UNIQUENESS OF SELF-SIMILAR SHRINKERS WITH ASYMPTOTICALLY CONICAL ENDS

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1. INTRODUCTION

Self-shrinkers are a special class of solutions to the mean curvature flow in $\mathbb{R}^{n+1}$, in which a later time slice is a scaled down copy of an earlier slice. More precisely, a hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ is said to be a self-shrinker if it satisfies

$$H = \frac{1}{2} \langle x, n \rangle.$$

Here $H = \text{div}(n)$ is the mean curvature, $n$ is the outward unit normal, $x$ is the position vector and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. One reason that self-shrinking solutions to the mean curvature flow are particularly interesting is that they provide singularity models of the flow; see [20,21], [24] and [48].

It is a fundamental problem to classify smooth complete embedded self-shrinkers in $\mathbb{R}^{n+1}$. And the new families of examples constructed by Kapouleas, Kleene and Møller [28], Nguyen [38–40], and Møller [36], as well as the numerical examples produced by Chopp [4], indicate that it is very difficult to establish a full classification. However, under certain conditions, several results in this direction have been obtained since the 1980s. For $n = 1$, Abresch and Langer [1] had already shown that the circle is the only simple closed self-shrinking curve. For higher dimensions, Colding and Minicozzi [5], proved that the only smooth complete embedded self-shrinkers in $\mathbb{R}^{n+1}$ that are mean convex and have polynomial volume growth are $S^k(\sqrt{2}k) \times \mathbb{R}^{n-k}$, $0 \leq k \leq n$, which generalized an earlier result of Huisken [20,21]. Here $S^k(\sqrt{2}k)$ denotes the $k$-dimensional round sphere centered at the origin with radius $\sqrt{2}k$. Moreover, in the same paper, they showed that those self-shrinkers are the only entropy stable ones under the mean curvature flow. Also, in [47], we established a Bernstein type theorem for smooth entirely graphical self-shrinkers in $\mathbb{R}^{n+1}$, which generalized a result of Ecker and Huisken [11]. Besides, by imposing symmetries, Kleene and Møller [29] classified all smooth complete embedded self-shrinking hypersurfaces of revolution in $\mathbb{R}^{n+1}$.

Our specific interest is in smooth noncompact embedded self-shrinkers. Excluding products, to the author’s knowledge, the only smooth complete noncompact properly embedded self-shrinkers in the literature belong to the family of examples in [28] and [38–40], each of which has a conical structure at infinity. In fact, it is a...
well-known conjecture (see page 39 of [23]) that

**Conjecture 1.1.** Let $\Sigma \subset \mathbb{R}^3$ be a smooth complete properly embedded self-shrinker of finite genus. Then $\Sigma$ outside a sufficiently large ball decomposes into a finite number of ends $U_j$ and for each $j$, either

(a) As $\lambda \to 0^+$, $\lambda U_j$ converges locally smoothly to a cone $C_j$ such that $C_j$ is smooth except at its vertex.

(b) There is a vector $v_j$ such that, as $\tau \to +\infty$, $U_j - \tau v_j$ converges to the cylinder $\{x : \text{dist}(x, \text{span}(v_j)) = \sqrt{2}\}$.

Moreover, there is another conjecture on the rigidity of self-shrinking cylinders (see page 39 of [23]), which asserts that

**Conjecture 1.2.** Let $\Sigma \subset \mathbb{R}^3$ be a smooth complete properly embedded self-shrinker of finite genus. If one end of $\Sigma$ is asymptotic to a cylinder (see (b) of Conjecture 1.1), then $\Sigma$ is isometric to the self-shrinking cylinder.

It is the previously mentioned noncompact examples and Conjectures 1.1 and 1.2 that motivate our investigation of the rigidity of asymptotically conical structures. We approach this by proving the uniqueness for smooth properly embedded self-shrinking ends asymptotic to a given cone.

We first fix some notations and conventions to present the main results of the paper. Throughout, $O$ is the origin of $\mathbb{R}^{n+1}$; $B_R$ denotes the open ball in $\mathbb{R}^{n+1}$ centered at $O$ with radius $R$ and $S_R = \partial B_R$; $D_R$ denotes the open disk in $\mathbb{R}^n \times \{0\}$ centered at the origin with radius $R$. We say that $C \subset \mathbb{R}^{n+1}$ is a regular cone with vertex at $O$, if

\[ C = \{l \Gamma, \ 0 \leq l < \infty\}, \]

where $\Gamma$ is a smooth connected closed (compact without boundary) embedded sub-manifold of $S_1$ with codimension one. Note that the normal component of the position vector on $C$ vanishes and $C\setminus\{O\}$ is smooth.

The main theorem of this paper states that

**Theorem 1.3.** Let $C \subset \mathbb{R}^{n+1}$ be a regular cone with vertex at $O$ and $R_0$ a positive constant. Suppose that $\Sigma$ and $\tilde{\Sigma}$ are smooth, connected, properly embedded self-shrinkers in $\mathbb{R}^{n+1}\setminus B_{R_0}$ with their boundaries in $S_{R_0}$. If $\Sigma$ and $\tilde{\Sigma}$ are asymptotic to the same cone $C$, i.e., $\lambda \Sigma$ and $\lambda \tilde{\Sigma}$ converge to $C$ locally smoothly as $\lambda \to 0^+$, then $\Sigma$ coincides with $\tilde{\Sigma}$.

There are several interesting applications of our uniqueness theorem. For instance, by Theorem 1.3, it is not hard to show that not every regular cone with vertex at $O$ has a smooth complete properly embedded self-shrinker asymptotic to it. Consider a rotationally symmetric regular cone $C \subset \mathbb{R}^{n+1}$ with vertex at $O$. In [29], Kleene and Møller constructed a smooth embedded self-shrinking end of revolution in $\mathbb{R}^{n+1}$ that is asymptotic to $C$. Thus, Theorem 1.3 implies that any smooth, connected, properly embedded self-shrinker that is asymptotic to $C$ must

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1These assumptions on self-shrinkers in Conjectures 1.1 and 1.2 are implied from the content of Lecture 3 in [23] and the equivalence of Euclidean volume growth and properness of self-shrinkers demonstrated in [9] and [3].

2More precisely, we mean that, for $\forall R > 0$ and $k \in \mathbb{N}$, $\lambda \Sigma \cap (\overline{B_R \setminus B_{1/R}})$ and $\lambda \tilde{\Sigma} \cap (\overline{B_R \setminus B_{1/R}})$ converge to $C \cap (\overline{B_R \setminus B_{1/R}})$ in the $C^k$ topology, as $\lambda \to 0^+$.
have a rotational symmetry. Hence, using the classification result for smooth complete embedded self-shrinkers with a rotational symmetry (see [29]), we conclude that

**Corollary 1.4.** Let \( C \subset \mathbb{R}^{n+1} \) be a rotationally symmetric regular cone with vertex at \( O \). Assume that \( C \) is not a hyperplane. Then there do not exist smooth complete properly embedded self-shrinkers in \( \mathbb{R}^{n+1} \) with an end asymptotic to \( C \).

On the other hand, by Huisken’s monotonicity formula (see [20,21]), self-shrinkers are hypersurfaces in the Euclidean space that are minimal with respect to the Gaussian conformally changed metric; see [2] and [5,6]. It is well known that one can construct many smooth complete properly embedded minimal surfaces with asymptotically planar ends, e.g., Costa-Hoffman-Meeks minimal surfaces (see [8], [27] and [19]). However, in contrast to the minimal surfaces theory, it follows from Theorem 1.3 that the only smooth complete properly embedded self-shrinkers with ends asymptotic to hyperplanes are hyperplanes. This gives an explanation of the dramatic change of asymptotics of the noncompact ends occurring in the desingularization construction of self-shrinkers in [28] and [38–40].

Also, assuming that Conjecture 1.1 is true, a gap theorem for self-shrinkers can be established using Theorem 1.3.

**Corollary 1.5.** Let \( \Sigma \subset \mathbb{R}^3 \) be a smooth complete properly embedded self-shrinker. Assume that one end of \( \Sigma \) is asymptotic to a regular cone and Conjecture 1.1 is true. Then any smooth complete properly embedded self-shrinker in \( \mathbb{R}^3 \) that has sublinear growth of Hausdorff distance from \( \Sigma \) must coincide with \( \Sigma \).

We give an overview of the proof of Theorem 1.3 in the following. Broadly speaking, our strategy is a reduction of the problem in Theorem 1.3 to a backwards uniqueness problem for parabolic differential inequalities. First, outside some compact set \( K \subset \mathbb{R}^{n+1} \), we write \( \tilde{\Sigma} \setminus K \) as the graph of a function \( v \), which is defined over \( \Sigma \) and vanishes at infinity with a certain rate; see Section 2. Next, we derive the differential equation for the function \( v \) (see (2.24)) which involves the Ornstein-Uhlenbeck type operator. One main difficulty comes from the fact that, when changing infinity to the origin, the resulting equation is elliptic but highly singular and degenerate at the origin; see the discussion in the beginning of Section 3. Besides, under our assumptions, \( v \) need not vanish to infinite order asymptotically. Thus, we cannot apply the strong unique continuation theorems in [17,18], [25], [26] or [41] to conclude that \( v \) is identically zero. Instead, we introduce a new function \( w \) (see (3.4)), which is defined by a suitable scaling of \( v \) on a domain of the space-time. Then the decay of \( v \) in space is converted to that of \( w \) as time approaches zero. Thus it suffices to show the backwards uniqueness for the parabolic differential inequality for \( w \) (see (3.5)) on a noncompact incomplete domain.

The subtlety of our backwards uniqueness problem is that the values of \( w \) at the parabolic boundary of the domain are not controlled by the assumptions of Theorem 1.3. We overcome this issue by extending the Carleman inequalities of Escauriaza, Seregin and Šverák [15] to the special cases of mean curvature flow in question, in which the backwards uniqueness was proved for a class of linear perturbations.

\[^3\]In those papers, one starts with surfaces by desingularizing the intersection of a sphere and a plane, but the resulting self-similar surfaces are asymptotic to cones (not planes). This phenomenon does not happen in the desingularization construction of minimal surfaces.
of the standard heat equation in the Euclidean space with a ball removed. As applications, they settled a long-standing question concerning sufficient conditions for regularity of solutions for the Navier-Stokes equations in $\mathbb{R}^3$; see [15]. For more bibliographic information on the backwards uniqueness for parabolic equations, we recommend the readers refer to [13–16], [31–35], [37], and [42–45].

Our approach above can be embedded into an abstract scheme for general geometric flows. Namely, let us consider two geometric objects moving by the same flow of second order parabolic type. Assume that they have the same limit at some terminal time. One might expect to address that the two must coincide with each other for all times, using similar Carleman type estimates as in our cases of mean curvature flow. However, besides the usual gauge degeneracy of geometric evolution equations, which requires different techniques to deal with, there are possible serious obstacles to constructing weight functions with desired convexity in the Carleman inequalities. Such constructions for the mean curvature flow discussed in the current paper depend heavily on the asymptotically conical structure of the flow and the fact that the equation for the evolving distance function is similar to the Euclidean distance. Indeed, in [46], we proved a weaker but optimal uniqueness theorem for asymptotically cylindrical self-shrinkers, on which we had to impose the infinite order rate assumption (i.e., (1.4) in Theorem 1.1 of [46]).

2. Notations and auxiliary lemmas

In this section, we set up the notations for the rest of the paper and prove several auxiliary lemmas. In particular, we deduce from the assumption that $\Sigma$ and $\tilde{\Sigma}$ are asymptotic to the same cone that, outside some compact set, $\tilde{\Sigma}$ can be written as the graph of a function $v$ over $\Sigma$ decaying with a certain rate, and we derive the differential equation for $v$.

Throughout, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_{|\alpha|})$, $\partial^{|\alpha|}_\alpha$ denotes the $|\alpha|$-th order partial derivative with respect to the $\alpha_i$-th, $1 \leq i \leq |\alpha|$, coordinates of the Euclidean space; the meaning of $\partial_t$ may vary in lemmas and propositions, and we will clarify it in the content; $\nabla$, $\nabla^2$ and $\Delta$ denote the gradient, Hessian and Laplacian on the hypersurfaces appearing in the subscripts, respectively; $A$ and $\nabla_i A$ are the second fundamental form and its $i$th covariant derivative, respectively; constants in lemmas and propositions depend only on $n$, $\Sigma$, $\tilde{\Sigma}$ and $C$, unless it is specified; constants in proofs are not preserved when crossing lemmas and propositions.

