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THE BUZZARD–DIAMOND–JARVIS CONJECTURE FOR UNITARY GROUPS

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1. Introduction

Let p be a prime number. Classically, given a continous, odd, irreducible representation $\bar{r} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, the weight part of Serre's conjecture predicts the set of weights k such that $\bar{r}$ is isomorphic to the mod p representation $\bar{r}_{f,p}$ attached to some eigenform of weight k (and prime-to-p level). In recent years, generalizations of the weight part of Serre's conjecture have taken on an increasing importance, at least in part because they can be viewed as statements about local-global compatibility in a possible mod p Langlands correspondence, as we now (briefly) recall.

Let F be a number field and $\bar{r} : G_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ a representation that is modular in a suitable sense. For simplicity, suppose that F has a single place w lying above p; in this context a Serre weight is an isomorphism class of irreducible mod p representations of $\mathrm{GL}_n(\mathcal{O}_{F_w})$. One may hope that there exists a mod p local Langlands correspondence that attaches to $\bar{r}|_{G_{F_w}}$ a mod p representation $\overline{\Pi}$ of $\mathrm{GL}_n(F_w)$. Although our present understanding of the putative representation $\overline{\Pi}$ is rather limited, one ultimately expects that $\bar{r}$ should be modular of Serre weight a if and only if a is a subrepresentation of $\overline{\Pi}|_{\mathrm{GL}_n(\mathcal{O}_{F_w})}$.

In this paper we establish the weight part of Serre's conjecture for rank two unitary groups in the case where F is unramified at p. To be precise, we prove the following.

Theorem A (Theorem 2.13). *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that each place of F^+ above p splits in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose $p > 2$ and that $\bar{r}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is an irreducible modular representation with split ramification such that $\bar{r}(G_{F(\zeta_p)})$ is adequate. Assume that p is unramified in F.*

Let a be a Serre weight. Then $a \in W^{\mathrm{BDJ}}(\bar{r})$ if and only if $\bar{r}$ is modular of weight a.

Here $W^{\mathrm{BDJ}}(\bar{r})$ is the set of Serre weights in which $\bar{r}$ is predicted to be modular. We will recall the definition of $W^{\mathrm{BDJ}}(\bar{r})$ in Section 2 below (as well as what we mean for $\bar{r}$ to be modular of weight a, and any other unfamiliar terminology in the statement of the theorem), but for now we give some motivation and context. Theorem A is the natural variant of the Buzzard–Diamond–Jarvis conjecture for

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unitary groups; recall that the original conjecture [BDJ10] was formulated for automorphic forms on indefinite quaternion algebras. Note that strictly speaking, this is not the most general result that one could hope to prove, because of the (mild) assumption that $\bar{r}(G_{F(\zeta_p)})$ is adequate. In fact we prove unconditionally that if $\bar{r}$ is modular of weight a, then $a \in W^{\mathrm{BDJ}}(\bar{r})$; see Proposition 2.11 and Theorem 2.12. The assumption that $\bar{r}(G_{F(\zeta_p)})$ is adequate is needed for the converse, which is proved in [BLGG] via automorphy lifting theorems.

To explain this in greater depth, suppose for simplicity that F^+ has a single place v above p, write the factorization of v in F as ww^c, and assume now that $F_w/\mathbb{Q}_p$ is unramified. If $\bar{r}$ is modular of weight a, then $\bar{r} \simeq \bar{r}_\pi$ for some cuspidal automorphic representation π whose infinitesimal character is determined by the weight a. In particular, the local representation $\bar{r}|_{G_{F_w}}$ has a lift $r_\pi|_{G_{F_w}}$ that is crystalline with specific Hodge–Tate weights: to be precise, the lift $r_\pi|_{G_{F_w}}$ has Hodge type a in the sense of Definition 2.2 below.

One plausible definition for the set of predicted weights $W^{\mathrm{BDJ}}(\bar{r})$ (which is not the definition that we will use, although the main result of this paper shows that it is in fact equivalent to our definition) would be the set of Serre weights a such that $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type a. (There is a natural modification of this definition in the case where $F_w/\mathbb{Q}_p$ is ramified.) Under this description of the set of predicted weights, it would be essentially automatic that if $\bar{r}$ is modular of weight a, then $a \in W^{\mathrm{BDJ}}(\bar{r})$, and the problem would be to prove that every predicted weight actually occurs. Significant progress towards establishing this result was made (irrespective of any ramification conditions on F) in [BLGG]. In particular, [BLGG] shows that under the hypotheses of Theorem A, if $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type a that furthermore is *potentially diagonalizable* in the sense of [BLGGT], then $\bar{r}$ is modular of weight a.

Temporarily adopting this definition of $W^{\mathrm{BDJ}}(\bar{r})$, our task, therefore, is to remove the potential diagonalizability hypothesis; or in other words, we are left with the purely local problem of showing that if $\bar{r}|_{G_{F_w}}$ has a crystalline lift of Hodge type a, then it has a potentially diagonalizable such lift. This is a consequence of the following theorem, which is our main local result.

Theorem B (Theorem 9.1). *Suppose that $p > 2$ and $K/\mathbb{Q}_p$ is a finite unramified extension. Let $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ be a crystalline representation whose κ-labeled Hodge–Tate weights for each embedding $\kappa\colon K \hookrightarrow \overline{\mathbb{Q}}_p$ are $\{0, r_\kappa\}$ with $r_\kappa \in [1, p]$. If $\overline{\rho}$ is reducible, then there exists a reducible crystalline representation $\rho'\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ with the same labeled Hodge–Tate weights as ρ such that $\overline{\rho} \simeq \overline{\rho}'$.*

Before discussing the proof of Theorem B, we make a few additional comments about the global setting of our paper and about the actual definition of $W^{\mathrm{BDJ}}(\bar{r})$ with which we work.

Remark on the definition of $W^{\mathrm{BDJ}}(\bar{r})$. One often builds the potential diagonalizability hypothesis into the definition of $W^{\mathrm{BDJ}}(\bar{r})$. In fact this is what is done in [BLGG], and for consistency we will adopt the same definition here. In this optic, the results of [BLGG] prove that if $a \in W^{\mathrm{BDJ}}(\bar{r})$, then $\bar{r}$ is modular of weight a (assuming of course that $\bar{r}$ is modular to begin with); but then it becomes non-trivial to show that if $\bar{r}$ is modular of weight a, then a is a predicted weight, and that is what is done in the present paper. One advantage of this alternative definition is that it is relatively easier to make completely explicit. Such a description of the set of Serre weights in the unramified case was made in [BDJ10], and it is in these explicit terms that we define the set $W^{\mathrm{BDJ}}(\bar{r})$ in Section 2 below. (In the case that $\bar{r}$ is reducible but indecomposable, the description is in terms of certain crystalline extension classes.)

Remarks on related papers. Theorem A had previously been established in the case of *generic* (or *regular*) weights in [Gee11], by a rather different method. In particular, the regularity hypothesis allowed the author to avoid the difficulties that arise when dealing with Hodge–Tate weights outside the Fontaine–Laffaille range, i.e., the Hodge–Tate weight range $[0, p-2]$. The main contribution of this paper is a method for addressing these difficulties. It is perhaps also worth emphasizing that for many applications (for instance the work of the first author and Kisin [GK12] on the Breuil-Mézard conjecture for potentially Barsotti–Tate representations) it is essential that one know the weight part of Serre's conjecture in its entirety, rather than generically.

We also recall that our previous paper [GLS12] established the weight part of Serre's conjecture for unitary groups in the totally ramified case. In that paper we used a mixture of local and global techniques to complete the proof. These techniques relied on a combinatorial relationship between Serre weights and the existence of potentially Barsotti–Tate lifts, which does not hold in general; in particular we were able to avoid having to prove the analogue of Theorem B in that setting.

Finally, we remark that Theorem A is rather more general than anything that has been proved directly for inner forms of GL_2 over totally real fields, where there is a parity obstruction due to the unit group: algebraic Hilbert modular forms must have paritious weight, which prevents one from applying the techniques of [BLGG] for non-paritious mod p weights. However, there are now two proofs (due to Newton [New13] and to Gee–Kisin [GK12]) that the weight part of Serre's conjecture for inner forms of GL_2 is equivalent to the conjecture for unitary groups. In combination with the results in this paper and in [BLGG], the conjecture for inner forms of GL_2 (that is, the original Buzzard–Diamond–Jarvis conjecture) has thus been established, under a mild Taylor–Wiles hypothesis on the image of the global representation.

Discussion of our approach to proving Theorem B. In the special case that the Hodge–Tate weights r_κ are all contained in the interval $[1, p-2]$, Theorem B follows easily from Fontaine–Laffaille theory. However, Fontaine–Laffaille theory cannot be extended to the required range, and so new methods are needed.

Perhaps the most direct approach to Theorem B would be to write down all the filtered φ-modules corresponding to crystalline representations ρ of the sort considered in the theorem and to attempt to compute each $\overline{\rho}$ explicitly, for instance using the theory of (ϕ, Γ)-modules and Wach modules. Some partial results towards Theorem B have been obtained by other authors working along these lines (cf. [CD11], [Dou10], [Zhu08]; the results of [CD11] are limited primarily to the case that $[K : \mathbb{Q}_p] = 2$, whereas the other two references consider only semi-simple $\overline{\rho}$ and restricted classes of representations). However, the general case has so far been resistant to these methods.

Instead, our idea is to proceed indirectly, by characterizing the mod p representations $\overline{\rho}$ that arise in Theorem B without actually computing the reduction mod

p of any specific ρ. The key technical innovation in our paper is that it is possible to carry out such an approach using the theory of $(\varphi, \hat{G})$-modules introduced in [Liu10b]. In particular, we are able to prove a structure theorem for $(\varphi, \hat{G})$-modules attached to crystalline Galois representations of *arbitrary dimension* with Hodge–Tate weights in $[0, p]$ (Theorem 4.1); this result is best possible, in the sense that it does not extend to any wider Hodge–Tate weight range. We expect this structure theorem to be of broader interest. For instance, it can be used to study the possible reductions mod p of n-dimensional crystalline representations with Hodge–Tate weights in the range $[0, p]$; we hope to report on this in a future paper.

The proof of the structure theorem is rather delicate and relies on a close study of the monodromy operator; the result does not extend to a wider range of Hodge–Tate weights, nor do we know how to extend it to the ramified case.

Now assume that ρ is as in Theorem B. We use our structure theorem and an elementary argument to determine the list of possible subcharacters of $\overline{\rho}$ (Corollary 7.11). This essentially completes the proof in the completely decomposable case, but in the indecomposable case we need to show that we have a lift of $\overline{\rho}$ to a particular crystalline extension of characters. To do this, we begin by making a careful study of the possible extensions of rank one Kisin modules. We then examine the possibility of extending these Kisin modules to $(\varphi, \hat{G})$-modules and show that in most cases such an extension is unique. Together with some combinatorial arguments, this enables us to show that all of the Galois representations resulting from these $(\varphi, \hat{G})$-modules have reducible crystalline lifts with the desired Hodge–Tate weights, completing the proof of Theorem B. Finally, note that Theorem B addresses only the case where $\overline{\rho}$ is reducible; we conclude by deducing the irreducible case of Theorem A from the reducible one, using the fact that an irreducible $\overline{\rho}$ becomes reducible after restriction to an unramified quadratic extension, together with another combinatorial argument.

It is natural to ask whether our methods could be extended to handle the general case, where $F_w/\mathbb{Q}_p$ is an arbitrary extension. Unfortunately we do not know how to do this, because the proof of the key Theorem 4.1 relies on the assumption that the base field is unramified.

Outline of the paper. In Section 2 we recall some material from [BLGG] and in particular explain the precise local results that we will need to prove in the remainder of the paper. The next three sections are concerned with the general theory of Kisin modules and $(\varphi, \hat{G})$-modules attached to crystalline representations. In Section 3 we review what we will need of the theory of Kisin modules from [Kis06]. In Section 4, which is the technical heart of the paper, we prove our structure theorem for the $(\varphi, \hat{G})$-modules attached to crystalline Galois representations (of arbitrary dimension) with Hodge–Tate weights in $[0, p]$. Section 5 proves a variety of foundational results on the $(\varphi, \hat{G})$-modules associated to crystalline representations.

With our technical foundations established, we then begin the proofs of Theorems A and B. Section 6 contains basic results about rank one Kisin modules and $(\varphi, \hat{G})$-modules. In Section 7 a detailed study of the possible extensions of rank one torsion Kisin modules is carried out; crucially, thanks to our work in Section 4 we are able to specialize these results for Kisin modules coming from the reduction mod p of crystalline representations with Hodge–Tate weights in $[0, p]$. This work is extended to the case of $(\varphi, \hat{G})$-modules in Section 8. Finally, we deduce our main results in Sections 9 and 10.

1.1. **Notation and conventions.**

1.1.1. *Galois theory.* If M is a field, we let G_M denote its absolute Galois group. If M is a global field and v is a place of M, let M_v denote the completion of M at v. If M is a finite extension of $\mathbb{Q}_\ell$ for some ℓ, we let M_0 denote the maximal unramified extension of $\mathbb{Q}_\ell$ contained in M, and we write I_M for the inertia subgroup of G_M. If R is a local ring, we write $\mathfrak{m}_R$ for the maximal ideal of R.

Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$ and residue field k. Fix a uniformizer π of K, let $E(u)$ denote the minimal polynomial of π over K_0, and set $e = \deg E(u)$. We also fix an algebraic closure $\overline{K}$ of K. The ring of Witt vectors $W(k)$ is the ring of integers in K_0.

Our representations of G_K will have coefficients in $\overline{\mathbb{Q}}_p$, another fixed algebraic closure of $\mathbb{Q}_p$, whose residue field we denote $\overline{\mathbb{F}}_p$. Let E be a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and containing the image of every embedding of K into $\overline{\mathbb{Q}}_p$; let $\mathcal{O}_E$ be the ring of integers in E, with uniformizer ϖ and residue field $k_E \subset \overline{\mathbb{F}}_p$.

We write $\mathrm{Art}_K \colon K^\times \to W_K^{\mathrm{ab}}$ for the isomorphism of local class field theory, normalized so that uniformizers correspond to geometric Frobenius elements. For each $\sigma \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ we define the fundamental character ω_σ corresponding to σ to be the composite

$$I_K \longrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} \mathcal{O}_K^\times \longrightarrow k^\times \xrightarrow{\ \sigma\ } \overline{\mathbb{F}}_p^\times.$$

In the case that $k \simeq \mathbb{F}_p$, we will sometimes write ω for ω_σ. Note that in this case we have $\omega^{[K:\mathbb{Q}_p]} = \bar{\varepsilon}$; here ε denotes the p-adic cyclotomic character, and $\bar{\varepsilon}$ the mod p cyclotomic character.

We fix a compatible system of p^nth roots of π: that is, we set $\pi_0 = \pi$ and for all $n > 0$ we fix a choice of π_n satisfying $\pi_n^p = \pi_{n-1}$. Similarly fix a compatible system of primitive p^nth roots of unity ζ_{p^n}. Define the following fields:

$$K_\infty = \bigcup_{n=0}^{\infty} K(\pi_n), \qquad K_{p^\infty} = \bigcup_{n=1}^{\infty} K(\zeta_{p^n}), \qquad \hat{K} = \bigcup_{n=1}^{\infty} K_\infty(\zeta_{p^n}).$$

Note that $\hat{K}$ is the Galois closure of K_∞ over K. Write $G_\infty = \mathrm{Gal}(\overline{K}/K_\infty)$, $\hat{G}_{p^\infty} := \mathrm{Gal}(\hat{K}/K_{p^\infty})$, $\hat{G} = \mathrm{Gal}(\hat{K}/K)$, and $H_K := \mathrm{Gal}(\hat{K}/K_\infty)$.

If $p > 2$, then $\hat{G} \simeq \hat{G}_{p^\infty} \rtimes H_K$ and $\hat{G}_{p^\infty} \simeq \mathbb{Z}_p(1)$ (see e.g. [Liu08, Lem. 5.1.2] for a proof), and so we can (and do) fix a topological generator $\tau \in \hat{G}_{p^\infty}$. In that case, we take our choice of ζ_{p^n} to be $\tau(\pi_n)/\pi_n$ for all n.

1.1.2. *Hodge–Tate weights.* If W is a de Rham representation of G_K over $\overline{\mathbb{Q}}_p$ and κ is an embedding $K \hookrightarrow \overline{\mathbb{Q}}_p$, then the multiset $\mathrm{HT}_\kappa(W)$ of Hodge–Tate weights of W with respect to κ is defined to contain the integer i with multiplicity

$$\dim_{\overline{\mathbb{Q}}_p} (W \otimes_{\kappa,K} \widehat{\overline{K}}(-i))^{G_K},$$

with the usual notation for Tate twists. (Here $\widehat{\overline{K}}$ is the completion of $\overline{K}$.) Thus for example $\mathrm{HT}_\kappa(\varepsilon) = \{1\}$. We will refer to the elements of $\mathrm{HT}_\kappa(W)$ as the "κ-labeled Hodge–Tate weights of W", or simply as the "κ-Hodge–Tate weights of W".

1.1.3. *p-adic period rings.* Define $\mathfrak{S} = W(k)[\![u]\!]$. The ring $\mathfrak{S}$ is equipped with a Frobenius endomorphism φ via $u \mapsto u^p$ along with the natural Frobenius on $W(k)$.

We denote by S the p-adic completion of the divided power envelope of $W(k)[u]$ with respect to the ideal generated by $E(u)$. Let $\mathrm{Fil}^r S$ be the closure in S of the ideal generated by $E(u)^i/i!$ for $i \geq r$. Write $S_{K_0} = S[1/p]$ and $\mathrm{Fil}^r S_{K_0} = (\mathrm{Fil}^r S)[1/p]$. There is a unique Frobenius map $\varphi\colon S \to S$ which extends the Frobenius on $\mathfrak{S}$. We write N_S for the K_0-linear derivation on S_{K_0} such that $N_S(u) = -u$.

Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$ where the transition maps are the pth power map. The ring R is a valuation ring with valuation defined by $v_R((x_n)_{n\geq 0}) = \lim_{n\to\infty} p^n v_p(x_n)$, where $v_p(p) = 1$; the residue field of R is $\overline{k}$, the residue field of $\overline{K}$.

By the universal property of the Witt vectors $W(R)$ of R, there is a unique surjective projection map $\theta\colon W(R) \to \widehat{\mathcal{O}}_{\overline{K}}$ to the p-adic completion of $\mathcal{O}_{\overline{K}}$ which lifts the projection $R \to \mathcal{O}_{\overline{K}}/p$ onto the first factor in the inverse limit. We denote by A_{cris} the p-adic completion of the divided power envelope of $W(R)$ with respect to $\ker(\theta)$. Write $\underline{\pi} = (\pi_n)_{n\geq 0} \in R$ and let $[\underline{\pi}] \in W(R)$ be the Teichmüller representative. We embed the $W(k)$-algebra $W(k)[u]$ into $W(R) \subset A_{\mathrm{cris}}$ by the map $u \mapsto [\underline{\pi}]$. This embedding extends to embeddings $\mathfrak{S} \hookrightarrow S \hookrightarrow A_{\mathrm{cris}}$ which are compatible with Frobenius endomorphisms. As usual, we write $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p]$. As a subring of A_{cris}, the ring S is not stable under the action of G; however, S is the subring of G_∞-invariants in A_{cris} (see [Bre97, §4]).

Let $\mathcal{O}_{\mathcal{E}}$ denote the p-adic completion of $\mathfrak{S}[\frac{1}{u}]$, a discrete valuation ring with residue field $k(\!(u)\!)$. Write $\mathcal{E}$ for the field of fractions of $\mathcal{O}_{\mathcal{E}}$. The inclusion $\mathfrak{S} \hookrightarrow W(R)$ extends to an inclusion $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}\, R)$, and thus to $\mathcal{E} \hookrightarrow W(\mathrm{Fr}\, R)[1/p]$. We let $\mathcal{E}^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathcal{E}$ in $W(\mathrm{Fr}\, R)[1/p]$, with ring of integers $\mathcal{O}^{\mathrm{ur}}$. Write $\widehat{\mathcal{E}^{\mathrm{ur}}}$ for the p-adic completion of $\mathcal{E}^{\mathrm{ur}}$, with ring of integers $\widehat{\mathcal{O}^{\mathrm{ur}}}$. Write $\mathfrak{S}^{\mathrm{ur}} = \widehat{\mathcal{O}^{\mathrm{ur}}} \cap W(R) \subset W(\mathrm{Fr}\, R)$.

Set $\underline{\epsilon} := (\zeta_{p^i})_{i\geq 0} \in R$ and $t = -\log([\underline{\epsilon}]) \in A_{\mathrm{cris}}$. For any $g \in G_K$, write $\underline{\varepsilon}(g) = g(\underline{\pi})/\underline{\pi}$, which is a cocycle from G_K to $R^\times$. Note that $\underline{\varepsilon}(\tau) = \underline{\varepsilon}$.

By [Liu07b, Ex. 5.3.3] (see also the discussion before Theorem 3.2.2 of ibid.) there exists an element $\mathfrak{t} \in W(R)$ such that $t = c\varphi(\mathfrak{t})$ with $c \in S^\times$. It is shown in the course of the proof of [Liu10b, Lem 3.2.2] that the image of $\mathfrak{t}$ in R has valuation $\frac{1}{p-1}$. Following [Fon94, §5] we define

$$I^{[m]}B^+_{\mathrm{cris}} = \{x \in B^+_{\mathrm{cris}} : \varphi^n(x) \in \mathrm{Fil}^m B^+_{\mathrm{cris}} \text{ for all } n > 0\}.$$

(See [Fon94, §5] for the definition of the filtration on B^+_{cris}.) For any subring $A \subset B^+_{\mathrm{cris}}$ write $I^{[m]}A = A \cap I^{[m]}B^+_{\mathrm{cris}}$. By [Fon94, Prop 5.1.3] the ideal $I^{[m]}W(R)$ is principal, generated by $\varphi(\mathfrak{t})^m$.

2. Serre weight conjectures

In this section we explain the definition of the sets of weights $W^{\mathrm{BDJ}}(\bar{r})$ and recall some results from [BLGG]. We refer the reader to Section 4 of [BLGG] for a detailed discussion of these definitions and their relationship with other definitions in the literature.

2.1. Local definitions.

Let K be a finite unramified extension of $\mathbb{Q}_p$ of degree f with residue field k, and let $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation.

Definition 2.1. A *Serre weight* is an isomorphism class of irreducible representations of $\mathrm{GL}_2(k)$ over $\overline{\mathbb{F}}_p$. Up to isomorphism, any such representation is of the form

$$F_a := \bigotimes_{\sigma\colon k\hookrightarrow\overline{\mathbb{F}}_p} \det{}^{a_{\sigma,2}} \otimes \mathrm{Sym}^{a_{\sigma,1}-a_{\sigma,2}} k^2 \otimes_{\sigma,k} \overline{\mathbb{F}}_p$$

where $0 \le a_{\sigma,1} - a_{\sigma,2} \le p-1$ for each σ. We recall that $F_a \simeq F_b$ as representations of $\mathrm{GL}_2(k)$ if and only if we have $a_{\sigma,1} - a_{\sigma,2} = b_{\sigma,1} - b_{\sigma,2}$ for all σ, and the character $k^\times \to \overline{\mathbb{F}}_p^\times$, $x \mapsto \prod_{\sigma\colon k\hookrightarrow\overline{\mathbb{F}}_p} \sigma(x)^{a_{\sigma,2}-b_{\sigma,2}}$ is trivial.

Write $\mathbb{Z}^2_+$ for the set of pairs of integers (n_1, n_2) with $n_1 \ge n_2$. We also use the term Serre weight to refer to tuples $a = (a_{\sigma,1}, a_{\sigma,2})_\sigma \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ with the property that $a_{\sigma,1} - a_{\sigma,2} \le p-1$ for all $\sigma \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$, and we identify the Serre weight $a \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ with the Serre weight represented by F_a. (Note that a Serre weight in the latter sense will be represented by infinitely many Serre weights in the former sense.) Since there is a natural bijection between $\mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ and $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$, we will feel free to regard a Serre weight as an element of $(\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$. (In the terminology of [BLGG] we are regarding the Serre weight as a *lift* of itself; as such lifts are unique in the unramified case, we choose not to use this terminology in this paper.)

Definition 2.2. Let $K/\mathbb{Q}_p$ be a finite extension, let $\lambda \in (\mathbb{Z}^2_+)^{\mathrm{Hom}_{\mathbb{Q}_p}(K,\overline{\mathbb{Q}}_p)}$, and let $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be a de Rham representation. Then we say that ρ has *Hodge type* λ if for each $\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ we have $\mathrm{HT}_\kappa(\rho) = \{\lambda_{\kappa,1} + 1, \lambda_{\kappa,2}\}$.

Following [BDJ10] (as explained in [BLGG, §4]), we define an explicit set of Serre weights $W^{\mathrm{BDJ}}(\overline{\rho})$.

Definition 2.3. If $\overline{\rho}$ is reducible, then a Serre weight $a \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ is in $W^{\mathrm{BDJ}}(\overline{\rho})$ if and only if $\overline{\rho}$ has a crystalline lift of the form

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

which has Hodge type a. In particular, if $a \in W^{\mathrm{BDJ}}(\overline{\rho})$, then by [GS11, Lem. 6.2] (or by Lemma 6.3 and Proposition 6.7 below) it is necessarily the case that there is a decomposition $\mathrm{Hom}(k, \overline{\mathbb{F}}_p) = J \coprod J^c$ such that

$$\overline{\rho}|_{I_K} \simeq \begin{pmatrix} \prod_{\sigma\in J} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma\in J^c} \omega_\sigma^{a_{\sigma,2}} & * \\ 0 & \prod_{\sigma\in J^c} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma\in J} \omega_\sigma^{a_{\sigma,2}} \end{pmatrix}.$$

Let K_2 denote the quadratic unramified extension of K inside $\overline{K}$, with residue field k_2.

Definition 2.4. If $\overline{\rho}$ is irreducible, then a Serre weight $a \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ is in $W^{\mathrm{BDJ}}(\overline{\rho})$ if and only if there is a subset $J \subset \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$ containing exactly one element extending each element of $\mathrm{Hom}(k, \overline{\mathbb{F}}_p)$, such that if we write $\mathrm{Hom}(k_2, \overline{\mathbb{F}}_p) = J \coprod J^c$, then

$$\overline{\rho}|_{I_K} \simeq \begin{pmatrix} \prod_{\sigma\in J} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma\in J^c} \omega_\sigma^{a_{\sigma,2}} & 0 \\ 0 & \prod_{\sigma\in J^c} \omega_\sigma^{a_{\sigma,1}+1} \prod_{\sigma\in J} \omega_\sigma^{a_{\sigma,2}} \end{pmatrix}.$$

We remark that by [BLGG, Lem. 4.1.19], if $a \in W^{\mathrm{BDJ}}(\overline{\rho})$ and $\overline{\rho}$ is irreducible, then $\overline{\rho}$ necessarily has a crystalline lift of Hodge type a.

It is worth stressing that in all cases, if $a \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{F}}_p)}$ is a Serre weight, then whether or not $a \in W^{\mathrm{BDJ}}(\overline{\rho})$ depends only on the representation F_a; this can be seen by twisting by suitable crystalline characters. It is also worth remarking again (cf. the discussion in the introduction) that there are other definitions one could make of a set of conjectural weights. For example, one could define the set of conjectural weights for $\overline{\rho}$ to be the set of weights a for which $\overline{\rho}$ has a crystalline lift of Hodge type a; this would be the most natural definition from the perspective of local-global compatibility, cf. Proposition 2.11, which shows that any set of conjectural weights should be contained in this set. We choose our definition of $W^{\mathrm{BDJ}}(\overline{\rho})$ in order to be consistent with [BLGG]; ultimately, it follows from the results of this paper that these two definitions are equivalent.

