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AN ASYMMETRIC CONVEX BODY WITH MAXIMAL SECTIONS OF CONSTANT VOLUME

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1. Introduction

As usual, a *convex body* $K \subset \mathbb{R}^d$ is a compact convex subset of $\mathbb{R}^d$ with non-empty interior. We assume that the origin is contained in the interior of K. We consider the *maximal section function* M_K:

$$M_K(u) = \max_{t\in\mathbb{R}} \mathrm{vol}_{d-1}(K \cap (u^\perp + tu)), \qquad u \in \mathbb{S}^{d-1}.$$

Here $u^\perp$ stands for the hyperplane passing through the origin and orthogonal to the unit vector u; $K \cap (u^\perp + tu)$ is the section of K by the affine hyperplane $u^\perp + tu$. It is well known ([Ga]) that for origin-symmetric convex bodies the maximal sections are central (i.e., correspond to $t = 0$) and the condition

$$M_{K_1}(u) = M_{K_2}(u) \qquad \forall u \in \mathbb{S}^{d-1}$$

implies $K_1 = K_2$.

It is also well known ([BF]) that on the plane there are convex bodies K that are *not* Euclidean discs but nevertheless satisfy $M_K(u) = 1$ for all $u \in \mathbb{S}^1$. These are the *bodies of constant width* 1.

In 1969 V. Klee asked whether the condition $M_{K_1} \equiv M_{K_2}$ implies that K_1 and K_2 are essentially the same (i.e., differ by translation and/or reflection about the origin) in general, or, at least, whether the condition $M_K \equiv c$ implies that K is a Euclidean ball ([Kl]).

Recently, R. Gardner and V. Yaskin, together with the second and the third authors, gave a negative answer to the first question of V. Klee by constructing two bodies of revolution K_1, K_2 such that K_1 is origin-symmetric, K_2 is not origin-symmetric, but $M_{K_1} \equiv M_{K_2}$ (see [GRYZ]). In [NRZ] this result was strengthened for all *even* dimensions. It was shown that there exist two essentially different convex bodies of revolution $K_1, K_2 \subset \mathbb{R}^d$ such that $A_{K_1} \equiv A_{K_2}$, $M_{K_1} \equiv M_{K_2}$, and $P_{K_1} \equiv P_{K_2}$, where, for $u \in \mathbb{S}^{d-1}$,

$$A_K(u) = \mathrm{vol}_{d-1}(K \cap u^\perp), \qquad P_K(u) = \mathrm{vol}_{d-1}(K|u^\perp),$$

and $K|u^\perp$ is the projection of K to $u^\perp$.

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In this paper we answer the second question of V. Klee. Our main result is the following

Theorem 1. *If $d \geq 3$, there exists a convex body of revolution $K \subset \mathbb{R}^d$ satisfying $M_K \equiv const$ that is not a Euclidean ball.*

Remark 1. An alternative proof in the case $d = 4$ has been given in [NRZ]. Unfortunately, the elementary techniques used there fail for other dimensions.

Our bodies will be *small perturbations of the Euclidean ball.* The proofs of Theorem 1 for even and odd d are different. For even d we can get away with elementary algebra and solve a finite moment problem to obtain a *local* perturbation of the ball. The case $d \geq 3$, d odd, is much more involved. To control the perturbation, we use the properties of the *spherical radon transform*, together with the *Borsuk-Ulam Theorem* asserting that any continuous map from $\mathbb{S}^m$ to $\mathbb{R}^m$, taking antipodal points to antipodal points, contains zero in its image. The reader can find all necessary information in [He] and [Mat].

The paper is organized as follows. In Section 2 we reduce the problem to finding a non-trivial solution to two integral equations. In Section 3 we prove Theorem 1 for even d. Section 4 is devoted to the most difficult part of the proof when d is odd. The appendix contains technical parts of the proofs and some auxiliary statements.

2. Reduction to a system of integral equations

From now on, we assume that $d \geq 3$. We will be dealing with convex bodies of revolution

$$K_f = \{x \in \mathbb{R}^d : x_1 \in [-\lambda, \mu],\ x_2^2 + x_3^2 + \cdots + x_d^2 \leq f^2(x_1)\},$$

obtained by the rotation of a smooth (except for the endpoints) strictly concave function $f : [-\lambda, \mu] \to [0, \infty)$ about the x_1-axis, where λ and μ are some positive real numbers.

Note that K is rotation invariant[1]; thus every hyperplane section is equivalent to a section by a hyperplane with normal vector in the second quadrant of the (x_1, x_2)-plane.

Lemma 1. *Let $L(\xi) = L(s, h, \xi) = s\xi + h$ be a linear function with slope s, and let $H(L) = \{x \in \mathbb{R}^d : x_2 = L(x_1)\}$ be the corresponding hyperplane. Then the section $K \cap H(L)$ is of maximal volume if and only if*

$$\int_{-x}^{y} (f^2 - L^2)^{(d-4)/2} L = 0, \tag{1}$$

where $-x$ and y are the first coordinates of the points at which L intersects the graphs of $-f$ and f, respectively (see Figure 1*).*

Proof. Fix $s > 0$. Observe that the slice $K \cap H(L) \cap H_\xi$ of $K \cap H(L)$ by the hyperplane $H_\xi = \{x \in \mathbb{R}^d : x_1 = \xi\}$, $-x(s,h) < \xi < y(s,h)$, is the $(d-2)$-dimensional Euclidean ball $\{(\xi, L(\xi), x_3, x_4, \ldots, x_d) : x_3^2 + \cdots + x_d^2 \leq r^2\}$ of radius

[1] Here and below "rotation invariance" means invariance with respect to rotations fixing the x_1-axis.

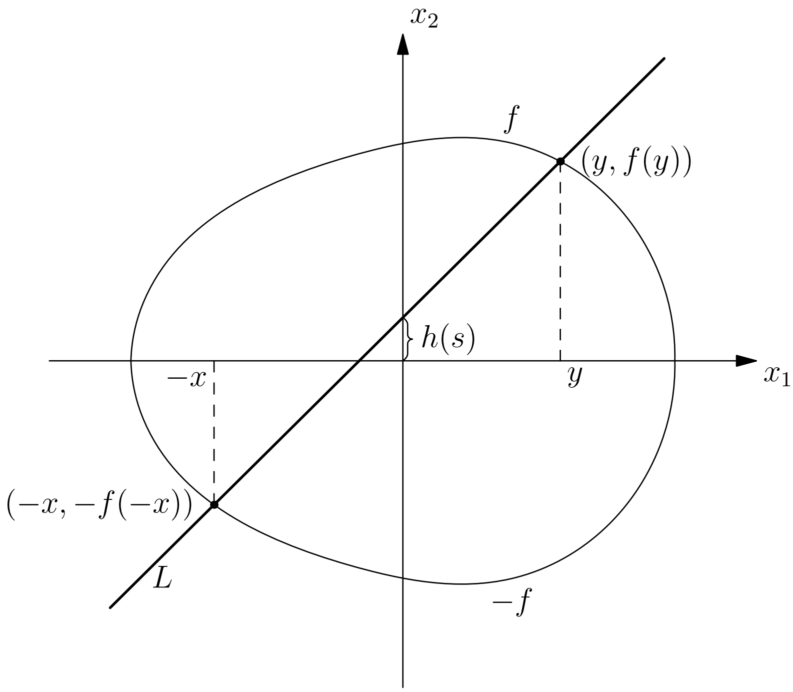

FIGURE 1. View of K and $H(L)$ in (x_1, x_2)-plane.

$r = \sqrt{f^2(\xi) - L^2(\xi)}$. Hence,

$$\text{vol}_{d-1}(K \cap H(L)) = v_{d-2}\sqrt{1+s^2} \int_{-x(s,h)}^{y(s,h)} (f^2(\xi) - L^2(\xi))^{(d-2)/2} d\xi, \tag{2}$$

where v_{d-2} is the volume of the unit ball in $\mathbb{R}^{d-2}$.

The section $K \cap H(L)$ is of maximal volume if and only if

$$\frac{d}{dh}\text{vol}_{d-1}(K \cap H(L)) = 0,$$

where in the *only if* part we use the theorem of Brunn ([Ga]). Computing the derivative, we conclude that for a given $s \in \mathbb{R}$, the section $K \cap H(L)$ is of maximal volume if and only if (1) holds. Note that $y(s,h)$ is determined as the solution of the equation $f(s) - sy - h = 0$ and thereby is differentiable in s and h as long as f is smooth and $f'(y) - s \neq 0$, i.e., the line L intersects the graph of f transversally, which is always the case for non-degenerate sections of a convex body. The same observation applies to $x(s,h)$. □

Lemma 2. *Let $L(s,\xi) = s\xi + h(s)$ be a family of linear functions parameterized by the slope s. For each L in our family, define the hyperplane $H(L)$ by $H(L) = \{x \in \mathbb{R}^d : x_2 = L(x_1)\}$ (see Figure 1). The corresponding family of sections is of* *constant* $(d-1)$*-dimensional volume if and only if*

$$\int_{-x}^{y} (f^2 - L^2)^{(d-2)/2} = \frac{const}{\sqrt{1+s^2}} \qquad \textit{for all} \qquad s > 0. \tag{3}$$

In the case of the unit Euclidean ball, the constant is equal to $\frac{v_{d-1}}{v_{d-2}}$.

Proof. The right-hand side in (2) is constant if and only if (3) holds. □

In what follows we will choose the *perturbation function* h that is infinitely smooth, not identically zero, supported on $[1-2\delta, 1-\delta]$ for some small $\delta > 0$, and which is small together with sufficiently many derivatives. The convex body corresponding to any such function will be automatically asymmetric since not all its maximal sections will pass through a single point.

3. The case of even d

Note that in this case $p = \frac{d-2}{2} \in \mathbb{N}$. Then (3) and (1) take the form

$$\int_{-x}^{y} (f^2 - L^2)^p = \int_{-x_0}^{y_0} (f_0^2 - L_0^2)^p = \frac{const}{\sqrt{1+s^2}}, \tag{4}$$

$$\int_{-x}^{y} (f^2 - L^2)^{p-1} L = 0, \tag{5}$$

where $f_0(\xi) = \sqrt{1-\xi^2}$, $L_0(s,\xi) = s\xi$, and $y_0(s) = x_0(s) = 1/\sqrt{1+s^2}$.

Our body of revolution K_f will be constructed as a *local* perturbation of the Euclidean ball (see Figure 2).

The equations

$$f(y(\sigma)) = L(\sigma, y(\sigma)), \qquad f(-x(\sigma)) = L(\sigma, -x(\sigma)) \tag{6}$$

show that to define f, it is enough to define two decreasing functions $x(\sigma)$ and $y(\sigma)$ on $[0, +\infty)$ (see Figure 3).

For the unperturbed case of the unit ball, $h \equiv 0$ and these functions are just $y_0(s) = x_0(s) = 1/\sqrt{1+s^2}$. Our new functions $x(\sigma)$ and $y(\sigma)$ will coincide with x_0 and y_0 for all $\sigma \notin [1-2\delta, 1-\delta]$. Since the curvature of the semicircle is strictly positive, the resulting function f will be strictly concave if x and y are close to x_0 and y_0 in C^2.

