\rho).$$

For the sum over torus indices we find $\frac{i}{3}\gamma^t\sigma_t = \frac{1}{3}\gamma(\rho)$, so altogether we have

$$D_\mu = \gamma(\mu + \lambda + \rho) \qquad \text{on } K_{-\lambda} \otimes S_{-\rho}.$$

Squaring, we obtain

$$D_\mu^2 = -|\mu + \lambda + \rho|^2 \qquad \text{on } K_{-\lambda} \otimes S_{-\rho},$$

from which the proposition follows. □

The kernels fit together into a G^σ-equivariant vector bundle

$$\operatorname{Ker} D(V) \mid \mathcal{O} \longrightarrow \mathcal{O}$$

over a G-coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. That $\mathcal{O}$ is a single G-orbit follows from the irreducibility of V, since G/G_1 acts transitively on the weights occurring in (1.18). Then (1.20) gives a decomposition

$$\operatorname{Ker} D(V) \mid \mathcal{O} \;=\; K \otimes S_{\mathrm{Ker}} \;\subset\; \mathcal{O} \times (V \otimes S),$$

The bundle $K \to \mathcal{O}$ is G-equivariant. If G is connected, then as a nonequivariant bundle $S_{\mathrm{Ker}} \to N$ is the spinor bundle of the normal bundle of $\mathcal{O}$ in $\mathfrak{g}^*$; see (1.16). As an equivariant bundle it differs by a character of T, and if G is not connected, possibly by a finite twist, as we now explain.

First, the normal bundle $N \to \mathcal{O}$ is G-equivariant and is equivariantly trivial as a G_1-bundle, since the identity component T_μ of the stabilizer of μ acts trivially on the fiber $\mathfrak{t}_\mu$ of N at μ. The adjoint representation

$$Z_\mu \longrightarrow Z_\mu/T_\mu \longrightarrow \mathrm{O}(\mathfrak{t}_\mu)$$

factors through the finite group Z_μ/T_μ of components of the stabilizer. The pullback of (1.1) for $H = \mathfrak{t}_\mu$ gives a central extension $Z_\mu^{\sigma(N)}$ of Z_μ by $\mathbb{T}$. Fix an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded complex $\mathrm{Cliff}^c(\mathfrak{t}_\mu^*)$-module $S(\mathfrak{t}_\mu^*)$ and compatible metric; it is then a unitary representation of $Z_\mu^{\sigma(N)}$. Note that $S_{-\rho(\mu)}$ is also an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded complex $\mathrm{Cliff}^c(\mathfrak{t}_\mu^*)$-module, but the extension Z_μ^σ acts. We fix the "sign" (§1.1) of the $\mathrm{Cliff}^c(\mathfrak{t}_\mu^*)$-module $S(\mathfrak{t}_\mu^*)$ by asking that it agree with that of $S_{-\rho}$. Let $\mathrm{Cliff}^c(N)$ be the bundle of Clifford algebras on the normal bundle. Define[11]

$$L = \mathrm{Hom}_{\mathrm{Cliff}^c(N)}\big(S(N), S_{\mathrm{Ker}}\big). \tag{1.23}$$

Then $L \to \mathcal{O}$ is an even line bundle. We summarize in the following proposition, which interprets the twistings in K-theory language.

Proposition 1.24. *Let σ be the twisting[12] of $K_G(\mathfrak{g}^*)$ induced by the central extension $G^\sigma \to G$; it restricts to a twisting of $K_G(\mathcal{O})$. Over the orbit $\mathcal{O} \cong G/Z_\mu$ the graded central extension $Z_\mu^{\sigma(N)}$ defines a second twisting of $K_G(\mathcal{O})$,[13] denoted $\sigma(N)$, and the bundle $S(N) \to \mathcal{O}$ induced from $S(\mathfrak{t}_\mu^*)$ is a $\sigma(N)$-twisted bundle. The equivariant tangent bundle $T\mathcal{O} \to \mathcal{O}$ determines the twisting $\sigma(\mathcal{O}) = \sigma - \sigma(N)$ of $K_G(\mathcal{O})$. The bundle $L \to \mathcal{O}$, and so $K \otimes L \to \mathcal{O}$, is $\sigma(\mathcal{O})$-twisted and is even. If $G = G_1$ is connected, then $\sigma(N)$ is canonically trivial, there is a canonical isomorphism $\sigma(\mathcal{O}) \cong \sigma$, and L is canonically the G^σ-bundle given by the character $-\rho$.*

The second statement follows from the G-equivariant isomorphism of $N \oplus T\mathcal{O} \to \mathcal{O}$ with the trivial bundle $\mathcal{O} \times \mathfrak{g}^* \to \mathcal{O}$. Notice that $L \to \mathcal{O}$, a rank one bundle, may be viewed as a trivialization of the twisting $\sigma(\mathcal{O})$. The latter is trivializable since the coadjoint orbit admits spin^c, in fact spin, structures; see the last paragraph of §1.4.

Proposition 1.19 is an inverse to the Kirillov correspondence: starting with an irreducible representation we construct a coadjoint orbit $\mathcal{O}$, the support of the Dirac family. Note that the orbit is ρ-shifted from the lowest weight of the representation (in the connected case) and so is a regular orbit. Furthermore, we construct a

[11] If rank N is odd, then elements of L are required to commute with the C_1^c action.

[12] In terms of the notation (0.2) for twistings

$$\sigma = \big(\mathfrak{g}^* \xrightarrow{\{1\}} \mathfrak{g}^*, G^\sigma, \epsilon^\sigma\big),$$

where ϵ^σ is the composition $G \to \mathrm{O}(\mathfrak{g}) \to \mathbb{Z}/2\mathbb{Z}$ of the adjoint map and the nontrivial grading of the orthogonal group.

[13] In terms of our model (0.2) for equivariant twistings

$$\sigma(N) = \big(G \xrightarrow{Z_\mu} G/Z_\mu, G \times (Z_\mu^{\sigma(N)})^{\mathrm{op}}, \epsilon^{\sigma(N)}\big),$$

where $G \times Z_\mu^{\mathrm{op}}$ acts on G by left $\times$ right multiplication and $\epsilon^{\sigma(N)}$ is the grading of Z_μ induced from the adjoint action.

vector bundle $K \otimes L \to \mathcal{O}$. As we will see in the next section, we can recover V from this bundle.

§1.4. K-theory interpretation. A G-equivariant family of odd skew-adjoint $\mathbb{Z}/2\mathbb{Z}$-graded Fredholm operators parametrized by a G-space X represents a class in $K_G^0(X)$. If there is a commuting C_1^c-structure, then it represents an element of $K_G^1(X)$. We apply this to $X = \mathfrak{g}^*$ with two variations to obtain a class

$$\big[D(V)\big] \in K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{\mathrm{cpt}},$$

where we use Bott periodicity to identify K_G^n with K_G^0 or K_G^1 according to the parity of n. First variation: the family of operators $D(V)$ is compactly supported (Proposition 1.19), that is, invertible outside a compact subset of $\mathfrak{g}^*$. Second variation: $V \otimes S$ is a representation of the central extension G^σ of G, which accounts for the twisting σ. Note that if $\dim G$ is odd there is a C_1^c-structure on S, hence also one on $V \otimes S$, and it commutes with $D_\mu(V)$ for all $\mu \in \mathfrak{g}^*$.

The Thom isomorphism in equivariant K-theory for the inclusion $j : \{0\} \hookrightarrow \mathfrak{g}^*$ is usually implemented as the map

$$\begin{aligned} j_* : K_G^0 &\longrightarrow K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{\mathrm{cpt}} \\ V &\longrightarrow (V \otimes S, 1 \otimes \gamma), \end{aligned} \tag{1.25}$$

which assigns to each finite dimensional unitary representation the family of Clifford multiplication operators. This family is supported at the origin. The Dirac family (1.13) is a compactly supported perturbation of (1.25), so implements the same map on K-theory.

Let $i : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ be the inclusion, and consider the diagram

$$\begin{tikzcd} & K_G^{\sigma(\mathcal{O})+\dim \mathcal{O}}(\mathcal{O}) \arrow[dl, "\pi_*^{\mathcal{O}}"'] \arrow[dr, "i_*"] & \\ K_G^0 \arrow[rr, shift left, "j_*"] & & K_G^{\sigma+\dim G}(\mathfrak{g}^*)_{\mathrm{cpt}} \arrow[ll, shift left, "(\pi^{\mathfrak{g}^*})_*"] \end{tikzcd} \tag{1.26}$$

of twisted G-equivariant K-groups.[14] Since $\mathcal{O}$ is a regular orbit, $\dim \mathcal{O} = \dim G - \dim T$ for any maximal torus T. (There is a similar diagram for any G-coadjoint orbit, but for our purposes it suffices to consider regular orbits.) For any G-space X we denote $\pi^X : X \to \mathrm{pt}$ the unique map. Then S, $S(N)$ represent equivariant Thom classes and so define the indicated pushforward maps $j_*, (\pi^{\mathfrak{g}^*})_*, i_*$; we choose a compatible twisted equivariant Thom class to define $\pi_*^{\mathcal{O}}$. Then

$$i_* = j_* \circ \pi_*^{\mathcal{O}},$$

since by naturality and the compatibility of the Thom classes,

$$\begin{gathered} (\pi^{\mathfrak{g}^*})_* \circ i_* = \pi_*^{\mathcal{O}}, \\ j_* \text{ is the inverse of } (\pi^{\mathfrak{g}^*})_*. \end{gathered} \tag{1.27}$$

[14]The (twisted) K-cohomology groups in the diagram are isomorphic to untwisted G-equivariant K-homology groups in degree zero. Also, recall that we use Bott periodicity so only keep track of the parity of the K-groups.

Theorem 1.28. (*i*) *For an irreducible representation V of G,*

$$j_*[V] = \big[D(V)\big]. \tag{1.29}$$

(*ii*) *Let $[K \otimes L] \in K_G^{\sigma(\mathcal{O})}(\mathcal{O})$ be the class of the vector bundle $K \otimes L \to \mathcal{O}$. Then*

$$i_*[K \otimes L] = \big[D(V)\big], \tag{1.30}$$

$$\pi_*^{\mathcal{O}}[K \otimes L] = [V]. \tag{1.31}$$

The twisting of $K \otimes L$ was described in Proposition 1.24.

Proof. See the remarks following (1.25) for (i).

The map i_* is multiplication by the Thom class of the normal bundle to $\mathcal{O}$ in $\mathfrak{g}^*$. This gives a class in a tubular neighborhood of $\mathcal{O}$ which is compactly supported; we extend by zero to a compactly supported class on $\mathfrak{g}^*$. To see that $\big[D(V)\big]$ represents this class, as asserted in (1.30), note first that $[D(V)]$ is supported on $\mathcal{O}$ by Proposition 1.19. Hence it suffices to restrict to a tubular neighborhood U of $\mathcal{O}$, which we identify with a neighborhood of the zero section in the normal bundle $N \to \mathcal{O}$ via orthogonal projection. Write $V \otimes S \mid U$ as the direct sum of $K \otimes L \otimes S(N)$, pulled back from $\mathcal{O}$ using the tubular neighborhood structure, and its orthogonal complement. We may ignore the latter since $D(V)$ is invertible on it if U is sufficiently small. For any $\mu \in \mathcal{O}$ and $\theta \in \mathfrak{g}^*$ perpendicular to μ we have

$$D_{\mu+\theta} = D_\mu + \gamma(\theta).$$

Restricted to $K \otimes L \otimes S(N)$ it acts as $1 \otimes 1 \otimes \gamma(\theta)$. Since $\theta \mapsto \gamma(\theta)$ on $S(N)$ is the Thom class of the normal bundle, we have proved (1.30).

Equation (1.31) is a formal consequence of (1.29) and (1.30) using (1.27). □

The Atiyah-Singer index theorem identifies $\pi_*^{\mathcal{O}}$ as the equivariant index of a Dirac operator on $\mathcal{O}$. Recall that the G-coadjoint orbit $\mathcal{O}$ is a finite union of regular G_1-coadjoint orbits, each of which is diffeomorphic to G_1/T. The latter carries a canonical spin structure up to isomorphism;[15] the twist $\sigma(\mathcal{O})$ in (1.26) means that we consider bundles on $\mathcal{O}$ which are equivariant for a covering group of G. In case G is connected and simply connected, both σ and $\sigma(N)$ are isomorphic to the trivial twisting, whence $\pi_*^{\mathcal{O}}$ is the usual G-equivariant Dirac index. The map $\pi_*^{\mathcal{O}}$ is termed *Dirac induction* in representation theory. Our construction in this section is an explicit inverse to Dirac induction. Equation (1.31) is part of the Borel-Weil-Bott theorem, but we do not prove the vanishing theorem necessary to show that the kernel of the Dirac operator is precisely V. (Note that the usual version of this theorem uses holomorphic induction in place of Dirac induction.)

§1.5. Variation: projective representations. In the loop group version of the Dirac family we encounter the finite dimensional construction considered here, but in a projective version which we now sketch.

[15]Write $G_1/T \cong \tilde{G}_1/\tilde{T}$ for $\tilde{G}_1$ the simply connected cover of G_1. A choice of Weyl chamber gives a $\tilde{G}_1$-invariant complex structure on $\tilde{G}_1/\tilde{T}$ whose first Chern class in $H^2(\tilde{G}_1/\tilde{T}) \cong \tilde{\mathfrak{t}}^*$ is identified with 2ρ. Since ρ is a character of $\tilde{G}_1$, we conclude that $\tilde{G}_1/\tilde{T}$ is spin, and since it is simply connected the spin structure is unique up to isomorphism. More precisely, the component of any $\mu \in \mathcal{O}$ is canonically G_1/T_μ and $\mathfrak{t}_\mu$ has a distinguished Weyl chamber which gives a distinguished complex structure on $\mathcal{O}$. In particular, there is a canonical orientation of $\mathcal{O}$.

Let $1 \to \mathbb{T} \to G^\tau \to G \to 1$ be a central extension of G.[16] A *projective representation of* G *at level* τ is a representation of G^τ on which the center $\mathbb{T}$ acts by scalar multiplication. The free abelian group K_G^τ generated by the irreducibles is a K_G-module via tensor product. In this situation $\mathfrak{g}^*$ is replaced by an affine space $\mathcal{A}_G^\tau$ for $\mathfrak{g}^*$ which has two equivalent definitions: it is the space of linear splittings of the Lie algebra extension $0 \to i\mathbb{R} \to \mathfrak{g}^\tau \to \mathfrak{g} \to 0$, or equivalently it is the subspace of $(\mathfrak{g}^\tau)^*$ consisting of functionals $\mu : \mathfrak{g}^\tau \to \mathbb{R}$ with $\mu(i) = 1,\ i \in i\mathbb{R}$. The group G acts on $\mathcal{A}_G^\tau$ compatibly with the coadjoint action on $\mathfrak{g}^*$. Note that the action of G on $\mathcal{A}_G^\tau$ has a fixed point, the center of mass of a G-orbit, from which $\mathcal{A}_G^\tau \cong \mathfrak{g}^*$ as G-spaces. (This is in contrast to the infinite dimensional situation we study later.)

Let $T \subset G$ be a maximal torus. By restriction we obtain a central extension $T^\tau \to T$, and T^τ is a maximal torus of G^τ. The affine space $\mathcal{A}_T^\tau$ sits inside $\mathcal{A}_G^\tau$ as the subspace of functionals which vanish on the root spaces. The infinitesimal characters of projective representations of T at level τ form a subset $\Lambda^\tau \subset \mathcal{A}_T^\tau$ which is a torsor for the weight lattice $\Lambda \subset \mathfrak{t}^*$. The Weyl group acts on $\mathcal{A}_T^\tau$ preserving Λ^τ; the action is conjugation by the normalizer of T^τ in G^τ. Choose a positive Weyl chamber in $\mathfrak{t}$, and so by pullback a positive Weyl chamber in $\mathfrak{t}^\tau$. (The Weyl groups of G and G^τ are canonically isomorphic.)

Given central extensions $G^{\tau_1} \to G$ and $G^{\tau_2} \to G$ we can form their product $G^{\tau_1+\tau_2} \to G$, which is also a central extension. Similarly, we define an inverse to a central extension. Now the Lie algebra extension (1.5) associated to the canonical spin central extension $G^\sigma \to G$ is canonically split,[17] so for any τ we identify $\mathcal{A}_G^\tau$ with $\mathcal{A}_G^{\tau-\sigma}$.

The projective version of our Dirac family proceeds as follows. Fix a central extension $G^\tau \to G$ and a projective representation V at level $\tau - \sigma$. So $V \otimes S$ is a projective representation at level τ. The Dirac family (1.12) is parametrized by $\mu \in \mathcal{A}_G^\tau \cong \mathcal{A}_G^{\tau-\sigma}$:

$$D_\mu = i\gamma^a (R_a)_\mu + \frac{i}{2}\gamma(\Omega), \tag{1.32}$$

where $(R_a)_\mu$ denotes the infinitesimal action of the basis element e_a on V defined using the splitting $\mu : \mathfrak{g} \to \mathfrak{g}^{\tau-\sigma}$. Then

$$D_{\mu+\nu} = D_\mu + \gamma(\nu), \qquad \nu \in \mathfrak{g}^*. \tag{1.33}$$

The discussion proceeds as before, and in particular Proposition 1.19 holds, but with the understanding that $\mu \in \mathcal{A}_G^\tau$, $\rho \in \mathcal{A}_G^\sigma$, and $\lambda \in \mathcal{A}_G^{\tau-\sigma}$. Also, in the K-theory discussion we use the fact that the Lie algebra extension $\mathfrak{g}^\tau \to \mathfrak{g}$ is split to find a fixed point of the G-action on $\mathcal{A}_G^\tau$ and so define the Thom isomorphism.

§1.6. Examples.

Example 1.34 ($G = \mathbb{T}$)**.** The irreducible representations are all one-dimensional and are labeled by an integer n: the representation $L_n = \mathbb{C}$ is $\lambda \mapsto \lambda^n$. Identify $\mathfrak{g} = i\mathbb{R}$ and set[18] $\langle ia, ia' \rangle = aa'$, thereby identifying $\mathfrak{g}^* \cong i\mathbb{R}$ as well. Set $S = \mathbb{C} \oplus \mathbb{C}$ with $\gamma = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. Then for $\mu = ia \in \mathfrak{g}^*$ the operator D_μ on $L_n \otimes S$ is the matrix

[16] Such extensions are classified by $H^3(BG;\mathbb{Z})$, which is torsion for a compact Lie group G. Therefore, a central extension by $\mathbb{T}$ factors through a central extension by a finite cyclic group.

[17] In the infinite dimensional case (§3) there is no such canonical splitting.

[18] This inner product generalizes for $G = U(n)$ to the inner product $\langle A, A' \rangle = -\operatorname{Tr}(AA')$ on skew-Hermitian matrices.

$D_{ia} = \left(\begin{smallmatrix} 0 & i(a-n) \\ i(a-n) & 0 \end{smallmatrix}\right)$. The kernel of the family $D(L_n)$ is supported at $\mu = in$. Note that the coadjoint action is trivial, since G is abelian, so the coadjoint orbits are points.

Example 1.35 ($G = \mathrm{O}_2$)**.** Now G is not connected. We identify $\mathfrak{g}, \mathfrak{g}^*$ as in the previous example. Reflections—elements of G not in the identity component—act as multiplication by -1 on $\mathfrak{g}^*$, so the coadjoint orbits are $\{0\}$ and $\{\pm ia\}_{a\neq 0}$. There is a one-dimensional irreducible representation V_ϵ which sends a matrix to multiplication by its determinant, and there are two-dimensional irreducibles V_n labeled by *positive* integers: the restriction of V_n to $\mathrm{SO}_2 \subset \mathrm{O}_2$ is $L_n \oplus L_{-n}$ in the notation of Example 1.34. For V_ϵ the infinitesimal representation vanishes, and $D_{ia} = \left(\begin{smallmatrix} 0 & ia \\ ia & 0 \end{smallmatrix}\right)$ acting on $V_\epsilon \otimes S$. The kernel is supported at $\mu = 0$ and is obviously $V_\epsilon \otimes S$. We can identify (1.23) with V_ϵ, and the stabilizer group $Z_0 = \mathrm{O}_2$ acts via the representation V_ϵ. For V_n we identify

$$D_{ia} = \begin{pmatrix} 0 & 0 & 0 & i(a-n) \\ 0 & 0 & i(a+n) & 0 \\ 0 & i(a-n) & 0 & 0 \\ i(a+n) & 0 & 0 & 0 \end{pmatrix}.$$

So $\operatorname{Ker} D(V_n)$ is supported on the coadjoint orbit $\mu = \pm in$.

Example 1.36 ($G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{T}$)**.** Again G is not connected, but now G is abelian so the irreducibles are one-dimensional and the coadjoint orbits are points. Let K denote the sign representation of $\mathbb{Z}/2\mathbb{Z}$; then the irreducibles of G are L_n and $L_n \otimes K$, where n ranges over all integers. As in Example 1.34 the kernel of the Dirac family is supported at $\mu = in$ for both $D(L_n)$ and $D(L_n \otimes K)$. They are distinguished by the action of the stabilizer $Z_\mu = G$ on the kernel.

Example 1.37 ($G = \mathrm{SU}_2$)**.** There is a single irreducible representation in each positive dimensions, and in a notation adapted to the quotient group SO_3 we label the representation V_j of dimension $2j+1$ by $j = 0, \frac{1}{2}, 1, \dots$. The Lie algebra is $\mathfrak{g} = \left\{ \left(\begin{smallmatrix} ia & b+ic \\ -b+ic & -ia \end{smallmatrix}\right) \right\}$, and using the inner product $\langle A, A' \rangle = -\frac{1}{2}\operatorname{Tr}(AA')$ we identify $\mathfrak{g} \cong \mathfrak{g}^* \cong i\mathbb{R}^3$. We take as maximal torus the group of diagonal matrices with Lie algebra $i\mathbb{R}$. Then the positive root is identified as $2i$ from which $\rho = i$. The coadjoint orbits are spheres centered at the origin. Proposition 1.19 states that $\operatorname{Ker} D(V_j)$ is supported on the sphere of radius $2j+1$. Notice that even for $j = 0$ the support is a *regular* coadjoint orbit, due to the ρ-shift.

Example 1.38 ($G = \mathrm{SO}_3$)**.** Only the representations of SU_2 with j integral pass to the quotient SO_3. Take as maximal torus T the set of rotations which fix a particular axis. The inner product is $\langle A, A' \rangle = -\frac{1}{2}\operatorname{Tr}(AA')$. With the identification $\mathfrak{t} = i\mathbb{R}$ we find $\rho = \frac{i}{2}$. Note that ρ is *not* a weight of G, which is possible as G is not simply connected. Also, $\operatorname{Ker} D(V_j)$ is supported on the sphere of radius $j + \frac{1}{2}$, where now $j \in \mathbb{Z}^{\geq 0}$.