2.2. **Global definitions.** The point of the local definitions above is to allow us to formulate global Serre weight conjectures. Following [BLGG], we work with rank two unitary groups which are compact at infinity. As we will not need to make any arguments that depend on the particular definitions made in [BLGG] and as our methods are purely local, we simply recall some notation and basic properties of the definitions, referring the reader to [BLGG] for precise formulations.

We emphasise that our conventions for Hodge–Tate weights are the opposite of those of [BLGG]; for this reason, we must introduce a dual into the definitions.

Fix an imaginary CM field F in which p is unramified, and let F^+ be its maximal totally real subfield. We define a global notion of Serre weight by taking a product of local Serre weights in the following way.

For each place $w|p$ of F, let k_w denote the residue field of F_w. If w lies over a place v of F^+, write $v = ww^c$. Write $S := \coprod_{w|p} \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$, and let $(\mathbb{Z}^2_+)^S_0$ denote the subset of $(\mathbb{Z}^2_+)^S$ consisting of elements a such that for each $w|p$, if $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$, then

$$a_{\sigma,1} + a_{\sigma c,2} = 0.$$

If $a \in (\mathbb{Z}^2_+)^S$ and $w|p$ is a place of F, then let a_w denote the element $(a_\sigma)_{\sigma\in\mathrm{Hom}(k_w,\overline{\mathbb{F}}_p)}$ of $(\mathbb{Z}^2_+)^{\mathrm{Hom}(k_w,\overline{\mathbb{F}}_p)}$.

Definition 2.5. We say that an element $a \in (\mathbb{Z}^2_+)^S_0$ is a *Serre weight* if for each $w|p$ and $\sigma \in \mathrm{Hom}(k_w, \overline{\mathbb{F}}_p)$ we have

$$p - 1 \geq a_{\sigma,1} - a_{\sigma,2}.$$

Let $\bar{r} \colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous irreducible representation. We refer the reader to [BLGG, Def. 2.1.9] for an explanation of what it means for $\bar{r}$ to be modular, and more precisely for $\bar{r}$ to be modular of some Serre weight a; roughly speaking, $\bar{r}$ is modular of weight a if there is a cohomology class on some unitary group with coefficients in a certain local system corresponding to a whose Hecke eigenvalues are determined by the characteristic polynomials of $\bar{r}$ at Frobenius elements. Since our conventions for Hodge–Tate weights are the opposite of those of [BLGG], we make the following definition.

Definition 2.6. Suppose that $\bar{r} \colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible modular representation. Then we say that $\bar{r}$ *is modular of weight* $a \in (\mathbb{Z}^2_+)^S_0$ if $\bar{r}^\vee$ is modular of weight a in the sense of [BLGG, Def. 2.1.9].

We remark that if $\bar{r}$ is modular, then $\bar{r}^c \simeq \bar{r}^\vee \otimes \overline{\varepsilon}$. We globalize the definition of the set $W^{\mathrm{BDJ}}(\overline{\rho})$ in the following natural fashion.

Definition 2.7. If $\bar{r}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous representation, then we define $W^{\mathrm{BDJ}}(\bar{r})$ to be the set of Serre weights $a \in (\mathbb{Z}^2_+)^S_0$ such that for each place $w|p$ the corresponding Serre weight $a_w \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k_w,\overline{\mathbb{F}}_p)}$ is an element of $W^{\mathrm{BDJ}}(\bar{r}|_{G_{F_w}})$.

One then has the following conjecture.

Conjecture 2.8. *Suppose that $\bar{r}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous irreducible modular representation and that $a \in (\mathbb{Z}^2_+)^S_0$ is a Serre weight. Then $\bar{r}$ is modular of weight a if and only if $a \in W^{\mathrm{BDJ}}(\bar{r})$.*

If $\bar{r}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous representation, then we say that $\bar{r}$ has *split ramification* if any finite place of F at which $\bar{r}$ is ramified is split over F^+. For the remainder of this section, we place ourselves in the following situation.

Hypothesis 2.9. *Let F be an imaginary CM field with maximal totally real subfield F^+, and let $\bar{r}\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Assume that:*

- *$p > 2$,*
- *$[F^+ : \mathbb{Q}]$ is even,*
- *F/F^+ is unramified at all finite places,*
- *p is unramified in F,*
- *each place of F^+ above p splits in F, and*
- *$\bar{r}$ is an irreducible modular representation with split ramification.*

The following result is [BLGG, Thm. 5.1.3], one of the main theorems of that paper, specialized to the case of interest to us where p is unramified in F. (Note that in [BLGG], the set of Serre weights $W^{\mathrm{BDJ}}(\bar{r})$ is often denoted $W^{\mathrm{explicit}}(\bar{r})$. Note also that the assumption that p is unramified in F implies that $\zeta_p \notin F$.)

Theorem 2.10. *Suppose that Hypothesis* 2.9 *holds. Suppose further that $\bar{r}(G_{F(\zeta_p)})$ is adequate. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. Assume that $a \in W^{\mathrm{BDJ}}(\bar{r})$. Then $\bar{r}$ is modular of weight a.*

Here *adequacy* is a group-theoretic condition, introduced in [Tho12]. For subgroups of $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ with $p > 5$, adequacy is equivalent to the usual condition that the action is irreducible; for $p = 3$ it is equivalent to irreducibility and the requirement that the projective image is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_3)$, and for $p = 5$ it is equivalent to irreducibility and the requirement that the projective image is not conjugate to $\mathrm{PSL}_2(\mathbb{F}_5)$ or $\mathrm{PGL}_2(\mathbb{F}_5)$. (See [BLGG, Prop. A.2.1].) We also remark that the hypotheses that F/F^+ is unramified at all finite places, that every place of F^+ dividing p splits in F, and that $[F^+ : \mathbb{Q}]$ is even are in fact part of the definition of "modular" made in [BLGG].

Theorem 2.10 establishes one direction of Conjecture 2.8, and we are left with the problem of "elimination," i.e., the problem of proving that if $\bar{r}$ is modular of weight a, then $a \in W^{\mathrm{BDJ}}(\bar{r})$. The following is [BLGG, Cor. 4.1.8].

Proposition 2.11. *Suppose that Hypothesis* 2.9 *holds. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. If $\bar{r}$ is modular of weight a, then for each place $w|p$ of F, there is a crystalline representation $\rho_w\colon G_{F_w} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\bar{r}|_{G_{F_w}}$, such that ρ_w has Hodge type a_w.*

We stress that Proposition 2.11 does not already complete the proof of Conjecture 2.8, because the representation ρ_w may for example be irreducible when $\overline{\rho}_w$ is reducible (compare with Definition 2.3). However, in light of this result, it is natural to conjecture that the following result holds.

Theorem 2.12. *Let $K/\mathbb{Q}_p$ be a finite unramified extension, and let $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a continuous representation. Let $a \in (\mathbb{Z}^2_+)^{\mathrm{Hom}(k,\overline{\mathbb{Q}}_p)}$ be a Serre weight, and suppose that there is a crystalline representation $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ lifting $\overline{\rho}$, such that ρ has Hodge type a. Then $a \in W^{\mathrm{BDJ}}(\overline{\rho})$.*

Theorem 2.12 is the main local result of this paper, and the remainder of the paper is concerned with its proof. In the case that $\overline{\rho}$ is irreducible, this is Theorem 10.1 below; and in the reducible case it follows immediately from Theorem 9.1. Our methods are purely local. We have the following global consequence, which essentially resolves Conjecture 2.8.

Theorem 2.13. *Suppose that Hypothesis* 2.9 *holds. Suppose further that $\bar{r}(G_{F(\zeta_p)})$ is adequate. Let $a \in (\mathbb{Z}^2_+)^S_0$ be a Serre weight. Then $\bar{r}$ is modular of weight a if and only if $a \in W^{\mathrm{BDJ}}(\bar{r})$.*

Proof. This is an immediate consequence of Theorem 2.10, Proposition 2.11, and Theorem 2.12. ☐

3. Kisin modules with coefficients

We begin to work towards the proof of Theorem 2.12 by recalling some facts about the theory of Kisin modules (or Breuil–Kisin modules) as initiated by [Bre00], [Bre98] and developed in [Kis06] and by giving some (essentially formal) extensions of these results in order to allow for non-trivial coefficients. Throughout this section we allow K to be an arbitrary finite extension of $\mathbb{Q}_p$ and recall that $e = e(K/\mathbb{Q}_p)$ is the ramification index of K. Recall also that our coefficient field E is a finite extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and containing the image of every embedding of K into $\overline{\mathbb{Q}}_p$.

Definition 3.1. A *φ-module over* $\mathfrak{S}$ is an $\mathfrak{S}$-module $\mathfrak{M}$ equipped with a φ-semi-linear map $\varphi_{\mathfrak{M}}\colon \mathfrak{M} \to \mathfrak{M}$. The subscript on $\varphi_{\mathfrak{M}}$ will generally be omitted. A morphism between two φ-modules $(\mathfrak{M}_1, \varphi_1)$ and $(\mathfrak{M}_2, \varphi_2)$ is an $\mathfrak{S}$-linear morphism compatible with the maps φ_i. The map $1 \otimes \varphi\colon \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ is $\mathfrak{S}$-linear, and we say that $(\mathfrak{M}, \varphi)$ has *height* r if the cokernel of $1 \otimes \varphi$ is killed by $E(u)^r$; we say that $(\mathfrak{M}, \varphi)$ has finite height if it has height r for some $r \geq 0$.

Denote by $'\,\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}}$ the category of φ-modules of height r. By definition, a *finite free Kisin module* (of height r) is a φ-module (of height r) $\mathfrak{M}$ such that the underlying $\mathfrak{S}$-module is finite free. *A torsion Kisin module* $\mathfrak{M}$ is a φ-module of height r which is killed by p^n for some $n \geq 0$ and such that the natural map $\mathfrak{M} \to \mathfrak{M}[\frac{1}{u}]$ is injective. By [Liu07b, Prop 2.3.2], this is equivalent to asking that $\mathfrak{M}$ can be written as the quotient of two finite free Kisin modules of equal $\mathfrak{S}$-rank.

Throughout this article, a *Kisin module* $\mathfrak{M}$ is either a finite free Kisin module or a torsion Kisin module, of some height r. We denote by $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}}$ the category of finite free Kisin modules, and by $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S},\mathrm{tor}}$ the category of torsion Kisin modules.

Define contravariant functors $T_{\mathfrak{S}}$ from $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S}}$ and $\mathrm{Mod}^{\varphi,r}_{\mathfrak{S},\mathrm{tor}}$ to the category $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ of $\mathbb{Z}_p[G_\infty]$-modules as follows:

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R)) \text{ if } \mathfrak{M} \text{ is a finite free Kisin module}$$

and

$$T_{\mathfrak{S}}(\mathfrak{M}) := \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) \text{ if } \mathfrak{M} \text{ is a torsion Kisin module.}$$

These definitions are slightly different from the ones that are sometimes given (e.g. [Kis06, Lem. 2.1.2, Cor. 2.1.4]), but in fact the various definitions are equivalent by [Fon90, Prop. B.1.8.3]. We summarize some important properties of the functor $T_{\mathfrak{S}}$.

Theorem 3.2 ([Kis06, Liu08]). (1) *The functor $T_{\mathfrak{S}}$ from $\mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ to $\mathrm{Rep}_{\mathbb{Z}_p}(G_\infty)$ is exact and fully faithful.*

(2) *For any $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$, the functor $T_{\mathfrak{S}}$ restricts to a bijective equivalence of categories between the set of φ-stable $\mathfrak{S}$-submodules $\mathfrak{N} \subset \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ of finite height and the set of G_∞-stable finite free $\mathbb{Z}_p$-submodules of $V = T_{\mathfrak{S}}(\mathfrak{M})[1/p]$.*

(3) *If V is a semi-stable representation of G_K with non-negative Hodge–Tate weights in some range $[0, r]$ and $L \subset V$ is a G_K-stable $\mathbb{Z}_p$-lattice, there exists $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^{\varphi,r}$ such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq L|_{G_\infty}$.*

(4) *With notation as in (3), if D is the filtered (φ, N)-module corresponding to the representation V, then there is a canonical isomorphism*

$$S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \cong S_{K_0} \otimes_{K_0} D$$

compatible with φ and filtrations, as well as with the monodromy operator (whose definition on the left-hand side we will not discuss).

Proof. Exactness in (1) is [Kis06, Lem. (2.1.2), Cor. (2.1.4)], while full faithfulness is [Kis06, Prop. (2.1.12)] or [Liu07b, Cor. 4.2.6]. Part (2) follows from [Kis06, Lem. (2.1.15)] together with the full faithfulness of (1). Part (3) is [Kis06, Cor. (1.3.15), Lem. (2.1.15)]. Finally, part (4) is [Liu08, Cor. 3.2.3]. □

Definition 3.3. With notation as in Theorem 3.2(3), we say that $\mathfrak{M}$ is the Kisin module attached to the lattice L; by Theorem 3.2(1) this is well-defined up to isomorphism.

Let A be a finite commutative $\mathbb{Z}_p$-algebra, by which we mean a commutative $\mathbb{Z}_p$-algebra that is finitely generated as a $\mathbb{Z}_p$-module. We say $\mathfrak{M}$ has a *natural A-action* (or *A-coefficients*) if $\mathfrak{M}$ is an A-module such that the A-action commutes with the $\mathfrak{S}$-action and φ-action on $\mathfrak{M}$ and such that the $\mathbb{Z}_p$-module structures on $\mathfrak{M}$ arising from $\mathbb{Z}_p \subset \mathfrak{S}$ and $\mathbb{Z}_p \to A$ are the same. If $\mathfrak{M}$ has a natural A-action, then it is easy to see that $T_{\mathfrak{S}}(\mathfrak{M})$ is an $A[G_\infty]$-module.

Proposition 3.4. *Let A be a finite commutative $\mathbb{Z}_p$-algebra.*

(1) *Suppose V is a semi-stable representation of G_K with non-negative Hodge–Tate weights and $L \subset V$ is a G_K-stable $\mathbb{Z}_p$-lattice. If L is an A-module such that the A-action commutes with the action of G_K, then the Kisin module attached to $\mathfrak{M}$ has a natural A-action.*

(2) *If L_1, L_2 are lattices with A-action as in (1) and $f\colon L_1 \to L_2$ is an $A[G_\infty]$-module homomorphism, then the map $g\colon \mathfrak{M}_2 \to \mathfrak{M}_1$ such that $T_{\mathfrak{S}}(g) = f$ is a morphism of Kisin modules with natural A-action.*

(3) *If $\mathfrak{M}$ is a Kisin module with $\mathcal{O}_E$-coefficients, then $\mathfrak{M}$ is free as a $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-module. Furthermore there is a natural isomorphism of $\mathbb{Z}_p[G_\infty]$-modules*

$$T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\varphi,\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) \simeq \mathrm{Hom}_{\varphi,\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}_E}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E).$$

Proof. The existence of the natural A-action on $\mathfrak{M}$ in (1) follows from the equivalence of categories in Theorem 3.2(2), and then the full faithfulness of $T_{\mathfrak{S}}$ gives (2). The first part of (3) follows from the fact that $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ is a semi-local ring whose maximal ideals are permuted transitively by φ together with the injectivity of the map $(1\otimes\varphi)\colon \mathfrak{S}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}\to\mathfrak{M}$. See [Kis09, Lemma (1.2.2)] for details.

The remainder of the proof concerns the last part of (3). The argument that we give is motivated by the proof of [Kis08, Lem. (1.4.1)]. Fix once and for all an isomorphism $\eta\colon \mathcal{O}_E\simeq\mathcal{O}_E^\vee:=\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_E,\mathbb{Z}_p)$ of $\mathcal{O}_E$-modules; our natural isomorphism will depend on this choice. Write $\mathfrak{S}_{\mathcal{O}_E}:=\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$, $\mathfrak{S}^{\mathrm{ur}}_{\mathcal{O}_E}:=\mathfrak{S}^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$, and $\mathcal{O}_{\mathcal{E},E}:=\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$. Further define

$$M=\mathcal{O}_{\mathcal{E}}\otimes_{\mathfrak{S}}\mathfrak{M},\qquad \varphi^*M=\mathcal{O}_{\mathcal{E}}\otimes_{\varphi,\mathcal{O}_{\mathcal{E}}}M\simeq\mathcal{O}_{\mathcal{E},E}\otimes_{\varphi,\mathcal{O}_{\mathcal{E},E}}M$$

and

$$M^\vee=\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}}(M,\mathcal{O}_{\mathcal{E}}),\qquad M_E^\vee=\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},E}}(M,\mathcal{O}_{\mathcal{E},E}).$$

Define a φ-action on $M_E^\vee$ as follows. For any $f\in M_E^\vee$, let $f^*\in\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},E}}(\varphi^*M,\mathcal{O}_{\mathcal{E},E})$ be the map sending the basic tensor $a\otimes m$ to $a\varphi(f(m))$. Note that $\varphi^*=1\otimes\varphi\colon\varphi^*M\to M$ is an $\mathcal{O}_{\mathcal{E},E}$-linear bijection, since $E(u)\in\mathcal{O}_{\mathcal{E}}^\times$. Then we can define $\varphi(f)=f^*\circ(\varphi^*)^{-1}$.

It is routine to check that φ on $M_E^\vee$ is a φ-semi-linear map and that $\varphi(f)\circ\varphi=\varphi\circ f$. (In particular, beware that $\varphi(f)\neq\varphi\circ f$.) Similarly, we have a φ-action on $M^\vee$ that also satisfies $\varphi(f)\circ\varphi=\varphi\circ f$.

Extend our fixed isomorphism η to isomorphisms $\eta_{\mathcal{E}}\colon\mathcal{O}_{\mathcal{E},E}\simeq\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E^\vee$ and $\eta^*\colon\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},E}}(M,\mathcal{O}_{\mathcal{E},E})\simeq\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},E}}(M,\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E^\vee)$ of $\mathcal{O}_{\mathcal{E},E}$-modules. If $g=\sum_i x_i\otimes\lambda_i\in\mathcal{O}_{\mathcal{E}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E^\vee$, we write $\theta(g)=\sum_i x_i\lambda_i(1)\in\mathcal{O}_{\mathcal{E}}$. Now we can construct a map $\iota\colon M_E^\vee\to M^\vee$ as follows: for each $f\in M_E^\vee$ we set

$$\iota(f)(m)=\theta(\eta^*(f)(m))$$

for all $m\in M$. Equivalently, $\iota(f)=\theta\circ\eta_{\mathcal{E}}\circ f$. It is easy to see that $\iota(f)$ is $\mathcal{O}_{\mathcal{E}}$-linear. We claim that ι is an isomorphism of $\mathcal{O}_{\mathcal{E},E}$-modules, compatible with φ-actions. To see the former, it suffices to assume that $M=\mathcal{O}_{\mathcal{E},E}$ because M is a finite free $\mathcal{O}_{\mathcal{E},E}$-module. Identifying $\mathcal{O}_E\simeq\mathrm{Hom}_{\mathcal{O}_E}(\mathcal{O}_E,\mathcal{O}_E)$ identifies η with an isomorphism $\mathrm{Hom}_{\mathcal{O}_E}(\mathcal{O}_E,\mathcal{O}_E)\simeq\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_E,\mathbb{Z}_p)$ sending a to $\theta\circ\eta\circ a$ (where θ again denotes evaluation at 1), and so the special case $M=\mathcal{O}_{\mathcal{E},E}$ follows by tensoring this isomorphism with $\mathcal{O}_{\mathcal{E}}$ over $\mathbb{Z}_p$. Checking that ι is φ-compatible boils down to checking that $\iota(f)^*=\theta\circ\eta_{\mathcal{E}}\circ f^*$, which follows directly from the definition since φ commutes with θ and η_E.

Set $\widehat{\mathcal{O}^{\mathrm{ur}}_E}:=\widehat{\mathcal{O}^{\mathrm{ur}}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$. We claim that the injection $\mathrm{Hom}_{\varphi,\mathfrak{S}_{\mathcal{O}_E}}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_{\mathcal{O}_E})\hookrightarrow\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E},E}}(M,\widehat{\mathcal{O}^{\mathrm{ur}}_E})$ is a bijection. To see this, first observe that the $\mathcal{O}_E$-linear map

$$\mathrm{Hom}_{\varphi,\mathfrak{S}}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}}_{\mathcal{O}_E})\hookrightarrow\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E}}}(M,\widehat{\mathcal{O}^{\mathrm{ur}}_E})\tag{3.5}$$

is a bijection: if g is an element of the right-hand side, then the image of $g(\mathfrak{M})$ under any $\widehat{\mathcal{O}^{\mathrm{ur}}}$-linear projection $\widehat{\mathcal{O}^{\mathrm{ur}}_E}\to\widehat{\mathcal{O}^{\mathrm{ur}}}$ must lie in $\mathfrak{S}^{\mathrm{ur}}$ by [Fon90, Proposition B 1.8.3], hence $g(\mathfrak{M})\subset\mathfrak{S}^{\mathrm{ur}}_{\mathcal{O}_E}$. Then the claim follows by taking $\mathcal{O}_E$-invariants on both sides of (3.5). Similarly, we have $\mathrm{Hom}_{\varphi,\mathfrak{S}}(\mathfrak{M},\mathfrak{S}^{\mathrm{ur}})=\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E}}}(M,\widehat{\mathcal{O}^{\mathrm{ur}}})$.

Since $M_E^\vee$ is finite $\mathcal{O}_{\mathcal{E},E}$-free, we have a canonical isomorphism $\widehat{\mathcal{O}^{\mathrm{ur}}_E}\otimes_{\mathcal{O}_{\mathcal{E},E}}M_E^\vee\simeq\mathrm{Hom}_{\mathcal{O}_{\mathcal{E},E}}(M,\widehat{\mathcal{O}^{\mathrm{ur}}_E})$ sending $\sum_i a_i\otimes f_i\mapsto\sum_i a_if_i$. We will now check that this isomorphism identifies $(\widehat{\mathcal{O}^{\mathrm{ur}}_E}\otimes_{\mathcal{O}_{\mathcal{E},E}}M_E^\vee)^{\varphi=1}$ with $\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E},E}}(M,\widehat{\mathcal{O}^{\mathrm{ur}}_E})$. The element $\lambda=\sum_i a_i\otimes f_i\in\widehat{\mathcal{O}^{\mathrm{ur}}_E}\otimes_{\mathcal{O}_{\mathcal{E},E}}M_E^\vee$ is φ-invariant if and only if

$$\sum_i\varphi(a_i)\otimes\varphi(f_i)=\sum_i a_i\otimes f_i,$$

and this is equivalent to the identity $\sum_i \varphi(a_i)(\varphi(f_i))(\varphi(x)) = \sum_i a_i f_i(\varphi(x))$ for all $x \in M$; it suffices to test equality on elements of the form $\varphi(x)$ since $\varphi(M)$ spans M. Recalling that $\varphi(f) \circ \varphi = \varphi \circ f$, we see that λ is φ-invariant if and only if $\sum_i \varphi(a_i)\varphi(f_i(x)) = \sum_i a_i f_i(\varphi(x))$; but this is precisely the condition that $f = \sum_i a_i f_i$ is in $\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E},E}}(M, \widehat{\mathcal{O}^{\mathrm{ur}}_E})$, as desired. Similarly, we obtain an identification of $(\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M^\vee)^{\varphi=1}$ with $\mathrm{Hom}_{\varphi,\mathcal{O}_{\mathcal{E}}}(M, \widehat{\mathcal{O}^{\mathrm{ur}}})$ as $\mathcal{O}_E$-modules.

From what we have proved above, it suffices to show that there is a natural isomorphism $(\widehat{\mathcal{O}^{\mathrm{ur}}_E} \otimes_{\mathcal{O}_{\mathcal{E},E}} M_E^\vee)^{\varphi=1} \simeq (\widehat{\mathcal{O}^{\mathrm{ur}}} \otimes_{\mathcal{O}_{\mathcal{E}}} M^\vee)^{\varphi=1}$ of $\mathcal{O}_E[G_\infty]$-modules. But since we have constructed a natural $\mathcal{O}_{\mathcal{E},E}$-module isomorphism $\iota\colon M_E^\vee \simeq M^\vee$ compatible with φ, we see that

$$M_E^\vee \otimes_{\mathcal{O}_{\mathcal{E},E}} \widehat{\mathcal{O}^{\mathrm{ur}}_E} \simeq M^\vee \otimes_{\mathcal{O}_{\mathcal{E},E}} \widehat{\mathcal{O}^{\mathrm{ur}}_E} \simeq M^\vee \otimes_{\mathcal{O}_{\mathcal{E},E}} (\mathcal{O}_{\mathcal{E},E} \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}^{\mathrm{ur}}}) \simeq M^\vee \otimes_{\mathcal{O}_{\mathcal{E}}} \widehat{\mathcal{O}^{\mathrm{ur}}}$$

and the result follows. □

Remark 3.6. We stress that because of the choice of isomorphism $\eta\colon \mathcal{O}_E \simeq \mathcal{O}_E^\vee$, the isomorphism of Proposition 3.4(3) is natural but not canonical. In fact the functor $T_{\mathfrak{S},\mathcal{O}_E}\colon \mathfrak{M} \rightsquigarrow \mathrm{Hom}_{\varphi,\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}_E}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E)$ is in some sense the correct version of $T_{\mathfrak{S}}$ for use with coefficients; for instance it is evidently compatible with extension of the coefficient field, whereas $T_{\mathfrak{S}}$ is not. It will be convenient for us to use $T_{\mathfrak{S}}$ for the most part, e.g. so that we can directly apply results from certain references. Thanks to Proposition 3.4(3), on the occasions when we need to calculate $T_{\mathfrak{S}}$ we can use $T_{\mathfrak{S},\mathcal{O}_E}$ instead (see e.g. Lemmas 6.3 and 6.4).

4. The shape of Kisin modules with Hodge–Tate weights in $[0, p]$

Let T be a G_K-stable $\mathbb{Z}_p$-lattice in a semi-stable representation V of dimension d with Hodge–Tate weights in $[0, r]$, and let $\mathfrak{M}$ be the Kisin module attached to T. Write $0 \le r_1 \le \cdots \le r_d \le r$ for the Hodge–Tate weights of V.

We will write $[x_1, \dots, x_d]$ for the $d \times d$ diagonal matrix with diagonal entries $x_1, \dots, x_d$. The aim of this section is to prove the following:

Theorem 4.1. *Assume that K is unramified, V is crystalline, $r \le p$, and $p \ge 3$. Then there exists an $\mathfrak{S}$-basis $e_1, \dots, e_d$ of $\mathfrak{M}$ such that the matrix of φ is $X\Lambda Y$ where X and Y are invertible matrices such that Y is congruent to the identity matrix modulo p and where Λ is the matrix $[E(u)^{r_1}, \dots, E(u)^{r_d}]$.*

We proceed in several (progressively less general) steps.