Denote $\lambda \Sigma \cap (\bar{B}_R \backslash B_1/R)$ by $\Sigma_{R,\lambda}$ and $C \cap (\bar{B}_R \backslash B_1/R)$ by $C_R$. We recall that $\Sigma_{R,\lambda}$ converges to $C_R$ in the $C^k$ topology as $\lambda \to 0+$, if $\Sigma_{R,\lambda}$ converges to $C_R$ in the Hausdorff topology, and for any $x \in C_R$ and $\lambda > 0$ small, $\Sigma_{R,\lambda}$ locally (near $x$) is a graph over the tangent hyperplane $T_x C_R$ and the graph of $\lambda_{R,\lambda}$ converges to the graph of $C_R$ in the usual $C^k$ topology.

Since $\Sigma$ is a self-shrinker under the mean curvature flow, $\{\Sigma_t = \sqrt{t} \Sigma\}_{t \in (0,1]}$ is a solution to the backwards mean curvature flow, which means that, parametrizing $\Sigma_t$ in a suitable way, for $x \in \Sigma_t$ and $t \in (0,1],

$$\partial_t x = H n.$$  

Here, $\partial_t x$ stands for the normal velocity of the hypersurface. We recommend the readers refer to Chapter 2 of [10] for more details and other equivalent definitions. We begin with the following elementary lemma on the geometry of $\Sigma_t$.  

Lemma 2.1. There exist $C_1 > 0$ and $R_1 \geq R_0$ such that for $x \in \Sigma_t \setminus B_{R_1}$, $t \in (0, 1]$ and $0 \leq i \leq 2$,

$$|\nabla^i A(x)| \leq C_1 |x|^{-i-1}. \tag{2.2}$$

Proof. By [1.2] and the fact that $\Gamma$ is a smooth closed embedded submanifold of $S_1$, the second fundamental form and all its covariant derivatives of $C$ are bounded inside the annulus $\bar{B}_2 \setminus B_{1/2}$. Furthermore, since $\lambda \Sigma \rightarrow C$ locally smoothly as $\lambda \rightarrow 0+$, there exist $\delta_1, \lambda_1 > 0$ such that if $0 < \lambda < \delta_1$, then for $0 \leq i \leq 2$, $|\nabla^i A| \leq \lambda_1$ on $\lambda \Sigma \cap (\bar{B}_2 \setminus B_{1/2})$. Thus, for $x \in \Sigma$ with $|x| \geq R = 2 \max\{R_0, 1/\delta_1\}$, we can choose $\lambda = 1/|x|$ such that for $0 \leq i \leq 2$,

$$|\nabla^i A(x)| = \lambda^{i+1} |\nabla^i A(\lambda x)| \leq \lambda_1 |x|^{-i-1}, \tag{2.3}$$

where we note that $\lambda x \in \lambda \Sigma$ is inside $\bar{B}_2 \setminus B_{1/2}$.

Since $\Sigma_t = \sqrt{t} \Sigma$, we conclude that for $x \in \Sigma_t \setminus B_R$ and $0 \leq i \leq 2$,

$$|\nabla^i A(x)| = t^{\frac{i-1}{2}} |\nabla^i A\left(\frac{x}{\sqrt{t}}\right)| \leq \lambda_1 |x|^{-i-1}. \tag{2.4}$$

□

Next, it follows from the assumption of Theorem [1.3] that, outside a compact set, $\Sigma_t$ is given by the graph of a smooth function over $C$. On the other hand, using the proof of Lemma 2.2 on page 30 of [7], we can write $\Sigma_t$ locally as the graph of a smooth function over a fixed hyperplane.

Lemma 2.2. There exist $R_2 > R_1$ and compact sets $K_t \subset \Sigma_t$, $0 < t \leq 1$, such that $K_t \subset \Sigma_t \cap B_{R_2}$, and $\Sigma_t \setminus K_t$ is given by the graph of a smooth function $U(\cdot, t) : C \setminus B_{R_2} \rightarrow \mathbb{R}$. Moreover, there exist $0 < \varepsilon_0 < 1$ and $C_2 > 0$ such that for $z_0 \in C \setminus B_{R_2}$ and $t \in (0, 1]$, the component of $\Sigma_t \cap B_{\varepsilon_0 |z_0|}(z_0)$ containing $z_0 + U(z_0, t) \mathbf{n}(z_0)$ can be written as the graph of a smooth function $u(\cdot, t)$ over the tangent hyperplane $T_{z_0} C$ of $C$ at $z_0$ satisfying that, for $i = 0, 1, 2$,

$$|D^{i+1} u| \leq C_2 |z_0|^{-i} \quad \text{and} \quad |D^i \partial_t u| \leq C_2 |z_0|^{-1-i}. \tag{2.5}$$

Here $D$ and $D^2$ are the Euclidean gradient and Hessian on $\mathbb{R}^n$, respectively, and $\partial_t$ denotes the partial derivative with respect to $t$ fixing points in $T_{z_0} C$.

Proof. Given $\delta > 0$, by the assumption of Theorem [1.3] there exists $t_0 = t_0(\delta) \in (0, R_1^2]$ such that if $0 < t \leq t_0$, then $\Sigma_t \cap B_{3/2} \setminus B_{1/3}$ can be written as the graph of a function $V(\cdot, t) : \Omega_t \rightarrow \mathbb{R}$ such that $\|V(\cdot, t)\|_{C^1} \leq \delta$. Here, the domain $\Omega_t$ of $C$ satisfies that $B_{3/2} \setminus B_{1/2} \subset \Omega_t \subset B_{3} \setminus B_{1/4}$ and $\Omega_t \rightarrow C \cap (B_3 \setminus B_{1/3})$ as $t \rightarrow 0$. Thus, we can choose $\delta$ sufficiently small, depending on $C$ inside the annulus $B_4 \setminus B_{1/4}$, such that for $z \in C \cap (B_3 \setminus B_{1/3})$, $x_t = z + V(z, t) \mathbf{n}(z) \in \Sigma_t$ and $0 < t \leq t_0$, the distance $\text{dist}(x_t, C)$ from $x_t$ to $C$, which is achieved uniquely at $z$ and is equal to $|V(z, t)|$, is less than $1/100$, and $\langle \mathbf{n}(x_t), \mathbf{n}(z) \rangle > 99/100$. Hence, if $x \in \Sigma_t$ with $|x|^2 > t_0^{-1}$ and $0 < t \leq 1$, then, by the homogeneity of $C$, $\text{dist}(x, C) < |x|/100$, which is attained at a unique point $z$ on $C$, and $\langle \mathbf{n}(x), \mathbf{n}(z) \rangle > 99/100$. This implies that the nearest point projection $\Pi_t : \Sigma_t \setminus B_{3/\sqrt{t_0}} \rightarrow C$ is well defined for each $t \in (0, 1]$ and $|x|/2 < |\Pi_t(x)| < 2 |x|$ for $x \in \Sigma_t \setminus B_{2/\sqrt{t_0}}$. Moreover, the map $\Pi_t$ is injective and the image of $\Pi_t$ contains $C \setminus B_{4/\sqrt{t_0}}$. This proves the first part of Lemma 2.2 with $R_2 = 4/\sqrt{t_0}$ and $K_t = \Sigma_t \setminus \Pi_t^{-1}(C \setminus B_{4/\sqrt{t_0}})$. 


It follows from the discussion in the previous paragraph that, given \( \delta > 0 \), there exists \( r = r(\delta) > 0 \) such that for \( z_0 \in C \backslash B_r \) and \( 0 < t \leq 1 \), \( |\Pi_t^{-1}(z_0) - z_0| < \delta |z_0| \). Thus, by the proof of Lemma 2.2 in [7] and Lemma 2.1, there exist \( \delta_1 \in (0,1) \) and \( r_1, \lambda_1 > 0 \), depending only on \( C_1 \), such that for \( z_0 \in C \backslash B_{r_1} \) and \( t \in (0,1] \), the component of \( \Sigma_t \cap B_{\delta_1|z_0|}(z_0) \) containing \( \Pi_t^{-1}(z_0) \) can be written as the graph of a smooth function \( u(\cdot, t) \) over the tangent hyperplane of \( C \) at \( z_0 \) and \( u(\cdot, t) \) satisfies that \( |Du(\cdot, t)| \leq \lambda_1 |z_0|^{-1} \) and \( |D^2u(\cdot, t)| \leq \lambda_1 |z_0|^{-3} \). In fact, Lemma 2.1 implies that \( |D^3u(\cdot, t)| \leq \lambda_2 |z_0|^{-2} \) and \( |D^4u(\cdot, t)| \leq \lambda_2 |z_0|^{-3} \), where \( \lambda_2 > 0 \) depends only on \( C_1 \) and \( \lambda_1 \). Moreover, since \( \Sigma_t \) varies smoothly with respect to \( t \), the function \( U \) in this lemma depends on \( t \) smoothly and so does \( u \). In the following, we parametrize \( T_{z_0}C \) by \( F : \mathbb{R}^n \to T_{z_0}C, F(p) = z_0 + \sum_i p_i e_i \), where \( p = (p_1, \ldots, p_n) \) and \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( T_{z_0}C - z_0 \). And we identify \( u(p, t) \) with \( u(F(p), t) \). Note that \( \partial_t \) means the partial derivative with respect to \( t \) fixing \( p \). Since \( \{\Sigma_t\}_{t \in (0,1]} \) moves by mean curvature backwards, we derive the differential equation for \( u \):

\[
-\partial_t u = \sqrt{1 + |Du|^2} \div \left( \frac{Du}{1 + |Du|^2} \right).
\]

Thus, \( |\partial_t u| \leq \lambda_3 |z_0|^{-1} \), where \( \lambda_3 > 0 \) depends only on \( \lambda_1 \). Next, differentiating once with respect to the \( i \)th coordinate on both sides of the equation (2.6) gives

\[
-\partial_i \partial_t u = \frac{\sum_k \partial_k u \partial^2_{ik} u}{\sqrt{1 + |Du|^2}} \div \left( \frac{Du}{1 + |Du|^2} \right) + \sqrt{1 + |Du|^2} \partial_t \div \left( \frac{Du}{1 + |Du|^2} \right).
\]

Note that the divergence term on the right-hand side of the equation (2.6) is the mean curvature of \( \Sigma_t \) at \( \text{Graph}(u) \). Thus, Lemma 2.1 implies that

\[
|\partial_t \div \left( \frac{Du}{1 + |Du|^2} \right)| \leq \lambda_4 |z_0|^{-2},
\]

where \( \lambda_4 > 0 \) depends only on \( \lambda_1 \) and \( C_1 \). Hence, there exists \( \lambda_5 > 0 \), depending only on \( \lambda_1, \lambda_2 \) and \( \lambda_4 \), such that \( |\partial_i \partial_t u| \leq \lambda_5 |z_0|^{-2} \). Similarly, differentiating twice with respect to the \( i \)th and \( j \)th coordinates on the both sides of the equation (2.6), we get

\[
-\partial^2_{ij} \partial_t u = \left( \frac{\sum_k \partial^2_{ik} u \partial^2_{jk} u + \partial_k u \partial^3_{ijk} u}{\sqrt{1 + |Du|^2}} - \frac{\sum_{k,l} \partial_k u \partial_l u \partial^2_{ik} u \partial^2_{jk} u}{(1 + |Du|^2)^{3/2}} \right) \div \left( \frac{Du}{1 + |Du|^2} \right)
\]

\[
+ \frac{\sum_k \partial_k u \partial^2_{ik} u}{\sqrt{1 + |Du|^2}} \partial_j \div \left( \frac{Du}{1 + |Du|^2} \right)
\]

\[
+ \frac{\sum_k \partial_k u \partial^2_{jk} u}{\sqrt{1 + |Du|^2}} \partial_i \div \left( \frac{Du}{1 + |Du|^2} \right)
\]

\[
+ \sqrt{1 + |Du|^2} \partial^2_{ij} \div \left( \frac{Du}{1 + |Du|^2} \right).
\]
Note that, by Lemma 2.1, we have that
\begin{equation}
(2.8) \quad \left| \frac{\partial^2_i \text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) }{} \right| \leq \lambda_6 |z_0|^{-3},
\end{equation}
where $\lambda_6 > 0$ depends on $\lambda_1$ and $\lambda_2$. Hence, there exists $\lambda_7 > 0$, depending only on $\lambda_1$, $\lambda_2$, $\lambda_3$ and $\lambda_6$, such that $|\partial^2_i \partial_t u| \leq \lambda_7 |z_0|^{-3}$.

