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AUTOMORPHIC PERIOD AND THE CENTRAL VALUE OF RANKIN-SELBERG L-FUNCTION

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Contents

1. Introduction

In this article, as a sequel of [51], we prove a conjectural refinement of the global Gan-Gross-Prasad conjecture [7] for unitary groups under some local conditions. This refinement is modeled on the pioneering work of Waldspurger [40] on toric periods and the central values of L-functions on GL_2. In an influential paper [23], Ichino and Ikeda first formulated the refinement for orthogonal groups. After the Ichino-Ikeda formulation, R. N. Harris considered the case of unitary groups in his Ph.D. thesis at the University of California, San Diego [21].

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1.1. **The conjecture of Ichino-Ikeda and R. N. Harris.** We now recall the conjectural refinement. Let E/F be a quadratic extension of number fields with adeles denoted by $\mathbb{A} = \mathbb{A}_F$ and $\mathbb{A}_E$, respectively. Let V be a Hermitian space of dimension $n+1$ and W a (nondegenerate) subspace of codimension one. Denote the unitary groups by $U(V)$ and $U(W)$, respectively. Let $G = U(W) \times U(V)$ be the product and H the diagonal embedding of $U(W)$ into G. Let $\pi = \pi_n \otimes \pi_{n+1}$ be a cuspidal automorphic representation of $G(\mathbb{A})$ and let $\pi_{i,E}$ be the base change of π_i to $\mathrm{GL}_i(\mathbb{A}_E)$, $i = n, n+1$. Denote by $L(s, \pi_E)$ the Rankin-Selberg convolution L-function $L(s, \pi_{n,E} \times \pi_{n+1,E})$ due to Jacquet–Piatetskii-Shapiro–Shalika [27]. It is known to be the same as the one defined by the Langlands-Shahidi method. The reader may consult the introduction of [11] for an overview of the study of the central value $L(1/2, \pi_{n,E} \times \pi_{n+1,E})$. We also consider the adjoint L-function of π (cf. [7, §7],[21, Remark 1.4]),

$$L(s, \pi, Ad) = L(s, \pi_n, Ad)L(s, \pi_{n+1}, Ad).$$

We refer to Remark 7 for the definition of the adjoint L-function at bad places (also cf. Remark 1 after Conjecture 1.1 below).

Denote the constant

$$\Delta_{n+1} = \prod_{i=1}^{n+1} L(i, \eta^i) = L(1, \eta)L(2, 1_F)L(3, \eta) \cdots L(n+1, \eta^{n+1}),$$

where η is the quadratic character of $F^\times \backslash \mathbb{A}^\times$ associated to E/F by class field theory. Note that here $\Delta_{n+1} = L(M^\vee(1))$ where $M^\vee$ is the motive dual to the motive M associated to the quasi-split reductive group $U(n+1)$ defined by Gross [15]. We will be interested in the following combination of L-functions:

$$\mathscr{L}(s, \pi) = \Delta_{n+1} \frac{L(s, \pi_E)}{L(s+1/2, \pi, Ad)}. \tag{1.1}$$

We also write $\mathscr{L}(s, \pi_v)$ for the local factor at v.

Let $[H]$ denote the quotient $H(F)\backslash H(\mathbb{A})$ and similarly for G. We endow $H(\mathbb{A})$ ($G(\mathbb{A})$, *resp.*) with their Tamagawa measures[1] and $[H]$ ($[G]$, *resp.*) with the quotient measure by the counting measure on $H(F)$ $[G(F)]$; cf. §2. In [7], Gan, Gross, and Prasad propose to study an automorphic period integral

$$\mathscr{P}(\phi) = \mathscr{P}_H(\phi) := \int_{[H]} \phi(h)\, dh, \qquad \phi \in \pi.$$

They conjecture that the nonvanishing of the linear functional $\mathscr{P}$ on π [possibly by varying the Hermitian spaces (W, V) and switching to another member in the Vogan L-packet[2] of π] is equivalent to the nonvanishing of the central value $L(1/2, \pi_E)$ of the Rankin-Selberg L-function. This conjectural equivalence is proved for π satisfying some local conditions in our previous paper [51]. One direction of the equivalence had also been proved by Ginzburg-Jiang-Rallis (cf. [11], [12]).

For arithmetic application, it is necessary to have a more precise relation between the automorphic period integral $\mathscr{P}$ and the L-value $\mathscr{L}(1/2, \pi_E)$. To state the

[1] Since the unitary group H has a nontrivial central torus, we need to introduce a convergence factor: $dh = L(1, \eta)^{-1} \prod_v L(1, \eta_v)|\omega|_v$ for a nonzero invariant differential ω of top degree on H. Similarly for G.

[2] For the term "Vogan L-packet," cf. [7, §9-11].

precise refinement of the Gan-Gross-Prasad conjecture, we need to introduce more notations. Let $\langle\cdot,\cdot\rangle_{\mathrm{Pet}}$ be the Peterson inner product

$$\langle\phi,\varphi\rangle_{\mathrm{Pet}}=\int_{[G]}\phi(g)\overline{\varphi}(g)\,dg,\qquad \phi,\varphi\in\pi. \tag{1.2}$$

Fix a decomposition as a product

$$\langle\cdot,\cdot\rangle_{\mathrm{Pet}}=\prod_v\langle\cdot,\cdot\rangle_v$$

under the decomposition $\pi=\otimes\pi_v$. In this way we fix an invariant inner product on π_v. Ichino and Ikeda first consider the following integration of the matrix coefficient: for $\phi_v,\varphi_v\in\pi_v$, we define when π_v is tempered,

$$\alpha_v(\phi_v,\varphi_v)=\int_{H_v}\langle\pi_v(h)\phi_v,\varphi_v\rangle_v\,dh,\qquad H_v=H(F_v). \tag{1.3}$$

It has the following nice properties for *tempered* π_v:

(1) It converges absolutely, and it is positive definite: $\alpha_v(\phi_v,\phi_v)\geq 0$.
(2) When π_v is unramified,[3] and the vectors ϕ_v,φ_v are fixed by K_v such that $\langle\phi_v,\varphi_v\rangle_v=1$, we have
$$\alpha_v(\phi_v,\varphi_v)=\mathscr{L}\left(\frac{1}{2},\pi_v\right)\cdot\mathrm{vol}(H(\mathcal{O}_v)).$$
(3) If $\mathrm{Hom}_{H(F_v)}(\pi_v,\mathbb{C})\neq 0$, then the form α_v does not vanish identically.

The first two were proved by Ichino and Ikeda (R. N. Harris in the unitary group case). The third property was conjectured by them and proved by Sakellaridis and Venkatesh [36, §6.4] in a more general setting. Waldspurger also proved the third property in the p-adic orthogonal case. Because of the second property, we normalize the form α_v as follows:

$$\alpha_v^{\natural}(\phi_v,\varphi_v)=\frac{1}{\mathscr{L}(1/2,\pi_v)}\int_{H_v}\langle\pi_v(h)\phi_v,\varphi_v\rangle_v\,dh. \tag{1.4}$$

Clearly $\alpha_v^{\natural}$ is invariant under $H_v\times H_v$, and we may call it the "local canonical invariant form."

We are now ready to state the conjecture of Ichino-Ikeda and R. N. Harris (cf. [23], [21, Conjecture 1.3]) that refines the global Gan-Gross-Prasad conjecture for unitary groups. Assume that the measure on $H(\mathbb{A})$ defining $\mathscr{P}$ and the measures on $H(F_v)$ defining α_v satisfy

$$dh=\prod_v dh_v.$$

Conjecture 1.1. *Assume that π is* tempered*; i.e., π_v is tempered for all v. For any decomposable vector $\phi=\otimes\phi_v\in\pi=\otimes\pi_v$, we have*

$$\frac{|\mathscr{P}(\phi)|^2}{\langle\phi,\phi\rangle_{\mathrm{Pet}}}=\frac{1}{|S_\pi|}\mathscr{L}\left(\frac{1}{2},\pi\right)\prod_v\frac{\alpha_v^{\natural}(\phi_v,\phi_v)}{\langle\phi_v,\phi_v\rangle_v}, \tag{1.5}$$

where S_π is a finite elementary 2-group: the component group associated to the L-parameter of $\pi=\pi_n\otimes\pi_{n+1}$.

[3]For a non-Archimedean place v we say that π_v is unramified if the quadratic extension E/F is unramified at v, the group $G(F_v)$ has a hyperspecial subgroup $K_v=G(\mathcal{O}_v)$, and π_v has a nonzero K_v-fixed vector.

Remark 1. The right hand side of the conjectural formula is insensitive to the definition of local L-factors at the finitely many bad places, as long as we choose the same definition in $\mathscr{L}(s,\pi_v)$ and in the local canonical invariant form $\alpha_v^\natural$.

The conjectural formula of this kind goes back to the celebrated work of Waldspurger [40] for the central values of L-functions of GL(2) [more or less equivalent to the case $U(1)\times U(2)$ in the unitary setting]. An arithmetic geometric version generalizing the formula of Gross-Zagier and S. Zhang [16],[44] is also formulated in [7, §27], [49], and [50, §3.2]. More explicit formulae were obtained by Gross [14], S. Zhang [48], and many others. The formula of Waldspurger and the formula of Gross-Zagier and S. Zhang [16], [46],[47],[44] play an important role in the spectacular development in application to the Birch and Swinnerton-Dyer conjecture for elliptic curves in the past 30 years. More recently, Tian [39] applies both formulae together to a classical Diophantine question and proves the infinitudes of square-free congruent numbers with an arbitrary number of prime factors.

The refined global conjecture for $\mathrm{SO}(3)\times\mathrm{SO}(4)$, concerning "the triple product L-function," was established after the work by Garrett [9], Piatetski-Shapiro–Rallis, Harris-Kudla [19], Gross-Kudla, Watson [43], and Ichino [22]. Recently Gan and Ichino [8] established some new cases for $\mathrm{SO}(4)\times\mathrm{SO}(5)$ [for endoscopic L-packets on SO(5)]. All of the known cases utilize the theta correspondence in an ingenious way.

The Waldspurger formula was also reproved by Jacquet and Jacquet-Chen [4] using relative trace formulae.

1.2. **Main results.** We now state our main result. Throughout this paper, we will assume two hypothesis, denoted by **RH(I)** and **RH(II)**.

The first one is about some expected properties of the (global and local) L-packets of unitary groups (for all Hermitian spaces W,V), analogous to the work of Arthur on orthogonal groups (cf. [34], [42] for the progress toward the unitary group case).

> **RH(I)**: Let E/F be a quadratic extension of number fields. For $i=1,2$, let V_i be a Hermitian space of dimension N, $U(V_i)$ the unitary group, and π_i an irreducible cuspidal automorphic representation of $U(V_i)$. We further assume that at one place v_0 split in E/F, and the representation π_{i,v_0} ($i=1,2$) is supercuspidal. Then we have
> (i) The (weak) base change $\pi_{i,E}$ to $\mathrm{Res}_{E/F}\mathrm{GL}(N)$ exists, and $\pi_{i,E}$ is cuspidal with a unitary central character, while the Asai L-function $L(s,\pi_E, As^{(-1)^{N-1}})$ (cf. Remark 6) has a simple pole at $s=1$.
> (ii) The multiplicity of π_i in $L^2([U(V_i)])$ is one.
> (iii) Assume that π_1 and π_2 are nearly equivalent (i.e., $\pi_{1,v}\simeq\pi_{2,v}$ with respect to fixed isomorphisms $V_{1,v}\simeq V_{2,v}$, for all but finitely many places v of F). Then for every place v of F, $\pi_{1,v}$ and $\pi_{2,v}$ are in the same local Vogan L-packet, and this local Vogan L-packet is generic.

The second one is a part of the local Gan-Gross-Prasad conjecture in the unitary group case.

> **RH(II)**: Let E/F be a quadratic extension of local fields, and (W_0,V_0) a pair of Hermitian spaces of dimension n and $n+1$. Then in a generic local Vogan L-packet Π_ψ of $U(W_0)\times U(V_0)$, there is at most one representation π of a relevant pure inner form $G=U(W)\times U(V)$ that admits a nonzero invariant linear form, i.e.,

$$\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0.$$

We refer to [7, Conj. 17.1] and [7, §9] for the detailed description. A proof (of an even stronger version) for tempered L-packets for p-adic fields is recently posted by Beuzart-Plessis [2, Theorem 1]; the local conjecture in the orthogonal case for p-adic fields has earlier been proved by Waldspurger.

We need the fundamental lemma for the Jacquet-Rallis relative trace formulae. In [45] and its appendix, this fundamental lemma is proved when the residue characteristic $p \geq c(n)$ for a constant $c(n)$ depending only on n (cf. Theorem 4.1).

Theorem 1.2. *Let π be a tempered (i.e., π_v is tempered for every place v) cuspidal automorphic representation of $G(\mathbb{A})$. Assume that the running hypothesis* **RH(I)** *and* **RH(II)** *holds. Denote by Σ the finite set of nonsplit places v of F where π_v is not unramified. Assume that*

(i) *There exists a split place v_0 such that the local component π_{v_0} is supercuspidal.*
(ii) *If $v \in \Sigma$, then either H_v is compact or π_v is supercuspidal.*
(iii) *The set Σ contains all nonsplit v whose residue characteristic is smaller than the constant $c(n)$.*

Then we have the following two cases:

(1) *(the totally split case) when every Archimedean place v of F is split in the extension E/F [i.e., $G_{F_\infty} \simeq (\mathrm{GL}_n \times \mathrm{GL}_{n+1})_{F_\infty}$], we have*

$$\frac{|\mathscr{P}(\phi)|^2}{\langle \phi, \phi \rangle_{Pet}} = 2^{-2} \mathscr{L}\left(\frac{1}{2}, \pi\right) \prod_v \frac{\alpha_v^\natural(\phi_v, \phi_v)}{\langle \phi_v, \phi_v \rangle_v}.$$

(2) *(the totally definite case) if $G(F_\infty)$ is compact where $F_\infty = \prod_{v|\infty} F_v$, then there is a nonzero constant c_{π_∞} depending only on the Archimedean component π_∞ of π such that*

$$\frac{|\mathscr{P}(\phi)|^2}{\langle \phi, \phi \rangle_{Pet}} = c_{\pi_\infty} 2^{-2} \mathscr{L}\left(\frac{1}{2}, \pi\right) \prod_v \frac{\alpha_v^\natural(\phi_v, \phi_v)}{\langle \phi_v, \phi_v \rangle_v}.$$

Remark 2. Under our assumptions (i), the base change of π_E of π to the general linear group is cuspidal, and hence

$$|S_\pi| = |S_{\pi_n}| \cdot |S_{\pi_{n+1}}| = 4.$$

Remark 3. The condition (i) is due to the fact that currently we do not have a complete spectral decomposition of the Jacquet-Rallis relative trace formulae. The condition (ii) seems to be only a technical restriction for our approach and will be discussed in §9. We have the restriction for the Archimedean place because (1) we have not proved the existence of smooth transfer at Archimedean places (cf. §5), and (2) it is probably a more technical problem to evaluate the constant c_{π_∞}.

Remark 4. For a non-Archimedean place v, the unitary group H_v is possibly compact only when $n \leq 2$. When $n = 1$, H_v is always compact for a nonsplit v. In this case, our proof is essentially the same as the one in [4].

We also make a local conjecture (Conjecture 4.4) for each place v. Together with a suitable spectral decomposition of the relative trace formulae, this conjecture would imply Conjecture 1.1 for those π with cuspidal base change π_E.

1.3. **Some applications.** We have the following application to the positivity of some central L-values. The positivity is also predicted by the grand Riemann hypothesis. Lapid has obtained a more general result for Rankin-Selberg central L-values by a different method ([32]; cf. also [31] for the positivity of the central value of the L-function of symplectic type).

Theorem 1.3. *Assume that π satisfies the conditions of Theorem 1.2 and E/F is split at all Archimedean places. Then we have*

$$L\left(\frac{1}{2},\pi_E\right)\geq 0.$$

Proof. It suffices to show this when $L(\frac{1}{2},\pi_E)\neq 0$. Then by [51], there exists π' in the same Vogan L-packet of π such that the period $\mathscr{P}$ on π' does not vanish. By replacing π by π', we may assume that the space $\mathrm{Hom}_{U(W)(F_v)}(\pi_v,\mathbb{C})$ does not vanish for every v. Then the local terms α'_v do not vanish. Now the positivity follows from the fact that the α'_v are all positive definite, and the other L-values appearing in $\mathscr{L}(1/2,\pi_E)$ except $L(\frac{1}{2},\pi_E)$ are all positive. □

Remark 5. As another application, M. Harris showed that Conjecture 1.1 would imply the algebraicity of the L-value $\mathscr{L}(1/2,\pi)$ up to some simple constant when $G(F_\infty)$ is compact and $\alpha_\infty\neq 0$ (cf. [20, §4.1]).

1.4. **Outline of proof.** We now sketch the main ideas of the proof, following the strategy of Jacquet and Rallis [28]. First of all, by the multiplicity one result [1],[38], we know a priori that there is a constant denoted by $\mathscr{C}_\pi$ depending on π such that for all decomposable $\phi,\varphi\in\pi$,

$$\mathscr{P}(\phi)\overline{\mathscr{P}(\varphi)}=\mathscr{C}_\pi\prod_v\alpha_v^\natural(\phi_v,\varphi_v). \tag{1.6}$$

Instead of working with an individual $\phi\in\pi$ as in the conjecture, we switch our point of view to a distribution attached to π.

Definition 1.4. We define the *(global) spherical character* J_π associated to a cuspidal automorphic representation π as the distribution

$$J_\pi(f):=\sum_\phi\mathscr{P}(\pi(f)\phi)\overline{\mathscr{P}(\phi)},\qquad f\in\mathscr{C}_c^\infty(G(\mathbb{A})), \tag{1.7}$$

where the sum of ϕ is over an orthonormal basis of π (with respect to the Petersson inner product).

The name "spherical character" is suggested by many early analogous distributions (cf. [35], etc.). We also have a local counterpart as follows.

Definition 1.5. We define the *(local) spherical character* $J^\natural_{\pi_v}$ associated to π_v as the distribution,

$$J^\natural_{\pi_v}(f_v):=\sum_{\phi_v}\alpha_v^\natural(\pi_v(f_v)\phi_v,\phi_v),\qquad f_v\in\mathscr{C}_c^\infty(G(F_v)), \tag{1.8}$$

where the sum of ϕ_v is over an orthonormal basis of π_v. Similarly we define an unnormalized one J_{π_v},

$$J_{\pi_v}(f_v):=\sum_{\phi_v}\alpha_v(\pi_v(f_v)\phi_v,\phi_v). \tag{1.9}$$

By (1.6), we clearly have for decomposable $f = \bigotimes_v f_v$

$$J_\pi(f) = \mathscr{C}_\pi \prod_v J^\natural_{\pi_v}(f_v), \tag{1.10}$$

where in the product in the right hand side, for a given π and f, the local term $J^\natural_{\pi_v}(f_v) = 1$ for all but finitely many v. Then we have the following consequence of Conjecture 1.1.

Conjecture 1.6. *Assume that π is a* tempered *cuspidal automorphic representation. For all $f = \bigotimes_v f_v \in \mathscr{C}_c^\infty(G(\mathbb{A}))$, we have*

$$J_\pi(f) = \frac{1}{|S_\pi|}\mathscr{L}\left(\frac{1}{2},\pi\right)\prod_v J^\natural_{\pi_v}(f_v).$$

Lemma 1.7. *The Conjecture* 1.6 *is equivalent to Conjecture* 1.1.

Proof. It suffices to show that Conjecture 1.6 implies Conjecture 1.1. To see this, we note that the following are equivalent: (1) $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$, (2) $\alpha_v \neq 0$, (3) *the distribution $J^\natural_{\pi_v}$ does not vanish.*[4] Hence, Conjecture 1.1 holds if for some v the linear form α_v vanishes. Now assume that for all v, the linear forms α_v do not vanish. Then the distributions J_{π_v} do not vanish. Then by Conjecture 1.6, the constant $\mathscr{C}_\pi$ must be $\frac{1}{|S_\pi|}\mathscr{L}(1/2,\pi)$, which implies Conjecture 1.1. □

Note that there is a parallel question for the general linear group. This question can essentially be reduced to the celebrated theory of "Rankin-Selberg convolution" due to Jacquet–Piatetskii-Shapiro–Shalika [27]. The idea of Jacquet and Rallis is to transfer the question from the unitary group to the general linear group via (quadratic) base change. They [28] introduced two relative trace formulae (RTF), one on the unitary group and the other on the general linear group. This is the main tool of this paper and the previous one [51].

In the general linear group case, there is a decomposition of a global spherical character into a product of the local ones, analogous to Conjecture 1.6. But this time one may prove it without too much difficulty. Hence, to deduce Conjecture 1.6, it suffices to compare the two local spherical characters. Moreover, since we only need to find the constant $\mathscr{C}_\pi$, we may just choose some special test functions f, as long as the local spherical character on the unitary group does not vanish for our choice. Therefore the main innovation of this paper is a formula for the local spherical character evaluated at some special test functions. The formula can be viewed as a truncated *local expansion of the local spherical character*, analogous to the local expansion of a character due to Harish-Chandra. The result may be of independent interest in view of local harmonic analysis in the relative setting.

For comparison, let us recall briefly a result of Harish-Chandra. Let F be a p-adic field. We temporarily use the notation G for the F-points of a connected reductive group, and $\mathfrak{g}$ the Lie algebra of G. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$ and $\mathcal{N}/G$ the set of G-conjugacy classes in $\mathcal{N}$. The set $\mathcal{N}/G$ is finite. Let $\mu_\mathcal{O}$ be the nilpotent orbital integral associated to $\mathcal{O} \in \mathcal{N}/G$ for a suitable choice of measure. The exponential map defines a homeomorphism $\exp : \omega \to \Omega$ where ω (Ω resp.) is some neighborhood of 0 in $\mathfrak{g}$ (1 in G, resp.). Let π be an irreducible admissible representation of G. Then Harish-Chandra showed that there are constants $c_\mathcal{O}(\pi)$

[4] It is clear that (2) is equivalent to (3). The equivalence of (1) and (2) follows from the third property of α_v listed earlier.

indexed by $\mathcal{O} \in \mathcal{N}/G$ such that when Ω is sufficiently small, for all f supported in Ω,

$$\operatorname{tr}(\pi(f)) = \sum_{\mathcal{O}\in\mathcal{N}/G} c_{\mathcal{O}}(\pi)\mu_{\mathcal{O}}(\widehat{f_\natural}). \tag{1.11}$$

Here $f_\natural$ is the function on ω via the homeomorphism exp, and $\widehat{f_\natural}$ is its Fourier transform. The constants $c_{\mathcal{O}}(\pi)$ contain important information about π. For example, there is a distinguished nilpotent conjugacy class, namely the class of $0 \in \mathcal{N}$. If π is a discrete series representation, the constant $c_{\{0\}}(\pi)$ is equal to the formal degree of π for a suitable choice of Haar measure on G.

Now we return to our relative setting. We consider the local spherical character on the general linear group. Let $G' := \operatorname{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$, and let Π be an irreducible unitary generic representation of $G'(F)$. Then the local spherical character I_Π [cf. (3.31)] defines a distribution on $G'(F)$ with a certain invariance property. These distributions are related to distributions on the F-vector space,

$$\mathfrak{s}_{n+1} = \{X \in M_{n+1}(E) | X + \overline{X} = 0\}.$$

Here $X \mapsto \overline{X}$ denotes the Galois involution (entrywise). The group $\mathrm{GL}_{n+1}(F)$ acts on $\mathfrak{s}_{n+1}$ by conjugation. We will be interested in the restriction of this action to the subgroup $\mathrm{GL}_n(F)$ [as a factor of the Levi of the parabolic of the $(n,1)$-type]. We let ω be a small neighborhood of 0 in the F-vector space $\mathfrak{s}_{n+1}$. Then we have a natural way to pull back a function f' on a small neighborhood of 1 in G' to a function denoted by $f'_\natural$ on ω (cf. §8 for the precise definition). It is tempting to guess that there exists an analogous expansion of I_Π in terms of the [relative to $\mathrm{GL}_n(F)$ action] unipotent orbital integrals on $\mathfrak{s}_{n+1}$.[5] However, so far there are some difficulties. For example, when $n \geq 2$ there are infinitely many $\mathrm{GL}_n(F)$-nilpotent orbits in $\mathfrak{s}_{n+1}$, and these nilpotent orbital integrals often need to be regularized. We then restrict ourselves to a subspace of *admissible functions* (cf. Definition 8.1) supported on a small ω. The precise definition is very technical. We expect that admissible functions have vanishing nilpotent orbital integrals (however, generally not even defined so far), except for one of the two regular unipotent orbits denoted by ξ_-. An expansion such as (1.11) of $I_\Pi(f)$ would then tell us that there should be only one term left, corresponding to the regular unipotent orbit ξ_-. Though it seems challenging to prove something such as (1.11) in our setting, we nevertheless manage to establish a truncated version (see Theorem 8.5 for the detail).

Theorem 1.8. *Let Π be an irreducible unitary generic representation of $G'(F)$. Then for any small neighborhood ω of 0 in $\mathfrak{s}_{n+1}$, there exists an admissible function $f' \in \mathscr{C}_c^\infty(G'(F))$ such that $f'_\natural$ is supported in ω and*

$$I_\Pi(f') = (*)\mu_{\xi_-}(\widehat{f'_\natural}) \neq 0,$$

where $(*)$ is an explicit nonzero constant depending only on the central character of Π.

We have a similar result for a local spherical character J_π on the unitary group when either π is a supercuspidal representation or the group $U(W)$ is compact. See

[5] Relative to the $\mathrm{GL}_n(F)$-action, an $X \in \mathfrak{s}_{n+1}$ is "nilpotent" if the closure of its $\mathrm{GL}_n(F)$-orbit contains zero.

§9 for more details (Theorem 9.7). Then our main Theorem 1.2 follows from the local comparison of the two spherical characters (cf. §4 Conjecture 4.4).

The proof for the unitary group case seems to be harder and needs the full strength of our previous results in the companion paper [51]. Namely we have to make use of the following results (cf. §9):

(1) The existence of smooth transfer.
(2) Compatibility of smooth transfer with Fourier transform.
(3) Local (relative) trace formula on "Lie algebra."

Note that the proof in [51] of these ingredients is in the reverse order listed here.

1.5. **Structure of this paper.** After fixing some notations in §2, we review several global periods involving the general linear group in §3 and deduce the decomposition analogous to Conjecture 1.6. Then in §4 we recall the Jacquet-Rallis RTF and reduce the question to a comparison of local spherical characters. Then we give the proof of Theorem 1.2 assuming a local result (Theorem 4.6). In §5 we deal with the totally definite case (i.e., $G(\mathbb{R})$ compact). In §6 we prepare some (relative) harmonic analysis on Lie algebras. In §7 and §8, we prove the local character expansion for the general linear group. The two key ingredients are Lemma 7.6 and Lemma 8.8. In §9 we show the local character expansion for the unitary group under some conditions, and we complete the proof via the comparison of both spherical characters.

Finally, we warn the reader of the change of measures: Only in the introduction do we use the Tamagawa measures associated to a differential form ω on H normalized by

$$dh = L(1,\eta)^{-1}\prod_v L(1,\eta_v)|\omega|_v.$$

To have a natural local decomposition, below we will immediately switch to

$$dh = \prod_v dh_v, \qquad dh_v = \prod_v L(1,\eta_v)|\omega|_v.$$

Another change comes when we move to the local setting (cf. the paragraph before Lemma 4.7): there we consider the unnormalized local measure

$$dh_v = |\omega|_v.$$

A similar warning applies to other groups such as G and the general linear group.

Part 1. Global theory

2. Measures and notations

We always endow discrete groups with the counting measure.

2.1. **Measures and notations related to the general linear group.** We first list the main notations and conventions throughout this paper. We denote $H_n = \mathrm{GL}_n$, its standard Borel B_n with the diagonal torus A_n, the unipotent radical N_n of B_n. We denote by $B_{n,-}$ the opposite Borel subgroup, and $N_{n,-}$ its unipotent radical, and an open subvariety $H_n' = N_nA_nN_{n-}$ of H_n (essentially the open cell of Bruhat decomposition). Their Lie algebras are denoted by $\mathfrak{h}_n, \mathfrak{n}_n$, etc. We denote by $M_{n,m}(F)$ the F-vector space of all $n\times m$ matrices with coefficients in F; and if $n=m$ we write it as $M_n(F)$. Then we have a natural embedding $H_n\subset M_n$. We denote

$$e_n = (0,0,\dots,0,1)\in M_{1,n}(F),$$

and let $e_n^* \in M_{n,1}(F)$ be the transpose of e_n. The letter u (v, resp.) usually denotes an upper (lower, resp.) triangular unipotent matrix or a column (row, resp.) vector.

We usually understand H_{n-1} as a subgroup of H_n via the block-diagonal embedding

$$H_{n-1} \ni h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix} \in H_n.$$

We thus have a sequence of embeddings $\ldots \subset H_{n-2} \subset H_{n-1} \subset H_n$. Similarly we have a sequence of embeddings for the diagonal torus A_n, the unipotent N_n, etc.

For a quadratic extension E/F (local or global), we assume that

$$E = F[\tau],$$

where $\tau = \sqrt{\delta}$, $\delta \in F^\times$. We write $E^\pm$ the F-vector space where the nontrivial Galois automorphism in $\mathrm{Gal}(E/F)$ acts by ± 1, and $E^+ = F$.

Now let F be a local field. We will fix an additive character $\psi = \psi_F$ of F and then define a character ψ_E of E by

$$\psi_E(z) = \psi\left(\frac{1}{2}\mathrm{tr}_{E/F} z\right)$$

for the trace map $\mathrm{tr}_{E/F} : E \to F$. In particular, we have the compatibility $\psi_E|F = \psi$. We also say that ψ is unramified if F is non-Archimedean and the largest fractional ideal of F over which ψ is trivial is $\mathcal{O}_F$, and similarly for ψ_E. On $M_n(E)$ there is a bi-E-linear pairing valued in E given by

$$\langle X, Y\rangle := \mathrm{tr}(XY). \tag{2.1}$$

We then have a Fourier transform for $\phi \in \mathscr{C}_c^\infty(M_n(E))$,

$$\widehat{\phi}(X) := \int_{M_n(E)} \phi(Y)\psi_E(\langle X, Y\rangle)\, dY.$$

Here we use the self-dual measure on $M_n(E)$, i.e., the unique Haar measure characterized by

$$\widehat{\widehat{\phi}}(X) = \phi(-X). \tag{2.2}$$

Note that this is also the same measure obtained by identifying $M_n(E)$ with E^{n^2} and using the self-dual measure on $E = M_1(E)$. We now view both $M_n(F)$ and $M_n(E^-)$ as F-vector subspaces of $M_n(E)$. Then the restriction of the pairing $\langle\cdot,\cdot\rangle$ to each of them is nondegenerate F-valued pairing. In this way we may define the Fourier transform of $f \in \mathscr{C}_c^\infty(M_n(E^\pm))$ and we normalize the Haar measure on $M_n(E^\pm)$ as the self-dual one characterized by the analogous equation to (2.2). Set $n = 1$ and we have a measure for $F = E^+$ and E^-. Note that if we use the isomorphism $F \simeq E^-$ by $x \mapsto \sqrt{\delta}x$, then the measure on E^- is $|\delta|_F^{1/2} dx$ for the self-dual measure dx on F. Here our absolute values on F and E are normalized such that

$$d(ax) = |a|_F\, dx, \qquad a \in F,$$

and similarly for E.

On $F^\times$ we denote the *normalized* Tamagawa measure associated to the differential form $x^{-1}dx$,

$$d^\times x = \zeta_F(1)\frac{dx}{|x|_F},$$

and the *unnormalized* one,

$$d^*x = \frac{dx}{|x|_F},$$

and similarly for $E^\times$. On $H_n(F)$ we will take the Haar measure

$$dg = \zeta_F(1)\frac{\prod_{ij} dx_{ij}}{|\det(g)|_F^n}, \qquad g = (x_{ij}),$$

and similarly for $H_n(E)$ [where we replace $\zeta_F(1)$ by $\zeta_E(1)$]. Sometimes we also shorten $|\det(g)|$ by $|g|$ if no confusion arises.