4.1. **General properties of the Hodge filtration.** Let $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ be the Breuil module attached to $\mathfrak{M}$. Unless explicitly stated otherwise, we will regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathcal{D}$ from now on. By Theorem 3.2(4) (i.e., by [Liu08, Cor. 3.2.3]), $\mathcal{D}$ comes from the weakly admissible filtered (φ, N)-module $D_{\mathrm{st}}(V) = (D, \varphi, N, \mathrm{Fil}^i D_K)$, in the sense that there is a canonical isomorphism $\mathcal{D} \cong S_{K_0} \otimes_{K_0} D$ compatible with all structures. We write $f_\pi\colon \mathcal{D} \to D_K$ for the map induced by $u \mapsto \pi$. By [Bre97, §6], $\mathrm{Fil}^i \mathcal{D}$ is inductively defined by $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and

$$\mathrm{Fil}^i \mathcal{D} = \{x \in \mathcal{D} : f_\pi(x) \in \mathrm{Fil}^i D_K, N(x) \in \mathrm{Fil}^{i-1} \mathcal{D}\}. \tag{4.2}$$

Then the filtration $\mathrm{Fil}^i D_K$ coincides with $f_\pi(\mathrm{Fil}^i \mathcal{D})$, again by [Bre97, §6].

Let $\mathfrak{M}^*$ be the $\mathfrak{S}$-submodule $\mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \subset \mathcal{D}$. Recall that we have an $\mathfrak{S}$-linear map $1 \otimes \varphi\colon \mathfrak{M}^* \to \mathfrak{M}$. Define

$$\mathrm{Fil}^i \mathfrak{M}^* = \{x \in \mathfrak{M}^* | (1 \otimes \varphi)(x) \in E(u)^i \mathfrak{M}\}.$$

Lemma 4.3. *The filtration on $\mathfrak{M}^*$ has the following properties.*

(1) $\mathrm{Fil}^i\,\mathfrak{M}^* = \mathfrak{M}^* \cap \mathrm{Fil}^i\,\mathcal{D}$.
(2) $\mathrm{gr}^i\mathfrak{M}^*$ *is finite* $\mathcal{O}_K$*-free.*
(3) $\mathrm{rank}_{\mathcal{O}_K}\mathrm{gr}^i\mathfrak{M}^* = \dim_K \mathrm{gr}^i\mathcal{D}$.

Proof. Since $\mathcal{D} = S_{K_0} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, one can prove (see, for example, [Liu08, §3.2]) that

$$\mathrm{Fil}^i\,\mathcal{D} = \{x \in \mathcal{D} : (1 \otimes \varphi)(x) \in \mathrm{Fil}^i\,S_{K_0}\mathcal{D}\}.$$

Since $\mathfrak{M}$ is finite $\mathfrak{S}$-free, (1) then follows from the fact that $\mathrm{Fil}^i\,S_{K_0} \cap \mathfrak{S} = E(u)^i\mathfrak{S}$.

From (1) it follows that $\mathrm{gr}^i\mathfrak{M}^*$ injects in $\mathrm{gr}^i\mathcal{D}$, which is a K-vector space; this gives (2).

Finally, set $M := \mathfrak{M}^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathrm{Fil}^i\,M := \mathrm{Fil}^i\,\mathfrak{M}^* \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = M \cap \mathrm{Fil}^i\,\mathcal{D}$. Observe that $\mathcal{D} = M + (\mathrm{Fil}^{i+1}\,S_{K_0})\mathfrak{M}^*$, since $M \subset \mathcal{D}$ is finite $\mathfrak{S}[\frac{1}{p}]$-free and any $s \in S_{K_0}$ can be written as $s_0 + s_1$ with $s_0 \in K_0[u] \subset \mathfrak{S}[\frac{1}{p}]$ and $s_1 \in \mathrm{Fil}^{i+1}\,S_{K_0}$. From this we deduce that $\mathrm{Fil}^i\,\mathcal{D} = \mathrm{Fil}^i\,M + (\mathrm{Fil}^{i+1}\,S_{K_0})\mathfrak{M}^*$, so $\mathrm{gr}^i M \simeq \mathrm{gr}^i\mathcal{D}$ and (3) follows. □

Set $M_K := f_\pi(\mathfrak{M}^*) \subset D_K$ and $\mathrm{Fil}^i\,M_K = M_K \cap \mathrm{Fil}^i\,D_K$, so that $\mathrm{Fil}^i\,M_K$ is an $\mathcal{O}_K$-lattice in $\mathrm{Fil}^i\,D_K$. By Lemma 4.3(1), $f_\pi(\mathrm{Fil}^i\,\mathfrak{M}^*) \subset \mathrm{Fil}^i\,M_K$ for $i \in \mathbb{Z}_{\geq 0}$.

Consider the positive integers $1 = n_0 \leq n_1 \leq n_2 \leq \cdots \leq n_{r_d} \leq d$ such that $\dim_K \mathrm{Fil}^i\,D_K = d - n_i + 1$. Choose an $\mathcal{O}_K$-basis $e_1, \ldots, e_d$ of M_K such that $e_{n_i}, \ldots, e_d$ forms an $\mathcal{O}_K$-basis of $\mathrm{Fil}^i\,M_K$; the existence of such a basis follows by repeated application of the following lemma.

Lemma 4.4. *Let D_K be a finite K-vector space, M_K an $\mathcal{O}_K$-lattice in D_K, and $D'_K \subset D_K$ a K-subspace. Then there exists an $\mathcal{O}_K$-basis $e_1, \ldots, e_d$ of M_K such that $\{e_m, \ldots, e_d\}$ is a K-basis of D'_K for some integer m.*

Proof. Consider the exact sequence of K-vector spaces

$$0 \longrightarrow D'_K \longrightarrow D_K \xrightarrow{f} D''_K \longrightarrow 0.$$

Then we get an exact sequence $0 \to M'_K \to M_K \to M''_K \to 0$ where $M'_K = M_K \cap D'_K$ and $M''_K = f(M_K)$. Since D'_K is divisible, we see that M''_K is torsion free and thus finite $\mathcal{O}_K$-free, so there exists a section $s\colon M''_K \hookrightarrow M_K$ such that $M_K = M'_K \oplus s(M''_K)$. □

Proposition 4.5. *Assume that $f_\pi(\mathrm{Fil}^i\,\mathfrak{M}^*) = \mathrm{Fil}^i\,M_K$ for all $i \in \mathbb{Z}_{\geq 0}$.*

(1) *There exists an $\mathfrak{S}$-basis $\hat{e}_1, \ldots, \hat{e}_d$ of $\mathfrak{M}^*$ such that $f_\pi(\hat{e}_j) = e_j$ for all j and $\hat{e}_j \in \mathrm{Fil}^i\,\mathfrak{M}^*$ for $j \geq n_i$.*
(2) *For any basis as in* (1), *the module* $\mathrm{Fil}^{r_d}\,\mathfrak{M}^*$ *is generated by $(\hat{e}_1, \ldots, \hat{e}_d)\Lambda^*$, where Λ^* is the matrix $[E(u)^{r_d - r_1}, \ldots, E(u)^{r_d - r_d}]$.*

Proof. Since $f_\pi(\mathrm{Fil}^i\,\mathfrak{M}^*) = \mathrm{Fil}^i\,M_K$, there exist $\hat{e}_1, \ldots, \hat{e}_d \in \mathfrak{M}^*$ such that $f_\pi(\hat{e}_j) = e_j$ for all j and $\hat{e}_j \in \mathrm{Fil}^i\,\mathfrak{M}^*$ for $j \geq n_i$. One easily checks that $\{\hat{e}_i\}$ forms an $\mathfrak{S}$-basis of $\mathfrak{M}^*$; this proves (1). Now define $\widetilde{\mathrm{Fil}}^i\mathfrak{M}^*$ inductively as follows: $\widetilde{\mathrm{Fil}}^0\mathfrak{M}^* = \mathfrak{M}^*$ and $\widetilde{\mathrm{Fil}}^i\mathfrak{M}^*$ is the $\mathfrak{S}$-submodule generated by $E(u)\widetilde{\mathrm{Fil}}^{i-1}\mathfrak{M}^*$ and $\hat{e}_{n_i}, \ldots, \hat{e}_d$. It is immediate from this description that

$$\widetilde{\mathrm{Fil}}^i\mathfrak{M}^* = \bigoplus_{j=0}^{i-1}(E(u)^{i-j}\mathfrak{S}\hat{e}_{n_j} \oplus \cdots \oplus E(u)^{i-j}\mathfrak{S}\hat{e}_{n_{j+1}-1}) \oplus \bigoplus_{j=n_i}^{d} \mathfrak{S}\hat{e}_j. \tag{4.6}$$

Comparing (4.6) with the statement of the proposition, we see that we will be done if we can prove that $\mathrm{Fil}^{r_d}\mathfrak{M}^* = \widetilde{\mathrm{Fil}}^{r_d}\mathfrak{M}^*$. In fact we now show by induction on i that $\mathrm{Fil}^i\mathfrak{M}^* = \widetilde{\mathrm{Fil}}^i\mathfrak{M}^*$ for $0 \le i \le r_d$. The statement is clear for $i = 0$. Assume that the statement is true for $i = l$, and let us consider the case $i = l+1$. From the construction of $\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^*$ we see that $\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^* \subset \mathrm{Fil}^{l+1}\mathfrak{M}^*$, and so we get a surjection $\alpha\colon \mathrm{Fil}^l\mathfrak{M}^*/\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^* \to \mathrm{Fil}^l\mathfrak{M}^*/\mathrm{Fil}^{l+1}\mathfrak{M}^*$. By (4.6) it is clear that $\mathrm{Fil}^l\mathfrak{M}^*/\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^* = \widetilde{\mathrm{Fil}}^l\mathfrak{M}^*/\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^*$ is finite $\mathcal{O}_K$-free with rank $n_{l+1}-1 = d - \dim_K(\mathrm{Fil}^{l+1} D_K)$. By Lemma 4.3, we know that $\mathrm{gr}^l\mathfrak{M}^*$ is finite $\mathcal{O}_K$-free with $\mathrm{rank}_{\mathcal{O}_K}\mathrm{gr}^l\mathfrak{M}^* = \dim_K \mathrm{gr}^l\mathcal{D}$, so α is an isomorphism if and only if $\dim_K \mathrm{gr}^l\mathcal{D} = d - \dim_K(\mathrm{Fil}^{l+1} D_K)$. But this is immediate from the fact that $(\mathcal{D}, \mathrm{Fil}^i\mathcal{D})$ has a *base adaptée* in the sense of [Bre97, Déf. A.1], and indeed a *base adaptée* given as in the displayed equation in the middle of page 223 of ibid. Therefore α is an isomorphism and we have $\widetilde{\mathrm{Fil}}^{l+1}\mathfrak{M}^* = \mathrm{Fil}^{l+1}\mathfrak{M}^*$. □

4.2. **The range of monodromy.** We retain the notation of the previous subsection, except that we now let N denote the monodromy operator on $\mathcal{D}$. In this subsection, we always regard $\mathfrak{M}$ as an $\varphi(\mathfrak{S})$-submodule of $\mathcal{D}$. Select a $\varphi(\mathfrak{S})$-basis $\hat{e}_1, \dots, \hat{e}_d$ of $\mathfrak{M}$ (not necessarily related to the basis of Proposition 4.5). We have $N(\hat{e}_1, \dots, \hat{e}_d) = (\hat{e}_1, \dots, \hat{e}_d)U$ with U a matrix with coefficients in S_{K_0}. In this subsection, we would like to control the coefficients of U. Let $\tilde{S} = W(k)[\![u^p, \frac{u^{ep}}{p}]\!]$, so that $\varphi(\mathfrak{S}) \subset \tilde{S} \subset S$ and $N(\tilde{S}) \subset \tilde{S}$. Note that unlike S, the ring $\tilde{S}$ has the property that if $u^p x \in \tilde{S}$ for some $x \in K_0[\![u]\!]$, then $x \in \tilde{S}[\frac{1}{p}]$.

Proposition 4.7. *We have $U \in \mathrm{M}_{d\times d}(\tilde{S}[\frac{1}{p}])$. If V is crystalline and $p \ge 3$, then furthermore $U \in u^p(\mathrm{M}_{d\times d}(\tilde{S}[\frac{1}{p}] \cap S))$.*

Proof. Note that $\{\hat{e}_1, \dots, \hat{e}_d\}$ forms an S_{K_0}-basis of $\mathcal{D}$. Let e_i be the image of $\hat{e}_i$ under the natural map $\mathcal{D} \to \mathcal{D}/I_+S\mathcal{D} = D$, where $I_+S = uK_0[\![u]\!] \cap S$. Since D has a unique (φ, N)-equivariant section $s\colon D \to \mathcal{D}$ (see [Bre97, Prop. 6.2.1.1]), we just write e_i for $s(e_i)$; obviously $\{e_1, \dots, e_d\}$ forms an S_{K_0}-basis for $\mathcal{D}$. Let $X \in \mathrm{M}_{d\times d}(S_{K_0})$ be the matrix such that $(\hat{e}_1, \dots, \hat{e}_d) = (e_1, \dots, e_d)X$.

We claim that both X and X^{-1} are in $\mathrm{M}_{d\times d}(\tilde{S}[\frac{1}{p}])$. In fact this is a consequence of the proof of [Liu07a, Prop. 2.4.1], as we now explain. As in that proof, let $\tilde{A} \in \mathrm{M}_{d\times d}(\mathfrak{S})$ denote the matrix such that $\varphi(\hat{e}_1, \dots, \hat{e}_d) = (\hat{e}_1, \dots, \hat{e}_d)\tilde{A}$ in $\mathfrak{M}$; then the matrix of φ on $\mathcal{D}$ with respect to the same basis is $A = \varphi(\tilde{A}) \in \mathrm{M}_{d\times d}(\varphi(\mathfrak{S}))$. Again as in loc. cit. let $A_0 \in \mathrm{M}_{d\times d}(W(k))$ be the matrix of φ on $\mathcal{D}$ with respect to the basis $e_1, \dots, e_d$. Since $\varphi(E(u))/p \in \tilde{S}^\times$, observe that the next-to-last paragraph of loc. cit. actually shows that $p^r A^{-1} \in \mathrm{M}_{d\times d}(\tilde{S})$ and that $A_0A^{-1} = I_d + \frac{u^p}{p^r}Y$ with $Y \in \mathrm{M}_{d\times d}(\tilde{S})$ (note that the matrix Y' in loc. cit. is actually in $\mathrm{M}_{d\times d}(\varphi(\mathfrak{S}))$).

The main part of the argument in loc. cit. shows that $X = X_0 + \sum_{i=0}^{\infty} \frac{u^{p^{i+1}}}{p^r} Z_i$ where $X_0 = A_0A^{-1}$ and Z_i is defined by the formula

$$Z_i = A_0\varphi(A_0)\cdots\varphi^i(A_0)\varphi^{i+1}(Y)\varphi^i(A^{-1})\cdots\varphi(A^{-1})A^{-1}.$$

From the previous paragraph the matrices p^rX_0 and $p^{r(i+1)}Z_i$ are all in $\mathrm{M}_{d\times d}(\tilde{S})$. Choose any $i_0 \ge 1$ such that $p^i \ge er(i+2-i_0)$ for all $i \ge 0$. Then $p^{ri_0}\cdot\frac{u^{p^{i+1}}}{p^r}Z_i \in \tilde{S}$ for $i \ge 0$, and $p^{ri_0}X \in \mathrm{M}_{d\times d}(\tilde{S})$, as desired. The argument for X^{-1} is essentially the same, beginning from an analysis of AA_0^{-1} instead of A_0A^{-1}; cf. the last paragraph of loc. cit.

Since $N(\hat{e}_1, \ldots, \hat{e}_d) = N((e_1, \ldots, e_d)X)$, we compute that

$$U = X^{-1}BX + X^{-1}N(X)$$

where $B \in \mathrm{M}_{d\times d}(K_0)$ is the matrix of N acting on $e_1, \ldots, e_d$. Since $N(X) \in \mathrm{M}_{d\times d}(\tilde{S})$ (indeed it is contained in $u^p\mathrm{M}_{d\times d}(\tilde{S})$), this completes the argument in the semi-stable case.

Suppose for the rest of the argument that V is crystalline, so that $B = 0$, $U = X^{-1}N(X)$, and $U \in u^p\mathrm{M}_{d\times d}(\tilde{S}[\frac{1}{p}])$. Write $U = u^pU'$; we have to show that $U' \in \mathrm{M}_{d\times d}(S)$. Here we use the argument in the proof of [Liu12, Prop. 2.13][1], and we freely use the notation of that item; in particular for any $x \in \mathcal{D}$ we define

$$\tau(x) = \sum_{i=0}^{\infty} \gamma_i(t) \otimes N^i(x). \tag{4.8}$$

Recall that the element t is defined in Section 1.1.3; since the topological generator $\tau \in \hat{G}_{p^\infty}$ acts trivially on t, one can recursively define $\tau^n(x)$.

Suppose that $x \in \mathfrak{M}$. The formula (2.4.2) of ibid. and the comments immediately following it show that $(\tau-1)^n(x) \in u^pB^+_{\mathrm{cris}}\otimes_{\varphi,\mathfrak{S}}\mathfrak{M}$ and $(\tau-1)^n(x) \in I^{[n]}W(R)\otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. We claim that $(\tau - 1)^n(x) \in u^pI^{[n]}W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. In fact, if $y \in u^pB^+_{\mathrm{cris}} \cap I^{[n]}W(R)$, then by [Liu10a, Lem. 3.2.2] we have $y = u^pz$ with $z \in W(R)$. Since $uw \in \mathrm{Fil}^n W(R)$ with $w \in W(R)$ implies $w \in \mathrm{Fil}^n W(R)$, it follows from $u^pz \in I^{[n]}W(R)$ that $z \in I^{[n]}W(R)$, and this proves the claim.

Since $(\tau - 1)^n(x)/u^p$ is in $I^{[n]}W(R)$, it follows exactly as in the final paragraph of the proof of [Liu12, Prop. 2.13] that the elements $(\tau - 1)^n(x)/(ntu^p)$ lie in $A_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ and tend to 0 as $n \to \infty$. (Recall from Section 1.1.3 that $I^{[n]}W(R)$ is a principal ideal generated by $(\varphi(\mathfrak{t}))^n$.) Therefore the sum

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(\tau-1)^n}{ntu^p}(x)$$

converges in $A_{\mathrm{cris}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. But by (2.4.3) and (2.4.4) of ibid. this sum is precisely $N(x)/u^p$. Since $A_{\mathrm{cris}} \cap \tilde{S}[\frac{1}{p}] \subset A_{\mathrm{cris}} \cap S[\frac{1}{p}] = S$ (e.g. by recalling that S is the subring of G_∞-invariants in A_{cris}), we are done. □

Remark 4.9. It is possible that the matrices U and U' in the preceding proof are in $\mathrm{M}_{d\times d}(\tilde{S})$, but we do not know how to show it.

For later use, we record the conclusion of the next-to-last paragraph of the preceding proof (with $n = 1$) as a separate corollary.

Corollary 4.10. *If V is crystalline and $p \geq 3$, then for any $x \in \mathfrak{M}$ there exists $y \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ such that $\tau(x) - x = u^p\varphi(\mathfrak{t})y$, with $\tau(x)$ as in* (4.8) *and $\mathfrak{t}$ the element defined in the last paragraph of Section* 1.1.3.

Write $S' = \tilde{S}[\frac{1}{p}] \cap S$, and let $\mathcal{I}_l$ denote the ideal $\sum_{m=1}^{l} p^{l-m}u^{pm}S'$ in S'. If $x \in \mathfrak{M}$, write $x = (\hat{e}_1, \ldots, \hat{e}_d) \cdot v$ with v a column vector whose entries lie in $\varphi(\mathfrak{S})$ and with $(\hat{e}_1, \ldots, \hat{e}_d)$ viewed as a row vector. Let v_l be the column vector such that $N^l(x) = (\hat{e}_1, \ldots, \hat{e}_d)v_l$.

[1]The hypothesis that $p \geq 3$ is required by the argument in [Liu12, Prop. 2.13]. In fact this is the only place in the proof of Theorem 4.1 that the hypothesis $p \neq 2$ is used.

Corollary 4.11. *Suppose that V is crystalline and $p \geq 3$ .Then v_l has entries in $\mathcal{I}_l$.*

Proof. We proceed by induction on l. For $l = 1$ we have $v_1 = U \cdot v + N(v)$, and since $U \cdot v$ and $N(v)$ both have entries in $u^p S'$ (the former by Proposition 4.7), the base case follows.

Suppose the statement is true for l, and consider the case $l + 1$. We have the recursion formula

$$v_{l+1} = U \cdot v_l + N(v_l),$$

and it suffices to show that the two terms on the right-hand side of the recursion both have entries in $\mathcal{I}_{l+1}$. This is immediate for $U \cdot v_l$ since $u^p \mathcal{I}_l \subset \mathcal{I}_{l+1}$ and $U \in u^p \mathrm{M}_{d \times d}(S')$. For the other term we must show that $N(\mathcal{I}_l) \subset \mathcal{I}_{l+1}$.

Observe that if $z \in S'$, then $N(z) \in pS'$. Indeed, since $z \in K_0[\![u^p]\!]$ and $N(u^{pi}) = -piu^{pi}$, the valuation of the coefficient of u^j in $N(z)/p$ is at least the valuation of the coefficient of u^j in z for any $j \geq 0$, and we have $N(z)/p \in S'$. As a consequence we see that

$$N(p^{l-m}u^{pm}z) = p^{l-m}(-pmu^{pm}z + u^{pm}N(z)) \in p^{l+1-m}u^{pm}S' \subset \mathcal{I}_{l+1}$$

and the induction is complete. □

In the remainder of this subsection, we prove two technical lemmas for the next subsection. In Lemma 4.12 we assume for simplicity that $K = K_0$ is unramified, although the analogue for general K can be proved by exactly the same argument.

Lemma 4.12. *Assume that K is unramified. Suppose that $y \in \mathcal{I}_l$ for some $1 \leq l \leq p$, and write $y = \sum_{i=0}^{\infty} a_i(u - \pi)^i$ with $a_i \in K$. Then we have $a_i \in W(k)$ for $0 \leq i \leq p$. More precisely we have $p^{p+l-1} \mid a_0$, $p^{p+l-i} \mid a_i$ for $1 \leq i \leq p-1$ and $p^{l-1} \mid a_p$.*

Proof. By hypothesis we have $y = \sum_{m=1}^{l} p^{l-m}u^{pm}z_m$ with $z_m \in S'$. We can write $z_m = \sum_{j=0}^{\infty} \frac{b_{j,m}u^{pj}}{(pj)!}$ with $b_{j,m} \in W(k)$. Then

$$\begin{aligned} y &= \sum_{m=1}^{l} p^{l-m} \left(\sum_{j=0}^{\infty} b_{j,m} \frac{u^{p(j+m)}}{(pj)!} \right) \\ &= \sum_{m=1}^{l} \sum_{j=0}^{\infty} p^{l-m} b_{j,m} \frac{(u-\pi+\pi)^{p(j+m)}}{(pj)!} \\ &= \sum_{m=1}^{l} \sum_{j=0}^{\infty} p^{l-m} \frac{b_{j,m}}{(pj)!} \left(\sum_{i=0}^{p(j+m)} \binom{p(j+m)}{i} (u-\pi)^i \pi^{p(j+m)-i} \right) \\ &= \sum_{i=0}^{\infty} \left(\sum_{m=1}^{l} \sum_{j \geq s_{i,p,m}} \frac{b_{j,m}\pi^{p(j+m)-i}p^{l-m}}{(pj)!} \binom{p(j+m)}{i} \right) (u-\pi)^i, \end{aligned}$$

where $s_{i,p,m} = \max\{0, i/p - m\}$. Since we only consider a_i for $0 \leq i \leq p$, we have $s_{i,p,m} = 0$ in all our cases. Note that $p^{pj}/(pj)! \in \mathbb{Z}_p$ for all $j \geq 0$. We first observe that $v_p(a_0) \geq (p-1)m + l \geq p - 1 + l$ because $m \geq 1$. If $1 \leq i \leq p-1$, then p

divides $\binom{p(j+m)}{i}$. So we get $v_p(a_i) \geq pm - i + l - m + 1 \geq p + l - i$. Finally, we have $v_p(a_p) \geq pm + l - m - p \geq l - 1$. □

Lemma 4.13. *We have* $N^l((u-\pi)^k) = \sum_{m=0}^{k} c_m \pi^m (u-\pi)^{k-m}$ *for some* $c_m \in \mathbb{Z}$.

Proof. We induct on l, with trivial base case $l = 0$. Assume that the statement is true for l. Then

$$\begin{aligned} N^{l+1}((u-\pi)^k) &= N\left(\sum_{m=0}^{k} c_m \pi^m (u-\pi)^{k-m}\right) \\ &= \sum_{m=0}^{k} c_m \pi^m (k-m)(u-\pi)^{k-m-1}(-u+\pi-\pi) \end{aligned}$$

which rearranges to

$$\sum_{m=0}^{k} c_m (m-k)\pi^m (u-\pi)^{k-m} + \sum_{m=0}^{k} c_m (m-k)\pi^{m+1}(u-\pi)^{k-m-1}.$$

The induction follows. □

4.3. **The proof of Theorem 4.1.** Retain the notation of the previous subsections, but assume now that $K = K_0$ is unramified, V is crystalline, and $r \leq p$. Recall that π denotes our fixed choice of uniformizer in $W(k)$. The essential remaining input that we need for the proof of Theorem 4.1 is the statement that $f_\pi(\mathrm{Fil}^i \mathfrak{M}^*) = \mathrm{Fil}^i M_K$ for $i \in \mathbb{Z}$ when $p \geq 3$. The proof of that statement is the key point where the hypothesis $r \leq p$ is used (see Remark 4.18 below). We begin the proof with the following lemma.

Lemma 4.14. *Assume that K is unramified. There exists a $\varphi(\mathfrak{S})$-basis $\mathfrak{e}_1, \dots, \mathfrak{e}_d \in \mathfrak{M}$ such that for $0 \leq i \leq r$, $f_\pi(\mathfrak{e}_{n_i}), \dots, f_\pi(\mathfrak{e}_d)$ forms an $\mathcal{O}_K$-basis for* $\mathrm{Fil}^i M_K$.

Proof. There exists an $\mathfrak{S}$-basis $\mathfrak{e}'_1, \dots, \mathfrak{e}'_d$ of $\mathfrak{M}^*$ such that $f_\pi(\mathfrak{e}'_{n_i}), \dots, f_\pi(\mathfrak{e}'_d)$ forms an $\mathcal{O}_K$-basis of $\mathrm{Fil}^i M_K$ for all $0 \leq i \leq r$. (Choose any basis of M_K as in the sentence preceding Lemma 4.4, and lift it to $\mathfrak{M}^*$.) Select any $\varphi(\mathfrak{S})$-basis $\tilde{\mathfrak{e}}_1, \dots, \tilde{\mathfrak{e}}_d$ of $\mathfrak{M}$. We have $(\mathfrak{e}'_1, \dots, \mathfrak{e}'_d) = (\tilde{\mathfrak{e}}_1, \dots, \tilde{\mathfrak{e}}_d)B$ where $B \in \mathrm{M}_{d\times d}(\mathfrak{S})$ is an invertible matrix. Let $B_0 = f_\pi(B) \in \mathrm{GL}_d(W(k))$ and set $(\mathfrak{e}_1, \dots, \mathfrak{e}_d) := (\tilde{\mathfrak{e}}_1, \dots, \tilde{\mathfrak{e}}_d)B_0$. Evidently $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ is a $\varphi(\mathfrak{S})$-basis of $\mathfrak{M}$. Also note that $f_\pi(\mathfrak{e}_i) = f_\pi(\mathfrak{e}'_i)$, so $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ is just the basis we need. □

Remark 4.15. The fact that B_0 has entries in $W(k)$ in the above lemma makes essential use of the hypothesis that K is unramified. We are not aware of any way to extend this lemma to the case of a ramified base.