Thus, by Lemmas 2.1 and 2.2, we can write $\tilde{\Sigma}$ outside a compact set as the graph of a function over $\Sigma$. Namely,

**Lemma 2.3.** There exist $R_3 > R_2$ and $C_3 > 0$ such that, outside a compact set, $\tilde{\Sigma}$ is given by the graph of a smooth function $v : \Sigma \setminus \bar{B}_{R_3} \rightarrow \mathbb{R}$ satisfying that for $x \in \Sigma \setminus \bar{B}_{R_3}$ and $0 \leq i \leq 2$,
\begin{equation}
(2.9) \quad |\nabla^i_S v(x)| \leq C_3 |x|^{-1-i}.
\end{equation}

**Proof.** First, using the same arguments in Lemma 2.2, there exist $\tilde{R}_2 > R_1$ and compact sets $\tilde{K}_t \subset \Sigma_t$, $0 < t \leq 1$, such that $\tilde{K}_t \subset B_{2\tilde{R}_2}$, and $\Sigma_t \setminus \tilde{K}_t$ is given by the graph of a smooth function $\tilde{U}(\cdot, t) : \tilde{\Sigma} \setminus \tilde{B}_{\tilde{R}_2} \rightarrow \mathbb{R}$. Moreover, there exist $0 < \tilde{\varepsilon}_0 < 1$ and $\tilde{C}_2 > 0$ such that for $z_0 \in \tilde{\Sigma} \setminus \tilde{B}_{\tilde{R}_2}$ and $t \in (0, 1]$, the component of $\tilde{\Sigma} \cap B_{\tilde{\varepsilon}_0}|z_0|(|z_0|)$ containing $z_0 + \tilde{U}(z_0, t) \tilde{n}(z_0)$ can be written as the graph of a smooth function $\tilde{u}(\cdot, t)$ over the tangent hyperplane of $C$ at $z_0$ satisfying that, for $i = 0, 1, 2$,
\begin{equation}
(2.10) \quad |D^{i+1} \tilde{u}| \leq \tilde{C}_2 |z_0|^{-1-i} \quad \text{and} \quad |D^i \partial_t \tilde{u}| \leq \tilde{C}_2 |z_0|^{-1-i}.
\end{equation}

For $r_1 > 2 \max \{ R_2, \tilde{R}_2, \sqrt{C_2}, \sqrt{\tilde{C}_2} \}$, let $\Pi : \Sigma \setminus B_{r_1} \rightarrow C$ and $\tilde{\Pi} : \tilde{\Sigma} \setminus B_{r_1} \rightarrow C$ be the nearest point projections. Thus, if $y \in \tilde{\Sigma} \setminus B_{4r_1}$, then $z = \tilde{\Pi}(y) \in C \setminus B_{2r_1}$ and $\text{dist}(y, \Sigma) \leq |y - z| + |z - \Pi^{-1}(z)| \leq 2(C_2 + \tilde{C}_2) |y|^{-1}$. Hence, by Lemma 2.1, we can choose $r_1 > 1$, depending only on $C_1$, $C_2$ and $\tilde{C}_2$, such that for $y \in \tilde{\Sigma} \setminus B_{4r_1}$, there exists a unique $x \in \Sigma$ satisfying that $|y - x| = \text{dist}(y, \Sigma)$.

Next, for $x \in \Sigma \setminus B_{r_1}$, Lemma 2.2 implies that $z' = \Pi(x) \in C \setminus B_{r_1}$, $|x - z'| \leq C_2 |z'|^{-1}$ and $\langle \tilde{n}(x), n(z') \rangle > 1 - \mu_1 |z'|^{-4}$, where $\mu_1 > 0$ depends on $C_2$. Thus, if $r_1$ is sufficiently large, depending only on $C_2$ and $\tilde{C}_2$, then the component of $\Sigma \cap B_{\tilde{\varepsilon}_0}|z'|/2(z')$ containing $\Pi^{-1}(z')$ can be written as the graph of the function $f$ over the tangent hyperplane of $\Sigma$ at $x$. And $|f(x)| \leq \mu_2 |z'|^{-1}$ for some $\mu_2$, depending on $C_2$ and $\tilde{C}_2$. Hence, there exists $r_2 > 2r_1$ large, depending only on $\mu_2$ and $C_1$, such that the image of the nearest point projection from $\tilde{\Sigma} \setminus B_{4r_1} \rightarrow \Sigma$ contains $\Sigma \setminus B_{r_2}$.

Finally, it is easy to show that there exists $\delta_1 > 0$, depending only on $C$, such that if $z_1, z_2 \in C \setminus B_{r_1}$ and $|z_1 - z_2| < \delta_1$, then the geodesic distance between $z_1$ and $z_2$ on $C$, $\text{dist}_C(z_1, z_2)$, is less than $2|z_1 - z_2|$. Note that for $y \in \tilde{\Sigma} \setminus B_{2r_1}$, $\tilde{\Pi}(y) \in C \setminus B_{r_1}$, $|\tilde{U}(\tilde{\Pi}(y), 1)| \leq \tilde{C}_2 |\tilde{\Pi}(y)|$ and $|\nabla C \tilde{U}(\cdot, 1)|$ is uniformly small on $C \setminus B_{r_1}$. Thus, there exist $\delta_2 > 0$, $r_3 \gg 1$ and $\mu_3 > 1$, depending only on $\tilde{C}_2$, $\delta_1$ and $r_1$, such that if $y_1, y_2 \in \tilde{\Sigma} \setminus B_{r_3}$ and $|y_1 - y_2| \leq \delta_2$, then the geodesic distance
\[ \text{dist}_\Sigma(y_1, y_2) \leq \mu_3 |y_1 - y_2|. \]

Hence, by the discussion in the previous paragraph, the nearest point projection from \( \Sigma \setminus B_{r_4} \) to \( \Sigma \) is injective, for some large \( r_4 > 4 \max\{r_1, r_3\} \), depending on \( C_2 \) and \( \bar{C}_2 \). Therefore, choosing \( r_5 = 2 \max\{r_2, r_4\} \), we conclude that, outside a compact set, \( \Sigma \) is given by the graph of a smooth function \( v \) over \( \Sigma \setminus B_{r_5} \).

In the following, we derive the bounds of \( v \) and its derivatives up to the second order. Fix \( x \in \Sigma \setminus B_{r_5} \). It is clear that \( |v(x)| \leq \lambda_1 |x|^{-1} \), where \( \lambda_1 > 0 \) depends only on \( C_2 \) and \( \bar{C}_2 \). Let \( z' = \Pi(x) \) and \( y = x + v(x) n(x) \). Thus, by Lemma 2.2, \( |x - z'| \) and \( |y - \bar{\Pi}^{-1}(z')| \) are bounded by multiples of \( 1/|x| \). Hence, the geodesic distance \( \text{dist}_\Sigma(y, \bar{\Pi}^{-1}(z')) \) is also a multiple of \( 1/|x| \). Therefore, we can write the component of \( \Sigma \cap B_{\varepsilon_0|x|}(z') \) containing \( x \) and the component of \( \Sigma \cap B_{\varepsilon_0|x|}(z') \) containing \( y \) as the graphs of smooth functions \( u_1 = u(\cdot, 1) \) and \( \tilde{u}_1 = \tilde{u}(\cdot, 1) \) over the tangent hyperplane \( T_x C \) of \( C \) at \( z' \), respectively. We parametrize \( T_x C \) by \( F : \mathbb{R}^n \to T_x C \), \( F(p) = z' + \sum_i p_i e_i \), where \( p = (p_1, \ldots, p_n) \) and \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( T_x C - z' \). For \( p, q \in \mathbb{R}^n \), we identify \( u_1(p) \), \( \tilde{u}_1(q) \) and \( v(p) \) with \( u_1(F(p)) \), \( \tilde{u}_1(F(q)) \) and \( v(F(p)) \), respectively. Let \( e_{n+1} \) be the unit normal of the hyperplane \( T_x C \). Note that the unit normal \( n \) of \( \Sigma \) at a point \( F(p) + u_1(p)e_{n+1} \) is given by

\[
(2.11) \quad n(p) = \sum_{h=1}^n \frac{-\partial_h u_1}{\sqrt{1 + |Du_1|^2}} e_h + \frac{1}{\sqrt{1 + |Du_1|^2}} e_{n+1}.
\]

Thus, for \( h = 1, 2, \ldots, n \),

\[
(2.12) \quad q_h = p_h - \frac{\partial_h u_1}{\sqrt{1 + |Du_1|^2}} v,
\]

\[
(2.13) \quad \tilde{u}_1(q) = u_1(p) + \frac{1}{\sqrt{1 + |Du_1|^2}} v.
\]

Differentiating the equations (2.12) and (2.13) with respect to \( p_i \) gives

\[
\partial_i q_h = \partial_{i, h} - \frac{\partial^2_{h, i} u_1}{\sqrt{1 + |Du_1|^2}} v + \frac{\partial_h u_1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial_l u_1 \partial^2_{l, i} u_1 \right) v
\]

\[
(2.14) \quad - \frac{\partial_h u_1}{\sqrt{1 + |Du_1|^2}} \partial_i v,
\]

\[
(2.15) \quad \sum_k \partial_k \tilde{u}_1 \partial_k q_k = \partial_i u_1 + \frac{\partial_i v}{\sqrt{1 + |Du_1|^2}} - \frac{v}{(1 + |Du_1|^2)^{3/2}} \sum_l \partial_l u_1 \partial^2_{l, i} u_1.
\]

Furthermore, plugging the equation (2.14) into the equation (2.15) gives

\[
\frac{1 + \sum_k \partial_k \tilde{u}_1 \partial_k u_1}{\sqrt{1 + |Du_1|^2}} \partial_i v = (\partial_i \tilde{u}_1 - \partial_i u_1) - \frac{\sum_k \partial_k \tilde{u}_1 \partial^2_{k, i} u_1}{\sqrt{1 + |Du_1|^2}} v
\]

\[
+ \frac{1 + \sum_k \partial_k \tilde{u}_1 \partial_k u_1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial_l u_1 \partial^2_{l, i} u_1 \right) v.
\]
Note that $|q(0)| \leq \lambda_1 |x|^{-1}$ and $|x|, |y|$ and $|z'|$ are comparable. Thus, by the assumption of Theorem 1.3, (2.5) and (2.10), there exists $\lambda_2 > 0$, depending only on $\lambda_1, C_2$ and $\tilde{C}_2$ such that at $p = 0$,

\begin{equation}
|\partial_i \bar{u}_1 \circ q - \partial_i u_1| \leq |\partial_i \bar{u}_1 \circ q - \partial_i \tilde{u}_1| + |\partial_i \tilde{u}_1 - \partial_i u_1| \leq \lambda_2 |x|^{-2}.
\end{equation}

Hence, it follows from the upper bound of $v$, (2.5), (2.10) and (2.17) that $|\partial_i v(0)| \leq \lambda_3 |x|^{-2}$ for some $\lambda_3 > 0$ depending only on $\lambda_2, C_2$ and $\tilde{C}_2$, and thus the same holds true for $|\nabla \Sigma v|$ at $x$.