We shall make our construction in several steps. First, we *define* $x = x_0$, $y = y_0$ on $[1, \infty)$. Second, we will transfrom equations (4), (5) to obtain equations (9), (10) written purely in terms of x and y (see below). Then we will use these new equations to extend the functions x and y to $[1-3\delta, 1]$. Due to the existence and uniqueness lemma and the remark after it (Lemma 8 and Remark 2 in the appendix), we will be able to do it if δ and h are sufficiently small, and, moreover, the extensions will coincide with x_0 and y_0 on $[1-\delta, 1]$ and will be close to x_0 and y_0 with two derivatives on $[1-3\delta, 1-\delta]$. Then, we will show that a little miracle happens and our extensions automatically coincide with x_0 and y_0 on $[1-3\delta, 1-2\delta]$ as well. This will allow us to put $x = x_0$, $y = y_0$ on the remaining interval $[0, 1-3\delta]$ and get a nice smooth function (see Figure 2). At last, we will show that equations (4), (5) will hold up to $s = 0$, thus finishing the proof.

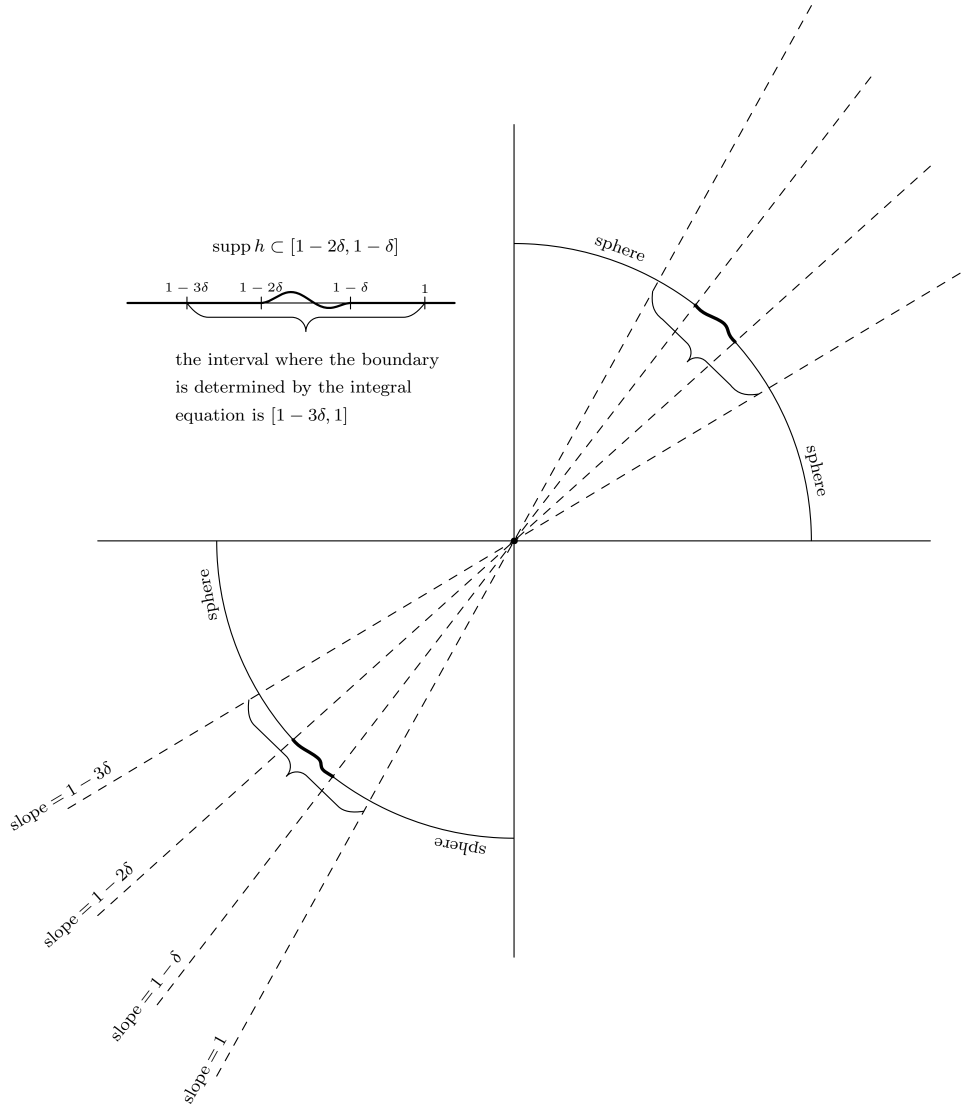

FIGURE 2. Graph of f (the case of an even dimension).

Step 1. We put $x = x_0$, $y = y_0$ on $[1, \infty)$.

Step 2. To construct x, y on $[1-3\delta, 1]$, we will make some technical preparations. First, we will differentiate equations (4), (5) a few times to obtain a system of *four* integral equations with *four* unknown functions x, y, x', y'. Next, we will apply Lemma 8 and Remark 2 to show that there exists a solution x, y, x', y' of the constructed system of integral equations on $[1-3\delta, 1]$, which coincides with x_0, y_0, $\frac{dx_0}{ds}$, $\frac{dy_0}{ds}$ on $[1-\delta, 1]$. Finally, we will prove that the x and y components of that solution give a solution of (4), (5) with f defined by (6).

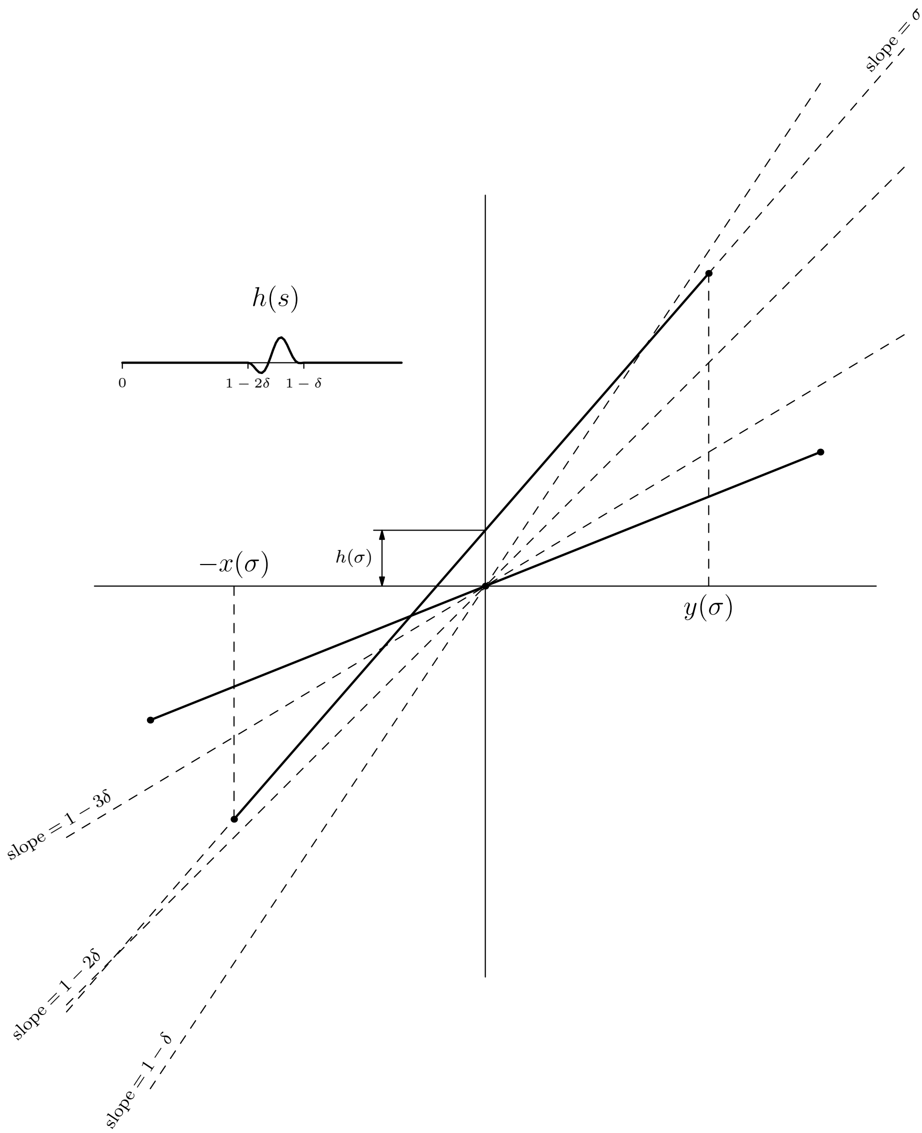

FIGURE 3. The functions $x(\sigma)$ and $y(\sigma)$.

Differentiating equation (4) $p+1$ times and equation (5) p times, we obtain

$$(-2)^p p!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}\Big)^p \frac{dx}{ds}(s) + \Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}\Big)^p \frac{dy}{ds}(s)\Big]$$

$$+\int_{-x(s)}^{y(s)} \Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big((f^2(\xi)-L^2(s,\xi))^p\Big)d\xi = \Big(\frac{d}{ds}\Big)^{p+1}\Big(\frac{const}{\sqrt{1+s^2}}\Big), \tag{7}$$

and

$$(-2)^{p-1}(p-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,-x(s))}\frac{dx}{ds}(s)+\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,y(s))}\frac{dy}{ds}(s)\Big]$$

$$+\int_{-x(s)}^{y(s)}\Big(\frac{\partial}{\partial s}\Big)^{p}\Big((f^2(\xi)-L^2(s,\xi))^{p-1}L(s,\xi)\Big)d\xi=0. \tag{8}$$

When $s \le 1$, the integral term I in (7) can be split as

$$I=\int_{-x(s)}^{y(s)}\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big((f^2(\xi)-L^2(s,\xi))^{p}\Big)d\xi$$

$$=\Big(\int_{-x(s)}^{-x_0(1)}+\int_{y_0(1)}^{y(s)}\Big)\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big((f^2(\xi)-L^2(s,\xi))^{p}\Big)d\xi\;+\;\Xi_1(s),$$

where

$$\Xi_1(s)=\int_{-x_0(1)}^{y_0(1)}\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big((f_0^2(\xi)-L^2(s,\xi))^{p}\Big)d\xi.$$

Making the change of variables $\xi=-x(\sigma)$ in the integral $\int_{-x(s)}^{-x_0(1)}$ and $\xi=y(\sigma)$ in the integral $\int_{y_0(1)}^{y(s)}$, we obtain

$$I=-\int_{s}^{1}\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,-x(\sigma))-L^2(s,-x(\sigma))\Big)^{p}\,\frac{dx}{ds}(\sigma)d\sigma$$

$$-\int_{s}^{1}\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,y(\sigma))-L^2(s,y(\sigma))\Big)^{p}\,\frac{dy}{ds}(\sigma)d\sigma\;+\;\Xi_1(s).$$