Example 1.39 ($G^\tau = \mathrm{U}_2 \to G = \mathrm{SO}_3$)**.** Notice that U_2 has double cover $\mathrm{SU}_2 \times \mathbb{T}$, and the projective representations of interest pull back to representations of $\mathrm{SU}_2 \times \mathbb{T}$ which are of half-integral spin on SU_2 and standard on $\mathbb{T}$. Let T be the maximal torus of SO_3 as in the previous example. We may identify the affine space $\mathcal{A}_T^\tau$ with $i\mathbb{R}$ and the nontrivial element of the Weyl group acts by $ia \mapsto -ia$. It has no fixed points on the torsor Λ^τ of imaginary half-integers. The Dirac family for the spin j representation has kernel supported on the sphere of radius $j+\frac{1}{2}$ in $\mathcal{A}^\tau_{\mathrm{SO}_3}$.

§2. Loop groups and energy

§2.1. Basic definitions. Let G be a compact Lie group and $P \to S^1$ a principal G-bundle with base the standard circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The bundle P is classified up to isomorphism by a conjugacy class in $\pi_0(G) = G/G_1$. Let $G[P] \subset G$ denote the corresponding union of components, which is a G-space under conjugation. Note that if G is connected, then P is necessarily trivializable and $G[P] = G$.

Notation 2.1. L_PG is the group of smooth gauge transformations of $P \to S^1$. Its Lie algebra is denoted $L_P\mathfrak{g}$. Let $\mathbb{T}_{\mathrm{rot}}$ be the group of rigid rotations of S^1 and $(L_PG)_{\mathrm{rot}}$ be the group of smooth automorphisms of $P \to S^1$ which cover elements of $\mathbb{T}_{\mathrm{rot}}$. For any connected finite covering $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$ we define $\widehat{L}_PG$ by pullback:

$$\begin{array}{ccc} \widehat{L}_PG & \longrightarrow & \widehat{\mathbb{T}}_{\mathrm{rot}} \\ \Big\downarrow & & \Big\downarrow \\ (L_PG)_{\mathrm{rot}} & \longrightarrow & \mathbb{T}_{\mathrm{rot}} \end{array} \tag{2.2}$$

Finally, $\mathcal{A}_P$ is the space of smooth connections on $P \to S^1$.

For the trivial bundle $P = S^1 \times G$ we omit the subscript "P". Then LG is naturally identified with the smooth loop group $LG = \mathrm{Map}(S^1, G)$. For topologically nontrivial bundles L_PG is often termed a *twisted loop group*. Its Lie algebra $L_P\mathfrak{g}$ may be identified with $\Omega^0_{S^1}(\mathfrak{g}_P)$, the space of smooth sections of the adjoint bundle $\mathfrak{g}_P \to S^1$, or equivalently as the space of smooth G-invariant vertical vector fields on $P \to S^1$. The space $\mathcal{A}_P$ of connections is affine with associated vector space $\Omega^1_{S^1}(\mathfrak{g}_P)$.

Fix a basepoint $p \in P$ in the fiber over $0 \in \mathbb{R}/2\pi\mathbb{Z}$. Then for each connection $A \in \mathcal{A}_P$ parallel transport around S^1 maps p to a point $p \cdot \mathrm{hol}(A)$ of the same fiber for some $\mathrm{hol}(A) \in G$. The *holonomy* is a surjective map

$$\mathrm{hol} : \mathcal{A}_P \longrightarrow G[P]. \tag{2.3}$$

Furthermore, the holonomy map is equivariant for the natural action of L_PG on $\mathcal{A}_P$ and the conjugation action of G on $G[P]$ as follows. For $\varphi : P \to P$ in L_PG and $A \in \mathcal{A}_P$ we have a new connection $(\varphi^{-1})^*(A)$ and a group element $g_\varphi \in G$ defined by $\varphi(p) = p \cdot g_\varphi^{-1}$. (Recall A is a 1-form on P.) Then

$$\mathrm{hol}\big((\varphi^{-1})^*(A)\big) = g_\varphi \,\mathrm{hol}(A) g_\varphi^{-1}.$$

The subgroup $\Omega_PG \subset L_PG$ of gauge transformations which fix $p \in P$ acts freely on $\mathcal{A}_P$, and (2.3) is a (left) Ω_PG-principal bundle.

There is an exact sequence of groups

$$1 \longrightarrow L_PG \longrightarrow \widehat{L}_PG \longrightarrow \widehat{\mathbb{T}}_{\mathrm{rot}} \longrightarrow 1$$

and a corresponding sequence of Lie algebras

$$0 \longrightarrow L_P\mathfrak{g} \longrightarrow \widehat{L}_P\mathfrak{g} \longrightarrow i\mathbb{R}_{\mathrm{rot}} \longrightarrow 0. \tag{2.4}$$

(The Lie algebras of $\widehat{\mathbb{T}}_{\mathrm{rot}}$ and $\mathbb{T}_{\mathrm{rot}}$ are canonically isomorphic.) Splittings of (2.4) are in 1:1 correspondence with $\mathcal{A}_P$. For $A \in \mathcal{A}_P$ we denote by d_A the lift to $\widehat{L}_P\mathfrak{g}$

of the canonical generator $i \in i\mathbb{R}_{\text{rot}}$. Geometrically, d_A is a G-invariant horizontal vector field on P which projects to the standard vector field on S^1. We remark that the extended loop group $\widehat{L}_P G$ may be used to eliminate the basepoint p in the map (2.3) above.[19]

The Lie algebra $\mathfrak{g}$ of a compact Lie group G has a canonical splitting

$$\mathfrak{g} = \text{center}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] = \mathfrak{z} \oplus \mathfrak{g}' \tag{2.5}$$

into its abelian and semisimple pieces; the decomposition is orthogonal with respect to any invariant inner product on $\mathfrak{g}$. This decomposition is Ad-invariant, so it leads to a decomposition of the adjoint bundle $\mathfrak{g}_P \to S^1$ of any principal G-bundle $P \to S^1$, whence to a decomposition of the loop algebra $L_P\mathfrak{g}$.

§2.2. Admissible central extensions. The representations of $L_P G$ of interest are projective, and so we study central extensions

$$1 \longrightarrow \mathbb{T} \longrightarrow (L_P G)^\tau \longrightarrow L_P G \longrightarrow 1 \tag{2.6}$$

of loop groups with corresponding central extensions

$$0 \longrightarrow i\mathbb{R} \longrightarrow (L_P \mathfrak{g})^\tau \longrightarrow L_P \mathfrak{g} \longrightarrow 0 \tag{2.7}$$

of loop algebras. If P is trivial we write LG^τ for the central extension. Recall that a *graded* central extension of $L_P G$ is a central extension (2.6) together with a homomorphism $\epsilon : L_P G \to \mathbb{Z}/2\mathbb{Z}$.

Definition 2.8. Let $((L_P G)^\tau, \epsilon)$ be a graded central extension of $L_P G$. The *associated twisting* of the G-equivariant K-theory of $G[P]$ is

$$\tau = (\mathcal{A}_P \xrightarrow{\Omega_P G} G[P], (L_P G)^\tau, \epsilon), \tag{2.9}$$

where G acts on $G[P]$ by conjugation.

Here $\mathcal{A}_P \to G[P]$ is the principal $\Omega_P G$-bundle (2.3) and the twisting has the form (0.2). This construction of a twisting from a graded central extension of the loop group is fundamental to our work.

Let[20] K denote the central element $i \in i\mathbb{R}$.

[19]Namely, if we include the choice of basepoint in the domain of the holonomy map, then

$$\text{hol} : \mathcal{A}_P \times P \longrightarrow G[P]$$

is a principal $\widehat{L}_P G$-bundle and admits a commuting G-action.

[20]Our K and d_A (below) are $i = \sqrt{-1}$ times real multiples of the K and d in [K1]. (Compare [K1, (7.2.2)] to (2.21) and (2.22) below.) This motivates the sign in (2.12).

Definition 2.10. A central extension $(L_PG)^\tau$ is *admissible* if:

(1) There exists a central extension $(\widehat{L}_PG)^\tau$ of $\widehat{L}_PG$ which fits into a commutative diagram

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & (L_PG)^\tau & \longrightarrow & L_PG & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{T} & \longrightarrow & (\widehat{L}_PG)^\tau & \longrightarrow & \widehat{L}_PG & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \widehat{\mathbb{T}}_{\mathrm{rot}} & = & \widehat{\mathbb{T}}_{\mathrm{rot}} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1. & & \end{array} \tag{2.11}$$

(2) There exists an $(\widehat{L}_PG)^\tau$-invariant symmetric bilinear form $\ll \cdot,\cdot \gg_\tau$ on $(\widehat{L}_P\mathfrak{g})^\tau$ such that

$$\ll K, d \gg_\tau = -1 \tag{2.12}$$

for all $d \in (\widehat{L}_P\mathfrak{g})^\tau$ which project to $i \in i\mathbb{R}_{\mathrm{rot}}$.

It follows immediately from (2.12) that

$$\ll K, (L_P\mathfrak{g})^\tau \gg_\tau = 0, \tag{2.13}$$

and in particular $\ll K, K \gg_\tau = 0$. Hence $\ll \cdot,\cdot \gg_\tau$ induces an L_PG-invariant symmetric bilinear form on $L_P\mathfrak{g}$, which we also denote with double angle brackets. For $A \in \mathcal{A}_P$ the element $d_A \in \widehat{L}_P\mathfrak{g}$ has a unique lift to a null element of $(\widehat{L}_P\mathfrak{g})^\tau$ which we denote d_A^τ:

$$\ll d_A^\tau, d_A^\tau \gg_\tau = 0. \tag{2.14}$$

In the sequel the phrase "$(L_PG)^\tau$ is an admissible central extension" implies that we have fixed $(\widehat{L}_PG)^\tau$ and $\ll\cdot,\cdot\gg_\tau$. If there is no possibility for confusion, we write $\ll\cdot,\cdot\gg$ for the invariant bilinear form. We remark that the finite covering map $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$ is often the identity; see the discussion of tori in §2.3 for a case in which it is a double cover. The choice of $(\widehat{L}_PG)^\tau$ and a bilinear form is additional structure, as reflected in Lemma 2.18 below, for example.

If $\mathfrak{g}$ is semisimple, then every central extension of L_PG is admissible.

Proposition 2.15. *Let G be a compact Lie group with $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, i.e., $\mathfrak{g}$ semisimple. Let $P \to S^1$ be a principal G-bundle. Then any central extension $(L_PG)^\tau$ is admissible. Furthermore, for each $(\widehat{L}_PG)^\tau$ which satisfies (2.11) there exists a unique $(\widehat{L}_PG)^\tau$-invariant symmetric bilinear form $\ll\cdot,\cdot\gg = \ll\cdot,\cdot\gg_\tau$ which satisfies (2.12).*

For connected and simply connected groups there is a direct construction, which we give in §2.3. The general proof is more complicated and is deferred to the appendix.

The Lie algebra $\mathfrak{g}$ of any compact Lie group G carries a canonical G-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle_\sigma$ defined by

$$\langle\xi,\eta\rangle_\sigma = -\frac{1}{2}\operatorname{Tr}(\operatorname{ad}\xi\circ\operatorname{ad}\eta),\qquad \xi,\eta\in\mathfrak{g}.$$

Note that $\langle\cdot,\cdot\rangle_\sigma$ is positive semidefinite, and is positive definite if $\mathfrak{g}$ is semisimple. Now if $P\to S^1$ is a principal G-bundle, then we define the L^2 metric

$$\ll\beta_1,\beta_2\gg_\sigma = \int_{S^1}\langle\,\beta_1(s),\beta_2(s)\,\rangle_\sigma\,\frac{|ds|}{2\pi},\qquad \beta_1,\beta_2\in L_P\mathfrak{g}. \tag{2.16}$$

We will prove (Proposition 3.13) that (2.16) is associated to a canonical admissible graded central extension of L_PG.

Let $(L_PG)^\tau\to L_PG$ be any admissible central extension. The space of linear splittings of (2.7) carries an affine action of $L_P\mathfrak{g}$. Namely, if

$$\begin{aligned} s\,:\,L_P\mathfrak{g}&\longrightarrow(L_P\mathfrak{g})^\tau\\ \beta&\longmapsto\beta^\tau_s\end{aligned}$$

is a splitting and $\beta'\in L_P\mathfrak{g}$, then we define a new splitting $\beta\mapsto\beta^\tau_{s+\beta'}$ by

$$\beta^\tau_{s+\beta'}=\beta^\tau_s-\ll\beta,\beta'\gg_\tau\,K.$$

Recall that we identify $\mathcal{A}_P$ as the space of G-invariant vector fields d_A on $P\to S^1$ which project to the standard vector field on S^1. In this form $\mathcal{A}_P$ is affine for the action of $L_P\mathfrak{g}$ by subtraction: for $\xi\in L_P\mathfrak{g}$ we have

$$d_{A+\xi ds}=d_A-\xi. \tag{2.17}$$

Lemma 2.18. *Let $(L_PG)^\tau$ be an admissible central extension of L_PG. Then the form $\ll\cdot,\cdot\gg_\tau$ determines an $L_P\mathfrak{g}$-equivariant map*

$$\begin{aligned}\mathcal{A}_P&\longrightarrow\{\textit{linear splittings of }(L_P\mathfrak{g})^\tau\to L_P\mathfrak{g}\}\\ A&\longmapsto(\beta\mapsto\beta^\tau_A)\end{aligned}$$

which is an isomorphism if $\ll\cdot,\cdot\gg_\tau$ is nondegenerate.

Therefore, a connection simultaneously splits all admissible central extensions.

Proof. Characterize β^τ_A by the condition

$$\ll\beta^\tau_A,d^\tau_A\gg_\tau=0 \tag{2.19}$$

for $d^\tau_A\in(\widehat{L}_P\mathfrak{g})^\tau$ defined by (2.14). □

Suppose $(L_PG)^{\tau_i}\to L_PG$, $i=1,2$ are admissible central extensions. Then the product $(L_PG)^{\tau_1+\tau_2}=(L_PG)^{\tau_1}\times_{L_PG}(L_PG)^{\tau_2}$ is also admissible: the bilinear forms on $L_P\mathfrak{g}$ satisfy

$$\ll\beta_1,\beta_2\gg_{\tau_1+\tau_2}=\ll\beta_1,\beta_2\gg_{\tau_1}+\ll\beta_1,\beta_2\gg_{\tau_2},\qquad\beta_1,\beta_2\in L_P\mathfrak{g}.$$

There is also an (admissible) inverse central extension $(L_PG)^{-\tau}$ to an (admissible) central extension $(L_PG)^\tau$; in the admissible case the form changes sign when passing to the inverse.

§2.3. Special cases. We treat the three prototypical classes of compact Lie groups: connected simply connected groups, tori, and finite groups.

Simply connected groups. If G is simply connected, then every central extension (2.6) is admissible and $\ll\cdot,\cdot\gg$ is uniquely determined by the extension. This is a special case of Proposition 2.15. To verify this we may as well assume P is trivial so that $L_PG = LG$ is the standard loop group. Then [PS, §4.2] central extensions of the loop algebra $L\mathfrak{g}$ correspond to G-invariant symmetric bilinear forms $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ as follows. For $\xi \in \mathfrak{g}_{\mathbb{C}}$ and $n \in \mathbb{Z}$ we write $z^n\xi$ for the loop $s \mapsto e^{ins}\xi$ in $L\mathfrak{g}_{\mathbb{C}}$. The algebraic direct sum $\bigoplus_{n\in\mathbb{Z}} z^n\mathfrak{g}_{\mathbb{C}}$ is dense in $L\mathfrak{g}_{\mathbb{C}}$. Define

$$(\widehat{L}\mathfrak{g})^\tau = \mathbb{R}K \oplus L\mathfrak{g} \oplus \mathbb{R}d \tag{2.20}$$

as a vector space. As for the Lie bracket, K is central, d acts as the derivation

$$[d, z^n\xi] = inz^n\xi \tag{2.21}$$

on $L\mathfrak{g}_{\mathbb{C}}$, and the bilinear form on $\mathfrak{g}$ enters into the bracket

$$[z^n\xi, z^m\eta] = z^{n+m}[\xi,\eta] + n\langle\xi,\eta\rangle\delta_{n+m=0}\,\frac{K}{i}, \qquad \xi,\eta \in \mathfrak{g}_{\mathbb{C}}. \tag{2.22}$$

Any $\widetilde{L\mathfrak{g}}^\tau$-invariant bilinear form $\ll\cdot,\cdot\gg$ satisfies

$$\ll[z^n\xi, z^{-n}\eta], d\gg = \ll z^n\xi, [z^{-n}\eta, d]\gg = in\ll z^n\xi, z^{-n}\eta\gg, \tag{2.23}$$

and from (2.22) and (2.12) we deduce that for $n \neq 0$,

$$\ll z^n\xi, z^{-n}\eta\gg = \langle\xi,\eta\rangle. \tag{2.24}$$

For $n = 0$ we derive (2.24) by consideration of $\ll [z\zeta_1, z^{-1}\zeta_2], \eta \gg$ and the fact that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Similarly, we find $\ll z^n\xi, z^m\eta\gg = 0$ if $n+m \neq 0$. Thus the bilinear form on $L\mathfrak{g}$ is necessarily the L^2 metric

$$\ll\beta_1,\beta_2\gg = \int_{S^1} \langle\, \beta_1(s), \beta_2(s)\,\rangle\, \frac{|ds|}{2\pi}, \qquad \beta_1,\beta_2 \in L\mathfrak{g}. \tag{2.25}$$

Notice that this form is symmetric and is also LG-invariant.

For G connected and simply connected

$$H^2(LG;\mathbb{R}) \cong \mathrm{Sym}^2(\mathfrak{g}^*)^G,$$

the correspondence given by associating to the G-invariant symmetric form $\langle\cdot,\cdot\rangle$ the left-invariant 2-form

$$\omega(\beta_1,\beta_2) = \int_{S^1} \langle\beta_1, d\beta_2\rangle, \qquad \beta_1,\beta_2 \in L\mathfrak{g},$$

on LG. Note that ω is the cocycle (2.22) which defines the central extension of $L\mathfrak{g}$, and for a central extension (2.6) of LG we interpret it as $1/i$ times the curvature of the left-invariant connection given by the splitting (2.20). In particular, it lies in a full lattice of forms determined by $H^2(LG; 2\pi\mathbb{Z}) \subset H^2(LG;\mathbb{R})$, and this lattice classifies central extensions of LG (see [PS, §4.4]). Any such extension extends to $\widehat{L}G = LG \rtimes \mathbb{T}_{\mathrm{rot}}$.

Tori. Let T be a torus group. Write $\mathfrak{t} = \mathrm{Lie}(T)$ and $\Pi = \exp^{-1}(1)/2\pi \subset \mathfrak{t}$, so that $T \cong \mathfrak{t}/\Pi$. The loop group has a canonical decomposition

$$LT \cong T \times \Pi \times U, \tag{2.26}$$

where

$$U = \exp V, \qquad V = \left\{\beta\,:\,S^1 \to \mathfrak{t} : \int_{S^1} \beta(s)\,ds = 0\right\}.$$

Note that the vector space $V \subset L\mathfrak{t}$ is isomorphic to $L\mathfrak{t}/\mathfrak{t}$. We can interpret $T \times \Pi$ as the Lie group of parametrized closed geodesics on T with respect to an invariant metric. An element $X \in \Pi$ corresponds to the one-parameter group $\varphi_X(s) = \exp(sX)$. Let $\Lambda = \mathrm{Hom}(T, \mathbb{T})$ be the $\mathbb{Z}$-dual of Π.

Proposition 2.27. *Suppose LT^τ is an admissible central extension of LT. Then:*

(i) *LT^τ is the product of central extensions $(T \times \Pi)^\tau$ and U^τ.*

(ii) *Write $[\beta_1, \beta_2] = \omega(\beta_1, \beta_2)K$ for $\beta_1, \beta_2 \in L\mathfrak{t}$. Then*

$$\omega(\beta_1, \beta_2) = \ll \beta_1, \dot{\beta}_2 \gg, \tag{2.28}$$

where $\dot{\beta}$ is the derivative $d\beta/ds$.

(iii) *The commutator in $(T \times \Pi)^\tau$ determines a homomorphism $\kappa : \Pi \to \Lambda$ and a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{t} \times \mathfrak{t}$. The form $\langle \cdot, \cdot \rangle$ is symmetric.*

The central extension U^τ of the vector space U is a *Heisenberg group* if the commutator pairing (2.28) is nondegenerate. Conditions (i) and (ii) are stated as *analytic regularity* in [FHT3, §2]; here they follow from the existence of $\ll \cdot, \cdot \gg$. For (iii) observe that the restriction $T^\tau \to T$ of $LT^\tau \to LT$ is necessarily split. Let $t \mapsto \tilde{t}$ be a splitting. Choose a lift $\tilde{\varphi}_X \in LT^\tau$ of $\varphi_X \in LT$ for each $X \in \Pi$. Then $\kappa_X \in \Lambda = \mathrm{Hom}(T, \mathbb{T})$ is defined by

$$\tilde{\varphi}_X \tilde{t} \tilde{\varphi}_X^{-1} \tilde{t}^{-1} = \kappa_X(t). \tag{2.29}$$

Let $\dot{\kappa}_X \in \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$ be $1/i$ times the derivative of κ_X; it defines a bilinear form

$$\langle X, \xi \rangle = -\dot{\kappa}_X(\xi)$$

which extends to all of $\mathfrak{t} \times \mathfrak{t}$ and takes integral values on $\Pi \times \Pi$. (The sign is for convenience.)

Proof. The assertion in (ii) follows immediately from the invariance of $\ll \cdot, \cdot \gg$, as in (2.23). This implies that the constant loops are orthogonal to loops which are derivatives, i.e., to elements of V.

By admissibility there is a central extension $\widehat{L}T^\tau \to LT \rtimes \widehat{\mathbb{T}}_{\mathrm{rot}}$ for a finite cover $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$ of degree δ. Choose a splitting $R_u \mapsto \tilde{R}_u$ for $R_u \in \widehat{\mathbb{T}}_{\mathrm{rot}}$, $0 \le u \le 2\pi\delta$. Then

$$\tilde{R}_u \tilde{\varphi}_X \tilde{R}_{-u} \tilde{\varphi}_X^{-1} = \widetilde{\exp(uX)} c_u(X) \tag{2.30}$$

in $\widehat{L}T$, where $c_u(X)$ is central. Write $c_u(X) = \exp\bigl(iu\dot{c}(X)\bigr)$. Imposing the group law we find for $X_1, X_2 \in \Pi$ that

$$\frac{c_u(X_1 + X_2)}{c_u(X_1) c_u(X_2)} = \kappa(X_2)(\exp uX_1),$$

from which $\langle \cdot, \cdot \rangle$ is symmetric and $\dot{c}(X) = -\frac{1}{2}\langle X, X - \eta_0 \rangle$ for some $\eta_0 \in \mathfrak{t}$. This proves (iii). Furthermore, from the condition $c_{2\pi\delta}(X) = 1$ we deduce $\frac{\delta}{2}\langle X, X - \eta_0 \rangle \in \mathbb{Z}$ for all $X \in \Pi$. Below we prove that $\eta_0 = 0$ and so this condition is always satisfied if $\delta = 2$ or if $\delta = 1$ and Π is *even*: $\langle X, X \rangle \in 2\mathbb{Z}$ for all $X \in \Pi$.