Next, differentiating the equations (2.14) and (2.15) with respect to $p_j$ gives that

\begin{equation}
\partial^2_{ij} q_h = -\frac{\partial^3_{hij} u_1}{\sqrt{1 + |Du_1|^2}} v + \frac{\partial^2_{hi} u_1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial_l u_1 \partial^2_{ij} u_1 \right) v
\end{equation}

\begin{equation}
- \frac{3 \partial_h u_1}{(1 + |Du_1|^2)^{5/2}} \left( \sum_l \partial_l u_1 \partial^2_{ij} u_1 \right) \left( \sum_m \partial_m u_1 \partial^2_{mj} u_1 \right) v
\end{equation}

\begin{equation}
+ \frac{\partial_h u_1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial^2_{ij} u_1 \partial^2_{ij} u_1 + \partial_l u_1 \partial^3_{ij} u_1 \right) v
\end{equation}

\begin{equation}
+ \frac{\partial^2_{ij} u_1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial_l u_1 \partial^2_{ij} u_1 \right) v - \frac{\partial^2_{hi} u_1 \partial_j v + \partial^2_{ij} u_1 \partial_i v}{\sqrt{1 + |Du_1|^2}}
\end{equation}

\begin{equation}
- \frac{\partial_h u_1}{\sqrt{1 + |Du_1|^2}} \partial^2_{ij} v,
\end{equation}

\begin{equation}
\sum_{k,h} \left( \partial^2_{kh} \bar{u}_1 \partial_i q_k \partial_j q_h + \partial_k \bar{u}_1 \partial^2_{ij} q_k \right)
\end{equation}

\begin{equation}
= \partial^2_{ij} u_1 + \frac{3}{(1 + |Du_1|^2)^{5/2}} \left( \sum_l \partial_l u_1 \partial^2_{ij} u_1 \right) \left( \sum_m \partial_m u_1 \partial^2_{mj} u_1 \right) v
\end{equation}

\begin{equation}
- \frac{1}{(1 + |Du_1|^2)^{3/2}} \left( \sum_l \partial^2_{ij} u_1 \partial_j u_1 + \partial_l u_1 \partial_{ij} u_1 \right) v
\end{equation}

\begin{equation}
- \frac{\sum_l \partial_l u_1 \partial^2_{ij} u_1}{(1 + |Du_1|^2)^{3/2}} \partial_i v - \frac{\sum_l \partial_l u_1 \partial^2_{ij} u_1}{(1 + |Du_1|^2)^{3/2}} \partial_j v + \frac{\partial^2_{ij} v}{\sqrt{1 + |Du_1|^2}}.
\end{equation}

It follows from Lemma 2.2 and the bounds for $|v(0)|$ and $|Dv(0)|$ that, for $h = 1, 2, \ldots, n$,

\begin{equation}
\left| \partial^2_{ij} q_h(0) + \frac{\partial_h u_1(0)}{\sqrt{1 + |Du_1(0)|^2}} \partial^2_{ij} v(0) \right| \leq \lambda_4 |x|^{-3},
\end{equation}

where $\lambda_4 > 0$ depends only on $C_2, \lambda_1$ and $\lambda_3$. Thus, by the assumption of Theorem 1.3, (2.5), (2.10) and $|q(0)| = \lambda_1 |x|^{-1}$, we estimate, at $p = 0$,

\begin{equation}
|\partial^2_{ij} \bar{u}_1 \circ q - \partial^2_{ij} u_1| \leq |\partial^2_{ij} \bar{u}_1 \circ q - \partial^2_{ij} \tilde{u}_1| + |\partial^2_{ij} \tilde{u}_1 - \partial^2_{ij} u_1| \leq \lambda_5 |x|^{-3}.
\end{equation}
Here $\lambda_5 > 0$ depends only on $C_2, \tilde{C}_2$ and $\lambda_1$. Since $|D\tilde{u}_1(q(0))| \leq \tilde{C}_2, |D^2\tilde{u}_1(q(0))| \leq \tilde{C}_2 |x|^{-1}$ and there exists $\lambda_6 > 0$, depending only on $C_2, \lambda_1$ and $\lambda_3$, such that

$$\tag{2.22} |\partial_i q_h(0) - \delta_{ih}| \leq \lambda_6 |x|^{-2},$$

and simplifying the equation (2.19) gives that

$$\tag{2.23} \left| \frac{1 + \sum_k \partial_k u_1(0) \partial_k \tilde{u}_1(0)}{1 + |Du_1(0)|^2} \partial_{ij}^2 v(0) - (\partial_{ij}^2 \tilde{u}_1(0) - \partial_{ij}^2 u_1(0)) \right| \leq \lambda_7 |x|^{-3}.$$

Here $\lambda_7 > 0$ depends only on $C_2, \tilde{C}_2, \lambda_1$ and $\lambda_3$. Therefore, it follows from (2.17), (2.21), (2.23) and Lemma 2.2 that $|\nabla^2_{\Sigma} v(x)| \leq \lambda_8 |x|^{-3}$, where $\lambda_8 > 0$ depends on $C_2, \lambda_2, \lambda_5$ and $\lambda_7$.

Finally, we conclude this section by deriving the differential equation for $v$ from the definition of self-shrinkers. Note that the operator $L_{\Sigma}$ in Lemma 2.4 is a small perturbation of the stability operator introduced in [5].

**Lemma 2.4.** There exists $C_4 > 0$ such that at $x \in \Sigma \setminus B_{R_3}$,

$$\tag{2.24} L_{\Sigma} v = \Delta_{\Sigma} v - \frac{1}{2} \langle x, \nabla_{\Sigma} v \rangle + \left( |A|^2 + \frac{1}{2} \right) v + Q(x, v, \nabla_v v, \nabla^2_{\Sigma} v) = 0,$$

where the function $Q$ satisfies that

$$\tag{2.25} |Q(x, v, \nabla_{\Sigma} v, \nabla^2_{\Sigma} v)| \leq C_4 |x|^{-2} (|v| + |\nabla_{\Sigma} v|).$$

**Proof.** Fix $x_0 \in \Sigma \setminus B_{R_3}$. We choose a local parametrization of $\Sigma$ in a neighborhood of $x_0$, $F : \Omega \rightarrow \Sigma$, satisfying that $F(0) = x_0$, $\langle \partial_i F(0), \partial_j F(0) \rangle = \delta_{ij}$ and $\partial_{ij}^2 F(0) = a_{ij}(0)n(x_0)$ with $a_{ij} = A(\partial_i F, \partial_j F)$ and $a_{ij}(0) = 0$ if $i \neq j$. Here $\Omega$ is a domain in $\mathbb{R}^n$ containing $0$. Thus, in a neighborhood of $y_0 = x_0 + v(x_0)n(x_0)$, there exists a local parametrization of $\tilde{\Sigma}$, $\tilde{F} : \Omega \rightarrow \tilde{\Sigma}$, such that for $p \in \Omega$,

$$\tag{2.26} \tilde{F}(p) = F(p) + v(p)n(p).$$

Here, we identify $v(p)$ and $n(p)$ with $v(F(p))$ and $n(F(p))$, respectively.

First, we calculate the tangent vectors $\partial_i \tilde{F}$ for $1 \leq i \leq n$. Namely,

$$\tag{2.27} \partial_i \tilde{F} = \partial_i F + (\partial_i v)n + v\partial_i n.$$

Thus, the unit normal vector $n(\tilde{F}(0))$ to $\tilde{\Sigma}$ at $\tilde{F}(0)$ parallels to the following vector, that is,

$$\tag{2.28} N = -\sum_k \left[ \prod_{l \neq k} (1 - a_{ll}v) \right] (\partial_k v) \partial_k F + \left[ \prod_k (1 - a_{kk}v) \right] n.$$
Next, we calculate the second derivatives of $\tilde{F}$ at $p = 0$. It follows from \[ (2.27) \] that
\begin{equation}
\partial_{ij}^2 \tilde{F} = \partial_{ij} \tilde{F} + (\partial_{ij} v) \mathbf{n} + (\partial_{i} v) \partial_{j} \mathbf{n} + (\partial_{j} v) \partial_{i} \mathbf{n} + v \partial_{ij}^{2} \mathbf{n}.
\end{equation}

Note that, at $p = 0$,
\begin{align}
\partial_{ij} \mathbf{n} &= \sum_{k} \langle \partial_{ij}^{2} \mathbf{n}, \partial_{k} \tilde{F} \rangle \partial_{k} \tilde{F} + \langle \partial_{ij}^{2} \mathbf{n}, \mathbf{n} \rangle \mathbf{n} \\
&= -\sum_{k} (\partial_{j} a_{ik}) \partial_{k} \tilde{F} - a_{ij} a_{jj} \delta_{ij} \mathbf{n}.
\end{align}

Plugging (2.30) into (2.29) gives that, at $p = 0$,
\begin{align}
\partial_{ij}^2 \tilde{F} &= -a_{ii} (\partial_{j} v) \partial_{i} \tilde{F} - a_{jj} (\partial_{i} v) \partial_{j} \tilde{F} - \sum_{k} (\partial_{j} a_{ik}) v \partial_{k} \tilde{F} \\
&\quad + (a_{ij} - a_{ii} a_{jj} \delta_{ij} v + \partial_{ij}^2 \mathbf{n}) \mathbf{n}.
\end{align}

Thus, making an inner product with the vector $\mathbf{N}$, we obtain that, at $p = 0$,
\begin{align}
\langle \partial_{ij}^2 \tilde{F}, \mathbf{N} \rangle &= a_{ii} (\partial_{j} v) \bigg( (1 - a_{kk} v) + a_{jj} (\partial_{i} v) \bigg) (1 - a_{kk} v) \\
&\quad + (a_{ij} - a_{ii} a_{jj} \delta_{ij} v + \partial_{ij}^2 \mathbf{n}) \prod_{k} (1 - a_{kk} v) \\
&\quad + v \sum_{k} \prod_{l \neq k} (1 - a_{ll} v) (\partial_{j} a_{ik}) \partial_{k} v.
\end{align}

Next, it is easy to calculate the pullback metric $g = (g_{ij})$ from $\tilde{\Sigma}$, that is, at $p = 0$,
\begin{equation}
g_{ij} = \langle \partial_{i} \tilde{F}, \partial_{j} \tilde{F} \rangle = (1 - a_{ii} v)(1 - a_{jj} v) \delta_{ij} + (\partial_{i} v)(\partial_{j} v).\end{equation}

Thus, at $p = 0$, the determinant $\det(g)$ and the inverse $g^{-1} = (g^{ij})$ of $g$ are
\begin{align}
\det(g) &= 1 - 2 \sum_{k} a_{kk} v + Q_{1}(p, v, Dv), \\
g^{ij} \det(g) &= \begin{cases} 
Q_{2ij}(p, v, Dv) \partial_{i} v 
& \text{if } i \neq j, \\
1 - 2 \sum_{k \neq i} a_{kk} v + Q_{2ij}(p, v, Dv) & \text{if } i = j.
\end{cases}
\end{align}

Here, by Lemmas 2.1 and 2.3 there exists $\lambda_{1} > 0$, depending only on $C_{1}$ and $C_{3}$, such that $|Q_{1}(0, v, Dv)| \leq \lambda_{1} |x_{0}|^{-2} (|v(0)| + |Dv(0)|)$ with similar bounds holding for $|Q_{2ij}(0, v, Dv)|$ for $1 \leq i, j \leq n$. 

Finally, we compute, at $p = 0$,
\begin{align}
\langle \tilde{F}, \mathbf{N} \rangle &= -\sum_{k} \prod_{l \neq k} (1 - a_{ll} v) \langle F, \partial_{k} \tilde{F} \rangle \partial_{k} v + (\langle F, \mathbf{n} \rangle + v) \prod_{k} (1 - a_{kk} v).
\end{align}

Since $\tilde{\Sigma}$ is a self-shrinker under the mean curvature flow, we have that
\begin{equation}
\sum_{i,j} g^{ij} \langle \partial_{ij}^2 \tilde{F}, \mathbf{N} \rangle = -\frac{1}{2} \langle \tilde{F}, \mathbf{N} \rangle.
\end{equation}
Substituting all the previous computation into the equation (2.37) gives that, at 
$p = 0$, 
\[
\sum_k \partial^2 k v - \frac{1}{2} \sum_k \langle F, \partial_k F \rangle \partial_k v + \left( |A|^2 + \frac{1}{2} \right) v = \tilde{Q} \left( p, v, Dv, D^2 v \right),
\]
where the function $\tilde{Q}$ satisfies that 
\[
| \tilde{Q} (0, v, Dv, D^2 v) | \leq \lambda_2 |x_0|^{-2} (|v(0)| + |Dv(0)|).
\]
Here $\lambda_2 > 0$ depends only on $C_1$ and $C_3$ in Lemma 2.3. Therefore, by the choice of 
the parametrization in a neighborhood of $x_0$, Lemma 2.4 follows immediately from 
(2.38). □

3. Proof of Theorem 1.3

By the discussion in Section 2, Theorem 1.3 can be restated as the unique con-
tinuation property for the elliptic equation (2.24). Let us consider the simplest case 
that $\Sigma = \mathbb{R}^n \times \{0\}$. We define $\bar{v}(r, \theta) = v(1/r, \theta)$, where $(r, \theta) \in (0, +\infty) \times S^{n-1}$ 
are the spherical coordinates of $\mathbb{R}^n \times \{0\}$. Then, the equation (2.24) gives that 
\[
\Delta \mathbb{R}^n \bar{v} + \frac{1}{2} \left[ r^{-3} + 4(2 - n)r^{-1} \right] \partial_r \bar{v} + \frac{1}{2} r^{-4} \bar{v} + \tilde{Q} \left( r, \theta, \bar{v}, D\bar{v}, D^2 \bar{v} \right) = 0,
\]
or equivalently, 
\[
\text{div} \left( r^{4-2n}e^{-\frac{1}{4r^2}} D\bar{v} \right) + \frac{1}{2} r^{-2n}e^{-\frac{1}{4r^2}} \bar{v} + r^{4-2n}e^{-\frac{1}{4r^2}} \tilde{Q} \left( r, \theta, \bar{v}, D\bar{v}, D^2 \bar{v} \right) = 0.
\]
Here the function $\tilde{Q}$ satisfies that 
\[
| \tilde{Q} (r, \theta, \bar{v}, D\bar{v}, D^2 \bar{v}) | \leq C_4 r^{-2} (|\bar{v}| + r^2 |D\bar{v}|).
\]
Hence, we cannot apply some well-known strong unique continuation theorems in 
[25, 26] and [17, 18] to the equations (3.1) or (3.2). In [41], Pan and Wolff proved 
the unique continuation results for operators in the general form, $\Delta_{\mathbb{R}^n} + \langle W, \nabla_{\mathbb{R}^n} \rangle + V$, where $|r^2V(\theta, \theta)|$ is bounded and $|rW(\theta, \theta)|$ is small. They also constructed 
counter examples when the conditions are violated. However, it seems impossible 
to transform the equation (3.2) to that of their form by changing coordinates. 
Besides, $\bar{v}$ need not vanish of infinite order at the origin under our assumption. For 
general $\Sigma$, global coordinates of $\Sigma$ near infinity may not even exist.