Similarly, we have

$$\int_{-x(s)}^{y(s)}\Big(\frac{\partial}{\partial s}\Big)^{p}\Big((f^2(\xi)-L^2(s,\xi))^{p-1}L(s,\xi)\Big)d\xi$$

$$=-\int_{s}^{1}\Big(\frac{\partial}{\partial s}\Big)^{p}\Big(\Big(L^2(\sigma,-x(\sigma))-L^2(s,-x(\sigma))\Big)^{p-1}L(s,-x(\sigma))\Big)\,\frac{dx}{ds}(\sigma)d\sigma$$

$$-\int_{s}^{1}\Big(\frac{\partial}{\partial s}\Big)^{p}\Big(\Big(L^2(\sigma,y(\sigma))-L^2(s,y(\sigma))\Big)^{p-1}L(s,y(\sigma))\Big)\,\frac{dy}{ds}(\sigma)d\sigma\;+\;\Xi_2(s),$$

where

$$\Xi_2(s)=\int_{-x_0(1)}^{y_0(1)}\Big(\frac{\partial}{\partial s}\Big)^{p}\Big((f_0^2(\xi)-L^2(s,\xi))^{p-1}L(s,\xi)\Big)d\xi.$$

To reduce the resulting system of integro-differential equations to a pure system of integral equations, we add two independent unknown functions x', y' and two new relations

$$x(s) = -\int_s^1 x'(\sigma)d\sigma + x_0(1), \qquad y(s) = -\int_s^1 y'(\sigma)d\sigma + y_0(1).$$

We rewrite our equations (7), (8) as follows:

$$(-2)^p p!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x(s))}\Big)^p x'(s) + \Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y(s))}\Big)^p y'(s)\Big]$$
$$-\int_s^1 \Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,-x(\sigma)) - L^2(s,-x(\sigma))\Big)^p x'(\sigma)d\sigma$$
$$-\int_s^1 \Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,y(\sigma)) - L^2(s,y(\sigma))\Big)^p y'(\sigma)d\sigma + \Xi_1(s) = \Big(\frac{d}{ds}\Big)^{p+1}\Big(\frac{const}{\sqrt{1+s^2}}\Big) \tag{9}$$

and

$$(-2)^{p-1}(p-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,-x(s))} x'(s) + \Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,y(s))} y'(s)\Big]$$
$$-\int_s^1 \Big(\frac{\partial}{\partial s}\Big)^{p}\Big(\Big(L^2(\sigma,-x(\sigma)) - L^2(s,-x(\sigma))\Big)^{p-1} L(s,-x(\sigma))\Big) x'(\sigma)d\sigma$$
$$-\int_s^1 \Big(\frac{\partial}{\partial s}\Big)^{p}\Big(\Big(L^2(\sigma,y(\sigma)) - L^2(s,y(\sigma))\Big)^{p-1} L(s,y(\sigma))\Big) y'(\sigma)d\sigma + \Xi_2(s) = 0. \tag{10}$$

Now we rewrite our system in the form

$$\mathbf{G}(s, Z(s)) = \int_s^1 \mathbf{\Theta}(s,\sigma,Z(\sigma))d\sigma + \mathbf{\Xi}(s). \tag{11}$$

Here

$$Z = \begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix},$$

$$\mathbf{G}(s,Z) = \begin{pmatrix} x \\ y \\ (-2)^p p!\Big[\Big(L\frac{\partial L}{\partial s}\Big|_{(s,-x)}\Big)^p x' + \Big(L\frac{\partial L}{\partial s}\Big|_{(s,y)}\Big)^p y'\Big] \\ (-2)^{p-1}(p-1)!\Big[\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,-x)} x' + \Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1}L\Big)\Big|_{(s,y)} y'\Big] \end{pmatrix},$$

$$\mathbf{\Theta}(s,\sigma,Z) = -\begin{pmatrix} x' \\ y' \\ \Theta_1 \\ \Theta_2 \end{pmatrix},$$

where

$$\Theta_1 = -\Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,-x) - L^2(s,-x)\Big)^p x' - \Big(\frac{\partial}{\partial s}\Big)^{p+1}\Big(L^2(\sigma,y) - L^2(s,y)\Big)^p y' ,$$

$$\Theta_2 = -\Big(\frac{\partial}{\partial s}\Big)^p\Big(\Big(L^2(\sigma,-x) - L^2(s,-x)\Big)^{p-1} L(s,-x)\Big)x' \\ -\Big(\frac{\partial}{\partial s}\Big)^p\Big(\Big(L^2(\sigma,y) - L^2(s,y)\Big)^{p-1} L(s,y)\Big)y'$$

and

$$\mathbf{\Xi}(s) = \begin{pmatrix} x_0(1) \\ y_0(1) \\ -\Xi_1(s) + \Big(\frac{d}{ds}\Big)^{p+1}\Big(\frac{const}{\sqrt{1+s^2}}\Big) \\ -\Xi_2(s) \end{pmatrix}.$$

Note that $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are well-defined and infinitely smooth for all $s, \sigma \in (0,1]$ and $Z \in \mathbb{R}^4$. Observe also that

$$D_Z\mathbf{G}\Big|_{(s,Z)} = \begin{pmatrix} \mathbf{I} & 0 \\ * & \mathbf{A} \end{pmatrix},$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{A} = \mathbf{A}(s,x,y) \\ = \begin{pmatrix} (-2)^p p!\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,-x)}\Big)^p & (-2)^p p!\Big(\Big(L\frac{\partial L}{\partial s}\Big)\Big|_{(s,y)}\Big)^p \\ (-2)^{p-1}(p-1)!\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1} L\Big)\Big|_{(s,-x)} & (-2)^{p-1}(p-1)!\Big(\Big(L\frac{\partial L}{\partial s}\Big)^{p-1} L\Big)\Big|_{(s,y)} \end{pmatrix}.$$

The function

$$Z_0(s) = \begin{pmatrix} x_0(s) \\ y_0(s) \\ \frac{dx_0}{ds}(s) \\ \frac{dy_0}{ds}(s) \end{pmatrix}$$

solves the system (11) with $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ corresponding to $h \equiv 0$ (we will denote them by $\mathbf{G}_0$, $\mathbf{\Theta}_0$, $\mathbf{\Xi}_0$) on $[\frac{1}{2}, 1]$, say.

We claim that

$$\text{(12)} \qquad \det\Big(D_Z\mathbf{G}_0\Big|_{(s,Z_0(s))}\Big) = \det(\mathbf{A}_0(s,x_0(s),y_0(s))) \neq 0 \qquad \text{for all } s \in (0,1].$$

Indeed, the matrix $\mathbf{A}_0(s,x_0(s),y_0(s))$ has the sign pattern

$$\begin{pmatrix} + & + \\ + & - \end{pmatrix} \text{ when } p \text{ is even} \quad \text{and} \quad \begin{pmatrix} - & - \\ - & + \end{pmatrix} \text{ when } p \text{ is odd.}$$

Thus, (12) follows. In particular,

$$\det\Big(D_Z\mathbf{G}_0\Big|_{(1,Z_0(1))}\Big) \neq 0.$$

Lemma 8 implies then that we can choose some small $\delta > 0$ and, for any fixed $k \in \mathbb{N}$, construct a C^k close to $Z_0(s)$ solution $Z(s)$ of (11) on $[1-3\delta, 1]$ whenever $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are sufficiently close to $\mathbf{G}_0$, $\mathbf{\Theta}_0$, $\mathbf{\Xi}_0$ in C^k on certain compact sets. Since $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ and their derivatives are some explicit (integrals of) polynomials in Z, s, σ, $h(s)$, and the derivatives of $h(s)$, this closeness assumption will hold if h is sufficiently close to zero with sufficiently many derivatives. Moreover, since h vanishes on $[1-\delta,1]$, the assumptions of Remark 2 are satisfied and we have $Z(s)=Z_0(s)$ on $[1-\delta,1]$.

To prove that the x and y components of the solution that we found give a solution of (4), (5) with f defined by (6), we consider the functions

$$F(s):=\int_{-x(s)}^{y(s)}\Big(f(s,\xi)^2-L^2(s,\xi)\Big)^p d\xi-\frac{const}{\sqrt{1+s^2}},$$

$$H(s):=\int_{-x(s)}^{y(s)}\Big(f(s,\xi)^2-L(s,\xi)^2\Big)^{p-1}L(s,\xi)d\xi.$$

Since equations (9) and (10) of our system (11) were obtained by the differentiation of equations (4), (5), we have

$$\left(\frac{d}{ds}\right)^{p+1}F(s)=0,\qquad \left(\frac{d}{ds}\right)^{p}H(s)=0$$

on $[1-3\delta,1]$. Hence, F and H are polynomials on $[1-3\delta,1]$. Since $h(s)=0$, $x(s)=x_0(s)$, $y(s)=y_0(s)$ on $[1-\delta,1]$, F and H vanish on $[1-\delta,1]$ and, therefore, identically. Thus, we conclude that the x and y components of the solutions of (9), (10) solve (4), (5) on $(1-3\delta,1]$. Step 2 is completed.

Step 3. We claim that $x=x_0$, $y=y_0$ on $[1-3\delta,1-2\delta]$; i.e., the *perturbed solution* returns to the semicircle. Since h is supported on $[1-2\delta,1-\delta]$, we have $L=L_0=s\xi$ and $\frac{\partial}{\partial s}L(s,\xi)=\xi$ for $s\in[1-3\delta,1-2\delta]$. It follows that every time we differentiate equation (4) (with respect to s) we can divide the result by s to obtain

$$\int_{-x(s)}^{y(s)}(f^2(\xi)-L_0^2(s,\xi))^{p-k}\xi^{2k}d\xi=\int_{-x_0(s)}^{y_0(s)}(f_0^2(\xi)-L_0^2(s,\xi))^{p-k}\xi^{2k}d\xi,\qquad k\le p. \tag{13}$$

If we take $k=p$ in (13), we get

$$\int_{-x(s)}^{y(s)}\xi^{2p}d\xi=\int_{-x_0(s)}^{y_0(s)}\xi^{2p}d\xi. \tag{14}$$

Similarly, for $k\le p-1$, equation (5) implies that

$$\int_{-x(s)}^{y(s)}(f^2(\xi)-L_0^2(s,\xi))^{p-1-k}\xi^{2k+1}d\xi=0. \tag{15}$$

Putting $k=p-1$ in (15), we get

$$\int_{-x(s)}^{y(s)}\xi^{2p-1}d\xi=0=\int_{-x_0(s)}^{y_0(s)}\xi^{2p-1}d\xi. \tag{16}$$

Equation (16) yields $x(s)=y(s)$, and the symmetry (with respect to 0) of intervals $(-x_0(s),y_0(s))$, $(-x(s),y(s))$, together with (14), yield $(-x_0(s),y_0(s))=(-x(s),y(s))$ for all $s\in[1-3\delta,1-2\delta]$. Step 3 is completed.

Step 4. We put $x = x_0$, $y = y_0$ on $[0, 1 - 3\delta]$, which will result in a function f defined on $[-1, 1]$ and coinciding with $\sqrt{1-\xi^2}$ outside small intervals around $\pm\frac{1}{\sqrt{2}}$. It remains to check that (4), (5) are valid for $s \in [0, 1 - 3\delta]$. We will prove the validity of (4). The proof for equation (5) is similar.

Since $h \equiv 0$ away from $(1 - 2\delta, 1 - \delta)$, we have $L(s, \xi) = s\xi$ for $s \in [0, 1-3\delta]$, so we need to check that

$$\int_{-x(s)}^{y(s)} (f^2(\xi) - (s\xi)^2)^p d\xi = \int_{-x(s)}^{y(s)} (f_0^2(\xi) - (s\xi)^2)^p d\xi, \qquad \forall s \in [0, 1 - 3\delta].$$

Recall that $x = x_0$ and $y = y_0$ everywhere on this interval, so we can write x and y instead of x_0 and y_0 on the right-hand side.