Proposition 4.16. *Suppose that K is unramified, V is crystalline, $r \leq p$, and $p \geq 3$. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ as in Lemma 4.14 . Then there exists an $\mathfrak{S}$-basis $\mathfrak{e}'_1, \dots, \mathfrak{e}'_d$ of $\mathfrak{M}^*$ with the properties that $f_\pi(\mathfrak{e}'_j) = f_\pi(\mathfrak{e}_j)$ and $\mathfrak{e}'_j - \mathfrak{e}_j \in p\sum_{j'=1}^{d} \mathfrak{S}\mathfrak{e}_{j'}$ for all $1 \leq j \leq d$, and moreover $\mathfrak{e}'_j \in \mathrm{Fil}^i \mathfrak{M}^*$ whenever $n_i \leq j < n_{i+1}$ (taking $i = r_d$ when $j \geq n_{r_d}$).*

In particular $f_\pi(\mathrm{Fil}^i \mathfrak{M}^*) = \mathrm{Fil}^i M_K$ for all $i \geq 0$.

Proof. Let $\mathfrak{e}_1, \dots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ as in Lemma 4.14, and set $e_j = f_\pi(\mathfrak{e}_j) \in M_K$. The desired statement is only non-trivial for $1 \le i \le r$. To prove $f_\pi(\mathrm{Fil}^i \mathfrak{M}^*) = \mathrm{Fil}^i M_K$ for $1 \le i \le r$, we consider the following assertion:

($\star$) For each $i = 1, \dots, r$, there exist $\mathfrak{e}_{n_i}^{(i)}, \dots, \mathfrak{e}_d^{(i)} \in \mathrm{Fil}^i \mathfrak{M}^*$ such that for all $n_i \le j \le d$ we have

$$\mathfrak{e}_j^{(i)} = \mathfrak{e}_j + \sum_{n=1}^{d} \sum_{s=1}^{i-1} \alpha_{j,n,s}^{(i)} p^{p-s} (u-\pi)^s \mathfrak{e}_n$$

with $\alpha_{j,n,s}^{(i)} \in W(k)$. (Recall that the integers n_i are defined immediately above the statement of Lemma 4.4.) Since $f_\pi(\mathfrak{e}_j^{(i)}) = f_\pi(\mathfrak{e}_j) = e_j$, this assertion is sufficient to establish the result, taking $\mathfrak{e}'_j = \mathfrak{e}_j^{(i)}$ whenever $n_i \le j < n_{i+1}$ (and using $i = r_d$ when $j \ge n_{r_d}$).

We will prove the statement ($\star$) by induction on i. Let us first treat the case that $i = 1$. We just set $\mathfrak{e}_j^{(1)} = \mathfrak{e}_j$ for $n_1 \le j \le d$. Note that $f_\pi(\mathfrak{e}_j^{(1)}) = f_\pi(\mathfrak{e}_j) = e_j \in \mathrm{Fil}^1 M_K \subset \mathrm{Fil}^1 D_K$. By the construction of $\mathrm{Fil}^1 \mathcal{D}$, we see that $\mathfrak{e}_j^{(1)} \in \mathrm{Fil}^1 \mathcal{D}$, and therefore $\mathfrak{e}_j^{(1)} \in \mathfrak{M}^* \cap \mathrm{Fil}^1 \mathcal{D} = \mathrm{Fil}^1 \mathfrak{M}^*$. This settles the case that $i = 1$.

Now assume that ($\star$) is valid for some $i < r$, and let us consider the case $i+1$. Set $H(u) = \frac{u-\pi}{\pi}$. If $n_{i+1} \le j \le d$, we set

$$\tilde{\mathfrak{e}}_j^{(i+1)} := \sum_{l=0}^{i} \frac{H(u)^l N^l(\mathfrak{e}_j^{(i)})}{l!}.$$

We claim that $\tilde{\mathfrak{e}}_j^{(i+1)} \in \mathrm{Fil}^{i+1} \mathcal{D}$. Since $f_\pi(\tilde{\mathfrak{e}}_j^{(i+1)}) = e_j \in \mathrm{Fil}^{i+1} D_K$, from (4.2) it suffices to check that $N(\tilde{\mathfrak{e}}_j^{(i+1)}) \in \mathrm{Fil}^i \mathcal{D}$. One computes, after rearranging, that

$$N(\tilde{\mathfrak{e}}_j^{(i+1)}) = \frac{H(u)^i N^{i+1}(\mathfrak{e}_j^{(i)})}{i!} + \sum_{l=1}^{i} \frac{(1 + N(H(u)))H(u)^{l-1} N^l(\mathfrak{e}_j^{(i)})}{(l-1)!}.$$

Now the claim follows from the facts that $N^l(\mathfrak{e}_j^{(i)}) \in \mathrm{Fil}^{i-l} \mathcal{D}$ (apply (4.2) again, together with the inductive assumption that $\mathfrak{e}_j^{(i)} \in \mathrm{Fil}^i \mathfrak{M}^* \subset \mathrm{Fil}^i \mathcal{D}$) and $H(u)^l \in \mathrm{Fil}^l S_{K_0}$, together with the observation that $1 + N(H(u)) = 1 - u/\pi \in \mathrm{Fil}^1 S_{K_0}$.

Now, by induction, we have

$$\tilde{\mathfrak{e}}_j^{(i+1)} - \mathfrak{e}_j^{(i)} = \sum_{l=1}^{i} \frac{(u-\pi)^l}{\pi^l l!} N^l \left(\mathfrak{e}_j + \sum_{n=1}^{d} \sum_{s=1}^{i-1} \alpha_{j,n,s}^{(i)} p^{p-s} (u-\pi)^s \mathfrak{e}_n \right)$$

which rearranges to

$$\begin{aligned} \tilde{\mathfrak{e}}_j^{(i+1)} - \mathfrak{e}_j^{(i)} &= \sum_{l=1}^{i} \frac{(u-\pi)^l}{\pi^l l!} N^l(\mathfrak{e}_j) \\ &+ \sum_{n=1}^{d} \sum_{l=1}^{i} \sum_{s=1}^{i-1} \frac{p^{p-s}(u-\pi)^l}{\pi^l l!} \alpha_{j,n,s}^{(i)} \sum_{t=0}^{l} \binom{l}{t} N^{l-t}((u-\pi)^s) N^t(\mathfrak{e}_n). \end{aligned} \tag{4.17}$$

Now write $N^t(\mathfrak{e}_n) = \sum_{k=1}^{d}\sum_{m=0}^{\infty} c^{n,t}_{m,k}(u-\pi)^m\mathfrak{e}_k$ with $c^{n,t}_{m,k} \in K_0$. Using Lemma 4.13 and noting that we always have $l \geq 1$, we can write

$$\tilde{\mathfrak{e}}_j^{(i+1)} = \mathfrak{e}_j^{(i)} + \sum_{k=1}^{d}\sum_{m=1}^{\infty} b_{m,k}(u-\pi)^m\mathfrak{e}_k$$

for some elements $b_{m,k} \in K_0$. Now we remove all terms of $(u-\pi)$-degree at least $i+1$ from this expression and define

$$\mathfrak{e}_j^{(i+1)} = \mathfrak{e}_j^{(i)} + \sum_{k=1}^{d}\sum_{m=1}^{i} b_{m,k}(u-\pi)^m\mathfrak{e}_k.$$

Since $(u-\pi)^{i+1} = E(u)^{i+1} \in \mathrm{Fil}^{i+1} S_{K_0}$, we see that $\mathfrak{e}_j^{(i+1)}$ is still in $\mathrm{Fil}^{i+1}\mathcal{D}$. Comparing with $(\star)$, it remains to prove that $p^{p-m} \mid b_{m,k}$, which we do by showing that every occurrence of $(u-\pi)^m\mathfrak{e}_k$ on the right-hand side of (4.17) has coefficient divisible by p^{p-m}. There are two cases to consider.

We begin with terms coming from the first sum $\sum_{l=1}^{i} \frac{(u-\pi)^l}{\pi^l l!}N^l(\mathfrak{e}_j)$ in (4.17). By Corollary 4.11 and Lemma 4.12, each term coming from this sum is of the form $\frac{(u-\pi)^l}{\pi^l l!}\cdot a_h(u-\pi)^h\mathfrak{e}_k$ with $l+h \leq i$ and with a_h as in Lemma 4.12 applied to $\mathcal{I}_l$. In all cases a_h is divisible by $p^{p+l-h-1}$, and so this occurrence of $(u-\pi)^{l+h}\mathfrak{e}_k$ has coefficient divisible by p^{p-h-1}. Since $l \geq 1$, the claim follows in this case.

For the large second sum in (4.17), by Corollary 4.11 and Lemmas 4.12 and 4.13, each term coming from this sum is of the form

$$\alpha^{(i)}_{j,n,s}\frac{p^{p-s}(u-\pi)^l}{\pi^l l!}\binom{l}{t}\cdot\left[c_m\pi^m(u-\pi)^{s-m}\right]\cdot\left[a_h(u-\pi)^h\right]\mathfrak{e}_k$$

with $l+(s-m)+h \leq i$, with $c_m \in \mathbb{Z}$ as in Lemma 4.13, with a_h as in Lemma 4.12 applied to $\mathcal{I}_t$ if $t \geq 1$, and with $a_h = \delta_{k,n}\delta_{h,0}$ if $t = 0$. (Here $\delta_{x,y}$ is 1 if $x = y$ and 0 otherwise.) In all cases we have $a_h \in W(k)$, which is all that we will need here. In particular this occurrence of $(u-\pi)^{l+s-m+h}$ has coefficient divisible by $p^{p-s}\pi^{m-l}$, or equivalently by $p^{p-s+m-l}$. Since $h \geq 0$, this gives what we need. ☐

Remark 4.18. If we had $r = p+1$, then the induction in the above argument would fail when trying to deduce the case $i = p+1$ from the case $i = p$. Indeed if we had $i = p$ in the last paragraph of the proof, then the term with $l = p$, $t = h = 0$, and $m = s$ and $k = n$ would have the form $\alpha^p_{j,n,s}c_s\frac{p^{p-s}(u-\pi)^p}{\pi^p p!}\pi^s\mathfrak{e}_n$, whose coefficient need not be in $W(k)$.

Combining Propositions 4.5 and 4.16 immediately gives the following.

Corollary 4.19. *Suppose that K is unramified, V is crystalline, $r \leq p$, and $p \geq 3$. There exists an $\mathfrak{S}$-basis $\hat{e}_1, \dots, \hat{e}_d$ of $\mathfrak{M}^*$ such that $\mathrm{Fil}^{r_d}\mathfrak{M}^*$ is generated by $(\hat{e}_1, \dots, \hat{e}_d)\Lambda^*$, where Λ^* is the matrix $[E(u)^{r_d-r_1}, \dots, E(u)^{r_d-r_d}]$.*

Finally, we can prove Theorem 4.1, which we restate here for the convenience of the reader.

Theorem 4.20. *Assume that K is unramified, V is crystalline, $r \leq p$, and $p \geq 3$. Then there exists an $\mathfrak{S}$-basis $e_1, \dots, e_d$ of $\mathfrak{M}$ such that the matrix of φ is $X\Lambda Y$*

where X and Y are invertible matrices such that Y is congruent to the identity matrix modulo p and where Λ is the matrix $[E(u)^{r_1}, \ldots, E(u)^{r_d}]$.

Proof. Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ as in Lemma 4.14. For each $1 \le j \le d$ choose i such that $n_i \le j < n_{i+1}$ (taking $i = r_d$ when $n_{r_d} \le j$) and set $\mathfrak{e}'_j = \mathfrak{e}_j^{(i)}$ as in the proof of Proposition 4.16. By construction we have $f_\pi(\mathfrak{e}'_j) = e_j$ and $\mathfrak{e}'_j \in \mathrm{Fil}^i \mathfrak{M}^*$, so Proposition 4.5(2) shows that $\mathrm{Fil}^{r_d} \mathfrak{M}^*$ is generated by $(\mathfrak{e}'_1, \ldots, \mathfrak{e}'_d)\Lambda^*$.

We now consider $\mathfrak{M}$ as an $\mathfrak{S}$-module in its own right, rather than as a $\varphi(\mathfrak{S})$-submodule of $\mathcal{D}$. Let A be the matrix of φ on $\mathfrak{M}$ with respect to the basis $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$. Then there exists a matrix B such that $AB = BA = E(u)^{r_d} I_d$. It follows straightforwardly from the definition of $\mathrm{Fil}^{r_d} \mathfrak{M}^*$ that $(\mathfrak{e}_1, \ldots, \mathfrak{e}_d)B$ forms a basis of $\mathrm{Fil}^{r_d} \mathfrak{M}^*$, and therefore there exists a matrix $X^{-1} \in \mathrm{GL}_d(\mathfrak{S})$ such that $(\mathfrak{e}'_1, \ldots, \mathfrak{e}'_d)\Lambda^* X^{-1} = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)B$. If we write $(\mathfrak{e}'_1, \ldots, \mathfrak{e}'_d) = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)Y^{-1}$, then we get $Y^{-1}\Lambda^* X^{-1} = B$. Hence $A = XE(u)^{r_d}(\Lambda^*)^{-1}Y$, and since $E(u)^{r_d}(\Lambda^*)^{-1} = [E(u)^{r_1}, \ldots, E(u)^{r_d}] = \Lambda$, we have $A = X\Lambda Y$.

Finally, observe from the formula for $\mathfrak{e}_j^{(i)}$ in $(\star)$ that $\mathfrak{e}'_j - \mathfrak{e}_j$ is divisible by p (since the index s in $(\star)$ is always at most $p-1$). It follows that Y is congruent to the identity modulo p, as claimed. □

4.4. **Coefficients.** We now prove an analogue of Theorem 4.1 for representations with non-trivial coefficients. Assume as before that $K = K_0$ is unramified, let E be a finite extension of $\mathbb{Q}_p$ containing the images of all the embeddings $K \hookrightarrow \overline{\mathbb{Q}}_p$, and let T be a G_K-stable $\mathcal{O}_E$-lattice in a crystalline representation V of E-dimension d with Hodge–Tate weights in $[0, p]$. Let $\mathfrak{M}$ be the Kisin module with coefficients attached to T, so that $\mathfrak{M}$ is a free module of rank d over $(W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E)[\![u]\!]$ by Proposition 3.4(3). Write $f = [K_0 : \mathbb{Q}_p]$, and assume that $p \ge 3$.

Let $\mathcal{S} = \{\kappa \colon K \hookrightarrow E\}$ be the set of embeddings of K into E. Fix one such embedding κ_0, and recursively define κ_{s+1} to be the embedding such that $\kappa_{s+1}^p \equiv \kappa_s \pmod p$; these subscripts are to be taken mod f, so that $\kappa_f = \kappa_0$. Let $\varepsilon_s \in W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ be the unique idempotent element such that $(x \otimes 1)\varepsilon_s = (1 \otimes \kappa_s(x))\varepsilon_s$ for all $x \in W(k)$. Then we have $\varepsilon_s(W(k) \otimes_{\mathbb{Z}_p} \mathcal{O}_E) \simeq \mathcal{O}_E$.

Definition 4.21. The filtered (φ, N)-module D is a $K \otimes E$-module and decomposes as a product $D = D_0 \times \cdots \times D_{f-1}$ with $D_s = \varepsilon_s D$ an E-vector space of dimension d. Since $D = D_K$, we have a similar decomposition of $\mathrm{Fil}^i D_K$ for all i. Write $0 \le r_{1,s} \le \cdots \le r_{d,s} \le p$ for the jumps in the filtration $\mathrm{Fil}^i D_s := \varepsilon_s(\mathrm{Fil}^i D_K)$ on D_s. The integers $r_{j,s}$ are the *κ_s-labeled Hodge–Tate weights* of V, as defined in Section 1.1. Note that the multiset $\{r_{j,s} : 1 \le j \le d, 0 \le s \le f-1\}$ taken $[E : K_0]$ times is precisely the set of Hodge–Tate weights of V regarded as a $\mathbb{Q}_p$-representation.

The object $\mathcal{D}$ can be formed from $\mathfrak{M}$ by the same formula as in the preceding section, and since $\mathfrak{M}$ is free as an $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-module, $\mathcal{D}$ is free as an $S_{K_0} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-module and has a decomposition $\mathcal{D} = \mathcal{D}_0 \times \cdots \times \mathcal{D}_{f-1}$ with $\mathcal{D}_s = \varepsilon_s \mathcal{D}$. Similar statements hold for $\mathfrak{M}$, $\mathfrak{M}^*$, and $M := M_K$ (with S_{K_0} replaced by $\mathfrak{S}$ and $W(k)$, respectively), so in particular each $M_s \subset D_s$ is an $\mathcal{O}_E$-lattice. However, note that when we regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\mathfrak{M}^*$, we are regarding $\mathfrak{M}_{s+1}$ (rather than $\mathfrak{M}_s$) as a submodule of $\mathfrak{M}_s^*$ because $\varphi(\varepsilon_s) = \varepsilon_{s+1}$.

Theorem 4.22. *Assume that K is unramified, V is crystalline with Hodge–Tate weights in $[0,p]$, and $p \geq 3$. Then there exists an $\mathcal{O}_E[\![u]\!]$-basis $\{e_{j,s}\}$ of $\mathfrak{M}$ such that*

- $e_{1,s},\dots,e_{d,s}$ *is an $\mathcal{O}_E[\![u]\!]$-basis of $\mathfrak{M}_s$ for each $0 \leq s \leq f-1$ and*
- *we have*

$$\varphi(e_{1,s},\dots,e_{d,s}) = (e_{1,s+1},\dots,e_{d,s+1})X_s\Lambda_s Y_s$$

where X_s and Y_s are invertible matrices, Y_s is congruent to the identity matrix modulo p, and Λ_s is the matrix $[E(u)^{r_{1,s}},\dots,E(u)^{r_{d,s}}]$.

Proof. Setting $\mathrm{Fil}^i M = M \cap \mathrm{Fil}^i D$ as before, we have $\varepsilon_s \mathrm{Fil}^i M = M_s \cap \mathrm{Fil}^i D_s$, which must therefore be an $\mathcal{O}_E$-lattice in $\mathrm{Fil}^i D_s$. Let $1 = n_{0,s} \leq n_{1,s} \leq \cdots \leq n_{r_d,s} \leq d$ be the positive integers such that $\dim_E \mathrm{Fil}^i D_s = d - n_{i,s} + 1$. By the same argument as in the paragraph before Lemma 4.4, there exists an $\mathcal{O}_E$-basis $e_{1,s},\dots,e_{d,s}$ of M_s such that $e_{n_{i,s}},\dots,e_d$ forms an $\mathcal{O}_E$-basis of $\varepsilon_s \mathrm{Fil}^i M$. Now the same argument as in Lemma 4.14 produces an $\mathcal{O}_E[\![u^p]\!]$-basis $\mathfrak{e}_{1,s},\dots,\mathfrak{e}_{d,s}$ of $\mathfrak{M}_{s+1}$ such that $f_\pi(\mathfrak{e}_{n_{i,s}}),\dots,f_\pi(\mathfrak{e}_{d,s})$ forms an $\mathcal{O}_E$-basis for $\mathrm{Fil}^i M_s$ and $f_\pi(\mathfrak{e}_{i,s}) = e_{i,s}$.

Choose any $\mathcal{O}_K$-basis $y_1,\dots,y_g$ of $\mathcal{O}_E$ with $y_1 = 1$. Then $\{y_m\mathfrak{e}_{j,s}\}_{m,j,s}$ is a $\varphi(\mathfrak{S})$-basis of $\mathfrak{M}$ as in Lemma 4.14, and so Proposition 4.16 produces an $\mathfrak{S}$-basis $\mathfrak{e}'_{m,j,s}$ of $\mathfrak{M}^*$ with the properties that $f_\pi(\mathfrak{e}'_{m,j,s}) = y_m e_{j,s}$, $\mathfrak{e}'_{m,j,s} - y_m\mathfrak{e}_{j,s} \in p\sum_{m',j',s'} \mathfrak{S} y_{m'}\mathfrak{e}_{j',s'}$, and $\mathfrak{e}'_{m,j,s} \in \mathrm{Fil}^i \mathfrak{M}^*$ for i as in the proposition.

Set $\mathfrak{e}''_{j,s} = \varepsilon_s \mathfrak{e}'_{1,j,s}$. From the above we see that $f_\pi(\mathfrak{e}''_{j,s}) = e_{j,s}$, $\mathfrak{e}''_{j,s} \in (\mathrm{Fil}^i \mathfrak{M}^*)_s$, and $\mathfrak{e}''_{j,s} - \mathfrak{e}_{j,s} \in p\sum_{j'} \mathcal{O}_E[\![u]\!]\mathfrak{e}_{j',s}$, and one checks easily that $\{\mathfrak{e}''_{j,s}\}$ forms an $\mathcal{O}_E[\![u]\!]$-basis of $\mathfrak{M}^*$. Let $r_d = \max_s\{r_{d,s}\}$. Now the argument of Proposition 4.5 proves that $\mathrm{Fil}^{r_d} \mathfrak{M}^*$ is generated over $\mathcal{O}_E[\![u]\!]$ by the elements of the form $E(u)^{r_d - i}\mathfrak{e}''_{j,s}$ where i is determined by $n_{i,s} \leq j < n_{i+1,s}$ (or $i = r_{d,s}$ when $n_{r_{d,s}} \leq j \leq d$), i.e., where $i = r_{j,s}$.

Let A be the matrix of φ on $\mathfrak{M}$ with respect to the $\mathcal{O}_E[\![u]\!]$-basis $\mathfrak{e}_{j,s}$. Since $\varphi(\varepsilon_{s+1}) = \varepsilon_{s+2}$, the map φ sends $\mathfrak{e}_{j,s}$ into the span of the elements $\mathfrak{e}_{1,s+1},\dots,\mathfrak{e}_{d,s+1}$. Let B be the matrix such that $AB = BA = E(u)^{r_d} I_{df}$. It follows as in the proof of Theorem 4.1 that the image of $\{\mathfrak{e}_{j,s}\}$ under B forms a basis of $\mathrm{Fil}^{r_d} \mathfrak{M}^*$, and moreover the matrix B maps $\mathfrak{e}_{j,s+1}$ into the span of the elements $\{\mathfrak{e}_{l,s}\}$. It follows as in the proof of Theorem 4.1 that the matrix A has the form $X\Lambda Y$, where the matrix X sends $\mathfrak{e}_{j,s}$ into the span of the elements $\mathfrak{e}_{1,s+1},\dots,\mathfrak{e}_{d,s+1}$, the matrix Y is congruent to I_{df} modulo p and preserves each block $\{\mathfrak{e}_{1,s},\dots,\mathfrak{e}_{d,s}\}$, and the matrix Λ sends $\mathfrak{e}_{j,s}$ to $E(u)^{r_{j,s}}\mathfrak{e}_{j,s}$ with i defined as above. Recalling that $\mathfrak{e}_{1,s},\dots,\mathfrak{e}_{d,s}$ are a basis of $\mathfrak{M}_{s+1}$ rather than $\mathfrak{M}_s$, the theorem follows. ☐

Remark 4.23. Theorem 4.22 is best possible, in the sense that it is false if the Hodge–Tate weight range $[0,p]$ is replaced with $[0,r]$ for any $r > p$; see Example 6.8 for an explanation.

5. $(\varphi, \hat{G})$-modules and crystalline representations

We recall that the theory of $(\varphi,\hat{G})$-modules, introduced by the second author in [Liu10b], has been used to classify lattices in semi-stable Galois representations. In this section we review the theory of $(\varphi,\hat{G})$-modules and discuss some properties of the $(\varphi,\hat{G})$-modules arising from crystalline representations. As in Section 3, we allow K to be an arbitrary finite extension of $\mathbb{Q}_p$ and recall that $e = e(K/\mathbb{Q}_p)$ is the ramification index of K.

5.1. $(\varphi, \hat{G})$**-modules.** Define a subring inside B^+_{cris}:

$$\mathcal{R}_{K_0} := \left\{ x = \sum_{i=0}^{\infty} f_i t^{\{i\}} : f_i \in S_{K_0} \text{ and } f_i \to 0 \text{ as } i \to +\infty \right\},$$

where $t^{\{i\}} = \frac{t^i}{p^{\tilde{q}(i)}\tilde{q}(i)!}$ and $\tilde{q}(i)$ satisfies $i = \tilde{q}(i)(p-1) + r(i)$ with $0 \le r(i) < p-1$. Define $\widehat{\mathcal{R}} = W(R) \cap \mathcal{R}_{K_0}$. One can show that $\mathcal{R}_{K_0}$ and $\widehat{\mathcal{R}}$ are stable under the action of G_K and that the G_K-action factors through $\hat{G}$ (see [Liu10b, §2.2]). Recall that the ring R is a valuation ring whose valuation we have denoted v_R, and let $I_+R = \{x \in R : v_R(x) > 0\}$ be the maximal ideal of R.

We have an exact sequence

$$0 \longrightarrow W(I_+R) \longrightarrow W(R) \stackrel{\nu}{\longrightarrow} W(\overline{k}) \longrightarrow 0.$$

By the discussion in the paragraphs leading up to [Liu10b, Lem. 2.2.1] one can naturally extend ν to a map $\nu\colon B^+_{\mathrm{cris}} \to W(\overline{k})[\frac{1}{p}]$.

For any subring A of B^+_{cris}, we write $I_+A = \ker(\nu) \cap A$, and we also write $I_+ = I_+\widehat{\mathcal{R}}$. Since $\nu(u) = 0$, it is not hard to see that $I_+\mathfrak{S} = u\mathfrak{S}$ and

$$I_+S = \left\{ x \in S : x = \sum_{i=1}^{\infty} a_i \frac{u^i}{q(i)!},\ a_i \in W(k) \right\},$$

where $q(i)$ satisfies $i = q(i)e + r(i)$ with $0 \le r(i) < e$. By [Liu10b, Lem. 2.2.1], one has $\widehat{\mathcal{R}}/I_+ \simeq S/I_+S \simeq \mathfrak{S}/u\mathfrak{S} \simeq W(k)$ and that $\widehat{\mathcal{R}}$ is φ-stable.

Definition 5.1. Following [Liu10b] and [CL11], a $(\varphi, \hat{G})$-*module of height* r is a triple $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{\mathcal{G}})$ in which:

(1) $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is an (either finite free or torsion) Kisin module of height r,
(2) $\hat{\mathcal{G}}$ is an $\widehat{\mathcal{R}}$-semi-linear $\hat{G}$-action on $\hat{\mathfrak{M}} := \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$,
(3) the $\hat{G}$-action commutes with $\varphi_{\hat{\mathfrak{M}}} := \varphi \otimes \varphi_{\mathfrak{M}}$ on $\hat{\mathfrak{M}}$, i.e., for any $g \in \hat{G}$ we have $g\varphi_{\hat{\mathfrak{M}}} = \varphi_{\hat{\mathfrak{M}}} g$,
(4) regarding $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule in $\hat{\mathfrak{M}}$, we have $\mathfrak{M} \subset \hat{\mathfrak{M}}^{H_K}$, and
(5) $\hat{G}$ acts on the $W(k)$-module $M := \hat{\mathfrak{M}}/I_+\hat{\mathfrak{M}} \simeq \mathfrak{M}/u\mathfrak{M}$ trivially.

A morphism between two $(\varphi, \hat{G})$-modules is a morphism of φ-modules that commutes with the $\hat{G}$-actions on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. We will generally allow $\hat{\mathfrak{M}}$ to denote the $(\varphi, \hat{G})$-module $(\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{\mathcal{G}})$, and (as usual) we will typically suppress the subscripts on $\varphi_{\mathfrak{M}}$ and $\varphi_{\hat{\mathfrak{M}}}$.