We will assign the measure on $N_n(F)$ the additive self-dual measure,

$$du = \prod_{1\le i<j\le n} du_{ij}, \qquad u = (u_{ij}) \in N_n(F).$$

We denote the modular character by

$$\delta_n(a) = \det(Ad(a) : \mathfrak{n}_n) = \prod_{i=1}^{n} a_i^{n+1-2i},$$

where $a = diag[a_1, a_2, \ldots, a_n] \in A_n(F)$ acts on $\mathfrak{n}$ by $Ad(a)X = aXa^{-1}$. Similarly, we have $\delta_{n,E}$ if we replace F by E. For $x \in M_{n,m}(F)$, we define

$$\|x\| = \max\{|x_{ij}|_F\}_{1\le i\le n, 1\le j\le m}.$$

Now let F be a number field and let $\psi = \prod_v \psi_v$ be a nontrivial character of $F\backslash\mathbb{A}$. We denote by $\mathbb{A}^1$ the subgroup of $\mathbb{A}^\times$ consisting of $x = (x_v)_v \in \mathbb{A}^\times$ with $|x| = \prod_v |x_v|_v = 1$. We endow the group $H_n(\mathbb{A})$ with the product measure

$$dg = \prod_v dg_v.$$

We denote by Z_n the center of H_n, and the measure is determined by the measure on $\mathbb{A}_F^\times$,

$$d^\times x = \prod_v d^\times x_v.$$

Note that under our measure, if ψ_v is unramified, the volume of the maximal compact subgroup of $H_n(F_v)$ is given by

$$\mathrm{vol}(H_n(\mathcal{O}_{F_v})) = \zeta_v(2)^{-1}\zeta_v(3)^{-1}\cdots\zeta_v(n)^{-1}.$$

If E is a quadratic extension of F, we take similar conventions for $H_n(\mathbb{A}_E)$, $Z_n(\mathbb{A}_E)$ et al.

2.2. **Measures and notations related to unitary groups.** In this paper, $W \subset V$ will denote an embedding of Hermitian spaces of dimension n and $n+1$, respectively, $U(W)$ and $U(V)$ the corresponding unitary group, $G = U(W) \times U(V)$ and its subgroup H being the diagonal embedding of $U(W)$.

Our method involves the comparison of orbital integrals between the unitary and general linear group cases, and between their Lie algebras, respectively. We thus need to choose compatible measures on them. Let θ be a nonsingular Hermitian matrix of size $n+1$. Then we may and will view the group $U(\theta)(F)$ as the subgroup of $\mathrm{GL}_{n+1}(E)$ consisting of $g \in \mathrm{GL}_{n+1}(E)$ such that

$$\overline{g}^t \cdot \theta g\theta^{-1} = 1.$$

We may and will view the Lie algebra $\mathfrak{u}(\theta)$ of $U(\theta)$ as the subspace of $M_{n+1}(E)$ consisting of $X \in M_{n+1}(E)$ such that

$$\overline{X}^t + \theta X \theta^{-1} = 0.$$

We denote by $\mathfrak{u}(\theta)^\dagger$ a companion space where the last equality is replaced by

$$\overline{X}^t = \theta X \theta^{-1}.$$

For any number $\tau \in E$ such that $\overline{\tau} = -\tau \neq 0$, we have an isomorphism (as F-vector spaces) from $\mathfrak{u}(\theta)$ to $\mathfrak{u}(\theta)^\dagger$ mapping X to $\tau^{-1}X$.

We will need to consider the symmetric space $S_{n+1}(F) \simeq H_{n+1}(F)\backslash H_{n+1}(E)$. We identify S_{n+1} with the subspace of $\mathrm{GL}_{n+1}(E)$ consisting of $g \in \mathrm{GL}_{n+1}(E)$ such that

$$\bar{g}g = 1. \tag{2.3}$$

We have its tangent space $\mathfrak{s} = \mathfrak{s}_{n+1}$ at $1 \in S_{n+1}$, which we call the *Lie algebra of* $S_{n+1}(F)$. Viewed as a subspace of $M_{n+1}(E)$, the vector space $\mathfrak{s}$ consists of $X \in M_{n+1}(E)$ such that

$$\overline{X} + X = 0. \tag{2.4}$$

Its companion is the space $M_{n+1}(F)$ [or $\mathfrak{gl}_{n+1}(F)$] viewed as a subspace of $M_{n+1}(E)$, namely consisting of $X \in M_{n+1}(E)$ such that

$$\overline{X} = X.$$

For any number $\tau \in E$ such that $\overline{\tau} = -\tau \neq 0$, we have an isomorphism (as F-vector spaces) from $\mathfrak{s}(F)$ to $M_{n+1}(F)$ mapping X to $\tau^{-1}X$.

We consider both $\mathfrak{s}$ and $\mathfrak{u}$ as F-vector subspaces of $M_{n+1}(E)$. The restrictions of the bilinear form $\langle \cdot, \cdot \rangle$ [cf. (2.1)] on $M_{n+1}(E)$ to $\mathfrak{s}$ and $\mathfrak{u} = \mathfrak{u}(\theta)$ take values in F and are nondegenerate. The additive characters ψ and ψ_E then determine self-dual measures on $M_{n+1}(\mathbb{A}_E)$, $\mathfrak{u}(\mathbb{A})$, $\mathfrak{s}(\mathbb{A})$, and the local analogues. Moreover, if we change the Hermitian matrix θ defining $\mathfrak{u}$ to an equivalent one, the subspace $\mathfrak{u}$ changes to its conjugate by an element in $\mathrm{GL}_{n+1}(E)$. Hence the measures are compatible with the change of θ. These measures can also be treated as Tamagawa measures associated to top degree invariant differential forms. Let ω_0 be a differential form on $\mathfrak{u}$ so that $|\omega_0|_v$ defines the self-dual measure for every place v. We also use the form ω_0 to normalize the differential form ω that defines the measure on $U(\theta)(F_v)$ as follows. We consider the Cayley map

$$\mathfrak{c}(X) := (1+X)(1-X)^{-1}. \tag{2.5}$$

It defines a birational map between $\mathfrak{u}$ and $U(\theta)$, and it is defined at $X = 0$. We normalize the invariant differential form ω on $U(\theta)$ by requiring that the pullback $\mathfrak{c}^*\omega$ evaluating at 0 is the same as ω_0 evaluated at 0. It follows that, when v is non-Archimedean, under the Cayley map, the restriction of the self-dual measure to a small neighborhood of 0 in $\mathfrak{u}$ is compatible with the restriction of the Tamagawa measure $|\omega|_v$ to a small neighborhood of 1 in $U(\theta)(F_v)$. In this way we choose the measure on $U(\theta)(\mathbb{A})$, globally and locally, as follows:

$$dh = \prod_v L(1, \eta_v)|\omega|_v.$$

Our global measure is therefore not the Tamagawa measure, which should be $L(1,\eta)^{-1}dh$. In particular, under our choice of measure, the volume of $[U(1)] = U(1)(F)\backslash U(1)(\mathbb{A})$ is given by

$$\mathrm{vol}([U(1)]) = 2L(1,\eta). \tag{2.6}$$

This is due to the fact that the Tamagawa number for $U(1) \simeq \mathrm{SO}(2)$ is equal to 2.

3. Explicit local factorization of some periods

In this section, we decompose several global linear forms on the general linear group into explicit products of local invariant linear forms. Nothing is original in this section, but we need to determine all constants in order to prove the main result of this paper.

3.1. **Invariant inner product.** Let $\Pi = \Pi_n$ be a cuspidal automorphic representation of $H_n(\mathbb{A}_E)$ with unitary central character ω_Π. We recall some basic facts on the Whittaker model of $\Pi = \otimes_w \Pi_w$. We extend the additive character ψ_E to a character of $N_n(E)$ by

$$\psi_E(u) = \psi_E\left(\sum_{i=1}^{n-1} u_{i,i+1}\right), \qquad u = (u_{i,j}) \in N_n(E).$$

Similar convention applies to the other unipotent matrices in $N_n(F)$ et al. We denote by $\mathscr{C}^\infty(N_n(\mathbb{A}_E)\backslash H_n(\mathbb{A}_E), \psi_E)$ the space of smooth functions f on $H_n(\mathbb{A}_E)$ such that

$$f(ug) = \psi(u)f(g), \qquad u \in N_n(\mathbb{A}_E), g \in H_n(\mathbb{A}_E).$$

Similarly we have the local counterpart $\mathscr{C}^\infty(N_n(E_w)\backslash H_n(E_w), \psi_w)$ for each place w of E. The Fourier coefficient of $\phi \in \Pi$ is defined as

$$W_\phi(g) = \int_{N(E)\backslash N(\mathbb{A}_E)} \phi(ug)\overline{\psi}_E(u)\, du.$$

Then we have $W_\phi \in \mathscr{C}^\infty(N_n(\mathbb{A}_E)\backslash H_n(\mathbb{A}_E), \psi_E)$. The map $\phi \mapsto W_\phi$ realizes an equivariant embedding $\Pi \hookrightarrow \mathscr{C}^\infty(N_n(\mathbb{A}_E)\backslash H_n(\mathbb{A}_E), \psi_E)$. The image, the Whittaker model of Π, is denoted by $\mathcal{W}(\Pi, \psi_E)$. For $\phi \in \Pi = \otimes_w \Pi_w$, we assume that W_ϕ is decomposable

$$W_\phi(g) = \prod_w W_{\phi,w}(g_w), \qquad W_{\phi,w} \in \mathscr{C}^\infty(N_n(E_w)\backslash H_n(E_w), \psi_{E,w}), \tag{3.1}$$

where w runs over all places of E, and $W_w(1) = 1$ for almost all places w.

We need to compare the unitary structure in the decomposition

$$\Pi \simeq \bigotimes_w \mathcal{W}(\Pi_w, \psi_{E,w}).$$

On Π we have the Petersson inner product, for $\phi, \phi' \in \Pi$,

$$\langle \phi, \phi' \rangle_{\mathrm{Pet}} = \int_{Z_n(\mathbb{A}_E)H_n(E)\backslash H_n(\mathbb{A}_E)} \phi(g)\overline{\phi'(g)}\, dg.$$

On $\mathcal{W}(\Pi_w, \psi_{E,w})$ we have an invariant inner product defined by

$$\vartheta_w(W_w, W_w') = \int_{N_{n-1}(E_w)\backslash H_{n-1}(E_w)} W_w\begin{pmatrix} h & \\ & 1\end{pmatrix} \overline{W}_w'\begin{pmatrix} h & \\ & 1\end{pmatrix} dh. \tag{3.2}$$

The integral ϑ_w converges absolutely if Π_w is generic unitary. When Π_w and $\psi_{E,w}$ are unramified, the vectors $W_w = W'_w$ are fixed by $K_{n,w} := H_n(\mathcal{O}_{E,w})$ and normalized by $W_w(1) = 1$, we have

$$\vartheta_w = \mathrm{vol}(K_{n,w}) L(1, \Pi_w \times \widetilde{\Pi}_w). \tag{3.3}$$

This can be deduced from [29, Prop. 2.3] [also a consequence of the proof of Prop. 3.1, particularly (3.8) and (3.9)]. Therefore we define a normalized invariant inner product

$$\vartheta^\natural_w(W_w, W'_w) = \frac{\vartheta_v(W_w, W'_w)}{L(1, \Pi_w \times \widetilde{\Pi}_w)}. \tag{3.4}$$

Then the product $\prod_w \vartheta^\natural_w$ converges and defines an invariant inner product on $\mathcal{W}(\Pi, \psi_E)$. It is a natural question to compare it with the Petersson inner product. We now recall a result of Jacquet-Shalika (implicitly in [29, §4]; cf. [5, p.265]).

Proposition 3.1. *We have the following decomposition of the Petersson inner product in terms of the local inner product $\vartheta^\natural_w$:*

$$\langle \phi, \phi' \rangle_{\mathrm{Pet}} = \frac{n \cdot \mathrm{Res}_{s=1} L(s, \Pi \times \widetilde{\Pi})}{\mathrm{vol}(E^\times \backslash \mathbb{A}^1_E)} \prod_v \vartheta^\natural_w(W_{\phi,w}, W_{\phi',w}), \tag{3.5}$$

where $W_\phi = \otimes_w W_{\phi,w}$ and $W_{\phi'} = \otimes_w W_{\phi',w}$.

Proof. Up to a constant this is proved by [29, §4]. We thus recall their proof in order to determine this constant, and the same idea of proof will also be used below to decompose the Flicker-Rallis period. We consider an Eisenstein series associated to a Schwartz-Bruhat function Φ on $\mathbb{A}^n_E$. We consider the action of $H_n(E)$ on the row vector space E^n from right multiplication. Then the stabilizer of $e_n = (0, 0, \ldots, 1) \in E^n$ is the mirabolic subgroup P_n of H_n. Set

$$f(g, s) = |g|^s \int_{\mathbb{A}^\times_E} \Phi(e_n a g) |a|^{ns} \, d^\times a, \qquad \mathrm{Re}(s) >> 0.$$

Consider the Epstein-Eisenstein series

$$E(g, \Phi, s) := \sum_{\gamma \in ZP(E) \backslash H_n(E)} f(\gamma g, s), \tag{3.6}$$

which is absolutely convergent when $\mathrm{Re}(s) > 1$. Equivalently, we have

$$E(g, \Phi, s) = |g|^s \int_{E^\times \backslash \mathbb{A}^\times_E} \sum_{\xi \in E^n - \{0\}} \Phi(\xi a g) |a|^{ns} \, d^\times a.$$

(Note: this corresponds to the case $\eta = 1$ in [29, §4].) It has meromorphic continuation to $\mathbb{C}$ and has a simple pole at $s = 1$ with residue [29, Lemma 4.2]

$$\frac{\mathrm{vol}(E^\times \backslash \mathbb{A}^1_E)}{n} \widehat{\Phi}(0).$$

Note that the only nonexplicit constant denoted by c in [29, Lemma 4.2] is the volume of $E^\times \backslash \mathbb{A}^1_E$. Now consider the zeta integral

$$I(s, \Phi, \phi, \phi') = \int_{Z_n(\mathbb{A}_E) H_n(E) \backslash H_n(\mathbb{A}_E)} E(g, \Phi, s) \phi(g) \overline{\phi'}(g) \, dg.$$

On the one hand, it has a pole at $s = 1$ with residue

$$\frac{\mathrm{vol}(E^\times \backslash \mathbb{A}_E^1)}{n} \widehat{\Phi}(0) \langle \phi, \phi' \rangle_{\mathrm{Pet}}.$$

On the other hand, when $\mathrm{Re}(s)$ is large, it is also equal to the following integral:

$$\Psi(s, \Phi, W_\phi, W_{\phi'}) = \int_{N_n(\mathbb{A}_E)\backslash H_n(\mathbb{A}_E)} \Phi(eg) W_\phi(g) \overline{W_{\phi'}}(g) |\det(g)|^s \, dg. \tag{3.7}$$

This is equal to the product

$$\prod_w \Psi(s, \Phi_w, W_{\phi,w}, W_{\phi',w}),$$

where the local integral is defined as

$$\Psi(s, \Phi_w, W_{\phi,w}, W_{\phi',w}) = \int_{N_n(E_w)\backslash H_n(E_w)} \Phi_w(eg) W_{\phi,w}(g) \overline{W_{\phi',w}}(g) |\det(g)|^s \, dg.$$

By [29, Prop. 2.3], we have, for unramified data of (Φ_w, W_w, W'_w) and $\psi_{E,w}$ at a place w, normalized such that $W(1) = W(1) = \Phi_w(0) = 1$,

$$\Psi(s, \Phi, W_w, W'_w) = \mathrm{vol}(K_{n,w}) L(s, \Pi_w \times \widetilde{\Pi}_w). \tag{3.8}$$

[Note: for our measure on $N_n(E_w)$, we have $\mathrm{vol}(N_n(E_w) \cap K_w) = 1$.] From this we may deduce that

$$\Psi(s, \Phi, W_\phi, W_{\phi'}) = L(s, \Pi \times \widetilde{\Pi}) \prod_w \frac{\Psi(s, \Phi_w, W_{\phi,w}, W_{\phi',w})}{L(s, \Pi_w \times \widetilde{\Pi}_w)},$$

where the local factors are entire functions of s and for almost all w they are equal to one. Moreover, all local factors converge absolutely in the half plane $\mathrm{Re}(s) > 1 - \epsilon$ for some $\epsilon > 0$ [29]. From this we deduce that its residue at $s = 1$ is given by another formula,

$$\mathrm{R}es_{s=1} L(s, \Pi \times \widetilde{\Pi}) \prod_w \frac{\Psi(1, \Phi_w, W_{\phi,w}, W_{\phi',w})}{L(1, \Pi_w \times \widetilde{\Pi}_w)}.$$

From the two formulae of the residue, we will first deduce that $\vartheta_w(W_{\phi,w}, W_{\phi',w})$ is $H_n(E_w)$ invariant and second that

$$\Psi(1, \Phi_w, W_{\phi,w}, W_{\phi',w}) = \widehat{\Phi_w}(0) \vartheta_w(W_{\phi,w}, W_{\phi',w}). \tag{3.9}$$

To see this, let $N_{n,1,+}(E_w)$ be the unipotent part of the mirabolic P_n and $N_{n,1,-}(E_w)$ the transpose of $N_{n,1,+}(E_w)$. We consider the open dense subset $N_{n,1,+}H_{n-1}N_{n,1,-}Z_n$ $= P_n N_{n,1,-} Z_n$. We may decompose the measure on H_n (or more precisely its restriction to the open subset)

$$dg = |\det(h)|^{-1} \, dn_+ \, dh \, dn_- \, d^*a,$$

where

$$g = n_+ h n_- a, \qquad h \in H_{n-1}, \qquad n_\pm \in N_{n,1,\pm}, \qquad a \in Z_n.$$

Note also that the embedding $H_{n-1} \hookrightarrow P_n$ induces an isomorphism $N_n\backslash P_n \simeq N_{n-1}\backslash H_{n-1}$. For an integrable function f on $N_n\backslash H_n$, we may write

$$\begin{aligned}&\int_{N_n(E_w)\backslash H_n(E_w)} f(g)dg \\ &\quad = \int_{Z_nN_{n,1,-}(E_w)} \left(\int_{N_{n-1}(E_w)\backslash H_{n-1}(E_w)} f(hn_-a)|\det(h)|^{-1}dh\right) dn_-\, d^*a.\end{aligned}$$

We now apply this formula to the integral $\Psi(1,\Phi_w,W_{\phi,w},W_{\phi',w})$. For simplicity we write $W_w = W_{\phi,w}$, and $W'_w = W_{\phi',w}$. Clearly ϑ_w is $P_n(E_w)$ invariant. Therefore we may write

$$\begin{aligned}&\Psi(1,\Phi_w,W_{\phi,w},W_{\phi',w}) \\ &\quad = \int_{Z_nN_{n,1,-}(E_w)} \Phi(ean_-)\vartheta_w(\Pi_w(an_-)W_w,\Pi_w(an_-)W'_w)|a|^n\, d^*a\, dn_-.\end{aligned}$$

For $X \in E_w^n$ (with last entry nonzero), let $n_-(X)$ be the element in $Z_nN_{n,1,-}(E_w)$ with the last row equal to X. We consider

$$\Gamma(X) := \vartheta_w(\Pi_w(n_-(X))W_w, \Pi_w(n_-(X))W'_w),$$

whenever it is defined. A suitable substitution yields

$$\Psi(1,\Phi_w,W_{\phi,w},W_{\phi',w}) = \int_{E_w^n} \Phi_w(X)\Gamma(X)\, dX.$$

Since by the other residue formula, we also know that this is equal to a constant multiple times $\widehat{\Phi_w}(0)$ times an invariant inner product on $\mathcal{W}(\Pi_w,\psi_{E,w})$, for all Φ_w and W_w, W'_w. We deduce that $\Gamma(X)$ is a constant function (whenever it is defined). Therefore $\Gamma(X) = \vartheta_w(W_w,W'_w)$ and ϑ_w is $H_n(E_w)$ invariant. Moreover, we now have

$$\Psi(1,\Phi_w,W_{\phi,w},W_{\phi',w}) = \vartheta_w(W_w,W'_w)\int_{E_w^n}\Phi(X)\, dX = \vartheta_w(W_w,W'_w)\widehat{\Phi_w}(0).$$

This completes the proof. We also note that if we use the $H_n(E_w)$ invariance of ϑ_w, which can be proved independently, then the proposition can be deduced immediately from the two residue formulae. □

For later use, as we will be dealing with the case of a quadratic extension E/F, we will consider $H_{n,E}$ as an algebraic group over the base field F. Therefore we rewrite the result as

$$\langle\phi,\phi'\rangle_{Pet} = \frac{n\cdot \mathrm{Res}_{s=1}L(s,\Pi\times\widetilde{\Pi})}{\mathrm{vol}(E^\times\backslash\mathbb{A}_E^1)}\prod_v \vartheta_v^\natural(W_v,W'_v), \tag{3.10}$$

where $\vartheta_v = \prod_{w|v}\vartheta_w$ for all (one or two) places w above v.

3.2. **Flicker-Rallis period.** Now let E/F be a quadratic extension of number fields and $\Pi = \Pi_n$ a cuspidal automorphic representation of $H_n(\mathbb{A}_E)$. Assume that its central character satisfies

$$\omega_\Pi|_{\mathbb{A}^\times} = 1.$$

We would like to decompose the Flicker-Rallis period [6],[10] explicitly. It can be viewed as a twisted version of the Petersson inner product (it indeed gives the

Petersson inner product if we allow $E = F \times F$ to be split globally). Therefore it is natural that the method is similar as well.

We first assume that n is odd. Then we have the global Flicker-Rallis period, an $H_n(\mathbb{A})$-invariant linear form on Π:

$$\beta(\phi) = \beta_n(\phi) := \int_{Z_n(\mathbb{A})H_n(F)\backslash H_n(\mathbb{A})} \phi(h)\, dh, \qquad \phi \in \Pi. \tag{3.11}$$

The global period β is related to the Asai L-function $L(s, \Pi, \mathrm{A}s^+)$ (for the definition of $\mathrm{A}s^{\pm}$; cf. [7, §7]). We set

$$\widetilde{\epsilon}_n = diag(\tau^{n-1}, \tau^{n-2}, \dots, 1) \in H_n(E) \tag{3.12}$$

and

$$\epsilon_{n-1} = \tau \cdot diag(\tau^{n-2}, \tau^{n-3}, \dots, 1) = \tau\widetilde{\epsilon}_{n-1} \in H_{n-1}(E). \tag{3.13}$$

(Note: $\tau \in E^-$.) Indeed, we may choose any $\widetilde{\epsilon}_n = diag(a_1, \dots, a_{n-1}, a_n)$ such that $a_i/a_{i+1} \in E^-$ for $i = 1, \dots, n-1$ and $a_n = 1$. We again use the Whittaker model of Π. We will again consider $H_{n,E}$ as an algebraic group over F. In particular, we consider Π_v as a representation of $H_n(E_v)$ where $E_v = E \otimes_F F_v$ is a semisimple F_v algebra of rank two. For a place v of F, we define the local Flicker-Rallis period β_v as follows: for $W_v \in \mathcal{W}(\Pi_v, \psi_{E_v})$

$$\beta_v(W_v) = \int_{N_{n-1}(F_v)\backslash H_{n-1}(F_v)} W_v \begin{pmatrix} \epsilon_{n-1}h & \\ & 1 \end{pmatrix} dh. \tag{3.14}$$

The integral β_v converges absolutely if Π_v is generic unitary. It depends on the choice of $\tau = \sqrt{\delta}$. For unramified data with normalization $W_v(1) = 1$, we have

$$\beta_v(W_v) = \mathrm{vol}(K_{n,v})L(1, \Pi_v, \mathrm{A}s^+). \tag{3.15}$$

We thus define a normalized linear form

$$\beta_v^\natural(W_v) = \frac{\beta(W_v)}{L(1, \Pi_v, \mathrm{A}s^+)}. \tag{3.16}$$

Remark 6. For bad places v, we may define the local factor $L(s, \Pi_v, \mathrm{A}s^+)$ as the greatest common divisor (GCD) of the local zeta integral in (3.19). Then the local factor $L(s, \Pi_v, \mathrm{A}s^+)$ has no pole or zero when at $s = 1$ for a unitary generic Π_v.

Proposition 3.2. *We have an explicit decomposition*

$$\beta(\phi) = \frac{n \cdot \mathrm{R}es_{s=1}L(s, \Pi, \mathrm{A}s^+)}{\mathrm{vol}(F^\times\backslash\mathbb{A}^1)} \prod_v \beta_v^\natural(W_v), \tag{3.17}$$

where $W = W_\phi = \otimes_v W_v \in \mathcal{W}(\Pi, \psi_E)$.

Proof. For a Schwartz-Bruhat function Φ on $\mathbb{A}^n$, we consider the Epstein-Eisenstein series $E(g, \Phi, s)$ [cf. (3.6)] replacing the field E by F. Then we define

$$I(s, \Phi, \phi) := \int_{Z(\mathbb{A})H_n(F)\backslash H_n(\mathbb{A})} E(g, \Phi, s)\phi(g)\, dg.$$

We then have [6, p.303]

$$I(s, \phi, \Phi) = \Psi(s, W_\phi, \Phi), \tag{3.18}$$

where

$$\Psi(s, W_\phi, \Phi) = \int_{N_n(\mathbb{A})\backslash H_n(\mathbb{A})} W_\phi(\widetilde{\epsilon}_n h)\Phi(e_n h)|h|^s\, dh.$$

(Note that we use a different choice of the additive character ψ_E.) Indeed, we have

$$\begin{aligned} I(s,\Phi,\phi) &= \int_{P_n(F)\backslash H_n(\mathbb{A})} \Phi(e_n g)\phi(g)|g|^s\,dg \\ &= \int_{P_n(F)N_n(\mathbb{A})\backslash H_n(\mathbb{A})} \Phi(e_n g)\left(\int_{N_n(F)\backslash N_n(\mathbb{A})} \phi(ng)\,dn\right)|g|^s\,dg. \end{aligned}$$

We have a Fourier expansion

$$\phi(g) = \sum_{\gamma\in N_n(E)\backslash P_n(E)} W_\phi(\gamma g).$$

Only those γ such that $\psi_E(\gamma n\gamma^{-1}) = 1$ for all $n \in N_n(\mathbb{A})$ contribute nontrivially. Therefore we may replace the sum by $\gamma \in \widetilde{\epsilon}_n P_n(F)$:

$$\begin{aligned} I(s,\Phi,\phi) &= \int_{P_n(F)N_n(\mathbb{A})\backslash H_n(\mathbb{A})} \Phi(e_n g)\left(\sum_{\gamma\in N_n(F)\backslash P_n(F)} W_\phi(\widetilde{\epsilon}_n\gamma g)\right)|g|^s\,dg \\ &= \int_{N_n(\mathbb{A})\backslash H_n(\mathbb{A})} \Phi(e_n g)W_\phi(\widetilde{\epsilon}_n g)|g|^s\,dg \\ &= \Psi(s,W_\phi,\Phi). \end{aligned}$$

[Note that $\mathrm{vol}(N_n(F)\backslash N_n(\mathbb{A})) = 1$.] We define for each place v of F,

$$\Psi(s,W_v,\Phi_v) = \int_{N_n(F_v)\backslash H_n(F_v)} W_v(\widetilde{\epsilon}_n h)\Phi_v(e_n h)|h|^s\,dh. \tag{3.19}$$

For unramified data, we have

$$\Psi(s,W_v,\Phi_v) = \mathrm{vol}(K_n(\mathcal{O}_{F_v}))L(s,\Pi_v,\mathrm{As}^+).$$

And we have [10, p.185]

$$\Psi(1,W_v,\Phi_v) = \beta_v(W_v)\widehat{\Phi}_v(0).$$

Alternatively we may prove this using (3.18), analogous to the proof of Prop. 3.1. Again, analogous to the proof of Prop. 3.1, we may take the residue of (3.18) to obtain

$$\frac{vol(F^\times\backslash\mathbb{A}^1)}{n}\widehat{\Phi}(0)\beta(\phi) = \mathrm{R}es_{s=1}L(s,\Pi,\mathrm{As}^+)\widehat{\Phi}(0)\prod_v \beta_v^\natural(W_v).$$

This completes the proof. □

When n is even, we insert the character η in the definition of β,

$$\beta(\phi) = \beta_n(\phi) := \int_{Z_n(\mathbb{A})H_n(F)\backslash H_n(\mathbb{A})} \phi(h)\eta(h)\,dh, \qquad \phi\in\Pi, \tag{3.20}$$

where, for simplicity, we denote $\eta(h) = \eta(\det(h))$.

The Asai L-function is then replaced by $L(s,\Pi,\mathrm{As}^-)$, or we may write it as $L(s,\Pi,\mathrm{As}^{(-1)^{n-1}})$. We also modify the definition

$$\beta_v(W_v) = \int_{N_{n-1}(F_v)\backslash H_{n-1}(F_v)} W_v\begin{pmatrix}\epsilon_{n-1}h & \\ & 1\end{pmatrix}\eta_v(h)\,dh. \tag{3.21}$$

The same argument shows that (3.17) still holds.

3.3. **Rankin-Selberg period.** We now follow [27]. Let $\Pi = \Pi_n \otimes \Pi_{n+1}$ and Π_i be a cuspidal automorphic representation of $H_i(\mathbb{A}_E)$, $i = n, n+1$. We define the global Rankin-Selberg period as

$$\lambda(\phi) = \int_{H_n(E)\backslash H_n(\mathbb{A}_E)} \phi(h)\, dh, \qquad \phi \in \Pi, \tag{3.22}$$

where H_n embeds diagonally into $H_n \times H_{n+1}$. To decompose it, we need the Whittaker model $\mathcal{W}(\Pi_n, \overline{\psi}_E)$ ($\mathcal{W}(\Pi_{n+1}, \psi_E)$, resp.) of Π_n (Π_{n+1}, resp.) with respect to the additive character $\overline{\psi}_E$ (ψ_E, resp.). We define a local Rankin-Selberg period on the local Whittaker model which associates with $W_w \in \mathcal{W}(\Pi_n, \overline{\psi}_E) \otimes \mathcal{W}(\Pi_{n+1}, \psi_E)$,

$$\lambda_w(s, W_w) = \int_{N_n(E_w)\backslash H_n(E_w)} W_w(h) |\det(h)|^s\, dh, \qquad s \in \mathbb{C}, \tag{3.23}$$

and a normalized one using the local Rankin-Selberg L-function $L(s, \Pi_{n,w} \times \Pi_{n+1,w})$ (cf. [27]),

$$\lambda_w^\natural(s, W_w) = \frac{\lambda_w(W_w)}{L(s+1/2, \Pi_{n,w} \times \Pi_{n+1,w})}. \tag{3.24}$$

When Π_w is generic, the integral $\lambda_w(s, \cdot)$ is absolutely convergent when $\mathrm{Re}(s)$ is large enough and extends to a meromorphic function in $s \in \mathbb{C}$. The normalized $\lambda_w^\natural(s, \cdot)$ extends to an entire function in $s \in \mathbb{C}$. Moreover, there exists W_w such that $\lambda_w^\natural(s, W_w) = 1$ (cf. [25, Theorem 2.1, 2.6] for Archimedean places). Therefore we will define

$$\lambda_w^\natural(W_w) = \lambda_w^\natural(0, W_w). \tag{3.25}$$

In particular, $\lambda_w^\natural$ defines a nonzero element of the (one-dimensional) space $\mathrm{Hom}_{H_n(E_w)}(\Pi_w, \mathbb{C})$ for generic Π_w.

If Π_w is tempered, then the integral $\lambda_w(s, \cdot)$ is absolutely convergent when $\mathrm{Re}(s) > -1/2$ (cf. [25, Lemma 5.3] for Archimedean places). Therefore in this case we may even define $\lambda_w(W_w) = \lambda(0, W_w)$ directly.

When Π_w and $\psi_{E,w}$ are unramified, the vector W_w is fixed by $K_{n,w} \times K_{n+1,w}$ and normalized by $W_w(1) = 1$, we have [30, p. 781]

$$\lambda_w(s, W_w) = \mathrm{vol}(K_{n,w}) L(s+1/2, \Pi_{n,w} \times \Pi_{n+1,w}), \tag{3.26}$$

and therefore

$$\lambda_w^\natural(W_w) = \mathrm{vol}(K_{n,w}).$$

We also form the global (complete) Rankin-Selberg L-function

$$L(s, \Pi_n \times \Pi_{n+1}) = \prod_w L(s, \Pi_{n,w} \times \Pi_{n+1,w}).$$

It is an entire function in $s \in \mathbb{C}$.

Proposition 3.3. *We have the following decomposition if* Π *is cuspidal unitary, and* $\phi \in \Pi$

$$\lambda(\phi) = L\left(\frac{1}{2}, \Pi_n \times \Pi_{n+1}\right) \prod_w \lambda_w^\natural(W_w), \tag{3.27}$$

where $W_\phi = \prod_w W_{\phi,w}$ *is as before.*

Proof. This is due to Jacquet, Piatetskii-Shapiro, and Shalika [27]. $\square$

For generic unitary Π, we need to use the completed L-function. Indeed, the local L-factor may have poles at $s = 1/2$ since we do not know the temperedness of each Π_w.