Opening the parentheses, we see that it suffices to check that

(17)
$$\int_{-x(s)}^{y(s)} f^{2j}(\xi)\xi^{2(p-j)} d\xi = \int_{-x(s)}^{y(s)} f_0^{2j}(\xi)\xi^{2(p-j)} d\xi, \quad \forall j = 1, \dots, p, \quad s \in [0, 1 - 3\delta].$$

Since $f \equiv f_0$ outside $[-x(1-3\delta), y(1-3\delta)]$, it is enough to check (17) for $s = 1-3\delta$.

To this end, we first take $s = 1 - 3\delta$, $k = p - 1$ in (13) and conclude that

$$\int_{-x(1-3\delta)}^{y(1-3\delta)} f^2(\xi)\xi^{2p-2} d\xi = \int_{-x(1-3\delta)}^{y(1-3\delta)} f_0^2(\xi)\xi^{2p-2} d\xi, \tag{18}$$

which is (17) for $j = 1$. Now we go "one step up", by taking $s = 1 - 3\delta$, $k = p - 2$ in (13), to get

$$\int_{-x(1-3\delta)}^{y(1-3\delta)} (f^2(\xi) - (s\xi)^2)^2 \xi^{2p-4} d\xi = \int_{-x(1-3\delta)}^{y(1-3\delta)} (f_0^2(\xi) - (s\xi)^2)^2 \xi^{2p-4} d\xi.$$

The last equality together with (18) yields

$$\int_{-x(1-3\delta)}^{y(1-3\delta)} f^4(\xi)\xi^{2p-4} d\xi = \int_{-x(1-3\delta)}^{y(1-3\delta)} f_0^4(\xi)\xi^{2p-4} d\xi,$$

which is (17) for $j = 2$. Proceeding in a similar way, we get (17) for $j = 1, \dots, p$. This finishes the proof of Theorem 1 in even dimensions.

4. The odd-dimensional case

Note that in this case $p = q + \frac{1}{2}$, $q \in \mathbb{Z}_+$. Then (3) and (1) take the form

$$\int_{-x}^{y} (f^2 - L^2)^{q+\frac{1}{2}} = \int_{-x_0}^{y_0} (f_0^2 - L_0^2)^{q+\frac{1}{2}} = \frac{const}{\sqrt{1+s^2}}, \tag{19}$$

$$\int_{-x}^{y} (f^2 - L^2)^{q-\frac{1}{2}} L = 0, \tag{20}$$

where $f_0(\xi) = \sqrt{1-\xi^2}$, $L_0(s,\xi) = s\xi$, and $y_0(s) = x_0(s) = 1/\sqrt{1+s^2}$.

Let $L = L(s,\xi) = s\xi + h(s)$ be a family of linear functions, parameterized by the slope s. Here the *perturbation function* h is infinitely smooth, supported on $[1-2\delta, 1-\delta]$ for some small $\delta > 0$ to be chosen later, and is small together with sufficiently many derivatives.

Our body of revolution K_f will be constructed as a perturbation of the Euclidean ball (see Figure 4, and compare it with Figure 2). Note that in the case of an odd dimension, the perturbation is not local.

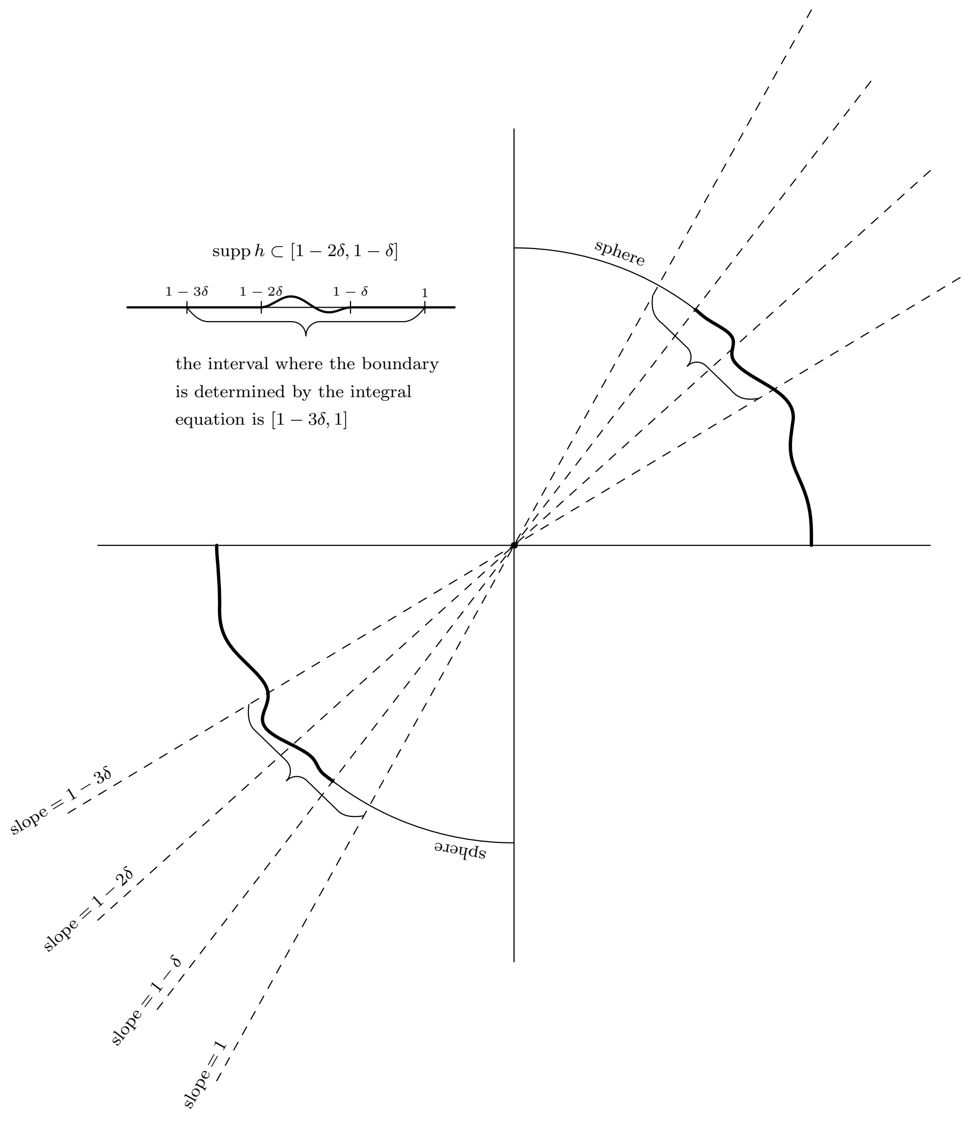

FIGURE 4. Graph of f (the case of an odd dimension).

We shall make our construction in several steps corresponding to the slope ranges $s \in [1,\infty)$, $s \in [1-3\delta, 1]$, and $s \in (0, 1-3\delta]$. We will use different ways to describe the boundary of K_f within those ranges. We will *define* $f(\xi) = f_0(\xi)$ for $\xi \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. We will differentiate (19), (20) and rewrite the resulting equations in terms of x and y to extend x and y to $[1-3\delta, 1]$ as we did in the even case. As before, f is related to x and y by (6). Finally, we will change the point of view and define the remaining part of f in terms of the radial functions $R(\alpha)$ and $r(\alpha)$, related to f by

$$f(R(\alpha)\cos\alpha) = R(\alpha)\sin\alpha, \qquad f(-r(\alpha)\cos\alpha) = r(\alpha)\sin\alpha, \qquad \alpha \in [0, \tfrac{\pi}{2}]. \tag{21}$$

Note that the *radial function*

$$\rho_K(u) = \sup\{t > 0 : \ tu \in K\}$$

of the resulting body K satisfies

$$\rho_K(u) = \begin{cases} R(\alpha) & \text{if } u_1 > 0, \\ r(\alpha) & \text{if } u_1 < 0, \end{cases} \tag{22}$$

where $u = (u_1, \dots) \in \mathbb{S}^{d-1}$ and $\alpha \in [0, \frac{\pi}{2}]$, $\cos\alpha = |u_1|$.

The solutions $R(\alpha)$ and $r(\alpha)$ of the equations that we will use during the last step may develop a singularity at $\alpha = 0$. To avoid this, we will impose several additional cancellation conditions on the perturbation function h. We will use the Borsuk-Ulam Theorem to show the existence of a non-zero function h satisfying these cancellation restrictions.

Step 1. We put $x = x_0$, $y = y_0$ on $[1, \infty)$, which is equivalent to putting $f(\xi) = \sqrt{1-\xi^2}$ for $\xi \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.

Step 2. Differentiating equation (19) $q+1$ times, we obtain

$$\left(\frac{\partial}{\partial s}\right)^{q+1} \int_{-x(s)}^{y(s)} (f^2(\xi) - L^2(s,\xi))^{q+\frac{1}{2}} \, d\xi \tag{23}$$

$$= \Big(\int_{-x(s)}^{-x_0(1)} + \int_{y_0(1)}^{y(s)} \Big) \left(\frac{\partial}{\partial s}\right)^{q+1} \Big((f^2(\xi) - L^2(s,\xi))^{q+\frac{1}{2}}\Big) d\xi + V_1(s) = \left(\frac{d}{ds}\right)^{q+1} \frac{const}{\sqrt{1+s^2}},$$

where

$$V_1(s) = \int_{-x_0(1)}^{y_0(1)} \left(\frac{\partial}{\partial s}\right)^{q+1} \Big((f_0^2(\xi) - L^2(s,\xi))^{q+\frac{1}{2}}\Big) d\xi.$$

Note that the main difference between $V_1(s)$ and the function Ξ_1 in the even-dimensional case is that the function V_1 is well-defined only for $s \le 1$ and only if $\|h\|_{C^1}$ is much less than 1. Also, even in that case, $V_1(s)$ is C^∞ on $[0, 1)$ but not at 1, where it is merely continuous.

Observe that

$$\left(\frac{\partial}{\partial s}\right)^{q+1} \Big((f^2(\xi) - L^2(s,\xi))^{q+\frac{1}{2}}\Big) = \frac{J_1(s, \xi, f(\xi))}{\sqrt{f^2(\xi) - L^2(\xi)}},$$

where $J_1(s, \xi, f)$ is some polynomial expression in s, ξ, f, $h(s)$, and the derivatives of h at s.