Let $U^\tau \to U$ be the restriction of $LT^\tau \to LT$ to U. Choose an embedding $L : V \to \mathrm{Lie}(L\mathfrak{t}^\tau)$ which splits the projection $\mathrm{Lie}(L\mathfrak{t}^\tau) \to \mathrm{Lie}(LT) \to V$, and for $v \in V$ set $\tilde{v} = \exp(Lv)$. For $X \in \Pi$ define the character $a_X : V \to \mathbb{T}$ by

$$\tilde{\varphi}_X \tilde{v} \tilde{\varphi}_X^{-1} \tilde{v}^{-1} = a_X(v). \tag{2.31}$$

Let $\dot{a}_X : V \to \mathbb{R}$ be $\frac{1}{i}$ times the derivative of a_X. Then use (2.29), (2.30), and (2.31) to see that for $X \in \Pi$, $\xi \in \mathfrak{t}$, and $v \in V$ we have

$$\mathrm{Ad}_{\tilde{\varphi}_X}(d) = d - X + \frac{1}{2}\langle X, X + \eta_0\rangle K, \tag{2.32}$$

$$\mathrm{Ad}_{\tilde{\varphi}_X}(\xi) = \xi + \dot{\kappa}_X(\xi)K, \tag{2.33}$$

$$\mathrm{Ad}_{\tilde{\varphi}_X}(v) = v + \dot{a}_X(v)K. \tag{2.34}$$

We apply the LT^τ-invariance of $\ll\cdot,\cdot\gg$ and (2.12), (2.13) to (2.32)–(2.34). Thus

$$\ll d, v\gg = \ll \mathrm{Ad}_{\tilde{\varphi}_X}(d), \mathrm{Ad}_{\tilde{\varphi}_X}(v)\gg = \ll d, v\gg - \dot{a}_X(v),$$

from which $\dot{a}_X(v) = 0$ and so $a_X(v) = 1$. Note that $\ll X, v\gg = 0$ by the argument in the first paragraph of the proof. Then (2.31) implies (i). Next,

$$\ll d, \xi\gg = \ll \mathrm{Ad}_{\tilde{\varphi}_X}(d), \mathrm{Ad}_{\tilde{\varphi}_X}(\xi)\gg = \ll d, \xi\gg - \ll X, \xi\gg - \dot{\kappa}_X(\xi),$$

from which $\ll X, \xi\gg = -\dot{\kappa}_X(\xi)$, and so

$$\ll \xi, \eta\gg = \langle \xi, \eta\rangle \quad \text{for all } \xi, \eta \in \mathfrak{t}.$$

From (2.32) and (2.14) we deduce

$$0 = \ll d, d\gg = \ll \mathrm{Ad}_{\tilde{\varphi}_X}(d), \mathrm{Ad}_{\tilde{\varphi}_X}(d)\gg = -\langle X, X + \eta_0\rangle + \ll X, X\gg,$$

and so $\eta_0 = 0$ as claimed earlier. □

Finite groups. For G finite and $P \to S^1$ a principal G-bundle, any central extension of L_PG is admissible. To see this, fix a basepoint $p \in P$ and suppose the holonomy is $h \in G$. Then we identify P with $(\mathbb{R} \times G)/\mathbb{Z}$, where $1 \in \mathbb{Z}$ acts as

$$(x, g) \longmapsto (x + 1, hg). \tag{2.35}$$

Then the "loop group" L_PG, or group of gauge transformations of P, is identified with $Z_h \subset G$, the centralizer of h. The extension $\widehat{L}_PG$ (with $\widehat{\mathbb{T}}_{\mathrm{rot}} = \mathbb{T}_{\mathrm{rot}}$) is identified with $(\mathbb{R} \times Z_h)/\mathbb{Z}$, where $1 \in \mathbb{Z}$ acts as in (2.35). Given a central extension Z_h^τ of Z_h, we fix a lift $\tilde{h} \in Z_h^\tau$ of h and define $(\widehat{L}_PG)^\tau$ as the quotient $(\mathbb{R} \times Z_h^\tau)/\mathbb{Z}$, with $\tilde{h}$ replacing h in (2.35). (Different choices of $\tilde{h}$ lead to isomorphic central extensions.)

§2.4. Finite energy loops. First, given $A \in \mathcal{A}_P$ we construct a decomposition of $L_P\mathfrak{g}$ into finite dimensional subspaces adapted to d_A. Denote by $Z_A \subset L_PG$ the stabilizer of $A \in \mathcal{A}_P$. Fix a G-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Using the basepoint $p \in P$ we identify the fiber $\mathfrak{g}_P$ at $0 \in S^1$ of the adjoint bundle with $\mathfrak{g}$. The holonomy $\mathrm{hol}(A)$ acts as an orthogonal transformation of $\mathfrak{g}$, thus as a unitary transformation of $\mathfrak{g}_{\mathbb{C}}$. Write the latter as $\exp(-2\pi i S_A)$ for some self-adjoint S_A with eigenvalues strictly between -1 and 1. Notice that S_A is not uniquely defined, nor can it be made to depend continuously on A. Decompose

$$\mathfrak{g}_{\mathbb{C}} \cong (\mathfrak{z}_A)_{\mathbb{C}} \oplus \mathfrak{p}_A \oplus \mathfrak{n}_A, \tag{2.36}$$

where $\mathfrak{z}_A$ is the Lie algebra of Z_A and S_A is positive on $\mathfrak{p}_A$, negative on $\mathfrak{n}_A$. It is useful to write

$$\mathfrak{p}_A = \bigoplus_{0<\epsilon<1} (\mathfrak{p}_A)_\epsilon, \tag{2.37}$$

where $S_A = \epsilon$ on $(\mathfrak{p}_A)_\epsilon$; there is a similar decomposition of $\mathfrak{n}_A$. By definition S_A vanishes on the Lie algebra $\mathfrak{z}_A$. Define

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &\longrightarrow L_P\mathfrak{g}_{\mathbb{C}} \\ \xi &\longmapsto \xi_A \end{aligned} \tag{2.38}$$

by letting $\xi_A(s)$ equal $\exp(isS_A)$ applied to the parallel transport of ξ using A. Then

$$[d_A, \xi_A] = iS_A(\xi)_A.$$

For $n \in \mathbb{Z}$ define

$$(z^n\xi_A)(s) = e^{ins}\xi_A(s), \tag{2.39}$$

so that

$$[d_A, z^n\xi_A] = iz^n(S_A + n)(\xi)_A.$$

The algebraic direct sum

$$(L_P\mathfrak{g}_{\mathbb{C}})_{\mathrm{fin}}(A) = \bigoplus_{n\in\mathbb{Z}} z^n\mathfrak{g}_{\mathbb{C}} \tag{2.40}$$

is dense in $L_P\mathfrak{g}_{\mathbb{C}}$. Elements of $(L_P\mathfrak{g}_{\mathbb{C}})_{\mathrm{fin}}(A)$ are said to have *finite energy*.

Let $(L_PG)^\tau \to L_PG$ be an admissible central extension. Recall from Lemma 2.18 that the connection defines a splitting $\beta \mapsto \beta^\tau_A$ of the loop algebra extension.

Lemma 2.41. *If $[d_A, \beta] = iE\beta$ in $L_P\mathfrak{g}$, then the lift of β to the central extension $(L_P\mathfrak{g})^\tau$ also has definite energy: $[d^\tau_A, \beta^\tau_A] = iE\beta^\tau_A$.*

Proof. Use the characterization (2.19) and the invariance of the bilinear form:

$$\ll d^\tau_A, [d^\tau_A, \beta^\tau_A] \gg_\tau = \ll [d^\tau_A, d^\tau_A], \beta^\tau_A \gg_\tau = 0.$$

Since $[d^\tau_A, \beta^\tau_A]$ is a lift of $iE\beta$ the result follows. □

§2.5. Positive energy representations. Following Segal [PS, §9] we define a distinguished class of projective representations of loop groups.

Definition 2.42. Fix $(L_PG)^\tau$ an admissible graded central extension of L_PG. Let $\rho : (L_PG)^\tau \to U(V)$ be a unitary representation on a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hilbert space. Assume that ρ is even and the center $\mathbb{T}$ acts by scalar multiplication. We say V has *positive energy* if:

(1) ρ extends to a unitary representation $\hat{\rho} : (\widehat{L}_PG)^\tau \to U(V)$. For $A \in \mathcal{A}_P$ denote the skew-adjoint operator $\dot{\hat{\rho}}(d^\tau_A)$ by iE_A.
(2) For all $A \in \mathcal{A}_P$ the *energy operator* E_A is self-adjoint with discrete spectrum bounded below.

A positive energy representation is said to be *finitely reducible* if it is a finite sum of irreducible representations.

We regard ungraded representations as $\mathbb{Z}/2\mathbb{Z}$-graded representations with odd part zero. For an irreducible positive energy representation the extension $\hat{\rho}$ is unique up to a character of $\widehat{\mathbb{T}}_{\mathrm{rot}}$, so the energy is determined up to a shift by n/δ, where $n \in \mathbb{Z}$ and δ is the degree of the cover $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$.

Let V be a positive energy representation of $(L_PG)^\tau$, and choose an extension to $(\widehat{L}_PG)^\tau$. Fix a connection A and decompose V as the Hilbert space direct sum of closed $\mathbb{Z}/2\mathbb{Z}$-graded subspaces $V_e(A)$, $e \in \mathbb{R}$ on which E_A acts as multiplication by e. The positive energy condition asserts that $V_e(A) \neq 0$ for a discrete set of e which is bounded below. The algebraic direct sum

$$V_{\mathrm{fin}}(A) = \bigoplus_{e\in\mathbb{R}} V_e(A) \tag{2.43}$$

consists of vectors of *finite energy*; it is dense in V. Now A defines a splitting of the Lie algebra central extension (see Lemma 2.18), and we use it to define the infinitesimal representation $\dot\rho_A$ on elements of $L_P\mathfrak{g}$; namely, $\dot\rho_A(\beta)$ denotes the infinitesimal action of β^τ_A, where $\beta \in L_P\mathfrak{g}$. The image of $\dot\rho_A$ consists of skew-Hermitian operators on V, which in general are unbounded. Note from Lemma 2.41 that the lift of $\beta \in L_P\mathfrak{g}$ with definite energy has the same energy. Suppose $\xi \in \mathfrak{g}_\mathbb{C}$ is an eigenvector of S_A of eigenvalue ϵ. It follows that

$$\dot\rho_A(z^n\xi_A)\big(V_e(A)\big) \subset V_{e+n+\epsilon}(A),$$

so in particular $\dot\rho_A(z^n\xi_A)\big(V_e(A)\big) = 0$ if $e+n+\epsilon$ is less than the minimal energy $e_{\min}$. A positive energy representation V has a minimal E_A-energy subspace $V_{e_{\min}}(A)$ which is a representation of Z^τ_A, the restriction of the central extension $(L_PG)^\tau$ over the stabilizer Z_A of $A \in \mathcal{A}_P$. Also, $\mathfrak{n}_A \oplus \bigoplus_{n>0} z^{-n}\mathfrak{g}_\mathbb{C}$ acts trivially on $V_{e_{\min}}(A)$. If V is irreducible, then $V_{\mathrm{fin}}(A)$ is the subspace spanned by vectors

$$\dot\rho_A\big(z^{n_r}(\xi_r)_A\big)\dots\dot\rho_A\big(z^{n_1}(\xi_1)_A\big)\Omega, \tag{2.44}$$

where $\Omega \in V_{e_{\min}}(A)$, $r \geq 0$, ξ_i is an eigenvector of S_A with eigenvalue ϵ_i, and $n_i \geq 0$ with equality only if $\epsilon_i > 0$. The energy of (2.44) is $e_{\min} + \sum(n_i+\epsilon_i)$. This proves the following.

Lemma 2.45. *Let V be a finitely reducible positive energy representation. Then for any connection A and $C > 0$ the sum of the E_A-eigenspaces for eigenvalue less than C is finite dimensional.*

In other words, E_A has the spectral characteristics of a first-order elliptic operator on the circle.

We use (2.14) to compute the dependence of the energy operator on the connection. Let A' denote the connection whose horizontal vector field is[21] $d_A - \beta$ for some $\beta \in L_P\mathfrak{g}$. Then

$$E_{A'} = E_A + i\dot\rho_A(\beta) + \frac{\ll\beta,\beta\gg}{2}. \tag{2.46}$$

The following proposition relates the existence of positive energy representations to the bilinear form on the central extension Definition 2.10.

Proposition 2.47. *Let $(L_PG)^\tau$ be an admissible central extension of L_PG which admits a nonzero positive energy representation V. Then the inner product $\ll\cdot,\cdot\gg_\tau$ on $L_P\mathfrak{g}$ is positive semidefinite.*

Proof. Fix a connection A. It suffices to prove that

$$\ll z^n\,\xi_A, z^{-n}\,\bar\xi_A \gg\ \geq\ 0 \qquad \text{for all } \xi \in (\mathfrak{z}_A)_\mathbb{C}\oplus\mathfrak{p}_A,\ n \geq 0. \tag{2.48}$$

(Recall the notation from (2.36), (2.38), and (2.39); also, we drop the subscript "τ" on the form, which is bilinear on the complexified loop algebra.) By Lemma 2.18

[21] The corresponding 1-form is $A + \beta\,ds$; see (2.17).

A defines a linear splitting $L_P\mathfrak{g} \to (L_P\mathfrak{g})^\tau$, which is characterized by (2.19); we denote it $\beta \mapsto \beta^\tau$. Then

$$[\beta_1^\tau, \beta_2^\tau] = [\beta_1, \beta_2]^\tau + \omega(\beta_1, \beta_2)\frac{K}{i}$$

for a real skew form ω, and

$$\ll d_A^\tau, [\beta_1^\tau, \beta_2^\tau] \gg \;=\; -\frac{\omega(\beta_1, \beta_2)}{i}.$$

For ξ, n as in (2.48) with ξ an eigenvector of S_A with eigenvalue $\epsilon > 0$ (see (2.37)), we find

$$\begin{aligned} \ll d_A^\tau, \big[(z^n\,\xi_A)^\tau, (z^{-n}\,\bar\xi_A)^\tau\big] \gg \;&=\; \ll [d_A^\tau, (z^n\,\xi_A)^\tau], (z^{-n}\,\bar\xi_A)^\tau \gg \\ &=\; i(n+\epsilon) \;\ll z^n\,\xi_A, z^{-n}\,\bar\xi_A \gg, \end{aligned}$$

whence

$$\omega(z^n\,\xi_A, z^{-n}\,\bar\xi_A) = (n+\epsilon) \;\ll z^n\,\xi_A, z^{-n}\,\bar\xi_A \gg .$$

Then for $\Omega \in V_{e_{\min}}(A)$ and $n > 0$, since $\dot\rho_A\big((z^n\,\xi_A)^\tau\big)^* = -\dot\rho_A\big((z^{-n}\,\bar\xi_A)^\tau\big)$ we have

$$\begin{aligned} 0 &\le \|\,\dot\rho_A\big((z^n\,\xi_A)^\tau\big)\Omega\,\|_V^2 \\ &= -\,\langle\, \dot\rho_A\big([(z^{-n}\,\bar\xi_A)^\tau, (z^n\,\xi_A)^\tau]\big)\Omega\,,\,\Omega\,\rangle_V \\ &= (n+\epsilon) \;\ll z^n\,\xi_A, z^{-n}\,\bar\xi_A \gg \;-\; \langle\, \dot\rho_A\big([\bar\xi_A, \xi_A]^\tau\big)\Omega\,,\,\Omega\,\rangle_V . \end{aligned} \tag{2.49}$$

Now we use the orthogonal decomposition (2.5). If $\xi \in \mathrm{center}(\mathfrak{g})$, then the last term of (2.49) vanishes and (2.48) follows immediately. On the other hand, if $\xi = [\eta, \eta']$, then $\xi_A = [\eta_A, \eta'_A]$ and

$$\begin{aligned} \ll z^n\,\xi_A\,,\, z^{-n}\,\bar\xi_A \gg \;&=\; \ll [\eta_A, z^n\eta'_A]\,,\, z^{-n}\,\bar\xi_A \gg \\ &=\; \ll \eta_A\,,\, [z^n\eta'_A, z^{-n}\,\bar\xi_A] \gg \\ &=\; \ll \eta_A\,,\, [\eta'_A, \bar\xi_A] \gg \\ &=\; \ll [\eta_A, \eta'_A]\,,\, \bar\xi_A \gg \\ &=\; \ll \xi_A\,,\, \bar\xi_A \gg . \end{aligned}$$

The argument applies to sums of terms $[\eta, \eta']$. If $\ll \xi_A, \bar\xi_A \gg < 0$, then for n large the right-hand side of (2.49) is negative, which is a contradiction. This completes the proof of (2.48). □

We state without proof some basic facts about positive energy representations. To avoid problems with abelian factors, we assume that the inner product $\ll\cdot,\cdot\gg$ is positive definite when restricted to $L_P(\mathfrak{z})$ for $\mathfrak{z} = \mathrm{center}(\mathfrak{g})$.

Positive energy representations are completely reducible [PS, §11.2].

For any irreducible positive energy representation, $V_e(A)$ is finite dimensional [PS, (9.3.4)].

If V is finitely reducible and $\hat\rho$ in Definition 2.42(1) is given, then if Definition 2.42(2) is satisfied for one $A \in \mathcal{A}_P$ it is satisfied for all $A \in \mathcal{A}_P$.

There is a finite number of isomorphism classes of irreducible positive energy representations of $(L_PG)^\tau$.

See [PS, (9.3.5)] and the remarks which follow for the last statement.

Following the usual procedure we make an abelian group out of positive energy representations.

Definition 2.50. Let $R^\tau(L_PG)$ denote the abelian group generated by isomorphism classes of finitely reducible $\mathbb{Z}/2\mathbb{Z}$-graded positive energy representations of $(L_PG)^\tau$ under direct sum, modulo the subgroup of isomorphism classes of representations which admit a commuting action of the Clifford algebra C_1^c.

We extend $R^\tau(L_PG)$ to a $\mathbb{Z}/2\mathbb{Z}$-graded free abelian group. The aforementioned representations are even. Odd representations in addition carry a (graded) commuting action of C_1^c. An odd representation is equivalent to zero if this action extends to a commuting action of C_2^c.

Example 2.51. To illustrate odd representations consider the cyclic group $\mathbb{Z}/2\mathbb{Z}$ of order two with generator x. With the trivial grading and no central extension there are two inequivalent irreducible even representations (of graded dimension $1|0$) and no nontrivial odd representations. Thus $K^0_{\mathbb{Z}/2\mathbb{Z}}$ has rank two and $K^1_{\mathbb{Z}/2\mathbb{Z}} = 0$. With the nontrivial grading ϵ and no central extension there is up to isomorphism one even irreducible representation on the $(1|1)$-dimensional space $\mathbb{C}\oplus\mathbb{C}$; the element x acts as $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. But this represents the trivial element of K-theory, as there is a commuting action of C_1^c where the generator acts as $\left(\begin{smallmatrix}0&i\\i&0\end{smallmatrix}\right)$. Together with this C_1^c-action this representation is nontrivial in odd K-theory, as there is no extension to an action of C_2^c. Therefore, $K^\epsilon_{\mathbb{Z}/2\mathbb{Z}} = 0$ and $K^{\epsilon+1}_{\mathbb{Z}/2\mathbb{Z}}$ has rank one.

§3. Dirac families and loop groups

§3.1. The spin representation (infinite dimensional case). For an algebraic approach, see [KS]; our geometric approach follows [PS, §12]. Let H be an infinite dimensional real Hilbert space. We define the Clifford algebra $\mathrm{Cliff}^c(H^*)$ as in finite dimensions (1.3), but now the construction of an irreducible Clifford module, and so of the spin representation, depends on a *polarization*. Recall that a complex structure on H is an orthogonal map $J : H \to H$ with $J^2 = -1$; it follows that J is skew-symmetric.

Definition 3.1. A *polarization* is a set $\mathcal{J}$ of complex structures maximal under the property that any two elements differ by a Hilbert-Schmidt operator. The *restricted orthogonal group* of $(H, \mathcal{J})$ is

$$\mathrm{O}_{\mathcal{J}}(H) = \{T \in \mathrm{O}(H) : TJT^{-1} \in \mathcal{J} \text{ for all } J \in \mathcal{J}\}.$$

The group $\mathrm{O}_{\mathcal{J}}(H)$ has two components, each simply connected with second homology free of rank one; in fact, $\mathrm{O}_{\mathcal{J}}(H)$ has the homotopy type of $\mathrm{O}_\infty / \mathrm{U}_\infty$. In particular, there is a unique nontrivial grading $\epsilon : \mathrm{O}_{\mathcal{J}}(H) \to \mathbb{Z}/2\mathbb{Z}$. Also, $\mathrm{O}_{\mathcal{J}}(H)$ acts transitively on $\mathcal{J}$ with contractible isotropy groups, so $\mathcal{J}$ has the same homotopy type as $\mathrm{O}_{\mathcal{J}}(H)$. There is a distinguished central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathrm{Pin}^c_{\mathcal{J}}(H) \longrightarrow \mathrm{O}_{\mathcal{J}}(H) \longrightarrow 1, \tag{3.2}$$

which, together with the grading, forms a graded central extension of $\mathrm{O}_{\mathcal{J}}(H)$. We remark that there is no infinite dimensional analog of the Pin group (1.1).

If $J_0 : H \to H$ is a real skew-adjoint Fredholm operator with nonzero eigenvalues $\pm i$ of infinite multiplicity—in other words, J_0 has finite dimensional kernel and is a complex structure on the orthogonal complement—then it determines a polarization

$$\mathcal{J} = \big\{J : J \text{ a complex structure, } J - J_0 \text{ Hilbert-Schmidt}\big\} \tag{3.3}$$

if $\dim\ker J_0$ is even. (Recall that the parity of $\dim\ker J_0$—the mod 2 index of J_0—is invariant under deformations.) If $\dim\ker J_0$ is odd, then define

$$\begin{aligned}\mathcal{J} = \big\{J : J \text{ real skew-adjoint Fredholm, } \dim\ker J = 1,\\ J \text{ a complex structure on } (\ker J)^{\perp},\ J - J_0 \text{ Hilbert-Schmidt}\big\}.\end{aligned} \tag{3.4}$$

(We require that the eigenvalues $\pm i$ have infinite multiplicity.) Call (3.4) an *odd* polarization; (3.3) is *even*. The kernels lead to an action of C_1^c on S which commutes with the $\mathrm{Pin}^c_{\mathcal{J}}(H)$ and $\mathrm{Cliff}^c(H^*)$-actions. As usual, we carry this extra structure implicitly.