3.4. **Decomposing the spherical character on the general linear group.** We now denote

$$G' = \mathrm{Res}_{E/F}\left(\mathrm{GL}_n \times \mathrm{GL}_{n+1}\right) \tag{3.28}$$

viewed as an F-algebraic group. We consider its two subgroups: H_1' is the diagonal embedding of $\mathrm{Res}_{E/F}\mathrm{GL}_n$ (where GL_n is embedded into GL_{n+1} by $g \mapsto diag[g, 1]$) and H_2' is $\mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$ embedded into G' in the obvious way.

Now we consider a cuspidal automorphic representation $\Pi = \Pi_n \otimes \Pi_{n+1}$ of $G'(\mathbb{A})$. Denote by $\beta = \beta_n \otimes \beta_{n+1}$ the (product of) Flicker-Rallis period on $\Pi = \Pi_n \otimes \Pi_{n+1}$.

Definition 3.4. We define the *global spherical character* I_Π as the following distribution on $H(\mathbb{A})$: for $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$,

$$I_\Pi(f') = \sum_\phi \frac{\lambda(\Pi(f')\phi)\overline{\beta(\phi)}}{\langle \phi, \phi \rangle_{Pet}}, \tag{3.29}$$

where the sum runs over an *orthogonal* basis of Π. Equivalently,

$$I_\Pi(f') = \sum_\phi \lambda(\Pi(f')\phi)\overline{\beta(\phi)},$$

where the sum runs over an *orthonormal* basis of Π for the Petersson inner product.

Note that the definition of $\Pi(f')$ involves a choice of the measure on $G'(\mathbb{A})$ to define the Petersson inner product. We could choose any one as long as then we use the same measure [quotient by the counting measure on $G'(F)$].

By definition of $\mathrm{A}s^\pm$, we have for $i = n, n+1$

$$L(s, \Pi_i \times \Pi_i^\sigma) = L(s, \Pi_i, \mathrm{A}s^+)L(s, \Pi_i, \mathrm{A}s^-).$$

Now recall that in the Introduction we have a product of unitary groups $G = U(W) \times U(V)$ for Hermitian spaces $W \subset V$ with $\dim W = n, \dim V = n+1$. Assume that $\Pi = \pi_E$ is the base change of a cuspidal automorphic representation $\pi = \pi_n \otimes \pi_{n+1}$ of $G(\mathbb{A})$. By [7, Prop. 7.4] we also have

$$L(s, \Pi_i, \mathrm{A}s^{(-1)^i}) = L(s, \pi_i, Ad).$$

Remark 7. For bad places v, we may define the local factor $L(s, \pi_i, Ad)$ by this formula. But note that for our purpose, it only matters to know the local L-factors at unramified places.

Since such Π must be conjugate self-dual—$\widetilde{\Pi} \simeq \Pi^\sigma$ where σ is the nontrivial element in $\mathrm{Gal}(E/F)$—we deduce that $L(s, \Pi_i \times \widetilde{\Pi}_i)$ has a simple pole at $s = 1$. By our running hypothesis **RH(I)**(i), the Asial $L(s, \Pi_i, \mathrm{A}s^{(-1)^{i-1}})$ has a simple pole. We conclude that $L(s, \Pi_i, \mathrm{A}s^{(-1)^i}) = L(s, \pi_i, Ad)$ is regular at $s = 1$ and

$$\frac{\mathrm{R}es_{s=1}L(1, \Pi_i \times \widetilde{\Pi}_i)}{\mathrm{R}es_{s=1}L(s, \Pi_i, \mathrm{A}s^{(-1)^{i-1}})} = L(1, \Pi_i, \mathrm{A}s^{(-1)^i}) = L(1, \pi_i, Ad). \tag{3.30}$$

We denote by $\mathcal{W}(\Pi, \psi)$ the Whittaker model $\mathcal{W}(\Pi_n, \overline{\psi}) \otimes \mathcal{W}(\Pi_{n+1}, \psi)$. Let $\Pi = \otimes_v \Pi_v$. Let $\lambda_v^\natural$, $\beta_v^\natural$ be the local Rankin-Selberg period (3.25) and the local Flicker-Rallis period (3.16). Let $\vartheta_v^\natural$ be the normalized local invariant inner product (3.4).

Definition 3.5. We define the *normalized local spherical character* $I^\natural_{\Pi_v}$ associated to a unitary generic representation Π_v,

$$I^\natural_{\Pi_v}(f'_v) = \sum_{W_v} \frac{\lambda^\natural_v(\Pi_v(f'_v)W_v)\overline{\beta^\natural_v}(W_v)}{\vartheta^\natural_v(W_v, W_v)}, \tag{3.31}$$

where the sum runs over an *orthogonal* basis $W_v \in \mathcal{W}(\Pi_v, \psi_v)$. We also define an unnormalized local spherical character $I_{\Pi_v,s}$ as the meromorphic function in $s \in \mathbb{C}$,

$$I_{\Pi_v,s}(f'_v) = \sum_{W_v} \frac{\lambda_v(s, \Pi_v(f'_v)W_v)\overline{\beta_v}(W_v)}{\vartheta_v(W_v, W_v)}. \tag{3.32}$$

We will write $I_{\Pi_v}(f'_v)$ for its value $I_{\Pi_v,0}(f'_v)$ at $s = 0$.

We now summarize to arrive at an analogue of the decomposition in Conjecture 1.6.

Proposition 3.6. *Assume that the cuspidal automorphic representation Π of $G'(\mathbb{A})$ is the base change π_E of a cuspidal automorphic representation π of $G(\mathbb{A})$. Then we have*

$$I_\Pi(f') = L(1, \eta)^2 \frac{L(1/2, \Pi)}{L(1, \pi, Ad)} \prod_v I^\natural_{\Pi_v}(f'_v). \tag{3.33}$$

Proof. By the assumption, Π is unitary generic. Note that

$$\frac{\mathrm{vol}(E^\times\backslash\mathbb{A}^1_E)}{\mathrm{vol}(F^\times\backslash\mathbb{A}^1_F)} = L(1, \eta).$$

Then the result follows from Prop. 3.1, 3.2, 3.3 and the relation (3.30). □

Remark 8. Note that we do not need to assume the temperedness of π at this moment.

4. Relative trace formulae of Jacquet and Rallis

4.1. **The construction of Jacquet and Rallis.**

We recall the Jacquet-Rallis relative trace formulae [28], and we refer to [51] for more details.

First we recall the construction of the RTF of Jacquet-Rallis in the unitary group case. For $f \in \mathscr{C}^\infty_c(G(\mathbb{A}))$ we consider a kernel function

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y),$$

and a distribution

$$J(f) := \int_{H(F)\backslash H(\mathbb{A})} \int_{H(F)\backslash H(\mathbb{A})} K_f(x, y)\, dx\, dy.$$

The integral converges when the test function f is nice in the sense of [51, §2.3] (the precise definition will not be used in this paper). Associated to the RTF we have two objects:

- the global spherical character J_π associated to a cuspidal automorphic representation π of $G(\mathbb{A})$ (Definition 1.4 in the Introduction), and
- the (relative) orbital integral associated to a *regular semisimple* element[6] $\delta \in G(F)$: for $f \in \mathscr{C}^\infty_c(G(\mathbb{A}))$, we define its orbital integral

[6] See [50, §2.1] for the definition, where "regular" corresponds to "regular semisimple" in this paper.

$$O(\delta, f) := \int_{H(\mathbb{A})\times H(\mathbb{A})} f(x^{-1}\delta y)\, dx\, dy. \tag{4.1}$$

We have a local counterpart associated to a regular semisimple element $\delta \in G(F_v)$: for $f_v \in \mathscr{C}_c^\infty(G(F_v))$ we define

$$O(\delta, f_v) = \int_{H(F_v)\times H(F_v)} f_v(x^{-1}\delta y)\, dx\, dy. \tag{4.2}$$

We now recall the RTF in the general linear group case. Recall that $G' = \mathrm{Res}_{E/F}(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$ as an F-algebraic group. We consider its two subgroups:

- H'_1 is the diagonal embedding of $\mathrm{Res}_{E/F}\mathrm{GL}_n$ (where GL_n is embedded into GL_{n+1} by $g \mapsto diag[g, 1]$), and
- $H'_2 = \mathrm{GL}_{n,F} \times \mathrm{GL}_{n+1,F}$ embedded into G' in the obvious way.

For $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$, we define a kernel function

$$K_{f'}(x, y) = \int_{Z_{H'_2}(\mathbb{A})} \sum_{\gamma \in G'(F)} f'(x^{-1}\gamma z y) dz.$$

We then consider a distribution on $G'(\mathbb{A})$,

$$I(f') = \int_{H'_1(F)\backslash H'_1(\mathbb{A})} \int_{Z_{H'_2}(\mathbb{A})H'_2(F)\backslash H'_2(\mathbb{A})} K_{f'}(h_1, h_2)\eta(h_2)\, dh_1\, dh_2,$$

where $\eta(h_2) := \eta^{n-1}(g_n)\eta^n(g_{n+1})$ if $h_2 = (g_n, g_{n+1}) \in H_n(\mathbb{A}) \times H_{n+1}(\mathbb{A})$. The integral converges when the test function f is nice in the sense of [51, §2.2]. Associated to the RTF we have two objects:

- the global spherical character I_Π (cf. [50, §2]) associated to a cuspidal automorphic representation Π of $G'(\mathbb{A})$ (Definition 3.29), and
- the (relative) orbital integral associated to a regular semisimple element (cf. [50, §2]) $\gamma \in G'(F)$: for $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$, we define its orbital integral:

$$O(\gamma, f') := \int_{H'_1(\mathbb{A})} \int_{H'_2(\mathbb{A})} f'(h_1^{-1}\gamma h_2)\eta(h_2)\, dh_1\, dh_2. \tag{4.3}$$

Similarly we have a local counterpart: a regular semisimple element $\gamma \in G'(F_v)$: for $f'_v \in \mathscr{C}_c^\infty(G'(F_v))$ we define

$$O(\gamma, f'_v) = \int_{H'_1(F_v)} \int_{H'_2(F_v)} f'_v(h_1^{-1}\gamma h_2)\eta(h_2)\, dh_1\, dh_2. \tag{4.4}$$

We now recall the comparison of the orbits (cf. [50, §2]). Denote by $(H'_1(F)\backslash G'(F)/H'_2(F))_{rs}$ the set of regular semisimple $(H'_1 \times H'_2)(F)$-orbits in $G'(F)$ and $(H(F)\backslash G(F)/H(F))_{rs}$ the set of regular semisimple $(H\times H)(F)$-orbits in $G(F)$. We need to vary the pair $W \subset V$ of Hermitian spaces of dimension n and $n+1$ modulo the equivalence relation: (W, V) is equivalent to (W', V') if there is a constant $\kappa \in F^\times$ such that $\kappa W \simeq W'$ and $\kappa V \simeq V'$ (here κW means that we multiply the Hermitian form by the constant κ). Without loss of generality, we may and will assume that V is an *orthogonal* sum of W and a one-dimensional Hermitian space Ee with a norm one vector,

$$V = W \oplus Ee, \qquad \langle e, e\rangle = 1. \tag{4.5}$$

In particular, V is determined by W so that we only need to vary the Hermitian space W.[7] To indicate the dependence on the Hermitian spaces W, we will write G_W for G and H_W for H. Then there is a natural bijection [50, Lemma 2.3]

$$(H_1(F)\backslash G'(F)/H_2(F))_{rs} \simeq \coprod_W (H_W(F)\backslash G_W(F)/H_W(F))_{rs}, \tag{4.6}$$

where on the right hand side the disjoint union runs over all Hermitian space W of dimension n. Moreover, the same holds if we replace F by F_v for every place v of F. When v is non-Archimedean, there are precisely two isomorphism classes of Hermitian spaces W_v.

In [51, §2.4] we defined an explicit *transfer factor* $\{\Omega_v\}_v$ on the regular semisimple locus of $G'(F_v)$ for any place v. It satisfies the following properties:

- If $\gamma \in G'(F)$ is regular semisimple, then we have a product formula $\prod_v \Omega_v(\gamma) = 1$.
- For any $h_i \in H_i'(F_v)$ and $\gamma \in G'(F_v)$, we have $\Omega(h_1\gamma h_2) = \eta(h_2)\Omega_v(\gamma)$.

The construction is as follows. It depends on an auxiliary character η',

$$\eta' : E^\times\backslash \mathbb{A}_E^\times \to \mathbb{C}^\times \tag{4.7}$$

(not necessarily quadratic) such that its restriction $\eta'|_{\mathbb{A}^\times} = \eta$. Let S_{n+1} be the subvariety of $\mathrm{Res}_{E/F}\mathrm{GL}_{n+1}$ defined by the equation $s\bar{s} = 1$. By Hilbert Satz-90, we have an isomorphism of two affine varieties

$$\mathrm{Res}_{E/F}\mathrm{GL}_{n+1}/\mathrm{GL}_{n+1,F} \simeq S_{n+1},$$

induced by the following morphism ν between F varieties,

$$\nu : \mathrm{Res}_{E/F}\mathrm{GL}_{n+1} \to S_{n+1} \tag{4.8}$$

$$g \mapsto g\bar{g}^{-1}, \tag{4.9}$$

and in the level of F-points,

$$\mathrm{GL}_{n+1}(E)/\mathrm{GL}_{n+1}(F) \simeq S_{n+1}(F). \tag{4.10}$$

Write $\gamma = (\gamma_1, \gamma_2) \in G'(F_v)$ and $s = \nu(\gamma_1^{-1}\gamma_2)$. We define for a regular semisimple $s \in S_{n+1}(F_v)$

$$\Omega_v(s) := \eta_v'(\det(s)^{-[(n+1)/2]}\det(e, es, \ldots, es^n)). \tag{4.11}$$

Here $e = e_{n+1} = (0, \ldots, 0, 1)$ and $(e, es, \ldots, es^n) \in M_{n+1}$ is the matrix whose ith row is es^{i-1}. If n is odd, we define

$$\Omega_v(\gamma) := \eta_v'(\det(\gamma_1^{-1}\gamma_2))\Omega_v(s), \tag{4.12}$$

and if n is even, we simply define

$$\Omega_v(\gamma) := \Omega_v(s). \tag{4.13}$$

For a place v of F, we say that the function $f' \in \mathscr{C}_c^\infty(G'(F_v))$ and the tuple $(f_W)_W, f_W \in \mathscr{C}_c^\infty(G_W(F_v))$, indexed by the set of all equivalence classes of Hermitian spaces W over $E_v = E \otimes F_v$, *are smooth transfers of each other* or *match* if

$$\Omega_v(\gamma)O(\gamma, f') = O(\delta, f_W), \tag{4.14}$$

whenever a regular semisimple $\gamma \in G'(F_v)$ matches $\delta \in G_W(F_v)$ via (4.6). One of the main local results in [51] is the existence of a smooth transfer at

[7] In terms of [7, §2], we only consider Hermitian pairs (W, V) that are *relevant* to each other.

non-Archimedean nonsplit places (cf. [51, Theorem 2.6]) and arbitrary split places (cf. [51, Prop. 2.5]).

In this paper, we usually need to consider a fixed W, and we say that f' and $f_W \in \mathscr{C}_c^\infty(G_W(F_v))$ match if there exist some $f_{W'}$ for each equivalence class $W' \neq W$ such that f' matches the completed tuple $f_W, f_{W'}$.

Moreover, the fundamental lemma of Jacquet-Rallis predicts a specific case of matching functions.

Theorem 4.1 ([45]). *Assume that the quadratic extension E_v/F_v is unramified. Denote by $\{W_v, W_v'\}$ the two isomorphism classes of Hermitian spaces of dimension n where W_v contains a self-dual (with respect to the Hermitian form) $\mathcal{O}_{E_v}$ lattice. Set*[8]

(4.15)

$$f_{W_v} = \frac{1}{\mathrm{vol}(H_{W_v}(\mathcal{O}_v))^2} 1_{G_{W_v}(\mathcal{O}_v)}, \quad f_{W_v'} = 0, \quad f_v' = \frac{1}{\mathrm{vol}(H_1'(\mathcal{O}_v))\mathrm{vol}(H_2'(\mathcal{O}_v))} 1_{G'(\mathcal{O}_v)}.$$

Then there is a constant $c(n)$ depending only on n such that, when the characteristic of the residue field of F_v is larger than $c(n)$, the function f_v' matches the pair $(f_{W_v}, f_{W_v'})$.

We need some simplification of orbital integrals [51, §2.1]. Identify $H_1'\backslash G'$ with $\mathrm{Res}_{E/F}\mathrm{GL}_{n+1}$. Now we write F for F_v for a fixed place v. We may integrate f' over $H_1'(F)$ to get a function on $\mathrm{Res}_{E/F}\mathrm{GL}_{n+1}(F)$,

$$\widetilde{f'}(g) := \int_{H_1'(F)} f'(h_1(1,g))\, dh_1, \qquad g \in \mathrm{Res}_{E/F}\mathrm{GL}_{n+1}(F). \tag{4.16}$$

Using the fiber integral of ν [cf. (2.3) and (4.10)] we define

$$\widetilde{\widetilde{f'}}(s) := \int_{H_{n+1}(F)} \widetilde{f'}(gh)\, dh, \qquad \nu(g) = s, \tag{4.17}$$

if n is even, and

$$\widetilde{\widetilde{f'}}(s) := \int_{H_{n+1}(F)} \widetilde{f'}(gh)\eta'(gh)\, dh, \qquad \nu(g) = s, \tag{4.18}$$

when n is odd (then this depends on the auxiliary character η'). Then $\widetilde{\widetilde{f'}} \in \mathscr{C}_c^\infty(S_{n+1}(F))$ and all functions in $\mathscr{C}_c^\infty(S_{n+1}(F))$ arise in this way.

Now it is easy to see that for $\gamma = (\gamma_1, \gamma_2)$

$$O(\gamma, f') = \eta'(\det(\gamma_1^{-1}\gamma_2)) \int_{H_n(F)} \widetilde{\widetilde{f'}}(h^{-1}sh)\eta(h)\, dh, \qquad s = \nu(\gamma_1^{-1}\gamma_2), \tag{4.19}$$

if n is odd, and

$$O(\gamma, f') = \int_{H_n(F)} \widetilde{\widetilde{f'}}_v(h^{-1}sh)\eta(h)\, dh, \qquad s = \nu(\gamma_1^{-1}\gamma_2), \tag{4.20}$$

if n is even. Up to a sign, the integral on the right hand side depends only on the orbit of s under the conjugation by $H_n(F)$. Therefore, we define the orbital integral associated to a regular semisimple element $s \in S_{n+1}(F)$,

$$O(s, \widetilde{\widetilde{f'}}) := \int_{H_n(F)} \widetilde{\widetilde{f'}}(h^{-1}sh)\eta(h)\, dh, \qquad \widetilde{\widetilde{f'}} \in \mathscr{C}_c^\infty(S_{n+1}(F)). \tag{4.21}$$

[8]Note that the measures in the fundamental lemma proved in [45] are different from ours.

Then we always have, for regular semisimple $\gamma = (\gamma_1, \gamma_2) \in G'(F_v)$ [cf. (4.11)],

$$\Omega(\gamma)O(\gamma, f') = \Omega(s)O(s, \widetilde{\widetilde{f'}}), \qquad s = \nu(\gamma_1^{-1}\gamma_2). \tag{4.22}$$

4.2. **A trace formula identity.** We are led to a comparison of the two RTFs and the two spherical characters I_Π and J_π when $\Pi = \pi_E$ is the base change of π.

Conjecture 4.2. *Let π be an irreducible cuspidal automorphic representation on $G(\mathbb{A})$ that admits the invariant linear functional,*

$$\mathrm{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C}) \neq 0.$$

Let π_E be the base change of π and assume that π_E is cuspidal. Then, for every $f \in \mathscr{C}_c^\infty(G(\mathbb{A}))$ and a smooth transfer $f' \in \mathscr{C}_c^\infty(G'(\mathbb{A}))$ of f, we have

$$2^{-2}L(1,\eta)^{-2}I_{\pi_E}(f') = J_\pi(f).$$

Remark 9. Note that we do not need to assume that π is tempered.

Theorem 4.3. *Assume the following:*

(1) *At a split place v_1, π_{v_1} is supercuspidal.*
(2) *The test functions f and f' are nice and f' is a smooth transfer of f.*

Then Conjecture 4.2 *holds for such π and the test functions f, f'.*

Proof. We would like to apply the result from [51]. But we need to compare the difference on the normalization of the Petersson inner product in the unitary group case (caused by the presence of the center). There implicitly we use a different Petersson inner product

$$\langle \phi, \phi' \rangle' = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \phi(g)\overline{\phi'}(g)\, dg = \mathrm{vol}(Z(F)\backslash Z(\mathbb{A}))^{-1}\langle \phi, \phi' \rangle.$$

Note that the center Z of G is isomorphic to $U(1) \times U(1)$. Hence the volume for our choice of measure is [cf. (2.6)]

$$\mathrm{vol}(Z(F)\backslash Z(\mathbb{A})) = (2L(1,\eta))^2.$$

Now taking into account this correction, we apply the trace formula identity [51, Prop. 2.11]: if a nice function f' matches a tuple (f_W) indexed by equivalence classes of W, we have

$$I_{\pi_E}(f') = (2L(1,\eta))^2 \sum_W \sum_{\pi_W} J_{\pi_W}(f_W),$$

where the sum is over all equivalence classes of W and all cuspidal automorphic representations π_W of $G_W(\mathbb{A})$ that are nearly equivalent to π and at v_1 all π_{W,v_1} are isomorphic to π_{v_1}. We denote by (W_0, V_0) the Hermitian spaces we started with, $\pi_{W_0} = \pi$, and by $f_{W_0} = f$ the function in the assumption of the theorem.

By our running hypothesis **RH(I)**, we have

(1) the multiplicity of each cuspidal π_W in $L^2([G_W])$ is one. Namely, for a fixed W, all π_W occurring in the sum are nonisomorphic.
(2) Note that for all W, all π_W occurring in the sum are in the same nearly equivalent class and π_{W,v_1} are supercuspidal (so π_E is cuspidal and particularly $\pi_{E,v}$ is generic for every v). Hence for every v, the $\pi_{W,v}$'s are in the same Vogan L-packet, and this L-packet is generic.

Now by our running hypothesis **RH(II)**, there exists at most one π_W and W in the sum such that $\mathrm{Hom}_{H_W(\mathbb{A})}(\pi_W,\mathbb{C})\neq 0$. By our assumption $\mathrm{Hom}_{H(\mathbb{A})}(\pi,\mathbb{C})\neq 0$. Hence the sum reduces to one term contributed by the π we started with,

$$I_{\pi_E}(f')=(2L(1,\eta))^2 J_\pi(f).$$

□

Remark 10. If π_E is not cuspidal, then we may reformulate the conjecture at least for tempered representation π. We also need to regularize the definition of I_{π_E} in the above equality, and the constant 2^2 should be replaced by $|S_\pi|$. Then the analogous conjecture should ultimately follow from the full spectral decomposition of the Jacquet-Rallis relative trace formulae.

4.3. **Reduction to a local question.** Our main ingredient is an identity between the two local distributions I_{Π_v} [cf. (3.31)] and J_{π_v} [cf. (1.8)]. Note that the distribution J_{π_v} does not depend on the choice of the inner product on π_v. Denote $d_n=\binom{n}{3}$, which satisfies

$$\tau^{d_n}=\delta_{n-1}(\epsilon_{n-1})=\det(Ad(\epsilon_{n-1}):N_{n-1}(E)). \tag{4.23}$$

We have a local conjecture.

Conjecture 4.4. *Let $\pi_v=\pi_{n,v}\otimes\pi_{n+1,v}$ be an irreducible tempered unitary representation of $G(F_v)$ with $\alpha_v\neq 0$. Assume that the base change $\Pi_v=\Pi_{n,v}\otimes\Pi_{n+1,v}$ of π_v is generic unitary (so that I_{Π_v} is well-defined). If the functions $f_v\in\mathscr{C}_c^\infty(G(F_v))$ and $f'_v\in\mathscr{C}_c^\infty(G'(F_v))$ match, then we have*

$$I_{\Pi_v}(f'_v)=\kappa_v L(1,\eta_v)^{-1}J_{\pi_v}(f_v), \tag{4.24}$$

where the constant κ_v is given by

$$\begin{aligned}\kappa_v&=\kappa_v(\eta',\tau,n,\psi)\\&=|\tau|_{E,v}^{(d_n+d_{n+1})/2}(\epsilon(1/2,\eta_v,\psi_v)/\eta'(\tau))^{n(n+1)/2}\eta_v(\mathrm{disc}(W))\omega_{\Pi_{n,v}}(\tau).\end{aligned}$$

Here $\omega_{\Pi_{n,v}}$ is the central character of $\Pi_{n,v}$, and $\mathrm{disc}(W)\in F^\times/\mathrm{N}E^\times$ is the discriminant of the Hermitian space W, I_{Π_v} (J_{π_v}, resp.) is defined by (3.32) ((1.9), resp.).

Proposition 4.5. *Let π be a tempered cuspidal automorphic representation of $G(\mathbb{A})$ with cuspidal base change $\Pi=\pi_E$. Assume that there exists a test function $f=\otimes f_v$ and a smooth transfer $f'=\otimes f'_v$ such that for every place v*

$$J^\natural_{\pi_v}(f_v)\neq 0.$$

Assume that

- *Conjecture 4.2 holds for π, f,f'.*
- *For every v, Conjecture 4.4 holds for π_v,f_v,f'_v.*

Then Conjecture 1.6 and 1.1 holds for π.

Proof. By Conjecture 4.2 and Prop. 3.6 we have

$$J_\pi(f)=2^{-2}L(1,\eta)^{-2}I_\Pi(f')=2^{-2}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)}\prod_v I^\natural_{\Pi_v}(f'_v).$$

Conjecture 4.4 is equivalent to the identity between the normalized distributions

$$I^\natural_{\Pi_v}(f'_v)=\kappa_v L(1,\eta_v)^{-1}\Delta_{n+1,v}J^\natural_{\pi_v}(f_v).$$

Since

$$\prod_v \epsilon\left(\frac{1}{2}, \eta_v, \psi_v\right) = \epsilon\left(\frac{1}{2}, \eta\right) = 1,$$

we have

$$\prod_v \kappa_v = 1.$$

Note that the product $\prod_v L(1,\eta_v)^{-1}\Delta_{n+1,v}$ converges absolutely to $L(1,\eta)^{-1}\Delta_{n+1}$. We thus obtain

$$J_\pi(f) = 2^{-2}L(1,\eta)^{-1}\Delta_{n+1}\frac{L(1/2,\pi_E)}{L(1,\pi,Ad)}\prod_v J^\natural_{\pi_v}(f_v).$$

Note that the global measure on H and G in the Introduction are normalized by $L(1,\eta)^{-1}$ and $L(1,\eta)^{-2}$, respectively. The correction of measures yields Conjecture 1.6 for the choice of test function f. Since $J^\natural_{\pi_v}(f_v) \neq 0$ for all v (and equal to one for almost all v), it follows that Conjecture 1.6 holds for all test functions $f \in \mathscr{C}_c^\infty(G(\mathbb{A}))$. We have shown in Lemma 1.7 that Conjecture 1.6 implies Conjecture 1.1. □

We have the following evidence of Conjecture 4.4.

Theorem 4.6. *Let v be a place of F and let π_v be a tempered representation as in Conjecture* 4.4.

(1) *Conjecture* 4.4 *holds if the place v is split in E/F.*
(2) *If v is a non-Archimedean place nonsplit in E/F, then under any one of the following conditions, there exists f_v and a smooth transfer f'_v, such that the equality (*4.24*) holds and $J_{\pi_v}(f_v) \neq 0$:*
   (i) *The representation π_v is unramified and the residue characteristic $p \geq c(n)$.*
   (ii) *The group $H(F_v)$ is compact.*
   (iii) *The representation π_v is supercuspidal.*

Below we first prove Theorem 4.6 when π_v is unramified [case (1)] or v is split in Corollary 4.11 [case (2)-(i)]. We postpone the proof of the cases (2)-(ii) and (2)-(iii) to the last part of §9.

We now give the proof of the first part of Theorem 1.2 assuming Theorem 4.6.

Proof of Theorem 1.2*: Case* (1). We may assume that $\mathrm{Hom}_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$ for all v (otherwise the formula holds trivially). This implies that the linear form α'_v does not vanish for all v. We then construct nice test functions $f = \otimes f_v$ on $G(\mathbb{A})$ and $f' = \otimes f'_v$ as follows:

- at each inert v with residue characteristic $p \geq c(n)$, f_v, f'_v are given by the fundamental lemma (Theorem 4.1).
- at each $v \in \Sigma$, we choose f_v, f'_v as in (2)-(ii) or (2)-(iii) of Theorem 4.6.
- at almost every split place, we choose the unit element in the spherical Hecke algebra.
- at the remaining finitely many split places including v_0 and the Archimedean ones, we choose suitable functions so that f, f' are nice and such that $J_{\pi_v}(f_v) \neq 0$.

Apply Theorem 4.3 to π and f, f' to obtain

$$2^{-2}L(1,\eta)^{-2}I_{\pi_E}(f') = J_\pi(f).$$

Now Theorem 1.2 case (1) follows from Prop. 4.5. $\square$

4.4. **Proof of Theorem 4.6: The Case of π_v Unramified and $p \geq c(n)$.** Then we may assume that

(1) The quadratic extension E/F is unramified at v.
(2) The number τ is a v-adic unit.
(3) The character ψ is unramified and hence so is ψ_E.

Indeed, it is easy to see how I_{Π_v} depends on τ: only the local period β_v involves the choice of τ, and we see that $|\tau|_{E,v}^{-(d_n+d_{n+1})/2}\eta'(\tau)^{n(n+1)/2}I_{\Pi_v}$ is independent of the choice of τ. If we twist ψ by $a \in F_v^\times$, it amounts to change τ by $a\tau$.

We need to utilize the fundamental lemma: by Theorem 4.1, we have a matching pair,

$$f_v = \frac{1}{\mathrm{vol}(H(\mathcal{O}_v))^2}1_{G(\mathcal{O}_v)}, \qquad f'_v = \frac{1}{\mathrm{vol}(H'_1(\mathcal{O}_v))\mathrm{vol}(H'_2(\mathcal{O}_v))}1_{G'(\mathcal{O}_v)}.$$

Let $W_0 \in \mathcal{W}(\Pi_v, \psi_E)$ be the unique spherical element normalized such that $W_0(1) = 1$. Then we have

$$\Pi_v(f'_v)W_0 = \frac{\mathrm{vol}(G'(\mathcal{O}_v))}{\mathrm{vol}(H'_1(\mathcal{O}_v))\mathrm{vol}(H'_2(\mathcal{O}_v))}W_0$$

and

$$I_{\Pi_v}(f'_v) = \frac{\lambda(\Pi_v(f'_v)W_0)\overline{\beta}(W_0)}{\vartheta_v(W_0,W_0)} = \frac{\mathrm{vol}(G'(\mathcal{O}_v))}{\mathrm{vol}(H'_1(\mathcal{O}_v))\mathrm{vol}(H'_2(\mathcal{O}_v))}\frac{\lambda(W_0)\overline{\beta}(W_0)}{\vartheta_v(W_0,W_0)}.$$

Note that by (3.26)

$$\lambda(W_0) = L(1/2,\Pi_v)\cdot \mathrm{vol}(H'_1(\mathcal{O}_v))$$

and by (3.3) and (3.15)

$$\frac{\overline{\beta}(W_0)}{\vartheta_v(W_0,W_0)} = L(1,\pi_v,Ad)^{-1}\frac{\mathrm{vol}(H'_2(\mathcal{O}_v))}{\mathrm{vol}(H_n(\mathcal{O}_{E,v}))\mathrm{vol}(H_{n+1}(\mathcal{O}_{E,v}))}.$$

We obtain

$$I_{\Pi_v}(f'_v) = \frac{L(1/2,\Pi_v)}{L(1,\pi_v,Ad)}\cdot\frac{\mathrm{vol}(G'(\mathcal{O}_v))}{\mathrm{vol}(H'_1(\mathcal{O}_v))\mathrm{vol}(H'_2(\mathcal{O}_v))}\cdot\frac{\mathrm{vol}(H'_1(\mathcal{O}_v))\mathrm{vol}(H'_2(\mathcal{O}_v))}{\mathrm{vol}(H_n(\mathcal{O}_{E,v}))\mathrm{vol}(H_{n+1}(\mathcal{O}_{E,v}))}.$$

In summary we have

$$I_{\Pi_v}(f'_v) = \frac{L(1/2,\Pi_v)}{L(1,\pi_v,Ad)}. \tag{4.25}$$

In the unitary group case, we take $\phi_0 \in \pi_v^{K_v}$ normalized by $\langle\phi_0,\phi_0\rangle = 1$,

$$\pi_v(f_v)\phi_0 = \frac{\mathrm{vol}(G(\mathcal{O}_v))}{\mathrm{vol}(H(\mathcal{O}_v))^2}\phi_0.$$

Therefore we have

$$J_{\pi_v}(f_v) = \alpha_v(\pi_v(f_v)\phi_0,\phi_0) = \frac{\mathrm{vol}(G(\mathcal{O}_v))}{\mathrm{vol}(H(\mathcal{O}_v))^2}\alpha_v(\phi_0,\phi_0).$$

By the unramified computation in [21]

$$\alpha_v(\phi_0, \phi_0) = \mathrm{vol}(H(\mathcal{O}_v))\Delta_{n+1,v}\frac{L(1/2, \Pi_v)}{L(1, \pi_v, Ad)}.$$

We thus obtain

$$J_{\pi_v}(f_v) = \frac{\mathrm{vol}(G(\mathcal{O}_v))}{\mathrm{vol}(H(\mathcal{O}_v))}\Delta_{n+1,v}\frac{L(1/2, \Pi_v)}{L(1, \pi_v, Ad)}.$$

Note that $\frac{\mathrm{vol}(G(\mathcal{O}_v))}{\mathrm{vol}(H(\mathcal{O}_v))}$ is equal to the volume of the hyperspecial compact open of $U(V)(F_v)$, which is equal to $L(1, \eta)\Delta_{n+1,v}^{-1}$. Therefore we obtain that

$$J_\pi(f) = L(1, \eta_v)\frac{L(1/2, \Pi_v)}{L(1, \pi_v, Ad)}. \tag{4.26}$$

By (4.25) and (4.26), we have

$$I_{\Pi_v}(f'_v) = L(1, \eta_v)^{-1}J_{\pi_v}(f_v). \tag{4.27}$$

This completes the proof of case (i) of Theorem 4.6.