Instead, we associate the elliptic differential equation (2.24) to the parabolic 
equation in the following lemma. Define a function $w : \cup_{t \in (0,1]} \Sigma_t \times \{t\} \rightarrow \mathbb{R}$ by 
\[
w(x, t) = \sqrt{t} v \left( \frac{x}{\sqrt{t}} \right).
\]
Let \( d/dt \) denote the total derivative with respect to time \( t \). Then, by Lemma 2.4, \( w \) satisfies the following equation:

**Lemma 3.1.** Given \( \varepsilon_1 > 0 \), there exists \( R > R_3 \) such that on \( \bigcup_{t \in (0, 1]} \Sigma_t \setminus B_R \times \{t\} \),

\[
\left| \frac{dw}{dt} + \Delta_{\Sigma_t} w \right| \leq \varepsilon_1 \left( |w| + |\nabla_{\Sigma_t} w| \right).
\]

**Proof.** By a straightforward computation, we get that

\[
\frac{dw}{dt} + \Delta_{\Sigma_t} w = \frac{1}{2\sqrt{t}} v \left( \frac{x}{\sqrt{t}} \right) + \sqrt{t} \left( \nabla_{\Sigma_t} v \left( \frac{x}{\sqrt{t}} \right), \frac{1}{\sqrt{t}} \frac{\partial x}{\partial t} - \frac{x}{2t\sqrt{t}} \right)
\]

\[+ \frac{1}{\sqrt{t}} \Delta_{\Sigma_t} v \left( \frac{x}{\sqrt{t}} \right).\]

Note that for \( x \in \Sigma_t \) and \( t \in (0, 1) \),

\[
\partial_t x = Hn \quad \text{and} \quad H = \frac{\langle x, n \rangle}{2t}.
\]

Hence, the equation (3.6) and (3.7) give that

\[
\frac{dw}{dt} + \Delta_{\Sigma_t} w = \frac{1}{\sqrt{t}} \left( \Delta_{\Sigma_t} v \left( \frac{x}{\sqrt{t}} \right) - \frac{1}{2} \langle x, \nabla_{\Sigma_t} v \left( \frac{x}{\sqrt{t}} \right) \rangle + \frac{1}{2} v \left( \frac{x}{\sqrt{t}} \right) \right).
\]

Therefore, Lemma 3.1 follows immediately from Lemmas 2.1 and 2.4. 

It is implied from (3.4) and (2.9) that \( w \) is asymptotic to zero as \( t \to 0^+ \). Thus, appealing to the strong unique continuation theorems in [17] [18], it is enough to show that there exist \( T \in (0, 1) \) and \( R > 0 \) such that \( w \equiv 0 \) in \( \bigcup_{t \in (0, T]} \Sigma_t \setminus B_R \times \{t\} \) so as to prove Theorem 1.3. One of the standard methods to prove backwards uniqueness is the so-called Carleman type estimates, i.e., a family of uniform weighted \( L^2 \) estimates. The key to derive such estimates is the construction of suitable weight functions satisfying a convexity property, which is closely related to the distance function and heat kernel on the underlying manifold.

As no control of the parabolic boundary value is made on \( w \), our backwards uniqueness problem is local near infinity and thus more delicate than the usual global ones. Fortunately, some precedent in the case of heat type inequalities in the Euclidean space with a ball removed has been obtained by Escauriaza, Seregin and Šverák in [15]. Since the radial distance function on any cone satisfies the same Hessian equation as that on the Euclidean space, we are able to modify their arguments to apply to asymptotically conical manifolds.

First, we prove the following key identity (see (3.9) below) which is a generalization of that in Lemma 1 of [15] to our geometric setting. Namely,

**Lemma 3.2.** Assume that \( \phi \) and \( G \) are smooth functions on \( \bigcup_{t \in (0, 1]} \Sigma_t \times \{t\} \) and that \( G \) is positive. And let \( F = (dG/dt - \Delta_{\Sigma_t} G)/G \). Then, the following identity

\[4\text{Under slightly different notations, a similar identity might have already been implicitly obtained in [15].} \]
holds on $\Sigma_t$:

$$\text{div}_{\Sigma_t} \left\{ 2G \frac{d\phi}{dt} \nabla_{\Sigma_t} \phi + |\nabla_{\Sigma_t} \phi|^2 \nabla_{\Sigma_t} G - 2 \left( \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} G \rangle \right) \nabla_{\Sigma_t} \phi + \mathcal{F} \phi \nabla_{\Sigma_t} \phi \\
+ \frac{1}{2} \phi^2 \mathcal{F} \nabla_{\Sigma_t} G - \frac{1}{2} \phi^2 \nabla_{\Sigma_t} \mathcal{F} \right\} - \frac{d}{dt} \left( |\nabla_{\Sigma_t} \phi|^2 G - \frac{1}{2} \phi^2 \mathcal{F} G \right) \right.$$  

$$\left. = -2\nabla_{\Sigma_t}^2 \log G \left( \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi \right) G - 2 \left( \frac{d\phi}{dt} - \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle + \frac{1}{2} \phi \mathcal{F} \right)^2 G \\
+ 2 \left( \frac{d\phi}{dt} - \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle + \frac{1}{2} \phi \mathcal{F} \right) \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right) G \\
- 2H A \left( \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi \right) G - \frac{1}{2} \phi^2 \left( \frac{d\mathcal{F}}{dt} + \Delta_{\Sigma_t} \mathcal{F} \right) G. \right.$$  

(3.9)

**Proof.** First, a straightforward calculation gives that

$$2 \left( \frac{d\phi}{dt} - \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle + \frac{1}{2} \phi \mathcal{F} \right) \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right) G$$  

$$- 2 \left( \frac{d\phi}{dt} - \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle + \frac{1}{2} \phi \mathcal{F} \right)^2 G$$  

$$= \text{div}_{\Sigma_t} \left( 2G \frac{d\phi}{dt} \nabla_{\Sigma_t} \phi + \phi \mathcal{F} \nabla_{\Sigma_t} \phi + \frac{1}{2} \phi^2 \mathcal{F} \nabla_{\Sigma_t} G - \frac{1}{2} \phi^2 \nabla_{\Sigma_t} \mathcal{F} \right)$$  

$$- \frac{d}{dt} \left( |\nabla_{\Sigma_t} \phi|^2 G + \frac{1}{2} \phi^2 \mathcal{F} G \right) + \frac{1}{2} \phi^2 \mathcal{F} \left( \frac{d\mathcal{F}}{dt} + \Delta_{\Sigma_t} \mathcal{F} \right) + |\nabla_{\Sigma_t} \phi|^2 \Delta_{\Sigma_t} G$$  

$$- 2 \langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \Delta_{\Sigma_t} \phi - 2 \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle \mathcal{F} G$$  

(3.10)  

$$+ \left( \frac{d}{dt} |\nabla_{\Sigma_t} \phi|^2 - 2 \left( \frac{d\phi}{dt} \right) \nabla_{\Sigma_t} \phi \right) G.$$  

Here we use the fact that, given functions $f$ and $g$ on a hypersurface $N$,

$$f \Delta_N g = g \Delta_N f + \text{div}_N \left( f \nabla_N g - g \nabla_N f \right).$$  

(3.11)

On the other hand, we note that

$$2 \left\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \right\rangle \Delta_{\Sigma_t} \phi = \text{div}_{\Sigma_t} \left( 2\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi \right) - 2 \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} G \rangle \nabla_{\Sigma_t} G$$  

$$= \text{div}_{\Sigma_t} \left( 2\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi \right) - 2 \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} G \rangle \nabla_{\Sigma_t} G$$  

$$- 2 \langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi$$  

$$= \text{div}_{\Sigma_t} \left( 2\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi \right) - 2 \nabla_{\Sigma_t}^2 G \nabla_{\Sigma_t} \phi$$  

$$- \left\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \right\rangle \nabla_{\Sigma_t} \phi$$  

$$= \text{div}_{\Sigma_t} \left( 2\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi \right) - 2 \nabla_{\Sigma_t}^2 G \nabla_{\Sigma_t} \phi$$  

$$- \left\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \right\rangle \nabla_{\Sigma_t} \phi$$  

$$= \text{div}_{\Sigma_t} \left( 2\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \rangle \nabla_{\Sigma_t} \phi \right) - 2 \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle \mathcal{F} G$$  

$$- \left\langle \nabla_{\Sigma_t} G, \nabla_{\Sigma_t} \phi \right\rangle \nabla_{\Sigma_t} \phi$$  

(3.12)  

$$- 2 \nabla_{\Sigma_t}^2 G \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi \rangle + |\nabla_{\Sigma_t} \phi|^2 \Delta_{\Sigma_t} G.$$  

Also, it is easy to check that

$$\nabla_{\Sigma_t}^2 \log G \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi \rangle G = \nabla_{\Sigma_t}^2 G \langle \nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi \rangle - \langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \rangle \mathcal{F} G.$$

(3.13)

Finally, let $\Omega \subset \mathbb{R}^n$ and $F : \Omega \times (0, 1) \rightarrow \mathbb{R}^{n+1}$ be the local parametrization of $\{\Sigma_t\}_{t \in (0, 1]}$ such that $F(\Omega, t) \subset \Sigma_t$ and $dF/dt = H n$. Let $g_t$ be the pullback metric
on $\Omega$ from $\Sigma_t$. Thus, we have that
\begin{equation}
\frac{d}{dt} |\nabla_{\Sigma_t} \phi|^2 - 2 \left\langle \nabla_{\Sigma_t} \left( \frac{d\phi}{dt} \right), \nabla_{\Sigma_t} \phi \right\rangle = \sum_{i,j} \partial_t g_{t}^{ij} \partial_i \phi \partial_j \phi = 2HA(\nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi),
\end{equation}
where $\partial_t g_{t}^{ij}$ denotes the time derivative fixing a point in $\Omega$, and we use the following equation in the second equality (see Appendix B in [10]):
\begin{equation}
\partial_t g_{t}^{ij} = 2H \sum_{k,l} g_{t}^{ik} g_{t}^{jl} A(\partial_k F, \partial_l F).
\end{equation}
Therefore, Lemma 3.2 follows from (3.10) and (3.12)–(3.15). $\square$

Let $g_t$ be the metric on $\Sigma_t$ induced from $\mathbb{R}^{n+1}$ and $d\mu_t$ the Hausdorff measure on $\Sigma_t$. Thus we obtain an integral inequality from the divergence identity (3.9).