Making the change of variables $\xi = -x(\sigma)$ in the integral $\int_{-x(s)}^{-x_0(1)}$ and $\xi = y(\sigma)$ in the integral $\int_{y_0(1)}^{y(s)}$, we can rewrite the sum of integrals on the left-hand side of (23) as

$$-\int_s^1 \left[\frac{J_1(s,-x(\sigma),L(\sigma,-x(\sigma)))}{\sqrt{L^2(\sigma,-x(\sigma))-L^2(s,-x(\sigma))}}\,\frac{dx}{ds}(\sigma)+\frac{J_1(s,y(\sigma),L(\sigma,y(\sigma)))}{\sqrt{L^2(\sigma,y(\sigma))-L^2(s,y(\sigma))}}\,\frac{dy}{ds}(\sigma)\right]d\sigma.$$

Now write

$$L^2(\sigma,\xi)-L^2(s,\xi)=(L(\sigma,\xi)-L(s,\xi))(L(\sigma,\xi)+L(s,\xi))$$

and

$$L(\sigma,\xi)-L(s,\xi)=\sigma\xi+h(\sigma)-s\xi-h(s)=(\sigma-s)(\xi+H(s,\sigma)),$$

where

$$H(s,\sigma)=\frac{h(\sigma)-h(s)}{\sigma-s}=\int_0^1 h'(s+(\sigma-s)\tau)d\tau$$

is an infinitely smooth function of s and σ. Denote

$$K_1(s,\sigma,\xi)=\frac{J_1(s,\xi,L(\sigma,\xi))}{\sqrt{(\xi+H(s,\sigma))(L(\sigma,\xi)+L(s,\xi))}}.$$

The function K_1 is well-defined and infinitely smooth for all s, σ, ξ satisfying $(\xi+H(s,\sigma))(L(\sigma,\xi)+L(s,\xi))>0$. If $\|h\|_{C^1}$ is small enough, this condition is fulfilled whenever s, $\sigma\in[\frac{1}{2},1]$ and $|\xi|>\frac{1}{2}$.

Now we can rewrite equation (23) in the form

$$-\int_s^1 \Big(K_1(s,\sigma,-x(\sigma))\,\frac{dx}{ds}(\sigma)\;+\;K_1(s,\sigma,y(\sigma))\,\frac{dy}{ds}(\sigma)\Big)\,\frac{d\sigma}{\sqrt{\sigma-s}} \tag{24}$$

$$=-V_1(s)+\Big(\frac{d}{ds}\Big)^{q+1}\frac{const}{\sqrt{1+s^2}}.$$

Similarly, we can differentiate (20) and transform the resulting equation into

$$-\int_s^1 \Big(K_2(s,\sigma,-x(\sigma))\,\frac{dx}{ds}(\sigma)\;+\;K_2(s,\sigma,y(\sigma))\,\frac{dy}{ds}(\sigma)\Big)\,\frac{d\sigma}{\sqrt{\sigma-s}}=-V_2(s), \tag{25}$$

where K_2 is well-defined and infinitely smooth in the same range as K_1. The function V_2 on the right-hand side of (25) is given by

$$V_2(s)=\int_{-x_0(1)}^{y_0(1)}\Big(\frac{\partial}{\partial s}\Big)^q\Big((f_0^2(\xi)-L^2(s,\xi))^{q-\frac{1}{2}}L(s,\xi)\Big)d\xi,$$

and everything that we said about V_1 applies to V_2 as well.

Equations (24) and (25) together can be written in the form

$$\int_s^1 \frac{K(s,\sigma,z(\sigma),\frac{dz}{ds}(\sigma))}{\sqrt{\sigma-s}}d\sigma=R(s), \tag{26}$$

where, for $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $z' = \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2$,

$$K(s,\sigma,z,z') = -\begin{pmatrix} K_1(s,\sigma,-x)\,x' + K_1(s,\sigma,y)\,y' \\ K_2(s,\sigma,-x)\,x' + K_2(s,\sigma,y)\,y' \end{pmatrix},$$

$$R(s) = \begin{pmatrix} -V_1(s) + \left(\frac{d}{ds}\right)^{q+1} \frac{const}{\sqrt{1+s^2}} \\ -V_2(s) \end{pmatrix}.$$

By Lemma 6 with $b = 1$ (see the appendix), equation (26) is equivalent to

$$-G_2(s,s,z,z') + \int_s^1 \frac{\partial}{\partial s} G_2(s,\sigma,z(\sigma),\frac{dz}{ds}(\sigma))d\sigma = \widetilde{R}(s), \tag{27}$$

where

$$G_2(s,\sigma,z,z') = \int_0^1 \frac{K(s+\tau(\sigma-s),\sigma,z,z')}{\sqrt{\tau(1-\tau)}} d\tau, \qquad \widetilde{R}(s) = \frac{d}{ds}\int_s^1 \frac{R(s')}{\sqrt{s'-s}} ds'.$$

Note that

$$G_2(s,s,z,z') = C \cdot K(s,s,z,z'), \qquad C = \int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}}.$$

To reduce the resulting system of integro-differential equations to a pure system of integral equations, we add two independent unknown functions x', y', denote $z' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, $z_0(s) = \begin{pmatrix} x_0(s) \\ y_0(s) \end{pmatrix}$, and add two new relations

$$z(s) = -\int_s^1 z'(\sigma)d\sigma + z_0(1).$$

Together with (27), they lead to the system

$$\mathbf{G}(s,Z(s)) = \int_s^1 \mathbf{\Theta}(s,\sigma,Z(\sigma))d\sigma + \mathbf{\Xi}(s), \qquad Z = \begin{pmatrix} z \\ z' \end{pmatrix} = \begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix}. \tag{28}$$

Here

$$\mathbf{G}(s,Z) = \begin{pmatrix} z \\ -G_2(s,s,z,z') \end{pmatrix}, \qquad \mathbf{\Theta}(s,\sigma,Z) = -\begin{pmatrix} z' \\ \frac{\partial}{\partial s} G_2(s,\sigma,z,z') \end{pmatrix},$$

and

$$\mathbf{\Xi}(s) = \begin{pmatrix} z_0(1) \\ \widetilde{R}(s) \end{pmatrix}.$$

In what follows, we will choose h so that $\|h\|_{C^1}$ is much less than 1. In this case, $\mathbf{G}$, $\mathbf{\Theta}$ are well-defined and infinitely smooth whenever s, $\sigma \in [\frac{1}{2}, 1]$, $|x|$, $|y| > \frac{1}{2}$, $z' \in \mathbb{R}^2$, and $\mathbf{\Xi}$ is well-defined and infinitely smooth on $[\frac{1}{2}, 1)$. Observe also that

$$D_Z\mathbf{G}\Big|_{(s,Z(s))} = \begin{pmatrix} \mathbf{I} & 0 \\ * & \mathbf{A} \end{pmatrix},$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{A}(s,z) = C \cdot \mathbf{E}(s,z),$$

and

$$\mathbf{E}(s,z) = \begin{pmatrix} K_1(s,s,-x) & K_1(s,s,y) \\ K_2(s,s,-x) & K_2(s,s,y) \end{pmatrix}.$$

The function

$$Z_0(s) = \begin{pmatrix} z_0(s) \\ \frac{dz_0}{ds}(s) \end{pmatrix} = \begin{pmatrix} x_0(s) \\ y_0(s) \\ \frac{dx_0}{ds}(s) \\ \frac{dy_0}{ds}(s) \end{pmatrix}$$

solves the system (28) with $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ corresponding to $h \equiv 0$ (we will denote them by $\mathbf{G}_0$, $\mathbf{\Theta}_0$, $\mathbf{\Xi}_0$) on $[\frac{1}{2}, 1]$, say.

We claim that

$$\det\left(D_Z \mathbf{G}_0\Big|_{(s,Z_0(s))}\right) = \det(\mathbf{A}_0(s, z_0(s))) \neq 0 \qquad \text{for all } s \in [\tfrac{1}{2}, 1]. \tag{29}$$

Indeed, since $K_{1,2}(s,s,\xi)$ have the same signs as $J_{1,2}(s,\xi,L(s,\xi))$ and since

$$J_1(s,\xi,L(s,\xi)) = (2q+1)!!\Big(-L(s,\xi)\frac{\partial}{\partial s}L(s,\xi)\Big)^{q+1},$$

$$J_2(s,\xi,L(s,\xi)) = (2q-1)!!\Big(-L(s,\xi)\frac{\partial}{\partial s}L(s,\xi)\Big)^{q} L(s,\xi),$$

we conclude that the matrix $\mathbf{A}_0(s, z_0(s))$ has the same sign pattern as the matrix

$$\begin{pmatrix} (-1)^{q+1} & (-1)^{q+1} \\ (-1)^q(-x_0(s)) & (-1)^q y_0(s) \end{pmatrix},$$

i.e., the signs in the first row are the same, and the signs in the second one are opposite.

Thus, (29) follows. In particular,

$$\det\left(D_Z \mathbf{G}_0\Big|_{(1,Z_0(1))}\right) \neq 0.$$

Lemma 8 implies then that we can choose some small $\delta > 0$ and construct a C^k close to $Z_0(s)$ solution $Z(s)$ of (28) on $[1-3\delta, 1]$ whenever $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are sufficiently close to $\mathbf{G}_0$, $\mathbf{\Theta}_0$, $\mathbf{\Xi}_0$ in C^k on certain compact sets. Since $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$, and their derivatives are (integrals of) some explicit elementary expressions in Z, s, σ, $h(s)$, and the derivatives of $h(s)$, this closeness assumption will hold if h is sufficiently close to zero together with sufficiently many derivatives. Moreover, since h vanishes on $[1-\delta, 1]$, the assumptions of Remark 2 are satisfied and we have $Z(s) = Z_0(s)$ on $[1-\delta, 1]$.

The x and y components of Z solve the equations obtained by differentiating (19) and (20). The passage to (19), (20) is now exactly the same as in the even case.

Step 3. From now on, we change the point of view and switch to the functions $R(\alpha)$ and $r(\alpha)$, $\alpha \in (0, \frac{\pi}{2})$, related to f by (21). The functions x and y, which we have already constructed, implicitly define C^∞-functions $R_h(\alpha)$ and $r_h(\alpha)$ for all α with $\tan\alpha > 1 - 3\delta$.

Instead of parameterizing hyperplanes by the slopes s of the corresponding linear functions, we will parameterize them by the angles β they make with the x_1-axis, where β is related to s by $\tan\beta = s$.

Our next task will be to derive the equations that would ensure that all central sections corresponding to angles β with $\tan\beta < 1-2\delta$ are maximal and of constant volume. Note that those sections are already defined and satisfy these properties when $\tan\beta \in (1-3\delta, 1-2\delta)$.

We remind the reader that if the volume of the central section $K\cap v^\perp$ of a convex body K is constant, then

$$\frac{1}{d-1}(\mathcal{R}\rho_K^{d-1})(v) = \mathrm{vol}_{d-1}(K\cap v^\perp) = const, \tag{30}$$

where $\mathcal{R}$ is the *spherical radon transform* and ρ_K is the radial function of the body K.

Since

$$\mathrm{vol}_{d-1}(K\cap(v^\perp + te_1)) = \mathrm{vol}_{d-1}((K-te_1)\cap v^\perp),$$

the central section corresponding to a unit vector v with $\langle v, e_1\rangle \neq 0$ is of maximal volume if and only if

$$\mathcal{R}(\rho_K^{d-2}(\cdot)\left.\frac{\partial}{\partial t}\right|_{t=0}\rho_{K-te_1}(\cdot))(v) = 0. \tag{31}$$

Here e_1 is the unit vector along the x_1-axis. We need these equations to hold for all unit vectors $v = (\pm\sin\alpha, \dots) \in \mathbb{S}^{d-1}$ corresponding to the angles α with $\tan\alpha < 1-2\delta$.

Note that when $K = K_f$ is the body of revolution that we are constructing, these equations hold if $\tan\alpha \in (1-3\delta, 1-2\delta)$, and the left-hand sides of (30), (31) are already defined on the cap

$$\{v\in\mathbb{S}^{d-1} : v = (\pm\sin\alpha,\dots), \alpha\in[0,\tfrac{\pi}{2}],\ \tan\alpha \geq 1-3\delta\}$$

and are smooth even rotation-invariant functions there. We will denote these functions by φ_h and ψ_h, respectively.