Change of measures. *From now on, all measures will be the unnormalized [namely, without the convergence factor $\zeta_v(1), L(1, \eta_v)$ etc.] Tamagawa measures with the natural invariant differential forms on the general linear groups, their subgroups, and Lie algebras.*

Lemma 4.7. *When using the unnormalized measures, the identity in Conjecture* 4.4 *becomes*

$$I_{\Pi_v}(f'_v) = \kappa_v J_{\pi_v}(f_v), \tag{4.28}$$

for matching functions f_v and f'_v (also under the unnormalized measures).

Proof. The old distribution I_{Π_v} is the new one times

$$\zeta_{E,v}(1)^2\frac{\zeta_{E,v}(1)\zeta_{F,v}(1)^2}{\zeta_{E,v}(1)^2},$$

where the first term comes from the measure on G' involving the definition of $\Pi_v(f'_v)$, and the fraction comes from the measures in λ_v, β_v, and ϑ_v. Similarly, the old distribution J_{π_v} is the new one times

$$L(1, \eta_v)^2 \cdot L(1, \eta_v),$$

where the first term comes from the measure on G involving the definition of $\pi_v(f_v)$, and the second from the measure on $H(F_v)$ in the definition of α_v. Moreover, the change of measures on $H'_1(F_v), H'_2(F_v)$, and $H(F_v)$ also changes the requirement of smooth matching: if f_v and f'_v match for the normalized measures, then $\zeta_{E,v}(1)\zeta_{F,v}(1)^2 f_v$ and $L(1, \eta_v)^2 f'_v$ match for the unnormalized measures. Therefore, when using the unnormalized measures, the identity in Conjecture 4.4 becomes the asserted one (4.28). ☐

4.5. **Proof of Theorem 4.6: The Case of a Split Place v.** Assume that $F = F_v$ is split. Let $\pi = \pi_n \otimes \pi_{n+1}$ be an irreducible unitary generic representation of $G(F)$. We may identify $H_n(E)$ with $\mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$ and identify $U(W)(F_v)$ with a subgroup consisting of elements of the form $(g, {}^t g^{-1})$, $g \in \mathrm{GL}_n(F)$ and ${}^t g$ is the transpose of g. Let p_1, p_2 be the two isomorphisms between $U(W)(F)$ with $\mathrm{GL}_n(F)$ induced by the two projections from $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ to $\mathrm{GL}_n(F)$. If π_n is an irreducible generic representation of $U(W)(F_v)$, the representation $\Pi_n = BC(\pi_n)$

can be identified with $p_1^*\pi_v \otimes p_2^*\pi_v$ of $H_n(E)$ where $p_i^*\pi_v$ is a representation of $\mathrm{GL}_n(F)$ obtained by the isomorphism p_i. For simplicity, we will write $\Pi_n = \pi_n \otimes \widetilde{\pi}_n$, and similarly for π_{n+1}, Π_{n+1}. We fix a Whittaker model $\mathcal{W}(\pi_i), i = n, n+1$ using the additive character ψ at v. We define an auxiliary element $\alpha' \in \mathrm{Hom}(\pi \otimes \widetilde{\pi}, \mathbb{C})$ by

$$\alpha'(W, W') = \lambda(W)\overline{\lambda(W')}, \qquad W, W' \in \mathcal{W}(\pi). \tag{4.29}$$

Again we have identified $\widetilde{\pi}$ with $\overline{\pi}$. Here we require that the invariant inner product on $\mathcal{W}(\pi)$ is the one defined by ϑ [cf. (3.2)]. Then we define a variant of the local spherical character,

$$J'_\pi(f) = \sum_W \alpha'(\pi(f)W, W), \qquad f \in \mathscr{C}_c^\infty(G(F)), \tag{4.30}$$

where the sum of W runs over an orthonormal basis of $\mathcal{W}(\pi)$. Similarly we have a normalized one $J_\pi'^\natural(f)$.

Lemma 4.8. *Let $f' = f_1 \otimes f_2 \in \mathscr{C}_c^\infty(G'(F))$ and $f = f_1 * f_2^* \in \mathscr{C}_c^\infty(G(F))$ [$f_2^*(x) = \overline{f}_2(x^{-1})$] matching f'. Then we have*

$$I_{\Pi_v}(f') = \kappa_v J'_\pi(f). \tag{4.31}$$

Proof. We identify Π_n with $\pi_n \otimes \widetilde{\pi}_n = \pi_n \otimes \overline{\pi}_n$. Then we have for $W, W' \in \mathcal{W}(\pi_n)$

$$\beta_n(W \otimes \overline{W'}) = \int_{N_{n-1}(F)\backslash H_{n-1}(F)} W\begin{pmatrix} \epsilon_{n-1}h & \\ & 1\end{pmatrix} \overline{W'}\begin{pmatrix} \epsilon_{n-1}h & \\ & 1\end{pmatrix} dh.$$

This yields

$$\beta_n(W \otimes \overline{W'}) = |\tau|^{d_n/2}\vartheta_n(W, W'),$$

and similarly for β_{n+1}. Then the desired equality follows by the definition of I_Π in terms of the linear functional λ, β, and ϑ (note that $\delta = \tau^2$ is indeed a square in F). ☐

Now it remains to identify the distribution J'_π with J_π, or equivalently, to prove that $\alpha' = \alpha$. The key ingredient is from [33]; in the non-Archimedean case, we could also use [41, §3.5].

Note that we may write α in terms of the Whittaker model $\mathcal{W}(\pi)$,

$$\alpha(W, W') = \int_{H_n(F)} \langle \pi(h)W, W'\rangle\, dh, \qquad \langle W, W'\rangle = \vartheta(W, W').$$

We temporarily denote $N_- = N_{n-}(F)$ and $N = N_n(F)$. Let $N^\circ = [N, N]$ be the commutator subgroup of N, $N^{ab} = N^\circ\backslash N$ the maximal Abelian quotient of N, and $\widehat{N^{ab}}$ the group of characters of N^{ab}. The diagonal subgroup A_n of H_n acts on N (by conjugation), on N^{ab}, and hence on $\widehat{N^{ab}}$. Moreover, A_n acts transitively on the subset $\widehat{N^{ab}}_{reg}$ of $\widehat{N^{ab}}$ consisting of regular characters (i.e., with minimal stabilizer under the action of A_n). The character ψ on N is regular, and we denote by ψ_t the character of N (equivalently, of N^{ab}) defined by

$$\psi_t(u) = \psi(tut^{-1}).$$

For $W_n, W_n' \in \mathcal{W}(\pi_n, \overline{\psi})$, we denote by $\Phi_{W_n, W_n'}$ the matrix coefficient

$$\Phi_{W_n, W_n'}(g) = \langle \pi_n(g)W_n, W_n'\rangle.$$

Lemma 4.9. *Assume that π_n is tempered.*

(i) *The integral*

$$\mathcal{F}_{W_n,W'_n}(u) := \int_{N^\circ} \Phi_{W_n,W'_n}(vu)dv \tag{4.32}$$

is absolutely convergent and defines a square integrable function $\mathcal{F}_{W_n,W'_n} \in L^2(N^{ab})$. Its Fourier transform $\widehat{\mathcal{F}}_{W_n,W'_n} \in L^2(\widehat{N^{ab}})$ is smooth on the open subset $\widehat{N^{ab}}_{reg}$ of $\widehat{N^{ab}}$.

(ii) *For all $t \in A_n$, and $W_n, W'_n \in \mathcal{W}(\pi_n, \overline{\psi})$, we have*

$$\widehat{\mathcal{F}}_{W_n,W'_n}(\psi_t) = |\delta_n(t)|^{-1} W_n(t)\overline{W'}_n(t). \tag{4.33}$$

Here the left hand side denotes the value of the Fourier transform at the character $\psi_t \in \widehat{N^{ab}}_{reg}$.

Proof. The first part of (i) follows from [33, Corollary 2.8]. The second part of (i) follows from [33, Lemma 3.2] for a special class of W_n and W'_n. The general case follows from this special case together with the Dixmier-Malliavin theorem (cf. [33, Remark 3.3]). The assertion in (ii) for $t = 1$ is [33, Prop. 3.4]. The general case of $t \in A_n$ follows easily from this. $\square$

Proposition 4.10. *Assume that $\pi = \pi_n \otimes \pi_{n+1}$ is tempered. Then we have, for all $W, W' \in \mathcal{W}(\pi_n, \overline{\psi}) \otimes \mathcal{W}(\pi_{n+1}, \psi)$,*

$$\alpha(W, W') = \lambda(W)\overline{\lambda(W')}.$$

Namely, $\alpha = \alpha'$ as nonzero elements in $\mathrm{Hom}(\pi \otimes \widetilde{\pi}, \mathbb{C})$.

Proof. The right hand side does not vanish by the nonvanishing of the local Rankin-Selberg integral [27], [25]. By the multiplicity one theorem for generic representations, $\dim \mathrm{Hom}_{H_n(F)}(\pi, \mathbb{C}) = 1$, the left hand side is a constant multiple of the right hand side for all W, W'. Hence it suffices to prove the identity for some choice of W, W' so that $\lambda(W)\lambda(W') \neq 0$.

Let $W = W_n \otimes W_{n+1}, W' = W'_n \otimes W'_{n+1}$. We choose W_{n+1}, W'_{n+1} as follows. Let φ be in $\mathscr{C}_c^\infty(B_-)$. Then there is a unique element in $\mathcal{W}(\pi_{n+1}, \psi)$, denoted by W_φ, such that the restriction $W_\varphi|_{H_n}$ is supported in NB_- and

$$W_\varphi \begin{pmatrix} ub & \\ & 1 \end{pmatrix} = \psi(u)\varphi(b), \qquad u \in N, b \in B_-.$$

Similarly we choose $\varphi' \in \mathscr{C}_c^\infty(B_-)$ and define $W_{\varphi'} \in \mathcal{W}(\pi_{n+1}, \psi)$.

We may and will consider the action of $\mathscr{C}_c^\infty(B_-)$ on $\mathcal{W}(\pi_n, \overline{\psi})$ by

$$\pi_n(\varphi)W(g) = \int_{B_-} W(gb)\varphi(b)db, \tag{4.34}$$

where db is the right invariant measure on B_- normalized so that the measure on H_n decomposes as $dg = du\,db$ where $g = ub, u \in N, b \in B_-$.

Let $c \in \mathbb{R}_+$ and consider the subset N_c of N consisting of elements $u = (u_{ij})_{1 \le i,j \le n}$ such that

$$|u_{i,i+1}| \le c, \qquad 1 \le i \le n-1.$$

We denote by N_c^{ab} the image of N_c in the Abelian quotient N^{ab}. Let us consider the integral parameterized by $t \in A_n$,

(4.35)
$$I_c(W, W'; \psi_t) := \int_{B_-} \int_{B_-} \int_{N_c} \Phi_{W_n, W_n'}(b'^{-1}ub)\psi_t(u) W_\varphi(b)\overline{W_{\varphi'}}(b')\, du\, db\, db'.$$

This is the same as

$$I_c(W, W'; \psi_t) = \int_{B_-} \int_{B_-} \int_{N_c} \Phi_{\pi_n(b)W_n, \pi_n(b')W_n'}(u)\psi_t(u)\varphi(b)\overline{\varphi'}(b')\, du\, db\, db'.$$

We *claim* that the triple integral (4.35) converges absolutely. Since $supp(\varphi)$ and $supp(\varphi')$ are compact, by [3, Theorem 2] and [37, Theorem 1.2], there exists a constant C such that, for all $b \in supp(\varphi), b' \in supp(\varphi')$, the matrix coefficients are bounded in terms of the Harish-Chandra spherical function Ξ (cf. [37])

$$|\Phi_{\pi_n(b)W_n, \pi_n(b')W_n'}(g)| \leq C \cdot \Xi(g), \qquad g \in H_n.$$

Hence the triple integral I_c is bounded above by

$$C \int_{B-} |\varphi(b)|\, db \int_{B-} |\varphi'(b')|\, db' \int_{N_c} \Xi(u)\, du.$$

It suffices to prove that $\int_{N_c} \Xi(u)\, du$ is finite. We may write it as

$$\int_{N_c} \Xi(u)\, du = \int_{N_c^{ab}} \left(\int_{N^\circ} \Xi(vu)\, dv \right) du. \tag{4.36}$$

Since Ξ is also a matrix coefficient of a tempered representation, the function $u \in N^{ab} \mapsto \int_{N^\circ} \Xi(vu)\, dv$ is in $L^2(N^{ab})$ by Lemma 4.9 (or rather directly, [33, Lemma 2.7]). Now the integral (4.36) is finite since N_c^{ab} is compact. This proves the claim.

For $\varphi \in \mathscr{C}_c^\infty(B_-)$ and $t \in A_n$, we define $\varphi_t \in \mathscr{C}_c^\infty(B_-)$ by $\varphi_t(b) = \varphi(t^{-1}b)$. For simplicity we denote $W_t = W_n \otimes W_{\varphi_t}$ and $W_t' = W_n' \otimes W_{\varphi_t'}$. Then we have

(4.37)
$$\pi_n(\varphi)W_n(t) = \int_{B-} W_n(tb)\varphi(b) db = |\delta_n(t)| \int_{B-} W_n(b)\varphi(t^{-1}b) db = |\delta_n(t)|\lambda(W_t).$$

We now study the integral I_c as $c \to \infty$. We first substitute $u \mapsto t^{-1}ut$ in (4.35),

$$\begin{aligned} &I_c(W, W'; \psi_t) \\ &= |\delta_n(t)|^{-1} \int_{B_-} \int_{B_-} \int_{N_{c,t}} \Phi_{W_n, W_n'}((tb')^{-1}utb)\psi(u) W_\varphi(b)\overline{W_{\varphi'}}(b')\, du\, db\, db', \end{aligned}$$

where $N_{c,t} := tN_ct^{-1}$. Substitute $b \mapsto t^{-1}b$ and $b' \mapsto t^{-1}b'$,

$$\begin{aligned} &I_c(W, W'; \psi_t) \\ &= |\delta_n(t)| \int_{B_-} \int_{B_-} \int_{N_{c,t}} \Phi_{W_n, W_n'}(b'^{-1}ub)\psi(u) W_\varphi(t^{-1}b)\overline{W_{\varphi'}}(t^{-1}b')\, du\, db\, db' \\ &= |\delta_n(t)| \int_{B_-} \int_{B_-} \int_{N_{c,t}} \Phi_{W_n, W_n'}(b'^{-1}ub)\psi(u) W_{\varphi_t}(b)\overline{W_{\varphi_t'}}(b')\, du\, db\, db'. \end{aligned}$$

Since the triple integral is absolutely convergent and $\psi(u)W_{\varphi_t}(b) = W_{\varphi_t}(ub)$, we could rewrite it by Fubini's theorem as

$$I_c(W, W'; \psi_t) = |\delta_n(t)| \int_{B_-} \int_{N_{c,t}B_-} \Phi_{W_n, W_n'}(b'^{-1}g) W_{\varphi_t}(g)\overline{W_{\varphi_t'}}(b')\, dg\, db'.$$

Now we make a substitution $g \mapsto b'g$ and then interchange the order of integration,

$$\begin{aligned} I_c(W, W'; \psi_t) &= |\delta_n(t)| \int_{B_-} \int_{b'^{-1} N_{c,t} B_-} \Phi_{W_n, W_n'}(g) W_{\varphi_t}(b'g) \overline{W_{\varphi_t'}}(b') \, dg \, db' \\ &= |\delta_n(t)| \int_{B_- N_{c,t} B_-} \Phi_{W_n, W_n'}(g) \left(\int_{B_-} W_{\varphi_t}(b'g) \overline{W_{\varphi_t'}}(b') \, db' \right) dg \\ &= |\delta_n(t)| \int_{B_- N_{c,t} B_-} \Phi_{W_n, W_n'}(g) \Phi_{W_{\varphi_t}, W_{\varphi_t'}}(g) \, dg. \end{aligned}$$

Since the integral

$$\alpha(W_t, W_t') = \int_{H_n(F)} \Phi_{W_n, W_n'}(g) \Phi_{W_{\varphi_t}, W_{\varphi_t'}}(g) \, dg$$

converges absolutely and $H_n \setminus \bigcup_{c \to \infty} B_- N_{c,t} B_-$ is of measure zero, we conclude that for all $t \in A_n$, the limit $\lim_{c \to \infty} I_c(W, W'; \psi_t)$ exists and is given by

$$\lim_{c \to \infty} I_c(W, W'; \psi_t) = |\delta_n(t)| \alpha(W_t, W_t'). \tag{4.38}$$

There is another way to evaluate the limit. We first interchange the order of integration in (4.35) and rewrite it as

$$I_c(W, W'; \psi_t) = \int_{N_c} \Phi_{\pi_n(\varphi) W_n, \pi_n(\varphi') W_n'}(u) \psi_t(u) \, du. \tag{4.39}$$

This integral is the same as [cf. (4.32)]

$$I_c(W, W'; \psi_t) = \int_{N_c^{ab}} \mathcal{F}_{\pi_n(\varphi) W_n, \pi_n(\varphi') W_n'}(u) \psi_t(u) \, du. \tag{4.40}$$

Note that $\mathcal{F}_{\pi_n(\varphi) W_n, \pi_n(\varphi') W_n'} \in L^2(N^{ab})$ by Lemma 4.9 (i). We now view $I_c(W, W'; \cdot)$ as a function of $\psi_t \in \widehat{N^{ab}}$. It follows that $\lim_{c \to \infty} I_c(W, W'; \cdot)$ converges in $L^2(\widehat{N^{ab}})$ to $\widehat{\mathcal{F}}_{\pi_n(\varphi) W_n, \pi_n(\varphi') W_n'}$. But we have proved that $\lim_{c \to \infty} I_c(W, W'; \cdot)$ converges pointwise (for regular characters) almost everywhere. Therefore, for almost all (i.e., except a measure zero set) $t \in A_n$, the pointwise limit is the same as the Fourier transform (cf. [13, Theorem 1.1.11]),

$$\lim_{c \to \infty} I_c(W, W'; \psi_t) = \widehat{\mathcal{F}}_{\pi_n(\varphi) W_n, \pi_n(\varphi') W_n'}(\psi_t).$$

By (ii) of Lemma 4.9, the right hand side is equal to

$$|\delta(t)|^{-1} \pi_n(\varphi) W_n(t) \overline{\pi_n(\varphi') W_n'}(t) = |\delta_n(t)| \lambda(W_t) \overline{\lambda(W_t')}, \tag{4.41}$$

where the equality follows from (4.37). Therefore we have for almost all $t \in A_n$

$$\lim_{c \to \infty} I_c(W, W'; \psi_t) = |\delta_n(t)| \lambda(W_t) \overline{\lambda(W_t')}. \tag{4.42}$$

Comparing (4.42) with (4.38), we have for almost all $t \in A_n$

$$\alpha(W_t, W_t') = \lambda(W_t) \overline{\lambda(W_t')}. \tag{4.43}$$

In particular, in any small open neighborhood of 1 in A_n, there exists t so that the equality (4.43) holds.

Finally, it remains to verify that for some choice of W_n, W_n' and φ, φ', the local period $\lambda(W_t) \overline{\lambda(W_t')}$ does not vanish for t in a small open neighborhood of 1 in A_n.

We choose W_n, W'_n such that $W_n(1)\neq 0, W'_n(1)\neq 0$. Since $W_n|_{B-}$ is a continuous function, there exists $\varphi\in\mathscr{C}_c^\infty(B_-)$ so that

$$\int_{B_-} W_n(b)\varphi(b)db\neq 0.$$

It is easy to see that $t\mapsto\int_{B_-} W_n(b)\varphi(tb)db$ is continuous. Hence for t in a small open neighborhood of 1 in A_n, the integral $\int_{B_-} W_n(b)\varphi(tb)db\neq 0$, or equivalently, by (4.37), $\lambda(W_t)\neq 0$ for $W_t=W_n\otimes W_{\varphi_t}$. Similarly we may achieve $\lambda(W'_t)\neq 0$ for t in a small open neighborhood of 1. This completes the proof. □

Corollary 4.11. *The case (1) (i.e., for a split v) of Theorem* 4.6 *holds.*

Proof. This follows from Lemma 4.8 and Prop. 4.10. □

5. The totally definite case

We now prove part (2) of Theorem 1.2, assuming Theorem 4.6. We hence assume that $G(F_\infty)$ is compact. Equivalently, F is a totally real field, E is a CM extension and the Hermitian spaces W, V are positive definite at every Archimedean place v of F. As in the proof of part (1) of Theorem 1.2, we may assume that the local invariant form $\alpha_v\neq 0$ for all v. We may further assume that the global period $\mathscr{P}\neq 0$ so that the global spherical character J_π does not vanish (otherwise $\mathscr{L}(1/2,\pi)=0$ and the result holds trivially). We then have

$$J_\pi(f)=\mathscr{C}_\pi\prod_v J^\natural_{\pi_v}(f_v) \tag{5.1}$$

for a nonzero constant $\mathscr{C}_\pi$.

Let us recall that in the disjoint union of (4.6) we take all isomorphism classes of n-dimensional Hermitian spaces W_v. When $F_v\simeq\mathbb{R}$ is non-Archimedean and $E_v\simeq\mathbb{C}$, the isomorphism classes of such W_v are indexed by the signature (p,q) of W_v. We denote them by $W_{(p,q),v}$. Then the two definite (positive or negative) spaces correspond to $(p,q)=(n,0),(0,n)$. Only when $W_v=W_{(n,0),v}$ is the positive definite one, is the space V_v also positive definite [by (4.5)], or equivalently the group $G_{W_v}(F_v)$ is compact. Let $G'(F_v)_{rs,(n,0)}$ be the open subset of the regular semisimple locus $G'(F_v)_{rs}$ corresponding to the positive definite one $G_{W_{(n,0),v}}(F_v)_{rs}$ in the disjoint union (4.6). In our case, our $G(F_v)$ is isomorphic to $G_{W_{(n,0),v}}(F_v)$ for all $v|\infty$.

For every $v|\infty$, we now choose a test function f_v supported in the regular semisimple locus $G_{W_{(n,0),v}}(F_v)_{rs}$. Then there exists a smooth transfer f'_v supported in $G'(F_v)_{rs,(n,0)}$. Since the representation π_v must be finite dimensional and we are assuming that $\alpha_v\neq 0$, our choice of f_v can be made so that $J^\natural_{\pi_v}(f_v)\neq 0$.

For non-Archimedean places v, we choose f_v and its smooth transfer f'_v as in the proof of case (1). Particularly, $J^\natural_{\pi_v}(f_v)\neq 0$ for all non-Archimedean v. Then for such test functions $f=\otimes f_v$ and $f'=\otimes f'_v$, we again have, by Theorem 4.3,

$$J_\pi(f)=2^{-2}L(1,\eta)^{-2}I_{\pi_E}(f').$$

By Prop. 3.6, the right hand side is equal to

$$c\cdot\mathscr{L}(1/2,\pi)\prod_v I^\natural_{\pi_{E,v}}(f'_v),$$

for some constant c independent of π. Now fix an arbitrary $v_0|\infty$, and we further assume that $J^\natural_{\pi_v}(f_v)\neq 0$ for $v\neq v_0$. By comparison with (5.1), there exists a constant $b_{\pi_{v_0}} \neq 0$, such that for all f_{v_0} with regular semisimple support and its smooth transfer f'_{v_0} supported in $G'(F_{v_0})_{rs,(n,0)}$, we have

$$I_{\Pi_{v_0}}(f'_{v_0}) = b_{\pi_{v_0}} J_{\pi_{v_0}}(f_{v_0}).$$

Now we do not know how to evaluate b_{π_v} for $v|\infty$. Nevertheless, the same argument as the proof of Prop. 4.5 shows that

$$\frac{|\mathscr{P}(\phi)|^2}{\langle\phi,\phi\rangle_{Pet}} = c_{\pi_\infty} 2^{-2} \mathscr{L}\left(\frac{1}{2},\pi\right) \prod_v \frac{\alpha_v^\natural(\phi_v,\phi_v)}{\langle\phi_v,\phi_v\rangle_v},$$

where the constant $c_{\pi_\infty} = \prod_{v|\infty} c_{\pi_v}$ and

$$c_{\pi_v} = b_{\pi_v} \kappa_v^{-1} L(1,\eta_v),$$

where κ_v is the constant in Conjecture 4.4.

Part 2. Local theory

In the rest of the paper, we prove the remaining parts of Theorem 4.6.

6. Harmonic analysis on Lie algebra

We establish some basic results to prove the identity between local characters, Theorem 4.6 for a nonsplit non-Archimedean place v.

6.1. **Relative regular nilpotent elements in M_{n+1}.** Let F be any field. The group H_n, viewed as a subgroup of H_{n+1}, acts on M_{n+1} by conjugation. Write

$$X = \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in M_{n+1}.$$

The ring of invariants for this action is freely generated by either

$$(6.1) \qquad (-1)^{i-1}\mathrm{tr} \wedge^i X, \quad e_{n+1}X^j e^*_{n+1}, \qquad 1 \le i \le n+1, 1 \le j \le n,$$

or

$$(6.2) \qquad (-1)^{i-1}\mathrm{tr} \wedge^i A, \quad vA^j u, \quad w, \quad 1 \le i \le n, 0 \le j \le n-1.$$

We define a matrix

$$(6.3) \qquad \delta_+(X) := (A^{n-1}u, A^{n-2}u, \ldots, u) \in M_n(F)$$

and its determinant

$$(6.4) \qquad \Delta_+(X) = \det(\delta_+(X)).$$

Similarly, we define

$$\delta_-(X) := (v, vA, \ldots, vA^{n-1}) \in M_n(F), \qquad \Delta_-(X) = \det(\delta_-(X)),$$

and

$$\Delta := \Delta_+\Delta_-.$$

Clearly we have for $X \in M_{n+1}(F)$ and $h \in \mathrm{GL}_n(F)$

$$(6.5) \qquad \delta_+(hXh^{-1}) = h\delta_+(X), \qquad \delta_-(hXh^{-1}) = \delta_-(X)h^{-1}.$$

The H_n-nilpotent cone $\mathcal{N}$ is defined to be the zeros of all of the above invariant functions on M_{n+1}. An element in M_{n+1} is called *H_n-regular* (or *regular* if no confusion arises) if its stabilizer is trivial. Denote

$$\xi_{n+1,+} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{n+1}(F), \tag{6.6}$$

and $\xi_{n+1,-}$ its transpose. If no confusion arises, we simply denote them by $\xi_{\pm}$. Clearly $\xi_{\pm}$ are regular nilpotent.

Lemma 6.1. *Let $X \in \mathcal{N}$. The following statements are equivalent:*

(1) *X is regular nilpotent.*
(2) *ξ is H_n-equivalent to ξ_+ or ξ_-.*
(3) *$\Delta_+(X) \neq 0$ or $\Delta_-(X) \neq 0$.*

In particular, the orbit of ξ_+ is open in $\mathcal{N}$.

Proof. Let $X = \begin{pmatrix} A & u \\ v & 0 \end{pmatrix}$ be an H_n-nilpotent element. We then have $A^n = 0$ and $vA^iu = 0$, for $i = 0, 1, \ldots, n-1$. It follows that $vA^iu = 0$ for all $i \in \mathbb{Z}_{\geq 0}$. Let r be the dimension of subspace of the $(n \times 1)$ column vectors spanned by $A^iu, i = 0, 1, \ldots, n-1$, and similarly r' the dimension of the subspace spanned by $vA^i, i = 0, 1, \ldots, n-1$. Clearly, we have an inequality $r + r' \leq n$.

$1 \Rightarrow 2$. It suffices to show that, if X is regular unipotent, then either r or r' is equal to n. Indeed, for example, if $r = n$, then $r' = 0$ (i.e., $v = 0$) and the column vectors $A^iu, i = 0, 1, \ldots, n-1$ form a basis of the n-dimensional space of column vectors. Then $\{e^*, Xe^*, \ldots, X^{n-1}e^*\}$ form a basis of the $(n+1)$-dimensional column vectors [recall that e^* is the transpose of $e = (0, \ldots, 0, 1) \in M_{1,n+1}(F)$]. In terms of this new basis we see that X becomes ξ_+.

Now suppose that $r, r' < n$. Clearly if $r = r' = 0$, X must have a positive dimensional stabilizer, hence is nonregular. We now assume that $0 < r < n$. Let L be the subspace spanned by $A^iu, i = 0, 1, \ldots, r-1$. It is easy to see that this is the same as the space spanned by $A^iu, i = 0, 1, \ldots, n-1$. We write the column vector spaces $F^n = L \oplus L'$ for a subspace L'. Then in terms of the basis of L given by $A^iu, i = 0, 1, \ldots, r-1$, we may write u as $(0, 0, \ldots, 1, 0, 0, \ldots, 0)^t$ where only the rth entry is nonzero and may be assumed to be equal to one, and

$$A = \begin{pmatrix} Y & B \\ 0 & Z \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_r(F).$$

Then $A^iu = (0, 0, \ldots, 1, 0, 0, \ldots, 0)^t$ where only the $(r-i)$th entry is one, $i = 0, 1, \ldots, r-1$. Hence the conditions $vA^iu = 0$ $(0 \leq i \leq n-1)$ imply that v is of the form $(0, 0, \ldots, 0, *, \ldots, *)$ where the first r entries are all zero.

Now we consider

$$h = \begin{pmatrix} 1_r & Q \\ 0 & 1_{n-r} \end{pmatrix} \in \mathrm{GL}_n(F).$$

Clearly the matrix $h^{-1}Xh$ is an element of the same form with B replaced by $B + YQ - QZ$. Hence the stabilizer of X at least contains all h with Q satisfying $YQ - QZ = 0$. Define $\varphi \in \mathrm{End}(M_{r,n-r}(F))$ by $Q \mapsto YQ - QZ$. We claim that the dimension of the kernel $\mathrm{Ker}(\varphi)$ is positive. Clearly the dimension of $\mathrm{Ker}(\varphi)$ depends only on the conjugacy class of Z in $M_{n-r}(F)$. Since A is nilpotent, so is Z. We thus can assume that Z is a Jordan canonical form. The endomorphism φ cannot be surjective since $M_{r,n-r}(F) \neq 0$ $(0 < r < n)$ and every $YQ - QZ$ must have zero as its lower left entry. This proves the claim, and hence the stabilizer of such X cannot be trivial.

$2 \Rightarrow 3$. This is clear since we have $\delta_+(hXh^{-1}) = h\delta_+(X)$ and $\delta_-(hXh^{-1}) = \delta_-(X)h^{-1}$ by (6.5).