The spin representation is a distinguished irreducible unitary representation

$$\chi : \mathrm{Pin}^c_{\mathcal{J}}(H) \to \mathrm{Aut}(S), \tag{3.5}$$

where $S = S^0 \oplus S^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded complex Hilbert space. It is constructed in [PS, §12] as the space of sections of a holomorphic line bundle over $\mathcal{J}$. There is a Clifford multiplication

$$\gamma : H^* \longrightarrow \mathrm{End}(S)$$

which is compatible with (3.5) in the sense that

$$\gamma(g\cdot\mu) = \chi(g)\gamma(\mu)\chi(g)^{-1}, \qquad g \in \mathrm{Pin}^c_{\mathcal{J}}(H), \quad \mu \in H^*. \tag{3.6}$$

We arrange that $\gamma(\mu)$ be skew-Hermitian. Clifford multiplication γ is odd, and χ is compatible with the gradings on $\mathrm{Pin}^c_{\mathcal{J}}(H)$ and S.

Let $P \to S^1$ be a principal G-bundle. Fix[22] a G-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, and so an L^2 inner product $\ll\cdot,\cdot\gg$ on $L_P\mathfrak{g}$ (as in (2.25)). Take H be the L^2 completion of $L_P\mathfrak{g}$. For $A \in \mathcal{A}_P$ the elliptic operator d_A extends to a skew-adjoint operator on H with discrete spectrum. There is an orthogonal decomposition of the complexification[23]

$$H_{\mathbb{C}} \cong \bigoplus_{e\in\mathbb{R}} H_e(A) \qquad (L^2 \text{ completed}), \tag{3.7}$$

where $d_A = ie$ on $H_e(A)$. Elements of $H_e(A)$ are said to have energy e. Define the *real* skew-adjoint Fredholm operator

$$J_A = \begin{cases} 0, & \text{on } H_0(A),\\ \dfrac{d_A}{|e|}, & \text{on } H_e(A).\end{cases} \tag{3.8}$$

The proofs of the next two lemmas are similar to [PS, (6.3.1)], so they are omitted. The loop group L_PG acts on the loop algebra $L_P\mathfrak{g}$ by the pointwise adjoint action of G on $\mathfrak{g}$.

Lemma 3.9. *The polarization $\mathcal{J}$ defined by J_A is independent of A.*

[22] Different choices lead to isomorphic constructions. In fact, we can use any invariant inner product $\ll\cdot,\cdot\gg$ on $L_P\mathfrak{g}$ with respect to which d_A is skew-Hermitian.

[23] Of course, (3.7) is the L^2 completion of (2.40). Below we denote the real points in the algebraic direct sum as H_{fin}. The finite dimensional subspace $H_e(A)$ contains smooth elements, i.e., lies in $L_P\mathfrak{g}_{\mathbb{C}}$.

Lemma 3.10. *Any $\varphi \in L_P G$ extends to a bounded orthogonal operator on H which preserves the polarization $\mathcal{J}$.*

Hence we have a homomorphism

$$L_P G \longrightarrow \mathrm{O}_{\mathcal{J}}(H), \tag{3.11}$$

and so by pullback of (3.2) a distinguished graded central extension

$$1 \longrightarrow \mathbb{T} \longrightarrow (L_P G)^\sigma \longrightarrow L_P G \longrightarrow 1 \tag{3.12}$$

and a graded unitary representation of $(L_P G)^\sigma$ on S.

Proposition 3.13. *The central extension $(L_P G)^\sigma$ is admissible with respect to the symmetric positive semidefinite bilinear form (2.16).*

Therefore, by Lemma 2.18 a connection $A \in \mathcal{A}_P$ defines a splitting

$$\begin{aligned} L_P\mathfrak{g} &\longrightarrow (L_P\mathfrak{g})^\sigma \\ \beta &\longmapsto \beta_A^\sigma \end{aligned}$$

of the spin central extension of the loop algebra.

Proof. The adjoint action factors through the semisimple adjoint group $\mathrm{Ad}\, G$, which leads to a factorization of (3.11) through a twisted loop group for $\mathrm{Ad}\, G$. Thus we may as well assume that G is semisimple. Fix a connection A on $P \to S^1$ and let φ_t be the one-parameter group in $\widehat{L}_P G$ generated by d_A. (We take $\widehat{\mathbb{T}}_{\mathrm{rot}} = \mathbb{T}_{\mathrm{rot}}$ in (2.2).) Then $\widehat{L}_P G$ is generated by $L_P G$ and $\{\varphi_t\}_{t\in\mathbb{R}}$. Now by (3.8) we have $\varphi_t^* J_A = J_A$, and so φ_t acts on $L_P\mathfrak{g}$ by an element of $\mathrm{O}_{\mathcal{J}}(H)$. Therefore (3.11) and (3.12) extend to $\widehat{L}_P G$. By Proposition 2.15 there is a unique bilinear form $\ll\cdot,\cdot\gg_\sigma$ on $(\widehat{L}_P\mathfrak{g})^\sigma$ compatible with the lifted rotation action.

We use an explicit model of S to compute $\ll\cdot,\cdot\gg_\sigma$. Dual to (3.7) is a decomposition

$$H_{\mathbb{C}}^* \cong \bigoplus_{e\in\mathbb{R}} H_e^*(A) \qquad (L^2 \text{ completed}), \qquad H_e^*(A) = \big(H_{-e}(A)\big)^*, \tag{3.14}$$

into finite dimensional subspaces of energy e. Then the dense finite energy subspace of the spin representation may be realized as (compare (1.16))

$$S_{\mathrm{fin}}(A) = S_0(A) \otimes {\textstyle\bigwedge}^\bullet\Big(\bigoplus_{e>0} H_e^*(A)\Big), \tag{3.15}$$

where $S_0(A)$ is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded module for the (finite-dimensional) Clifford algebra of $\mathfrak{z}_A^*$. (Recall that $\mathfrak{z}_A$ is the Lie algebra of the stabilizer Z_A of A.) Normalize the energy on S so elements of $S_0(A) \otimes \bigwedge^0$ have zero energy. Now the inner product $\ll\cdot,\cdot\gg$ on H induces a linear map $H_{\mathbb{C}} \to H_{\mathbb{C}}^*$, denoted $\beta \mapsto \beta^*$, which sends elements of energy e to elements of energy $-e$. Then for $\beta \in H_{-e}(A)$ we have $\beta^* \in H_e^*(A)$ and Clifford multiplication on $S_{\mathrm{fin}}(A)$ is given as

$$\gamma(\beta^*) = \begin{cases} 1 \otimes \epsilon(\beta^*), & e > 0, \\ \gamma_0(\beta^*) \otimes 1, & e = 0, \\ -1 \otimes \iota(\beta), & e < 0. \end{cases} \tag{3.16}$$

Here ϵ is exterior multiplication, ι is interior multiplication, and γ_0 is Clifford multiplication on $S_0(A)$. This satisfies the usual Clifford relation

$$\big[\gamma(\beta_1^*), \gamma(\beta_2^*)\big] = -2 \ll\beta_1, \beta_2\gg = -2 \ll\beta_1^*, \beta_2^*\gg \tag{3.17}$$

for β_1, β_2 of finite energy. On $S_{\mathrm{fin}}(A)$ the operator $\dot\chi\big((\beta)^\sigma_A\big)$ corresponding to β of finite energy is given by the first formula of (1.7), where now the sum is infinite as the indices range over a basis of finite energy vectors in $L_P\mathfrak{g}$; see (3.19) below. But only a finite number of terms are nonzero when acting on a fixed element of $S_{\mathrm{fin}}(A)$. (We formalize such infinite sums in §3.2.)

For the explicit computation we first treat the case $P \to S^1$ as trivial and choose $A = A_0$ as the trivial connection. Fix a basis e_a of $\mathfrak{g}_{\mathbb{C}}$, which we identify with constant loops in $L\mathfrak{g}_{\mathbb{C}}$. Then $\{z^n e_a\}_{n\in\mathbb{Z}}$ is an (algebraic) basis of $L\mathfrak{g}_{\mathbb{C}}$; let $\{z^{-n}e^a\}_{n\in\mathbb{Z}}$ be the dual basis. For any $n \in \mathbb{Z}$ let $\gamma^a(n)$ denote Clifford multiplication by $z^n e^a$; it has energy n. Now let $\xi, \eta \in \mathfrak{g}_{\mathbb{C}}$ be constant loops, and as usual $(z\xi)^\sigma_{A_0}, (z^{-1}\eta)^\sigma_{A_0}$ the lifts of the indicated loops to the central extension $(L\mathfrak{g})^\sigma_{\mathbb{C}}$. Then from the invariance of the bilinear form we see that it suffices to compute $\ll z^n\xi, z^m\eta \gg_\sigma$ for $n = -m = 1$ and also that

$$[(z\xi)^\sigma_{A_0}, (z^{-1}\eta)^\sigma_{A_0}] = [z\xi, z^{-1}\eta]^\sigma_{A_0} \;+\; \ll \xi, \eta \gg_\sigma \frac{K}{i}. \tag{3.18}$$

(The proof of Proposition 2.47 contains these assertions.) We evaluate the commutator using the infinitesimal spin representation, and it suffices to evaluate on a (vacuum) vector $\Omega \in S_0(A_0) \otimes \bigwedge^0$. Now the infinitesimal spin action is (cf. (1.7))

$$\dot\chi\big((z^n e_a)^\sigma_{A_0}\big) = \frac{1}{4} f_{abc} \sum_{k+\ell=n} \gamma^b(k)\gamma^c(\ell), \tag{3.19}$$

and as stated above the sum is finite on finite energy vectors. For example,

$$\dot\chi\big((z\xi)^\sigma_{A_0}\big)\Omega = \frac{1}{4} f_{abc}\xi^a[\gamma^b(1)\,\gamma^c(0) + \gamma^b(0)\,\gamma^c(1)]\,\Omega = \frac{1}{4} f_{abc}\xi^a[2\gamma^b(1)\,\gamma^c(0)]\Omega.$$

Hence

$$\begin{aligned} \big[\dot\chi((z\xi)^\sigma_{A_0}), \dot\chi((z^{-1}\eta)^\sigma_{A_0})\big]\Omega &= -\dot\chi\big((z^{-1}\eta)^\sigma_{A_0}\big)\dot\chi\big((z\xi)^\sigma_{A_0}\big)\Omega \\ &= -\frac{1}{4} f_{abc}f_{a'b'c'}\xi^a\eta^{a'}\gamma^{b'}(0)\,\gamma^{c'}(-1)\,\gamma^b(1)\,\gamma^c(0)\,\Omega \\ &= \frac{1}{2} f_{abc}f_{a'b'c'}g^{bc'}\xi^a\eta^{a'}\gamma^{b'}(0)\,\gamma^c(0)\,\Omega \\ &= \left[\frac{1}{4} f^b_{aa'}f_{bb'c}\gamma^{b'}(0)\,\gamma^c(0) - \frac{1}{2} f^c_{ab}f^b_{a'c}\right]\xi^a\eta^{a'}\Omega \\ &= \left[\dot\chi\big([\xi,\eta]^\sigma_{A_0}\big) - \frac{1}{2}\operatorname{Tr}(\operatorname{ad}\xi \circ \operatorname{ad}\eta)\right]\Omega. \end{aligned} \tag{3.20}$$

To pass from the third to fourth line we use the Jacobi identity and rearrange the terms. Recalling that K acts as multiplication by i in a representation, we see from (3.18) that

$$\ll \xi, \eta \gg_\sigma = -\frac{1}{2}\operatorname{Tr}(\operatorname{ad}\xi \circ \operatorname{ad}\eta), \tag{3.21}$$

which agrees with (2.16).

For the general (twisted) loop group the computation is similar, but now we use a generic connection A which satisfies (A.4), (A.5) in the appendix. (We freely use the notation of that appendix in this paragraph.) Then from (A.8) we have the basis $\{z^n\xi_i, z^n\chi_j, z^n\pi_{\lambda,e}, z^n\pi_{-\lambda,-e}\}_{n\in\mathbb{Z}}$ of $L_P\mathfrak{g}_{\mathbb{C}}$, but now the energies are not necessarily integers. Write this basis as $\{e_p\}$, let the structure constants be $F_{pqr} =$

$\ll [e_p, e_q], e_r \gg$, and let γ^p denote the Clifford multiplication. Then the usual formula $\dot\chi\big((e_p)^\sigma_A\big) = \frac14 F_{pqr}\gamma^q\gamma^r$ holds for basis elements e_p of *nonzero*[24] energy, so excludes the ξ_i. As before, it suffices to compute $\ll \xi,\xi \gg_\sigma$, $\ll \chi_j, z^{-1}\chi_{j'} \gg_\sigma$, and $\ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\sigma$, where $\xi \in \mathfrak{h}_0$ and $E(\chi_j) + E(\chi_{j'}) = 1$. Since $E(\xi) = 0$ and $[\xi,\xi] = 0$, invariance of the inner product implies

$$\big[\dot\chi\big((z\xi)^\sigma_A\big), \dot\chi\big((z^{-1}\xi)^\sigma_A\big)\big] = i \ll \xi,\xi \gg_\sigma K.$$

The computation of the bracket is similar to (3.20). For the remaining cases $\ll \chi_j, z^{-1}\chi_{j'} \gg_\sigma$ and $\ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\sigma$, which involve vectors of nonintegral energy, we compute the left-hand sides of

$$\begin{aligned}\big[\dot\chi\big((z\chi_j)^\sigma_A\big), \dot\chi\big((z^{-2}\chi_{j'})^\sigma_A\big)\big] \;-\; \big[\dot\chi\big((\chi_j)^\sigma_A\big), \dot\chi\big((z^{-1}\chi_{j'})^\sigma_A\big)\big]\\ = i \ll \chi_j, z^{-1}\chi_{j'} \gg_\sigma K\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\big[\dot\chi\big((z\pi_{\lambda,e})^\sigma_A\big), \dot\chi\big((z^{-1}\pi_{-\lambda,-e})^\sigma_A\big)\big] \;-\; \big[\dot\chi\big((\pi_{\lambda,e})^\sigma_A\big), \dot\chi\big((\pi_{-\lambda,-e})^\sigma_A\big)\big]\\ = i \ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\sigma K.\end{aligned} \tag{3.23}$$

In all cases we find (3.21), but we omit the details as the manipulations are similar to (3.20). □

We record some specific facts about the spin representation. Recall that $Z_A \subset L_PG$ is the stabilizer of a connection $A \in \mathcal{A}_P$ and $\mathfrak{z}_A$ its Lie algebra.

Lemma 3.24. (1) *The spin representation S of $(L_PG)^\sigma$ satisfies the positive energy condition. For any fixed $A \in \mathcal{A}_P$ we arrange that the minimal energy be zero and the minimal energy space $S_0(A)$ be an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded module for $\mathrm{Cliff}^c(\mathfrak{z}^*_A)$. The dense subspace $S_{\mathrm{fin}}(A)$ of finite energy vectors is generated by $\gamma(\beta^*_k)\cdots\gamma(\beta^*_2)\gamma(\beta^*_1)s_0$, for $\beta_i \in (L_P\mathfrak{g}_{\mathbb{C}})_{\mathrm{fin}}(A)$ and $s_0 \in S_0(A)$. Also, the restriction $Z^\sigma_A \to Z_A$ of $(L_PG)^\sigma \to L_PG$ acts on $S_0(A)$ compatibly with the $\mathrm{Cliff}^c(\mathfrak{z}^*_A)$-action, and $Z^\sigma_A \to Z_A$ is isomorphic to the graded central extension (1.4) (with $G = Z_A$). The splitting $\mathfrak{z}_A \to \mathfrak{z}^\sigma_A$ of the corresponding Lie algebra extension induced from Lemma 2.18 is a homomorphism of Lie algebras.*

(2) *Suppose $P \to S^1$ is trivial and A_0 is the trivial connection. Then $Z_A = G$ and the splitting $\mathfrak{g} \to \mathfrak{g}^\sigma$ from Lemma 2.18 agrees with the splitting in (1.5).*

Proof. Most of the statements in (1) are immediate from (3.15). For the penultimate assertion, note that Z^σ_A preserves energy, so acts on $S_0(A)$, and the isomorphism class of the graded central extension is determined by the action. As stated after (3.15), Z^σ_A acts via the finite dimensional spin representation, so the extension is isomorphic to (1.4). For the last assertion we use the characterization (2.19) of the splitting and note that if $(\zeta_1)^\sigma_A, (\zeta_2)^\sigma_A$ are lifts in $\mathfrak{z}^\sigma_A$, then

$$\ll d^\sigma_A, [(\zeta_1)^\sigma_A, (\zeta_2)^\sigma_A] \gg_\sigma = \ll [d^\sigma_A, (\zeta_1)^\sigma_A], (\zeta_2)^\sigma_A \gg_\sigma = 0,$$

[24]For the trivial connection (3.19) *is* valid for zero energy ($n = 0$). This may be verified from the $n \neq 0$ formula by computing

$$[\dot\chi((z^2\xi)^\sigma_{A_0}), \dot\chi((z^{-2}\eta)^\sigma_{A_0})] - [\dot\chi((z\xi)^\sigma_{A_0}), \dot\chi((z^{-1}\eta)^\sigma_{A_0})],$$

analogous to (3.22) and (3.23) below.

so that $[\zeta_1, \zeta_2]_A^\sigma = [(\zeta_1)_A^\sigma, (\zeta_2)_A^\sigma]$. Assertion (2) is the statement that (3.19) holds for $n = 0$; cf. the first equation of (1.7). $\square$

§3.2. The canonical 3-form on $L_P G$. Consider the 3-form

$$\Omega(\beta_1, \beta_2, \beta_3) = \ll (\beta_1, \beta_2), \beta_3 \gg_\sigma, \quad \beta_1, \beta_2, \beta_3 \in L_P\mathfrak{g}. \tag{3.25}$$

Since $\ll \cdot, \cdot \gg_\sigma$, defined in (2.16), is symmetric and $L_P G$-invariant, the form Ω is indeed skew-symmetric. Furthermore, Ω is d_A-invariant for all A, since $\ll \cdot, \cdot \gg_\sigma$ is $(\widehat{L}_P G)^\sigma$-invariant. Thus if $\beta_1, \beta_2, \beta_3$ have finite energy—i.e., live in the algebraic direct sum (2.40)—and are eigenvectors of d_A, then $\Omega(\beta_1, \beta_2, \beta_3)$ is nonzero only if the sum of the eigenvalues vanishes. Finally, Ω is $L_P G$-invariant as well. In infinite dimensions there is no natural notion of Clifford multiplication by a 3-form. Rather, to define the action of Ω on the spin representation S we use the energy operator of a connection A and work with finite energy spinor fields to define an operator Q_A on finite energy spinors. (See [L, §7] for another discussion.)

The subspace of finite energy vectors $H^*_{\mathrm{fin}}(A) \subset H^*$ has complexification the *algebraic* direct sum $H^*_{\mathrm{fin}}(A)_{\mathbb{C}} = \bigoplus_e H^*_e(A)$. Clifford multiplication on finite energy 1-forms (3.16) extends to an injective map

$$\mathrm{Cliff}^c(H^*_{\mathrm{fin}}(A)) \longrightarrow \mathrm{End}\big(S_{\mathrm{fin}}(A)\big). \tag{3.26}$$

Following [KS, §7] we endow $\mathrm{End}\big(S_{\mathrm{fin}}(A)\big)$ with the weak topology relative to the discrete topology on $S_{\mathrm{fin}}(A)$: a sequence $\{T_N\}_N \subset \mathrm{End}\big(S_{\mathrm{fin}}(A)\big)$ converges if and only if $T_N(s)$ stabilizes for all $s \in S_{\mathrm{fin}}(A)$ as $N \to \infty$. Furthermore, the standard filtration on $\mathrm{Cliff}^c(H^*_{\mathrm{fin}}(A))$ induces a filtration on the image of (3.26), and we let $E(A)_{\le p} \subset \mathrm{End}\big(S_{\mathrm{fin}}(A)\big)$ denote the weak closure of the image of $\mathrm{Cliff}^c(H^*_{\mathrm{fin}}(A))_{\le p}$. The $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathrm{End}\big(S_{\mathrm{fin}}(A)\big)$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading $E(A)^\bullet_{\le p} = E(A)^0_{\le p} \oplus E(A)^1_{\le p}$. Let $\mathrm{End}_e\big(S_{\mathrm{fin}}(A)\big)$ denote the endomorphisms of energy e, i.e., those which raise energy on S_{fin} by e, and set $E_e(A)^\bullet_{\le p} = E(A)^\bullet_{\le p} \cap \mathrm{End}_e\big(S_{\mathrm{fin}}(A)\big)$.

We now define an operator Q_A on $S_{\mathrm{fin}}(A)$ which plays the role of Clifford multiplication by Ω. Approximate (3.25) by truncation: let $\Omega_N(A) \in \bigwedge^3 H^*_{\mathrm{fin}}(A)$ be given by (3.25) on elements $\beta_1, \beta_2, \beta_3$ of definite energies not exceeding N in absolute value; $\Omega_N(A)$ vanishes if some absolute value of energy exceeds N. As in finite dimensions there is a vector space isomorphism[25] $\bigwedge^\bullet\big(H^*_{\mathrm{fin}}(A)\big) \cong \mathrm{Cliff}^c\big(H^*_{\mathrm{fin}}(A)\big)$, and the image of 3-forms lies in the odd Clifford algebra at filtration level ≤ 3. Noting that $\Omega_N(A)$ has energy zero, we see that its image $(Q_A)_N$ under (3.26) lies in $E_0(A)^1_{\le 3}$. It is straightforward[26] to check that for any $s \in S_{\mathrm{fin}}(A)$ the sequence $\{(Q_A)_N s\}_N$ stabilizes, so that $(Q_A)_N \to Q'_A \in E_0(A)^1_{\le 3}$ for some Q'_A. Also, $E_0(A)^1_{\le 1}$ is canonically the finite dimensional space $H^*_0 = (\mathfrak{z}_A)^*_{\mathbb{C}}$, acting by Clifford multiplication on $S_0(A) \otimes \bigwedge^0 \subset S_{\mathrm{fin}}(A)$.

Proposition 3.27. *There exists a unique $Q_A \in Q'_A + E_0(A)^1_{\le 1} \subset E_0(A)^1_{\le 3}$ such that for β of finite energy,*

$$-[\frac{1}{4} Q_A, \gamma(\beta^*)] = \dot{\chi}(\beta^\sigma_A), \qquad \beta \in H_{\mathrm{fin}}(A), \tag{3.28}$$

where $\dot{\chi} : (L_P\mathfrak{g})^\sigma_{\mathrm{fin}} \to \mathrm{End}\big(S_{\mathrm{fin}}(A)\big)$ is the infinitesimal spin representation. Furthermore, $A \mapsto Q_A$ is $(L_P G)^\sigma$-equivariant.