Let $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{\mathcal{G}})$ be a $(\varphi, \hat{G})$-module. We say that $(\mathfrak{M}, \varphi)$ is the *ambient Kisin module* of $\hat{\mathfrak{M}}$, and we say that a sequence of $(\varphi, \hat{G})$-modules is exact if the sequence of ambient Kisin modules is exact. It turns out that the natural map

$$\mathfrak{M} \simeq \mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M} \longrightarrow \mathfrak{S} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \longrightarrow \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

is always injective (see [CL11, Lem. 3.1.2] and the discussion preceding it); as a result we can regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S})$-submodule of $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, and we always do so.

To a $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{\mathcal{G}})$, we can attach a $\mathbb{Z}_p[G_K]$-module as follows:

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, W(R)) \text{ if } \mathfrak{M} \text{ is a finite free Kisin module}$$

and

$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R))$ if $\mathfrak{M}$ is a torsion Kisin module,

where G_K acts on $\hat{T}(\hat{\mathfrak{M}})$ via $g(f)(x) = g(f(g^{-1}(x)))$ for any $g \in G$ and $f \in \hat{T}(\hat{\mathfrak{M}})$. There is a natural map $\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})$ induced by $\mathfrak{f} \mapsto \varphi(\mathfrak{f})$.

Let A be a finite commutative $\mathbb{Z}_p$-algebra. We say $\hat{\mathfrak{M}}$ has a *natural A-action* if the ambient Kisin module $\mathfrak{M}$ has a natural A-action that also commutes with the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. If $\hat{\mathfrak{M}}$ has a natural A-action, then it is easy to see that $\hat{T}(\hat{\mathfrak{M}})$ is an $A[G_K]$-module. Now we summarize some useful results about the functor $\hat{T}$.

Theorem 5.2 ([Liu10b, CL11])**.** (1) *There is a natural isomorphism $\theta\colon T_{\mathfrak{S}}(\mathfrak{M}) \to \hat{T}(\hat{\mathfrak{M}})|_{G_\infty}$.*

(2) *The functor $\hat{T}$ is an anti-equivalence between the category of finite free $(\varphi, \hat{G})$-modules and the category of G_K-stable $\mathbb{Z}_p$-lattices in semi-stable representations with Hodge–Tate weights in $\{0, \dots, r\}$.*

(3) *The functor $\hat{T}$ is exact.*

(4) *Let A be a finite $\mathbb{Z}_p$-algebra that is free as a $\mathbb{Z}_p$-module, and let $L \subset L'$ be two finite free A-modules with an action of G_K such that $L[\frac{1}{p}] = L'[\frac{1}{p}]$ is a semi-stable representation with Hodge–Tate weights in $\{0, \dots, r\}$. Then there exists an exact sequence of $(\varphi, \hat{G})$-modules*

$$0 \longrightarrow \hat{\mathfrak{L}}' \longrightarrow \hat{\mathfrak{L}} \longrightarrow \hat{\mathfrak{M}} \longrightarrow 0$$

such that:

- *$\hat{\mathfrak{L}}$, $\hat{\mathfrak{L}}'$ are finite free $(\varphi, \hat{G})$-modules with natural A-actions,*
- *$\hat{\mathfrak{M}}$ is a torsion $(\varphi, \hat{G})$-module with a natural A-action,*
- *$\hat{T}(\hat{\mathfrak{L}}' \hookrightarrow \hat{\mathfrak{L}})$ is the inclusion $L \hookrightarrow L'$, and*
- *there is a natural isomorphism $L'/L = \hat{T}(\hat{\mathfrak{L}}')/\hat{T}(\hat{\mathfrak{L}}) \simeq \hat{T}(\hat{\mathfrak{M}})$.*

Proof. Parts (1) and (2) are proved in [Liu10b, Thm. 2.3.1]. The functor $T_{\mathfrak{S}}$ is exact from Theorem 3.2, and then (1) implies the exactness of $\hat{T}$. The proof of [CL11, Thm. 3.1.3(3), Lem. 3.1.4] gives (4) except for consideration of the natural A-actions. In particular if $p^n\hat{\mathfrak{M}} = 0$, then the snake lemma gives a natural exact sequence of torsion $(\varphi, \hat{G})$-modules [CL11, Eq. (3.1.4)]:

$$0 \to \hat{\mathfrak{M}} \to \hat{\mathfrak{L}}'/p^n\hat{\mathfrak{L}}' \to \hat{\mathfrak{L}}/p^n\hat{\mathfrak{L}} \to \hat{\mathfrak{M}} \to 0$$

and the isomorphism $\hat{T}(\hat{\mathfrak{L}}')/\hat{T}(\hat{\mathfrak{L}}) \simeq \hat{T}(\hat{\mathfrak{M}})$ is induced by applying $\hat{T}$ to the left-hand part of this sequence. For the A-actions, the proof of [Liu12, Prop. 3.13] shows that there exist natural A-actions on $\hat{\mathfrak{L}}$ and $\hat{\mathfrak{L}}'$ such that the injection $\iota\colon \hat{\mathfrak{L}}' \hookrightarrow \hat{\mathfrak{L}}$ is also a morphism of A-modules and $\hat{T}(\iota)\colon \hat{T}(\mathfrak{L}) \hookrightarrow \hat{T}(\mathfrak{L}')$ is just the injection $L \hookrightarrow L'$ as $A[G]$-modules. Hence $\hat{\mathfrak{M}}$ has a natural A-action and $\hat{T}(\hat{\mathfrak{M}}) \simeq L'/L$ as $A[G]$-modules. ☐

We highlight the following consequence of Theorem 5.2(4).

Proposition 5.3. *Let V be a semi-stable representation of G_K with E-coefficients and Hodge–Tate weights in $\{0, \dots, r\}$, and let $L \subset V$ be a G_K-stable $\mathcal{O}_E$-lattice inside V. Let $\hat{\mathfrak{L}}$ be a finite free $(\varphi, \hat{G})$-module with natural $\mathcal{O}_E$-action such that*

$\hat{T}(\hat{\mathfrak{L}}) \simeq L$. Then $\hat{\mathfrak{L}}/\mathfrak{m}_E\hat{\mathfrak{L}}$ is a torsion $(\varphi, \hat{G})$-module with natural k_E-action such that $\hat{T}(\hat{\mathfrak{L}}/\mathfrak{m}_E\hat{\mathfrak{L}}) \simeq L/\mathfrak{m}_E L$.

Proof. Write $L' = \frac{1}{\varpi}L$ and $\overline{L} = L'/L \simeq L/\mathfrak{m}_E L$. Let $\iota\colon \hat{\mathfrak{L}}' \hookrightarrow \hat{\mathfrak{L}}$ be the inclusion of $(\varphi, \hat{G})$-modules inducing $L \hookrightarrow L'$, as provided by Theorem 5.2(4).

Since $\hat{T}$ is an (anti-)equivalence of categories, there is an isomorphism $m\colon \hat{\mathfrak{L}} \simeq \hat{\mathfrak{L}}'$ such that $\hat{T}(m)$ is the multiplication-by-ϖ map $L' \simeq L$. Now $\hat{T}(m \circ \iota)$ is multiplication by ϖ on L', hence $m \circ \iota$ is multiplication by ϖ on $\hat{\mathfrak{L}}'$, and we deduce that $\hat{\mathfrak{L}}' = \varpi\hat{\mathfrak{L}} = \mathfrak{m}_E\hat{\mathfrak{L}}$.

Now the rest of Theorem 5.2(4) implies that $\hat{\mathfrak{M}} := \hat{\mathfrak{L}}/\mathfrak{m}_E\hat{\mathfrak{L}}$ is a $(\varphi, \hat{G})$-module with natural $\mathcal{O}_E$-action such that

$$\hat{T}(\hat{\mathfrak{M}}) = \hat{T}(\hat{\mathfrak{L}}/\mathfrak{m}_E\hat{\mathfrak{L}}) \simeq \tfrac{1}{\varpi}L/L \simeq \overline{L} \tag{5.4}$$

as $\mathcal{O}_E$-modules. The natural $\mathcal{O}_E$-action on the $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$ evidently induces a natural k_E-action, and the isomorphisms in (5.4) are k_E-module isomorphisms. □

Lemma 5.5. *Let $\hat{\mathfrak{M}}$ be a torsion $(\varphi, \hat{G})$-module with natural k_E-action, and assume further that $\hat{\mathfrak{M}}$ arises as a quotient $\hat{\mathfrak{M}} \simeq \hat{\mathfrak{L}}/\hat{\mathfrak{L}}'$ of finite free $(\varphi, \hat{G})$-modules with natural $\mathcal{O}_E$-action as in Theorem 5.2(4). Suppose that $\overline{L} := \hat{T}(\hat{\mathfrak{M}})$ sits in a short exact sequence of $k_E[G_K]$-modules*

$$\mathcal{L}\colon 0 \longrightarrow \overline{L}' \longrightarrow \overline{L} \longrightarrow \overline{L}'' \longrightarrow 0.$$

Then there exists a short exact sequence of $(\varphi, \hat{G})$-modules with natural k_E-action

$$\hat{\mathcal{M}}\colon 0 \longrightarrow \hat{\mathfrak{M}}'' \longrightarrow \hat{\mathfrak{M}} \longrightarrow \hat{\mathfrak{M}}' \longrightarrow 0$$

such that $\hat{T}(\hat{\mathcal{M}}) = \mathcal{L}$.

Proof. If G is a group, $H < G$ is a subgroup, and $\mathcal{N}$ is a short exact sequence of G-representations, let $\mathcal{N}|_H$ denote the short exact sequence of H-representations obtained from $\mathcal{N}$ by restriction.

Let $\mathfrak{M}$ be the ambient Kisin module of $\hat{\mathfrak{M}}$ and let $M = k(\!(u)\!) \otimes_{k[\![u]\!]} \mathfrak{M}$. By the theory of étale φ-modules ([Fon90, Proposition A.1.2.6], and see also the exposition in [Liu07b, §2.2]), there exists an exact sequence of étale φ-modules with natural k_E-actions

$$0 \longrightarrow M'' \longrightarrow M \xrightarrow{\mathfrak{f}} M' \longrightarrow 0 \tag{5.6}$$

which corresponds to $\mathcal{L}|_{G_\infty}$ under the functor T of [Liu07b, (2.2.4)]. Set $\mathfrak{M}' := \mathfrak{f}(\mathfrak{M})$ and $\mathfrak{M}'' := \ker(\mathfrak{f}|_{\mathfrak{M}})$. By [Liu07b, Lem. 2.3.6] applied to the map $\mathfrak{f}|_{\mathfrak{M}}\colon \mathfrak{M} \to M'$ we see that $\mathfrak{M}', \mathfrak{M}''$ are both Kisin modules with natural k_E-actions, and evidently

$$\mathcal{M}\colon 0 \longrightarrow \mathfrak{M}'' \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}' \longrightarrow 0$$

is a short exact sequence. It is easy to check that $k(\!(u)\!) \otimes_{k[\![u]\!]} \mathcal{M}$ is the short exact sequence of (5.6), so that by [Liu07b, Cor 2.2.2] the short exact sequence $T_{\mathfrak{S}}(\mathcal{M})$ is also isomorphic to $\mathcal{L}|_{G_\infty}$. It remains to show that the short exact sequence of Kisin modules $\mathcal{M}$ extends to a short exact sequence of $(\varphi, \hat{G})$-modules that yields $\mathcal{L}$.

By [Liu07b, Prop. 3.2.1], we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}'' & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M} & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathfrak{S}} \mathfrak{M}' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \iota_{\mathfrak{M}}} & & \big\downarrow & & \\ 0 & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \overline{L}''^{\vee} & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \overline{L}^{\vee} & \longrightarrow & \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \overline{L}'^{\vee} & \longrightarrow & 0 \end{array}$$

which is compatible with the G_∞-actions, φ-actions, and k_E-actions and where the superscript $^\vee$ denotes the $\mathbb{Q}_p/\mathbb{Z}_p$-dual; the vertical arrows are injective by [Liu07b, Thm. 3.2.2(2)]. Now tensoring with $W(R)$ and $\widehat{\mathcal{R}}$, respectively, we get another commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' & \longrightarrow & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \longrightarrow & \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'' & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \longrightarrow & W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle W(R)\otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{M}}} & & \big\downarrow & & \\ 0 & \longrightarrow & W(R) \otimes_{\mathbb{Z}_p} \overline{L}''^{\vee} & \longrightarrow & W(R) \otimes_{\mathbb{Z}_p} \overline{L}^{\vee} & \xrightarrow{f} & W(R) \otimes_{\mathbb{Z}_p} \overline{L}'^{\vee} & \longrightarrow & 0. \end{array}$$

The exactness of the rows and the vertical maps follow from the facts that $\mathfrak{M}''$, $\mathfrak{M}$, and $\mathfrak{M}'$ are all finite $k[\![u]\!]$-free modules and that $\widehat{\mathcal{R}}/p\widehat{\mathcal{R}}$ and $\mathfrak{S}^{\mathrm{ur}}/p\mathfrak{S}^{\mathrm{ur}}$ inject into R (the latter by [Fon90, Proposition B.1.8.3(iv)]), which is a domain.

Thanks to the hypothesis that $\hat{\mathfrak{M}}$ is the quotient of two finite free $(\varphi, \hat{G})$-modules, [Liu12, Lem. 3.10] shows that the map $W(R) \otimes_{\varphi,\mathfrak{S}^{\mathrm{ur}}} \iota_{\mathfrak{M}}$ is equal to the map $\hat{\iota}_{\hat{\mathfrak{M}}}$ of the diagram [Liu12, (3.4)] and so in particular is G_K-equivariant (see e.g. [Liu12, Thm. 2.5]). Note also that the G_K-actions in the middle column commute with the k_E-actions.

Regarding $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ and $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ as submodules of $W(R) \otimes_{\mathbb{Z}_p} \overline{L}'^{\vee}$ and $W(R) \otimes_{\mathbb{Z}_p} \overline{L}^{\vee}$, respectively, we have $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}' = (f \circ \hat{\iota}_{\mathfrak{M}})(\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$. So $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}'$ inherits a G_K-action which factors through $\hat{G}$, and then so does $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}''$; moreover these $\hat{G}$-actions commute with the k_E-actions. It is easy to check that these $\hat{G}$-actions satisfy the axioms for $(\varphi, \hat{G})$-modules, so we obtain an exact sequence of $(\varphi, \hat{G})$-modules that we call $\hat{\mathcal{M}}$.

It remains to check that $\hat{T}(\hat{\mathcal{M}}) \simeq \mathcal{L}$. To see this, we note that $\mathcal{M}$ is the sequence of ambient Kisin modules underlying $\hat{\mathcal{M}}$ and $T_{\mathfrak{S}}(\mathcal{M})$ is isomorphic to $\mathcal{L}|_{G_\infty}$. We therefore have $\hat{T}(\hat{\mathcal{M}})|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathcal{M}) \simeq \mathcal{L}|_{G_\infty}$, where the first isomorphism comes from Theorem 5.2(1). But by hypothesis the middle map $\hat{T}(\hat{\mathfrak{M}})|_{G_\infty} \to L|_{G_\infty}$ in that complex is actually a G_K-isomorphism, and it follows that $\hat{T}(\hat{\mathcal{M}}) \simeq \mathcal{L}$ as short exact sequences of $k_E[G]$-modules. (Suppose G is a topological group, $\mathcal{L}', \mathcal{M}'$ are short exact sequences of continuous G-representations, and $f\colon \mathcal{L}' \to \mathcal{M}'$ is an isomorphism between $\mathcal{L}'$ and $\mathcal{M}'$ regarded as short exact sequences of vector spaces. If the map in the middle of f is an isomorphism of continuous G-representations, it follows formally that the same is true of the two outer maps, and f is an isomorphism from $\mathcal{L}'$ to $\mathcal{M}'$.) ∎

Remark 5.7. Lemma 5.5 may well remain true without the assumption that $\hat{\mathfrak{M}}$ arises as a quotient $\hat{\mathfrak{M}} \simeq \hat{\mathfrak{L}}/\hat{\mathfrak{L}}'$ of finite free $(\varphi, \hat{G})$-modules with natural k_E-action, but the proof would require additional work and we will only need the weaker statement.

Before continuing, we note one additional consequence of the relationship between torsion Kisin modules and the theory of étale φ-modules.

Lemma 5.8. *Suppose that $\hat{f}\colon \hat{\overline{\mathfrak{M}}} \to \hat{\overline{\mathfrak{M}'}}$ is a map of torsion $(\varphi, \hat{G})$-modules with natural A-action, and let $\overline{\mathfrak{M}}, \overline{\mathfrak{M}'}$ be the ambient Kisin modules of $\hat{\overline{\mathfrak{M}}}, \hat{\overline{\mathfrak{M}'}}$, respectively. Then $\hat{T}(\hat{f})$ is injective (resp. surjective, an isomorphism) if and only if the induced map $\overline{\mathfrak{M}}[\frac{1}{u}] \to \overline{\mathfrak{M}'}[\frac{1}{u}]$ is injective (resp. surjective, an isomorphism).*

Proof. Note that $\overline{\mathfrak{M}}[\frac{1}{u}] = k(\!(u)\!) \otimes_{k[\![u]\!]} \overline{\mathfrak{M}}$ and similarly for $\overline{\mathfrak{M}'}$. Let f denote the map of Kisin modules underlying $\hat{f}$, and f_u the map of étale φ-modules $\overline{\mathfrak{M}}[\frac{1}{u}] \to \overline{\mathfrak{M}'}[\frac{1}{u}]$ obtained by inverting u. By Theorem 5.2(1), the map $\hat{T}(\hat{f})$ is injective (resp. surjective) if and only if the map $T_{\mathfrak{S}}(f)$ is injective (resp. surjective). By [Liu07b, Cor. 2.2.2] the map $T_{\mathfrak{S}}(f)$ is naturally isomorphic to $T(f_u)$. But the functor T is an equivalence of abelian categories. □

5.2. **τ-actions for crystalline representations.** We now restate Corollary 4.10 using the language of $(\varphi, \hat{G})$-modules.

Proposition 5.9. *Suppose $p > 2$. Let L be a G-stable $\mathbb{Z}_p$-lattice in a crystalline representation and let $\hat{\mathfrak{M}}$ be the $(\varphi, \hat{G})$-module corresponding to L, with ambient Kisin module $\mathfrak{M}$. Then for any $x \in \mathfrak{M}$ we have $\tau(x) - x \in \hat{\mathfrak{M}} \cap u^p\varphi(\mathfrak{t})(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})$.*

Proof. This is a direct consequence of Corollary 4.10, since by [Liu10b, Eq. (3.2.1) and Prop. 3.2.1] the formula (4.8) defines the action of τ on the $(\varphi, \hat{G})$-module $\mathfrak{L}$. □

We let $\mathrm{Rep}^{\mathrm{st},r}_E(G_K)$ denote the category of finite E-vector spaces V with an E-linear G_K-action such that V is a semi-stable representation with Hodge–Tate weights in $\{0, \ldots, r\}$. We denote by $\mathrm{Rep}^{\mathrm{st},r}_{\mathcal{O}_E}(G_K)$ the category of G-stable $\mathcal{O}_E$-lattices inside objects in $\mathrm{Rep}^{\mathrm{st},r}_E(G_K)$ and by $\mathrm{Rep}^{\mathrm{cris},r}_{\mathcal{O}_E}$ the subcategory of $\mathrm{Rep}^{\mathrm{st},r}_{\mathcal{O}_E}$ whose objects are crystalline.

Now assume that L is in $\mathrm{Rep}^{\mathrm{cris},r}_{\mathcal{O}_E}$ and let $\hat{\overline{\mathfrak{M}}}$ be the $(\varphi, \hat{G})$-module corresponding to the reduction $L/\mathfrak{m}_E L$ via Theorem 5.2(4), with ambient Kisin module $\overline{\mathfrak{M}}$.

Corollary 5.10. *For any $x \in \overline{\mathfrak{M}}$, there exist $\alpha \in R$ and $y \in R \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}$ such that $\tau(x) - x = \alpha y$ and $v_R(\alpha) \geq \frac{p}{p-1} + \frac{p}{e}$.*

Proof. Recall from Section 1.1.3 that the image of $\mathfrak{t}$ in R has valuation $\frac{1}{p-1}$, from which it follows that the image of $\varphi(\mathfrak{t})$ in R has valuation $\frac{p}{p-1}$. Since the image of u in R is $\underline{\pi}$ and since $v_R(\underline{\pi}) = \frac{1}{e}$, the result follows from Proposition 5.9. □

6. Kisin modules and $(\varphi, \hat{G})$-modules of rank one

We assume for the remainder of this article that $K/\mathbb{Q}_p$ is unramified, with $f = [K : \mathbb{Q}_p]$. In this section we determine the isomorphism classes of $(\varphi, \hat{G})$-modules of rank one, compute their corresponding Galois representations, and show that they arise as the reductions of crystalline characters with specified Hodge–Tate weights.

Recall that E is a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_E$ and residue field k_E. As in Section 4.4, we fix (again for the remainder of the article) an embedding $\kappa_0\colon K \hookrightarrow E$ and recursively define $\kappa_{s+1}\colon K \hookrightarrow E$ so that $\kappa_{s+1}^p \equiv \kappa_s \pmod p$. Let $\varepsilon_s \in W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}_E$ be the idempotent defined in Section 4.4, and if M is any module that can naturally be regarded as a module over $W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}_E$, we write M_s for $\varepsilon_s M$.

Definition 6.1. Suppose $r_0,\dots,r_{f-1}$ are non-negative integers and $a \in k_E^\times$. Let $\overline{\mathfrak{M}}(r_0,\dots,r_{f-1};a)$ be the Kisin module with natural k_E-action that is rank one over $\mathfrak{S}\otimes_{\mathbb{Z}_p} k_E$ and satisfies

- $\overline{\mathfrak{M}}(r_0,\dots,r_{f-1};a)_s$ is generated by e_s and
- $\varphi(e_i) = (a)_i u^{r_i} e_{i+1}$.

Here $(a)_i = a$ if $i \equiv 0 \pmod f$ and $(a)_i = 1$ otherwise. (For later use, we extend this notation as follows: if $S \subset \mathbb{Z}$, we write $(a)_S = a$ if S contains an integer divisible by f and $(a)_S = 1$ otherwise.)

The following fact is proved by a standard change-of-variables argument whose details we omit (but see for instance the paragraph before the statement of [Sav08, Thm. 2.1] for an analogous argument).

Lemma 6.2. *Any rank one φ-module over $\mathfrak{S}\otimes_{\mathbb{Z}_p} k_E$ is isomorphic to (exactly) one of the form $\overline{\mathfrak{M}}(r_0,\dots,r_{f-1};a)$.*

Now let $\hat{a} \in \mathcal{O}_E$ be a lift of a. Let $\mathfrak{M}(r_0,\dots,r_{f-1};\hat{a})$ be the rank one φ-module over $\mathfrak{S}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$ such that

(1) $\mathfrak{M}(r_0,\dots,r_{f-1};\hat{a})_s$ is generated by $\mathfrak{e}_s$ and
(2) $\varphi(\mathfrak{e}_i) = (\hat{a})_i (u-\pi)^{r_i}\mathfrak{e}_{i+1}$.

It is obvious that $\mathfrak{M} := \mathfrak{M}(r_0,\dots,r_{f-1};\hat{a})$ is a finite free Kisin module such that $\mathfrak{M}/\mathfrak{m}_E\mathfrak{M} = \overline{\mathfrak{M}}(r_0,\dots,r_{f-1};a)$. We would like to show that the G_∞-representation $T_{\mathfrak{S}}(\mathfrak{M})$ can be uniquely extended to a crystalline character of G_K.

Lemma 6.3. *There exists a unique $(\varphi,\hat{G})$-module $\hat{\mathfrak{M}} := \hat{\mathfrak{M}}(r_0,\dots,r_{f-1};\hat{a})$ such that the ambient Kisin module of $\hat{\mathfrak{M}}$ is $\mathfrak{M}(r_0,\dots,r_{f-1};\hat{a})$ and $\hat{T}(\hat{\mathfrak{M}})$ is a crystalline character. The κ_s-labeled Hodge–Tate weight of $\hat{T}(\hat{\mathfrak{M}})$ is r_s.*

Proof. The uniqueness is a general fact, combining Theorem 5.2(2) with [Kis06, Thm. (0.2)]. For existence, consider the Kisin module $\mathfrak{N}(j) = \mathfrak{M}(0,\dots,1,\dots,0;1)$ where $r_j = 1$ and $r_i = 0$ if $i \neq j$. This is a Kisin module of height 1, and it follows from [Kis06, Thm. (2.2.7)] that $T_{\mathfrak{S}}(\mathfrak{N}(j))$ can be uniquely extended to a crystalline character ψ_j with Hodge–Tate weights in $\{0,1\}$. By Theorem 4.22 (or, if one prefers, from Lemma 4.3(3) together with the existence of a *base adaptée* for $\mathcal{D}$), ψ_j has κ_s-labeled Hodge–Tate weights 0 if $s \neq j$ and 1 if $s = j$.

Next consider $\mathfrak{N}(\hat{a}) = \mathfrak{M}(0,\dots,0;\hat{a})$, and define $\lambda_{\hat{a}} = T_{\mathfrak{S}}(\mathfrak{N}(\hat{a}))$. Let $\mathbb{Z}_p^{\mathrm{ur}}$ denote the maximal unramified extension of $\mathbb{Z}_p$. Since there exists $x \in \mathbb{Z}_p^{\mathrm{ur}}\otimes_{\mathbb{Z}_p}\mathcal{O}_E$ with $\varphi^f(x) = (1\otimes\hat{a})x$, it is easy to check using the functor $T_{\mathfrak{S},\mathcal{O}_E}$ that $\mathfrak{N}(\hat{a})$ is the Kisin module attached to the unramified character of G_K sending arithmetic Frobenius to $\hat{a}$. Now it suffices to show that the Kisin module associated to the crystalline character $\lambda_{\hat{a}}\psi_0^{r_0}\cdots\psi_{f-1}^{r_{f-1}}$ is just $\mathfrak{M}(r_0,\dots,r_{f-1};\hat{a})$. This is a consequence of the following general fact. ∎

Lemma 6.4. *Let χ and χ' be two crystalline $\mathcal{O}_E$-characters of G_K whose Kisin modules $\mathfrak{N}$, $\mathfrak{N}'$ are defined by the conditions*

- *$\mathfrak{N}_i$, $\mathfrak{N}'_i$ are generated by $\mathfrak{e}_i$, $\mathfrak{e}'_i$, respectively, and*
- *$\varphi(\mathfrak{e}_i) = \alpha_i \mathfrak{e}_{i+1}$ and $\varphi(\mathfrak{e}'_i) = \alpha'_i \mathfrak{e}_{i+1}$ with $i = 0, \ldots, f-1$ and $\alpha_i, \alpha'_i \in \mathcal{O}_E[\![u]\!]$.*

Then the Kisin module $\tilde{\mathfrak{N}}$ of $\chi \cdot \chi'$ has the form $\varphi(\mathfrak{f}_i) = \alpha_i \alpha'_i \mathfrak{f}_{i+1}$, with $\mathfrak{f}_i$ a generator of $\tilde{\mathfrak{N}}_i$.

Proof. We compute using the functor $T_{\mathfrak{S},\mathcal{O}_E}$. Pick generators f, f' of the rank one $\mathcal{O}_E$-modules $T_{\mathfrak{S},\mathcal{O}_E}(\mathfrak{N})$ and $T_{\mathfrak{S},\mathcal{O}_E}(\mathfrak{N}')$, and write β_i, β'_i for the elements $f(\mathfrak{e}_i), f(\mathfrak{e}'_i)$ in $\mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$. Then $\varphi(\beta_{i-1}) = \alpha_{i-1}\beta_i$ and similarly for $\varphi(\beta'_{i-1})$.