$3 \Rightarrow 1$. Note that $\Delta_+(X) \neq 0$ is equivalent to $\delta_+(X) \in \mathrm{GL}_n(F)$. The latter property implies that the stabilizer (under the H_n action) of any $X \in M_{n+1,+}$ must be trivial. Indeed, if $hXh^{-1} = X$, we have $\delta_+(X) = \delta_+(hXh^{-1}) = h\delta_+(X)$; hence $h = 1$ and similarly for $\Delta_-(X) \neq 0$. □

6.2. **A regular section.** Denote

$$M_{n+1,+} := \{X \in M_{n+1} | \Delta_+(X) \neq 0\}.$$

Note that every element in $M_{n+1,+}$ is regular (cf. the proof of "$3 \Rightarrow 1$" of Lemma 6.1). We shall write $\mathscr{X} = \mathbb{A}^n \times \mathbb{A}^{n+1}$,[9] the affine space of dimension $2n+1$. Then the second set of generators (6.2) defines a morphism that is constant on H_n orbits

$$\pi : M_{n+1} \longrightarrow \mathscr{X} = \mathbb{A}^n \times \mathbb{A}^{n+1},$$
$$\begin{pmatrix} A & u \\ v & w \end{pmatrix} \mapsto (a, b),$$

where $a = (a_1, \dots, a_n), b = (b_0, \dots, b_n)$, $a_i = (-1)^{i-1}\mathrm{tr} \wedge^i A$, $b_0 = w$, and $b_i = vA^{i-1}u$ for $1 \leq i \leq n$. We say that $x \in \mathscr{X}$ is *regular semisimple* if one element (and hence all) in $\pi^{-1}(x)$ is H_n-regular semisimple.

Now we define a section of the morphism $\pi : M_{n+1} \to \mathscr{X}$,

$$\sigma : \mathscr{X} \longrightarrow M_{n+1}$$
$$(a, b) \mapsto \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 \\ a_2 & 0 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 1 & 0 \\ a_n & 0 & 0 & 0 & 1 \\ b_n & \cdots & \cdots & b_1 & b_0 \end{pmatrix}.$$

We note that ξ_+ is precisely the image of $0 \in \mathscr{X}$ under σ.

$$\begin{tikzcd} M_{n+1} \arrow[d, "\pi"] \\ \mathscr{X} = M_{n+1//H_n} \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

Lemma 6.2. *The morphism σ is a section of π, i.e.,*

$$\sigma \circ \pi = id.$$

The image of σ lies in $M_{n+1,+}$ (in particular, σ is a regular *section, in the sense that the image $\sigma(a, b)$ is always H_n-regular).*

[9]We use $\mathbb{A}$ in this section only to denote the affine line.

Proof. It is easy to check that

$$\det\left(T\cdot 1_n+\begin{pmatrix} a_1 & 1 & 0 & 0\\ a_2 & 0 & 1 & 0\\ \cdots & 0 & 0 & 1\\ a_n & 0 & 0 & 0\end{pmatrix}\right)=T^n+a_1T^{n-1}-a_2T^{n-2}+\cdots+(-1)^{n-1}a_n,$$

and the b invariants of $\sigma(a,b)$ are $(b_0,b_1,b_2,\dots,b_n)$. This shows that σ is a section of π. To see that the image of σ lies in $M_{n+1,+}$, we note that for any $(a,b)\in\mathscr{X}$ we have

$$\delta_+(\sigma(a,b))=1_n. \tag{6.7}$$

□

Proposition 6.3. *We have an H_n-equivariant morphism*

$$\begin{aligned}\iota:\mathrm{GL}_n\times\mathscr{X}&\to M_{n+1,+}\\ (h,(a,b))&\mapsto h\sigma(a,b)h^{-1},\end{aligned}$$

where the group H_n acts on the left hand side by a left translation on the first factor, and trivially on the second factor. Moreover, the morphism ι is an isomorphism with its inverse given by $(\delta_+,\pi|_{M_{n+1,+}})$.

Proof. It suffices to prove that $(\delta_+,\pi)\circ\iota=id$ and $\iota\circ(\delta_+,\pi)=id$. To show the first identity we note that the invariants of $h\sigma(a,b)h^{-1}$ [being the same as $\sigma(a,b)$] are (a,b). Hence it is enough to show that $\delta_+(\iota(h,(a,b))=h$. This follows from the fact that $\delta_+(\sigma(a,b))=1_n$ [cf. (6.7)] and $\delta_+(hXh^{-1})=h\delta_+(X)$ by (6.5).

Now we show the second identity. Let $X=\begin{pmatrix}A & u\\ v & w\end{pmatrix}\in M_{n+1,+}$. Let $(a,b)=\pi(X)$ and $h=\delta_+(X)=(A^{n-1}u,A^{n-2}u,\dots,u)\in H_n$. Denote $\iota\circ(\delta_+,\pi)(X)=\begin{pmatrix}A' & u'\\ v' & w'\end{pmatrix}$. Clearly $w=w'$. By the first identity, the elements $\iota\circ(\delta_+,\pi)(X)$ and X have the same invariants. In particular,

$$\det(T\cdot 1_n+A)=T^n+\sum_{i=1}^n(-1)^{i-1}a_iT^{n-i},$$

and therefore

$$A^n=\sum_{i=1}^n a_iA^{n-i}.$$

This implies that

$$A\delta_+(X)=(A^nu,A^{n-1}u,\dots,Au)=\delta_+(X)\begin{pmatrix} a_1 & 1 & 0 & 0\\ a_2 & 0 & 1 & 0\\ \cdots & 0 & 0 & 1\\ a_n & 0 & 0 & 0\end{pmatrix}.$$

Since $\delta_+(X)=h$ is invertible, we obtain

$$A=h\begin{pmatrix} a_1 & 1 & 0 & 0\\ a_2 & 0 & 1 & 0\\ \cdots & 0 & 0 & 1\\ a_n & 0 & 0 & 0\end{pmatrix}h^{-1}=A'.$$

Obviously we have $u = \delta_+(X)e_n^* = he_n^* = u'$ $[e_n^* = (0, 0, \ldots, 0, 1)^t]$. Finally since $b_i = vA^{i-1}u$, $(b_n, b_{n-1}, \ldots, b_1) = v\delta_+(X)$, we have

$$v = (b_n, b_{n-1}, \ldots, b_1)\delta_+(X)^{-1} = (b_n, b_{n-1}, \ldots, b_1)h^{-1} = v'.$$

This completes the proof of the second identity. □

Similarly we define a variant $\sigma' : \mathscr{X} = \mathbb{A}^n \times \mathbb{A}^{n+1} \to M_{n+1}$ by

$$\sigma'(a, b) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n-1} & \cdots & a_1 & 1 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 \end{pmatrix}. \tag{6.8}$$

To facilitate the exposition, we introduce the following.

Definition 6.4. Consider a morphism between two affine spaces:

$$\phi : \mathbb{A}^m = \mathrm{Spec}F[x_1, \ldots, x_m] \longrightarrow \mathbb{A}^m = \mathrm{Spec}F[y_1, \ldots, y_m]$$

with induced morphism $\phi^* : F[y_1, \ldots, y_m] \longrightarrow F[x_1, \ldots, x_m]$. We say that ϕ is *triangular* if we have, possibly after reordering the coordinates,

$$\phi^*(y_i) = \pm x_i + \varphi_i(x_1, \ldots, x_{i-1}), \qquad 1 \le i \le m,$$

where $\varphi_i(x_1, \ldots, x_{i-1}) \in F[x_1, \ldots, x_{i-1}]$ is a polynomial of $x_1, \ldots, x_{i-1}$.

It is easy to see that if ϕ is triangular, then it is an isomorphism and its inverse is triangular, too. Moreover, the Jacobian factor of a triangular morphism is equal to ± 1.

Corollary 6.5. *The following morphism is an isomorphism:*

$$\iota' : H_n \times \mathscr{X} \to M_{n+1,+}$$
$$(h, (a, b)) \mapsto h\sigma'(a, b)h^{-1}.$$

Moreover, the induced morphism $\pi \circ \sigma' : \mathscr{X} \to \mathscr{X}$ is triangular, and in particular an isomorphism.

Proof. The proof of the first part follows the same line as the previous one: it suffices to show that for an arbitrarily $X = \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in M_{n+1,+}$, we may solve for (a, b) and h uniquely in terms of the polynomials of the entries of X,

$$h\sigma'(a, b)h^{-1} = X. \tag{6.9}$$

We proceed in three steps.

Step 1. For the a component of $\sigma'(a, b)$, we have $a_i = (-1)^{i-1}\mathrm{tr} \wedge^i A$.

Step 2. By (6.5), we have $\delta_+(X) = h\delta_+(\sigma'(a, b))$. Note that the matrix $\delta_+(\sigma'(a, b))$ lies in $N_{n,-}$, and it depends only on a (but not on b). Combined with Step 1, we see that it can be expressed in terms of X,

$$h = \delta_+(X)\delta_+(\sigma'(a, b))^{-1}. \tag{6.10}$$

Step 3. In (6.9), the last row of $\sigma'(a, b)$, i.e., $(b_n, \ldots, b_1, b_0)$, is equal to (vh, w). Combining with Step 2 we complete the proof.

To show the second part, by Step 1 we may write $\pi \circ \sigma'(a, b') = (a, b)$. By computing the b invariants of $\sigma'(a, b')$, we say that $b_0 = b'_0$, and for each $i \geq 1$, $b_i - b'_i$ is a polynomial of $a, b'_1, \ldots, b'_{i-1}$. This also shows that b_i is a polynomial of $a, b'_1, \ldots, b'_{i-1}$. Therefore $\pi \circ \sigma' : \mathscr{X} \to \mathscr{X}$ is a triangular morphism. $\square$

We will need to consider the restriction of ι' to some closed subvarieties. We denote by $\mathcal{W}$ the subvariety of M_{n+1} consisting of matrices X of the following form:

$$(6.11) \qquad X = \begin{pmatrix} * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \\ * & * & * & * & * \end{pmatrix} \in M_{n+1}.$$

Denote by $\mathcal{V}$ the subvariety of $\mathcal{W}$ consisting of X of the same form but with the last row identically zero. Then we have a natural projection $p : \mathcal{W} \to \mathcal{V}$.

Lemma 6.6. (1) *The variety $\mathcal{W}$ is a subvariety of $M_{n+1,+}$ and the preimage of $\mathcal{W}$ under ι' is the product $N_{n,-} \times \mathscr{X}$.*

(2) *For every $(a, b) \in \mathscr{X}$, we define a morphism $\nu_{(a,b)} : N_{n,-} \longrightarrow \mathcal{W}$ by*

$$(6.12) \qquad \nu_{(a,b)}(u) = u\sigma'(a, b)u^{-1}, \qquad u \in N_{n,-}.$$

Then the composition $\nu'_{(a,b)} := p \circ \nu_{(a.b)} : N_{n,-} \longrightarrow \mathcal{V}$ is an isomorphism with Jacobian equal to ± 1.

Proof. Let $X = \begin{pmatrix} A & u \\ v & w \end{pmatrix}$ be in $\mathcal{W}$. It is easy to verify the following properties about $\delta_+(X)$:

(i) $\delta_+(X) \in N_{n,-}$.

(ii) $\delta_+(X)$ depends only on the last $n - 1$ columns of A, but not on the first column.

(iii) For each i, $0 \leq i \leq n - 1$, the $(n - i)$th column of $\delta_+(X)$ is equal to the sum of the $(n - i + 1)$th column of A plus a column vector whose entries are polynomials depending only on the last $(i - 1)$ columns of A.

By (i), such X lies in $M_{n+1,+}$, and hence $\mathcal{W} \subset M_{n+1}$. Setting $X = \sigma'(a, b)$ shows that $\delta_+(\sigma'(a, b))$ lies in $N_{n,-}$ and depends only on a. Let $(h, (a, b))$ be the preimage $\iota'^{-1}(X)$. By (6.10) in the proof of Prop. 6.3, we have

$$h = \delta_+(X)\delta_+(\sigma'(a, b))^{-1} \in N_{n,-}.$$

Hence the preimage of $\mathcal{W}$ is contained in $N_{n,-} \times \mathscr{X}$. Since $\mathcal{W}$ is preserved under the conjugation by $N_{n,-}$ and contains the image of σ, it follows that the preimage of $\mathcal{W}$ is exactly $N_{n,-} \times \mathscr{X}$. This proves part (1) of the lemma. Alternatively, we may identify $\mathcal{W}$ with the variety consisting of $X \in M_{n+1}$ such that $\delta_+(X) \in N_{n,-}$.

To show part (2), we denote by $\mathcal{W}_{(a,b)}$ the image of $N_{n,-} \times \{(a, b)\}$ under ι'. Consider an auxiliary subvariety $\mathcal{V}'$ of $\mathcal{W}$ with the first column and the last row both being zero. Let $p' : \mathcal{W} \to \mathcal{V}$ be the natural projection. By the property (iii) above, the composition $p' \circ \nu : N_{n,-} \to \mathcal{V}'$ is a triangular morphism. Thus the restriction of the projection p' to $\mathcal{W}_{(a,b)}$ induces an isomorphism $p'_{(a,b)} : \mathcal{W}_{(a,b)} \to \mathcal{V}'$. To prove part (2), it remains to show that the morphism $p|_{\mathcal{W}_{(a,b)}} \circ (p'_{(a,b)})^{-1} : \mathcal{V}' \to \mathcal{V}$ is triangular.

Now we let $X = \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in \mathcal{W}_{(a,b)}$ with

$$A = \begin{pmatrix} \alpha_1 & 1 & 0 & 0 & 0 \\ \alpha_2 & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 \\ * & * & * & * & 1 \\ \alpha_n = \beta_n & \beta_{n-1} & * & * & \beta_1 \end{pmatrix} \in M_n.$$

We denote by $\widetilde{A}$ the square matrix obtained by deleting the first and the last row/column of A. By computing the coefficients of the characteristic polynomial of A, we have the following:

(♠) *For each i, $1 \le i \le n$, the sum $\alpha_i + \beta_i$ is a polynomial of $\alpha_1, \dots, \alpha_{i-1}$, $\beta_1, \dots, \beta_{i-1}$, $a_1, \dots, a_i$ and the entries of $\widetilde{A}$.*

By induction on i, α_i is a polynomial of $\beta_1, \dots, \beta_i$, $a_1, \dots, a_i$ and the entries of $\widetilde{A}$. The same statement holds if we replace α by β everywhere. Note that the last row of $X \in \mathcal{W}_{(a,b)}$ is also determined by the entries $\alpha_1, \dots, \alpha_n, a_1, \dots, a_n$ and $\widetilde{A}$. It follows that the morphism $p|_{\mathcal{W}_{(a,b)}} \circ (p'_{(a,b)})^{-1} : \mathcal{V}' \to \mathcal{V}$ is triangular. This completes the proof. $\square$

A by-product of the proof is the following corollary.

Corollary 6.7. *Let $X = \begin{pmatrix} A & u \\ v & w \end{pmatrix} \in \mathcal{W}$. Then every entry of the last row of A is a polynomial of the first $n-1$ rows of A and the coefficients of the characteristic polynomial of A.*

Proof. In the proof of the previous Lemma 6.6, the β_i's are polynomials of the first $n-1$ rows of A, and the coefficient a_i's of the characteristic polynomial of A. $\square$

We also have an easier statement about the upper unipotent N_n acting on $\xi_{n+1,+}$.

Lemma 6.8. *Denote by $\mathcal{V}_+$ the subvariety of M_{n+1} consisting of X of the following form:*

$$X = \begin{pmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ \cdots & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{n+1}.$$

Define a morphism

$$\begin{aligned} \nu_+ : N_n &\to \mathcal{V}_+ \\ u &\mapsto u\xi_{n+1,+}u^{-1}. \end{aligned}$$

Then ν_+ is triangular.

Proof. Similar to the previous one. We omit the detail. $\square$

For later use in §8, we take the transpose of the morphism σ' and denote it by ϱ,

$$\varrho(a,b) = \sigma'(a,b)^t. \tag{6.13}$$

6.3. **Regular nilpotent orbital integral.** We now assume that F is a *p-adic local field.* We now define the (H_n, η)-orbital integral of a regular nilpotent orbit. Since the orbits of $\xi_\pm$ are not closed, we need to regularize the orbital integral. We consider the following integral for $s \in \mathbb{C}$, $X \in M_{n+1}(F)$:

$$O(X, f, s) = \int_{H_n(F)} f(X^h)\eta(h)|\det(h)|^s \, dh, \qquad f \in \mathscr{C}_c^\infty(M_{n+1}(F)). \tag{6.14}$$

It is absolutely convergent for all $s \in \mathbb{C}$ if X is regular semisimple in which case we denote

$$O(X, f) = O(X, f, 0). \tag{6.15}$$

Lemma 6.9. *The integral $O(\xi_\pm, f, s)$ converges absolutely when $\mathrm{Re}(s) > 1 - \frac{1}{n}$ and extends to a meromorphic function in $s \in \mathbb{C}$ with at most simple poles at*

$$s = 1 - \frac{1}{\ell} + \frac{2\pi i}{\log q}\mathbb{Z},$$

for even integers ℓ with $1 < \ell \leq n$. Here q is the cardinality of the residue field $\mathcal{O}_F/(\varpi)$.

Proof. We use the Iwasawa decomposition of $H_n(F) = KAN$. Define

$$f_K(X) = \int_K f(kXk^{-1}) \, dk.$$

By Iwasawa decomposition on $H_n(F)$, we have

$$\int_{A(F)} \int_{N(F)} f_K(au\xi_+ u^{-1}a^{-1})\eta(a)|a|^s|\delta(a)| \, du \, da,$$

where δ is the modular character

$$\delta(a) = a_1^{n-1}a_2^{n-3}\cdots a_n^{-(n-1)}, \qquad a = diag[a_1, \ldots, a_n].$$

By Lemma 6.8, this is

$$\int_{A(F)} \int_{\mathcal{V}_+(F)} f_K(axa^{-1})\eta(a)|a|^s \, dx \, da.$$

Replacing $x_{\ell j}$ by $x_{\ell j}a_j a_\ell^{-1}$, $1 \leq \ell \leq j-2 \leq (n+1)-2$, and setting $a_{n+1} = 1$, we may partially cancel the factor $|\delta(a)|$,

$$\int_{A(F)} \int_{\mathcal{V}_+(F)} f_K \begin{pmatrix} 0 & \frac{a_1}{a_2} & x_{13} & x_{14} & * \\ 0 & 0 & \frac{a_2}{a_3} & * & * \\ 0 & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & 0 & 0 & \frac{a_n}{a_{n+1}} \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \eta(a)|a|^s|a_2a_3\cdots a_n|^{-1} \, dx \, da.$$

Substitute $b_\ell := a_\ell/a_{\ell+1}, 1 \leq \ell \leq n$,

$$\int_{A(F)} \int_{\mathcal{V}_+(F)} f_K \begin{pmatrix} 0 & b_1 & x_{13} & x_{14} & * \\ 0 & 0 & b_2 & * & * \\ 0 & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & 0 & 0 & b_n \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \eta(b_1b_3\cdots)|\prod_{\ell=1}^n b_\ell^\ell|^{-1+s} \prod_{\ell=1}^n db_\ell \, dx.$$

(Note db_ℓ is the additive Haar measure; cf. §2.) Now it is clear that the integral converges absolutely if $\mathrm{Re}(\ell(-1+s)) > -1$ for all $\ell = 1, 2, \ldots, n$, or equivalently

$\mathrm{Re}(s) > 1 - \frac{1}{n}$. By Tate's local zeta integral, the integral extends meromorphically to $s \in \mathbb{C}$ with at most simple poles at those s modulo $\frac{2\pi i}{\log q}\mathbb{Z}$ satisfying one of the following:

$$\ell(-1+s) = -1, \qquad \ell = 2, 4, \ldots, 2[n/2].$$

Namely $s = 1 - \frac{1}{\ell} + \frac{2\pi i}{\log q}\mathbb{Z}$, for even ℓ with $1 < \ell \le n$. □

Definition 6.10. For $f \in \mathscr{C}_c^\infty(M_{n+1}(F))$, we define the regular nilpotent orbital integral $O(\xi_\pm, f)$, also denoted by $\mu_{\xi_\pm}(f)$, as

$$\mu_{\xi_\pm}(f) = O(\xi_\pm, f) := O(\xi_\pm, f, 0).$$

This defines an (H_n, η)-invariant distribution on $M_{n+1}(F)$.

The two propositions below will not be used later on. They are interesting in their own right and provide heuristics for the admissible functions in the remaining sections of this paper.

Proposition 6.11. *The intersection $M_{n+1,+}(F) \cap \mathcal{N}$ is equal to the H_n orbit of ξ_+. In particular, for a function $f \in \mathscr{C}_c^\infty(M_{n+1}(F))$ supported on $M_{n+1,+}(F)$, any distribution on $M_{n+1}(F)$ supported in the closed subset $\mathcal{N} \setminus (H_n \cdot \xi_+)$ of $M_{n+1}(F)$ vanishes on f.*

Proof. Since ξ_+ is precisely the image of $0 \in \mathscr{X}$ under σ, the H_n orbit of ξ_+ is then the image of $H_n \times \{0\}$ under ι. We also have $M_{n+1,+} \cap \mathcal{N} = (\pi|_{M_{n+1,+}})^{-1}(0)$, the fiber of $0 \in \mathscr{X}$ under $\pi|_{M_{n+1,+}}$. By Proposition 6.3, the fiber $(\pi|_{M_{n+1,+}})^{-1}(0)$ is precisely the image of $H_n \times \{0\}$ under ι. This proves the first assertion. The "In particular" part is clear from the definition of the support of a distribution. □

Proposition 6.12. *For any $f \in \mathscr{C}_c^\infty(M_{n+1,+}(F)) \subset \mathscr{C}_c^\infty(M_{n+1}(F))$, the orbital integral*

$$\phi_f(x) := O(\sigma(x), f)$$

defined for regular semisimple $x \in \mathscr{X}$ [cf. (6.15)] extends to a locally constant function with compact support on $\mathscr{X}$ [i.e., $\phi_f \in \mathscr{C}_c^\infty(\mathscr{X})$]. Conversely, given any function ϕ in $\mathscr{C}_c^\infty(\mathscr{X})$, there exists $f \in \mathscr{C}_c^\infty(M_{n+1,+}(F))$ such that $O(\sigma(x), f) = \phi(x)$ for all regular semisimple x.

Proof. The orbital integral (6.15) is given by

$$O(\sigma(x), f) = \int_{H_n} f(h\sigma(x)h^{-1})\eta(h)\, dh.$$

By the H_n-equivariant isomorphism $\iota : H_n(F) \times \mathscr{X}(F) \to M_{n+1,+}(F)$, corresponding to f we have an element denoted by f' in $\mathscr{C}_c^\infty(H_n \times \mathscr{X})$, defined by

$$f'(h, x) = f(h\sigma(x)h^{-1}), \qquad h \in H_n, x \in \mathscr{X},$$

with the property that

$$O(\sigma(x), f) = \int_{H_n} f'(h, x)\eta(h)\, dh.$$

The integral on the right hand side is clearly absolutely convergent for all $x \in \mathscr{X}$ and defines an element in $\mathscr{C}_c^\infty(\mathscr{X})$. The converse is clearly now by the isomorphism ι. □

Remark 11. The proof also shows that if f is supported on $M_{n+1,+}(F)$, the integral $O(\xi_+, f, s)$ [cf. (6.14)] converges absolutely for all $s \in \mathbb{C}$.

6.4. **Orbital integrals on $\mathfrak{s}_{n+1}$.** We will need to consider the induced H_n action on the tangent space $\mathfrak{s}_{n+1}$ at 1 of the symmetric space S_{n+1},

$$\mathfrak{s}_{n+1}(F) = \{X \in M_{n+1}(E) | X + \overline{X} = 0\}.$$

Fixing a choice of nonzero $\tau \in E^-$, we have an isomorphism

$$M_{n+1}(F) \simeq \mathfrak{s}_{n+1}(F), \tag{6.16}$$

defined by $X \mapsto \tau X$. In particular, we will abuse the notation $\xi_\pm$ to denote $\tau\xi_\pm$ if we want to consider the regular unipotent orbit on $\mathfrak{s}_{n+1}$. We may extend the definitions of σ, ϱ and the orbital integrals to the setting of $\mathfrak{s}_{n+1}$ via the isomorphism (6.16). Then it is clear how to extend the results from the setting of $M_{n+1}(F)$ to the setting of $\mathfrak{s}_{n+1}$.

7. Smoothing local periods

7.1. **Convolutions.** We introduce some abstract notions for the use of this and the next section. Let F be a p-adic local field and G the F-points of some reductive group. We consider the space of $\mathscr{C}_c^\infty(G)$ with an (anti-)involution $*$ defined by

$$f^*(g) := \overline{f(g^{-1})}, \qquad f \in \mathscr{C}_c^\infty(G). \tag{7.1}$$

We will also use the other (anti-)involution defined by

$$f^\vee(g) = f(g^{-1}). \tag{7.2}$$

Let dg be a Haar measure on G. Let H be a unimodular (closed) subgroup of G and dh a Haar measure H. We define a left and a right action of $\mathscr{C}_c^\infty(H)$ on $\mathscr{C}_c^\infty(G)$ as follows: for $f \in \mathscr{C}_c^\infty(G)$ and $\phi \in \mathscr{C}_c^\infty(H)$, we define *convolutions* $f * \phi$ and $\phi * f$ both in $\mathscr{C}_c^\infty(G)$,

$$(f * \phi)(g) = \int_H f(gh^{-1})\phi(h)\, dh \tag{7.3}$$

and

$$(\phi * f)(g) = \int_H \phi(h) f(h^{-1}g)\, dh = \int_H \phi(h^{-1}) f(hg)\, dh.$$

This also applies to the case $H = G$. Then we have

$$(f * \phi)^* = \phi^* * f^*, \qquad (\phi * f)^* = f^* * \phi^*.$$

If we have two closed unimodular subgroups H_1, H_2, we could iterate the definition: for example, for $f \in \mathscr{C}_c^\infty(G)$ and $\phi_i \in \mathscr{C}_c^\infty(H_i)$, we define

$$\phi_1 * \phi_2 * f := \phi_1 * (\phi_2 * f) \in \mathscr{C}_c^\infty(G).$$

Now if we have a smooth representation π of G (hence its restriction to H is a smooth representation as well), as usual we define $\pi(f)$ and $\pi(\phi)$ to be the endomorphisms of π,

$$\pi(f) = \int_G f(g)\pi(g)\, dg, \qquad \pi(\phi) = \int_H \phi(h)\pi(h)\, dh.$$

Then we have $\pi(f * \phi) = \pi(f)\pi(\phi) \in \mathrm{End}(\pi)$ and so on. If π has a G-invariant inner product $\langle \cdot, \cdot \rangle$, we then have

$$\langle \pi(f)u, u' \rangle = \langle u, \pi(f^*)u' \rangle, \qquad u, u' \in \pi.$$

The questions addressed in this section can be abstracted as follows (cf. [26, §2]). Let π be a smooth admissible representation of G. The algebraic dual space $\pi^* := \mathrm{Hom}(\pi, \mathbb{C})$ is usually much larger than the congragredient $\widetilde{\pi}$ (the subspace of π^* consisting of smooth linear functional, i.e., K-finite vectors in π^* for some open compact K). Very often we will be interested in some distinguished element called ℓ in $\pi^* \setminus \widetilde{\pi}$ (the set of nonsmooth linear functionals). Then the question is to find some $\phi \in \mathscr{C}_c^\infty(H)$, for suitable subgroup H (smaller than G in order to be useful), such that $\pi^*(\phi)\ell$ is nonzero and smooth (i.e., in $\widetilde{\pi}$). In this section we will study this question for the local Flicker-Rallis period and the local Rankin-Selberg period.

7.2. **A compactness lemma.** Now we return to our setting. Let E/F be a quadratic extension of non-Archimedean local fields of characteristic zero with residue characteristic p. We denote by η the quadratic character associated to E/F, and set

$$\eta_n = \eta^{n-1}. \tag{7.4}$$

Let $\varphi_{n-1} \in \mathscr{C}_c^\infty(H_{n-1}(E))$ and $\phi_{n-1} \in \mathscr{C}_c^\infty(M_{n-1,1}(E))$. We consider the Fourier transform of ϕ_{n-1} as a function on $M_{1,n-1}(E)$ by

$$\widehat{\phi}_{n-1}(X) = \int_{M_{1,n-1}(E)} \phi_{n-1}(Y)\psi_E(\mathrm{tr}(XY))\, dY.$$

With the pair $(\varphi_{n-1}, \phi_{n-1})$ we associate a new function on $H_{n-1}(E)$ by

$$\widetilde{W}_{\varphi_{n-1},\phi_{n-1}}(g) := \widehat{\phi}_{n-1}(-e_{n-1}g) \iint \varphi_{n-1}(g^{-1}u\epsilon_{n-1}h)\overline{\psi}_E(u)\eta_n(h)\, du\, dh, \tag{7.5}$$

where $u \in N_{n-1}(E), h \in N_{n-1}(F)\backslash H_{n-1}(F)$, and the integral is iterated. Note that the integral converges absolutely. Clearly we have

$$W_{\varphi_{n-1},\phi_{n-1}}(ug) = \overline{\psi}_E(u) W_{\varphi_{n-1},\phi_{n-1}}(g)$$

for $u \in N_{n-1}(E)$.

We would like to obtain a function with compact support modulo $N_{n-1}(E)$ by imposing suitable conditions on $(\varphi_{n-1}, \phi_{n-1})$. To simplify the notation, we will denote, when $p > 2$,

$$\Lambda = \mathcal{O}_E.$$

It decomposes as $\Lambda = \Lambda^+ \oplus \Lambda^-$ where $\Lambda^\pm = \mathcal{O}_{E^\pm}$. If $p = 2$, in the rest of the paper we define $\mathcal{O}_E$ as $\mathcal{O}_{E^+} \oplus \mathcal{O}_{E^-}$, which may be a nonmaximal order of E. Then the Fourier transform of $1_\Lambda \in \mathscr{C}_c^\infty(E)$ is a nonzero multiple of 1_{Λ^*} for a lattice (i.e., an $\mathcal{O}_E$ module) $\Lambda^* \subset E$ (depending on ψ). Let ϖ be a uniformizer of F. For an integer $m > 0$, we naturally view $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]$ as the subspace of $\mathscr{C}_c^\infty(E)$ consisting of functions supported in $\varpi^m\Lambda$ and invariant by $\varpi^{2m}\Lambda$.

Definition 7.1. We define a *dagger* space of level m, denoted by $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]^\dagger$ or $\mathscr{C}_c^\infty(E)_m^\dagger$, as the subspace of $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]$ spanned by functions $\theta = \theta^+ \otimes \theta^-, \theta^\pm \in \mathscr{C}_c^\infty(E^\pm)$, satisfying the following:

- θ^+ is a multiple of $1_{\varpi^m\Lambda^+}$.
- The Fourier transform $\widehat{\theta} \in \mathbb{C}[\varpi^{-2m}\Lambda^*/\varpi^{-m}\Lambda^*]$ is supported in $\varpi^{-2m}\Lambda^* - \varpi^{-2m+1}\Lambda^*$. With the condition on θ^+, this is equivalent to that the function $\widehat{\theta^-}$ is supported in $\varpi^{-2m}\Lambda^{-*} - \varpi^{-2m+1}\Lambda^{-*}$ where $\Lambda^{-*} = \Lambda^* \cap E^-$.

In particular, every element in $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]^\dagger$ is invariant under multiplication by $1+\varpi^m\mathcal{O}_E$. Heuristically, such θ has a constant real part θ^+ but a highly oscillating imaginary part θ^-.

Definition 7.2. We denote by $\mathscr{C}_c^\infty(M_{n-1,1}(E))_m^\dagger$ the space spanned by functions on $M_{n-1,1}(E)$ of the form $\phi_{n-1}=\bigotimes_{1\leq i\leq n-1}\phi^{(i)}$ in the way that $\phi_{n-1}(x)=\prod_i\phi^{(i)}(x_i)$ if $x=(x_1,\dots,x_{n-1})^t\in M_{n-1,1}(E)$, satisfying

- When $1\leq i\leq n-2$, $\phi^{(i)}$ is the characteristic function of $\varpi^m\Lambda$; $\phi^{(n-1)}$ is an element of $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]^\dagger$.