Now we put $\varphi_h(v) = const$ and $\psi_h(v) = 0$ for $v = (\pm\sin\alpha,\dots)$, $\tan\alpha\in[0, 1-2\delta]$. This definition agrees with the one we already have when $\tan\alpha\in[1-3\delta, 1-2\delta]$, so φ_h and ψ_h are even rotation-invariant infinitely smooth functions on the entire sphere.

Recall that the values of $\mathcal{R}g(v)$ for all $v = (\pm\sin\alpha,\dots)$ with $\tan\alpha > 1-3\delta$ are completely defined by the values of the even function $g(u)$ for all $u = (\pm\cos\alpha,\dots)$, $\tan\alpha > 1-3\delta$ and *for bodies of revolution* (but not in general) the converse is also true (see the explicit inversion formula in [Ga, p. 433, formula (C.17)]).

Since the equation $\mathcal{R}g = \widetilde{g}$ with even C^∞ right-hand side $\widetilde{g}$ is equivalent to

$$\frac{g(v)+g(-v)}{2} = \mathcal{R}^{-1}\widetilde{g}(v),$$

we can rewrite our equations (30) and (31) as

$$\rho_K^{d-1}(u) + \rho_K^{d-1}(-u) = 2(d-1)(\mathcal{R}^{-1}\varphi_h)(u)$$

and

$$\rho_K^{d-2}(u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\rho_{K-te_1}\Big)(u) + \rho_K^{d-2}(-u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\rho_{K-te_1}\Big)(-u) = 2(d-1)(\mathcal{R}^{-1}\psi_h)(u).$$

The already-constructed part of ρ_K satisfies these equations for $u = (\pm\cos\alpha, \dots)$ with $\tan\alpha > 1 - 3\delta$.

Since the spherical radon transform commutes with rotations and our initial ρ_K was rotation invariant, the even functions $2(d-1)\mathcal{R}^{-1}\varphi_h(u)$, $2(d-1)\mathcal{R}^{-1}\psi_h(u)$ are rotation invariant as well and can be written as $\Phi_h(\alpha)$ and $\Psi_h(\alpha)$, where $u = (\pm\cos\alpha, \dots)$, $\alpha \in [0, \frac{\pi}{2}]$. Note that the mappings $h \mapsto \Phi_h$, $h \mapsto \Psi_h$ are continuous from C^{k+d} to C^k, say. Thus, for all h sufficiently close to zero in C^{k+d}, Φ_h and Ψ_h will be close to $\Phi_0 \equiv 2$ and $\Psi_0 \equiv 0$ in C^k.

We will be looking for a rotation-invariant solution ρ_K, which will be described in terms of two functions $R(\alpha)$ and $r(\alpha)$ related to it by (22). Equation (30) translates into

$$r^{d-1}(\alpha) + R^{d-1}(\alpha) = \Phi_h(\alpha). \tag{32}$$

To rewrite equation (31) , observe that

$$\rho_K^{d-2}(u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\,\rho_{K-te_1}\Big)(u) + \rho_K^{d-2}(-u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\,\rho_{K-te_1}\Big)(-u) \tag{33}$$
$$= -\Big[R^{d-3}(\alpha)(R(\alpha)\sin\alpha)' - r^{d-3}(\alpha)(r(\alpha)\sin\alpha)'\Big]$$

(see Lemma 9 in the appendix). Thus, equation (31) translates into

$$R^{d-3}(\alpha)(R(\alpha)\sin\alpha)' - r^{d-3}(\alpha)(r(\alpha)\sin\alpha)' = -\Psi_h(\alpha).$$

Multiplying it by $(d-2)\sin^{d-3}\alpha$, we obtain

$$\Big((R(\alpha)\sin\alpha)^{d-2} - (r(\alpha)\sin\alpha)^{d-2}\Big)' = -(d-2)\Psi_h(\alpha)\sin^{d-3}\alpha. \tag{34}$$

Taking into account the condition $R(\frac{\pi}{2}) = r(\frac{\pi}{2})$ and integrating, we see that (34) can also be written as

$$R^{d-2}(\alpha) - r^{d-2}(\alpha) = \frac{\Theta_h(\alpha)}{\sin^{d-2}\alpha}, \tag{35}$$

where

$$\Theta_h(\alpha) = (d-2)\int_{\alpha}^{\pi/2} \Psi_h(\beta)\sin^{d-3}\beta d\beta.$$

Note that equations (32), (35) together with conditions $R(\alpha) > 0$ and $r(\alpha) > 0$ determine $R(\alpha)$ and $r(\alpha)$ uniquely, and the originally constructed functions R_h and r_h satisfy these equations for all $\alpha \in [0, \frac{\pi}{2}]$ with $\tan\alpha \geq 1 - 3\delta$. Thus, any solution R, r of this system will satisfy $R(\alpha) = R_h(\alpha)$, $r(\alpha) = r_h(\alpha)$ in this range.

It remains to show that the solutions R and r do exist and to define a *convex body* if h is chosen appropriately. Note that when h is small with several derivatives the functions $\Phi_h - 2$ and Ψ_h are close to zero uniformly with several derivatives. The only problem is that the right-hand side of (35) can blow up as $\alpha \to 0^+$. To prevent this, we will choose the perturbation function h close to 0 in C^{2d+k} so that

$$\Theta_h(0) = \Theta_h'(0) = \dots = \Theta_h^{(d+k-1)}(0) = 0. \tag{36}$$

Then, the right hand side of (35) will be close to zero in $C^k([0, \frac{\pi}{2}])$. Since the map

$$\mathbf{D} : (R, r) \;\mapsto\; (R^{d-1} + r^{d-1},\, R^{d-2} - r^{d-2})$$

is smoothly invertible near the point $(1, 1)$ by the inverse function theorem, the functions R, r exist in this case on the entire interval $[0, \frac{\pi}{2}]$ and are close to 1 in C^2.

Moreover, $R'(0) = r'(0) = 0$, because $\Phi_h'(0) = 0$ (otherwise the function $\mathcal{R}^{-1}\varphi_h$ would not be smooth at $(1, 0, \ldots, 0)$), and the right-hand side of (35) is $o(\alpha)$ as $\alpha \to 0^+$. This is enough to ensure that the body given by R and r is convex and corresponds to some strictly concave function f defined on $[-r(0), R(0)]$.

Finally, to prove the existence of a perturbation function h for which the cancellation conditions (36) hold, we use the Borsuk-Ulam Theorem, stating that a continuous map from $\mathbb{S}^m$ to $\mathbb{R}^m$, taking the antipodal points to antipodal points, contains zero in its image. For $x = (x_1, \ldots, x_{d+k+1}) \in \mathbb{S}^{d+k}$ we define $h_x = \sum\limits_{j=1}^{d+k+1} x_j h_j$, where the h_j are not identically zero smooth functions on $[1-2\delta, 1-\delta]$ with pairwise disjoint supports. We define the map $\mathbf{B} : \mathbb{S}^{d+k} \to \mathbb{R}^{d+k}$ by

$$\mathbf{B} : (x_1, \ldots, x_{d+k+1}) \;\mapsto\; h_x \;\mapsto\; (\Theta_{h_x}(0), \Theta_{h_x}'(0), \ldots, \Theta_{h_x}^{(d+k-1)}(0)).$$

Observing that $R_{(-h)} = r_h$ and $r_{(-h)} = R_h$ and using the linearity of the inverse spherical radon transform, we conclude that the map $h \mapsto \Psi_h$ is odd. Hence, $\mathbf{B}$ maps antipodal points to antipodal points, and there exists some not identically zero h for which (36) holds. This completes the proof of Theorem 1 in the case of an odd dimension.

5. Appendix

All results collected in this appendix are well known. However, since we targeted this article not exclusively at specialists in integral equations and since in many cases it was much harder to find an exact reference than to write a full proof, we decided to present them here. The reader should also keep in mind that we tailored the exact statements to our particular needs, so we cut corners whenever possible to reduce the presentation to the bare minimum.

Lemma 3. *Let g, $q \in C([a, b])$ be two functions with values in $\mathbb{R}^m$. Then, $g \equiv q$ if and only if for every $s \in [a, b)$ the equality*

$$\int_s^b \frac{g(s')}{\sqrt{s'-s}}\, ds' = \int_s^b \frac{q(s')}{\sqrt{s'-s}}\, ds'$$

holds.

Proof. The only non-trivial statement here is that the equality of integrals implies the equality of functions. To prove it, take $t < b$, multiply both parts by $\frac{1}{\sqrt{s-t}}$, and integrate with respect to s from t to b. Making the change of variables $s = t + \tau(s' - t)$, performing the integration with respect to τ first, and canceling the common factor $\int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} > 0$, we obtain

$$\int_t^b g(s')\, ds' = \int_t^b q(s')\, ds' \quad \text{for all } t \in [a, b).$$

The result follows now from the fundamental theorem of calculus and the continuity assumption. ∎

Lemma 4. *Let $R : [a, b] \to \mathbb{R}^m$. If $R \in C([a, b]) \cap C^\infty([a, b))$, then the function*

$$s \mapsto \int_s^b \frac{R(s')}{\sqrt{s'-s}}\, ds'$$

belongs to $C^\infty([a, b))$ and tends to zero as $s \to b^-$.

Proof. The second statement follows from the crude bound

$$\int_s^b \frac{|R(s')|}{\sqrt{s'-s}}\, ds' \le 2\|R\|_{C([a,b])}\sqrt{b-s} \to 0$$

as $s \to b^-$. To prove the first one, fix $\delta > 0$. For $s < b - \delta$, we have

$$\int_s^b \frac{R(s')}{\sqrt{s'-s}}\, ds' = \int_s^{s+\delta} \frac{R(s')}{\sqrt{s'-s}}\, ds' + \int_{s+\delta}^b \frac{R(s')}{\sqrt{s'-s}}\, ds'.$$

The kernel $\frac{1}{\sqrt{s'-s}}$ is C^∞-smooth in s for $s' > s + \delta$, so the second integral is infinitely smooth on $[a, b)$ because R is infinitely smooth there. The first integral can be rewritten as

$$\int_0^\delta \frac{R(s+\tau)}{\sqrt{\tau}}\, d\tau.$$

Since $s+\tau$ stays away from b when s stays away from $b-\delta$, this integral also defines a C^∞-function on $[a, b-\delta)$. Since δ is arbitrary, the lemma follows. □

Lemma 5. *Let $U \in C([a, b]^2)$. Then the function*

$$s \mapsto \int_s^b \frac{U(s,\sigma)}{\sqrt{\sigma-s}}\, d\sigma$$

extended by zero to $s = b$ is continuous.