Let $\tilde{\mathfrak{N}}$ be as in the statement of the lemma, and consider the map $\tilde{f}\colon \tilde{\mathfrak{N}} \to \mathfrak{S}^{\mathrm{ur}} \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ which sends $\mathfrak{f}_i$ to $\beta_i\beta'_i$. Evidently $\tilde{f} \in T_{\mathfrak{S},\mathcal{O}_E}(\tilde{\mathfrak{N}})$, and the latter is an $\mathcal{O}_E$-character of G_∞. As $\tilde{f} = f \cdot f'$, we see that $T_{\mathfrak{S},\mathcal{O}_E}(\tilde{\mathfrak{N}}) = (\chi\chi')|_{G_\infty}$ as $\mathcal{O}_E[G_\infty]$-modules. That is, $\tilde{\mathfrak{N}}$ is the Kisin module associated to $\chi \cdot \chi'$. ☐

Corollary 6.5. *There is a unique $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}} := \hat{\overline{\mathfrak{M}}}(r_0, \ldots, r_{f-1}; a)$ whose ambient Kisin module is $\overline{\mathfrak{M}}(r_0, \ldots, r_{f-1}; a)$. Furthermore, $\hat{T}(\hat{\overline{\mathfrak{M}}})$ is the reduction of the crystalline character $\hat{T}(\hat{\mathfrak{M}}(r_0, \ldots, r_{f-1}; \hat{a}))$ for any lift $\hat{a} \in \mathcal{O}_E$ of a.*

Proof. The existence of $\hat{\overline{\mathfrak{M}}}$ follows from Lemma 6.3 and Theorem 5.2(4). For uniqueness, it suffices to see that the action of τ on $\overline{\mathfrak{M}}$ is uniquely determined. Write $\tau(e_i) = \alpha_i e_i$ with $\alpha_i \in R$. We see that $\alpha_{i+1} = \underline{\epsilon}^{-pr_i}\varphi(\alpha_i)$, and it follows that $\varphi^f(\alpha_i) = \alpha_i \underline{\epsilon}^{m_i}$ for some integer m_i which is determined by the r_j. Lemma 6.6 below shows that $\alpha_i = c\eta^{m_i}$ for some $c \in k$, where the element $\eta \in R$ is defined in Lemma 6.6(2). Since $\eta - 1 \in I_+ R$ and $\hat{G}$ must act trivially on $\overline{\mathfrak{M}}/u\overline{\mathfrak{M}}$, we have $c = 1$ and α_i is uniquely determined for all i. ☐

Lemma 6.6. *Recall that $\underline{\epsilon} = \underline{\epsilon}(\tau)$ is the image in R of $\tau(u)/u$.*

1. *Write $m = p^s m_0 \in \mathbb{Z}_p$ with $m_0 \in \mathbb{Z}_p^\times$ and $s \in \mathbb{Z}_{\geq 0}$ a non-negative integer. Then $v_R(\underline{\epsilon}^{-m} - 1) = p^s(\frac{p}{p-1})$.*
2. *If $m \in \mathbb{Z}$, then the solutions to the equation $\varphi^f(x) = x\underline{\epsilon}^m$ with $x \in R$ are precisely the $c\eta^m$ where $\eta = \prod_{n=0}^{\infty} (\underline{\epsilon}^{-1})^{p^{nf}}$ and $c \in k$.*
3. *If $m \in \mathbb{Z}$, then $v_R(\eta^m - 1) = v_R(\underline{\epsilon}^m - 1)$.*

Proof. (1) It suffices to prove that $v_R(\underline{\epsilon}^m - 1) = p^s(\frac{p}{p-1})$. If $m = m_0 \in \mathbb{Z}_p^\times$, then $v_R(\underline{\epsilon}^m - 1) = \lim_{n\to\infty} p^n v_p(\zeta_{p^n}^m - 1) = \frac{p}{p-1}$ where $\zeta_{p^n}^m$ is defined in the usual way for $m \in \mathbb{Z}_p^\times$. For the general case, note that $\underline{\epsilon}^m - 1 = (\underline{\epsilon}^{m_0})^{p^s} - 1 = (\underline{\epsilon}^{m_0} - 1)^{p^s}$.

(2) One checks that (1) implies the convergence of η in R and that $c\eta^m$ is a solution to the equation. Comparing valuations on both sides of the equation $\varphi^f(x) = x\underline{\epsilon}^m$, one sees that if $x \neq 0$, then $v_R(x) = 0$; it follows that if x, y are two solutions with the same image in $\overline{k} \simeq R/\mathfrak{m}_R$, then $x - y = 0$. Also note that since $\underline{\epsilon} \equiv 1 \pmod{\mathfrak{m}_R}$ and k is the fixed field of φ^f in $\overline{k}$, the image of x in $\overline{k}$ must lie in k. It is easy to see that $\eta \equiv 1 \pmod{\mathfrak{m}_R}$, and we conclude that if $c \in k$, then $c\eta^m$ is the unique solution with image c in $R/\mathfrak{m}_R$.

(3) Write $\varphi^f(\eta^m - 1) = \eta^m(\underline{\epsilon}^m - 1) + (\eta^m - 1)$. Since $\eta^m - 1$ has positive valuation, the term on the left-hand side has greater valuation than the second term on the right-hand side; therefore the two terms on the right-hand side must have equal valuation. ☐

Recall that in Section 1.1.1, for each $\sigma \in \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ we have defined the fundamental character $\omega_\sigma \colon I_K \to \overline{\mathbb{F}}_p^\times$ corresponding to σ. Let $\overline{\kappa}_s \colon k \hookrightarrow \overline{\mathbb{F}}_p$ be the embedding obtained by reducing κ_s modulo p, and for brevity we write ω_s for $\omega_{\overline{\kappa}_s}$ (throughout the rest of the paper).

Proposition 6.7. *Write* $\hat{\overline{\mathfrak{M}}} = \hat{\overline{\mathfrak{M}}}(r_0, \ldots, r_{f-1}; a)$ *and* $\hat{\overline{\mathfrak{M}}}' = \hat{\overline{\mathfrak{M}}}(r'_0, \ldots, r'_{f-1}; a')$ *for some* $a, a' \in k_E$ *and non-negative integers* $r_0, r'_0, \ldots, r_{f-1}, r'_{f-1}$*. Let* $\overline{\mathfrak{M}}, \overline{\mathfrak{M}}'$ *denote the ambient Kisin modules of* $\hat{\overline{\mathfrak{M}}}, \hat{\overline{\mathfrak{M}}}'$*.*

1. *We have* $\hat{T}(\hat{\overline{\mathfrak{M}}})|_{I_K} \simeq \omega_0^{r_0} \cdots \omega_{f-1}^{r_{f-1}}$*.*
2. *We have* $\hat{T}(\hat{\overline{\mathfrak{M}}}) \simeq \hat{T}(\hat{\overline{\mathfrak{M}}}')$ *if and only if* $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}}')$*.*
3. *The isomorphism in (2) occurs if and only if* $a = a'$ *and* $\sum_{i=0}^{f-1} p^{f-i-1} r_i \equiv \sum_{i=0}^{f-1} p^{f-i-1} r'_i \pmod{p^f - 1}$*.*

Proof. (1) By Lemma 6.3 and Corollary 6.5, it suffices to check that $\overline{\psi}_s|_{I_K} = \omega_s$, where $\overline{\psi}_s$ is the reduction modulo p of the character ψ_s whose κ_j-labeled Hodge–Tate weight is 1 if $j = s$ and 0 otherwise. By [Con11, Prop. B.3] we have $(\psi_s \circ \mathrm{Art}_K)|_{\mathcal{O}_K^\times} \simeq \kappa_s|_{\mathcal{O}_K^\times}$; comparing with the definition of ω_s, the result follows.

(2) Since K_∞/K is totally wildly ramified but the kernels of mod p characters of G_K correspond to tame extensions, a mod p character of G_K that is trivial on G_∞ must be trivial.

(3) Let us first check that the given conditions are sufficient. Choose any integers $r''_0, \ldots, r''_{f-1}$ such that $r''_i \geq \max(r_i, r'_i)$ and $\sum_{i=0}^{f-1} p^{f-i-1} r''_i \equiv \sum_{i=0}^{f-1} p^{f-i-1} r_i \pmod{p^f - 1}$, and define $\overline{\mathfrak{M}}'' = \overline{\mathfrak{M}}(r''_0, \ldots, r''_{f-1}; a)$. It is enough to check that $T_{\mathfrak{S}}(\overline{\mathfrak{M}}) \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}}'')$ (for then we must have $T_{\mathfrak{S}}(\overline{\mathfrak{M}}') \simeq T_{\mathfrak{S}}(\overline{\mathfrak{M}}'')$ by the same argument). Set $m_i = \frac{1}{p^f - 1} \sum_{i=0}^{f-1} p^{f-i-1}(r''_i - r_i)$, which by construction is a non-negative integer for all i. Then there is a map $f \colon \overline{\mathfrak{M}}'' \to \overline{\mathfrak{M}}$ sending $e''_i \mapsto u^{m_i} e_i$ (with the obvious meaning for e''_i). Since f is an isomorphism after inverting u, it follows from the theory of étale φ-modules (as in Section 5) that $T_{\mathfrak{S}}(f)$ is an isomorphism.

In the reverse direction, it follows from (1) that the condition $\sum_{i=0}^{f-1} p^{f-i-1} r_i \equiv \sum_{i=0}^{f-1} p^{f-i-1} r'_i \pmod{p^f - 1}$ is necessary. The calculation of the unramified character $\lambda_{\hat{a}}$ in the proof of Lemma 6.3, together with Lemma 6.4 and Corollary 6.5, shows that changing a' must change $\hat{T}(\hat{\overline{\mathfrak{M}}}')$. Thus for fixed values of $r_0, r'_0, \ldots, r_{f-1}, r'_{f-1}$ and a the isomorphism in (2) holds for at most one value of a', and so the necessity of $a = a'$ follows from the result of the previous paragraph. ∎

Example 6.8. We can now show that Theorem 4.22 is best possible. Suppose that V is a two-dimensional crystalline representation of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$ for some $r > 0$, and assume that the reduction mod p of V is reducible. Possibly after extending the coefficients of V, it is possible to choose a lattice $T \subset V$ with associated Kisin module $\mathfrak{M}$ such that $\overline{\mathfrak{M}}$ is a direct sum $\overline{\mathfrak{M}}(h; a) \oplus \overline{\mathfrak{M}}(h'; a')$ for some h, h' with $h + h' = r$. (This follows by essentially the same argument by which it is possible to choose a lattice in V whose reduction is split, again after possibly extending the coefficients.)

If the conclusion of Theorem 4.22 were to hold for the Kisin module $\mathfrak{M}$, then φ on $\overline{\mathfrak{M}}$ would be non-trivial mod u. It would then follow that $\{h, h'\} = \{0, r\}$ and

$\overline{V}^{\text{ss}} \cong 1 \oplus \overline{\varepsilon}^r$. But if $r = p+1$, it is well known that there exists V as above with $\overline{V}^{\text{ss}} \cong \overline{\varepsilon} \oplus \overline{\varepsilon}$, a contradiction.

7. Extensions of rank one φ-modules

Recall that we have assumed that $K/\mathbb{Q}_p$ is unramified. In this section we consider possible extensions of Kisin modules. Our analysis in this section, combined with the results of Section 4, is already sufficient to prove our main results for semi-simple representations; in Section 8, we will extend this analysis to $(\varphi, \hat{G})$-modules, in order to be able to handle extension classes.

Before we begin our analysis of extensions of rank one φ-modules, we give some combinatorial lemmas, which will be used to determine when an extension of Kisin modules corresponds to a Galois representation with scalar semi-simplification. (See Remark 7.10 below, and see also the discussion in the opening pages of [BDJ10, §3.2].)

Lemma 7.1. *Suppose that* $r_0, \dots, r_{f-1}$ *are integers in the range* $[-p, p]$ *that satisfy* $\sum_{i=0}^{f-1} p^{f-1-i} r_i \equiv 0 \pmod{p^f - 1}$. *Then either:*

(1) $(r_0, \dots, r_{f-1}) = \pm(p-1, \dots, p-1)$,

(2) the numbers $r_0, \dots, r_{f-1}$*, considered as a cyclic list, can be broken up into strings of the form* $\pm(-1, p-1, \dots, p-1, p)$ *(where there may not be any occurrences of* $p-1$*) and strings of the form* $(0, \dots, 0)$*, or else*

(3) $p = 2$ *and* $(r_0, \dots, r_{f-1}) = \pm(2, \dots, 2)$.

Proof. First suppose that none of the r_i are equal to $\pm p$. Then $|\sum_{i=0}^{f-1} p^{f-1-i} r_i| \leq p^f - 1$; so the only possibilities for that sum are 0 and $\pm(p^f - 1)$, and the latter can occur only for $(r_0, \dots, r_{f-1}) = \pm(p-1, \dots, p-1)$. If instead $\sum_{i=0}^{f-1} p^{f-1-i} r_i = 0$, then considering divisibility by p we have $r_{f-1} = 0$. Dividing by p and repeating, we see that $r_i = 0$ for all i in this case.

Next suppose that $r_i = \pm p$ for some i. We perform a "carrying" operation, by adding $\mp p$ to r_i and adding ± 1 to r_{i-1}; this preserves the given congruence. Now move left, and if the new $|r_{i-1}|$ is at least p, we perform the carrying operation there. Continue this process with $r_{i-2}, \dots, r_0, r_{f-1}, \dots, r_{i+1}$ until we have returned to r_i again. Note that if we have had to carry for both r_j and r_{j-1}, then the two carries necessarily had the same sign; so a string of consecutive carries has the effect of subtracting $\pm(-1, p-1, \dots, p-1, p)$ from a subsequence of the r_j's, or else $\pm(p-1, \dots, p-1)$ from the full list.

At the end of this carrying process, we have a new sequence $r'_0, \dots, r'_{f-1}$ satisfying the original congruence condition, but with all $r'_j \in [-(p-1), (p-1)]$. Note also that $r_i \in \{0, \pm 1\}$ at our starting point. If $p > 2$, then the first paragraph implies that $r'_i = 0$ for all i, and the last sentence of the second paragraph shows that $(r_0, \dots, r_{f-1})$ has the desired shape. If $p = 2$, then it is also possible that $r'_i = 1$ for all i or that $r'_i = -1$ for all i. But note that if we add some number of (non-overlapping) strings of the form $(1, -1, \dots, -1, -2)$ to $(1, \dots, 1)$, the result actually has the form (2) again; so the only new possibility when $p = 2$ is (3). ☐

Definition 7.2. Let $\mathcal{P}$ be the set of f-tuples $(r_0, \dots, r_{f-1})$ with $r_i \in \{1, p-1, p\}$ for all i and such that

- if $r_i = p$, then $r_{i+1} = 1$, and
- if $r_i \in \{1, p-1\}$, then $r_{i+1} \in \{p-1, p\}$,

conventionally taking $r_f = r_0$. (If $p > 2$, these conditions are equivalent to $r_i = p$ if and only if $r_{i+1} = 1$.)

The preceding definition is motivated by the following lemma.

Lemma 7.3. *Let $r_0, \dots, r_{f-1}$ be integers in the range $[1, p]$. Let J be a subset of $\{0, \dots, f-1\}$, and set $h_i = r_i$ if $i \in J$ and $h_i = 0$ if $i \notin J$. Then*

$$\sum_{i=0}^{f-1} p^{f-1-i} h_i \equiv \sum_{i=0}^{f-1} p^{f-1-i}(r_i - h_i) \pmod{p^f - 1}$$

if and only if $(r_0, \dots, r_{f-1}) \in \mathcal{P}$ and J satisfies:

- *if $(r_{i-1}, r_i) = (p, 1)$, then $i + 1 \in J$ if and only if $i \notin J$,*
- *if $(r_{i-1}, r_i) = (1, p-1)$ or $(p-1, p-1)$, then $i+1 \in J$ if and only if $i \in J$,*

or else $p = 2$, $(r_0, \dots, r_{f-1}) = (2, \dots, 2)$, and $J = \varnothing$ or $\{0, 1, \dots, f-1\}$.

Proof. The congruence is equivalent to $\sum_{i=0}^{f-1}(-1)^{[i\in J]} p^{f-1-i} r_i \equiv 0 \pmod{p^f - 1}$, where we write $[i \in J] = 1$ if $i \in J$ and $[i \in J] = 0$ otherwise. Since none of the r_i are zero, by Lemma 7.1 we see that if $p > 2$, the sequence $((-1)^{[i\in J]} r_i)_{0 \le i \le f-1}$ must either be $\pm(p-1, \dots, p-1)$ or else break up into subsequences of the form $\pm(-1, p-1, \dots, p-1, p)$; when $p = 2$, we have the additional possibilities $\pm(2, \dots, 2)$. This is equivalent to the description in the statement of the lemma. □

The following result gives a structure theorem for extensions of Kisin modules; we will build on it in the following section to prove Proposition 8.8, which is the main result we will need on extensions of $(\varphi, \hat{G})$-modules.

Proposition 7.4. *Let $r_0, \dots, r_{f-1}$ be integers in the range $[1, p]$. Let J be a subset of $\{0, \dots, f-1\}$, and set $h_i = r_i$ if $i \in J$ and $h_i = 0$ if $i \notin J$. Fix $a, b \in k_E^\times$. Let $\overline{\mathfrak{M}}$ be an extension of $\overline{\mathfrak{M}}(h_0, \dots, h_{f-1}; a)$ by $\overline{\mathfrak{M}}(r_0 - h_0, \dots, r_{f-1} - h_{f-1}; b)$; then we can choose bases e_i, f_i of the $\overline{\mathfrak{M}}_i$ so that φ has the form*

$$\begin{aligned} \varphi(e_i) &= (b)_i u^{r_i - h_i} e_{i+1}, \\ \varphi(f_i) &= (a)_i u^{h_i} f_{i+1} + x_i e_{i+1} \end{aligned}$$

with $x_i \in k_E[\![u]\!]$ a polynomial with $\deg(x_i) < h_i$, except in the following cases:

- *$(r_0, \dots, r_{f-1}) \in \mathcal{P}$, $J = \{i : r_{i-1} \neq p\}$, and $a = b$ or*
- *$p = 2$, $(r_0, \dots, r_{f-1}) = (2, \dots, 2)$, $J = \{0, \dots, f-1\}$, and $a = b$.*

In that case fix $i_0 \in J$; then x_i may be taken to be a polynomial of degree $\deg(x_i) < h_i$ for all i except $i = i_0$, where x_{i_0} is the sum of a polynomial of degree less than h_{i_0} and a (possibly trivial) term of degree p (for the first exceptional case) or degree 4 (for the second exceptional case).

Proof. Let $\overline{\mathfrak{M}}$ be an extension of $\overline{\mathfrak{M}}(h_0, \dots, h_{f-1}; a)$ by $\overline{\mathfrak{M}}(r_0 - h_0, \dots, r_{f-1} - h_{f-1}; b)$; then we can choose bases e_i, f_i of the $\overline{\mathfrak{M}}_i$ so that φ has the form

$$\begin{aligned} \varphi(e_i) &= (b)_i u^{r_i - h_i} e_{i+1}, \\ \varphi(f_i) &= (a)_i u^{h_i} f_{i+1} + x_i e_{i+1}. \end{aligned}$$

We wish to determine to what extent the x_i's can be simultaneously simplified via a change of basis of the form $f_i' = f_i + \alpha_{i-1} e_i$ for some elements $\alpha_{i-1} \in k_E[\![u]\!]$. If $\alpha = \alpha(u) \in k_E[\![u]\!]$, let $\varphi(\alpha) = \alpha(u^p)$. Observing that

$$\varphi(f_i + \alpha_{i-1} e_i) = (a)_i u^{h_i}(f_{i+1} + \alpha_i e_{i+1}) + (x_i + (b)_i u^{r_i - h_i} \varphi(\alpha_{i-1}) - (a)_i u^{h_i} \alpha_i) e_{i+1},$$

we see that such a change of basis replaces each x_i with

$$x_i' = x_i + (b)_i u^{r_i - h_i}\varphi(\alpha_{i-1}) - (a)_i u^{h_i}\alpha_i.$$

Observe that we may make $x_i' = 0$ if $i \notin J$ (at least for any individual such i) by choosing

$$\alpha_i = (a)_i^{-1}(x_i + (b)_i u^{r_i}\varphi(\alpha_{i-1})). \tag{7.5}$$

If $J \neq \varnothing$, then we can take $x_i' = 0$ simultaneously for all $i \notin J$ by choosing α_i arbitrarily for each $i \in J$ and determining α_i recursively by the formula (7.5) for $i \notin J$. If $J = \varnothing$, then the preceding sentence shows that we can at least have $x_i' = 0$ for $i \neq f-1$ by choosing α_{f-1} arbitrarily and choosing α_i recursively for $i = 0, \dots, f-2$ using (7.5). Suppose now that $x_0 = \cdots = x_{f-2} = 0$. Taking α_{f-1} arbitrary and choosing $\alpha_i = (b/a)_i u^{r_i}\varphi(\alpha_{i-1})$ for $i = 0, \dots, f-2$, one computes that

$$x_{f-1}' = x_{f-1} + (b/a)u^{r_{f-1}+pr_{f-2}+\cdots+p^{f-1}r_0}\varphi^f(\alpha_{f-1}) - \alpha_{f-1}.$$

It is possible to choose α_{f-1} in the above equation so that $x_{f-1}' = 0$: indeed, if we set the right-hand side of the above expression equal to zero, the resulting equation

$$\alpha_{f-1} = x_{f-1} + (b/a)u^{r_{f-1}+pr_{f-2}+\cdots+p^{f-1}r_0}\varphi^f(\alpha_{f-1})$$

can be regarded as a system of equations for the coefficients of α_{f-1}. Since $r_{f-1} + pr_{f-2} + \cdots + p^{f-1}r_0 > 0$, the coefficient of u^i on the left-hand side depends only on lower-degree coefficients of α_{f-1} on the right-hand side, and so this system can be solved recursively. With such a choice of α_{f-1} we have $x_i' = 0$ for all i.

The preceding paragraph shows that in all cases, we can assume (possibly after a change of variables) that $x_i = 0$ if $i \notin J$. At this point we are done with the case $J = \varnothing$, so we assume from now on that $J \neq \varnothing$. For the remainder of the argument, whenever we consider a simultaneous change of basis of the form $f_i' = f_i + \alpha_{i-1}e_i$, we will make some choice of α_i's for $i \in J$ and then (without further comment) define α_i for $i \notin J$ by the recursive formula $\alpha_i = (b/a)_i u^{r_i}\varphi(\alpha_{i-1})$; then the resulting change of variables preserves the property that $x_i = 0$ if $i \notin J$.

If $i \in J$, let δ_i be the least positive integer such that $i + \delta_i \in J$ (taken modulo f, as usual); then a simultaneous change of basis of the form $f_i' = f_i + \alpha_{i-1}e_i$ has the effect

$$x_{i+\delta_i}' = x_{i+\delta_i} + \frac{(b)_{\{i+1,\dots,i+\delta_i\}}}{(a)_{\{i+1,\dots,i+\delta_i-1\}}} u^{\sum_{j=1}^{\delta_i-1} r_{i+j}p^{\delta_i-j}}\varphi^{\delta_i}(\alpha_i) - (a)_{i+\delta_i}u^{r_{i+\delta_i}}\alpha_{i+\delta_i}. \tag{7.6}$$

If $i \in J$ and $d \geq r_i$, we shall say that the u^d-term in x_i *affects* the $u^{d'}$-term in $x_{i+\delta_i}$ if the change of variables $f_{i+1}' = f_{i+1} + cu^{d-r_i}e_{i+1}$ (for just the single $i \in J$) alters the term of degree $u^{d'}$ in $x_{i+\delta_i}'$, or in other words if

$$d' = p^{\delta_i}(d - r_i) + \sum_{j=1}^{\delta_i-1} r_{i+j}p^{\delta_i-j}. \tag{7.7}$$

In that case, for brevity we will write that (i, d) affects $(i + \delta_i, d')$.

Observe that each pair (i, d) affects exactly one pair (i', d') (though possibly with $d' < r_{i'}$) and similarly is affected by at most one pair (though often by none). Observe also, e.g. from (7.7), that if (i, d) affects $(i + \delta_i, d')$, then $(i, d + 1)$ affects $(i + \delta_i, d' + p^{\delta_i})$; one deduces that there are at most finitely many pairs (i, d) that affect a pair (i', d') with $d' \leq d$. It follows that the set of all pairs (i, d) with $i \in J$ and $d \geq r_i$ is partitioned into:

- a finite number of *loops* $(i_0, d_0), \ldots, (i_{|J|-1}, d_{|J|-1})$ in which (i_j, d_j) affects (i_{j+1}, d_{j+1}) (and $(i_{|J|-1}, d_{|J|-1})$ affects (i_0, d_0)),
- a finite number of *stubs* $(i_0, d_0), \ldots, (i_m, d_m)$ in which (i_0, d_0) is not affected by any (i, d), while (i_m, d_m) affects some (i', d') with $d' < r_{i'}$,
- a collection of *paths* $(i_0, d_0), \ldots, (i_j, d_j), \ldots$ in which (i_0, d_0) is not affected by any (i, d) and in which (i_j, d_j) affects (i_{j+1}, d_{j+1}).

It is straightforward to see that by making a suitable choice of $u^{d_0 - r_{i_0}}$-coefficient in α_{i_0} (in the second and third cases) or an arbitrary choice of $u^{d_0 - r_{i_0}}$-coefficient in α_{i_0} (in the first case), recursively making suitable choices for $u^{d_j - r_{i_j}}$-coefficient in α_{i_j} for $j > 0$ (stopping at $j = |J| - 1$ in the first case and at $j = m$ in the second case), and doing this simultaneously for all loops, stubs, and paths, the resulting change of basis ensures that x'_i has degree less than r_i for all $i \in J$, with the exception that for each loop, x'_{i_0} may also have a term of degree d_0.

Assume that we have made such a change of basis, so that now x_i is a polynomial of degree less than h_i for all i, except possibly for a term of degree d_0 in x_{i_0} for each loop as above.

It remains to analyze any possible loops more closely. It follows immediately from (7.7) that in a loop $(i_0, d_0), \ldots, (i_{|J|-1}, d_{|J|-1})$ we have $p \mid d_j$ for all j. But note that if $d \geq 2p$ and (i, d) affects $(i + \delta_i, d')$, then since $d' \geq p^{\delta_i}(d - p)$ we have $d' > d$ unless $p = 2$, $d = 4$, and $\delta_i = 1$. It follows that there is at most one loop, and in any loop we either have $d_i = p$ for all i or else $p = 2$ and $(\delta_i, r_i, d_i) = (1, 2, 4)$ for all i.

The latter is the second exceptional case described in the statement (except for the condition that $a = b$); now consider the former. If $\delta_i > 2$, then $\sum_{j=1}^{\delta_i - 1} r_{i+j} p^{\delta_i - j} > p$ since $r_{i+1} > 0$, so any loop with $d_i = p$ for all i requires $\delta_i \leq 2$ for all $i \in J$. The possibilities, then, are either $\delta_i = 1$ and $r_i = p - 1$ or else $\delta_i = 2$ and

$$p = p^2(p - r_i) + p r_{i+1},$$

i.e., $r_i = p$ and $r_{i+1} = 1$. Conversely, if $\delta_i \in \{1, 2\}$ for all $i \in J$, with $r_i = p - 1$ whenever $\delta_i = 1$ and $(r_i, r_{i+1}) = (p, 1)$ whenever $\delta_i = 2$, we indeed have a loop $\{(i, p) : i \in J\}$. Observe that this is precisely the first exceptional case described in the statement of the proposition, again modulo the condition that $a = b$.