Definition 7.3. We denote by $\mathscr{C}_c^\infty(H_{n-1}(E))_m^\dagger$ the space spanned by functions on $H_{n-1}(E)$ of the form $\varphi_{n-1}=\bigotimes_{1\leq i,j\leq n-1}\varphi^{(ij)}$ in the way that $\varphi_{n-1}(g)=\prod_{i,j}\varphi^{(ij)}(g_{ij})$ if $g=(g_{ij})$, satisfying

- When $1\leq j<i\leq n-1$, $\varphi^{(ij)}$ is the characteristic function of $\varpi^m\mathcal{O}_E$.
- When $1\leq i=j\leq n-1$, $\varphi^{(ij)}$ is the characteristic function of $1+\varpi^m\mathcal{O}_E$.
- When $1\leq i<j\leq n-1$, $j-i\neq 1$, $\varphi^{(ij)}$ is the characteristic function of $\varpi^m\Lambda$.
- When $1\leq i=j-1\leq n-2$, $\varphi^{(ij)}$ is an element of $\mathbb{C}[\varpi^m\Lambda/\varpi^{2m}\Lambda]^\dagger$.

Remark 12. A function $\varphi_{n-1}\in\mathscr{C}_c^\infty(H_{n-1}(E))_m^\dagger$ has the following property:

$$\varphi_{n-1}(u_{n-2}\cdots u_1av_1\cdots v_{n-2})=\varphi_{n-1}(u_{\sigma'(n-2)}\cdots u_{\sigma'(1)}av_{\sigma'(1)}\cdots v_{\sigma'(n-2)}),$$

where $a\in A_{n-1},u_i\in N_{n-1}$ ($v_i\in N_{n-1,-}$, resp.), u_i-1 (v_i-1, resp.) has nonzero entries only in the $(i+1)$th column (row, resp.), and σ,σ' are any permutations.

Definition 7.4. We say that the pair $(\varphi_{n-1},\phi_{n-1})$ is *m-admissible* if $\phi_{n-1}\in\mathscr{C}_c^\infty(M_{n-1,1}(E))_m^\dagger$, and $\varphi_{n-1}\in\mathscr{C}_c^\infty(H_{n-1}(E))_m^\dagger$.

Remark 13. The pair $(\varphi_{n-1},\phi_{n-1})$ defines a function denoted by $\varphi_{n-1}\otimes\phi_{n-1}$ on the mirabolic subgroup P_n of $H_n(E)$,

$$\varphi_{n-1}\otimes\phi_{n-1}\left[\begin{pmatrix}x&\\&1\end{pmatrix}\begin{pmatrix}1_{n-1}&u\\&1\end{pmatrix}\right]=\varphi_{n-1}(x)\phi_{n-1}(u).$$

For *m-admissible* $(\varphi_{n-1},\phi_{n-1})$ as above, we define recursively $\varphi_i,\phi_i,\phi'_{i+1}$ for $i=n-2,\dots,1$, such that

$$\varphi_{i+1}=\varphi_i\otimes\phi_i\otimes\phi'_{i+1}, \tag{7.6}$$

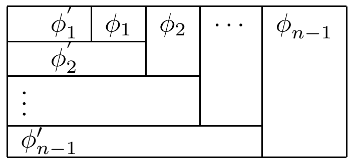
where

$$\varphi_i\in\mathscr{C}_c^\infty(M_i(E)),\qquad \phi_i\in\mathscr{C}_c^\infty(M_{i,1}(E)),\qquad \phi'_{i+1}\in\mathscr{C}_c^\infty(M_{1,i+1}(E)).$$

Here the function φ_i is viewed as a function on $M_i(E)$ [though it is supported in $H_i(E)$]. The tensor product is understood as

$$\varphi_{i+1}(X_{i+1})=\varphi_i(X_i)\phi_i(u_i)\phi'_{i+1}(v_{i+1}),$$

where $X_{i+1} = \begin{pmatrix} X_i & u_i \\ & v_{i+1} \end{pmatrix} \in M_{i+1}(E), X_i \in M_i(E), u_i \in M_{i,1}(E), v_{i+1} \in M_{1,i+1}(E)$. Set $\phi_1' = \varphi_1$ so that we have the following decomposition of $\varphi_{n-1} \otimes \phi_{n-1}$:



To facilitate the exposition, we list the properties of admissible functions that will be used later in our proof.

Proposition 7.5. *Let $(\varphi_{n-1}, \phi_{n-1})$ be m-admissible (Definition 7.4), and we decompose it according to (7.6). Then we have the following properties:*

(i) *The function ϕ_i' is the characteristic function of $(0, \dots, 0, 1) + \varpi^m M_{1,i}(\mathcal{O}_E)$, and $\phi_i \in \mathscr{C}_c^\infty(M_{i,1}(E))_m^\dagger$.*
(ii) *The function φ_{n-1} is left and right invariant under $N_{n-1,-}(\varpi^m \mathcal{O}_E) = 1 + \varpi^m \mathfrak{n}_{n-1,-}(\mathcal{O}_E)$.*
(iii) *With respect to the decomposition $M_{n-1}(E) = M_{n-1}(F) \oplus M_{n-1}(E^-)$, the function $\varphi_{n-1} = \varphi_{n-1}^+ \otimes \varphi_{n-1}^-$ is decomposable and the "real" part φ_{n-1}^+ is a multiple of the characteristic function of $1 + \varpi^m M_{n-1}(\mathcal{O}_F)$.*
(iv) *The function φ_{n-1} is left and right invariant under the compact open subgroup $1 + \varpi^m M_{n-1}(\mathcal{O}_F)$ (i.e., the support of the real part φ_{n-1}^+ of φ_{n-1}).*

Proof. They all follow from Definitions 7.2, 7.3, and 7.4. □

Property (iii) and (iv) of admissible functions will not be used until the next section. Our key result of this section is the following compactness lemma.

Lemma 7.6. *Assume that $(\varphi_{n-1}, \phi_{n-1})$ is m-*admissible *for some $m > 0$ and we have the derived functions ϕ_i, ϕ_i' as above.*

(1) *Then the support of the function $\widetilde{W}_{\varphi_{n-1}, \phi_{n-1}}$ is compact modulo $N_{n-1}(E)$; i.e., it defines an element in*
$$\mathscr{C}_c^\infty(N_{n-1}(E) \backslash H_{n-1}(E), \overline{\psi}_E).$$
Furthermore, $\widetilde{W}_{\varphi_{n-1}, \phi_{n-1}}(\epsilon_{n-1} g)$ is nonzero only when
$$g \in H_{n-1}'(E) = N_{n-1}(E) A_{n-1}(E) N_{n-1,-}(E).$$
(2) *View $\phi' := \otimes_{i=1}^{n-1} \phi_i'$ as a function on $B_{n-1,-}(E)$ or its Lie algebra $\mathfrak{b}_{n-1,1}(E)$ (this is possible due to the special feature of the function ϕ'). Denote by $d_n = \binom{n}{3}$ so that*
$$\tau^{d_n} = \delta_{n-1}(\epsilon_{n-1}) = \det(Ad(\epsilon_{n-1}) : N_{n-1}(E)).$$
Then the value of $\widetilde{W}_{\varphi_{n-1}, \phi_{n-1}}(\epsilon_{n-1} g)$ at $g = yv \in A_{n-1} N_{n-1,-}(F)$,
$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_1 y_2 & \\ & & & y_1 \end{pmatrix} \in A_{n-1}(F),$$

and

$$v=\prod_{i=1}^{n-2}\begin{pmatrix}1_i & \\ v_i & 1\end{pmatrix}\in N_{n-1,-}(F),\qquad v_i\in M_{1,i}(F),$$

is given by the product of the constant

$$|\tau|_E^{d_n}\int_{B_{n-1,-}(F)}\phi'(b)\,db \tag{7.7}$$

and

$$\eta_n(y)|\delta_{n-1}(y)|_F\prod_{i=1}^{n-1}\widehat{\phi_{n-i}}(-y_i(v_{n-i-1},1)\tau). \tag{7.8}$$

Here the measure db on $B_{n-1,-}(F)$ is either the left or the right invariant one; they give the same value to the integral since the support of ϕ' is contained in $1+\varpi\mathfrak{b}_{n-1,-}(\mathcal{O}_E)$.

Proof. It suffices to consider the following absolutely convergent integral:

$$w(g)=\int_{B_{n-1,-}(F)}\int_{N_{n-1}(E)}\varphi_{n-1}(g^{-1}uh)\overline{\psi}_\tau(u)\eta_n(h)\,du\,dh, \tag{7.9}$$

where we have replaced $N_{n-1}(F)\backslash H_{n-1}(F)$ by $B_{n-1,-}$ (with the right invariant measure) and

$$\psi_\tau(u)=\psi_{E,\tau}(u)=\psi_E(\epsilon_{n-1}u\epsilon_{n-1}^{-1}).$$

Indeed, by a suitable substitution we have

$$\widetilde{W}_{\varphi_{n-1},\phi_{n-1}}(\epsilon_{n-1}g)=|\tau|_E^{d_n}\widehat{\phi}_{n-1}(-e_{n-1}\tau g)w(g).$$

By the condition on the support of $\widehat{\phi}_{n-1}$, we know that $\widehat{\phi}_{n-1}(e_{n-1}g)$ is zero unless the $(n-1,n-1)$th entry of $g\in H_{n-1}(E)$ is nonzero. Up to the left translation by $N_{n-1}(E)$, such $g\in H_{n-1}(E)$ is of the form

$$g=y_1\begin{pmatrix}x_{n-2} & \\ & 1\end{pmatrix}\begin{pmatrix}1_{n-2} & \\ v_{n-2} & 1\end{pmatrix}, y_1\in E^\times, x_{n-2}\in H_{n-2}(E), v_{n-2}\in M_{1,n-2}(E).$$

By the support condition on $\widehat{\phi}_{n-1}$ [noting that $\phi_{n-1}\in\mathscr{C}_c^\infty(M_{n-1,1}(E))_m^\dagger$; cf. Definition 7.2], for $w(g)$ in (7.9) to be nonzero, y_1 must lie in a compact set of $E^\times$ and $v_{n-2}\in\varpi^m M_{1,n-2}(\mathcal{O}_E)$. By the property (ii) in Prop. 7.5, the function φ_{n-1} is invariant under left multiplication by such $\begin{pmatrix}1_{n-2} & \\ v_{n-2} & 1\end{pmatrix}$. Hence we have

$$\widetilde{W}_{\varphi_{n-1},\phi_{n-1}}(\epsilon_{n-1}g)=|\tau|_E^{d_n}\widehat{\phi}_{n-1}(\tau y_1(v_{n-2},1))w\left[y_1\begin{pmatrix}x_{n-2} & \\ & 1\end{pmatrix}\right]. \tag{7.10}$$

Therefore it is enough to consider $w(g)$ when $g=y_1\begin{pmatrix}x_{n-2} & \\ & 1\end{pmatrix}$ for $x_{n-2}\in H_{n-2}(E)$.

We write $h\in B_{n-1,-}(F)=A_{n-1}(F)N_{n-1,-}(F)$ as

$$h=b_1\begin{pmatrix}a_{n-2} & \\ & 1\end{pmatrix}\begin{pmatrix}1_{n-2} & \\ c_{n-2} & 1\end{pmatrix},$$

where $b_1\in F^\times, a_{n-2}\in B_{n-2,-}(F), c_{n-2}\in M_{1,n-2}(F)$. The measure can be chosen as

$$|a_{n-2}|^{-1}|b_1|^{-1}\,db_1\,dc_{n-2}\,da_{n-2},$$

where da_{n-2} is the right invariant measure on $B_{n-2,-}(F)$. For the integration over $u \in N_{n-1}(E)$, we write

$$u = \begin{pmatrix} 1_{n-2} & u'_{n-2} \\ & 1 \end{pmatrix} \begin{pmatrix} u_{n-2} & \\ & 1 \end{pmatrix} \in N_{n-1}(E).$$

Then the product $g^{-1}uh$ is equal to

$$\text{(7.11)} \qquad b_1 y_1^{-1} \begin{pmatrix} x_{n-2}^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-2} & u'_{n-2} \\ & 1 \end{pmatrix} \begin{pmatrix} u_{n-2} & \\ & 1 \end{pmatrix} \begin{pmatrix} a_{n-2} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-2} & \\ c_{n-2} & 1 \end{pmatrix}.$$

Then the last row of the product (7.11) is equal to $y_1^{-1} b_1(c_{n-2}, 1) \in M_{1,n-1}(E)$. By the condition on the support of ϕ'_{n-1} [cf. Property (i) of Prop. 7.5], we can assume that

$$c_{n-2} \in \varpi^m M_{1,n-2}(\mathcal{O}_E),$$

so that φ_{n-1} is invariant under the right translation by such $\begin{pmatrix} 1_{n-2} & \\ c_{n-2} & 1 \end{pmatrix}$ [cf. Property (ii) of Prop. 7.5]. The product of the first four matrices in (7.11) is then equal to

$$y_1^{-1} b_1 \begin{pmatrix} x_{n-2}^{-1} u_{n-2} a_{n-2} & x_{n-2}^{-1} u'_{n-2} \\ & 1 \end{pmatrix}.$$

The integrations on c_{n-2} and u'_{n-2} yield, respectively,

$$\int_{M_{n-2,1}(F)} \phi'_{n-1}(y_1^{-1} b_1(c_{n-2}, 1))\, dc_{n-2},$$

$$\begin{aligned} &\int_{M_{n-2,1}(E)} \phi_{n-2}(b_1 y_1^{-1} x_{n-2}^{-1} u'_{n-2}) \overline{\psi}_\tau(u'_{n-2})\, du'_{n-2} \\ &\qquad = |b_1^{-1} y_1|_E^{n-2} |x_{n-2}|_E \widehat{\phi}_{n-2}(-e_{n-2} \tau y_1 b_1^{-1} x_{n-2}). \end{aligned}$$

(Here we note that the Fourier transform of ϕ_{n-2} is defined by the character ψ_E.) Therefore $w(g)$ is equal to the integration of the function of $b_1 \in F^\times$ given by the product of the above two terms and

$$\int_{B_{n-2,-}(F)} \int_{N_{n-2}(E)} \varphi_{n-2}(y_1^{-1} b_1 x_{n-2}^{-1} u_{n-2} a_{n-2}) \overline{\psi}_\tau(u_{n-2}) |a_{n-2}|^{-1} \eta_n(h)\, du_{n-2}\, da_{n-2}$$

with respect to the measure $|b_1|^{n-2} db_1$. We may repeat the process and hence may assume that the function on $H_{n-2}(E)$ defined by

$$\begin{aligned} g_{n-2} \mapsto \widehat{\phi}_{n-2}(-e_{n-2} g_{n-2}) &\int_{B_{n-2,-}(F)} \int_{N_{n-2}(E)} \varphi_{n-2}(g_{n-2}^{-1} u_{n-2} a_{n-2}) \\ &\times \overline{\psi}_\tau(u_{n-2}) |a_{n-2}|^{-1} \eta_n(a_{n-2})\, du_{n-2}\, da_{n-2} \end{aligned}$$

is zero unless $g_2 \in H'_{n-2}(E)$ and its support is compact modulo $N_{n-2}(E)$. By the support condition of ϕ'_{n-1} [cf. Property (i) of Prop. 7.5], we know that $y_1^{-1} b_1 \in 1 + \varpi^m \mathcal{O}_E$. Since y_1 is in a compact region of $E^\times$, the integration of b_1 must also be in a compact region. This implies that $w(g) \neq 0$ only when $x_{n-2} \in H'_{n-2}(E)$ and in a region compact modulo $N_{n-2}(E)$. By the boundedness of v_{n-2} as shown in Eq. (7.10), we complete the proof of part (1).

To show part (2), we need to keep track of the computation above. Since we are now assuming that $g \in H_{n-1}(F)$, we have $y_1 \in F^\times$, and hence we may substitute $b_1 \mapsto b_1 y_1$. Then for $\phi'_{n-1}(b_1(c_{n-2}, 1))$ to be nonzero, we must have $b_1 \in 1 + \varpi^m \mathcal{O}_F$

[cf. Property (i) of Prop. 7.5]. Note that $\widehat{\phi_{n-2}}$ and φ_{n-2} are invariant under multiplication by scalars in $1+\varpi^m\mathcal{O}_F$. We see that $w(g)$ is given by the product of

$$\eta_n(y_1^{n-1})\int_{M_{n-1,1}(F)}\phi'_{n-1}(b_1(c_{n-2},1))dc_{n-2}\eta(b_1^{n-1})|b_1|^{-1}\,db_1,$$

$$|x_{n-2}|_E\widehat{\phi}_{n-2}(-e_{n-2}\tau x_{n-2}),$$

and

$$\int_{B_{n-2,-}(F)}\int_{N_{n-2}(E)}\varphi_{n-2}(x_{n-2}^{-1}u_{n-2}a_{n-2})\overline{\psi}_\tau(u_{n-2})|a_{n-2}|^{-1}\eta_n(a_{n-2})\,du_{n-2}\,da_{n-2}.$$

When $y=y_1\cdot diag(x_{n-2},1), x_{n-2}\in B_{n-2,-}(F)$, we have

$$\delta_{n-1}(y)=\delta_{n-2}(x_{n-2})\det(x_{n-2}). \tag{7.12}$$

Note that

$$|x_{n-2}|_E=|x_{n-2}|_F^2.$$

Now we may repeat this process to complete the proof. □

For admissible $\varphi_{n-1}\otimes\phi_{n-1}\in\mathscr{C}_c^\infty(H_{n-1}(E)\times M_{n,1}(E))$, the function $\widetilde{W}_{\varphi_{n-1},\phi_{n-1}}$ lies in $\mathscr{C}_c^\infty(N_{n-1}(E)\backslash H_{n-1}(E),\psi_E)$ and therefore determines a unique element in $\mathcal{W}$, denoted by $W_{\varphi_{n-1},\phi_{n-1}}$, characterized by

$$W_{\varphi_{n-1},\phi_{n-1}}\begin{pmatrix} g & \\ & 1\end{pmatrix}=\widetilde{W}_{\varphi_{n-1},\phi_{n-1}}(g),\qquad g\in H_{n-1}(E). \tag{7.13}$$

7.3. **Smoothing local Flicker-Rallis period $\boldsymbol{\beta_n}$.** Let Π_n be an irreducible unitary generic representation of $H_n(E)$. We now consider the local period Flicker-Rallis β_n [cf. §3, (3.11), (3.20)]. We let $\mathcal{W}=\mathcal{W}(\Pi_n,\overline{\psi}_E)$ be the Whittaker model of Π_n with respect to the complex conjugate of ψ_E for later convenience. As earlier we have endowed $\mathcal{W}$ with a nondegenerate positive definite invariant Hermitian structure [cf. (3.2)],

$$\langle W,W'\rangle=\vartheta(W,W')=\int_{N_{n-1}(E)\backslash H_{n-1}(E)}W\begin{pmatrix} g & \\ & 1\end{pmatrix}\overline{W'\begin{pmatrix} g & \\ & 1\end{pmatrix}}\,dg.$$

We also consider its Kirillov model denoted by $\mathcal{K}=\mathcal{K}(\Pi_n,\overline{\psi}_E)$, which is a certain subspace of smooth functions $\mathscr{C}^\infty(N_{n-1}(E)\backslash H_{n-1}(E),\overline{\psi}_E)$. Moreover, it is well-known that the Kirillov model always contains the subspace $\mathscr{C}_c^\infty(N_{n-1}(E)\backslash H_{n-1}(E),\overline{\psi}_E)$ of smooth compactly supported functions.

Let $\mathcal{W}^*$ be the (conjugate) algebraic dual space of $\mathcal{W}$ and $\mathcal{H}$ be the Hilbert space underlying the unitary representation Π_n. Then $\mathcal{W}$ is the space of smooth vectors in $\mathcal{H}$ and we have inclusions,

$$\mathcal{W}\subset\mathcal{H}\subset\mathcal{W}^*.$$

The Hermitian pairing on $\mathcal{W}\times\mathcal{W}$ extends to $\mathcal{H}\times\mathcal{H}$ and $\mathcal{W}\times\mathcal{W}^*$. A similar discussion also appears in [25, §2.1]. We still denote by Π_n the representation of $H_n(E)$ on $\mathcal{W}^*$ so for any $W\in\mathcal{W}, W'\in\mathcal{W}^*$,

$$\langle\Pi_n(g)W,W'\rangle=\langle W,\Pi_n(g^{-1})W'\rangle.$$

Then the local Flicker-Rallis period β_n is an element in $\mathcal{W}^*$ defined by (3.11). To ease notation, we write this as

$$\beta_n(W)=\int_{N_{n-1}(F)\backslash H_{n-1}(F)}W\left(\epsilon_{n-1}h\right)\eta_n(h)\,dh, \tag{7.14}$$

where η_n is as in (7.4). We also write this as

$$\beta_n(W) = \langle W, \beta_n \rangle.$$

It is $(H_n(F), \eta_n)$ invariant [10],

$$\beta_n \in \mathrm{Hom}_{H_n(F)}(\mathcal{W}, \mathbb{C}_{\eta_n}).$$

We would like to smoothen the local period β_n by applying some sort of "mollifier."

Proposition 7.7. *Let Π_n be an irreducible unitary generic representation of $H_n(E)$. Assume that the pair $\varphi_{n-1} \in \mathscr{C}_c^\infty(H_{n-1}(E)), \phi_{n-1} \in \mathscr{C}_c^\infty(M_{n,1}(E))$ is m-admissible for $m > 0$. Let $W_{\varphi_{n-1},\phi_{n-1}} \in \mathcal{W}$ be the element determined by (7.13). Then for every $W \in W(\Pi_n, \overline{\psi}_E)$, we have*

$$\langle \Pi_n(\varphi_{n-1}^*)\Pi_n(\phi_{n-1}^*)W, \beta_n \rangle = \langle W, W_{\varphi_{n-1},\phi_{n-1}} \rangle.$$

In other words, the linear functional $\Pi_n(\phi_{n-1})\Pi_n(\varphi_{n-1})\beta_n$, a priori only in $\mathcal{W}^*$, is indeed a smooth vector and is represented by $W_{\varphi_{n-1},\phi_{n-1}} \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$.

Proof. The left hand side is given by

$$\int_{N_{n-1}(F)\backslash H_{n-1}(F)} \int_{H_{n-1}(E)} \Pi_n(\phi_{n-1}^*)W(\epsilon_{n-1}hg_{n-1})\overline{\varphi}_{n-1}(g_{n-1}^{-1})dg_{n-1}\eta_n(h)\,dh.$$

Substitute $g_{n-1} \mapsto h^{-1}\epsilon_{n-1}^{-1}g_{n-1}$,

$$\int_{N_{n-1}(F)\backslash H_{n-1}(F)} \int_{H_{n-1}(E)} \Pi_n(\phi_{n-1}^*)W(g_{n-1})\overline{\varphi}_{n-1}(g_{n-1}^{-1}\epsilon_{n-1}h)dg_{n-1}\eta_n(h)\,dh.$$

Since $W(ug_{n-1}) = \overline{\psi}_E(u)W(g_{n-1})$ for $u \in N_{n-1}(E)$, we may rewrite the integral as

$$\int \Pi_n(\phi_{n-1}^*)W(g_{n-1}) \left(\int_{N_{n-1}(E)} \overline{\varphi}_{n-1}(g_{n-1}^{-1}u^{-1}\epsilon_{n-1}h)\overline{\psi}_E(u)du \right) \eta_n(h)\,dh\,dg_{n-1},$$

where the outer integral is over

$$h \in N_{n-1}(F)\backslash H_{n-1}(F), \qquad g_{n-1} \in N_{n-1}(E)\backslash H_{n-1}(E).$$

Now we also note that

$$\begin{aligned}
\Pi_n(\phi_{n-1}^*)W(g_{n-1}) &= \int_{M_{n-1,1}(E)} W\left[g_{n-1}\begin{pmatrix} 1_{n-1} & u \\ & 1 \end{pmatrix}\right] \phi_{n-1}^*(u)\,du \\
&= W(g_{n-1}) \int_{M_{n-1,1}(E)} \overline{\phi}_{n-1}(-u)\overline{\psi}_E(e_{n-1}^*g_{n-1}u)\,du \\
&= W(g_{n-1})\,\widehat{\overline{\phi}}_{n-1}(-e_{n-1}^*g_{n-1}).
\end{aligned}$$

This completes the proof. □

7.4. **Smoothing local Rankin-Selberg period $\boldsymbol{\lambda}$.** Now let $\Pi = \Pi_n \otimes \Pi_{n+1}$ be an irreducible unitary generic representation of $G'(F) = H_n(E) \times H_{n+1}(E)$. We now need another compactness lemma. For an integer $m' > 0$, we consider $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))_{m'}^\dagger$ (cf. Definition 7.2). Note that by definition, the Fourier transform $\widehat{\phi}_n$ is supported in the domain

$$\{(x_1, \ldots, x_n) \in M_{1,n}(E) \mid |x_i/x_n| \le |\varpi|^{m'}, i = 1, 2, \ldots, n-1\}. \tag{7.15}$$

Lemma 7.8. *Let $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))^\dagger_{m'}$. Let $W \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$ be such that its restriction to $H_{n-1}(E)$ has support compact modulo $N_{n-1}(E)$. If m' is sufficiently large (depending on the vector W), the map*

$$H_n(E) \ni g \mapsto \widehat{\phi}_n(e_n g)W(g)$$

defines an element in $\mathscr{C}_c^\infty(N_n(E)\backslash H_n(E), \overline{\psi}_E)$.

Proof. Clearly $\widehat{\phi}_n(e_n g)$ is $N_n(E)$ invariant and smooth. Hence the product defines an element in $\mathscr{C}^\infty(N_n(E)\backslash H_n(E), \overline{\psi}_E)$. It remains to show that the product has support compact modulo $N_n(E)$. By the support condition of $\widehat{\phi}_n(e_n g)$ [cf. (7.15)], we may assume that the lower right entry of g is nonzero. Therefore we may write $g = xu \begin{pmatrix} h & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & \\ v & 1 \end{pmatrix}$ where $h \in H_{n-1}(E), u \in N_n(E), v \in M_{1,n-1}(E)$, and $x \in E^\times$. Again by the assumption on $\widehat{\phi}_n$, we may assume that $||v|| < |\varpi|^{m'}$ [otherwise $\widehat{\phi}_n(e_n g)$ vanishes]. We may choose m' sufficiently large so that W is invariant under right multiplication by $1 + \varpi^{m'} M_n(\mathcal{O}_E)$ (such m' exists since $W \in \mathcal{W}$ is smooth). We now have

$$\begin{aligned} \widehat{\phi}_n(e_n g)W(g) &= \psi_E(u)\widehat{\phi}_n(x(v,1))W\left[x\begin{pmatrix} h & \\ & 1\end{pmatrix}\right] \\ &= \psi_E(u)\omega_{\Pi_n}(x)\widehat{\phi}_n(x(v,1))W\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right], \end{aligned}$$

where ω_{Π_n} is the central character of Π_n. Since the last factor has support compact modulo $N_{n-1}(E)$, and x lies in a compact region, the desired compactness follows. $\square$

Since the Kirillow model $\mathcal{K}(\Pi_{n+1}, \overline{\psi}_E)$ contains $\mathscr{C}_c^\infty(N_n(E)\backslash H_n(E), \overline{\psi}_E)$ as a subspace, we may view the product $\widehat{\phi}_n(e_n g)W(g)$ as an element in $\mathcal{K}(\Pi_{n+1}, \overline{\psi}_E)$. This determines uniquely an element in the Whittaker model $\mathcal{W}(\Pi_{n+1}, \overline{\psi}_E)$, denoted by W_{ϕ_n}. In other words, with each $W \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$ whose restriction to $H_{n-1}(E)$ lies in $\mathscr{C}_c^\infty(N_{n-1}(E)\backslash H_{n-1}(E), \overline{\psi}_E)$ and a $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))^\dagger_{m'}$ for m' sufficiently large, we associate an element $W_{\phi_n} \in \mathcal{W}(\Pi_{n+1}, \overline{\psi}_E)$ characterized by

$$W_{\phi_n}\begin{pmatrix} g & \\ & 1\end{pmatrix} = \widehat{\phi}_n(e_n g)W(g), \qquad g \in H_n(E). \tag{7.16}$$

Its complex conjugate $\overline{W}_{\phi_n}$ defines an element in $\mathcal{W}(\Pi_{n+1}, \psi_E)$.

Now we recall that the local Rankin-Selberg period is defined by [cf. (3.23)]

$$\lambda(s, W \otimes W') = \int_{N_n(E)\backslash H_n(E)} W(g)W'\begin{pmatrix} g & \\ & 1\end{pmatrix}|g|^s\,dg,$$

where $W \in \mathcal{W}(\Pi_n, \overline{\psi}_E), W' \in \mathcal{W}(\Pi_{n+1}, \psi_E)$. It has a meromorphic continuation to $s \in \mathbb{C}$ [with possible poles at those poles of the local Rankin-Selberg L-factor $L(s + 1/2, \Pi_v)$]. For $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))$, we have an action on $\mathcal{W}(\Pi_{n+1}, \psi_E)$ by

$$\Pi_{n+1}(\phi_n)W'(g) = \int_{M_{n-1,1}(E)} W'\left[g\begin{pmatrix} 1_{n-1} & u \\ & 1\end{pmatrix}\right]\phi_n(u)\,du, \quad g \in H_{n+1}(E).$$

Proposition 7.9. *Let $\Pi = \Pi_n \otimes \Pi_{n+1}$ be an irreducible unitary generic representation of $H_n(E) \times H_{n+1}(E)$. Assume that the restriction of $W \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$ to $H_{n-1}(E)$ lies in $\mathscr{C}_c^\infty(N_{n-1}(E)\backslash H_{n-1}(E), \overline{\psi}_E)$ and that $\widehat{\phi}_n \in \mathscr{C}_c^\infty(M_{n,1}(E))^\dagger_{m'}$*

for m' sufficiently large (depending on W). Then for every $W' \in \mathcal{W}(\Pi_{n+1}, \psi_E)$, $\lambda(s, W \otimes \Pi_{n+1}(\phi_n)W')$ is an entire function in $s \in \mathbb{C}$ and we have

$$\lambda(0, W \otimes \Pi_{n+1}(\phi_n)W') = \langle W', \overline{W}_{\phi_n} \rangle.$$

Proof. For $g \in H_n(E)$, we have

$$\begin{aligned}\Pi_{n+1}(\phi_n)W' \begin{pmatrix} g & \\ & 1 \end{pmatrix} &= \int_{M_{n-1,1}(E)} W' \left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_{n-1} & u \\ & 1 \end{pmatrix} \right] \phi_n(u)\, du \\ &= \int_{M_{n-1,1}(E)} W' \left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right] \psi_E(e_n g u) \phi(u)\, du \\ &= \widehat{\phi}_n(e_n g) W' \left[\begin{pmatrix} g & \\ & 1 \end{pmatrix} \right].\end{aligned}$$

Return to the local Rankin-Selberg period, and

$$\lambda(s, W \otimes \Pi_{n+1}(\phi_n)W') = \int_{N_n(E)\backslash H_n(E)} W(g) \widehat{\phi}_n(e_n g) W' \begin{pmatrix} g & \\ & 1 \end{pmatrix} |g|^s\, dg. \tag{7.17}$$

By Lemma 7.8, $W(g)\widehat{\phi}_n(e_n g)$ has compact support modulo $N_n(E)$. It follows that the last integral converges absolutely for all $s \in \mathbb{C}$ and hence defines an entire function in $s \in \mathbb{C}$. Moreover the value at $s = 0$ is given by (cf. 7.16)

$$\lambda(0, W \otimes \Pi_{n+1}(\phi_n)W') = \langle W', \overline{W}_{\phi_n} \rangle.$$

This completes the proof. □

8. Local character expansion in the general linear group case

In this section we prove a "limit" formula for the (local) spherical character in the general linear group case. We will choose a subspace of test functions supported in a small neighborhood of the origin of the symmetric space S_{n+1}. Then we may treat them as functions on the "Lie algebra" $\mathfrak{s}_{n+1}$ of S_{n+1}, the tangent space at the origin. The key property for these functions is that their Fourier transforms have vanishing unipotent orbital integrals except for the regular unipotent element. The intermediate steps are messy and somehow ugly; but the final outcome seems to be miraculously neat.

8.1. **A "limit" formula for the spherical character $I_\Pi(f)$.** Now let $\Pi = \Pi_n \otimes \Pi_{n+1}$ be an irreducible unitary generic representation of $H_n(E) \times H_{n+1}(E)$. Assume that the central characters of both Π_n and Π_{n+1} are trivial on $F^\times$. Let $I_{\Pi,s}(f)$ be the (unnormalized) local spherical character defined by (3.32). We now combine Prop. 7.7 and 7.9 to obtain a formula of $I_{\Pi,s}(f)$ for a special class of test functions f.