\textbf{Lemma 3.3.} Given $R > 2R_2$ and $0 < T \leq 1$, let $\mathcal{G}$ be a smooth positive function in
\begin{equation}
Q_{R,T} = \left\{ (x,t) \mid x \in \Sigma_t \setminus \bar{B}_R, t \in (0,T) \right\}.
\end{equation}
As in Lemma 3.2, set $\mathcal{F} = (dG/dt - \Delta_{\Sigma_t} G)/G$. Then the following inequality holds for all $\phi \in C^\infty_c(Q_{R,T})$:
\begin{equation}
\begin{aligned}
&\int_0^T \int_{\Sigma_t} \left( 2\nabla_{\Sigma_t}^2 \log G + 2HA + H^2 g_t \right) (\nabla_{\Sigma_t} \phi, \nabla_{\Sigma_t} \phi) \mathcal{G} d\mu_t dt \\
&\quad + \int_0^T \int_{\Sigma_t} \frac{1}{2} \left( \frac{d\mathcal{F}}{dt} + \Delta_{\Sigma_t} \mathcal{F} + H^2 \mathcal{F} \right) \phi^2 \mathcal{G} d\mu_t dt \\
&\leq \int_0^T \int_{\Sigma_t} \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right)^2 \mathcal{G} d\mu_t dt + \int_{\Sigma_T} \left( |\phi|^2 \mathcal{F} + |\nabla_{\Sigma_T} \phi|^2 \right) \mathcal{G} d\mu_T.
\end{aligned}
\end{equation}

\textbf{Proof.} Since $\phi \in C^\infty_c(Q_{R,T})$, $\phi(\cdot, t)$ is compactly supported in $\Sigma_t \setminus \bar{B}_R$ for each $t \in (0,T]$ and $\phi(\cdot, t) \equiv 0$ when $t < t_0$ for some small $t_0 > 0$. Thus, integrating against the measure $d\mu_t dt$, the inequality (3.17) follows from the divergence theorem and integration by parts together with applications of the Cauchy-Schwarz inequality and the basic property for backwards mean curvature flow (see Appendix B of [10]).

\begin{equation}
\frac{d}{dt} d\mu_t = H^2 d\mu_t.
\end{equation}

Namely, it follows from the Cauchy-Schwarz inequality, i.e., $2ab \leq \varepsilon^2 b^2 + \varepsilon^2 a^2$, that
\begin{equation}
2 \left( \frac{d\phi}{dt} - \left\langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \right\rangle + \frac{1}{2} \phi \mathcal{F} \right) \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right) \\
\leq 2 \left( \frac{d\phi}{dt} - \left\langle \nabla_{\Sigma_t} \log G, \nabla_{\Sigma_t} \phi \right\rangle + \frac{1}{2} \phi \mathcal{F} \right)^2 + \frac{1}{2} \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right)^2.
\end{equation}
Recall that, in our notations, \(d/dt\) is the total of the time derivatives, and thus, by the evolution of the volume element of mean curvature flow,

\[
\int_0^T \int_{\Sigma_t} \frac{d}{dt} \left( |\nabla_{\Sigma_t}\phi|^2 G + \frac{1}{2} \phi^2 FG \right) d\mu_t dt
\]

\[
= \int_0^T \frac{d}{dt} \left( \int_{\Sigma_t} \left( |\nabla_{\Sigma_t}\phi|^2 + \frac{1}{2} \phi^2 F \right) G d\mu_t \right) dt
\]

\[
- \int_0^T \int_{\Sigma_t} \left( |\nabla_{\Sigma_t}\phi|^2 + \frac{1}{2} \phi^2 F \right) H^2 G d\mu_t dt
\]

\[
= \int_{\Sigma_T} (|\nabla_{\Sigma_T}\phi|^2 + |\phi|^2 F) G d\mu_T - \int_0^T \int_{\Sigma_t} \left( |\nabla_{\Sigma_t}\phi|^2 + \frac{1}{2} \phi^2 F \right) H^2 G d\mu_t dt.
\]

Therefore Lemma 3.3 follows from collecting terms involving \(\phi\) and \(\nabla_{\Sigma_t}\phi\) in the integrand. \(\square\)

Next, we choose the function \(G\) in the previous lemma, which is a suitable variation of the choice in [15]. From now on, we fix a \(\delta \in (0,1)\). Given \(\alpha > 0\), \(0 < T \leq 1\) and \(R > 2R_2\), we set \(G\) in Lemma 3.3 to be

\[
G_{\alpha,T,R}(x,t) = \exp \left[ 2\alpha (T-t) (|x|^{1+\delta} - R^{1+\delta}) + 2 |x|^2 \right].
\]

Similarly, set \(F_{\alpha,T,R} = (dG_{\alpha,T,R}/dt - \Delta_{\Sigma_t} G_{\alpha,T,R})/G_{\alpha,T,R}\). Then, by straightforward calculations, we have that

**Lemma 3.4.** There exist \(\alpha_0 > 0\), \(R_4 > 2R_2\) and \(C_5 > 0\) such that for \(\alpha \geq \alpha_0\), \(0 < T \leq 1\), \(R > R_4\) and \((x,t) \in \cup_{t \in (0,T]} \Sigma_t \setminus B_R \times \{t\}\),

\[
F_{\alpha,T,R}(x,T) < 0,
\]

\[
\frac{d}{dt} F_{\alpha,T,R} + \Delta_{\Sigma_t} F_{\alpha,T,R} + H^2 F_{\alpha,T,R} \geq 2,
\]

\[
2 \nabla_{\Sigma_t}^2 \log G_{\alpha,T,R} + 2 HA + H^2 g_t \geq C_5 \Id.
\]

**Proof.** To prove the first property (3.21), we compute \(F_{\alpha,T,R}\) explicitly as follows. First, note that

\[
F_{\alpha,T,R} = \frac{d}{dt} \log G_{\alpha,T,R} - \Delta_{\Sigma_t} \log G_{\alpha,T,R} - |\nabla_{\Sigma_t} \log G_{\alpha,T,R}|^2.
\]

Then, we calculate each term on the right-hand side of the equation (3.24) accordingly. Since \(\Sigma\) is a self-shrinker of the mean curvature flow, we have that, for \(x \in \Sigma_t\) and \(0 < t \leq 1\),

\[
\partial_t x = H n = \frac{\langle x, n \rangle}{2t} n.
\]

Thus,

\[
\frac{d}{dt} \log G_{\alpha,T,R} = -2\alpha \left( |x|^{1+\delta} - R^{1+\delta} \right) + 2\alpha(1+\delta)(T-t) |x|^{\delta-1} \langle x, \partial_t x \rangle + 4 \langle x, \partial_t x \rangle
\]

\[
= -2\alpha \left( |x|^{1+\delta} - R^{1+\delta} \right) + 2\alpha(1+\delta)(T-t) |x|^{\delta-1} H \langle x, n \rangle + 4H \langle x, n \rangle
\]

\[
= -2\alpha \left( |x|^{1+\delta} - R^{1+\delta} \right) + 4\alpha(1+\delta) t(T-t) |x|^{\delta-1} H^2 + 8tH^2.
\]

Note that

\[
\nabla_{\Sigma_t} |x|^\beta = \beta |x|^{\beta-2} x^T,
\]
where \(x^T\) is the tangential part of \(x \in \Sigma_t\). Thus, by the equation (3.25),
\[
|\nabla_{\Sigma^t} \log G_{\alpha,T,R}|^2 = \left[ 2\alpha(1+\delta)(T-t) |x|^\delta + 4 \right] |x^T|^2
\]
(3.28)
\[
= \left[ 2\alpha(1+\delta)(T-t) |x|^\delta + 4 \right] \left( |x|^2 - 4t^2H^2 \right).
\]
It follows from (3.25) that
\[
\text{div}_{\Sigma^t} x^T = n - \langle x, n \rangle \text{div}_{\Sigma^t} n = n - 2tH^2.
\]
(3.29)
Thus, by (3.25), (3.27) and (3.29),
\[
\Delta_{\Sigma^t} |x|^\beta = \text{div}_{\Sigma^t} \left( \beta |x|^{\beta-2} x^T \right)
\]
\[
= \beta(\beta - 2) |x|^{\beta-4} |x^T|^2 + \beta |x|^{\beta-2} \text{div}_{\Sigma^t} x^T
\]
(3.30)
\[
= \beta(\beta - 2 + n) |x|^{\beta-2} - 2\beta t |x|^{\beta-2} H^2 - 4\beta(\beta - 2)t^2 |x|^{\beta-4} H^2.
\]
Hence, (3.30) with \(\beta = 1 + \delta\) or \(2\) gives that
\[
\Delta_{\Sigma^t} \log G_{\alpha,T,R} = 2\alpha(T-t) \Delta_{\Sigma^t} |x|^{1+\delta} + 2\Delta_{\Sigma^t} |x|^2
\]
\[
= 2\alpha(1+\delta)(\delta - 1 + n)(T-t) |x|^\delta - 4\alpha(1+\delta)t(T-t) |x|^\delta H^2
\]
\[
+ 8\alpha(1-\delta^2)t^2(T-t) |x|^\delta-3 H^2 + 4n - 8tH^2.
\]
(3.31)
Therefore, combining the equations (3.26), (3.28) and (3.31), we conclude that
\[
F_{\alpha,T,R}(x,t) = -2\alpha \left( |x|^{1+\delta} - R^{1+\delta} \right) + 2\alpha(1+\delta)(\delta - 1 + n)(T-t) |x|^\delta - 4n
\]
\[
- \left[ 2\alpha(1+\delta)(T-t) |x|^\delta + 4 |x| \right]^2 + 16tH^2
\]
\[
+ 8\alpha(1+\delta)t(T-t) |x|^\delta-1 H^2 - 8\alpha(1-\delta^2)t^2(T-t) |x|^\delta-3 H^2
\]
\[
+ 4t^2 \left[ 2\alpha(1+\delta)(T-t) |x|^\delta + 4 |x| \right]^2 H^2.
\]
(3.32)
In particular, by (3.32) and Lemma 2.1 there exists \(\lambda_1 > 0\), depending only on \(C_1\) and \(n\), such that when \(t = T, R > 2R_2\) and \(x \in \Sigma_T \backslash B_R\),
\[
F_{\alpha,T,R}(x,T) < -4n - 16 |x|^2 + 16T H^2 + 64T^2 H^2 \leq -\lambda_1 R^2.
\]
(3.33)
Next, in order to estimate derivatives of \(F_{\alpha,T,R}\), we group the terms in \(F_{\alpha,T,R}\) according to their powers of \(\alpha\). Thus we write
\[
F_{\alpha,T,R} = B_0 + \alpha B_1 + \alpha^2 B_2,
\]
(3.34)
and we will analyze each \(B_i\) in the following. Namely,
\[
B_0 = -4n - 16 |x|^2 + 16tH^2 + 64t^2 H^2,
\]
(3.35)
and we observe that
\[
\frac{d |x|}{dt} = 2t |x|^{-1} H^2 \quad \text{and} \quad \frac{dH}{dt} + \Delta_{\Sigma^t} H = - |A|^2 H.
\]
(3.36)
Here the evolution equation for \(H\) is obtained by the computation in Appendix B of [10]. Thus, it follows from Lemma 2.1 that there exists \(\lambda_2 > 0\), depending only on \(R_2\) and \(C_1\), such that for \(x \in \Sigma_t \backslash B_{2R_2}\),
\[
\left| \frac{d |x|}{dt} + \frac{dH}{dt} \right| \leq \lambda_2 |x|^{-3},
\]
(3.37)
together with (3.35), implies that

\[(3.38) \quad \left| \frac{dB_0}{dt} \right| \leq O(|x|^{-2}).\]

For \(B_1\), we extract out the constant term and the term with the highest power of \(|x|\) and denote the rest terms by \(P_1\). That is,

\[(3.39) \quad B_1 = -2(1 + 8(1 + \delta)(T - t))|x|^{1+\delta} + 2R^{1+\delta} + P_1,\]

where the highest power of \(|x|\) in \(P_1\) is equal to \(\delta - 1\). Note that \(P_1\) can be viewed as a polynomial of \(t, |x|\) and \(H\). Also, by Lemma 2.1 and (3.36), the time derivatives of \(|x|\) and \(H\) are uniformly bounded in some neighborhood of infinity. Thus, \(|dP_1/dt|\) decays at least as fast as \(|x|^{\delta-1}\). On the other hand,

\[(3.40) \quad \frac{d}{dt} \left\{ -2(1 + 8(1 + \delta)(T - t))|x|^{1+\delta} \right\} = 16(1 + \delta)|x|^{1+\delta} - 4(1 + \delta)(1 + 8(1 + \delta)(T - t)) t|x|^{\delta-1}H^2.\]

Hence there exists \(r_1 > 2R_2\), depending only on \(R_2\) and \(C_1\), such that for \(R > r_1\) and \(x \in \Sigma_t \setminus B_R\),

\[(3.41) \quad \frac{dB_1}{dt} \geq 8(1 + \delta)|x|^{1+\delta}.\]

By similar reasoning, one can enlarge \(r_1\) (if necessary) to conclude from (3.36) that for \(R > r_1\) and \(x \in \Sigma_t \setminus B_R\),

\[(3.42) \quad \frac{dB_2}{dt} \geq 4(1 + \delta)^2(T - t)|x|^{2\delta},\]

where \(B_2\) is given by

\[(3.43) \quad B_2 = -4(1 + \delta)^2(T - t)^2|x|^{2\delta} + 16(1 + \delta)^2(T - t)^2|x|^{2\delta-2}H^2.\]

Therefore, combining (3.38), (3.41) and (3.42), there exist \(\alpha_1 > 1\) and \(r_1 > 2R_2\), depending only on \(C_1\) and \(R_2\), such that for \(\alpha > \alpha_1\), \(R > r_1\) and \(x \in \Sigma_t \setminus B_R\),

\[(3.44) \quad \frac{d}{dt} F_{\alpha,T,R}(x,t) \geq 4\alpha(1 + \delta)|x|^{1+\delta} \left[ \alpha(1 + \delta)(T - t)|x|^{\delta-1} + 1 \right].\]