In fact one checks without difficulty for the first exceptional case in the statement (with $d_i = p$ for all i) that making the change of variables $\alpha_{i_0} = c u^{p - r_{i_0}}$ (and choosing α_{i_j} accordingly for $1 \leq j < |J|$ to ensure that x_{i_j} does not acquire a non-zero term of degree p), we find that $x'_{i_0} = x_{i_0} + (a)_{i_0}(b/a - 1) c u^p$. Thus if $a \neq b$, we can always choose c to kill the term of degree p in x_{i_0}, and the exceptional case only occurs when $a = b$. The argument in the second exceptional case is analogous. $\square$

Note that in Proposition 7.4 we made no assumption about $\overline{\mathfrak{M}}$ having a lift to some $\mathfrak{M}$ of characteristic zero (let alone having a lift to some $\mathfrak{M}$ coming from a crystalline representation). We now examine what happens when we make such an assumption. For the remainder of this section we reassume the notation of Section 4.4, so that $p > 2$, T is a G_K-stable $\mathcal{O}_E$-lattice in a crystalline representation V of E-dimension d with Hodge–Tate weights in $[0, p]$, and $\mathfrak{M}$ is the associated Kisin module. Write $r_{1,s},\dots,r_{d,s}$ for the κ_s-labeled Hodge–Tate weights of V, and let $\overline{\mathfrak{M}} := \mathfrak{M}\otimes_{\mathcal{O}_E} k_E$, with k_E the residue field of E.

Proposition 7.8. *With notation as above (in particular $p > 2$), suppose that $\overline{\mathfrak{N}} \subset \overline{\mathfrak{M}}$ is a sub-φ-module such that $\overline{\mathfrak{M}}/\overline{\mathfrak{N}}$ is free of rank one as a $W(k)[\![u]\!]\otimes_{\mathbb{Z}_p} k_E$-module. Then $\overline{\mathfrak{N}} \simeq \overline{\mathfrak{M}}(r_0,\dots,r_{f-1};a)$ with $r_s \in \{r_{1,s},\dots,r_{d,s}\}$ for all s and for some $a\in k_E^\times$.*

Proof. Choose a basis $\{e_{i,s}\}$ for $\mathfrak{M}$ as in Theorem 4.22. Since we will work in $\overline{\mathfrak{M}}$ for the remainder of the proof, no confusion will arise if we write $\{e_{i,s}\}$ also for the image of that basis in $\overline{\mathfrak{M}}$.

A generator f_s of $\overline{\mathfrak{N}}_s$ has the form $(e_{1,s},\dots,e_{d,s})\cdot(v_{1,s},\dots,v_{d,s})^T$ for some $v_{1,s},\dots,v_{d,s}\in k_E[\![u]\!]$; by hypothesis at least one of which is a unit. We know from Theorem 4.22 that

$$\varphi(f_s) = (e_{1,s+1},\dots,e_{d,s+1})\overline{X}_s\overline{\Lambda}_s\cdot(\varphi(v_{1,s}),\dots,\varphi(v_{d,s}))^T$$

where $\overline{X}_s$ is the reduction mod $\mathfrak{m}_E$ of X_s and $\overline{\Lambda}_s = [u^{r_{1,s}},\dots,u^{r_{d,s}}]$. Now, observe that each entry of $(\varphi(v_{1,s}),\dots,\varphi(v_{d,s}))^T$ is either a unit or is divisible by u^p, and at least one is a unit. Since we have $r_{i,s}\le p$ for all i, it follows that the largest power of u dividing the column vector $\overline{\Lambda}_s\cdot(\varphi(v_{1,s}),\dots,\varphi(v_{d,s}))^T$ is $u^{r_{i,s}}$ for some i. Noting that $\overline{X}_s$ is invertible, the same is true of $\varphi(f_s)$, and the proposition follows. □

Theorem 7.9. *Suppose that $K/\mathbb{Q}_p$ is unramified and $p>2$. Let T be a G_K-stable $\mathcal{O}_E$-lattice in a crystalline representation V of E-dimension 2 whose κ_s-labeled Hodge–Tate weights are $\{0,r_s\}$ with $r_s\in[1,p]$ for all s. Let $\mathfrak{M}$ be the Kisin module associated to T, and let $\overline{\mathfrak{M}} := \mathfrak{M}\otimes_{\mathcal{O}_E} k_E$.*

Assume that the $k_E[G_K]$-module $\overline{T} := T/\mathfrak{m}_E T$ is reducible. Then $\overline{\mathfrak{M}}$ is an extension of two φ-modules of rank one, and there exist $a,b\in k_E^\times$ and a subset $J\subset\{0,\dots,f-1\}$ so that $\overline{\mathfrak{M}}$ is as follows.

Set $h_i = r_i$ if $i\in J$ and $h_i=0$ if $i\notin J$. Then $\overline{\mathfrak{M}}$ is an extension of $\overline{\mathfrak{M}}(h_0,\dots,h_{f-1};a)$ by $\overline{\mathfrak{M}}(r_0-h_0,\dots,r_{f-1}-h_{f-1};b)$, and we can choose bases e_i,f_i of the $\overline{\mathfrak{M}}_i$ so that φ has the form

$$\begin{aligned}\varphi(e_i) &= (b)_i u^{r_i-h_i}e_{i+1},\\ \varphi(f_i) &= (a)_i u^{h_i} f_{i+1} + x_i e_{i+1}\end{aligned}$$

with $x_i=0$ if $i\notin J$ and $x_i\in k_E$ constant if $i\in J$, except in the following case:

- $(r_0,\dots,r_{f-1})\in\mathcal{P}$,
- $J=\{i : r_i = p-1, p\}$, *and*
- $a=b$.

In that case fix $i_0\in J$; then x_i may be taken to be 0 for all $i\notin J$, to be a constant for all i except $i=i_0$, and to be the sum of a constant and a term of degree p if $i=i_0$.

Finally, $\overline{T}|_{I_K} \simeq \begin{pmatrix}\prod_{i\in J}\omega_i^{r_i} & * \\ 0 & \prod_{i\notin J}\omega_i^{r_i}\end{pmatrix}.$

Proof. It follows from (for example) Lemma 5.5 that $\overline{\mathfrak{M}}$ is an extension of two rank one φ-modules. Then Proposition 7.8 guarantees that if $\overline{\mathfrak{M}}$ is an extension of $\overline{\mathfrak{M}}'$ by $\overline{\mathfrak{M}}''$, then $\overline{\mathfrak{M}}''$ has the form $\overline{\mathfrak{M}}(r_0',\dots,r_{f-1}';b)$ with $r_i'\in\{0,r_i\}$ for all i. Taking $i\in J$ if $r_i'=0$ and $i\notin J$ if $r_i'=r_i$ puts $\overline{\mathfrak{M}}''$ into the correct form; considering the determinant of φ in Theorem 4.22 one finds that $\overline{\mathfrak{M}}'$ then also has the correct form.

Now $\overline{\mathfrak{M}}$ can be taken to have the form given by Proposition 7.4, and it remains to show that each x_i with $i \in J$ cannot have any non-zero terms of degree between 1 and $r_i - 1$. But Theorem 4.22 implies that the image $\varphi(\overline{\mathfrak{M}}_{i-1}) \subset \overline{\mathfrak{M}}_i$ is spanned over $k_E[\![u^p]\!]$ by an element divisible exactly by u^0 and an element divisible exactly by u^{r_i}. On the other hand, if x_i were to have a term of degree between 1 and $r_i - 1$, then neither $(b)_i e_{i+1} + \varphi(c)((a)_i u^{r_i} f_{i+1} + x_i e_{i+1})$ nor $(a)_i u^{r_i} f_{i+1} + x_i e_{i+1} + \varphi(c)(b)_i e_{i+1}$ would be divisible exactly by u^{r_i} for any $c \in k_E[\![u]\!]$. This is a contradiction.

Finally, that $\overline{T}|_{I_K}$ is as claimed follows from parts (1) and (2) of Proposition 6.7, together with the fact that two mod p characters of G_K that are equal on G_∞ must be equal. □

Remark 7.10. It follows easily from Proposition 6.7 and Lemma 7.3 that the exceptional case of Theorem 7.9 in which we allow a term of degree p can only occur if $\overline{T}$ is an extension of a character by itself.

Corollary 7.11. *Suppose that $K/\mathbb{Q}_p$ is unramified and $p > 2$. Let $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the reduction* mod p *of a G_K-stable $\overline{\mathbb{Z}}_p$-lattice in a crystalline $\overline{\mathbb{Q}}_p$-representation of dimension* 2 *whose κ-labeled Hodge–Tate weights are $\{0, r_\kappa\}$ with $r_\kappa \in [1,p]$ for all κ.*

Assume that $\overline{\rho}$ is reducible. Let $S = \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$, and identify the set S with $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$. Then there is a subset $J \subset S$ such that

$$\overline{\rho}|_{I_K} \simeq \begin{pmatrix} \prod_{\sigma\in J} \omega_\sigma^{r_\sigma} & * \\ 0 & \prod_{\sigma\notin J} \omega_\sigma^{r_\sigma} \end{pmatrix}.$$

Proof. This follows immediately from Theorem 7.9, since ρ is necessarily defined over some finite extension $E/\mathbb{Q}_p$. □

Note that Corollary 7.11 does not suffice to prove Theorem 2.12 in the reducible case, because it says nothing about the extension classes. In the following sections we will improve on this result by making a more detailed study of the full $(\varphi, \hat{G})$-modules, rather than just the underlying Kisin modules. However, Corollary 7.11 can be combined with a combinatorial argument to deduce Theorem 2.12 in the irreducible case (see Theorem 10.1 below).

8. Extensions of rank one $(\varphi, \hat{G})$-modules

8.1. **From Kisin modules to $(\varphi, \hat{G})$-modules.** We will now study how (and whether) the rank two φ-modules of Section 7 can be extended to $(\varphi, \hat{G})$-modules. Return to the situation of the previous section: that is, suppose $K = K_0$ and $p > 2$, and let T be a G_K-stable $\mathcal{O}_E$-lattice as in Theorem 7.9. Let $\hat{\overline{\mathfrak{M}}} = (\bar{\mathfrak{M}}, \varphi, \hat{G})$ be the $(\varphi, \hat{G})$-module associated to $\overline{T} = T/\mathfrak{m}_E T$ via Theorem 5.2(4). We *further assume* that $\overline{T}$ is reducible and sits in an exact sequence

$$0 \to \overline{\psi}_1 \to \overline{T} \to \overline{\psi}_2 \to 0.$$

By Lemma 5.5, the $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}$ sits in an exact sequence of $(\varphi, \hat{G})$-modules, whose ambient Kisin module is an exact sequence $0 \to \overline{\mathfrak{M}}_2 \to \overline{\mathfrak{M}} \to \overline{\mathfrak{M}}_1 \to 0$. In the notation of Theorem 7.9, it follows from that result that $\overline{\mathfrak{M}}_1 = \overline{\mathfrak{M}}(h_0, \ldots, h_{f-1}; a)$ and $\overline{\mathfrak{M}}_2 = \overline{\mathfrak{M}}(r_0 - h_0, \ldots, r_{f-1} - h_{f-1}; b)$ for some choice of a, b, and J.

Lemma 8.1. *Except possibly for the case that $r_i = h_i = p$ for all $i = 0, \dots, f-1$, there is at most one way to extend the exact sequence*

$$0 \longrightarrow \overline{\mathfrak{M}}_2 \longrightarrow \overline{\mathfrak{M}} \longrightarrow \overline{\mathfrak{M}}_1 \longrightarrow 0$$

to an exact sequence of $(\varphi, \hat{G})$-modules with natural k_E-action satisfying the conclusion of Corollary 5.10. *In particular the $\hat{G}$-action on $\hat{\overline{\mathfrak{M}}}$ is uniquely determined by $\overline{\mathfrak{M}}$, except possibly for the case that $r_i = h_i = p$ for all $i = 0, \dots, f-1$.*

Proof. Since $\hat{\overline{\mathfrak{M}}}$ is assumed to come from a crystalline representation, the conclusion of Corollary 5.10 holds for $\hat{\overline{\mathfrak{M}}}$. Since by definition $\overline{\mathfrak{M}}$ is contained in the H_K-invariants of $\hat{\overline{\mathfrak{M}}}$, it suffices to show that the τ-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}$ is uniquely determined by the condition of Corollary 5.10. Since $\hat{\overline{\mathfrak{M}}}$ is reducible, we can write

$$\tau(e_i, f_i) = (e_i, f_i) \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \gamma_i \end{pmatrix}$$

with $\alpha_i, \beta_i, \gamma_i \in (\widehat{\mathcal{R}}/p\widehat{\mathcal{R}}) \otimes_{\mathbb{F}_p} k_E \subset R \otimes_{\mathbb{F}_p} k_E$. If $\zeta \in R \otimes_{\mathbb{F}_p} k_E$ is written as $\zeta = \sum_{i=1}^n y_i \otimes z_i$ with $z_1, \dots, z_n \in k_E$ linearly independent over $\mathbb{F}_p$, write $v_R(\zeta) = \min_i \{v_R(y_i)\}$. One checks without difficulty that this is independent of the sum representing ζ, so it is well-defined and satisfies the usual inequality $v_R(\zeta_1 + \zeta_2) \geq \min(v_R(\zeta_1), v_R(\zeta_2))$. The condition of Corollary 5.10 implies that $v_R(\alpha_i - 1)$, $v_R(\gamma_i - 1), v_R(\beta_i) \geq \frac{p^2}{p-1}$ for all i.

Recalling that $\overline{\mathfrak{M}}$ is regarded as a $\varphi(\mathfrak{S})$-submodule of $R \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}$, by Theorem 7.9 we may write $\varphi(e_i, f_i) = (e_{i+1}, f_{i+1})\varphi(A_i)$ with $A_i = \begin{pmatrix} (b)_i u^{r_i - h_i} & x_i \\ 0 & (a)_i u^{h_i} \end{pmatrix}$. Since φ and τ commute, we have

$$\varphi(A_i) \begin{pmatrix} \varphi(\alpha_i) & \varphi(\beta_i) \\ 0 & \varphi(\gamma_i) \end{pmatrix} = \begin{pmatrix} \alpha_{i+1} & \beta_{i+1} \\ 0 & \gamma_{i+1} \end{pmatrix} \tau(\varphi(A_i)).$$

Recall that $\tau(u) = \underline{\epsilon} u$, and once again let $\eta \in R$ be the element defined in Lemma 6.6(2), so that $\varphi^f(\eta) = \underline{\epsilon}\eta$. We obtain the following formulas:

$$u^{p(r_i - h_i)} \varphi(\alpha_i) = \alpha_{i+1} (\underline{\epsilon} u)^{p(r_i - h_i)}, \quad u^{ph_i} \varphi(\gamma_i) = (\underline{\epsilon} u)^{ph_i} \gamma_{i+1}, \tag{8.2}$$

and

$$(b)_i u^{p(r_i - h_i)} \varphi(\beta_i) + \varphi(x_i)\varphi(\gamma_i) = \alpha_{i+1} \tau(\varphi(x_i)) + (a)_i (\underline{\epsilon} u)^{ph_i} \beta_{i+1} \tag{8.3}$$

where for succinctness we have written $(a)_i$, $(b)_i$ in lieu of $1 \otimes (a)_i$, $1 \otimes (b)_i$ in the preceding equation.

From (8.2) we see that $\varphi^f(\alpha_i) = \alpha_i \underline{\epsilon}^{\sum_{j=0}^{f-1} p^{f-j}(r_{i+j} - h_{i+j})}$, and now Lemma 6.6(2) together with the requirement that $v_R(\alpha_i - 1) > 0$ implies that

$$\alpha_i = \eta^{\sum_{j=0}^{f-1} p^{f-j}(r_{i+j} - h_{i+j})} \otimes 1$$

for all i. Similarly we must have $\gamma_i = \eta^{\sum_{j=0}^{f-1} p^{f-j} h_{i+j}} \otimes 1$ for all i. So at least the α_i, γ_i are uniquely determined.

Now suppose that there exists some other extension of $\overline{\mathfrak{M}}$ to a $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}'$. Then the τ-action on $\hat{\overline{\mathfrak{M}}}'$ is given by some α_i', β_i', and γ_i' that also satisfy (8.2) and (8.3), and indeed we have already seen that $\alpha_i' = \alpha_i$ and $\gamma_i' = \gamma_i$.

Let $\tilde{\beta}_i = \beta_i - \beta'_i$. Taking the difference between (8.3) for $\hat{\overline{\mathfrak{M}}}$ and $\hat{\overline{\mathfrak{M}}}'$ gives

$$(b)_i u^{p(r_i-h_i)} \varphi(\tilde{\beta}_i) = (a)_i (\underline{\epsilon} u)^{ph_i} \tilde{\beta}_{i+1},$$

which implies that

$$b u^{\sum_{j=0}^{f-1} u^{p^{f-j}(r_{i+j}-h_{i+j})}} \varphi^f(\tilde{\beta}_i) = a(\underline{\epsilon} u)^{\sum_{j=0}^{f-1} u^{p^{f-j} h_{i+j}}} \tilde{\beta}_i.$$

Considering the valuations of both sides, we see that if $\widetilde{\beta}_i \neq 0$, then

$$v_R(\tilde{\beta}_i) = \frac{1}{p^f - 1} \sum_{j=0}^{f-1} p^{f-j}(2h_{i+j} - r_{i+j}). \tag{8.4}$$

But since $2h_i - r_i \in \{\pm r_i\}$ is at most p with equality if and only if $h_i = r_i = p$, the right-hand side of (8.4) is at most $\frac{p^2}{p-1}$ with equality if and only if $h_i = r_i = p$ for all i. In particular, since $v_R(\beta_i), v_R(\beta'_i) \geq \frac{p^2}{p-1}$, either $\beta_i = \beta'_i$ for all i or else $h_i = r_i = p$ for all i, as required. □

Remark 8.5. In the case that each r_i is at most $p-2$, the results of [Bre99] show that there is a canonical $\hat{G}$-action on $\hat{\overline{\mathfrak{M}}}$.

8.2. **Comparison of extensions of rank one $(\varphi, \hat{G})$-modules.** We are now in a position to prove our main result on extensions of rank one $(\varphi, \hat{G})$-modules, namely Proposition 8.8 below, which we will use in the following section to prove our main local result (Theorem 9.1), showing that (under appropriate hypotheses) the existence of a crystalline lift implies the existence of a reducible crystalline lift with the same Hodge–Tate weights.

Definition 8.6. Write $\vec{r} := (r_0, \ldots, r_{f-1})$ with $r_i \in [1, p]$ for all i. We say that a Kisin module $\overline{\mathfrak{M}}$ is an extension of type $(\vec{r}, a, b, J)$ if it has the same shape as the Kisin modules described by Theorem 7.9; that is, $\overline{\mathfrak{M}}$ sits in a short exact sequence

$$0 \longrightarrow \overline{\mathfrak{M}}(r_0 - h_0, \ldots, r_{f-1} - h_{f-1}; b) \longrightarrow \overline{\mathfrak{M}} \longrightarrow \overline{\mathfrak{M}}(h_0, \ldots, h_{f-1}; a) \longrightarrow 0$$

in which the extension parameters x_i satisfy $x_i = 0$ if $i \notin J$ and $x_i \in k_E$ if $i \in J$ (except that x_{i_0} is allowed to have a term of degree p for one $i_0 \in J$ when $\vec{r} \in \mathcal{P}$, $J = \{i : r_i = p-1, p\}$, and $a = b$). We say that a $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}$ with natural k_E-action is of type $(\vec{r}, a, b, J)$ if it is an extension

$$0 \longrightarrow \hat{\overline{\mathfrak{M}}}'' \longrightarrow \hat{\overline{\mathfrak{M}}} \longrightarrow \hat{\overline{\mathfrak{M}}}' \longrightarrow 0$$

such that the ambient short exact sequence of Kisin modules is an extension of type $(\vec{r}, a, b, J)$ and if for all $x \in \overline{\mathfrak{M}}$ there exist $\alpha \in R$ and $y \in R \otimes_{\varphi, \mathfrak{S}} \overline{\mathfrak{M}}$ such that $\tau(x) - x = \alpha y$ and $v_R(\alpha) \geq \frac{p^2}{p-1}$.

Remark 8.7. Thanks to our work in previous sections, we have the following.

(1) Unless $\vec{r} = (p, p, \ldots, p)$ and $J = \{0, \ldots, f-1\}$, we know from Lemma 8.1 that each extension $\overline{\mathfrak{M}}$ of type $(\vec{r}, a, b, J)$ extends to an extension $\hat{\overline{\mathfrak{M}}}$ of type $(\vec{r}, a, b, J)$ in at most one way.

(2) Suppose $K = K_0$, $p > 2$, and T is a G_K-stable $\mathcal{O}_E$-lattice as in Theorem 7.9. If $\overline{T} = T/\mathfrak{m}_E T$ is reducible, then the $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}}}$ associated to $\overline{T} = T/\mathfrak{m}_E T$ via Theorem 5.2(4) is of type $(\vec{r}, a, b, J)$ for some a, b, and J.

Suppose $p > 2$, fix integers $r_0, \dots, r_{f-1} \in [1,p]$, and let $r = \sum_{i=0}^{f-1} p^{f-1-i} r_i \pmod{p^f - 1}$, so that r is an element of $\mathbb{Z}/(p^f-1)\mathbb{Z}$. If $J \subset \{0,\dots,f-1\}$, let the integers h_i be defined as in Theorem 7.9, and write $h(J) := \sum_{i=0}^{f-1} p^{f-1-i} h_i \pmod{p^f-1}$.

Fix $h \in \mathbb{Z}/(p^f-1)\mathbb{Z}$ and suppose that there exists a subset $J \subset \{0,\dots,f-1\}$ such that $h = h(J)$. For fixed h there may be several such choices of J, as we now describe. Let $i_1, \dots, i_\delta$ be the distinct integers in the range $\{0,\dots,f-1\}$ such that:

- $(r_{i_j}, \dots, r_{i_j+s_j}) = (1, p-1, \dots, p-1, p)$ for some $s_j > 0$ and
- either $i_j \in J$ and $i_j+1, \dots, i_j+s_j \notin J$ or vice versa.

According to Lemma 7.1, all other sets J' such that $h = h(J')$ are obtained from J by choosing some integers $j \in [1,\delta]$ and removing i_j from J and adding $i_j+1, \dots, i_j+s_j$ to J (if $i_j \in J$ to begin with) or vice versa (if $i_j \notin J$ to begin with); or else $r_i = p-1$ for all i and J, $J' = \varnothing$ or $\{0,\dots,f-1\}$.

In particular, for each h such that $h = h(J)$ for at least one J, we can define $J_{\max}$ to be the unique subset of $\{0,\dots,f-1\}$ such that

- $h = h(J_{\max})$ and
- $i_j \notin J$ and $i_j+1,\dots,i_j+s_j \in J$ for all $1 \le j \le \delta$.

When $r_i = p-1$ for all i and $J = \varnothing$ or $\{0,\dots,f-1\}$, we set $J_{\max} = \{0,\dots,f-1\}$. (Strictly speaking we should write $J_{\max}(h)$ instead of $J_{\max}$, but h will always be fixed in any discussion involving $J_{\max}$.)

The main result of this subsection is the following.

Proposition 8.8. *Let $\hat{\overline{\mathfrak{M}}}$ be a $(\varphi,\hat{G})$-module of type $(\vec{r},a,b,J)$, and set $h = h(J)$. Then there exists a $(\varphi,\hat{G})$-module $\hat{\overline{\mathfrak{N}}}$ of type $(\vec{r},a,b,J_{\max})$ such that $\hat{T}(\hat{\overline{\mathfrak{N}}}) \simeq \hat{T}(\hat{\overline{\mathfrak{M}}})$.*

Proof. If $\vec{r} = (p-1,\dots,p-1)$ and $J = \varnothing$, then the ambient Kisin module $\overline{\mathfrak{M}}$ is split, and by Corollary 6.5 and Remark 8.7(1) the extension $\hat{\overline{\mathfrak{M}}}$ is also split. So in this case there is nothing to prove. If $J = J_{\max}$ (e.g. if $\delta = 0$), there is again nothing to prove, so we assume for the remainder of the proof that $\vec{r} \neq (p-1,\dots,p-1)$ and $J \neq J_{\max}$. In particular $\delta > 0$ and there exists some j such that $i_j \in J$ and $i_j+1,\dots,i_j+s_j \notin J$. Let $J' = J \cup \{i_j+1,\dots,i_j+s_j\} \setminus \{i_j\}$. By induction on δ it suffices to prove that there exists an extension $\hat{\overline{\mathfrak{M}}}'$ of type $(\vec{r},a,b,J')$ with $T(\hat{\overline{\mathfrak{M}}}') \simeq T(\hat{\overline{\mathfrak{M}}})$. For simplicity write i,s for i_j, s_j.

Take a basis of $\overline{\mathfrak{M}}$ with notation as in Theorem 7.9, so that in particular we have

$$\varphi(e_i)=(b)_i e_{i+1}, \qquad \varphi(e_{i+t})=(b)_{i+t}u^{p-1}e_{i+t+1}, \qquad \varphi(e_{i+s})=(b)_{i+s}u^p e_{i+s+1},$$
$$\varphi(f_i)=(a)_i u f_{i+1} + x_i e_{i+1}, \qquad \varphi(f_{i+t})=(a)_{i+t} f_{i+t+1}, \qquad \varphi(f_{i+s})=(a)_{i+s} f_{i+s+1}$$

with the middle set of equations holding for $1 \le t \le s-1$.

We will now construct two φ-submodules $\overline{\mathfrak{M}}'$ and $\overline{\mathfrak{M}}''$ of $\overline{\mathfrak{M}}[1/u]$ and check that they are the ambient Kisin modules of $(\varphi,\hat{G})$-modules that satisfy the conclusion of Corollary 5.10.

Set $e'_j = e_j$ and $f'_j = f_j$ for all j except $i+1 \le j \le i+s$, and take $e'_j = u^{-1}e_j$ and $f'_j = u f_j$ for $i+1 \le j \le i+s$. Let $\overline{\mathfrak{M}}'$ be the $\mathfrak{S} \otimes_{\mathbb{Z}_p} k_E$-submodule of $\overline{\mathfrak{M}}[1/u]$ spanned by the e'_j's and f'_j's. Then $\overline{\mathfrak{M}}'$ is a φ-submodule of $\overline{\mathfrak{M}}[1/u]$ with

$$\varphi(e'_i)=(b)_i u e'_{i+1}, \qquad \varphi(e'_{i+t})=(b)_{i+t} e'_{i+t+1}, \qquad \varphi(e'_{i+s})=(b)_{i+s} e'_{i+s+1},$$
$$\varphi(f'_i)=(a)_i f'_{i+1} + x_i u e'_{i+1}, \qquad \varphi(f'_{i+t})=(a)_{i+t}u^{p-1} f'_{i+t+1}, \qquad \varphi(f'_{i+s})=(a)_{i+s}u^p f'_{i+s+1}$$

together with defining equations for φ on $\overline{\mathfrak{M}'_j}$ with $j \notin \{i, \ldots, i+s\}$ that are identical to those of φ on $\overline{\mathfrak{M}}_j$.