Consider $f_n \in \mathscr{C}_c^\infty(H_n(E))$ and $f_{n+1} \in \mathscr{C}_c^\infty(H_{n+1}(E))$. Let $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))$. We consider a *perturbation* of f_{n+1} by ϕ_n

$$f_{n+1}^{\phi_n}(g) = \int_{M_{n,1}(E)} f_{n+1} \left[\begin{pmatrix} 1_n & u \\ & 1 \end{pmatrix} g \right] \phi_n(u)\, du, \qquad g \in H_{n+1}(E).$$

This is the same as $\phi_n * f_{n+1}$—the convolution introduced in §7.1, (7.3)—where we view $M_{n,1}(E)$ as a subgroup of $H_{n+1}(E)$. Similarly, for $\varphi_{n-1} \in \mathscr{C}_c^\infty(H_{n-1}(E))$, $\phi_{n-1} \in \mathscr{C}_c^\infty(M_{n-1,1}(E))$, we define a *perturbation* of f_n: for $g \in H_n(E)$,

$$f_n^{\varphi_{n-1},\phi_{n-1}}(g)$$
$$:= \int_{M_{n-1,1}(E)} \int_{H_{n-1}(E)} f_n \left[g \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & -u \\ & 1 \end{pmatrix} \right] \varphi_{n-1}(x)\phi_{n-1}(u)\, dx\, du.$$

Equivalently

$$f_n^{\varphi_{n-1},\phi_{n-1}} = f_n * \phi_{n-1} * \varphi_{n-1} \in \mathscr{C}_c^\infty(H_n(E)).$$

We will consider functions of the following form:

$$f = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n} \in \mathscr{C}_c^\infty(H_n(E) \times H_{n+1}(E)).$$

Definition 8.1. Fix Π. Let (m, m', r) be positive integers with $r > m' > m > 0$. We say that $f = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n}$ is (m, m', r)-*admissible* or *admissible* for Π if it satisfies the following:

- The function $\varphi_{n-1} \otimes \phi_{n-1}$ is m-admissible. Hence it determines an element $\widetilde{W}_{\varphi_{n-1},\overline{\phi}_{n-1}} \in \mathscr{C}_c^\infty(N_{n-1}\backslash H_{n-1}(E), \psi_E)$ (cf. Lemma 7.6) and $W_{\varphi_{n-1},\phi_{n-1}} \in \mathcal{W}(\Pi_n, \psi_E)$.
- The function $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))^\dagger_{m'}$ for sufficiently large m' (depending on Π_n, $\varphi_{n-1} \otimes \phi_{n-1}$, and hence on the integer m). More precisely, m' is large enough such that Lemma 7.8 holds for $W = W_{\varphi_{n-1},\phi_{n-1}}$. Let $W_{\varphi_{n-1},\phi_{n-1},\phi_n} \in \mathcal{W}(\Pi_{n+1}, \overline{\psi}_E)$ be the function characterized by the equation (7.16) for the choice $W = W_{\varphi_{n-1},\phi_{n-1}}$.
- The function f_n (f_{n+1}, resp.) is a multiple of the characteristic function of $1 + \varpi^r M_n(\mathcal{O}_E)$ (of $1 + \varpi^r M_{n+1}(\mathcal{O}_E)$). We normalize f_n, f_{n+1} by

$$\int_{H_n(E)} f_n(g)\, dg = \int_{H_{n+1}(E)} f_{n+1}(g)\, dg = 1. \tag{8.1}$$

  We require that r is sufficiently large (depending on Π, $\varphi_{n-1}, \phi_{n-1}, \phi_n$, and hence on m, m') so that

$$\Pi_n(f_n) W_{\varphi_{n-1},\phi_{n-1}} = W_{\varphi_{n-1},\phi_{n-1}}$$

  and

$$\Pi_{n+1}(f_{n+1}) W_{\varphi_{n-1},\phi_{n-1},\phi_n} = W_{\varphi_{n-1},\phi_{n-1},\phi_n}.$$

Proposition 8.2. *Fix Π and assume that $f = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n}$ is (m, m', r)-admissible for Π. Then we have*

$$I_{\Pi,s}(f) = \beta_{n+1}(W_{\varphi_{n-1},\phi_{n-1},\phi_n}).$$

In particular, it is independent of $s \in \mathbb{C}$.

Proof. First we have

$$\begin{aligned} I_{\Pi,s}(f_n \otimes f_{n+1}) &= \sum_{W,W'} \lambda(s, \Pi_n(f_n)W \otimes \Pi_{n+1}(f_{n+1})W')\overline{\beta_n(W)} \cdot \overline{\beta_{n+1}(W')} \\ &= \sum_{W,W'} \lambda(s, W \otimes \Pi_{n+1}(f_{n+1})W')\overline{\beta_n(\Pi_n(f_n^*)W)} \cdot \overline{\beta_{n+1}(W')}, \end{aligned}$$

where the sum of W (W', resp.) runs over an orthonormal basis of Π_n (Π_{n+1}, resp.).

For simplicity, we write ϕ for ϕ_{n-1} and φ for φ_{n-1}. Now we replace f_n by $f_n^{\varphi,\phi} = f * \phi * \varphi$. Note that

$$\Pi_n((f_n^{\varphi,\phi})^*) = \Pi_n(\varphi^*)\Pi_n(\phi^*)\Pi_n(f^*).$$

By Prop. 7.7, we have for all $W \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$

$$\beta_n(\Pi_n((f_n^{\varphi,\phi})^*)W) = \langle \Pi_n(f_n^*)W, W_{\varphi,\phi}\rangle = \langle W, \Pi_n(f_n)W_{\varphi,\phi}\rangle.$$

For any $W_0 \in \mathcal{W}(\Pi_n, \overline{\psi}_E)$, we have an orthonormal expansion

$$W_0 = \sum_W \langle W_0, W\rangle W,$$

where the sum of W runs over an orthonormal basis of Π_n. Hence we may fold the sum over W to obtain

$$I_{\Pi,s}(f_n^{\varphi,\phi} \otimes f_{n+1}) = \sum_{W'} \lambda(s, \Pi_n(f_n)W_{\varphi,\phi} \otimes \Pi_{n+1}(f_{n+1})W')\overline{\beta_{n+1}(W')}.$$

Now we further replace f_{n+1} by $f_{n+1}^{\phi_n} = \phi_n * f_{n+1}$ and assume that $f = f_n^{\varphi,\phi} \otimes f_{n+1}^{\phi_n}$ is admissible. By the admissibility, f_n and f_{n+1} have small support so that we have

$$\Pi_n(f_n)W_{\varphi,\phi} = W_{\varphi,\phi}$$

and

$$\Pi_{n+1}(f_{n+1}^*)\overline{W}_{\varphi,\phi,\phi_n} = \overline{W}_{\varphi,\phi,\phi_n}.$$

Now note that the function $W_{\varphi,\phi}(g)\widehat{\phi}_n(e_n g)$ is supported in $N_n(E)H_n(\mathcal{O}_E)$. Hence in (7.17), we have $|g|^s = 1$ independent of $s \in \mathbb{C}$. Now by Prop. 7.9 and $\int_{H_n(E)} f_n(g)dg = \int_{H_{n+1}(E)} f_{n+1}(g)dg = 1$, we have for all $W' \in \mathcal{W}(\Pi_{n+1}, \psi_E)$

$$\begin{aligned}\lambda(s, \Pi_n(f_n)W_{\varphi,\phi} \otimes \Pi_{n+1}(f_{n+1}^{\phi_n})W') &= \langle \Pi_{n+1}(f_{n+1})W', \overline{W}_{\varphi,\phi,\phi_n}\rangle \\ &= \langle W', \Pi_{n+1}(f_{n+1}^*)\overline{W}_{\varphi,\phi,\phi_n}\rangle \\ &= \langle W', \overline{W}_{\varphi,\phi,\phi_n}\rangle.\end{aligned}$$

Then we may fold the sum over W' to obtain

$$I_{\Pi,s}(f_n^{\varphi,\phi} \otimes f_{n+1}^{\phi_n}) = \overline{\beta_{n+1}(\overline{W}_{\varphi,\phi,\phi_n})} = \beta_{n+1}(W_{\varphi,\phi,\phi_n}).$$

This completes the proof. $\square$

We need to simplify the formula. Define for $a \in A_n(F)$

$$\overline{\delta}_n(a) := \det(Ad(a) : \mathfrak{n}_n))/\det(Ad(a) : \mathfrak{n}_{n-1}). \tag{8.2}$$

To simplify the exposition from now on we redefine $f_{n+1}^{\phi_n}$ as $\phi_n^\vee * f_{n+1}$.

Proposition 8.3. *Fix Π and assume that $f = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n}$ is (m, m', r)-admissible for Π. Use notations as in Lemma 7.6. Then we have*

$$I_{\Pi,s}(f) = \omega_{\Pi_n}(\tau)|\tau|_E^{d_n}\left(\int_{B_{n-1,-}(F)} \phi'(b)\, db\right)$$
$$\int \prod_{i=0}^{n-1} \widehat{\phi_{n-i}}(-y_i(v_{n-i-1}, 1)\tau)|\overline{\delta}_n(y)|^{-1}\eta(y)\, d^*y\, dv,$$

where the integral of d^*ydv is over $y \in A_n(F), v \in N_{n,-}(F)$,

$$y = \begin{pmatrix} y_0y_1y_2\cdots y_{n-1} & & & \\ & \ddots & & \\ & & y_0y_1 & \\ & & & y_0 \end{pmatrix} \in A_n(F), \tag{8.3}$$

and

$$v=\prod_{i=1}^{n-1}\begin{pmatrix}1_i & \\ v_i & 1\end{pmatrix}\in N_{n,-}(F),\qquad v_i\in M_{1,i}(F). \tag{8.4}$$

Proof. By (7.16), we have [cf. (7.14), and note to replace ϕ_n by $\phi_n^\vee$]

$$\beta_{n+1}(W_{\varphi,\phi,\phi_n})=\int_{N_n(F)\backslash H_n(F)}W_{\varphi,\phi}(\epsilon_n h)\widehat{\phi}_n(-e_n\epsilon_n h)\eta_{n+1}(h)\,dh.$$

Let $h=yv$ where

$$y=y_0\begin{pmatrix}y' & \\ & 1\end{pmatrix}\in A_n(F),\qquad y'\in A_{n-1}(F),$$

and

$$v=\begin{pmatrix}v' & \\ & 1\end{pmatrix}\begin{pmatrix}1_{n-1} & \\ v_{n-1} & 1\end{pmatrix},\qquad v'\in N_{n-1,-}(F).$$

Then we may replace the integral on the quotient $N_n(F)\backslash H_n(F)$ by the integral over y,v as in (8.3) and (8.4) for the measure

$$|\delta_n(y)|^{-1}\prod_{i=0}^{n-1}d^*y_i\prod_{j=1}^{n-1}dv_j.$$

Since we have $\delta_n(y)=\delta_{n-1}(y')\det(y')=\delta_{n-1}(y')\overline{\delta}_n(y')$ [cf. (7.12), (8.2)], we may also write this as a product

$$\left(|\delta_{n-1}(y')|^{-1}|y'|^{-1}\,d^*y'\,dv'\right)\,d^*y_0\,dv_{n-1},$$

where $d^*y'=\prod_{i=0}^{n-2}d^*y_i$ and $dv'=\prod_{j=1}^{n-2}dv_j$. Since the central character of Π_n is trivial on $F^\times$, we have

$$W_{\varphi,\phi}(\epsilon_n y_0 h)=W_{\varphi,\phi}(\epsilon_n h).$$

By the admissibility, the value $\widehat{\phi}_n(-e_n\epsilon_n h)=\widehat{\phi}_n(-y_0(v_{n-1},1)\tau)$ is nonzero only if $\|v_{n-1}\|\le|\varpi^{m'}|$ is very small so that $\begin{pmatrix}1_{n-1} & \\ v_{n-1} & 1\end{pmatrix}$ acts trivially on $W_{\varphi,\phi}$. This allows us to write the integral as the product of

$$\int\widehat{\phi}_n(-y_0(v_{n-1},1)\tau)\eta_{n+1}(y_0^n)\,d^*y_0\,dv_{n-1}$$

and

$$\omega_{\Pi_n}(\tau)\int_{A_{n-1}(F)}\int_{N_{n-1,-}(F)}W_{\varphi,\phi}\begin{pmatrix}\epsilon_{n-1}y'v' & \\ & 1\end{pmatrix}|\delta_{n-1}(y')|^{-1}|y'|^{-1}\eta_{n+1}(y')\,d^*y'\,dv',$$

where we have used the equality

$$\epsilon_n=\tau\begin{pmatrix}\epsilon_{n-1} & \\ & 1\end{pmatrix}.$$

By definition $W_{\varphi,\phi}\begin{pmatrix}\epsilon_{n-1}y'v' & \\ & 1\end{pmatrix}=\widetilde{W}_{\varphi,\phi}(\epsilon_{n-1}y'v')$ [cf. (7.13)], we may apply the formula of the latter by Lemma 7.6,

$$\begin{aligned}&W_{\varphi,\phi}\begin{pmatrix}\epsilon_{n-1}y'v' & \\ & 1\end{pmatrix}\\&\quad=|\tau|_E^{d_n}\left(\int_{B_{n-1,-}(F)}\phi'(b)\,db\right)\eta_n(y')|\delta_{n-1}(y')|_F\prod_{i=1}^{n-1}\widehat{\phi_{n-i}}(y_i(v_{n-i-1},1)\tau).\end{aligned}$$

Finally we note $\eta_n\eta_{n+1} = \eta$ and $\eta_{n+1}(y_0^n) = \eta(y_0^{n^2}) = \eta(y_0^n)$ [cf. (7.4)]. This completes the proof. $\square$

8.2. **Truncated local expansion of the spherical character I_Π.** We are now ready to deduce a truncated local expansion of I_Π around the origin. As we have alluded to in the Introduction, this expansion is the relative version of a theorem of Harish-Chandra. His result is a local expansion of the character of an admissible representation of a p-adic reductive group in terms of the Fourier transform of nilpotent orbital integrals. Here we obtain a truncated expansion that only involves the regular unipotent element.

First we need to associate with $f \in \mathscr{C}_c^\infty(H_n(E) \times H_{n+1}(E))$ with small support around 1 a function on the Lie algebra $\mathfrak{s}$ with small support around 0. To $f = f_n \otimes f_{n+1} \in \mathscr{C}_c^\infty(H_n(E)\times H_{n+1}(E))$ we have associated a function $\widetilde{f} \in \mathscr{C}_c^\infty(H_{n+1}(E))$ by (4.16) and $\widetilde{\widetilde{f}} \in \mathscr{C}_c^\infty(S_{n+1}(F))$ by (4.17), (4.18). It is easy to see that $\widetilde{f} = f_n^\vee * f_{n+1}$ [cf. (7.2)]. The Cayley map [cf. (2.5)] defines a local homeomorphism near a neighborhood of $0 \in \mathfrak{s}$

$$\begin{aligned} \mathfrak{c} = \mathfrak{c}_{n+1} : \mathfrak{s} &\to S_{n+1} \\ X &\mapsto (1+X)(1-X)^{-1}, \end{aligned}$$

and its inverse is given by

$$\mathfrak{c}^{-1}(x) = -(1-x)(1+x)^{-1}.$$

In particular, for a function $\Phi \in \mathscr{C}_c^\infty(S_n)$ ($\phi \in \mathscr{C}_c^\infty(\mathfrak{s})$, resp.) with support in a small neighborhood of $1 \in S_{n+1}$ ($0 \in \mathfrak{s}$, resp.), we may consider it as a function on $\mathfrak{s}$ (S_{n+1}, resp.) denoted by $\mathfrak{c}^{-1}(\Phi)$ ($\mathfrak{c}(\phi)$, resp.). We also have a morphism,

$$\begin{aligned} \iota = \iota_{n+1} : \mathfrak{s} &\to H_{n+1}(E), \\ X &\mapsto 1 + X, \end{aligned}$$

such that, wherever they are all defined, we have $\nu \circ \iota = \mathfrak{c}$,

$$\begin{tikzcd} & H_{n+1}(E) \arrow[d, "\nu"] \\ \mathfrak{s} \arrow[ur, "\iota"] \arrow[r, "\mathfrak{c}"] & S_{n+1} \end{tikzcd}$$

where ν is as in (4.8).

Definition 8.4. We associate with $f \in \mathscr{C}_c^\infty(H_n(E) \times H_{n+1}(E))$ a function on $\mathfrak{s}$ denoted by $f_\natural$,

$$f_\natural(X) := \int_{H_n(F)} \widetilde{f}((1+X)h)\, dh, \qquad X \in \mathfrak{s}, \tag{8.5}$$

if n is even, and

$$f_\natural(X) := \int_{H_n(F)} \widetilde{f}((1+X)h)\eta'((1+X)h)\, dh, \qquad X \in \mathfrak{s}, \tag{8.6}$$

if n is odd, where $\widetilde{f}$ is defined by (4.16).

Then we have when $\det(1-X)\neq 0$

$$\mathfrak{c}^{-1}(\widetilde{\widetilde{f}})(X)=f_\natural(X),\qquad X\in\mathfrak{s}.$$

From now on, all the test functions at hand will be supported in suitable neighborhoods of the obvious distinguished points of $H_n(E)\times H_{n+1}(E), S_{n+1}(F)$ and $\mathfrak{s}(F)$ so that $\mathfrak{c}$ is well-defined. In particular, we have $f_\natural\in\mathscr{C}_c^\infty(\mathfrak{s})$.

We then consider $\mathfrak{s}(F)$ as a subspace of $M_{n+1}(E)$. On $M_{n+1}(E)$ we have a bilinear pairing $\langle X,Y\rangle:=\mathrm{tr}(XY)$, under which the decomposition $M_{n+1}(E)=M_{n+1}(F)\oplus\mathfrak{s}(F)$ is orthogonal. We then define the Fourier transform on $\mathfrak{s}(F)$ with respect to the restriction of the above pairing,

$$\widehat{f_\natural}(X):=\int_{\mathfrak{s}}f_\natural(Y)\psi(\mathrm{tr}(XY))\,dY.$$

Here the measure is normalized so that $\widehat{\widehat{f_\natural}}(X)=f_\natural(-X)$ [cf. §2 (2.2)]. We will consider the orbital integral (Definition 6.10) of the regular unipotent element

$$\xi_-=\xi_{n+1,-}=\tau\begin{pmatrix}0&0&\cdots&0\\1&0&0&\cdots\\\cdots&1&0&0\\0&\cdots&1&0\end{pmatrix}\in\mathfrak{s}(F).\tag{8.7}$$

Theorem 8.5. *Let Π be an irreducible unitary generic representation. Then for any small neighborhood Ω of $1\in G'(F)$, there exists admissible $f\in\mathscr{C}_c^\infty(\Omega)$ such that for all $s\in\mathbb{C}$*

$$I_{\Pi,s}(f)=|\tau|_E^{(d_n+d_{n+1})/2}\omega_{\Pi_n}(\tau)\cdot\mu_{\xi_-}(\widehat{f_\natural}),$$

where ω_{Π_n} is the central character of Π_n, the constant $d_n=\binom{n}{3}$ is as in (4.23).

Remark 14. Note that $\mu_{\xi_-}(\widehat{f_\natural})$ depends on the choice of τ, but $\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f_\natural})$ does not.

The proof will occupy the rest of this section by several steps.

8.3. **Determine $f_\natural$.** We consider admissible functions of the form $f_n^{\varphi_{n-1},\phi_{n-1}}\otimes f_{n+1}^{\phi_n^\vee}$—*note the change to $\phi_n^\vee$ to simplify our later exposition [cf. (7.2)]*. Let P—not to be confused with the mirabolic P_n—be the subgroup P of $H_{n+1}(E)$ consisting of elements

$$\begin{pmatrix}*&*&*&*\\ *&\cdots&*&*\\0&0&1&*\\0&0&0&1\end{pmatrix}\in H_{n+1}(E)\subset M_{n+1}(E).$$

The triple $(\varphi_{n-1},\phi_{n-1},\phi_n)$ then defines a function Ψ on P by

$$\Psi\left[\begin{pmatrix}x&u&\\&1&u'\\&&1\end{pmatrix}\right]:=\varphi_{n-1}(x)\phi_{n-1}(u)\phi_n(u'),\tag{8.8}$$

where $x\in H_{n-1}(E),u\in M_{n-1,1}(E),u'\in M_{n,1}(E)$. For simplicity, we write

$$f^\Psi=f_n^{\varphi_{n-1},\phi_{n-1}}\otimes f_{n+1}^{\phi_n^\vee}.\tag{8.9}$$

We also consider the Lie algebra $\mathfrak{p}$ of P, consisting of

$$\begin{pmatrix} * & * & * & * \\ * & \cdots & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M_{n+1}(E).$$

Both $\mathfrak{p}$ and P are considered as subsets of $M_{n+1}(E)$. The map $p \mapsto 1+p$ from $\mathfrak{p}$ to P defines a local homeomorphism from $\varpi\mathfrak{p}(\mathcal{O}_E)$ to its image. The function Ψ is supported in the subgroup of $P(E)$,

$$P(\varpi\mathcal{O}) \oplus \mathfrak{p}(\varpi\mathcal{O}_E^-) = 1 + \mathfrak{p}(\varpi\mathcal{O}_E).$$

Note that $1+\mathfrak{p}(\varpi\mathcal{O}_E)$ is a compact subgroup of $M_{n+1}(E)$ and hence is unimodular. So both the left and the right invariant measures on $P(E)$ are restricted to a Haar measure on it. Let $(\varphi_{n-1}, \phi_{n-1}, \phi_n)$ be (m, m')-admissible. We may write $\phi_i = \phi^+ \otimes \phi_i^-$ $(i = n-1, n)$ according to the decomposition $M_{i,1}(E) = M_{i,1}(F) \oplus M_{i,1}(E^-)$ and similarly for φ_{n-1} viewed as a function on $M_{n-1}(E)$. Write $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ for $\mathfrak{p}^{\pm} = \mathfrak{p} \cap M_{n+1}(E^{\pm})$. Then we define Ψ^+ and Ψ^- as functions on $1 + \mathfrak{p}^+(\mathcal{O}_F)$ and $\mathfrak{p}^-$, respectively,

$$\Psi^? = \varphi_{n-1}^? \otimes \phi_{n-1}^? \otimes \phi_n^?, \qquad ? = \pm,$$

as follows:

$$\Psi^+\left[\begin{pmatrix} x & u & \\ & 1 & u' \\ & & 1 \end{pmatrix}\right] := \varphi_{n-1}^+(x)\phi_{n-1}^+(u)\phi_n^+(u'), \tag{8.10}$$

where $x \in H_{n-1}(F), u \in M_{n-1,1}(F), u' \in M_{n,1}(F)$, and

$$\Psi^-\left[\begin{pmatrix} x & u & \\ & 0 & u' \\ & & 0 \end{pmatrix}\right] := \varphi_{n-1}^-(x)\phi_{n-1}^-(u)\phi_n^-(u'), \tag{8.11}$$

where $x \in H_{n-1}(E^-), u \in M_{n-1,1}(E^-), u' \in M_{n,1}(E^-)$. We have

$$\Psi(p_+ + p_-) = \Psi^+(p_+)\Psi^-(p_-), \qquad p_+ \in 1+\mathfrak{p}^+, p_- \in \mathfrak{p}_-. \tag{8.12}$$

Lemma 8.6. *Assume that $f^\Psi = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n^\vee}$ is admissible. Then the function Ψ^- has the following invariance property:*

$$\Psi^-(p_+p_-) = \Psi^-(p_-) \tag{8.13}$$

whenever $p_+ \in supp(\Psi^+)$.

Proof. By the admissibility (8.1), we may and will assume that Ψ^+ is a multiple of the characteristic function of the subset of $1 + \mathfrak{p}^+(\mathcal{O}_F)$ consisting of the matrices (x_{ij}) where $x_{ij} \equiv \delta_{ij} (\mathrm{mod}\, \varpi^m)$ when $j \leq n$ and $x_{i,n+1} \equiv \delta_{i,n+1} (\mathrm{mod}\, \varpi^{m'})$. Any $p^+ \in supp(\Psi^+)$ is of the form

$$\begin{pmatrix} x & u & \\ & 1 & u' \\ & & 1 \end{pmatrix}, \qquad x \in 1 + \varpi^m M_{n-1}(\mathcal{O}_F), \quad u \in \varpi^m M_{n-1,1}(\mathcal{O}_F).$$

To show (8.13), we may assume that $p_- \in supp(\Psi^-)$. Now we note that

$$p_+p_- = \begin{pmatrix} x & u & \\ & 1 & u' \\ & & 1 \end{pmatrix} \begin{pmatrix} x_- & u_- & v_- \\ & 0 & v'_- \\ & & 0 \end{pmatrix} = \begin{pmatrix} xx_- & xu_- & xv_- + uv'_- \\ & 0 & v'_- \\ & & 0 \end{pmatrix}.$$

Now the invariance follows from Definition 7.2 (for ϕ_n), and Properties (iii), (iv) of $\varphi_{n-1}\otimes\phi_{n-1}$ in Prop. 7.5. $\square$

Lemma 8.7. *Fix* Π *and assume that* $f^\Psi=f_n^{\varphi_{n-1},\phi_{n-1}}\otimes f_{n+1}^{\phi_n^\vee}$ *is admissible for* Π. *Then we have*

$$f_\natural^\Psi(X)=c(\Psi^+)\int_{\mathfrak{p}^-}f_\natural(X+p)\Psi^-(p)\,dp,$$

where $f_\natural$ *is the function on* $\mathfrak{s}$ *associated to* $f_n\otimes f_{n+1}$ *and*

$$c(\Psi^+)=\int_{1+\mathfrak{p}^+(\varpi\mathcal{O})}\Psi^+(p)\,dp \tag{8.14}$$

is a constant.

Proof. We consider the case when n is odd; the case n even is similar and only requires us to change notations in several places. By definition we have

$$\widetilde{f^\Psi}=(f_n*\phi_{n-1}*\varphi_{n-1})^\vee*f_{n+1}^{\phi_n^\vee}=\varphi_{n-1}^\vee*\phi_{n-1}^\vee*f_n^\vee*f_{n+1}^{\phi_n^\vee}.$$

Since f_i $(i=n,n+1)$ is a multiple of $1_{1+\varpi_E^r M_i(\mathcal{O}_E)}$ for some and $\int f(g)dg=1$, we may assume that $f_n^\vee*f_{n+1}^{\phi_n}$ is the same as $f_{n+1}^{\phi_n}$,

$$\widetilde{f^\Psi}=\varphi_{n-1}^\vee*\phi_{n-1}^\vee*f_{n+1}^{\phi_n^\vee}=\varphi_{n-1}^\vee*\phi_{n-1}^\vee*\phi_n^\vee*f_{n+1}.$$

Explicitly, this reads

$$\widetilde{f^\Psi}(g)=\int\varphi_{n-1}(x)\phi_{n-1}(u)f_{n+1}^{\phi_n^\vee}(uxg)\,dx\,du=\int_P\Psi(p)f_{n+1}(pg)\,dp.$$

Let us denote the right hand side by $f_{n+1}^\Psi(g)$. Our choice of f_{n+1} also implies that f_{n+1} is conjugate invariant under $1+\varpi M_{n+1}(\mathcal{O}_E)$,

$$f_{n+1}(hgh^{-1})=f_{n+1}(g),\qquad h\in 1+\varpi M_{n+1}(\mathcal{O}_E). \tag{8.15}$$

By the support condition, $f_\natural^\Psi(X)$ vanishes unless $X\in\mathfrak{s}(\mathcal{O}_F)$. We thus assume that $X\in\mathfrak{s}(\mathcal{O}_F)$. Then by definition and the support condition of Ψ, we have

$$\begin{aligned}f_\natural^\Psi(X)&=\int_{H_{n+1}(F)}\int_{P(E)}\Psi(p)f_{n+1}(p(1+X)h)\,dp\,dh\\&=\int_{H_{n+1}(F)}\int_{\mathfrak{p}^-}\int_{P(F)}\Psi(p_+(1+p_-))f_{n+1}(p_+(1+p_-)(1+X)h)\,dp_+\,dp_-\,dh.\end{aligned}$$

Note that $\Psi(p_+(1+p_-))=\Psi^+(p_+)\Psi^-(p_+p_-)$ [cf. (8.12)] and $\Psi^-(p_+p_-)=\Psi^-(p_-)$ [cf. (8.13)]. Together with the conjugate invariance (8.15), we have

$$\begin{aligned}f_\natural^\Psi(X)&=\int_{H_{n+1}(F)}\int_{\mathfrak{p}^-}\int_{P(F)}\Psi^+(p_+)\Psi^-(p_-)f_{n+1}((1+p_-)(1+X)hp_+)\,dp_+\,dp_-\,dh\\&=\int_{H_{n+1}(F)}\int_{\mathfrak{p}^-}\int_{P(F)}\Psi^+(p_+)\Psi^-(p_-)f_{n+1}((1+p_-)(1+X)h)\,dp_+\,dp_-\,dh\\&=c(\Psi^+)\int_{H_{n+1}(F)}\int_{\mathfrak{p}^-}\Psi^-(p_-)f_{n+1}((1+p_-)(1+X)h)\,dp_-\,dh.\end{aligned}$$

Note that

$$(1+p_-)(1+X)=(1+p_-X)(1+(1+p_-X)^{-1}(p_-+X))$$

and $(1+p_-X) \in 1+\varpi M_{n+1}(\mathcal{O}_F) \subset H_{n+1}(F)$. We have

$$\int_{H_{n+1}(F)} \int_{\mathfrak{p}^-} \Psi^-(p_-) f_{n+1}((1+p_-)(1+X)h)\, dp_-\, dh$$
$$= \int_{H_{n+1}(F)} \int_{\mathfrak{p}^-} \Psi^-(p_-) f_{n+1}((1+(1+p_-X)^{-1}(p_-+X))h)\, dp_-\, dh.$$

Compared to the definition of $f_\natural$ for $f = f_n \otimes f_{n+1}$, we obtain

$$f_\natural^\Psi(X) = c(\Psi^+) \int_{\mathfrak{p}^-} \Psi^-(p_-) f_\natural((1+p_-X)^{-1}(p_-+X))\, dp_-.$$

Finally note that $f_\natural(X)$ is a multiple of $1_{\varpi^r \mathfrak{s}(\mathcal{O}_F)}$ for some $r > 1$. It follows that $f_\natural(X) = f_\natural(hX)$ for any $h \in 1+\varpi M_n(\mathcal{O}_F)$. Since $p_- \in supp(\Psi^-) \subset \mathfrak{p}^-(\varpi\mathcal{O})$ and $X \in \mathfrak{s}(\mathcal{O})$, we therefore obtain

$$f_\natural^\Psi(X) = c(\Psi^+) \int_{\mathfrak{p}^-} \Psi^-(p_-) f_\natural(p_-+X)\, dp_-.$$

This completes the proof. $\square$

8.4. **Local constancy of the orbital integral and a formula for the regular nilpotent orbital integral.** To compare with the unitary group case in the next section, we need to understand the orbital integral of $\widehat{f_\natural^\Psi}$ in Lemma 8.7, at least around zero. We show that the orbital integral is locally constant around zero. This constant is essentially given by the regular unipotent orbital integral [for ξ_- defined by (8.7)] of Fourier transform $\widehat{f_\natural^\Psi}$ of $f_\natural^\Psi$ on $\mathfrak{s}$.

Lemma 8.8. *Fix any m admissible function $(\varphi_{n-1}, \phi_{n-1})$.*

(1) *For an arbitrarily large compact neighborhood $\mathscr{X}$ of $0 \in \mathfrak{s}$, there exists large enough (m, m', r) and an (m, m', r)-admissible function $f^\Psi = f_n^{\varphi_{n-1},\phi_{n-1}} \otimes f_{n+1}^{\phi_n^\vee}$, such that the orbital integral $\eta'(\Delta_-(X)) O(X, \widehat{f_\natural^\Psi})$ is a (nonzero) constant for regular semisimple $X \in \mathscr{X}$ and this constant is equal to $\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f_\natural^\Psi})$.*

(2) *Let f^Ψ be as in (1). Then the regular unipotent orbital integral $\mu_{\xi_-}(\widehat{f_\natural^\Psi})$ is equal to*

$$c(\Psi^+)|\overline{\delta}_{n,E}(\epsilon_n)|^{-1/2} \prod_{i=1}^{n-1} \phi_i^{'-}(0)$$
$$\times \int_{A_n(F)N_{n,-}(F)} \prod_{i=1}^{n} \widehat{\phi_i^-}(-a_i(v_{i-1},1)\tau) |\overline{\delta}_n(a)|^{-1} \eta(a)\, d^*a\, dv,$$

where $v_i \in M_{i,1}(F)$, and $\overline{\delta}_n(a)$ is defined by (8.2).