To estimate \(\Delta_{\Sigma_t} F_{\alpha,T,R}\), we first note that

\[(3.45) \quad \Delta_{\Sigma_t} H^2 = 2H \Delta_{\Sigma_t} H + 2|\nabla_{\Sigma_t} H|^2,\]

and for \(\beta \neq 0\),

\[(3.46) \quad \Delta_{\Sigma_t}(|x|^\beta H^2) = H^2 \Delta_{\Sigma_t} |x|^\beta + 2 \left( \nabla_{\Sigma_t} H^2, \nabla_{\Sigma_t} |x|^\beta \right) + |x|^\beta \Delta_{\Sigma_t} H^2.\]

Assuming that \(R > 2R_2\) and \(x \in \Sigma_t \setminus B_R\), then Lemma 2.1 (3.27) and (3.30) give that, at \(x\),

\[(3.47) \quad \left| \Delta_{\Sigma_t} H^2 \right| \leq \lambda_3 |x|^{-4} \quad \text{and} \quad \left| \Delta_{\Sigma_t} \left( |x|^\beta H^2 \right) \right| \leq \lambda_4 |x|^{\beta - 4},\]

where \(\lambda_3 > 0\) depends only on \(C_1\) and \(\lambda_4 > 0\) depends on \(C_1, n\) and \(\beta\). Since each nonconstant term in \(F_{\alpha,T,R}\) is either \(|x|^\beta, H^2\), or \(|x|^\beta H^2\), a straightforward computation and (3.47) give that, for \(x \in \Sigma_t \setminus B_{2R_2}\),

\[(3.48) \quad |\Delta_{\Sigma_t} F_{\alpha,T,R}| \leq \lambda_5 \alpha^2 (T - t)|x|^{2\delta - 2} + \lambda_5 \alpha,\]

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where $\lambda_5 > 0$ depends only on $C_1$, $R_2$, $\lambda_3$ and $\lambda_4$. Also, for $x \in \Sigma_t \setminus B_{2R_2}$, it follows from Lemma 2.1 that
\[
(3.49) \quad |H^2 \mathcal{F}_{\alpha,T,R}| \leq \lambda_6 \alpha^2 (T-t) |x|^{2\delta-2} + \lambda_6 \alpha,
\]
where $\lambda_6 > 0$ depends on $R_2$ and $C_1$. Hence, the second property \(3.22\) is verified by \(3.44\), \(3.48\) and \(3.49\), when $t \in (0,1]$.

Finally, we estimate $\nabla^2_{\Sigma_t} |x|^\beta$ for $\beta \in (1,2]$. Fix $x \in \Sigma_t$, $0 < t \leq 1$. We choose a local geodesic orthonormal frame $\{e_1, \ldots, e_n\}$ of $\Sigma_t$ at $x$. Thus, \(3.25\) gives that, at $x$,
\[
\nabla^2_{\Sigma_t} |x|^\beta (e_i, e_j)
= \left\langle \nabla_{e_i} \nabla_{\Sigma_t} |x|^\beta, e_j \right\rangle = \beta \left\langle \nabla_{e_i} \left(|x|^{\beta-2} x^T\right), e_j \right\rangle
= \beta \left\langle \nabla_{e_i} \left(|x|^{\beta-2} x\right), e_j \right\rangle - \beta |x|^{\beta-2} \left\langle x, n \right\rangle \left\langle \nabla_{e_i} n, e_j \right\rangle
= \beta \left\langle D |x|^{\beta-2}, e_i \right\rangle \left\langle x, e_j \right\rangle + \beta |x|^{\beta-2} \left\langle \nabla_{e_i} x, e_j \right\rangle + 2\beta t |x|^{\beta-2} HA(e_i, e_j)
(3.50)
= \beta (\beta - 2) |x|^{\beta-4} \left\langle x, e_i \right\rangle \left\langle x, e_j \right\rangle + \beta |x|^{\beta-2} \delta_{ij} + 2\beta t |x|^{\beta-2} HA(e_i, e_j).
\]

Now, given $\eta \in T_x \Sigma_t$, $\eta = \sum_k \eta_k e_k$ and at $x$, it follows from \(3.50\) that
\[
\nabla^2_{\Sigma_t} |x|^\beta (\eta, \eta)
\geq \beta |x|^{\beta-4} \sum_{i,j} \left[ (\beta - 2) \left\langle x, e_i \right\rangle \left\langle x, e_j \right\rangle + |x|^T |\delta_{ij}| \eta_i \eta_j + 2\beta t |x|^{\beta-2} HA(\eta, \eta) \right]
= \beta (\beta - 2) |x|^{\beta-4} \left( \sum_i \left\langle x, e_i \right\rangle \eta_i \right)^2 + \beta |x|^{\beta-4} |x|^T |\eta|^2 + 2\beta t |x|^{\beta-2} HA(\eta, \eta)
\geq \beta (\beta - 1) |x|^{\beta-4} |x|^T |\eta|^2 + 2\beta t |x|^{\beta-2} HA(\eta, \eta)
= \beta (\beta - 1) |x|^{\beta-2} |\eta|^2 - 4\beta (\beta - 1) t^2 |x|^{\beta-4} H^2 |\eta|^2 + 2\beta t |x|^{\beta-2} HA(\eta, \eta).
(3.51)
\]
Hence, by Lemma 2.1 and the assumption that $\beta \in (1,2]$, there exists $r_2 > 2R_2$, depending only on $\beta$, $C_1$ and $R_2$, such that for $x \in \Sigma_t \setminus B_{r_2}$,
\[
(3.52) \quad \nabla^2_{\Sigma_t} |x|^\beta (\eta, \eta) \geq \frac{1}{2} \beta (\beta - 1) |x|^{\beta-2} |\eta|^2.
\]
Hence, the third property \(3.23\) follows immediately from \(3.52\) with $\beta = 1 + \delta$ or 2, and Lemma 2.1 when $t \in (0,1]$. \qed

Finally, combining Lemma 3.3 and Lemma 3.4, we establish the following Calleman inequality.

**Proposition 3.5.** Let $\alpha > \alpha_0$, $0 < T \leq 1$, $R > R_1$ and $Q_{R,T}$ as in Lemma 3.3. Then for all $\phi \in C^\infty_c (Q_{R,T})$,
\[
(3.53) \quad \int_0^T \int_{\Sigma_t} \left( |\phi|^2 + C_5 |\nabla_{\Sigma_t} \phi|^2 \right) G_{\alpha,T,R} d\mu_t dt \leq \int_0^T \int_{\Sigma_t} \left( \frac{d\phi}{dt} + \Delta_{\Sigma_t} \phi \right)^2 G_{\alpha,T,R} d\mu_t dt + \int_{\Sigma_T} |\nabla_{\Sigma_T} \phi|^2 G_{\alpha,T,R} d\mu_T.
\]
Since the weight function $G_{\alpha,T,R}$ in Proposition 3.5 grows very fast in space, in order to apply Proposition 3.5 to conclude backwards uniqueness, we need to show that $|w|$ and its gradient $|\nabla \Sigma_t w|$ decay at least as rapidly as $G_{\alpha,T,R}$.

**Lemma 3.6.** There exist $R_5 > R_3$ and $M > 0$ such that for $x \in \Sigma_t \setminus B_{R_5}$ and $t \in (0,1]$,

$$|w(x,t)| + |\nabla \Sigma_t w(x,t)| \leq \exp\left(-\frac{M|w|^2}{t}\right).$$  

**Proof.** Fix $z_0 \in C \setminus B_{R_3}$. By Lemma 2.2 for each $t \in (0,1]$, the component of $\Sigma_t \cap B_{\epsilon_t}(z_0)$ containing $z_0 + U(z_0,t)\mathbf{n}(z_0)$ can be written as the graph of a smooth function $u(\cdot,t)$ over the tangent plane $T_{z_0}C$ of $C$ at $z_0$ satisfying the property (2.5). Moreover, there exists $\delta_1 \in (0,\epsilon_0]$, depending only on $C_2$, such that the image of the orthogonal projection of $\Sigma_t \cap B_{\epsilon_t}(z_0)$ to $T_{z_0}C$, $0 < t \leq 1$ contains the disk in $T_{z_0}C$ centered at $z_0$ with radius $\delta_1 |z_0|$. We parametrize $T_{z_0}C$ by $F : \mathbb{R}^n \rightarrow T_{z_0}C$, $F(p) = z_0 + \sum_i p_i e_i$, where $p = (p_1, \ldots, p_n)$ and $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_{z_0}C - z_0$. And we identify $u(p,t)$ with $u(F(p),t)$ and define $\bar{w}(p,t) = w(F(p) + u(p,t)\mathbf{n}(z_0), t)$. Let $g_t$ be the pullback metric on $D_{\delta_1|z_0|}$ from $\Sigma_t$ via the map $p \mapsto F(p) + u(p,t)\mathbf{n}(z_0)$. In the following, $\partial_t$ denotes the partial derivative with respect to time $t$ fixing $p$. Then, it follows from Lemma 2.4 that, on $D_{\delta_1|z_0|} \times (0,1]$,

$$\partial_t \bar{w} + \Delta_{g_t} \bar{w} = \frac{1}{2\sqrt{t}} v\left(\frac{x}{\sqrt{t}}\right) + \sqrt{t} \left(\nabla v\left(\frac{x}{\sqrt{t}}\right) + \frac{1}{2} \frac{\partial x}{\partial t} - \frac{x}{2t\sqrt{t}}\right) + \frac{1}{\sqrt{t}} \Delta v\left(\frac{x}{\sqrt{t}}\right)$$

$$= \frac{1}{\sqrt{t}} \left[\frac{1}{2} v\left(\frac{x}{\sqrt{t}}\right) - \frac{1}{2} \left(\nabla v\left(\frac{x}{\sqrt{t}}\right) + \frac{x}{\sqrt{t}}\right) + \Delta v\left(\frac{x}{\sqrt{t}}\right)\right]$$

$$+ \left(\nabla v\left(\frac{x}{\sqrt{t}}\right), \partial x\right)$$

(3.55)  

$$\leq \frac{1}{\sqrt{t}} \left[ |A|^2 v\left(\frac{x}{\sqrt{t}}\right) + Q\left(\frac{x}{\sqrt{t}}, v, \nabla v, \nabla^2 v\right) + \left(\nabla v\left(\frac{x}{\sqrt{t}}\right), \partial x\right)\right].$$

Here, we use the backwards mean curvature flow equation for graphs, that is, for $x = F(p) + u(p,t)\mathbf{n}(z_0)$ and $p \in D_{\delta_1|z_0|}$,

$$\langle \partial_t x, \mathbf{n} \rangle = H,$$

and thus equivalently,

(3.56)  

$$\partial_t u = -\sqrt{1 + |Du|^2} \div \left(\frac{Du}{\sqrt{1 + |Du|^2}}\right).$$

Hence, by the calculation (3.55), Lemmas 2.1, 2.2 and 2.4 we have that, on $D_{\delta_1|z_0|} \times (0,1]$

(3.57)  

$$|\partial_t \bar{w} + \Delta_{g_t} \bar{w}| \leq \lambda_1 |z_0|^{-2} \left(|\bar{w}| + |z_0| |\nabla g_t \bar{w}|\right),$$
where \( \lambda_1 > 0 \) depends only on \( C_1, C_2, C_4 \) and \( \delta_1 \). It is easy to verify that \( g_t \) has the following properties: \( g_t(0) = \text{Id} \),

\[
\begin{align*}
\lambda_1^{-1} |\xi|^2 & \leq \sum_{i,j} g_{ij}^t \xi_i \xi_j \leq \lambda_2 |\xi|^2 \quad \text{for all } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \\
|Dg_{ij}^t| & \leq \lambda_2 |z_0|^{-1} \quad \text{and} \quad |\partial_t g_{ij}^t| \leq \lambda_2 |z_0|^{-2} \quad \text{for } 1 \leq i, j \leq n,
\end{align*}
\]

where \( \lambda_2 > 0 \) depends only on \( n \) and \( C_2 \). Thus, (3.58) gives that

\[
|\partial_t \bar{w} + \text{div} (g_t^{-1} D\bar{w})| \leq \lambda_3 |z_0|^{-2} (|\bar{w}| + |z_0||D\bar{w}|),
\]

where \( \lambda_3 > 0 \) depends only on \( \lambda_1 \) and \( \lambda_2 \).