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[25] In fact, an isomorphism of left $\mathrm{Cliff}^c\big(H^*_{\mathrm{fin}}(A)\big)$-modules [KS, §4].

[26] Because of the skew-symmetry, there is no normal ordering necessary.

Proof. We compute the left-hand side of (3.28) for β of fixed energy (after complexification) and act on $s\in S_{\mathrm{fin}}(A)$ of fixed energy. Thus we can replace Q'_A by $(Q_A)_N$ for N sufficiently large and compute $-[\frac14(Q_A)_N,\gamma(\beta^*)]$ in $\mathrm{Cliff}^c\big(H^*_{\mathrm{fin}}(A)\big)$. This computation involves only finite sums, and in the notation of §1 (3.28) becomes the equation

$$-[\frac{1}{24}\gamma^a\gamma^b\gamma^c,g_{de}\gamma^e]=\sigma_d, \tag{3.29}$$

which follows immediately from (1.7). In our present context the indices $a,b,c,\dots$ now correspond to a basis of a finite dimensional subspace of the loop algebra. If β has nonzero energy, then β^σ_A also has a definite nonzero energy Lemma 2.41, and this is true of the commutator (3.28) as well. In this case shifting Q_A by an element of $E_0(A)^1_{\le 1}$ does not affect the commutator. For β of energy zero we restrict to $S_0(A)\otimes\bigwedge^0$, where the commutator $-[\frac14(Q_A)_0,\gamma(\beta^*)]$ gives the standard finite dimensional splitting $\mathfrak{z}_A\to\mathfrak{z}^\sigma_A$, by (3.29). The splitting $\beta\mapsto\beta^\sigma_A$ on the right-hand side of (3.28) differs[27] by a functional on $\mathfrak{z}_A$, and this specifies the difference $Q_A-Q'_A$ in $E_0(A)^1_{\le 1}$.

For the equivariance, if $\varphi\in(L_PG)^\sigma$, then $\chi(\varphi)\in\mathrm{Aut}(S)$ maps $S_{\mathrm{fin}}(A)$ to $S_{\mathrm{fin}}\big((\varphi^{-1})^*A\big)$, and also maps $\Omega_N(A)$ to $\Omega_N\big((\varphi^{-1})^*A\big)$. The splitting $\beta\mapsto\beta^\sigma_A$ is mapped to the splitting $\beta\mapsto\beta^\sigma_{(\varphi^{-1})^*A}$, and the equivariance of $A\mapsto Q_A$ follows:

$$Q_{(\varphi^{-1})^*A}=\chi(\varphi)\,Q_A\,\chi(\varphi)^{-1},\qquad \varphi\in(L_PG)^\sigma, \tag{3.30}$$

as operators on $S_{\mathrm{fin}}\big((\varphi^{-1})^*A\big)$.

□

§3.3. A family of cubic Dirac operators on loop groups; main theorem. Let $(L_PG)^\tau$ be an admissible graded central extension which is *positive definite*[28] in the sense that $\ll\cdot,\cdot\gg_\tau$ is positive definite on $L_P\mathfrak{g}$. Then we use $\ll\cdot,\cdot\gg=\ll\cdot,\cdot\gg_\tau$ to build the spin representation; see the text and footnote preceding (3.7). Suppose V is a $\mathbb{Z}/2\mathbb{Z}$-graded irreducible positive energy representation of $(L_PG)^{\tau-\sigma}$. Then $W=V\otimes S$ is a positive energy representation of $(L_PG)^\tau$.

Fix a connection $A\in\mathcal{A}_P$. Assume the minimal energy on V is $e_{\min}$; then $e_{\min}$ is the minimal energy on W and

$$W_{e_{\min}}(A)=V_{e_{\min}}(A)\otimes S_0(A) \tag{3.31}$$

is the minimal energy subspace. Since V is irreducible, $W_{e_{\min}}(A)$ is finite dimensional. Also, $S_0(A)$ is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cliff}^c(\mathfrak{z}^*_A)$-module, by Lemma 3.24. For $\beta\in L_P\mathfrak{g}$ let $\dot\rho_A(\beta)$ be the infinitesimal action of $\beta^\tau_A\in L_P\mathfrak{g}^\tau$ on W. Note $\dot\rho_A(\beta)$ is typically unbounded. If β has finite energy, then $\dot\rho_A(\beta)$ preserves the dense subspace $W_{\mathrm{fin}}(A)\subset W$ of finite energy vectors.

We first define the Dirac operator D_A as an unbounded (formally) skew-Hermitian operator on the (algebraic, incomplete) inner product space $W_{\mathrm{fin}}(A)$. Fix a Hilbert space basis $\{e_p\}$ of $H_{\mathbb{C}}$ such that each e_p has a definite energy; see (3.7). Let γ^p be the dual Clifford multiplication (3.16) on $S_{\mathrm{fin}}(A)$. Recall from Lemma 2.18 that

[27]For $P\to S^1$ trivial and $A=A_0$ the trivial connections, these splittings agree; see Lemma 3.24(2).

[28]In [FHT3] we consider more general central extensions which instead satisfy a nondegeneracy condition. Our assumption here implies that $\tau-\sigma$ is positive definite on $L_P\mathfrak{z}$, where $\mathfrak{z}=\operatorname{center}\mathfrak{g}$.

A defines a splitting of $(L_P\mathfrak{g})^{\tau-\sigma} \to L_P\mathfrak{g}$, and so an infinitesimal action $(R_p)_A \in \operatorname{End}\big(V_{\mathrm{fin}}(A)\big)$. Set

$$D_A = i\gamma^p (R_p)_A + \frac{i}{2}Q_A \qquad \text{on } W_{\mathrm{fin}}(A), \tag{3.32}$$

where Q_A is defined in Proposition 3.27. The sum in (3.32) is finite on any element of $W_{\mathrm{fin}}(A)$, so D_A is well defined. Let E_A be the energy operator on W.

Proposition 3.33. (1) *D_A is odd (formally) skew-Hermitian on $W_{\mathrm{fin}}(A)$ and $[D_A, E_A] = 0$.*

(2) *The dependence of D_A on A is*

$$D_{A+\beta ds} = D_A + \gamma(\beta^*), \qquad \beta \in L_P\mathfrak{g}. \tag{3.34}$$

(3) *The operator*

$$\mathcal{F}_A = D_A(1 - D_A^2)^{-1/2} \tag{3.35}$$

extends to a bounded odd skew-adjoint operator on W with closed range.

(4) *The assignment $A \to \mathcal{F}_A$ is $(L_PG)^\tau$-invariant.*

(5) *We have $\operatorname{Ker}\mathcal{F}_A \subseteq W_{e_{min}}(A)$. In particular, $\mathcal{F}_A$ is an odd skew-adjoint Fredholm operator.*

Proof. For (1) note from (3.14) that γ^p has energy opposite to that of $(R_p)_A$. Equation (3.34) follows from Lemma 2.18 and Proposition 3.27. Now (1) implies that D_A preserves the eigenspaces of E_A, and $1 - D_A^2$ is positive self-adjoint on each eigenspace. Hence (3.35) defines a bounded odd skew-adjoint operator on the dense subspace $W_{\mathrm{fin}}(A) \subset W$, so it extends by continuity to W. The L_PG^τ-invariance of $A \mapsto D_A$ follows from Lemma 2.18, (3.6), and (3.30); whence $A \mapsto \mathcal{F}_A$ is $(L_PG)^\tau$-invariant as well. This proves (1)–(4).

The extension guaranteed in Definition 2.42(1) is unique up to a character of $\mathbb{T}_{\mathrm{rot}}$, so the energy operator E_A is determined up to a constant independent of A. We claim

$$D_A^2 + 2E_A \text{ is constant on } W, \text{ independent of } A. \tag{3.36}$$

Then (5) follows immediately, since $\operatorname{Ker}\mathcal{F}_A = \operatorname{Ker} D_A^2$ and D_A^2 is nonpositive. As a first step to proving (3.36) we compute that for any $\beta \in L_P\mathfrak{g}$,

$$[D_A, \gamma(\beta^*)] = -2i\dot{\rho}_A(\beta). \tag{3.37}$$

To derive this use (3.17) for the first term of (3.32) and (3.28) for the second term. Next, square (3.34) and combine (3.37) with (3.17) and (2.46) to deduce that $D_A^2 + 2E_A$ is independent of A. Now we show that $D_A^2 + 2E_A$ commutes with the action of $(L_PG)^\tau$ and with any Clifford multiplication $\gamma(\beta^*)$. In fact, $g \in (L_PG)^\tau$ conjugates $D_A^2 + 2E_A$ to $D_{A'}^2 + 2E_{A'}$ for A' the transform of A under the image of g in L_PG. For D_A^2 this follows from (4) in the proposition; for E_A we use the fact that the connections d_A and $d_{A'}$ are conjugate and the nullity condition (2.14) is $(L_PG)^\tau$-invariant, since $\ll\cdot,\cdot\gg_\tau$ is $(L_PG)^\tau$-invariant. By the previous argument, $D_A^2 + 2E_A = D_{A'}^2 + 2E_{A'}$. As for the commutator with Clifford multiplication, we first use the infinitesimal version of the $(L_PG)^\tau$-invariance of $A \mapsto D_A$, obtained from (3.34), which asserts that

$$[D_A, \dot{\rho}_A(\beta)] = -\gamma\big([d_A, \beta]^*\big) = -i[E_A, \gamma(\beta^*)], \qquad \beta \in L_P\mathfrak{g}. \tag{3.38}$$

Iterating (3.37) and (3.38) we find

$$[D_A^2, \dot\rho_A(\beta)] = [-2E_A, \dot\rho_A(\beta)],$$
$$[D_A^2, \gamma(\beta^*)] = [-2E_A, \gamma(\beta^*)].$$

In particular, $[D_A^2 + 2E_A, \gamma(\beta^*)] = 0$, as desired. Finally, since S is an irreducible Clifford module and V an irreducible representation of $(L_PG)^\tau$, it follows that $D_A^2 + 2E_A$, which commutes with all operators from $(L_PG)^\tau$ and the Clifford algebra, is a constant on $W = V \otimes S$. □

Recall that the graded central extension $(L_PG)^\tau$ of L_PG defines a twisting τ of $K_G(G[P])$; see (2.9).

Corollary 3.39. *Let $(L_PG)^\tau$ be a positive definite admissible graded central extension of L_PG, and let V be a finitely reducible $\mathbb{Z}/2\mathbb{Z}$-graded positive energy representation of $(L_PG)^{\tau-\sigma}$. Then*

$$\begin{aligned} D(V) \,:\, \mathcal{A}_P &\longrightarrow \mathrm{Fred}(V\otimes S) \\ A &\longmapsto \mathcal{F}_A = D_A(1-D_A^2)^{-1/2} \end{aligned} \tag{3.40}$$

represents an element of $K_G^{\tau+\dim G}(G[P])$. Furthermore, it only depends on V up to isomorphism and is additive, so defines a map

$$\Phi \,:\, R^{\tau-\sigma}(L_PG) \longrightarrow K_G^{\tau+\dim G}(G[P]). \tag{3.41}$$

In the odd case the C_1^c-action comes from that on $S_0(A) \subset S$, since $\dim Z_A = \dim G$ (mod 2).

Proof. Recall from [FHT1, Appendix A.5] that the family $\{\mathcal{F}_A\}$ of skew-adjoint Fredholm operators represents an element of twisted K-theory if $\mathcal{F}_A^2 + 1$ is compact and $A \mapsto (\mathcal{F}_A, \mathcal{F}_A^2 + 1)$ is continuous as a map into the product $\mathcal{B}(W) \times \mathcal{K}(W)$ of bounded and compact operators. Here $\mathcal{B}(W)$ has the compact open topology and $\mathcal{K}(W)$ the norm topology. By the remarks in [AS, Appendix 1] for this purpose we can replace the compact open topology on $\mathcal{B}(W)$ with the strong operator topology. From (3.36) we see that $\mathcal{F}_A^2 + 1 = (1 - D_A^2)^{-1}$ is the inverse of $2E_A + C$ for some constant C, and the latter is a positive operator. By Lemma 2.45 the energy operator of a finitely reducible positive energy representation V has eigenvalues of finite multiplicity tending to infinity. The same is true for the spin representation S by explicit construction (3.15), so it is also true for the energy operator of the tensor product W, whence the inverse of $2E_A + C$ is compact.

For the continuity we prove first that $A \mapsto (\mathcal{F}_A^2 + 1) = (1 - D_A^2)^{-1}$ is norm continuous. To ease the notation set $y = D_A$ and $x = D_{A+\beta\,ds} = y + b$, where $b = \gamma(\beta^*)$ is a bounded operator continuously varying with β; see (3.34). Then

$$(1-x^2)^{-1} - (1-y^2)^{-1} = \left[(1-x^2)^{-1}x\right] b(1-y^2)^{-1} + (1-x^2)^{-1} b \left[y(1-y^2)^{-1}\right].$$

Consideration of finite energy vectors leads to the bounds $\|(1-x^2)^{-1}\| \le 1$, $\|x(1-x^2)^{-1}\| \le 1$ on the operator norms. The same estimates hold for y replacing x, so that

$$\|(1-x^2)^{-1} - (1-y^2)^{-1}\| \le 2\|b\| \le C\|\beta\|$$

for some constant C, which proves $A \mapsto \mathcal{F}_A^2 + 1$ is norm continuous into $\mathcal{K}(W)$. To show the strong continuity of $A \mapsto \mathcal{F}_A$ we write

$$(1-x^2)^{-1/2}x - (1-y^2)^{-1/2}y = \Big[(1-x^2)^{-1/2} - (1-y^2)^{-1/2}\Big]\, y + (1-x^2)^{-1/2}b. \tag{3.42}$$

Now the square root map $z \mapsto z^{1/2}$ on bounded positive operators is continuous (even analytic) in the uniform topology [H2]. Thus since $A \mapsto (1-D_A^2)^{-1}$ is norm continuous, so is $A \mapsto (1-D_A^2)^{-1/2}$. It then follows directly from (3.42) that $A \mapsto \mathcal{F}_A(v)$ is continuous for $v \in W_{\mathrm{fin}}(A)$. The strong continuity of $A \mapsto \mathcal{F}_A$ is a consequence of the following lemma, since $\|\mathcal{F}_A\| \le 1$.

Lemma 3.43. *Let W be a Hilbert space, $V \subset W$ a dense subspace, and $T : X \to B(W)$ a family of bounded operators parametrized by a space X. Assume that $x \mapsto T_x v$ is continuous for all $v \in V$ and that $\|T_x\| \le C$ for all $x \in X$. Then T is strongly continuous.*

Proof. For $w \in W$ we must show that $x \mapsto T_x w$ is continous. Fix $x_0 \in X$ and a sequence $v_n \to w$ with $v_n \in V$. Now given $\epsilon > 0$ choose N so that $\|w - v_N\| < \epsilon/4C$ and a neighborhood U of x_0 so that $\|(T_x - T_{x_0})v_N\| < \epsilon/2$ for all $x \in U$. Then

$$\|(T_x - T_{x_0})w\| \le \|(T_x - T_{x_0})(w - v_N)\| + \|(T_x - T_{x_0})v_N\| < 2C\frac{\epsilon}{4C} + \frac{\epsilon}{2} = \epsilon$$

if $x \in U$, which proves the continuity. □

□

We can now assert our main theorem.

Theorem 3.44. *Let $(L_P G)^\tau$ be a positive definite admissible graded central extension of $L_P G$. The map $\Phi : R^{\tau-\sigma}(L_P G) \to K_G^{\tau + \dim G}(G[P])$ is an isomorphism of graded free abelian groups.*

The proof of Theorem 3.44 in the general case is in [FHT3, §13]. In the next section we present the proof in case G is connected and $\pi_1 G$ is torsion-free.

§4. Proofs

Let G be a connected Lie group. A principal G-bundle $P \to S^1$ is then trivializable, so in the sequel we take it to be the trivial bundle $P = S^1 \times G$. Let $A_0 \in \mathcal{A}_P$ be the trivial connection. Since $P \to S^1$ is trivial the extended loop group is $\widehat{L}_P G \cong LG \rtimes \widehat{\mathbb{T}}_{\mathrm{rot}}$ for some finite cover of $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$ of degree δ. An admissible central extension necessarily has the form $(\widehat{L}_P G)^\tau \cong LG^\tau \rtimes \widehat{\mathbb{T}}_{\mathrm{rot}}$. It follows that on any positive energy representation the energy operator E_{A_0} satisfies $\exp(2\pi i \delta E_{A_0}) = \mathrm{id}$, so its eigenvalues are $1/\delta$ times integers. By tensoring with a character of $\widehat{\mathbb{T}}_{\mathrm{rot}}$ we normalize the minimal E_{A_0}-energy to be zero. We allow the central extension LG^τ to have a nontrivial grading. Now the stabilizer $Z_{A_0} \subset LG$ of the trivial connection A_0 is the group of constant loops G. Let $G^\tau \to G$ be the restriction of the graded central extension $LG^\tau \to LG$ to the constant loops. It has a trivial grading since G is connected.

Assume the admissible graded central extension $LG^\tau \to LG$ is positive definite, i.e., the form $\ll\cdot,\cdot\gg_\tau$ is positive definite on $L\mathfrak{g}$. Thus it restricts to a positive definite form $\langle\cdot,\cdot\rangle_\tau$ on constant loops $\mathfrak{g}\subset L\mathfrak{g}$, which in turn induces an isomorphism

$$\begin{aligned}\kappa^\tau\,:\,\mathfrak{g}&\longrightarrow\mathfrak{g}^*\\ \xi&\longmapsto\xi^*,\end{aligned}\tag{4.1}$$

where $\xi^*(\eta)=\langle\xi,\eta\rangle_\tau$ for all $\eta\in\mathfrak{g}$.

Fix a maximal torus $T\subset G$ with Lie algebra $\mathfrak{t}\subset\mathfrak{g}$, let W be the associated Weyl group, and define the dual lattices $\Pi=\mathrm{Hom}(\mathbb{T},T)$ and $\Lambda=\mathrm{Hom}(T,\mathbb{T})$. Recall from §1.5 the affine space $\mathcal{A}_T^\tau$ and the subset $\Lambda^\tau\subset\mathcal{A}_T^\tau$ which is a Λ-torsor. The lattice $\Pi=\exp^{-1}(1)/2\pi\subset\mathfrak{t}$ acts by translation on Λ^τ via the map κ^τ. The Weyl group W acts as well, and these actions fit together into an action of the *extended affine Weyl group* $W_{\mathrm{aff}}^e=\Pi\rtimes W$. Let $(\Lambda^\tau)^{\mathrm{reg}}\subset\Lambda^\tau$ be the subset of *affine regular*[29] weights.

To prove Theorem 3.44 for G connected with π_1G torsion-free we first identify the set of isomorphism classes of irreducible positive energy representations of $LG^{\tau-\sigma}$ with certain W_{aff}^e-orbits of projective weights. This step is essentially a quotation from [PS, §9]. There is a correspondence between conjugacy classes in G and W_{aff}^e-orbits in $\mathcal{A}_T^{\tau-\sigma}$, so each irreducible representation determines a conjugacy class. Collectively these are the *Verlinde conjugacy classes*. The main step in the proof is the assertion that the kernel of the Dirac family (3.40) associated to a given irreducible representation is supported on the conjugacy class which corresponds to the given irreducible. Then the following results from [FHT1] (Theorem 4.27 and Proposition 4.41) complete the proof.

Theorem 4.2. (i) *For G connected of rank n with π_1G torsion-free,*

$$K_G^{\tau+q}(G)\cong\begin{cases}\mathrm{Hom}_{W_{\mathit{aff}}^e}\big(\Lambda^\tau,H_c^n(\mathfrak{t})\otimes\mathbb{Z}(\epsilon)\big), & q\equiv n\pmod 2,\\ 0, & q\not\equiv n\pmod 2.\end{cases}\tag{4.3}$$

The W_{aff}^e-action on Λ^τ is described in the previous paragraph; $W\subset W_{\mathit{aff}}^e$ acts on $H_c^n(\mathfrak{t})$ naturally via the sign representation; the grading of τ determines a homomorphism $\epsilon:\Pi\to\mathbb{Z}/2\mathbb{Z}$, and $\Pi\subset W_{\mathit{aff}}^e$ acts on $\mathbb{Z}(\epsilon)$ via this sign.

(ii) *If $k:H\hookrightarrow G$ also has rank n and is also connected with π_1H torsion-free, then $k^*:K_G^{\tau+q}(G)\to K_H^{k^*\tau+q}(H)$ is, for $q\equiv n\pmod 2$, the natural inclusion*

$$\mathrm{Hom}_{W_{\mathit{aff}}^e(G)}\big(\Lambda^\tau,H_c^n(\mathfrak{t})\otimes\mathbb{Z}(\epsilon)\big)\longhookrightarrow\mathrm{Hom}_{W_{\mathit{aff}}^e(H)}\big(\Lambda^\tau,H_c^n(\mathfrak{t})\otimes\mathbb{Z}(\epsilon)\big).\tag{4.4}$$

(iii) *Orient T and let $j:\{t\}\hookrightarrow T$ be the inclusion of a point. Then the pushforward j_* for $q\equiv\dim T\pmod 2$ is computed as the map*

$$\begin{tikzcd} K_T^{i^*k^*\tau+q}\big(\{t\}\big) \arrow[r, "j_*"] \arrow[d] & K_T^{k^*\tau+q}(T) \arrow[d] \\ \mathbb{Z}[\Lambda^\tau] \arrow[r] & \mathrm{Hom}_\Pi\big(\Lambda^\tau,\mathbb{Z}(\epsilon)\big) \end{tikzcd}\tag{4.5}$$

[29]A weight $\mu\in\Lambda^\tau$ is called *affine regular* if it has trivial stabilizer under the action of W_{aff}^e. Equivalently, by Lemma 4.9(1) below, if $\mu=\kappa^\tau(\xi)$ for $\xi\in\mathfrak{t}$, then μ is affine regular if and only if $\exp(2\pi\xi)\in T$ is regular in the sense that no nontrivial element of the Weyl group W fixes it.

which takes $\mu \in \Lambda^\tau$ to the Π-equivariant function $f : \Lambda^\tau \to \mathbb{Z}$ which is supported on the Π-orbit of μ and satisfies $f(\mu) = 1$.

Notice that an orientation of T identifies the compactly supported cohomology $H^n_c(\mathfrak{t})$ with $\mathbb{Z}$. The nonzero abelian group in (4.3) is free, and the set of *regular* W^e_{aff}-orbits in Λ^τ provides a generating set.

We carry out the proof separately for tori and simply connected groups, then combine them to prove the general case.