Proof. Let $s < b$. Fix $\delta \in (0, b-s)$ and let $|s - s'| < \frac{\delta}{2}$. Write

$$\Big|\int_{s'}^b \frac{U(s',\sigma)}{\sqrt{\sigma-s'}}\, d\sigma - \int_s^b \frac{U(s,\sigma)}{\sqrt{\sigma-s}}\, d\sigma\Big|$$

$$\le \int_{s'}^{s+\delta} \frac{|U(s',\sigma)|}{\sqrt{\sigma-s'}}\, d\sigma + \int_s^{s+\delta} \frac{|U(s,\sigma)|}{\sqrt{\sigma-s}}\, d\sigma + \int_{s+\delta}^b \Big|\frac{U(s',\sigma)}{\sqrt{\sigma-s'}} - \frac{U(s,\sigma)}{\sqrt{\sigma-s}}\Big|\, d\sigma.$$

The first two integrals are bounded by $2\|U\|_C\sqrt{\delta+|s'-s|} \le 2\|U\|_C\sqrt{2\delta}$ and $2\|U\|_C\sqrt{\delta}$, respectively. The third one tends to zero as $s' \to s$ because the integrand tends to zero uniformly. The continuity at b follows from the inequality

$$\Big|\int_s^b \frac{U(s,\sigma)}{\sqrt{\sigma-s}}\, d\sigma\Big| \le 2\|U\|_C\sqrt{b-s}.$$

□

Lemma 6. *Let $U \in C^\infty([a,b]^2)$, and let $R \in C([a,b]) \cap C^\infty([a,b))$. Then the equation*

$$\int_s^b \frac{U(s,\sigma)}{\sqrt{\sigma - s}}\, d\sigma = R(s) \tag{37}$$

holds for all $s \in [a,b)$ if and only if the equation

$$-V(s,s) + \int_s^b \frac{\partial V}{\partial s}(s,\sigma) d\sigma = \widetilde{R}(s)$$

holds, where

$$V(s,\sigma) = \int_0^1 \frac{U(s+\tau(\sigma - s),\sigma)}{\sqrt{\tau(1-\tau)}}\, d\tau, \qquad \widetilde{R}(s) = \frac{d}{ds}\int_s^b \frac{R(s')}{\sqrt{s'-s}}\, ds'.$$

Proof. Observe that our assumption on U implies that $V \in C^\infty([a,b]^2)$. Also, by the previous lemma, $\widetilde{R}(s)$ is well-defined for $s \in [a,b)$. By Lemmata 5 and 4 both parts of (37) are continuous functions. Therefore, by Lemma 3, equation (37) is equivalent to

$$\iint_{s<s'<\sigma<b} \frac{U(s',\sigma)}{\sqrt{\sigma - s'}\sqrt{s'-s}}\, ds' d\sigma = \int_s^b \frac{R(s')}{\sqrt{s'-s}}\, ds'.$$

Making the change of variables $s' = t + \tau(\sigma - s)$ and performing the integration with respect to τ first, we can rewrite the left-hand side as $\int_s^b V(s,\sigma)d\sigma$. Observe that both parts tend to zero as $s \to b^-$ and are differentiable functions on $[a,b)$. Thus, their equality is equivalent to the equality of their derivatives. □

Lemma 7 (Banach Fixed Point Theorem ([Ba]))**.** *Suppose that (X,d) is a complete metric space and T is a mapping from X to X satisfying*

$$d(T(x),T(y)) \le k\, d(x,y)$$

for all $x,y \in X$ with some $k \in (0,1)$. Then

(1) *there exists a unique fixed point z_* of the mapping T,*

(2) *the sequence of Picard iterations $z_{k+1} = T(z_k)$ starting with any point $z_0 \in X$ converges to z_*,*

(3) *for every point $z \in X$, we have $d(z,z_*) \le \dfrac{d(T(z),z)}{1-k}$.*

Lemma 8. *Let Ω be a domain in $\mathbb{R}^m$, let $[a,b] \subset \mathbb{R}$, and let $\mathbf{G}_0 : [a,b] \times \Omega \to \mathbb{R}^m$, $\mathbf{\Theta}_0 : [a,b]^2 \times \Omega \to \mathbb{R}^m$, $\mathbf{\Xi}_0 : [a,b) \to \mathbb{R}^m$, $Z_0 : [a,b] \to \Omega$. Assume that $\mathbf{G}_0$, $\mathbf{\Theta}_0$, Z_0 are infinitely smooth, $\mathbf{\Xi}_0$ is continuous, and*

$$\mathbf{G}_0(s, Z_0(s)) = \int_s^b \mathbf{\Theta}_0(s,\sigma,Z_0(\sigma))d\sigma + \mathbf{\Xi}_0(s)$$

for all $s \in [a,b]$.

Then $\mathbf{\Xi}_0$ extends to a C^∞-function on $[a,b]$. If, in addition,

$$\det\left(D_Z \mathbf{G}_0\Big|_{(b,Z_0(b))}\right) \ne 0,$$

then there exist $\varepsilon > 0$, $\delta > 0$, such that on the interval $[b-3\delta, b]$, every perturbed equation

$$\mathbf{G}(s, Z(s)) = \int_s^b \mathbf{\Theta}(s, \sigma, Z(\sigma))d\sigma + \mathbf{\Xi}(s) \tag{38}$$

has a unique continuous solution $Z(s)$ satisfying $\|Z - Z_0(b)\|_{C([b-3\delta,b])} < \varepsilon$, provided $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ are infinitely smooth and

$$\|\mathbf{G} - \mathbf{G}_0\|_{C^1([b-3\delta,b]\times B)}, \qquad \|\mathbf{\Theta} - \mathbf{\Theta}_0\|_{C^1([b-3\delta,b]^2\times B)}, \qquad \|\mathbf{\Xi} - \mathbf{\Xi}_0\|_{C^1([b-3\delta,b])}$$

are small enough. This solution is infinitely smooth and close to Z_0 in C^k, provided that

$$\|\mathbf{G} - \mathbf{G}_0\|_{C^k([b-3\delta,b]\times B)}, \qquad \|\mathbf{\Theta} - \mathbf{\Theta}_0\|_{C^k([b-3\delta,b]^2\times B)}, \qquad \|\mathbf{\Xi} - \mathbf{\Xi}_0\|_{C^k([b-3\delta,b])}$$

are small enough. Moreover, the solutions corresponding to two different triples $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ that are close in corresponding C^k are C^k close to each other.

Here B is the closed ball of radius ε centered at $Z_0(b)$.

Proof. The first statement is obvious because all terms in the unperturbed equation except $\mathbf{\Xi}_0$ are defined and infinitely smooth on the entire interval $[a, b]$.

Next, denote $\mathbf{Q} = D_Z\mathbf{G}_0(b, Z_0(b))$ and observe that there exist $\varepsilon > 0$, $\delta_1 > 0$ such that $B \subset \Omega$, and for all $s \in [b-3\delta_1, b]$ and all Z such that $|Z - Z_0(b)| < \varepsilon$, we have

$$|Z_0(s) - Z_0(b)| < \frac{\varepsilon}{8} \quad \text{and} \quad \|D_Z\mathbf{G}_0(s, Z) - \mathbf{Q}\| \le \frac{1}{8\|\mathbf{Q}^{-1}\|}.$$

The perturbed equation (38) can be rewritten as $Z(s) = (TZ)(s)$, where

$$(TZ)(s) = Z(s) - \mathbf{Q}^{-1}\Big[\mathbf{G}(s, Z(s)) - \int_s^b \mathbf{\Theta}(s, \sigma, Z(\sigma))d\sigma - \mathbf{\Xi}(s)\Big].$$

We will show that if $\delta \in (0, \delta_1)$ is small enough and $\|\mathbf{G}-\mathbf{G}_0\|_{C^1([b-3\delta,b])} \le \dfrac{1}{8\|\mathbf{Q}^{-1}\|}$, then T is a contraction on the set X of continuous functions mapping $C([b-3\delta, b])$ to B. Take two functions Z_1 and Z_2 in $C([b-3\delta, b])$ with values in B and notice that

$$\Big|\int_s^b \Big(\mathbf{\Theta}(s, \sigma, Z_1(\sigma)) - \mathbf{\Theta}(s, \sigma, Z_2(\sigma))\Big)d\sigma\Big|$$

$$\le 3\delta\Big[\max_{s',\sigma'\in[b-3\delta,b], |Z-Z_0(b)|\le\varepsilon} \|D_Z\mathbf{\Theta}(s', \sigma', Z)\|\Big]\|Z_1 - Z_2\|_{C([b-3\delta,b])}$$

for all $s \in [b-3\delta, b]$. So, if δ is chosen so small that

$$3\delta\,\|\mathbf{Q}^{-1}\|\,\big(\|\mathbf{\Theta}_0\|_{C^1([b-3\delta,b])} + 1\big) \le \frac{1}{4}$$

and if $\|\mathbf{\Theta} - \mathbf{\Theta}_0\|_{C^1([b-3\delta,b])} < 1$, we have

$$3\delta\,\|\mathbf{Q}^{-1}\| \max_{s',\sigma'\in[b-3\delta,b], |Z-Z_0(b)|\le\varepsilon} \|D_Z\mathbf{\Theta}(s', \sigma', Z)\|$$

$$\le 3\delta\,\|\mathbf{Q}^{-1}\|\Big(\|\mathbf{\Theta}_0\|_{C^1([b-3\delta,b])} + \|\mathbf{\Theta} - \mathbf{\Theta}_0\|_{C^1([b-3\delta,b])}\Big) < \frac{1}{4}$$

and

$$\Big|\mathbf{Q}^{-1}\Big(\int_s^b \mathbf{\Theta}(s,\sigma,Z_1(\sigma))d\sigma - \int_s^b \mathbf{\Theta}(s,\sigma,Z_2(\sigma))d\sigma\Big)\Big| \le \frac{1}{4}\|Z_1 - Z_2\|_{C([b-3\delta,b])}$$

for all $s \in [b-3\delta, b]$.

Consider the function $\mathbf{H}(s,Z) = Z - \mathbf{Q}^{-1}\mathbf{G}(s,Z)$. Note that

$$\|D_Z\mathbf{H}\| = \|\mathbf{I} - \mathbf{Q}^{-1}D_Z\mathbf{G}\| = \|\mathbf{Q}^{-1}(D_Z\mathbf{G} - \mathbf{Q})\|$$

$$\le \|\mathbf{Q}^{-1}\|\Big[\|D_Z\mathbf{G}_0 - \mathbf{Q}\| + \|D_Z\mathbf{G} - \mathbf{G}_0\|\Big] \le \frac{1}{4}$$

when $s \in [b-3\delta, b]$ and $|Z - Z_0(b)| \le \varepsilon$. Thus,

$$\Big|\Big(Z_1(s) - \mathbf{Q}^{-1}\mathbf{G}(s,Z_1(s))\Big) - \Big(Z_2(s) - \mathbf{Q}^{-1}\mathbf{G}(s,Z_2(s))\Big)\Big| = \Big|\mathbf{H}(s,Z_1) - \mathbf{H}(s,Z_2)\Big|$$

$$\le \Big[\max_{s'\in[b-3\delta,b],\,|Z-Z_0(b)|\le\varepsilon} \|D_Z\mathbf{H}(s',Z)\|\Big]\|Z_1 - Z_2\|_{C([b-3\delta,b])} \le \frac{1}{4}\|Z_1 - Z_2\|_{C([b-3\delta,b])}.$$

Bringing these estimates together, we see that

$$\Big|(TZ_1)(s) - (TZ_2)(s)\Big| \le \frac{1}{2}\|Z_1 - Z_2\|_{C([b-3\delta,b])}$$

for all $s \in [b-3\delta, b]$. To apply the Banach fixed point theorem, it remains to show that T maps X to itself. To this end, notice that

$$\Big|(TZ_0)(s) - Z_0(s)\Big| = \Big|\mathbf{Q}^{-1}\Big[\Big(\mathbf{G}(s,Z_0(s)) - \mathbf{G}_0(s,Z_0(s))\Big)$$

$$- \int_s^b \Big(\mathbf{\Theta}(s,\sigma,Z_0(\sigma)) - \mathbf{\Theta}_0(s,\sigma,Z_0(\sigma))\Big)d\sigma - \Big(\mathbf{\Xi}(s) - \mathbf{\Xi}_0(s)\Big)\Big]\Big|$$

$$\le \|\mathbf{Q}^{-1}\|\Big(\|\mathbf{G} - \mathbf{G}_0\|_{C([b-3\delta,b]\times B)} \tag{39}$$

$$+3\delta\|\mathbf{\Theta} - \mathbf{\Theta}_0\|_{C([b-3\delta,b]^2\times B)} + \|\mathbf{\Xi}(s) - \mathbf{\Xi}_0(s)\|_{C([b-3\delta,b])}\Big) < \frac{\varepsilon}{4},$$

provided that the C-norms in the last two lines are small enough.