Next set $e''_j = e_j$ for all j, set $f''_j = f_j$ for all j except $i+1 \le j \le i+s$, and take $f''_j = uf_j$ for $i+1 \le j \le i+s$. Let $\overline{\mathfrak{M}''}$ be the $\mathfrak{S} \otimes_{\mathbb{Z}_p} k_E$-submodule of $\overline{\mathfrak{M}}[1/u]$ spanned by the e''_j's and f''_j's. Then $\overline{\mathfrak{M}''}$ is a φ-submodule of $\overline{\mathfrak{M}}[1/u]$ with

$$\begin{aligned} \varphi(e''_i) &= (b)_i e''_{i+1}, & \varphi(e''_{i+t}) &= (b)_{i+t} u^{p-1} e''_{i+t+1}, & \varphi(e''_{i+s}) &= (b)_{i+s} u^p e''_{i+s+1}, \\ \varphi(f''_i) &= (a)_i f''_{i+1} + x_i e''_{i+1}, & \varphi(f''_{i+t}) &= (a)_{i+t} u^{p-1} f''_{i+t+1}, & \varphi(f''_{i+s}) &= (a)_{i+s} u^p f''_{i+s+1} \end{aligned}$$

and defining equations for φ on $\overline{\mathfrak{M}''_j}$ with $j \notin \{i, \ldots, i+s\}$ that are identical to those of φ on $\overline{\mathfrak{M}}_j$.

Let us check that the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}}[1/u]$ preserves $\overline{\mathfrak{M}'}$ and makes it into a $(\varphi, \hat{G})$-module of type $(\vec{r}, a, b, J')$. Since H_K acts trivially on u and $\overline{\mathfrak{M}}$ and since $\underline{\epsilon} - 1 \in I_+$, the only non-trivial part of this claim is that $\overline{\mathfrak{M}'}$ is preserved by τ. This is immediate for the action of τ on (e'_j, f'_j) if $j \notin [i+1, i+s]$. If $i+1 \le j \le i+s$ and

$$\tau(e_j, f_j) = (e_j, f_j) \begin{pmatrix} \alpha_j & \beta_j \\ 0 & \gamma_j \end{pmatrix},$$

then an easy calculation shows that

$$\tau(e'_j, f'_j) = (e'_j, f'_j) \begin{pmatrix} \alpha_j \underline{\epsilon}^{-p} & \beta_j u^{2p} \underline{\epsilon}^p \\ 0 & \gamma_j \underline{\epsilon}^p \end{pmatrix}$$

and again the conclusion is clear. In fact, since $v_R(\beta_j u^{2p} \underline{\epsilon}^p) \ge v_R(\beta_j) \ge p^2/(p-1)$, not only do we obtain a $(\varphi, \hat{G})$-module $\hat{\overline{\mathfrak{M}'}}$ with ambient Kisin module $\overline{\mathfrak{M}'}$, we have also shown that for all $x \in \overline{\mathfrak{M}'}$ there exist $\alpha \in R$ and $y \in R \otimes_{\varphi,\mathfrak{S}} \overline{\mathfrak{M}'}$ such that $\tau(x) - x = \alpha y$ and $v_R(\alpha) \ge \frac{p^2}{p-1}$. The argument for $\hat{\overline{\mathfrak{M}''}}$ is essentially the same, with the same conclusion.

By construction we have natural inclusions $\hat{\overline{\mathfrak{M}''}} \hookrightarrow \hat{\overline{\mathfrak{M}}}$ and $\hat{\overline{\mathfrak{M}''}} \hookrightarrow \hat{\overline{\mathfrak{M}'}}$. It follows from Lemma 5.8 that $\hat{T}(\hat{\overline{\mathfrak{M}'}}) \simeq \hat{T}(\hat{\overline{\mathfrak{M}''}}) \simeq \hat{T}(\hat{\overline{\mathfrak{M}}})$.

Note that we have not quite finished showing that $\hat{\overline{\mathfrak{M}'}}$ is an extension of type $(\vec{r}, a, b, J')$: because of the presence of the term $x_i u e'_{i+1}$ in $\varphi(f'_i)$, the presentation for $\overline{\mathfrak{M}'}$ that we have given does not have exactly the same shape as in Theorem 7.9. To conclude, we must show that there is a change of variables as in the proof of Proposition 7.4 that puts $\overline{\mathfrak{M}'}$ into the correct form. First replace f'_{i+1} with $f'_{i+1} + x_i u (a)_i^{-1} e'_{i+1}$, so that now $\varphi(f'_i) = (a)_i f'_{i+1}$; this introduces a term of the form $cu^p e'_{i+2}$ into the formula for $\varphi(f'_{i+1})$, with $c \in k_E$. Noting that $i+1 \in J'$, we can now use the terminology of the proof of Proposition 7.4: since $p \ge h_{i+1} \in \{p-1, p\}$, the pair $(i+1, p)$ affects some other pair. We now distinguish three possibilities.

- If the pair $(i+1, p)$ is part of a *loop*, then J' must be as in the exceptional case of Proposition 7.4; in that case $\overline{\mathfrak{M}'}$ is already written as an extension of type $(\vec{r}, a, b, J')$, because the term $cu^p e'_{i+2}$ is permitted.
- If the pair $(i+1, p)$ is part of a *stub*, suppose that the last term (i_m, d_m) in the stub affects (i', d') with $d' < r_{i'}$. Then in fact $d' = 0$, because $p \mid d'$ by (7.7) and $r_{i'} \le p$. It follows that there is a change of variables as in the proof of Proposition 7.4 that removes the term $cu^p e'_{i+2}$ from $\varphi(f'_{i+1})$ and adds a term of the form $x e'_{j+1}$ into some $\varphi(f'_j)$ with $j \in J'$ and $x \in k_E$.

After such a change of variables, $\overline{\mathfrak{M}'}$ is written as an extension of type $(\vec{r}, a, b, J')$.

- If the pair $(i+1, p)$ is part of a *path*, then just as in the proof of Proposition 7.4 there is a change of variables which eliminates the $cu^p e'_{i+2}$ term. After such a change of variables, $\overline{\mathfrak{M}'}$ is written as an extension of type $(\vec{r}, a, b, J')$.

This completes the proof. □

9. The reducible case

We now prove Theorem 2.12 in the reducible case. Let $K/\mathbb{Q}_p$ be a finite unramified extension with residue field k. As usual, we will identify $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$ with $\mathrm{Hom}(k, \overline{\mathbb{F}}_p)$. From Definition 2.3, we see that we need to prove the following result, whose proof will occupy the remainder of this section.

Theorem 9.1. *Suppose $p > 2$. Let $\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ be a continuous representation such that $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is reducible. Suppose that ρ is crystalline with κ-Hodge–Tate weights $\{b_{\kappa,1}, b_{\kappa,2}\}$ for each $\kappa \in \mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)$, and suppose further that $1 \le b_{\kappa,1} - b_{\kappa,2} \le p$ for each κ.*

Then there is a reducible crystalline representation $\rho'\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ with the same κ-Hodge–Tate weights as ρ for each κ, such that $\overline{\rho} \simeq \overline{\rho}'$.

Write $\overline{\rho} \simeq \begin{pmatrix} \overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$. Note that by Corollary 7.11 there is a decomposition $\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p) = J \coprod J^c$ such that $\overline{\psi}_1|_{I_K} = \prod_{\kappa \in J} \omega_{\overline{\kappa}}^{b_{\kappa,1}} \prod_{\kappa \in J^c} \omega_{\overline{\kappa}}^{b_{\kappa,2}}$ and $\overline{\psi}_2 = \prod_{\kappa \in J^c} \omega_{\overline{\kappa}}^{b_{\kappa,1}} \prod_{\kappa \in J} \omega_{\overline{\kappa}}^{b_{\kappa,2}}$. In fact there may be several such J; temporarily fix one choice.

Let $\psi_1, \psi_2\colon G_K \to \overline{\mathbb{Z}}_p^\times$ be crystalline lifts of $\overline{\psi}_1$, $\overline{\psi}_2$, respectively, with the properties that $\mathrm{HT}_\kappa(\psi_1) = b_{\kappa,1}$ if $\kappa \in J$ and $b_{\kappa,2}$ otherwise, and $\mathrm{HT}_\kappa(\psi_2) = b_{\kappa,2}$ if $\kappa \in J$ and $b_{\kappa,1}$ otherwise. (The characters ψ_1 and ψ_2 exist by Corollary 6.5 and Proposition 6.7 and are easily seen to be unique up to an unramified twist.)

We naturally identify $\mathrm{Ext}_{G_K}(\overline{\psi}_2, \overline{\psi}_1)$ with $H^1(G_K, \overline{\psi}_1 \overline{\psi}_2^{-1})$ from now on.

Definition 9.2. Let L_{ψ_1,ψ_2} be the subset of $H^1(G_K, \overline{\psi}_1 \overline{\psi}_2^{-1})$ consisting of all elements such that the corresponding representation has a crystalline lift of the form

$$\begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

We have the following variant of [GLS12, Lem. 4.2.2] (which is in turn a variant of [BDJ10, Lem. 3.12]).

Lemma 9.3. *L_{ψ_1,ψ_2} is an $\overline{\mathbb{F}}_p$-vector subspace of $H^1(G_K, \overline{\psi}_1 \overline{\psi}_2^{-1})$ of dimension $|J|$, unless $\overline{\psi}_1 = \overline{\psi}_2$, in which case it has dimension $|J| + 1$.*

Proof. Let $\psi = \psi_1 \psi_2^{-1}$. Recall that $H^1_f(G_K, \overline{\mathbb{Z}}_p(\psi))$ is by definition the preimage of $H^1_f(G_K, \overline{\mathbb{Q}}_p(\psi))$ under the natural map $\eta\colon H^1(G_K, \overline{\mathbb{Z}}_p(\psi)) \to H^1(G_K, \overline{\mathbb{Q}}_p(\psi))$, so that L_{ψ_1,ψ_2} is the image of $H^1_f(G_K, \overline{\mathbb{Z}}_p(\psi))$ in $H^1(G_K, \overline{\psi})$. The kernel of η is precisely the torsion part of $H^1(G_K, \overline{\mathbb{Z}}_p(\psi))$. Since $\psi \ne 1$, e.g. by examining Hodge–Tate weights, this torsion is non-zero if and only if $\overline{\psi} = 1$, in which case it has the form $\lambda^{-1}\overline{\mathbb{Z}}_p/\overline{\mathbb{Z}}_p$ for some $\lambda \in \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. (To see this, note that if $\psi \ne 1$ is defined over

E, then the long exact sequence associated to $0 \to \mathcal{O}_E(\psi) \stackrel{\varpi}{\to} \mathcal{O}_E(\psi) \to k_E(\overline{\psi}) \to 0$ identifies $k_E(\overline{\psi})^{G_K}$ with the ϖ-torsion in $\ker(\eta)$.)

By [Nek93, Prop. 1.24(2)] and the assumption that $b_{\kappa,1} > b_{\kappa,2}$ for each κ, we see that $\dim_{\overline{\mathbb{Q}}_p} H^1_f(G_K, \overline{\mathbb{Q}}_p(\psi)) = |J|$, again using $\psi \neq 1$. Since $H^1(G_K, \overline{\mathbb{Z}}_p(\psi))$ is a finitely generated $\overline{\mathbb{Z}}_p$-module, the result follows. □

The following lemma is a slight variant of [BLGG12, Lem. 6.1.6] and [GLS12, Prop. 5.2.9] and has an almost identical proof.

Lemma 9.4. *Suppose that for each κ we have $b_{\kappa,1} - b_{\kappa,2} = p$ and that $(\overline{\psi}_1 \overline{\psi}_2^{-1})|_{I_K} = \overline{\varepsilon}$. Then $\overline{\rho}$ has a reducible crystalline lift ρ' with $\mathrm{HT}_\kappa(\rho') = \{b_{\kappa,1}, b_{\kappa,2}\}$ for each κ.*

Proof. Suppose firstly that $\overline{\psi}_1 \neq \overline{\psi}_2 \overline{\varepsilon}$. By assumption, we can take $J = S$ in the above. Then for any choice of ψ_1, ψ_2, we have $L_{\psi_1,\psi_2} = H^1(G_K, \overline{\psi}_1 \overline{\psi}_2^{-1})$ by Lemma 9.3 and the local Euler characteristic formula, completing the proof in this case.

Assume now that $\overline{\psi}_1 \overline{\psi}_2^{-1} = \overline{\varepsilon}$. By twisting we can reduce to the case $(b_{\kappa,1}, b_{\kappa,2}) = (p, 0)$ for each κ. Let L be a given line in $H^1(G_K, \overline{\varepsilon})$, and choose an unramified character χ with trivial reduction. Let $E/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$, uniformizer ϖ, and residue field $\mathbb{F}$, such that χ is defined over E and L is defined over $\mathbb{F}$ (that is, there is a basis for L which corresponds to an extension defined over $\mathbb{F}$). Since any extension of 1 by $\chi\varepsilon^p$ is automatically crystalline, it suffices to show that we can choose χ so that L lifts to $H^1(G_K, \mathcal{O}(\chi\varepsilon^p))$.

Let H be the hyperplane in $H^1(G_K, \mathbb{F})$ which annihilates L under the Tate pairing. Let $\delta_1 \colon H^1(G_K, \mathbb{F}(\overline{\varepsilon})) \to H^2(G_K, \mathcal{O}(\chi\varepsilon^p))$ be the map coming from the exact sequence $0 \to \mathcal{O}(\chi\varepsilon^p) \stackrel{\varpi}{\to} \mathcal{O}(\chi\varepsilon^p) \to \mathbb{F}(\overline{\varepsilon}) \to 0$ of G_K-modules. We need to show that $\delta_1(L) = 0$ for some choice of χ.

Let δ_0 be the map $H^0(G_K, (E/\mathcal{O})(\chi^{-1}\varepsilon^{1-p})) \to H^1(G_K, \mathbb{F})$ coming from the exact sequence $0 \to \mathbb{F} \to (E/\mathcal{O})(\chi^{-1}\varepsilon^{1-p}) \stackrel{\varpi}{\to} (E/\mathcal{O})(\chi^{-1}\varepsilon^{1-p}) \to 0$ of G_K-modules. By Tate local duality, the condition that L vanish under the map δ_1 is equivalent to the condition that the image of the map δ_0 be contained in H. Let $n \geq 1$ be the largest integer with the property that $\chi^{-1}\varepsilon^{1-p} \equiv 1 \pmod{\varpi^n}$. Then we can write $\chi^{-1}\varepsilon^{1-p}(x) = 1 + \varpi^n \alpha_\chi(x)$ for some function $\alpha_\chi \colon G_K \to \mathcal{O}$. Let $\overline{\alpha}_\chi \colon G_K \to \mathbb{F}$ denote $\alpha_\chi \pmod{\varpi}$. Then $\overline{\alpha}_\chi$ is a group homomorphism (i.e., a 1-cocycle), and the choice of n ensures that it is non-trivial. It is straightforward to check that the image of the map δ_0 is the line spanned by $\overline{\alpha}_\chi$. If $\overline{\alpha}_\chi$ is in H for some χ, we are done. Suppose this is not the case. We break the rest of the proof into two cases.

Case 1: *L is très ramifié.* To begin, we observe that it is possible to have chosen χ so that $\overline{\alpha}_\chi$ is ramified. To see this, let m be the largest integer with the property that $(\chi^{-1}\varepsilon^{1-p})|_{I_K} \equiv 1 \pmod{\varpi^m}$. Note that m exists since the Hodge–Tate weights of $\chi^{-1}\varepsilon^{1-p}$ are not all 0. If $m = n$, then we are done, so assume instead that $m > n$. Let $g \in G_K$ be a fixed lift of Frob_K. We claim that $\chi^{-1}\varepsilon^{1-p}(g) = 1 + \varpi^n \alpha_\chi(g)$ such that $\alpha_\chi(g) \not\equiv 0 \pmod{\varpi}$. In fact, if $\alpha_\chi(g) \equiv 0 \pmod{\varpi}$, then $\chi^{-1}\varepsilon^{1-p}(g) \in 1 + \varpi^{n+1}\mathcal{O}_K$. Since $m > n$, we see that $\chi^{-1}\varepsilon^{1-p}(G_K) \subset 1 + \varpi^{n+1}\mathcal{O}_K$, and this contradicts the selection of n. Now let χ' be the unramified character sending our fixed g to $1 + \varpi^n \alpha_\chi(g)$. Then χ' has trivial reduction, and after replacing χ by $\chi\chi'$ we see that n has increased but m has not changed. After finitely many iterations of this procedure we have $m = n$, completing the claim.

Suppose, then, that $\overline{\alpha}_\chi$ is ramified. The fact that L is très ramifié implies that H does not contain the unramified line in $H^1(G_K, \mathbb{F})$. Thus there is a unique $\overline{x} \in \mathbb{F}^\times$

such that $\overline{\alpha}_\chi + u_{\overline{x}} \in H$ where $u_{\overline{x}}\colon G_K \to \mathbb{F}$ is the unramified homomorphism sending Frob_K to $\overline{x}$. Replacing χ with χ times the unramified character sending Frob_K to $(1+\varpi^n x)^{-1}$, for x a lift of $\overline{x}$, we are done.

Case 2: *L is peu ramifié.* Making a ramified extension of $\mathcal{O}$ if necessary, we can and do assume that $n \geq 2$ (for example, replacing E by $E(\varpi^{1/2})$ has the effect of replacing n by $2n$). The fact that L is peu ramifié implies that H contains the unramified line. It follows that if we replace χ with χ times the unramified character sending Frob_K to $1+\varpi$, then we are done (as the new $\overline{\alpha}_\chi$ will be unramified). □

Proof of Theorem 9.1. We maintain the notation established above, so that in particular we have $\overline{\rho} \simeq \begin{pmatrix} \overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$. If $(\overline{\psi}_1\overline{\psi}_2^{-1})|_{I_K} = \overline{\varepsilon}$ and $b_{\kappa,1} - b_{\kappa,2} = p$ for all κ, then the result follows from Lemma 9.4, so assume from now on that either $(\overline{\psi}_1\overline{\psi}_2^{-1})|_{I_K} \neq \overline{\varepsilon}$ or $b_{\kappa,1} - b_{\kappa,2} \neq p$ for some κ. Twisting, we can and do assume in addition that $b_{\kappa,2} = 0$ for each κ. Write $r_\kappa := b_{\kappa,1}$ for each κ.

Choose a finite extension $E/\mathbb{Q}_p$ which is sufficiently large. In particular, choose E such that ρ is defined over $\mathcal{O}_E$ and such that for each tuple of integers $\{s_\kappa\}$ in the range $[0,p]$ such that if $\overline{\psi}_i$ $(i=1,2)$ has a crystalline lift ψ_i with $\mathrm{HT}_\kappa(\psi_i) = s_\kappa$ for all κ, it has such a lift defined over $\mathcal{O}_E$. Fixing one choice for each possible ψ_i (for each choice of Hodge–Tate weights) in the previous clause, further enlarge E so that each space $H^1_f(G_K, \overline{\mathbb{Z}}_p(\psi_1\psi_2^{-1}))$ is defined over $\mathcal{O}_E$.

From now on, we will allow ρ (and thus $\overline{\rho}$) to vary over all crystalline representations $G_K \to \mathrm{GL}_2(\mathcal{O}_E)$ which have $\overline{\rho} \simeq \begin{pmatrix} \overline{\psi}_1 & * \\ 0 & \overline{\psi}_2 \end{pmatrix}$ (where the extension class $*$ is allowed to vary) and which have κ-labelled Hodge–Tate weights $\{0, r_\kappa\}$ for each κ. By Theorem 7.9 together with Remark 8.7(2), Proposition 8.8, and the discussion between them, we see that there exist a, $b \in k_E$ and a subset $J_{\max} \subset \{0,\ldots,f-1\}$ so that for any such ρ, there is a $(\varphi,\hat{G})$-module $\hat{\overline{\mathfrak{N}}}$ of type $(\vec{r}, a, b, J_{\max})$ such that $\hat{T}(\hat{\overline{\mathfrak{N}}}) \simeq \overline{\rho}$. (Apply Proposition 6.7 to see that a, b are uniquely determined.) By Theorem 7.9 and the assumption that we are not in the case that $(\overline{\psi}_1\overline{\psi}_2^{-1})|_{I_K} = \overline{\varepsilon}$ and each $r_\kappa = p$, we see that we are not in the exceptional case in Lemma 8.1; there are thus at most $(\#k_E)^{|J_{\max}|}$ isomorphism classes of $(\varphi,\hat{G})$-modules $\hat{\overline{\mathfrak{N}}}$ of type $(\vec{r}, a, b, J_{\max})$, and thus (by Theorem 7.9 and Remark 7.10) at most $(\#k_E)^{|J_{\max}|}$ elements of $H^1(G_K, \overline{\psi}_1\overline{\psi}_2^{-1})$ corresponding to representations $\overline{\rho}$, *unless* $\overline{\psi}_1 = \overline{\psi}_2$, in which case $(\#k_E)^{|J_{\max}|}$ must be replaced with $(\#k_E)^{|J_{\max}|+1}$.

Now apply the discussion at the beginning of this section with $J = J_{\max}$; that is, choose (as we may, by, for example, Proposition 7.8) crystalline characters ψ_1, ψ_2 lifting $\overline{\psi}_1$, $\overline{\psi}_2$, respectively, such that $\mathrm{HT}_\kappa(\psi_1) = r_\kappa$ if $\kappa \in J_{\max}$ and 0 otherwise, and $\mathrm{HT}_\kappa(\psi_2) = 0$ if $\kappa \in J_{\max}$ and r_κ otherwise. Note that by our choice of E we may further suppose that ψ_1, ψ_2, and $H^1_f(G_K, \overline{\mathbb{Z}}_p(\psi_1\psi_2^{-1}))$ are all defined over $\mathcal{O}_E$.

By Lemma 9.3 we see that there are $(\#k_E)^{|J_{\max}|}$ extension classes which arise as the reductions of crystalline representations which are extensions of ψ_2 by ψ_1, unless $\overline{\psi}_1 = \overline{\psi}_2$, in which case there are $(\#k_E)^{|J_{\max}|+1}$ extension classes. Since we have already shown that there are at most $(\#k_E)^{|J_{\max}|}$ (or $(\#k_E)^{|J_{\max}|+1}$ if $\overline{\psi}_1 = \overline{\psi}_2$) extension classes arising from the reduction of crystalline representations with κ-labelled Hodge–Tate weights $\{0, r_\kappa\}$, the result follows. □

10. The irreducible case

We now explain how to deduce the irreducible case of Theorem 2.12 from the reducible one. A usual, let $K = K_0$ be the unramified extension of $\mathbb{Q}_p$ of degree f, and let

$$\rho\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

be a continuous irreducible representation such that $\overline{\rho}\colon G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is also irreducible. Suppose that ρ is crystalline with κ-Hodge–Tate weights $\{b_{\kappa,1}, b_{\kappa,2}\}$ for each $\kappa \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_p)$, and suppose further that $1 \le b_{\kappa,1} - b_{\kappa,2} \le p$ for each κ.

Let k denote the residue field of K, and let K_2 be the quadratic unramified extension of K, with residue field k_2. We write $S = \mathrm{Hom}(k, \overline{\mathbb{F}}_p)$ and $S_2 = \mathrm{Hom}(k_2, \overline{\mathbb{F}}_p)$. We say that $J \subset S_2$ is a *balanced subset* if it consists of precisely one element of S_2 extending each element of S. If $\sigma \in S$ is the reduction mod p of $\kappa \in \mathrm{Hom}(K, \overline{\mathbb{Q}}_p)$, we write $b_{\sigma,i}$ for $b_{\kappa,i}$. Recalling the definition of $W^{\mathrm{BDJ}}(\overline{\rho})$ when $\overline{\rho}$ is irreducible (Definition 2.4), we see that in order to complete the proof of Theorem 2.12, we need to prove the following result.

Theorem 10.1. *There is a balanced subset $J \subset S_2$ such that*

$$\overline{\rho}|_{I_K} \simeq \begin{pmatrix} \prod_{\sigma\in J} \omega_\sigma^{b_{\sigma|_k,1}} \prod_{\sigma\notin J} \omega_\sigma^{b_{\sigma|_k,2}} & 0 \\ 0 & \prod_{\sigma\in J} \omega_\sigma^{b_{\sigma|_k,2}} \prod_{\sigma\notin J} \omega_\sigma^{b_{\sigma|_k,1}} \end{pmatrix}.$$

Proof. Since $\overline{\rho}|_{G_{K_2}}$ is reducible, by Corollary 7.11 we certainly have a decomposition as in the statement of the theorem for some $J \subset S_2$, but we do not know that J is balanced. Indeed, this is not completely automatic, but we will show that a balanced choice of J always exists.

To see this, note that since $\overline{\rho}|_{I_K}$ is irreducible, we must have

$$\prod_{\sigma\in J} \omega_\sigma^{b_{\sigma|_k,1}} \prod_{\sigma\notin J} \omega_\sigma^{b_{\sigma|_k,2}} = \prod_{\sigma\in J} \omega_{\sigma\circ\varphi^f}^{b_{\sigma|_k,2}} \prod_{\sigma\notin J} \omega_{\sigma\circ\varphi^f}^{b_{\sigma|_k,1}}.$$

Write J_1 for the set of places in S both of whose extensions to S_2 are in J, and J_2 for the set of places in S neither of whose extensions to S_2 are in J. Then we see that we have

$$\prod_{\sigma\in J_1} \omega_\sigma^{b_{\sigma,1}-b_{\sigma,2}} = \prod_{\sigma\in J_2} \omega_\sigma^{b_{\sigma,1}-b_{\sigma,2}}.$$

If both J_1, J_2 are empty, then J is balanced, and we are done. Assume therefore that this is not the case.

Define x_σ as follows: $x_\sigma = b_{\sigma,1} - b_{\sigma,2}$ if $\sigma \in J_1$, $x_\sigma = b_{\sigma,2} - b_{\sigma,1}$ if $\sigma \in J_2$, and $x_\sigma = 0$ otherwise. Note that since $\overline{\rho}$ is irreducible, there is at least one place σ with $x_\sigma = 0$. We have $\prod_{\sigma\in S} \omega_\sigma^{x_\sigma} = 1$, and each $x_\sigma \in [-p, p]$. Choose an element $\sigma_0 \in S$, and recursively define $\sigma_i = \sigma_{i+1}^p$. Writing ω_i for ω_{σ_i}, we have $\omega_{i+1}^p = \omega_i$. From now on, we identify S with $\{0, \dots, f-1\}$ by identifying σ_i with i. By Lemma 7.1, the cyclic set of those i with $x_i \neq 0$ must break up as a disjoint union of sets of the form $(i, i+1, \dots, i+j)$ with $(x_i, x_{i+1}, \dots, x_{i+j}) = \pm(-1, p-1, p-1, \dots, p-1, p)$ (where there may not be any occurrences of $p-1$). For each such interval (i, j), we may choose a lift of i to S_2 and replace J with $J\Delta\{i, \dots, i+j\}$. It is easy to see that this choice does not change $\overline{\rho}|_{I_K}$ and results in a balanced choice of J, as required. ∎

Remark 10.2. It is perhaps worth illustrating the proof of Theorem 10.1 with an example. Take $f = 4$, and consider a representation of the form

$$\begin{pmatrix} \omega_1^{p-1}\omega_2^p\omega_3^b\omega_5^{p-1}\omega_6^p & 0 \\ 0 & \omega_0\omega_4\omega_7^b \end{pmatrix},$$

with $0 < b \leq p-1$. This is certainly a possible restriction to inertia of an irreducible representation, but it is not written in the balanced form of the statement of Theorem 10.1. However, if we write it as

$$\begin{pmatrix} \omega_1^{p-1}\omega_2^p\omega_3^b\omega_4 & 0 \\ 0 & \omega_0\omega_5^{p-1}\omega_6^p\omega_7^b \end{pmatrix},$$

then we obtain a balanced expression, as required.

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