§4.1. The proof for tori. Let $G = T$ be a torus with Lie algebra $\mathfrak{t}$ and $\Pi = \mathrm{Hom}(\mathbb{T}, T)$. Recall the decomposition $LT \cong T \times \Pi \times U$ in (2.26). By Proposition 2.27(1) any admissible central extension LT^τ is a product of Heisenberg central extensions $(T \times \Pi)^\tau$ and U^τ. We first assume the grading of τ is trivial; below we discuss the modifications for nontrivial grading. Since τ is positive definite, $L\mathfrak{t}/\mathfrak{t}$ is a symplectic vector space and its Heisenberg extension U^τ has a unique irreducible positive energy representation $\mathcal{H}$. (See [PS, §9.5].) Fix an orbit $\tilde{\mathcal{O}}$ of the Π-action on Λ^τ and consider the constant vector bundle with fiber $\mathcal{H}$ over $\tilde{\mathcal{O}}$. The space $V_{\tilde{\mathcal{O}}}$ of L^2 sections (with respect to the measure which assigns unit mass to each point of $\tilde{\mathcal{O}}$) is an irreducible representation $V_{\tilde{\mathcal{O}}}$ of LT^τ: the subgroup U^τ acts on each fiber; Π^τ permutes the fibers; and T^τ acts on the fiber at $\lambda \in \Lambda^\tau \subset \mathrm{Hom}(T^\tau, \mathbb{T})$ by scalar multiplication, the character λ defining the multiplication. These are all of the irreducible positive energy representations up to isomorphism.

Proposition 4.6 [PS, (9.5.11)]. *The isomorphism classes of irreducible positive energy representations of LT^τ are in 1:1 correspondence with the orbits of Π on Λ^τ, i.e., with the points of $\Lambda^\tau/\kappa^\tau(\Pi)$.*

Turning to the Dirac family associated to the irreducible representation $V_{\tilde{\mathcal{O}}}$, let S be the spin representation of LT and set $W_{\tilde{\mathcal{O}}} = V_{\tilde{\mathcal{O}}} \otimes S$. Any connection on the trivial bundle over S^1 is gauge equivalent to $A_0 + \xi\, ds$ for some constant $\xi \in \mathfrak{t}$. Let D_ξ denote the corresponding Dirac operator (3.32) and E_ξ the associated energy operator. According to (3.36) if $D_\xi(w) = 0$ for some $w \in W_{\tilde{\mathcal{O}}}$ of unit norm, then $\langle E_\xi(w), w\rangle_W$ is a global minimum of E_ξ over all $\xi \in \mathfrak{t}$ and all unit norm $w \in W_{\tilde{\mathcal{O}}}$. The energy operator on $\mathcal{H}$ has discrete nonnegative spectrum and a one-dimensional kernel spanned by a unit norm vector $\Omega \in \mathcal{H}$. Let Ω_λ denote the copy of Ω in the fiber at $\lambda \in \tilde{\mathcal{O}}$. Then (2.32), (2.33), and (2.46) imply

$$\big\langle E_\xi(\Omega_\lambda), \Omega_\lambda \big\rangle_{\mathcal{H}} = \frac{1}{2}\left|\lambda - \kappa^\tau(\xi)\right|^2_\tau + C \tag{4.7}$$

for some real constant C. We can take $C = 0$ so that the global minimum of energy is zero. Let S_0 denote the zero energy subspace of S; it is an irreducible Clifford module for the algebra $\mathrm{Cliff}(\mathfrak{t}^*)$. For each $\lambda_0 \in \tilde{\mathcal{O}}$ there is a unique $\xi_0 \in \mathfrak{t}$ which makes (4.7) vanish—it satisfies $\kappa^\tau(\xi_0) = \lambda_0$—and these ξ_0 form a Π-orbit in $\mathfrak{t}$. For each such ξ_0 the kernel of D_{ξ_0} is $\mathbb{C} \cdot \Omega_{\lambda_0} \otimes S_0$. Furthermore, (3.34) shows that D_ξ acts on $\mathbb{C} \cdot \Omega_{\lambda_0} \otimes S_0$ as Clifford multiplication by $\kappa^\tau(\xi - \xi_0)$. Now Π-orbits in $\mathfrak{t}$ correspond to elements of T, and from this point of view the support of the kernel is a single element in T. More precisely, the linear splitting of Lemma 2.18 for the trivial connection $A = A_0$, restricted to constant loops, is a basepoint in the affine space $\mathcal{A}^\tau_T$. Hence (4.1) may be regarded as an isomorphism $\kappa^\tau : \mathfrak{t} \to \mathcal{A}^\tau_T$, which in turn induces an isomorphism

$$\bar{\kappa}^\tau : T = \mathfrak{t}/\Pi \longrightarrow \mathcal{A}^\tau_T/\kappa^\tau(\Pi).$$

The preceding proves the following.

Proposition 4.8. *Let $V_{\tilde{\mathcal{O}}}$ be the irreducible positive energy representation of LT^τ which corresponds to $\tilde{\mathcal{O}} \in \Lambda^\tau/\kappa^\tau(\Pi) \subset \mathcal{A}_T^\tau/\kappa^\tau(\Pi)$. Define $t \in T$ by $\bar{\kappa}^\tau(t) = \tilde{\mathcal{O}}$. Then the kernel of the Dirac family associated to $V_{\tilde{\mathcal{O}}}$ is supported on $\{t\} \subset T$. Let $i : \{t\} \to T$ be the inclusion, $i_* : K_T^{i^*\tau}(\{t\}) \to K_T^{\tau+\dim T}(T)$ the induced pushforward, and $K \to \{t\}$ the T^τ-line with character λ, where λ is some element of $\tilde{\mathcal{O}} \subset \Lambda^\tau$. Then the K-theory class of the Dirac family is $i_*[K]$.*

Theorm 3.44 for $G = T$ is now a direct consequence of Proposition 4.8 and Theorem 4.2.

A grading of LT is a homomorphism $LT \to \mathbb{Z}/2\mathbb{Z}$, and it necessarily factors through a homomorphism $\epsilon : \Pi \to \mathbb{Z}/2\mathbb{Z}$ since $\Pi \cong \pi_0(LT)$. Suppose now the admissible central extension LT^τ includes the grading ϵ. The representation $V_{\tilde{\mathcal{O}}}$ associated to a Π-orbit $\tilde{\mathcal{O}} \subset \Lambda^\tau$ depends mildly on a basepoint $\lambda \in \tilde{\mathcal{O}}$. Namely, define a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle over $\tilde{\mathcal{O}}$ whose fiber at $\pi \cdot \lambda$, $\pi \in \Pi$, is $\mathcal{H}$ with grading $\epsilon(\pi)$. Then $V_{\tilde{\mathcal{O}},\lambda}$ is the $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space of L^2 sections. A change of basepoint may replace $V_{\tilde{\mathcal{O}},\lambda}$ with the oppositely graded representation, whose equivalence class in the ring $R^\tau(LT)$ is negative to that of $V_{\tilde{\mathcal{O}},\lambda}$. (See Definition 2.50.) The analysis of the Dirac operator proceeds as before, but now the kernel of the Dirac family is a $\mathbb{Z}/2\mathbb{Z}$-graded line bundle over the orbit $\tilde{\mathcal{O}}$. The function f which represents $i_*(\lambda)$ in (4.5) is the rank of this kernel; it takes values ± 1 due to the nontrivial action of Π on $\mathbb{Z}(\epsilon)$ in $\operatorname{Hom}_\Pi\big(\Lambda^\tau, \mathbb{Z}(\epsilon)\big)$.

§4.2. The proof for simply connected groups. Let G be a simply connected compact Lie group and fix a positive definite admissible central extension LG^τ of its loop group. The grading ϵ is trivial since LG is connected. Fix a maximal torus T and choice of Weyl chamber in $\mathfrak{t}$. The set of conjugacy classes in G is isomorphic to the set of W-orbits in T. Let Δ be the set of roots of G and Δ^+ the set of positive roots. The following lemma is standard; see [PS, §5.1] for example.

Lemma 4.9. (1) $\exp(2\pi\ \cdot) : \{W_{\mathit{aff}}^e\text{-}\mathit{orbits\ in}\ \mathfrak{t}\} \longrightarrow \{W\text{-}\mathit{orbits\ in}\ T\}$ *is a bijection.*

(2) *Define the* alcove

$$\mathfrak{a} = \{\xi \in \mathfrak{t} : -1 < \alpha(\xi) < 0 \text{ for all } \alpha \in \Delta^+\}. \tag{4.10}$$

Then $\mathfrak{a}$ is contained in a Weyl chamber of $\mathfrak{t}$ and its closure $\bar{\mathfrak{a}}$ is a fundamental domain for the W_{aff}^e-action on $\mathfrak{t}$.

We do not review here the construction of positive energy representations, but merely quote the classification. Recall that LG^σ is the central extension defined by the spin representation. Let E_0 be the energy operator of the connection A_0.

Proposition 4.11 [PS, (9.3.5)]. (i) *If $\tau - \sigma$ is not positive semidefinite on $L\mathfrak{g}$, then there are no nonzero positive energy representations of $LG^{\tau-\sigma}$.*

(ii) *If $\tau - \sigma$ is positive semidefinite, then the isomorphism classes of irreducible positive energy representations of $LG^{\tau-\sigma}$ are in 1:1 correspondence with the orbits of W_{aff}^e on $\Lambda^{\tau-\sigma}$, i.e., with the points of $\Lambda^{\tau-\sigma}/\kappa^{\tau-\sigma}(W_{\mathit{aff}}^e)$. Furthermore, the representation V which corresponds to $-\lambda \in \kappa^{\tau-\sigma}(\bar{\mathfrak{a}})$ has zero E_0-energy space V_0 the irreducible representation of G with lowest weight $-\lambda$.*

Assertion (i) is Proposition 2.47. There is a ρ-shifted restatement of (ii) as follows. First, σ restricts on point loops to the finite dimensional spin extension by Lemma 3.24(i), and Lemma 1.15 shows $\rho \in \Lambda^\sigma$, where ρ is half the sum of the positive roots. The map $-\lambda \mapsto -(\lambda+\rho)$ maps $\Lambda^{\tau-\sigma} \cap \kappa^{\tau-\sigma}(\bar{\mathfrak{a}})$ isomorphically to $\Lambda^\tau \cap \kappa^\tau(\mathfrak{a})$, which consists of *affine regular* weights [A, 5.62]. Then the isomorphism classes of irreducible positive energy representations of $LG^{\tau-\sigma}$ are in 1:1 correspondence with the set of affine regular W^e_{aff}-orbits in Λ^τ, i.e., with the points of $(\Lambda^\tau)^{\mathrm{reg}}/\kappa^\tau(W^e_{\mathrm{aff}})$.

We turn now to the Dirac family (3.40), assuming that $\tau - \sigma$ is positive semidefinite. Let V be a $\mathbb{Z}/2\mathbb{Z}$-graded irreducible positive energy representation V of $LG^{\tau-\sigma}$ whose zero E_0-energy space has lowest weight $-\lambda_0$. The minimal E_0-energy subspace (3.31) of $W = V \otimes S$ is

$$W_0 = V_0 \otimes S_0,$$

where S_0 is the zero E_0-energy subspaces. Note that V_0 is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded representation of $G^{\tau-\sigma}$ and S_0 is a finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded representation of G^σ. Lemma 1.15 implies that a lowest weight of W_0 is $-\mu_0 = -(\lambda_0 + \rho)$. Write $-\mu_0 = \kappa^\tau(\xi_0)$ for $\xi_0 \in \mathfrak{a}$. Since $\bar{\mathfrak{a}}$ is a fundamental domain for the W^e_{aff}-action on $\mathfrak{t}$, any connection on the trivial bundle is gauge equivalent to $A_0 + \xi\, ds$ for $\xi \in \bar{\mathfrak{a}}$. Denote the associated Dirac and energy operators as D_ξ, E_ξ.

Proposition 4.12. (1) $\operatorname{Ker} D_\xi \subset W_0$ *if* $\xi \in \bar{\mathfrak{a}}$.

(2) *The restriction of D_ξ to W_0 may be identified with the operator $D_\mu(V_0)$ of (1.32) with $\mu = \kappa^\tau(\xi)$.*

Recall that the operators D_μ in (1.32) are parametrized by $\mu \in \mathcal{A}^\tau_G \cong \mathcal{A}^{\tau-\sigma}_G$, where the isomorphism is given by the canonical splitting (1.5) of the Lie algebra extension at level σ. Observe that the linear splitting of Lemma 2.18 for the trivial connection $A = A_0$, restricted to constant loops, gives a basepoint in $\mathcal{A}^\tau_G$ and $\mathcal{A}^{\tau-\sigma}_G$, and by Lemma 3.24 these correspond under the isomorphism $\mathcal{A}^\tau_G \cong \mathcal{A}^{\tau-\sigma}_G$. We use the basepoint to regard κ^τ as a map $\mathfrak{g} \to \mathcal{A}^\tau_G$.

Proof. As in the discussion preceding (4.7) $D_\xi = 0$ precisely on vectors in W which realize the global minimum of E_ξ (over all ξ). Now W is an irreducible representation of the semidirect product of $\mathrm{Cliff}^c(L\mathfrak{g}^*)$ and $L\mathfrak{g}^\tau$, so the set of vectors of the form

$$w = \gamma\big((z^{-n_{r+s}}\eta_{r+s})^*\big)\dots\gamma\big((z^{-n_{r+1}}\eta_{r+1})^*\big)\dot{\rho}_{A_0}(z^{n_r}\eta_r)\dots\dot{\rho}_{A_0}(z^{n_1}\eta_1)\Omega,$$
$$r, s \geq 0, \quad \eta_i \in \mathfrak{g}, \quad n_i > 0, \quad \Omega \in W_0,$$

is dense in W. Here γ is the Clifford action (3.16) and $\dot{\rho}_{A_0}$ the action of $L\mathfrak{g}$ defined after (2.43). Assume that η_i lies in the α_i-root space of $\mathfrak{g}$ or in $\mathfrak{t}$, in which case $\alpha_i = 0$. Since $\dot{\rho}_A(z^{n_i}\eta_i) = \dot{\rho}_{A_0}(z^{n_i}\eta_i)$ (see Lemma 2.18 and observe $\ll z^{n_i}\eta_i, \xi \gg = 0$ since $n_i > 0$) we see that the E_ξ-energy of w minus the E_ξ-energy of Ω is

$$\sum_{i=1}^{r+s}\big(n_i + \alpha_i(\xi)\big). \tag{4.13}$$

For $\xi \in \bar{\mathfrak{a}}$ this is nonnegative, by (4.10), which implies: For $\xi \in \bar{\mathfrak{a}}$, if $\operatorname{Ker} D_\xi \neq 0$, then $\operatorname{Ker} D_\xi \cap W_0 \neq 0$. Suppose Ω lies in the μ-weight space of W_0. Then by (2.46) its E_ξ-energy is

$$\mu(\xi) + \frac{\|\xi\|_\tau^2}{2},$$

which has a minimum value of $-\|\mu\|_\tau^2/2$ uniquely realized at $\xi \in \mathfrak{t}$ which satisfies $\kappa^\tau(\xi) = -\mu$. So the global minimum occurs for the lowest weight $\mu = -\mu_0$ at the point $\xi = \xi_0 \in \bar{\mathfrak{a}}$. Furthermore, since $|\alpha_i(\xi_0)| < 1$ by (4.10) we see from (4.13) that E_{ξ_0} does not realize its minimum on any E_0-eigenspace W_n for $n > 0$.

For (2) consider first $\xi = 0$. In the restriction of the first term in (3.32) to W_0 only terms of zero energy contribute to the infinite sum. That this is the first term of (1.32) follows from the remarks preceding the proof. As for the second term in (3.32), in view of Lemma 3.24(2) the characterization (3.28) of Q_{A_0}, restricted to S_0, also characterizes the second term in (1.32). This proves (2) for $\xi = 0$. For $\xi \neq 0$ it follows from (3.34) that $D_\xi = D_0 + \gamma(\xi^*)$, which matches (1.33) for $\nu = \xi^*$. $\square$

Proposition 1.19 and Proposition 1.24, or rather their projective analogs described in §1.5, translate to an explicit description of $\operatorname{Ker} D_\xi$, $\xi \in \bar{\mathfrak{a}}$, hence of the K-theory class $\Phi(V)$ in (3.41); we use Proposition 4.12 to reduce to consideration of the family of operators $D_\mu(V_0)$. (The Dirac operator D_ξ is invertible on the orthogonal complement $W_0^\perp$ to W_0, so the Dirac family restricted to $W_0^\perp$ represents the zero element of twisted K-theory.) Denote by $k : T \hookrightarrow G$ the inclusion of the maximal torus, and let $N \subset G$ be the normalizer of T. There is a restriction map

$$k^* : K_G^{\tau+\bullet}(G) \longrightarrow K_T^{k^*\tau+\bullet}(T)$$

computed topologically in Theorem 4.2(ii). The family of operators $D(V)$ parametrized by $\mathcal{A}_P$ represents a class $[D(V)] \in K_G^{\tau+\dim G}(G)$ (see (3.40)); its restriction to $\operatorname{hol}^{-1}(T)$—connections with holonomy in T—is denoted $k^*D(V)$ and it represents $k^*\big[D(V)\big] \in K_T^{k^*\tau+\dim T}(T)$. (We use periodicity and $\dim G \equiv \dim T$ (mod 2).)

Proposition 4.14. *Let V be an irreducible positive energy representation of $LG^{\tau-\sigma}$ whose zero E_0-energy space has lowest weight $-\lambda_0$. Define $\xi_0 \in \mathfrak{a}$ by $\kappa^\tau(\xi_0) = -(\lambda_0 + \rho)$. Then the kernel of $k^*D(V)$ is supported on* $\operatorname{hol}^{-1}(\mathcal{O})$ *for $i : \mathcal{O} \hookrightarrow T$ the* regular *W-orbit which contains $t = \exp(\xi_0)$. Also,*

$$k^*\big[D(V)\big] = i_*[K \otimes L] \tag{4.15}$$

is the image of the K-theory class of an N^τ-equivariant line bundle $K \otimes L \to \mathcal{O}$ under the pushforward

$$i_* : K_N^{i^*k^*\tau}(\mathcal{O}) \longrightarrow K_T^{i^*k^*\tau}(\mathcal{O}) \longrightarrow K_T^{k^*\tau+\dim T}(T). \tag{4.16}$$

The stabilizer $T^{\tau-\sigma} \subset N^{\tau-\sigma}$ acts on the fiber K_t by the character $-\lambda_0$ and T^σ acts on L by the character $-\rho$.

Proof. The pushforward (4.16) is realized as multiplication by spinors on the tangent space $\mathfrak{t}$, which is identified with $\mathfrak{t}^*$ via κ^τ (4.1). The description of the kernel follows directly from Proposition 1.19 and (4.15) follows from (1.30). For the last statement, see Proposition 1.24. $\square$

Proof of Theorem 3.44 (*G simply connected*). Consider the composition

$$R^{\tau-\sigma}(LG) \xrightarrow{\Phi} K_G^{\tau+\dim G}(G) \xrightarrow{k^*} K_T^{k^*\tau+\dim T}(T). \tag{4.17}$$

First, the distinguished basepoint of $\mathcal{O}$ in $\mathcal{O}\cap\kappa^\tau(\mathfrak{a})$ leads to an isomorphism $K_N^{i^*k^*\tau}(\mathcal{O}) \cong \mathbb{Z}[\Lambda^\tau]$ and to a function $\mathcal{O}\to\mathbb{Z}/2\mathbb{Z}$ which is the sign representation of the Weyl group after identifying $\mathcal{O}\cong W$ using the basepoint. Then if $t\in\mathcal{O}$ the restriction map $K_N^{i^*k^*\tau}(\mathcal{O}) \to K_T^{i^*k^*\tau}(\{t\})$ is $\mu\mapsto\pm\mu$ according to this sign, and so by Theorem 4.2(iii) the map (4.16) is $\mu\mapsto f_\mu$, where $f_\mu : \Lambda^\tau\to\mathbb{Z}$ is the Π-invariant function supported on the W_{aff}^e-orbit of μ with $f_\mu(w\cdot\mu) = (-1)^{\text{sign}(w)}$ for $w\in W$. Proposition 4.11 identifies $R^{\tau-\sigma}(LG)$ as the free abelian group on $\kappa^{\tau-\sigma}(\bar{\mathfrak{a}})\cap\Lambda^{\tau-\sigma}$, and by Proposition 4.14 the composition (4.17) sends $-\lambda_0\in\kappa^{\tau-\sigma}(\bar{\mathfrak{a}})\cap\Lambda^{\tau-\sigma}$ to f_μ for $\mu = -(\lambda_0+\rho)$. It follows from Theorem 4.2(ii) that these f_μ form a basis for the image of k^* in (4.17), and so Φ is an isomorphism. $\square$

We remark that if $\tau-\sigma$ is not positive semidefinite, then both sides of (3.41) vanish.

§4.3. The proof for connected G with π_1 torsion-free. If G is connected with $\pi_1 G$ free, then there is a finite cover

$$1\to A\to\tilde{G}\to G\to 1$$

such that $\tilde{G} = Z_1\times G'$ is the product of a torus and a simply connected compact Lie group G'. This follows easily from the proof of [FHT1, Lemma 4.1]. There is an induced exact sequence of groups

$$1\to A\to L\tilde{G}\xrightarrow{i} LG\to A\to 1, \tag{4.18}$$

where A is included in $L\tilde{G}$ as the point loops, $L\tilde{G}$ maps onto a union of components $i(L\tilde{G})\subset LG$, and the last map factors as $LG\to\pi_0 LG\cong\pi_1 G\to A$. Note also $L\tilde{G}\cong LZ_1\times LG'$.

Lemma 4.19. *The pullback $L\tilde{G}^{\tilde{\tau}-\tilde{\sigma}}$ factors as $\tilde{\tau}-\tilde{\sigma} = (\tau_Z)\times(\tau'-\sigma')$, i.e., the spin extension σ_Z is trivial and there are no cross terms. Similarly, $\tilde{\tau} = \tau_Z\times\tau'$.*

Proof. The vanishing of σ_Z is immediate from the triviality of the pointwise adjoint action (3.11). The vanishing of the cross terms is equivalent to the triviality of the homomorphism $LZ_1\to\operatorname{Hom}(LG',\mathbb{T})$ defined by conjugation in the central extension. But a homomorphism $\phi : LG'\to\mathbb{T}$ is determined by its derivative $\dot{\pi} : L\mathfrak{g}'\to i\mathbb{R}$, which vanishes on commutators. Since commutators span $L\mathfrak{g}'$, we deduce that ϕ is trivial. $\square$

Fix a maximal torus $T'\subset G'$, set $\tilde{T} = Z_1\times T'$ a maximal torus of $\tilde{G}$, let $T\subset G$ be its image under the projection $\tilde{G}\to G$, and denote the extended affine Weyl groups as $W_{\text{aff}}^{e\,\prime}, \tilde{W}_{\text{aff}}^e, W_{\text{aff}}^e$. The fundamental groups of $T',\tilde{T},T$ are $\Pi',\tilde{\Pi},\Pi$. The weight lattices are $\Lambda = \operatorname{Hom}(T,\mathbb{T})$ and $\tilde{\Lambda} = \operatorname{Hom}(Z_1,\mathbb{T})\times\operatorname{Hom}(T',\mathbb{T})$. Orient T and Z_1. Note the exact sequence $0\to\tilde{\Pi}\to\Pi\to A\to 0$. For readability we delete the map $\kappa : \Pi\to\Lambda$ and its superscripted cousins from the notation.