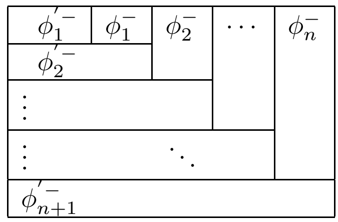
Proof. Without loss of generality we may assume $c(\Psi^+) = 1$. Assume that f^Ψ is (m, m', r)-admissible. By Lemma 8.7, when r is large enough, the function $f_\natural^\Psi$ on $\mathfrak{s}_{n+1}$ is of the form $\Psi^- \otimes \phi_n^{'-} \otimes \phi_{n+1}^-$ corresponding to the decomposition

$$\mathfrak{s}_{n+1} = \mathfrak{p}^- \oplus M_{1,n}(E^-) \oplus M_{1,n+1}(E^-).$$

More precisely we may write the function on $\mathfrak{s}$ defined by

$$X \mapsto f_\natural^\Psi(-X)$$

in the following form:



We also write $f_\natural^\Psi$ as the tensor product

$$\varphi_n^- \otimes \phi_n^- \otimes \phi_{n+1}^{'-},$$

where $\varphi_n^- \in \mathscr{C}_c^\infty(\mathfrak{s}_n)$. We recall their key properties for our computation below:

(i) For $i = 1, \dots, n-1$, $\phi_i^{'-}$ is a nonzero multiple of the characteristic function of $\varpi^m \mathcal{O}_{E^-}$; each of $\phi_n^{'-}$ and $\phi_{n+1}^{'-}$ is a nonzero multiple of the characteristic function of $\varpi^r \mathcal{O}_{E^-}$, normalized so that $\widehat{\phi_n^{'-}}(0) = \widehat{\phi_{n+1}^{'-}}(0) = 1$.

(ii) For $i = 1, \dots, n-1$, ϕ_i is in $\mathscr{C}_c^\infty(M_{i,1}(E))_m^\dagger$; ϕ_n is in $\mathscr{C}_c^\infty(M_{i,1}(E))_{m'}^\dagger$.

(iii) The function $\widehat{\varphi_n^-}$ on $\mathfrak{s}_n$ is invariant under left and right multiplication by $1 + \varpi^m M_{n-1}(\mathcal{O}_F)$ [as a subgroup of $H_{n-1}(F)$, and hence of $H_n(F)$].

Back to the orbital integral, we may fix an arbitrarily large compact neighborhood $\mathscr{X}$ of 0 in the quotient of $\mathfrak{s}_{n+1}$ by H_n. We use the following regular elements [cf. §6, (6.13)]:

$$\varrho(x,y) = \tau \begin{pmatrix} 0 & \cdots & 0 & x_n & y_n \\ 1 & 0 & \cdots & x_{n-1} & y_{n-1} \\ 0 & 1 & 0 & \cdots & y_{n-2} \\ \cdots & 0 & 1 & x_1 & \cdots \\ 0 & \cdots & 0 & 1 & y_0 \end{pmatrix} \in \mathfrak{s}_{n+1}, \tag{8.16}$$

where $(x, y) \in F^{2n+1}$ and we may assume that $\mathscr{X}$ is a compact neighborhood of 0 in F^{2n+1}. We also denote

$$\varrho(x) = \tau \begin{pmatrix} 0 & \cdots & 0 & x_n \\ 1 & 0 & \vdots & x_{n-1} \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & 1 & x_1 \end{pmatrix} \in \mathfrak{s}_n. \tag{8.17}$$

It suffices to show that, when we increase m, m', r suitably, the orbital integral of $\varrho(x, y)$ is a constant when (x, y) lies in the fixed compact set $\mathscr{X}$. We proceed in three steps.

Step 1. To ease notation, we denote

$$f' = \widehat{f_\natural^\Psi}.$$

By definition we have

(8.18)

$$O(\varrho(x,y), \widehat{f_\natural^\Psi}) = \int_{H_n(F)} \widehat{f_\natural^\Psi}(h^{-1}\varrho(x,y)h)\eta(h)\, dh = \int_{H_n(F)} f'(h^{-1}\varrho(x,y)h)\eta(h)\, dh.$$

We may write $h \in H'_n(F)$ as

$$h = a_n k v, \qquad k \in N_n H_{n-1}, \quad v = \begin{pmatrix} 1_{n-1} & \\ v_{n-1} & 1 \end{pmatrix}, \quad a_n \in F^\times.$$

Then the last row of $h^{-1}\xi_- h$ is of the form

$$(a_n(v_{n-1}, 1), y_0)\tau.$$

For the integrand $f'(h^{-1}\varrho(x,y)h)$ to be nonzero, $a_n(v_{n-1},1)$ must be in the support of $\widehat{\phi_n^-}$. Since $\phi_n \in \mathscr{C}_c^\infty(M_{n,1}(E))^\dagger_{m'})$, we may assume that [cf. (7.15)]

$$\|v_{n-1}\| \le |\varpi^{m'}|.$$

Now we *claim that if $h^{-1}\varrho(x)h$ lies in the support of $\widehat{\varphi_n^-}$, then the last column of $h^{-1}\varrho(x)h \in \mathfrak{s}_n$ is bounded by a polynomial of the norms of x_i $(1 \le i \le n)$ and $|\varpi|^{-m}$, independent of r.* To show this, suppose that $g = h^{-1}\varrho(x)h$ lies in the support of $\widehat{\varphi_n^-}$. The same argument as in the previous paragraph shows that we may find $v = \begin{pmatrix} 1_{n-2} & \\ v_{n-2} & 1 \end{pmatrix} \in N_{n-1,-} \in N_{n,-}$ with $\|v_{n-2}\| \le |\varpi|^m$, such that $v^{-1}gv$ is of the form

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & \alpha_{n-1} & * \end{pmatrix}.$$

Since $\widehat{\varphi_n^-}$ is invariant under both left and right multiplication by $1+\varpi^m M_{n-1}(\mathcal{O}_F)$, the above $v^{-1}gv$ again lies in the support of $\widehat{\varphi_n^-}$. Continuing this process, we may find an element v of $N_{n-1,-}(F) \subset N_n(F)$ whose off-diagonal entries all lie in $\varpi^m\mathcal{O}_F$, such that $v^{-1}gv$ is of the form

$$\begin{pmatrix} * & * & * & * & * \\ \alpha_1 & * & * & * & * \\ 0 & \alpha_2 & * & * & * \\ 0 & 0 & \ddots & * & * \\ 0 & 0 & 0 & \alpha_{n-1} & * \end{pmatrix}$$

and remains in the support of $\widehat{\varphi_n^-}$. Since the functions ϕ_i, $1 \le i \le n-1$, lie in $\mathscr{C}_c^\infty(M_{i,1}(E))^\dagger_m$, all $|\alpha_i|$ $(1 \le i \le n-1)$ are equal to some constant multiple of $|\varpi|^{-2m}$. Note that the x_i's are the coefficients of the characteristic polynomial of g and hence of $v^{-1}gv$. Now by Corollary 6.7 (taking transpose), it is easy to see that each entry of the last column of $v^{-1}gv$ is a polynomial of the first $n-1$ columns of $v^{-1}gv$, α_i, α_i^{-1}'s and the x_i's. This shows that the claim holds for the last column of $v^{-1}gv$ and hence also for g itself since the off-diagonal entries of v are bounded by $|\varpi|^m$. This proves the claim.

Now by the claim, there exists large enough m'_0 and r_0 such that once $m' \ge m'_0$ and $r \ge r_0$, we have the following invariance property when $(x,y) \in \mathscr{X}$:

$$f'(h^{-1}\varrho(x,y)h) = \widehat{\varphi_n^-}(-k^{-1}\varrho(x)k)\widehat{\phi_n^-}(-a_n(v_{n-1},1)\tau)\widehat{\phi'^-_{n+1}}(h^{-1}y\tau) \tag{8.19}$$

if the left hand side is not zero [i.e., at least $h^{-1}\varrho(x,y)h$ lies in the support of f'].

Step 2. We now may repeat the process and utilize the property (iii) [i.e., the invariance of $\widehat{\varphi_n^-}$ under $1+\varpi^m M_{n-1}(\mathcal{O}_F)$]. We write

$$h = uav, \quad a = \begin{pmatrix} a_1a_2\cdots a_n & & \\ & \ddots & \\ & & a_n \end{pmatrix}, \quad u \in N_n(F),$$

and write $v \in N_{n-}(F)$ as the product of $\begin{pmatrix} 1_{n-i} & \\ v_{n-i} & 1 \end{pmatrix} \in N_{n-i+1,-}(F) \subset N_{n-}(F)$ for $1 \leq i \leq n-1$. In Step 1 we have seen that $\|v_{n-1}\| \leq |\varpi|^{m\prime}$. Since the functions ϕ_i, $1 \leq i \leq n-1$, lie in $\mathscr{C}_c^\infty(M_{i,1}(E))_m^\dagger$, and by the property (iii), we may inductively show that

$$\|v_i\| \leq |\varpi|^m, \qquad 1 \leq i \leq n-2.$$

We now view $\bigotimes_{i=1}^{n+1} \widehat{\phi_i'^-}$ as a function on the set of upper triangular elements and then consider it as a function on $\mathfrak{s}_{n+1}$ via the natural projection from $\mathfrak{s}_{n+1}$ to the upper triangular elements. Then, when $(x,y) \in \mathscr{X}$, the orbital integral $O(\varrho(x,y), \widehat{f_\natural^\Psi})$ is equal to [cf. (8.18)]

(8.20)
$$\int \left(\bigotimes_{i=1}^{n+1} \widehat{\phi_i'^-}\right)(-(ua)^{-1}\varrho(x,y)ua)\,du \prod_{i=1}^{n} \widehat{\phi_i^-}(-a_i(v_{i-1},1)\tau)|\delta_n(a)|^{-1}\eta(a)\,d^*a\,dv,$$

where $u \in N_n(F), v \in N_{n,-}(F)$.

Step 3. Note that $u^{-1}\varrho(x,y)u$ is of the form

$$u^{-1}\varrho(x,y)u = \tau \begin{pmatrix} * & * & * & * & * \\ 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{pmatrix}.$$

By Lemma 6.6, we may make a substitution to replace the integral over u in (8.20) by an integral over u' of elements of the form

$$u' = \begin{pmatrix} * & * & * & 0 \\ 1 & * & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in H_n(F), \tag{8.21}$$

and the measure du' is induced by du,

$$\int \left(\bigotimes_{i=1}^{n+1} \widehat{\phi_i'^-}\right)((ua)^{-1}\varrho(x,y)ua)\,du = \int \left(\bigotimes_{i=1}^{n+1} \widehat{\phi_i'^-}\right)\left(a^{-1}\begin{pmatrix} u' & * & * \\ 1 & * & * \\ 0 & 1 & * \end{pmatrix} a\tau\right)du',$$

where the last two columns are polynomials of entries of u', x_i, y_j, and n, by Lemma 6.6 (taking transpose). Now we may increase r suitably (i.e., increase the support of $\widehat{\phi_n'^-}$ and $\widehat{\phi_{n+1}'^-}$) so that the constraints of the last columns on u' are superfluous. In particular, there exists $r_1 > r_0$ large enough (depending on m, m', and $\mathscr{X}$) such that when $r > r_1$ we have

$$\int \left(\bigotimes_{i=1}^{n+1} \widehat{\phi_i'^{-}} \right) ((ua)^{-1} \varrho(x,y) ua)\, du = \widehat{\phi_{n+1}'^{-}}(0) \widehat{\phi_n'^{-}}(0) \int \left(\bigotimes_{i=1}^{n-1} \widehat{\phi_i'^{-}} \right) \left(a^{-1} u' a \tau \right) du'.$$

This is independent of $(x,y) \in \mathscr{X}$. By (8.20), we conclude that the orbital integral $O(\varrho(x,y), \widehat{f_\natural^\Psi})$ is a constant and the constant is equal to the regular unipotent orbital integral $O(\varrho(0,0), \widehat{f_\natural^\Psi}) = \mu_{\xi_-}(\widehat{f_\natural^\Psi})$. This finishes the proof of the first part of the lemma.

To prove the second part of the lemma, it remains to evaluate the regular unipotent orbital integral $\mu_{\xi_-}(\widehat{f_\natural^\Psi})$. First we note that $\widehat{\phi_n'^{-}}(0) = \widehat{\phi_{n+1}'^{-}}(0) = 1$ by our normalization. We make the substitution $u' \mapsto a u' a^{-1} = Ad(a) u'$ (equivalently, $u'_{ij} \mapsto u'_{ij} \prod_{i<\ell\le j} a_\ell$ for $1 \le i \le n-1$). This yields

$$\begin{aligned} \int \left(\bigotimes_{i=1}^{n-1} \widehat{\phi_i'^{-}} \right) (a^{-1} u' a \tau)\, du' &= \det(Ad(a) : \mathfrak{n}_{n-1}) \int \left(\bigotimes_{i=1}^{n-1} \widehat{\phi_i'^{-}} \right) (u' \tau)\, du' \\ &= \det(Ad(a) : \mathfrak{n}_{n-1}) |\tau|_E^{-(1+2+\cdots+(n-2))/2} \prod_{i=1}^{n-1} \phi_i'^{-}(0). \end{aligned}$$

[Or more explicitly $\det(Ad(a) : \mathfrak{n}_{n-1}) = \prod_{i=1}^{n-1} |a_i|^{(i-1)(n-i)}$.] Here the factor

$$|\tau|_E^{(1+2+\cdots+(n-2))/2} = |\overline{\delta}_{n,E}(\epsilon_n)|^{1/2}$$

is caused by the difference between the measures on $F\tau$ and E^-. We thus proved that $\mu_{\mathcal{O}_0}(\widehat{f_\natural})$ is equal to

$$|\overline{\delta}_{n,E}(\epsilon_n)|^{-1/2} \prod_{i=1}^{n-1} \phi_i'^{-}(0) \int_{A_n(F) N_{n,-}(F)} \prod_{i=1}^{n} \widehat{\phi_i^-}(-a_i(v_{i-1}, 1)\tau) |\overline{\delta}_n(a)|^{-1} \eta(a)\, d^*a\, dv.$$

[Or more explicitly, $\overline{\delta}_n(a) = \det(Ad(a) : \mathfrak{n}_n)) / \det(Ad(a) : \mathfrak{n}_{n-1}) = \prod_{i=1}^{n-1} |a_i|^{n-i}$.] $\square$

8.5. **Proof of Theorem 8.5.** We choose an admissible function $f^\Psi = f_n^{\varphi_{n-1}, \phi_{n-1}} \otimes f_{n+1}^{\phi_n^\vee}$ for Π so that it also verifies the conditions Lemma 8.8. We decompose $\varphi_{n-1}, \phi_{n-1}$ as in Lemma 7.6 and similarly ϕ_n. Note that

$$\begin{aligned} &\int \prod_{i=0}^{n-1} \widehat{\phi_{n-i}}(-y_i(v_{n-i-1}, 1)\tau) |\overline{\delta}_n(y)|^{-1} \eta(y)\, d^*y\, dv, \\ &= \left(\prod_{i=0}^{n-1} \widehat{\phi_{n-i}^+}(0) \right) \int \prod_{i=0}^{n-1} \widehat{\phi_{n-i}^-}(-y_i(v_{n-i-1}, 1)\tau) |\overline{\delta}_n(y)|^{-1} \eta(y)\, d^*y\, dv, \\ &= \left(\prod_{i=1}^{n} \int_{M_{i,1}(F)} \phi_i^+(x_i)\, dx_i \right) \int \prod_{i=0}^{n-1} \widehat{\phi_{n-i}^-}(-y_i(v_{n-i-1}, 1)\tau) |\overline{\delta}_n(y)|^{-1} \eta(y)\, d^*y\, dv, \end{aligned}$$

where $y \in A_n(F), v \in N_{n,-}(F)$ are as in (8.3) and (8.4). Moreover,

$$\int_{B_{n-1,-}(F)} \phi'(b)\, db = \prod_{i=1}^{n-1} \phi_i'^{-}(0) \prod_{i=1}^{n-1} \int_{M_{1,i}(F)} \phi_i'^{+}(b_i)\, db_i.$$

It is clear that the constant in (8.14) is given by

$$c(\Psi^+) = \prod_{i=1}^{n} \int_{M_{i,1}(F)} \phi_i^+(x_i)\, dx_i \prod_{i=1}^{n-1} \int_{M_{1,i}(F)} \phi_i^{'+}(b_i)\, db_i.$$

Note that

$$|\tau|_E^{(d_n+d_{n+1})/2} = |\delta_{n-1,E}(\epsilon_{n-1})|^{1/2} |\delta_{n,E}(\epsilon_n)|^{1/2} = |\delta_{n-1,E}(\epsilon_{n-1})| |\overline{\delta}_{n,E}(\epsilon_n)|^{1/2}.$$

Then the identity in Theorem 8.5 follows by comparing Prop. 8.3 with Lemma 8.8. Moreover, we may choose such an admissible function with arbitrarily small support by increasing (m, m', r) and such that $\mu_{\xi_-}(f_\natural^\Psi) \neq 0$.

9. Local character expansion in the unitary group case

9.1. **Three ingredients from** [51]**.** Let F be a non-Archimedean local field of characteristic zero. We need to recall the main local results of [51]. There are two isomorphism classes of Hermitian spaces W_1, W_2 of dimension n. Denote by $H_{W_i} = U(W_i)$ the unitary group. We let $V_i = W_i \oplus Ee$ be the orthogonal sum of W_i and a one-dimensional space Ee with norm $\langle e, e\rangle = 1$. Denote by $\mathfrak{u}_i$ the Lie algebra of $U(V_i)$. We have a bijection of regular semisimple orbits (cf. (4.6) and [51, §3.1])

$$H_n(F)\backslash \mathfrak{s}(F)_{rs} \simeq H_{W_1}(F)\backslash \mathfrak{u}_1(F)_{rs} \coprod H_{W_2}(F)\backslash \mathfrak{u}_2(F)_{rs}.$$

A regular semisimple $X \in \mathfrak{s}$ matches some $Y \in \mathfrak{u}_i$ if and only if

$$\eta(\Delta(X/\tau)) = \eta(\mathrm{disc}(W_i)), \tag{9.1}$$

where $\mathrm{disc}(W_i) \in F^\times/\mathrm{N}E^\times$ is the discriminant of W_i. For an $f' \in \mathscr{C}_c^\infty(\mathfrak{s})$ and a pair (f_1, f_2), $f_i \in \mathscr{C}_c^\infty(\mathfrak{u}_i)$, we say that f' matches (f_1, f_2), if for all matching regular semisimple $X \in \mathfrak{s}, Y \in \mathfrak{u}_i$, we have [cf. (6.14)]

$$\eta'(\Delta_+(X)) O(X, f') = O(Y, f_i), \tag{9.2}$$

where η' is a fixed choice of character $E^\times \to \mathbb{C}^\times$ with restriction $\eta'|_{F^\times} = \eta$.

Let $W \in \{W_1, W_2\}$. Analogous to the general linear group case (cf. Definition 8.4), with a function in a small neighborhood of $1 \in G = U(W) \times U(V)$, we associate a function on the Lie algebra $\mathfrak{u}$ of $U(V)$. For $f = f_n \otimes f_{n+1} \in \mathscr{C}_c^\infty(G)$, we let $\widetilde{f}$ be the function on $U(V)$ defined by

$$\widetilde{f}(g) := \int_{U(W)} f_n(h) f_{n+1}(hg)\, dh, \qquad g \in U(V).$$

If f is supported in a small neighborhood Ω of 1 in G, then $\widetilde{f}$ is supported in a small neighborhood $\widetilde{\Omega}$ of 1 in $U(V)$. Since the Cayley map $\mathfrak{c} : \mathfrak{u} \to U(V)$ is a local homeomorphism around $0 \in \mathfrak{u}$, we may denote $\omega = \mathfrak{c}^{-1}(\widetilde{\Omega}) \simeq \widetilde{\Omega}$ and with $\widetilde{f}$ we associate a function denoted by $f_\natural = \mathfrak{c}^{-1}(\widetilde{f})$ on $\mathfrak{u}$ supported in ω. To connect the smooth transfer on the groups to the one on Lie algebras, we need the following.

Lemma 9.1. *Let $f' \in \mathscr{C}_c^\infty(G'(F))$ and $f_i \in \mathscr{C}_c^\infty(U(W_i) \times U(V_i))$ $(i = 1, 2)$ be matching functions [in the sense of §4, (4.14)] with support in a neighborhood of the identity where the Cayley map is well-defined. Then the functions $f'_\natural \in \mathscr{C}_c^\infty(\mathfrak{s})$ and $f_{i,\natural} \in \mathscr{C}_c^\infty(\mathfrak{u}_i)$ $(i = 1, 2)$ match.*

Proof. The support condition ensures that the associated functions $f'_\natural, f_{i,\natural}$ are well-defined. Then it remains to show the transfer factors are compatible,

$$\eta'(\Delta_+(X))O(X, f'_\natural) = \Omega(g)O(g, f'),$$

where $\nu(g) = \mathfrak{c}(X) \in S_{n+1}(F)$. This follows from the proof of [51, Lemma 3.5]. $\square$

Now we consider only one Hermitian space $W \in \{W_1, W_2\}$, with the corresponding groups $U(W), U(V)$, and the Lie algebra $\mathfrak{u}$. We say that $f \in \mathscr{C}_c^\infty(\mathfrak{u})$ and $f' \in \mathscr{C}_c^\infty(\mathfrak{s})$ match if the equality (9.2) holds for all regular semisimple X matching $Y \in \mathfrak{u}$.

In [51, Theorem 2.6] the following result is proved.

Theorem 9.2. *For any $f \in \mathscr{C}_c^\infty(\mathfrak{u})$ there exists a matching $f' \in \mathscr{C}_c^\infty(\mathfrak{s})$ and conversely.*

Moreover, we have [51, Theorem 4.17].

Theorem 9.3. *If the functions f and f' match, then so do $\epsilon(1/2, \eta, \psi)^{n(1+n)/2}\widehat{f}$ and $\widehat{f}'$.*

An important ingredient of the proof of both theorems above is a local relative trace formula on Lie algebra [51, Theorem 4.6]. Now we only need the one in the unitary group case.

Theorem 9.4. *For $f_1, f_2 \in \mathscr{C}_c^\infty(\mathfrak{u})$, we have*

$$\int_{\mathfrak{u}} f_1(X)O(X, \widehat{f_2})\, dX = \int_{\mathfrak{u}} O(X, \widehat{f_1})f_2(X)\, dX,$$

where the integrals are absolutely convergent.

9.2. **A Hypothesis.** We now return to the local spherical character in the unitary group case. Let π be an irreducible admissible representation of $G = U(V) \times U(W)$. We use the measure on $U(V)$ determined by the self-dual measure on $\mathfrak{u}$ via the Cayley map. We call a subset $\Omega \subset G$ a $U(W) \times U(W)$-*domain* (associated to ω) if there is an open and closed subset ω in the F points $(H\backslash\mathfrak{u})(F)$ of categorical quotient $H\backslash\mathfrak{u}$[10] such that

- the Cayley map is defined on the preimage of ω in $\mathfrak{u}$ and takes the preimage of ω to $\Omega' \subset U(V)$.
- Ω is the preimage of Ω' under the contraction map $U(W) \times U(V) \to U(V)$ [given by $(g_n, g_{n+1}) \mapsto g_n g_{n+1}$].

In particular, Ω is $U(W) \times U(W)$-invariant, open and closed.

We consider the following:

Hypothesis $(\star)$ *for* π: there exist a neighborhood $\Omega \subset G$ of $1 \in G$ that is a $U(W) \times U(W)$-domain, and a function $\Phi \in \mathscr{C}_c^\infty(\mathfrak{u})$, such that

$$\Phi(0) = 1, \tag{9.3}$$

and for all $f \in \mathscr{C}_c^\infty(\Omega) \subset \mathscr{C}_c^\infty(G)$,

$$J_\pi(f) = \int_{\mathfrak{u}} f_\natural(X)O(X, \Phi)\, dX.$$

[10] In our case, the categorical quotient $H\backslash\mathfrak{u} := \operatorname{Spec}\mathcal{O}_{\mathfrak{u}}^H$ is an affine space, and the natural morphism $\mathfrak{u} \to H\backslash\mathfrak{u}$ induces a continuous map on the F-points: $\mathfrak{u}(F) \to (H\backslash\mathfrak{u})(F)$.

Theorem 9.5. *Assume that π is tempered and $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$. Let ϕ be a matrix coefficient of π such that*

$$\int_H \phi(h)\, dh = 1.$$

Then the distribution J_π is represented by the orbital integral of ϕ, as a function on G,

$$G \ni g \mapsto O(g, \phi).$$

Moreover, the orbital integral $g \mapsto O(g, \phi)$ is a bi-H-invariant function that is locally L^1 on G.

Proof. When π is tempered and $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$, we have $\alpha \neq 0$ [cf. Property (iii) after (1.3)]. Hence there exists ϕ such that $\int_H \phi(h)dh \neq 0$. Up to a scalar multiplication, we may assume that $\int_H \phi(h)dh = 1$. Then the theorem is proved in [24]. $\square$

Proposition 9.6. *Assume that $\mathrm{Hom}_H(\pi, \mathbb{C}) \neq 0$. If the group $H = U(W)$ is compact or π is supercuspidal, then π verifies Hypothesis $(\star)$.*

Proof. Assume first that π is supercuspidal. We choose an open and closed neighborhood ω of 0 in the categorical quotient of $\mathfrak{u}$. Clearly we may choose such ω so that the Cayley map is defined on the preimage of ω in $\mathfrak{u}$. Then the associated $U(W) \times U(W)$-domain Ω is an open and closed neighborhood of $1 \in G$. Let ϕ be a matrix coefficient as in Theorem 9.5. We consider $\phi_\Omega = \phi \cdot 1_\Omega$ where 1_Ω is the characteristic function of Ω. Since $\phi \in \mathscr{C}_c^\infty(G)$ and Ω is open and closed, the function ϕ_Ω also lies in $\mathscr{C}_c^\infty(G)$. We now consider the function $\widetilde{\phi_\Omega}$ which lies in $\mathscr{C}_c^\infty(U(V))$ and let $\Phi = \phi_{\Omega,\natural} \in \mathscr{C}_c^\infty(\mathfrak{u})$ be the corresponding function on $\mathfrak{u}$ via the Cayley map. It is important to note that we still have

$$\Phi(0) = 1. \tag{9.4}$$

Moreover, the measure on $\mathfrak{u}$ is transferred to the measure on $U(V)$. Then $\Phi \in \mathscr{C}_c^\infty(\mathfrak{u})$ and for all functions $f \in \mathscr{C}_c^\infty(G)$ with small support around 1,

$$J_\pi(f) = \int_{\mathfrak{g}} f_\natural(X) O(X, \Phi)\, dX.$$

If H is compact (so that $dim\, W \leq 2$ or $E/F = \mathbb{C}/\mathbb{R}$), then there is a nonzero vector $\phi_0 \in \pi$ fixed by H. Then for all f, we have

$$J_\pi(f) = \mathrm{vol}(H) \int_G f(g) \langle \pi(g)\phi_0, \phi_0 \rangle\, dg$$

for a norm one $\phi_0 \in \pi^H$. Set $\Phi(g) = \mathrm{vol}(H)^{-1} \langle \pi(g)\phi_0, \phi_0 \rangle$. Then the same truncation as above completes the proof. $\square$

We say that f is admissible if there is an admissible f' matching f.

Theorem 9.7. *Assume that π verifies Hypothesis $(\star)$. Then there exist an admissible functions $f \in \mathscr{C}_c^\infty(G(F))$ and a matching function $f' \in \mathscr{C}_c^\infty(G'(F))$ such that*

$$J_\pi(f) = (\eta'(\tau)/\epsilon(1/2, \eta, \psi))^{n(n+1)/2} \eta(\mathrm{disc}(W)) \widehat{\mu_{\xi_-}}(f'_\natural) \neq 0.$$

Proof. Suppose that in Hypothesis $(*)$ we have a $U(W)\times U(W)$-domain Ω associated to ω in the categorical quotient of $\mathfrak{u}$. Since the categorical quotient of $\mathfrak{s}$ and that of $\mathfrak{u}$ are isomorphic, we also view ω as an open and closed set in the quotient of $\mathfrak{s}$. Let $f'\in\mathscr{C}_c^\infty(G')$ be an (m,m',r) admissible function and $f\in\mathscr{C}_c^\infty(G)$ be a matching function. We claim that we may choose an f supported in Ω. Indeed, we may choose (m,m',r) very large so that the support of f' is very small, say, so that the image of the support of $f'_\natural$ in the categorical quotient of $\mathfrak{s}$ is contained in ω. Now we choose any f_0 that matches f'. Then we set $f=f_0\cdot 1_\Omega$. Clearly the function f has the same orbital integral as f_0 and is supported in Ω. This proves the claim.

Now we apply the local trace formula (Theorem 9.4)

$$\int_{\mathfrak{u}} f_\natural(Y)O(Y,\Phi)\,dY=\int_{\mathfrak{u}} O(Y,\widehat{f_\natural})\check{\Phi}(Y)\,dY,$$

where $\check{\Phi}$ is the inverse of the Fourier transform. By the compatibility between the Fourier transform and the smooth transfer (Theorem 9.3) and (9.2), we have

$$\epsilon(1/2,\eta,\psi)^{n(n+1)/2}O(Y,\widehat{f_\natural})=\eta'(\Delta_+(X))O(X,\widehat{f'_\natural}) \tag{9.5}$$

for matching regular semisimple X and Y. Since $\check{\Phi}$ has compact support, we may choose a compact neighborhood $\mathscr{X}$ of $0\in\mathfrak{s}$ so that the image of $\mathscr{X}$ in the quotient $H_n\backslash\mathfrak{s}(F)\simeq H\backslash\mathfrak{u}(F)$ contains the image of $supp(\check{\Phi})$. By Lemma 8.8, we may choose an admissible function f' such that $\eta'(\Delta_-(X))O(X,\widehat{f'_\natural})$ is equal to a nonzero constant $\eta'(\Delta_-(\xi_-))O(\xi_-,\widehat{f'_\natural})$ when $X\in\mathscr{X}$. Thus for regular semisimple $Y\in supp(\check{\Phi})$ we have

$$O(Y,\widehat{f_\natural})=\epsilon(1/2,\eta,\psi)^{-n(n+1)/2}\eta'(\Delta_+(X)/\Delta_-(X))\eta'(\Delta_-(\xi_-))\mu_{\xi_-}(\widehat{f'_\natural})\neq 0.$$

By comparison with (9.1), we know that

$$\eta(\Delta_+(X)/\Delta_-(X))=\eta(\Delta_+(X/\tau)/\Delta_-(X/\tau))=\eta(\Delta(X/\tau))=\eta(\mathrm{disc}(W)).$$

We note that $\eta'(\Delta_-(\xi_-))=\eta'(\tau)^{n(n+1)/2}$. Therefore for all regular semisimple $Y\in supp(\check{\Phi})$, we have

$$O(Y,\widehat{f_\natural})=(\eta'(\tau)/\epsilon(1/2,\eta,\psi))^{n(n+1)/2}\eta(\mathrm{disc}(W))\mu_{\xi_-}(\widehat{f'_\natural})\neq 0.$$

We obtain

$$\begin{aligned}J_\pi(f)&=\int_{\mathfrak{u}} O(X,\widehat{f_\natural})\check{\Phi}(X)\,dX\\&=(\eta'(\tau)/\epsilon(1/2,\eta,\psi))^{n(n+1)/2}\eta(\mathrm{disc}(W))\mu_{\xi_-}(\widehat{f'_\natural})\cdot\int_{\mathfrak{u}}\check{\Phi}(X)\,dX\\&=(\eta'(\tau)/\epsilon(1/2,\eta,\psi))^{n(n+1)/2}\eta(\mathrm{disc}(W))\mu_{\xi_-}(\widehat{f'_\natural})\cdot\Phi(0).\end{aligned}$$

By Hypothesis $(\star)$ we have

$$\Phi(0)=1.$$

The theorem now follows. □

9.3. **Completion of the Proof of Theorem 4.6: Cases (2)-(ii) and (2)-(iii).** It remains to prove cases (2)-(ii) and (2)-(iii), i.e., when v is nonsplit. If we choose a suitable admissible function f' and a smooth transfer f, then the equality holds for f,f' by Theorem 8.5, Prop. 9.6, and Theorem 9.7.

9.4. **Concluding remarks.** Note that we only deal with π_v which appears as a local component of a global π. But we expect Conjecture 4.4 to hold in general (as long as Π_v is generic in order to define I_{Π_v}).

We conclude with the following.

Conjecture 9.8. *The spherical characters I_Π and J_π are representable by a locally L^1 function which is smooth (locally constant in the non-Archimedean case) on an open subset.*

There should be a more complete analogue of the Harish-Chandra local character expansion in our relative setting. Moreover, it seems that the spherical character (if nonzero) should contain as much information as the usual character of the representation.

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