Let $Z \in X$. It is obvious that TZ is a continuous function on $[b-3\delta, b]$. Also,

$$|Z(s) - Z_0(s)| \le |Z(s) - Z_0(b)| + |Z_0(s) - Z_0(b)| \le \varepsilon + \frac{\varepsilon}{8} = \frac{5\varepsilon}{8}$$

for all $s \in [b-3\delta, b]$, so $\|Z - Z_0\|_{C([b-3\delta,b])} \le \frac{5\varepsilon}{8}$, and

$$|TZ(s) - Z_0(b)| \le |(TZ)(s) - (TZ_0)(s)| + |(TZ_0)(s) - Z_0(s)| + |Z_0(s) - Z(b)|$$

$$\le \frac{1}{2}\cdot\frac{5\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon.$$

Thus, $TZ \in X$ as well. This completes the proof of the existence and uniqueness part of the lemma.

To show that Z is smooth, notice that the right-hand side of equation (38) is a C^1-function for every $Z \in X$. Thus, the left-hand side $\mathbf{G}(s, Z(s))$ is also C^1. Since $\mathbf{G} \in C^\infty([a,b]\times\Omega)$ and $D_Z\mathbf{G}(s,Z) \ne 0$ whenever $s \in [b-3\delta, b]$, $|Z - Z_0(b)| < \varepsilon$, we conclude by the implicit function theorem that $Z \in C^1$ and, moreover,

$$\frac{dZ}{ds}(s) =$$
$$\Big(D_Z\mathbf{G}(s,Z(s))\Big)^{-1}\Big(-\frac{\partial \mathbf{G}}{\partial s}(s,Z(s)) - \mathbf{\Theta}(s,s,Z(s) + \int_s^b \frac{\partial \mathbf{\Theta}}{\partial s}(s,\sigma,Z(\sigma))d\sigma + \frac{d\mathbf{\Xi}}{ds}(s)\Big).$$

Differentiating this identity again and again and plugging the expression for the derivative $\frac{dZ}{ds}$ into the right-hand side after every differentiation, we see that Z is infinitely smooth, and, moreover, $(\frac{d}{ds})^k Z$ can be written as some explicit expression involving only Z itself and various partial derivatives of the functions $\mathbf{G}$, $\mathbf{\Theta}$, $\mathbf{\Xi}$ of orders up to k. We see from this that to show that Z is close to Z_0 in C^k under the condition that the norms

$$\|\mathbf{G}-\mathbf{G}_0\|_{C^k([b-3\delta,b]\times B)}, \qquad \|\mathbf{\Theta}-\mathbf{\Theta}_0\|_{C^k([b-3\delta,b]^2\times B)}, \qquad \|\mathbf{\Xi}-\mathbf{\Xi}_0\|_{C^k([b-3\delta,b])}$$

are small, it suffices to show that, under this condition, the norm $\|Z-Z_0\|_{C([b-3\delta,b])}$ is small. By the third part of the Banach fixed point theorem, this would follow from the smallness of $\|TZ_0 - Z_0\|_{C([b-3\delta,b])}$. But we have already estimated this difference by

$$\|\mathbf{Q}^{-1}\|\,\Big(\|\mathbf{G}-\mathbf{G}_0\|_{C([b-3\delta,b]\times B)}+3\delta\|\Theta-\Theta_0\|_{C([b-3\delta,b]^2\times B)}+\|\Xi(s)-\Xi_0(s)\|_{C([b-3\delta,b])}\Big)$$

in (39).

Exactly the same argument can be used to prove the last statement of the lemma. $\square$

Remark 2. If $\mathbf{\Xi} = \mathbf{\Xi}_0$ on $[b-\delta,b]$, then to check that $\|\mathbf{\Xi}-\mathbf{\Xi}_0\|_{C^k([b-3\delta,b])}$ is small, it suffices to check that $\|\mathbf{\Xi}-\mathbf{\Xi}_0\|_{C^k([b-3\delta,b-\delta])}$ is small. If, in addition, $\mathbf{G}(s,Z) = \mathbf{G}_0(s,Z)$ for all $s \in [b-\delta,b]$ and $\mathbf{\Theta}(s,\sigma,Z) = \mathbf{\Theta}_0(s,\sigma,Z)$ for all $s,\sigma \in [b-\delta,b]$, then the solution Z, whose existence and uniqueness is asserted in Lemma 8, coincides with Z_0 on $[b-\delta,b]$.

This follows from the fact that if $Z = Z_0$ on $[b-\delta,b]$, then $TZ = Z$ on $[b-\delta,b]$ as well, so if we start the Picard iterations with Z_0, the values on this interval will never change.

Lemma 9. *Let K be a body of revolution around the x_1-axis and let ρ_K be the radial function of K. Then*

$$\rho_K^{d-2}(u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\,\rho_{K-te_1}\Big)(u) + \rho_K^{d-2}(-u)\Big(\tfrac{\partial}{\partial t}\big|_{t=0}\,\rho_{K-te_1}\Big)(-u)$$
$$= -\Big[R^{d-3}(\alpha)(R(\alpha)\sin\alpha)' - r^{d-3}(\alpha)(r(\alpha)\sin\alpha)'\Big]$$

with R and r defined by

$$\rho_K(u) = \begin{cases} R(\alpha) & \text{if } u_1 > 0,\\ r(\alpha) & \text{if } u_1 < 0,\end{cases}$$

where $u = (u_1,\dots) \in \mathbb{S}^{d-1}$ and $\alpha \in (0,\frac{\pi}{2})$, $\cos\alpha = |u_1|$.

Proof. Denote by W the (x_1,x_2)-plane. Let l be the line $\{(x_1,x_2,\dots) \in W : x_2 = x_1\tan\alpha\}$, where $\alpha \in (0,\pi/2)$. For a small $t > 0$ we denote by l_t the line $\{x \in W : x_2 = (x_1 - t)\tan\alpha\}$.

Denote by A and B the "top" points of intersection of the boundary of K with l and l_t, respectively. Let C be the point of intersection of l_t with the hyperplane orthogonal to l and passing through A (see Figure 5). Observe that A, B, $C \in W$ and that $K \cap l_t$ is the one-dimensional *central section* of the shifted body $K - te_1$.

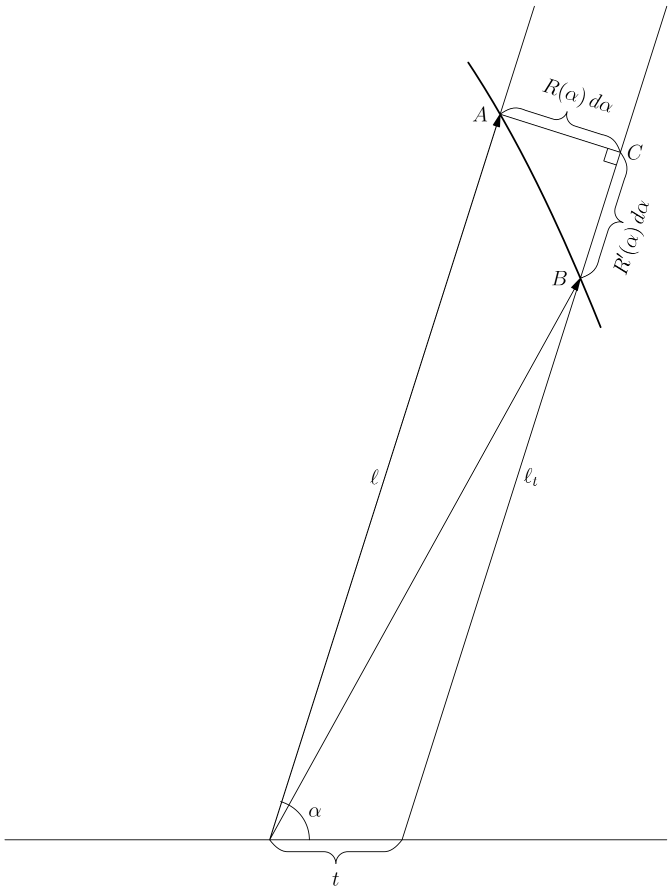

FIGURE 5. $\rho_{K-te_1}(u) = R(\alpha) - t\cos\alpha - t\frac{R'(\alpha)}{R(\alpha)}\sin\alpha$.

By elementary geometry,

$$\rho_{K-te_1}(u) = R(\alpha) - t\cos\alpha - t\sin\alpha\,\tan CAB,$$

where $u = (\cos\alpha, \dots) \in \mathbb{S}^{d-1}$ and $\alpha \in (0, \frac{\pi}{2})$. Observe that, up to terms of order t^2, we have

$$\tan CAB = \frac{R'(\alpha)}{R(\alpha)}.$$

Hence,

$$\rho_{K-te_1}(u) = R(\alpha) - t\cos\alpha - t\frac{R'(\alpha)}{R(\alpha)}\sin\alpha + o(t^2).$$

Similarly,

$$\rho_{K-te_1}(-u) = r(\alpha) + t\cos\alpha + t\frac{r'(\alpha)}{r(\alpha)}\sin\alpha + o(t^2).$$

Finally,

$$\rho_K^{d-2}(u)\Big(\tfrac{\partial}{\partial t}\Big|_{t=0}\rho_{K-te_1}\Big)(u) + \rho_K^{d-2}(-u)\Big(\tfrac{\partial}{\partial t}\Big|_{t=0}\rho_{K-te_1}\Big)(-u)$$
$$= R^{d-2}(\alpha)(-\cos\alpha - \frac{R'(\alpha)}{R(\alpha)}\sin\alpha) - r^{d-2}(\alpha)(-\cos\alpha - \frac{r'(\alpha)}{r(\alpha)}\sin\alpha)$$
$$= R^{d-3}(\alpha)(-R(\alpha)\cos\alpha - R'(\alpha)\sin\alpha) - r^{d-3}(\alpha)(-r(\alpha)\cos\alpha - r'(\alpha)\sin\alpha)$$
$$= -\Big[R^{d-3}(\alpha)(R(\alpha)\sin\alpha)' - r^{d-3}(\alpha)(r(\alpha)\sin\alpha)'\Big].$$

□

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