Lemma 4.20. *There is a commutative diagram*

$$\begin{tikzcd} \mathbb{Z}\big[\Lambda^{\tau-\sigma}/W^e_{\mathit{aff}}\big] \arrow[r, hook] & \mathbb{Z}\big[\tilde\Lambda^{\tilde\tau-\tilde\sigma}/\tilde W^e_{\mathit{aff}}\big] \\ R^{\tau-\sigma}(LG) \arrow[u, "\cong"] \arrow[r, hook] \arrow[d, "\Phi_G"'] & R^{\tilde\tau-\tilde\sigma}(L\tilde G) \arrow[u, "\cong"'] \arrow[d, "\Phi_{\tilde G}"] \\ K_G^{\tau+\dim G}(G) \arrow[r, hook] \arrow[d, "\cong"'] & K_{\tilde G}^{\tilde\tau+\dim G}(\tilde G) \arrow[d, "\cong"] \\ \mathrm{Hom}_{W^e_{\mathit{aff}}}\big(\Lambda^\tau, \mathbb{Z}(\epsilon)\big) \arrow[r, hook] & \mathrm{Hom}_{\tilde W^e_{\mathit{aff}}}\big(\tilde\Lambda^{\tilde\tau}, \mathbb{Z}(\epsilon)\big) \end{tikzcd} \tag{4.21}$$

in which the horizontal arrows are injective.

Proof. The top right vertical isomorphism follows from Proposition 4.6 and Proposition 4.11. The lower isomorphisms are (4.3). For the isomorphism in the upper left we factorize the second line as

$$R^{\tau-\sigma}(LG) \longrightarrow R^{\tau-\sigma}\big(i(L\tilde G)\big) \longrightarrow R^{\tilde\tau-\tilde\sigma}(L\tilde G). \tag{4.22}$$

The exact sequences (4.18) and $0 \to \Lambda \to \tilde\Lambda \to A^\vee \to 0$ imply

$$R^{\tau-\sigma}\big(i(L\tilde G)\big) \cong \mathbb{Z}\big[\Lambda^{\tau-\sigma}/\tilde W^e_{\mathrm{aff}}\big], \tag{4.23}$$

since a representation of $(L\tilde G)^{\tilde\tau-\tilde\sigma}$ drops to the quotient if and only if its restriction to $A \subset L\tilde G$ is trivial. Now $A \cong LG^{\tau-\sigma}/i(L\tilde G)^{\tau-\sigma}$ acts on the set of isomorphism classes of irreducible positive energy representations of $i(L\tilde G)^{\tau-\sigma}$, which under the isomorphism in (4.23) corresponds to the A-action on $\Lambda^{\tau-\sigma}/\tilde W^e_{\mathrm{aff}}$ induced from the Π-action on $\Lambda^{\tau-\sigma}$. We claim that this action is free. To see this recall first $\tilde W^e_{\mathrm{aff}} = \Pi_Z \times {W^e_{\mathrm{aff}}}'$. The Π_Z action on $\Lambda^{\tau-\sigma}/{W^e_{\mathrm{aff}}}'$ is free, as Π_Z acts by translations on $\tilde\Lambda^{\tilde\tau-\tilde\sigma}/{W^e_{\mathrm{aff}}}' \cong \Lambda_Z^{\tau_Z} \times (\Lambda')^{\tau'-\sigma'}/{W^e_{\mathrm{aff}}}' \supset \Lambda^{\tau-\sigma}/{W^e_{\mathrm{aff}}}'$. Since $\Pi/\Pi' \cong \pi_1 G$ is torsion-free, and $\Pi/\Pi' \supset \Pi_Z$ with finite quotient A, it follows that Π/Π' acts freely on $\Lambda^{\tau-\sigma}/{W^e_{\mathrm{aff}}}'$ as well. Hence A acts freely on $\Lambda^{\tau-\sigma}/\tilde W^e_{\mathrm{aff}}$. Since the set of A-orbits in $\Lambda^{\tau-\sigma}/\tilde W^e_{\mathrm{aff}}$ is isomorphic to $\Lambda^{\tau-\sigma}/W^e_{\mathrm{aff}}$, the upper left isomorphism follows.[30] Furthermore, this argument shows that each of the arrows in (4.22) is injective, and now the vertical isomorphisms in (4.21) prove that the top two horizontal arrows in that diagram are injective.

[30] That is, an irreducible representation of $LG^{\tau-\sigma}$ corresponds to an orbit of irreducible representations of $i(L\tilde G)^{\tau-\sigma}$. It may be that A carries a nontrivial grading even if the restriction of $\tau-\sigma$ to $i(L\tilde G)$ does not—see Example 4.28—but the conclusion is the same since the A-action is free: the irreducible representations in an orbit, when summed to construct an irreducible representation of $LG^{\tau-\sigma}$, may be even or odd.

The bottom half of (4.21) factors as

$$\begin{tikzcd} K_G^{\tau+\dim G}(G) \arrow[r] \arrow[d, "\cong"'] & K_{\tilde G}^{\tau+\dim G}(G) \arrow[r] \arrow[d, "\cong"'] & K_{\tilde G}^{\tilde\tau+\dim G}(\tilde G) \arrow[d, "\cong"] \\ \operatorname{Hom}_{W^e_{\mathrm{aff}}}\bigl(\Lambda^\tau, \mathbb{Z}(\epsilon)\bigr) \arrow[r] & \operatorname{Hom}_{\tilde W^e_{\mathrm{aff}}}\bigl(\Lambda^\tau, \mathbb{Z}(\epsilon)\bigr) \arrow[r] & \operatorname{Hom}_{\tilde W^e_{\mathrm{aff}}}\bigl(\tilde\Lambda^{\tilde\tau}, \mathbb{Z}(\epsilon)\bigr) \end{tikzcd} \tag{4.24}$$

The bottom left map is the obvious inclusion and the bottom right map is extension by zero, which is also injective. Hence the bottom two arrows in (4.21) are injective.

It remains to show that the center square commutes. For this we observe that the space of connections on the trivial G bundle over S^1 is naturally isomorphic to the space of connections on the trivial $\tilde G$ bundle over S^1. Furthermore, the map i in (4.18) induces an isomorphism of Lie algebras. A positive energy representation V of $LG^{\tau-\sigma}$ pulls back to a positive energy representation $\tilde V$ of $L\tilde G^{\tilde\tau-\tilde\sigma}$, and using these isomorphism we see that the associated Dirac families (3.32) coincide. □

Proof of Theorem 3.44 (*G connected, $\pi_1 G$ torsion-free*). The arguments in §4.1 and §4.2 prove that $\Phi_{\tilde G}$ in (4.21) is an isomorphism. It then follows immediately from Lemma 4.20 that Φ_G is injective. The groups at the top left and bottom left of (4.21) are free of equal rank (the number of regular W^e_{aff}-orbits in Λ^τ), and from the text following (4.17) we see that a natural set of generators of $R^{\tau-\sigma}(LG)$ maps under $\Phi_{\tilde G}$ to an independent set of indivisible elements in $K_{\tilde G}^{\tilde\tau+\dim G}(\tilde G)$. It now follows that Φ_G is surjective, hence an isomorphism. □

§4.4. Examples.

Example 4.25 (G simple, simply connected)**.** These have a well-known classification: $G = \mathrm{SU}_n$, Spin_n, Sp_n, G_2, F_4, E_6, E_7, E_8. A central extension LG^τ is determined up to isomorphism by the inner product $\langle\cdot,\cdot\rangle_\tau$ on $\mathfrak{g}$ (see §2.3), or equivalently by the induced map $\kappa^\tau : \mathfrak{t} \to \mathfrak{t}^*$. The one-dimensional real vector space of invariant inner products is naturally identified with $H^3(G;\mathbb{R})$ using a multiple of the formula in (1.6):

$$\begin{gathered} \langle\cdot,\cdot\rangle \longleftrightarrow \frac{1}{8\pi^2}\,\Omega, \\ \Omega(\xi_1,\xi_2,\xi_3) = \langle[\xi_1,\xi_2],\xi_3\rangle. \end{gathered} \tag{4.26}$$

The forms which arise from central extensions are *integral*, i.e., correspond to elements of $H^3(G;\mathbb{Z})$. Let $\langle\cdot,\cdot\rangle$ denote the distinguished positive integral generator. Then any other integral invariant form is an integer—the *level*—times the generator. For example, the level of the spin extension LG^σ is the *dual Coxeter number* $\check h(G)$. Suppose LG^τ is a positive definite admissible central extension, and let the level of $LG^{\tau-\sigma}$ be k. Then the positive definiteness means $k > -\check h(G)$. If $k < 0$, then there are no nonzero positive energy representations at level k (Proposition 4.12). Thus assume $k \ge 0$. Let $\alpha \in \mathfrak{t}^*$ be the highest root. Then the lowest weights $-\lambda$ in the image of $\kappa^{\tau-\sigma}(\bar{\mathfrak{a}})$ satisfy λ is dominant with $\langle\lambda,\alpha\rangle \le k$. The kernel of any Dirac family is supported on a finite set of *regular* conjugacy classes in G, namely those whose $k + \check h(G)$ power contains the identity element.

Example 4.27 ($G = \mathrm{SU}_2$)**.** This is the simplest case of Example 4.25, and we spell out the details a bit. First, the dual Coxeter number is $\check h(\mathrm{SU}_2) = 2$. Let

$T \subset \mathrm{SU}_2$ be the standard maximal torus of diagonal matrices, and identify its Lie algebra $\mathfrak{t}$ with $i\mathbb{R}$ as usual: $\left(\begin{smallmatrix} ia & 0 \\ 0 & -ia \end{smallmatrix}\right) \leftrightarrow ia$. Choose the alcove $\mathfrak{a} = \big(ia : -1/2 < a < 0\big) \subset \mathfrak{t}$. The affine Weyl group is $W^e_{\mathrm{aff}} \cong Z \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathfrak{t}$ as translation by i and the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts as the reflection $ia \mapsto -ia$. The generating integral inner product is[31] $\langle A, A'\rangle = -\operatorname{Tr}(AA')$, which restricts on $\mathfrak{t}$ to $\langle ia, ia'\rangle = 2aa'$. We use $\langle\cdot,\cdot\rangle$ to identify $\mathfrak{t}^* \cong i\mathbb{R}$. The fundamental group Π is identified with $i\mathbb{Z} \subset i\mathbb{R} \cong \mathfrak{t}$ and the weight lattice Λ with $\frac{i}{2}\mathbb{Z}$. A positive definite admissible central extension LSU_2^τ has $\langle\cdot,\cdot\rangle_\tau = (k+2)\langle\cdot,\cdot\rangle$ for some $k > -1$, and then $(\mathrm{LSU}_2)^{\tau-\sigma}$ is the extension at level k. The map $\kappa^{\tau-\sigma} : \mathfrak{t} \to \mathfrak{t}^*$ is identified with multiplication by k, so the image of $\bar{\mathfrak{a}}$ is the closed interval in $i\mathbb{Z}$ from $-ik/2$ to 0, which includes $k+1$ elements of Λ. So there are $k+1$ irreducible positive energy representations of $(\mathrm{LSU}_2)^{\tau-\sigma}$. The map κ^τ is identified with multiplication by $k+2$, and in the induced W^e_{aff}-action on $\mathfrak{t}^*$ the generator of $\mathbb{Z}$ translates by $i(k+2)$. There are $k+1$ *affine regular* W^e_{aff}-orbits on Λ; a slice is given by the set $i\{\frac{1}{2}, 1, \dots, \frac{k+1}{1}\}$. The corresponding conjugacy classes in SU_2 consist of matrices with eigenvalues $\pm\frac{2\pi i\ell}{2(k+2)}$, where $\ell = 1, 2, \dots, k+1$. The kernel of the Dirac family (3.40) for an irreducible positive energy representation has support on a single such conjugacy class.

Example 4.28 ($G = \mathrm{U}_2$)**.** The group of components of the loop group LU_2 is isomorphic to $\mathbb{Z}$: a generator is the loop $\varphi(z) = \left(\begin{smallmatrix} z & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We claim that the spin extension LU_2^σ, defined in (3.12), has the nontrivial grading, which is the nontrivial homomorphism $\mathrm{LU}_2 \to \pi_0(\mathrm{LU}_2) \to \mathbb{Z}/2\mathbb{Z}$. Furthermore, the loop group $L\tilde{G}$ of the double covering group $\tilde{G} = \mathrm{SU}_2 \times \mathbb{T}$ also has group of components isomorphic to $\mathbb{Z}$, and the induced map $\pi_0(L\tilde{G}) \to \pi_0(LG)$ is multiplication by 2. So the pullback $L\tilde{G}^{\tilde{\sigma}}$ central extension has trivial grading. Notice that for any central extension LU_2^τ, either τ or $\tau - \sigma$ has a nontrivial grading.

To verify the claim we write an arbitrary loop in $L\mathfrak{u}_2$ as

$$\begin{pmatrix} ix & \zeta \\ -\bar{\zeta} & iy \end{pmatrix} + \sum_{n>0} \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} z^n - \sum_{n>0} \begin{pmatrix} \bar{\alpha}_n & \bar{\gamma}_n \\ \bar{\beta}_n & \bar{\delta}_n \end{pmatrix} z^{-n}. \tag{4.29}$$

The operator (3.8) (for the trivial connection $A = A_0$) defines a polarization $\mathcal{J}$, and in that polarization (3.3) we choose the particular complex structure J which maps (4.29) to

$$\begin{pmatrix} iy & i\zeta \\ i\bar{\zeta} & -ix \end{pmatrix} + \sum_{n>0} \begin{pmatrix} i\alpha_n & i\beta_n \\ i\gamma_n & i\delta_n \end{pmatrix} z^n + \sum_{n>0} \begin{pmatrix} i\bar{\alpha}_n & i\bar{\gamma}_n \\ i\bar{\beta}_n & i\bar{\delta}_n \end{pmatrix} z^{-n}.$$

The only change to J under conjugation by φ is to its action on ζ, which changes sign. It follows that $\varphi J \varphi^{-1}$ and J are in opposite components of $\mathcal{J}$, and so under (3.11) the loop φ maps to the nonidentity component of $\mathrm{O}_{\mathcal{J}}(H)$.

Example 4.30. ($G = \mathrm{SO}_3$) Since $\pi_1 \mathrm{SO}_3$ is not torsion-free, this case is not covered by our work in this paper. There are new phenomena. The main point is that the centralizer subgroup of any rotation of $\mathbb{R}^3$ through angle π is not connected,

[31] Let $\zeta = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ be the coroot. Then the integral of the 3-form $\Omega(\xi_1, \xi_2, \xi_3) = \langle[\xi_1, \xi_2], \xi_3\rangle$ is $\int_{\mathrm{SU}_2} \Omega = 4\pi^2|\zeta|^2$. The properly normalized generator has $|\zeta|^2 = 2$. The normalization may be seen from the Chern-Weil formula for the second Chern class, for example, using the Chern-Simons form to transgress $[\Omega] \in H^3(\mathrm{SU}_2; \mathbb{R})$ to a class in $H^4(\mathrm{BSU}_2; \mathbb{R})$. It is one explanation for the factor $8\pi^2$ in (4.26).

which modifies Corollary 4.14 as well as the twisted equivariant K-theory. A closely related fact is that the argument after (4.23) fails: the action of $F \cong \mathbb{Z}/2\mathbb{Z}$ has fixed points and so the horizontal maps in (4.21) are not injective. Furthermore, the spin extension $\mathrm{U}_2 \to \mathrm{SO}_3$ is not split; its equivalence class in $H^3(BSO_3;\mathbb{Z})$ is the universal integral third Stiefel-Whitney class. This example is covered by [FHT3], which treats arbitrary (twisted) loop groups and central extensions. Detailed computations for $G = \mathrm{SO}_3$ appear in [FHT2, Appendix A].

Appendix: Central extensions in the semisimple case

Here we present the proof of Proposition 2.15, which we repeat for convenience.

Proposition 2.15. *Let G be a compact Lie group with $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, i.e., $\mathfrak{g}$ semisimple. Let $P \to S^1$ be a principal G-bundle. Then any central extension $(L_PG)^\tau$ is admissible. Furthermore, for each $(\widehat{L}_PG)^\tau$ which satisfies (2.11) there exists a unique $(\widehat{L}_PG)^\tau$-invariant symmetric bilinear form $\ll \cdot,\cdot \gg\; = \;\ll \cdot,\cdot \gg_\tau$ which satisfies (2.12).*

For G connected and simply connected, see §2.3. We remark that in the main text we only used the statement about the bilinear form (in the proof of Proposition 3.13).

Proof. Central extensions of L_PG by $\mathbb{T}$ are classified by the smooth group cohomology $H^2_{\mathrm{smooth}}(L_PG;\mathbb{T})$, as defined in [S1]. For G semisimple $H^q_{\mathrm{smooth}}(L_PG;\mathbb{R}) = 0$ for $q > 0$ by [PS, §14.6], [RCW], so the exponential sequence yields

$$H^2_{\mathrm{smooth}}(L_PG;\mathbb{T}) \cong H^3_{\mathrm{smooth}}(L_PG;\mathbb{Z}) \cong H^3(BL_PG;\mathbb{Z}).$$

In other words, the isomorphism class of a central extension of L_PG is characterized topologically. We must show that

$$H^3(B\widehat{L}_PG;\mathbb{Z}) \longrightarrow H^3(BL_PG;\mathbb{Z}) \quad \text{is surjective} \tag{A.1}$$

for some finite cover $\widehat{\mathbb{T}}_{\mathrm{rot}} \to \mathbb{T}_{\mathrm{rot}}$, and so construct $(\widehat{L}_PG)^\tau$ as in (2.11). For this consider the Leray spectral sequence $H^p\big(\mathrm{B}\widehat{\mathbb{T}}_{\mathrm{rot}}; H^q(BL_PG;\mathbb{Z})\big) \Longrightarrow H^{p+q}(B\widehat{L}_PG;\mathbb{Z})$. From the discussion surrounding (2.3) we see $H^q(BL_PG;\mathbb{Z}) \cong H^q_G\big(G[P];\mathbb{Z}\big)$. The first differential of interest is
(A.2)

$$d_2 \,:\, H^3_G\big(G[P];\mathbb{Z}\big) \cong H^0\big(\mathrm{B}\widehat{\mathbb{T}}_{\mathrm{rot}}; H^3_G(G[P];\mathbb{Z})\big) \longrightarrow H^2\big(\mathrm{B}\widehat{\mathbb{T}}_{\mathrm{rot}}; H^2_G(G[P];\mathbb{Z})\big).$$

The hypothesis that G has no torus factors implies $H^2_G\big(G[P];\mathbb{Z}\big)$ is a finite group—use the Leray spectral sequence for the homotopy quotient of $G[P]$ fibered over BG—and so for an appropriate cover the codomain of (A.2) vanishes. The next nonzero differential is

$$d_4 \,:\, H^0\big(\mathrm{B}\widehat{\mathbb{T}}_{\mathrm{rot}}; H^3_G(G[P];\mathbb{Z})\big) \longrightarrow H^4\big(\mathrm{B}\widehat{\mathbb{T}}_{\mathrm{rot}}; H^0_G(G[P];\mathbb{Z})\big). \tag{A.3}$$

For a suitable cover the bundle $\widehat{L}_PG \to \widehat{\mathbb{T}}_{\mathrm{rot}}$ has a section, in which case (A.3) vanishes. This completes the proof of (A.1).

Now we assume given a group $(\widehat{L}_PG)^\tau$ which fits into (2.11). Fix a connection A which is generic in the sense that its stabilizer $Z_A \subset L_PG$ has identity component a torus. We use the basepoint to identify it with a subgroup of G, and so the complexification $(\mathfrak{z}_A)_{\mathbb{C}}$ of its Lie algebra with an abelian subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{C}}$ be the centralizer of $\mathfrak{h}_0$; then $\mathfrak{h}$ is a Cartan subalgebra.[32] Decompose

$$\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{h} \;\oplus\; \bigoplus_{\lambda\in\Delta_0} \mathfrak{g}_\lambda, \qquad \lambda \in \mathfrak{h}_0^*,$$

into eigenspaces of $\mathrm{ad}(\mathfrak{h}_0)$, where $\mathfrak{g}_\lambda \neq 0$. Then $\lambda \neq 0$ for $\lambda \in \Delta_0$, since $\mathfrak{h}_0$ contains regular elements. The holonomy induces an automorphism of $\mathfrak{h}$ which fixes $\mathfrak{h}_0$, and since $\mathfrak{h}_0$ contains regular elements we can choose a Weyl chamber which is invariant under the holonomy. Now $\mathrm{ad}(\mathfrak{h})$ decomposes each $\mathfrak{g}_\lambda$ as a sum of root spaces, and by our choice of Weyl chamber the roots which occur are either all positive or all negative. In this way we partition Δ_0 into a positive set and a negative set. Use (2.38) to embed $\mathfrak{g}_{\mathbb{C}}$ in the complexified loop algebra. (We drop the subscript "A".) Note $[d_A, \mathfrak{h}_0] = 0$, but $[d_A, \mathfrak{h}] \neq 0$ in general, and so $\mathfrak{h}$ decomposes as a sum of eigenspaces of d_A. We write $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}'$, where d_A has nonzero eigenvalues on $\mathfrak{h}'$. By adding to d_A a suitable regular element of $\mathfrak{h}_0$ if necessary, and making an appropriate choice of the logarithm of the holonomy ("S_A" in the text preceding (2.36), we can ensure

$$\text{(A.4)} \qquad \mathrm{spec}(-i\,d_A) \subset (0,1) \qquad \text{on } \mathfrak{h}' \;\oplus\; \bigoplus_{\lambda>0} \mathfrak{g}_\lambda,$$

$$\text{(A.5)} \qquad \mathrm{spec}(-i\,d_A) \text{ on } \mathfrak{h} \text{ and } \bigoplus_{\lambda\in\Delta_0} \mathfrak{g}_\lambda \text{ are disjoint.}$$

Summarizing, we have a decomposition[33]

$$\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{h}_0 \;\oplus\; \bigoplus_j \mathbb{C}\cdot\chi_j \;\oplus\; \bigoplus_{\lambda,e>0} \mathbb{C}\cdot\pi_{\lambda,e} \;\oplus\; \bigoplus_{\lambda,e>0} \mathbb{C}\cdot\pi_{-\lambda,-e},$$

where the $\chi_j, \pi_{\lambda,e}$ are determined up to a nonzero scalar by

$$\begin{aligned} [\xi, \pi_{\lambda,e}] &= i\lambda(\xi)\pi_{\lambda,e}, \qquad \xi \in \mathfrak{h}_0,\\ [d_A, \pi_{\lambda,e}] &= ie\pi_{\lambda,e},\\ [d_A, \chi_j] &= iE(\chi_j)\chi_j, \end{aligned}$$

and $0 < e < 1$ if $\lambda > 0$. We arrange $\pi_{-\lambda,-e} = \overline{\pi_{\lambda,e}}$.

Observe that if $\eta \in L_P\mathfrak{g}_{\mathbb{C}}$ is an eigenvector of $\mathrm{ad}(d_A)$ with nonzero eigenvalue $iE(\eta)$, then it has a lift η_A^τ to $(L_P\mathfrak{g})_{\mathbb{C}}^\tau$ characterized by

$$\text{(A.6)} \qquad [d_A^\tau, \eta_A^\tau] = iE(\eta)\eta_A^\tau.$$

[32]In other words, the real points of $\mathfrak{h}$ form the Lie algebra of a maximal torus. To see this, we first show $\mathfrak{h}_0 \neq 0$. Identify the holonomy automorphism of $\mathfrak{g}_{\mathbb{C}}$ with Ad_g for some $g \in G$. Composing with Ad_{g_1} for a suitable g_1 in the identity component, we may assume that the automorphism of $\mathfrak{g}_{\mathbb{C}}$ fixes the Lie algebra of a maximal torus as well as a Weyl chamber. Then it permutes the positive roots, so fixes their sum and the line generated by the sum of the coroot vectors. Conjugating back by $\mathrm{Ad}_{g_1^{-1}}$ we conclude $\mathfrak{h}_0 \neq 0$. Now g^n lies in the identity component G_1 for suitable n, so may be written as the exponential of a real element of $\mathfrak{h}_0$. Multiply g by its inverse to obtain an automorphism of $\mathfrak{g}_{\mathbb{C}}$ of finite order. Then [K1, Lemma 8.1] applies to prove that $\mathfrak{h}$ is a Cartan subalgebra. Note for the untwisted case ($P \to S^1$ trivializable) we have $\mathfrak{h} = \mathfrak{h}_0$.

[33]We implicitly assume the joint eigenspaces of $\mathrm{ad}(\mathfrak{h}_0)$ and d_A with nonzero eigenvalues have dimension one, which may well be true in general. If not, the argument is only notationally more complicated.

Also, equation (A.6) implies that if η, η' are elements of $L_P\mathfrak{g}_{\mathbb{C}}$ of definite nonzero energy, and the sum of the energies is nonzero, then

$$[\eta, \eta']^\tau_A = [\eta^\tau_A, (\eta')^\tau_A], \qquad E(\eta), E(\eta'), E([\eta, \eta']) \neq 0. \tag{A.7}$$

Thus we have almost defined a decomposition of the finite energy vectors in the central extension:

$$\big((\widehat{L}_P\mathfrak{g})^\tau_{\mathbb{C}}\big)_{\mathrm{fin}}(A) \cong \mathbb{C}{\cdot}d^\tau_A \oplus \mathbb{C}{\cdot}K \oplus \mathfrak{h}_0 \oplus \bigoplus_j \mathbb{C}{\cdot}(\chi_j)^\tau_A \oplus \bigoplus_{\lambda, e\neq 0} \mathbb{C}{\cdot}(\pi_{\lambda,e})^\tau_A \oplus \bigoplus_{n\neq 0} z^n\mathfrak{g}_{\mathbb{C}}. \tag{A.8}$$

It remains to define the lift of $\mathfrak{h}_0$ to the central extension, which we do below.

The invariance of the desired form $\ll\cdot,\cdot\gg_\tau$ implies that it is nonzero only if the total energy of its arguments vanishes. (We implicitly use Lemma 2.41 here as well.) Similarly, eigenvectors for $\mathfrak{h}_0$ pair nontrivially only if their eigenvalues (as linear functionals on $\mathfrak{h}_0$) sum to zero. If $\eta, \eta' \in L_P\mathfrak{g}_{\mathbb{C}}$, then by semisimplicity write $\eta = \sum [\xi_i, \xi'_i]$ for some $\xi_i, \xi'_i \in L_P\mathfrak{g}_{\mathbb{C}}$. Any invariant form satisfies

$$\begin{aligned} \ll z^n\eta, z^{-n}\eta'\gg_\tau &= \sum_i \ll [\xi_i, z^n\xi'_i], z^{-n}\eta'\gg_\tau \\ &= \sum_i \ll \xi_i, [z^n\xi'_i, z^{-n}\eta']\gg_\tau \\ &= \sum_i \ll \xi_i, [\xi'_i, \eta']\gg_\tau \\ &= \sum_i \ll [\xi_i, \xi'_i], \eta'\gg_\tau \\ &= \ll \eta, \eta'\gg_\tau . \end{aligned} \tag{A.9}$$

Therefore, it suffices to define $\ll\cdot,\cdot\gg_\tau$ on $\mathfrak{g}_{\mathbb{C}}$; all other pairings are determined by (2.12), (2.13), (2.14), (2.19), and (A.9).

Set

$$\zeta_{\lambda,e} = i[\pi_{\lambda,e}, \pi_{-\lambda,-e}] \in \mathfrak{h}_0.$$

The $\zeta_{\lambda,e}$ span $\mathfrak{h}_0$, but in general there are linear relations among them. Define a sequence of lifts of $\zeta_{\lambda,e}$ by

$$\zeta^{(n)}_{\lambda,e} = i\big[(z^n\pi_{\lambda,e})^\tau_A, (z^{-n}\pi_{-\lambda,-e})^\tau_A\big] \in (L_P\mathfrak{g})^\tau, \qquad n \in \mathbb{Z}. \tag{A.10}$$

We verify below that $\zeta^{(n+1)}_{\lambda,e} - \zeta^{(n)}_{\lambda,e} \in \mathbb{R}{\cdot}K$ is independent of n, and we use it to define $\ll\pi_{\lambda,e}, \pi_{-\lambda,-e}\gg_\tau$:

$$\delta(\zeta_{\lambda,e}) = \zeta^{(n+1)}_{\lambda,e} - \zeta^{(n)}_{\lambda,e} = \ll\pi_{\lambda,e}, \pi_{-\lambda,-e}\gg_\tau\ K, \qquad n \in \mathbb{Z}. \tag{A.11}$$

This definition is forced by invariance:

$$\begin{aligned} \ll d^\tau_A, \delta(\zeta_{\lambda,e})\gg_\tau &= i\ \ll d^\tau_A, \big[(z^{n+1}\pi_{\lambda,e})^\tau_A, (z^{-(n+1)}\pi_{-\lambda,-e})^\tau_A\big]\gg_\tau \\ &\qquad - i \ll d^\tau_A, \big[(z^n\pi_{\lambda,e})^\tau_A, (z^{-n}\pi_{-\lambda,-e})^\tau_A\big]\gg_\tau \\ &= -(n+1+e) \ll z^{n+1}\pi_{\lambda,e}, z^{-(n+1)}\pi_{-\lambda,-e}\gg_\tau \\ &\qquad + (n+e) \ll z^n\pi_{\lambda,e}, z^{-n}\pi_{-\lambda,-e}\gg_\tau \\ &= - \ll\pi_{\lambda,e}, \pi_{-\lambda,-e}\gg_\tau, \end{aligned}$$

where at the last stage we use (A.9). We also define the lift

$$(\zeta_{\lambda,e})^\tau_A = \zeta^{(0)}_{\lambda,e} - e\delta(\zeta_{\lambda,e}). \tag{A.12}$$

This is also determined by invariance (and (2.19)): if $(\zeta_{\lambda,e})^\tau_A = \zeta^{(0)}_{\lambda,e} + cK$ for some constant c, then

$$\begin{aligned} 0 = \ll d^\tau_A, (\zeta_{\lambda,e})^\tau_A \gg_\tau &= i \ll d^\tau_A, \big[(\pi_{\lambda,e})^\tau_A, (\pi_{-\lambda,-e})^\tau_A\big] \gg_\tau - c \\ &= -e \ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\tau - c. \end{aligned}$$

For any $\xi \in \mathfrak{h}_0$ we set

$$\ll \zeta_{\lambda,e}, \xi \gg_\tau = -\lambda(\xi) \ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\tau, \tag{A.13}$$

as determined by invariance:

$$\begin{aligned} \ll \zeta_{\lambda,e}, \xi \gg_\tau &= i \ll [\pi_{\lambda,e}, \pi_{-\lambda,-e}], \xi \gg_\tau = i \ll \pi_{\lambda,e}, [\pi_{-\lambda,-e}, \xi] \gg_\tau \\ &= -\lambda(\xi) \ll \pi_{\lambda,e}, \pi_{-\lambda,-e} \gg_\tau . \end{aligned}$$

Finally, by semisimplicity we can write (nonuniquely)

$$\chi_j = \sum_k [\eta^{(k)}_j, {\eta'_j}^{(k)}], \tag{A.14}$$

where each $\eta^{(k)}_j, {\eta'_j}^{(k)}$ is a multiple of some $z^n\pi_{\lambda,e}$, and $E(\eta^{(k)}_j) + E({\eta'_j}^{(k)}) = E(\chi_j)$. Then define

$$\ll \chi_j, z^{-1}\chi_{j'} \gg_\tau = \sum_{k,\ell} \ll [[\eta^{(k)}_j, {\eta'_j}^{(k)}], \eta^{(\ell)}_{j'}], z^{-1}{\eta'_{j'}}^{(\ell')} \gg_\tau, \tag{A.15}$$

which is forced by invariance. Condition (A.5) implies that the triple bracket is a multiple of some $z^n\pi_{\lambda,e}$, and so the right-hand side of (A.15) is determined by (A.11) and (A.9).

The invariance arguments prove that $\ll \cdot, \cdot \gg_\tau$ is unique. We must check, though, that (A.12), (A.13), and (A.15) are consistent. In particular, this will show that (A.12) completes the definition of the splitting (A.8) (and so the form $\ll d^\tau_A, \zeta^\tau_A \gg = 0_\tau$ for all $\zeta \in \mathfrak{h}_0$).

First, we verify that (A.11) is independent of n. Choose $\xi \in \mathfrak{h}_0$ regular, i.e., with $\lambda(\xi) \neq 0$ for all λ. Then using (A.7) repeatedly we find

$$\begin{aligned} \zeta^{(n+1)}_{\lambda,e} &= \frac{1}{\lambda(\xi)}\big[[z\xi, z^n\pi_{\lambda,e}]^\tau_A, (z^{-(n+1)}\pi_{-\lambda,-e})^\tau_A\big] \\ &= \frac{1}{\lambda(\xi)}\big[[(z\xi)^\tau_A, (z^{-(n+1)}\pi_{-\lambda,-e})^\tau_A], (z^n\pi_{\lambda,e})^\tau_A\big] \\ &\qquad + \frac{1}{\lambda(\xi)}\big[(z\xi)^\tau_A, [(z^n\pi_{\lambda,e})^\tau_A, (z^{-(n+1)}\pi_{-\lambda,-e})^\tau_A]\big] \\ &= \zeta^{(n)}_{\lambda,e} - \frac{i}{\lambda(\xi)}[(z\xi)^\tau_A, (z^{-1}\zeta_{\lambda,e})^\tau_A], \end{aligned}$$

so

$$\delta(\zeta_{\lambda,e}) = \zeta^{(n+1)}_{\lambda,e} - \zeta^{(n)}_{\lambda,e} = -\frac{i}{\lambda(\xi)}[(z\xi)^\tau_A, (z^{-1}\zeta_{\lambda,e})^\tau_A] \tag{A.16}$$

is independent of n, as claimed.

Next, suppose $c^{\lambda,e}\zeta_{\lambda,e} = 0$ for some $c^{\lambda,e} \in \mathbb{R}$. Then we find using (A.11) and (A.16) that

$$\begin{aligned}\sum_{\lambda>0} c^{\lambda,e}\lambda(\xi) \ll\pi_{\lambda,e},\pi_{-\lambda,-e}\gg_\tau K &= -\sum_{\lambda>0} ic^{\lambda,e}\big[(z\xi)^\tau_A,(z^{-1}\zeta_{\lambda,e})^\tau_A\big]\\ &= -\sum_{\lambda>0} i\big[(z\xi)^\tau_A,(z^{-1}c^{\lambda,e}\zeta_{\lambda,e})^\tau_A\big]\\ &= 0.\end{aligned}$$

This demonstrates that (A.13) is a consistent definition of $\ll\cdot,\cdot\gg_\tau$ on $\mathfrak{h}_0$.

To check that (A.15) is consistent, we must check that any expression (A.14) leads to the same right-hand side of (A.15), or equivalently that if $\sum_k[\eta^{(k)},\eta'^{(k)}] = 0$, then the right-hand side of (A.15) vanishes. But this is obvious.

To see that the lift of (A.12) is consistent we must show that any linear relation among the $\zeta_{\lambda,e}$ is also satisfied by the lifts. Let $\lambda_1,\dots,\lambda_s \in \mathfrak{h}_0^*$ be the restrictions of the simple roots to $\mathfrak{h}_0$, after eliminating duplicates; call these the *simple* λ. Then $\{\zeta_{\lambda_1,e_1},\dots,\zeta_{\lambda_s,e_s}\}$ is a basis of $\mathfrak{h}_0$, where we choose $0 < e_1,\dots,e_s < 1$ to be minimal. There are two types of relation which generate them all. First, if λ_k is simple, then any $\pi_{\lambda_k,e}$ which occurs is of the form $\pi_{\lambda_k,e_k+E(\chi_j)}$ for some $\chi_j \in \mathfrak{h}'$ of definite (positive) energy, and for a suitable choice we can take $\pi_{\lambda_k,e} = [\chi_j,\pi_{\lambda_k,e_k}]$. Then for some constant c we have $[\pi_{\lambda_k,e},\bar\chi_j] = c\pi_{\lambda_k,e_k}$, and for any $n \in \mathbb{Z}$ we find

$$\begin{aligned}\zeta_{\lambda_k,e} &= i[z^n\pi_{\lambda_k,e}, z^{-n}[\bar\chi_j,\pi_{-\lambda_k,-e_k}]]\\ &= i[[z^n\pi_{\lambda_k,e},\bar\chi_j], z^{-n}\pi_{-\lambda_k,-e_k}] \;+\; i[\bar\chi_j,[z^n\pi_{\lambda_k,e},z^{-n}\pi_{-\lambda_k,-e_k}]]\\ &= c\,\zeta_{\lambda_k,e_k}.\end{aligned}\tag{A.17}$$

Now in the central extension an easy argument using invariance and (A.15) shows that

$$i[(\bar\chi_j)^\tau_A,[(z^n\pi_{\lambda_k,e})^\tau_A,(z^{-n}\pi_{-\lambda_k,-e_k})^\tau_A]] = cE(\chi_j)\delta(\zeta_{\lambda_k,e_k}).$$

By repeatedly applying (A.7), and using the definition (A.12) of the lifts, we verify the relation (A.17) for the lifts. The second type of relation comes from writing any $\lambda > 0$ as a sum of simple λ_k, and we proceed by induction on the length of such a relation. In the inductive step we take

$$(\lambda,e) = (\lambda',e') + (\lambda_k,e_k) - (0,\nu),$$

where $0 < e,e',e_k < 1$, which determines $\nu = 0$ or $\nu = 1$, and we already know the lift of $\zeta_{\lambda',e'}$ is in the linear span of the lifts of $\zeta_{\lambda_1,e_1},\dots,\zeta_{\lambda_s,e_s}$. Take

$$\pi_{\lambda,e} = z^{-\nu}[\pi_{\lambda',e'},\pi_{\lambda_k,e_k}].\tag{A.18}$$

Then $[\pi_{\lambda',e'},\pi_{-\lambda_k,-e_k}]$ is a (possibly vanishing) multiple of $\pi_{\lambda'-\lambda_k,e'-e_k}$. Substituting (A.18) and expanding we find a relation

$$\begin{aligned}\zeta_{\lambda,e} &= i[z^\nu\pi_{\lambda,e},\zeta^{-\nu}\pi_{-\lambda,-e}]\\ &= \lambda'(\zeta_{\lambda_k,e_k})\zeta_{\lambda',e'} + \lambda_k(\zeta_{\lambda',e'})\zeta_{\lambda_k,e_k} ic\Big[\operatorname{ad}(\pi_{\lambda',e'})\operatorname{ad}(\pi_{-\lambda_k,-e_k})\pi_{-(\lambda'-\lambda_k),-(e'-e_k)}\\ &\qquad\qquad - \operatorname{ad}(\pi_{\lambda_k,e_k})\operatorname{ad}(\pi_{-\lambda',-e'})\pi_{\lambda'-\lambda_k,e'-e_k}\Big]\\ &= \Big[\lambda'(\zeta_{\lambda_k,e_k}) + c_1\Big]\,\zeta_{\lambda',e'} \;+\; \Big[\lambda_k(\zeta_{\lambda',e'}) + c_2\Big]\,\zeta_{\lambda_k,e_k}\end{aligned}$$

for some (possibly vanishing) constants c_1, c_2. Using (A.10) and (A.12) we see that the lifts satisfy the same relation if and only if

$$(e+\nu)\delta(\zeta_{\lambda,e}) = \Big[\lambda'(\zeta_{\lambda_k,e_k}) + c_1\Big]\, e'\delta(\zeta_{\lambda',e'}) \;+\; \Big[\lambda_k(\zeta_{\lambda',e'}) + c_2\Big]\, e_k\delta(\zeta_{\lambda_k,e_k}).$$

But this follows from (A.16) after choosing $\xi \in \mathfrak{h}_0$ such that $e_k = \lambda_k(\xi)$ and $e' = \lambda'(\xi)$.

This completes the proof that $\ll\cdot,\cdot\gg_\tau$ is well defined and invariant under the adjoint action of $(\widehat{L}_P\mathfrak{g})^\tau$, so under the adjoint action of the identity component of $(\widehat{L}_PG)^\tau$. Finally, the uniqueness shows that it is invariant under the entire group $(\widehat{L}_PG)^\tau$. □

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References

[A] J. Frank Adams, *Lectures on Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR0252560 (40 #5780)

[AM] A. Alekseev and E. Meinrenken, *The non-commutative Weil algebra*, Invent. Math. **139** (2000), no. 1, 135–172, DOI 10.1007/s002229900025. MR1728878 (2001j:17022)

[AB] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. of Math. (2) **88** (1968), 451–491. MR0232406 (38 #731)

[ABS] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), no. suppl. 1, 3–38. MR0167985 (29 #5250)

[AS] Michael Atiyah and Graeme Segal, *Twisted K-theory*, Ukr. Mat. Visn. **1** (2004), no. 3, 287–330; English transl., Ukr. Math. Bull. **1** (2004), no. 3, 291–334. MR2172633 (2006m:55017)

[BW] A. Borel and A. Weil, *Representations lineaires et espaces homogenes Kählerians des groupes de Lie compactes*, Séminaire Bourbaki, May 1954, Exposé par J.-P. Serre.

[B] Raoul Bott, *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248. MR0089473 (19,681d)

[DK] J. J. Duistermaat and J. A. C. Kolk, *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000. MR1738431 (2001j:22008)

[F] Daniel S. Freed, *The geometry of loop groups*, J. Differential Geom. **28** (1988), no. 2, 223–276. MR961515 (89k:22036)

[FHT1] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, *Loop groups and twisted K-theory I*, J. Topol. **4** (2011), no. 4, 737–798, DOI 10.1112/jtopol/jtr019. MR2860342

[FHT2] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, *Twisted equivariant K-theory with complex coefficients*, J. Topol. **1** (2008), no. 1, 16–44, DOI 10.1112/jtopol/jtm001. MR2365650 (2009c:19011)

[FHT3] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, *Loop groups and twisted K-theory III*, Ann. of Math. (2) **174** (2011), no. 2, 947–1007, DOI 10.4007/annals.2011.174.2.5. MR2831111

[FHT4] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, *Consistent orientation of moduli spaces*, The many facets of geometry, Oxford Univ. Press, Oxford, 2010, pp. 395–419, DOI 10.1093/acprof:oso/9780199534920.003.0019. MR2681705 (2011h:57043)

[G] Sebastian Goette, *Equivariant η-invariants on homogeneous spaces*, Math. Z. **232** (1999), no. 1, 1–42, DOI 10.1007/PL00004757. MR1714278 (2001d:58022)

[H1] Nigel Hitchin, *Generalized geometry—an introduction*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lect. Math. Theor. Phys., vol. 16, Eur. Math. Soc., Zürich, 2010, pp. 185–208, DOI 10.4171/079-1/6. MR2681591 (2011j:53172)

[H2] Einar Hille, *On roots and logarithms of elements of a complex Banach algebra*, Math. Ann. **136** (1958), 46–57. MR0096137 (20 #2632)

[K1] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219 (92k:17038)

[K2] A. A. Kirillov, *Lectures on the orbit method*, Graduate Studies in Mathematics, vol. 64, American Mathematical Society, Providence, RI, 2004. MR2069175 (2005c:22001)

[K3] Bertram Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387. MR0142696 (26 #265)

[K4] Bertram Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501, DOI 10.1215/S0012-7094-99-10016-0. MR1719734 (2001k:22032)

[KS] Bertram Kostant and Shlomo Sternberg, *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, Ann. Physics **176** (1987), no. 1, 49–113, DOI 10.1016/0003-4916(87)90178-3. MR893479 (88m:58057)

[L] Gregory D. Landweber, *Multiplets of representations and Kostant's Dirac operator for equal rank loop groups*, Duke Math. J. **110** (2001), no. 1, 121–160, DOI 10.1215/S0012-7094-01-11014-4. MR1861090 (2002h:17028)

[M] Jouko Mickelsson, *Gerbes, (twisted) K-theory, and the supersymmetric WZW model*, Infinite dimensional groups and manifolds, IRMA Lect. Math. Theor. Phys., vol. 5, de Gruyter, Berlin, 2004, pp. 93–107. MR2104355 (2006c:58037)

[PS] Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications. MR900587 (88i:22049)

[RCW] Alvany Rocha-Caridi and Nolan R. Wallach, *Projective modules over graded Lie algebras. I*, Math. Z. **180** (1982), no. 2, 151–177, DOI 10.1007/BF01318901. MR661694 (83h:17018)

[S1] Graeme Segal, *Cohomology of topological groups*, Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69), Academic Press, London, 1970, pp. 377–387. MR0280572 (43 #6292)

[S2] Stephen Slebarski, *Dirac operators on a compact Lie group*, Bull. London Math. Soc. **17** (1985), no. 6, 579–583, DOI 10.1112/blms/17.6.579. MR813743 (87c:58115)

[T] C. Taubes, *Notes on the Dirac operator on loop space* unpublished manuscript (1989).

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