For the convenience of the readers, we present the arguments in pages 2877–2879 of \[37\] here to conclude the proof of Lemma 3.6. Let \( R = \delta_1 |z_0|/2 \) and \( \bar{w}_R \) be a rescaling of \( \bar{w}, \bar{w}_R(q, s) = \bar{w}(Rq, R^2s) \). Then \( \bar{w}_R \) satisfies

\[
|\partial_s \bar{w}_R + \text{div} (g_R^{-1} D\bar{w}_R)| \leq \lambda_3 (|\bar{w}_R| + |D\bar{w}_R|),
\]

where \( g_R^{-1}(q, s) = (g_{ij}^R)^{-1} = g_{ij}^{-1} \) \( R(q) \). It follows from (3.60) that, in \( D_2 \times [0, 1/R^2] \),

\[
|Dg_{ij}^R| \leq \lambda_2 \quad \text{and} \quad |\partial_s g_{ij}^R| \leq \lambda_2 \quad \text{for } 1 \leq i, j \leq n.
\]

Furthermore, by Lemma 2.3 and 3.59,

\[
|\bar{w}_R| + |D\bar{w}_R| \leq \lambda_4 |z_0|^{-1}
\]

in \( D_2 \times [0, 1/R^2] \), where \( \lambda_4 > 0 \) depends only on \( C_3 \) and \( \lambda_2 \). And \( \bar{w}_R, D\bar{w}_R \in C^0(D_2 \times [0, 1/R^2]) \) with \( \bar{w}_R(\cdot, 0) = 0 \), and \( \bar{w}_R \) is smooth in \( D_2 \times (0, 1/R^2] \). Define \( \phi = \psi(q)\eta(s)\bar{w}_R \), where \( \chi_{\delta_1} \leq \psi \leq \chi_{\delta_2} \) and \( \chi_{(0, 1/\alpha)} \leq \eta \leq \chi_{(0, 2/\alpha)} \) are bump functions, and \( \alpha \geq 2R^2 \) is a positive constant to be chosen. Then

\[
|\partial_s \phi + \text{div} (g_R^{-1} D\phi)| \leq \lambda_5 (|\phi| + |D\phi|) + \lambda_5 \alpha (|\bar{w}_R| + |D\bar{w}_R|) \chi_E.
\]

Here \( E = (D_2 \times [0, 2/\alpha]) \setminus (D_1 \times [0, 1/\alpha]) \), and \( \lambda_5 > 0 \) depends only on \( R_3, \delta_1, \lambda_2 \) and \( \lambda_3 \). Hence, using the Carleman inequality in \[13\] (see also Lemma 2.1 in \[37\]), there exists \( M_1 > 1 \), depending only on \( n \) and \( \lambda_2 \), such that for \( 0 < a < 1/\alpha \),

\[
\begin{align*}
\int_{\mathbb{R}^{n+1}} & \left( \alpha^2 \phi^2 + \alpha \sigma_a |D\phi|^2 \right) \sigma_a^{-\alpha} G_a dqds \\
& \leq \alpha^\alpha M_1^2 \sup_{s\geq0} \int_{\mathbb{R}^n \times \{s\}} \left( \phi^2 + |D\phi|^2 \right) dq + M_1 \lambda_3^2 \int_{\mathbb{R}^{n+1}} (|\phi| + |D\phi|)^2 \sigma_a^{1-\alpha} G_a dqds \\
& \quad + M_1 \lambda_3^2 \alpha^2 \int_E (|\bar{w}_R| + |D\bar{w}_R|)^2 \sigma_a^{1-\alpha} G_a dqds.
\end{align*}
\]

Here \( G_a(q, s) = (s + a)^{-n/2} \exp \left[ -|q|^2/4(s + a) \right], \sigma_a(s) = \sigma(s + a) \) and \( \sigma : (0, 4/\alpha) \rightarrow (0, +\infty) \) satisfying that \( M_1^{-1} s \leq \sigma(s) \leq s \). If \( \alpha \geq 2M_1 \lambda_3 \), then the second term on the right-hand side can be absorbed by the left-hand side. In \( E \), \( \sigma_a^{-\alpha} G_a \leq \alpha^{\alpha+\frac{n}{2}} M_1^\alpha \). Hence, by (3.64), we get

\[
\int_{\mathbb{R}^{n+1}} \left( \alpha^2 \phi^2 + \alpha \sigma_a |D\phi|^2 \right) \sigma_a^{-\alpha} G_a dqds \leq \alpha^{\alpha+\frac{n}{2}} M_2^\alpha,
\]

where \( M_2 > M_1 \) depends only on \( R_3, \lambda_4, \lambda_5 \) and \( M_1 \). Let \( \rho = 1/(M_2 e) \) and \( a = \rho^2/(2\alpha) \). Then, in \( D_2 \rho \times [0, \rho^2/(2\alpha)] \),

\[
\sigma_a^{1-\alpha} G_a \geq \alpha^{\alpha+\frac{n}{2}-1} M_2^{2\alpha+n-2}.
\]
Therefore, we deduce from (3.66) that
\[
\int_{D_{2\rho} \times [0,\rho^2/(2\alpha)]} \left( \phi^2 + |D\phi|^2 \right) dq ds \leq M_2^{2-\alpha-n}.
\]

We now choose \( \alpha = M_3R^2 \), where \( M_3 > 1 \) depends only on \( R_3, \delta_1, \lambda_3 \) and \( M_2 \), so that
\[
\int_{D_{2\rho} \times [0,\rho^2/(2M_3R^2)]} \left( \phi^2 + |D\phi|^2 \right) dq ds \leq M_2^{2-n}e^{-2R^2}.
\]

By Lemma 4.1 in [37], this implies that, in \( D_\rho \times [0,\rho^2/(4M_3R^2)] \),
\[
|\phi| + |D\phi| \leq \lambda_6 Rc e^{-R^2},
\]
where \( \lambda_6 > 0 \) depends on \( n, \lambda_2 \) and \( \lambda_3 \), and \( c > 0 \) depends on \( n \). Undoing the change of variables, we get
\[
|w(z_0, t)| + |\nabla_\Sigma w(z_0, t)| \leq \lambda_6 |z_0|^e \delta_t^2 |z_0|^2,
\]
if \( 0 \leq t \leq \rho^2/(4M_3) \).

Therefore, it follows from the definition of \( w \) that there exist \( M_4 > 0 \) and \( r_1 > 0 \), depending only on \( R_3, \delta_1, M_2, M_3, \lambda_6 \) and \( c \), such that for \( x \in \Sigma \setminus B_{r_1} \),
\[
|v(x)| + |\nabla_\Sigma v(x)| \leq \exp \left( -M_4 |x|^2 \right).
\]

It is clear that Lemma 3.6 follows immediately from the inequality (3.72).

Now, we are ready to conclude the proof of Theorem 1.3.

**Proof of Theorem 1.3** We basically follow the arguments in [15]. Choose \( \varepsilon_1 \) in Lemma 3.1 to be \((1 + C_5)/4 \). Then, choose \( R \) large such that each lemma and proposition can be applied, and \( T = M/16 \).

Given \( a \in (0, 1) \) and \( r \gg 1 \), we consider \( \psi_{a,r} \in C_c^\infty(\mathbb{R}^{n+1}) \) satisfying that \( \psi_{a,r} \equiv 1 \) when \((1 + 2a)R \leq |x| \leq r, \psi_{a,r} \equiv 0 \) when \( |x| \leq (1 + a)R \) or \( |x| \geq 2r \), \( 0 \leq \psi_{a,r} \leq 1 \) and \( |D\psi_{a,r}|, |D^2\psi_{a,r}| \) are bounded from above by a function of \( a \) and \( R \). We also need to introduce a cutoff function in the time direction: given \( 0 < \varepsilon < T/8 \), \( \eta_\varepsilon \in C_c^\infty(\mathbb{R}) \) satisfies that \( \eta_\varepsilon \equiv 1 \) when \( t \geq 2\varepsilon \), \( \eta_\varepsilon \equiv 0 \) when \( t \leq \varepsilon \), \( 0 \leq \eta_\varepsilon \leq 1 \) and \( |\eta_\varepsilon'| \leq 2\varepsilon^{-1} \). Then we choose the test function \( \phi \) in Proposition 3.5 to be \( \phi_{\varepsilon,a,r} = \eta_\varepsilon \psi_{a,r} w \). Thus, we get that
\[
\frac{d}{dt} \phi_{\varepsilon,a,r} + \Delta_{\Sigma_t} \phi_{\varepsilon,a,r} = \eta_\varepsilon \psi_{a,r} \left( \frac{dw}{dt} + \Delta_{\Sigma_t} w \right) + 2(\nabla_{\Sigma_t} \psi_{a,r}, \nabla_{\Sigma_t} w) \eta_\varepsilon + (D\psi_{a,r}, \partial_t x) + \Delta_{\Sigma_t} \psi_{a,r} \eta_\varepsilon w + \eta_\varepsilon' \psi_{a,r} w.
\]

Hence, by Lemma 3.1
\[
\left| \frac{d}{dt} \phi_{\varepsilon,a,r} + \Delta_{\Sigma_t} \phi_{\varepsilon,a,r} \right|^2 \leq \frac{1}{2} \left( |\phi_{\varepsilon,a,r}|^2 + C_5 |\nabla_{\Sigma_t} \phi_{\varepsilon,a,r}|^2 \right) + \lambda \left( |w|^2 + |\nabla_{\Sigma_t} w|^2 \right) \chi_{a,R,r} + 16\varepsilon^{-2} |w|^2 \tilde{\chi}_{\varepsilon,R,r},
\]
where \( \lambda > 0 \) depends only on \( n, C_1, a \) and \( R \); \( \chi_{a,R,r} \) is the characteristic function supported in \( (B_{2r} \setminus B_r) \cup (B_{(1+2a)R} \setminus B_{(1+a)R}) \); and \( \tilde{\chi}_{\varepsilon,R,r} \) is the characteristic function supported in \( (B_{2r} \setminus B_R) \times (\varepsilon, 2\varepsilon) \).
Hence, by Proposition 3.3 and the inequality (3.74), for $\alpha > \alpha_0$,
\[
\int_0^T \int_{\Sigma_t} |\phi_{\epsilon,a,r}|^2 \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
\leq \int_0^T \int_{\Sigma_t \cap B(1+2\alpha)R \setminus B(1+\alpha)R} \left( |w|^2 + |\nabla_{\Sigma_t} w|^2 \right) \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
+ \int_0^T \int_{\Sigma_t \cap B_2 \setminus B_r} \left( |w|^2 + |\nabla_{\Sigma_t} w|^2 \right) \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
+ \varepsilon^{-2} \int_\varepsilon^2 \int_{\Sigma_t \cap B_{2\varepsilon} \setminus B_R} |w|^2 \mathcal{G}_{\alpha,T,R} d\mu_t dt + \int_{\Sigma_T} |\nabla_{\Sigma_T} \phi_{\epsilon,a,r}|^2 \mathcal{G}_{\alpha,T,R} d\mu_T,
\] (3.75)
where $\lesssim$ stands for being less than some positive multiple of the quantities that follow, which depends only on $n$, $\lambda$, $a$, $R$ and $T$. By the assumption of Theorem 1.3, $\Sigma_t \setminus B_R$ flows backwards by mean curvature smoothly to the cone $C \setminus B_R$ as $t \to 0$. This implies that there exists $V_0 > 0$, depending only on the geometry of $C$, such that if $t$ is sufficiently small, then $\text{Vol}(\Sigma_t \cap B_2 \setminus B_R) \leq V_0 r^k$. Also, it follows from (2.9) that $|w| (x,t) \leq C t$ for $x \in \Sigma_t \cap B_2 \setminus B_R$. Thus, letting $\varepsilon \to 0$, the penultimate integral on the right-hand side of (3.75) converges to zero and the monotone convergence theorem implies that
\[
\int_0^T \int_{\Sigma_t} |\psi_{a,r,w}|^2 \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
\leq \int_0^T \int_{\Sigma_t \cap B(1+2\alpha)R \setminus B(1+\alpha)R} \left( |w|^2 + |\nabla_{\Sigma_t} w|^2 \right) \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
+ \int_0^T \int_{\Sigma_t \cap B_2 \setminus B_r} \left( |w|^2 + |\nabla_{\Sigma_t} w|^2 \right) \mathcal{G}_{\alpha,T,R} d\mu_t dt \\
+ \int_{\Sigma_T} |\nabla_{\Sigma_T} (\psi_{a,r,w})|^2 \mathcal{G}_{\alpha,T,R} d\mu_T.
\] (3.76)
Furthermore, it follows from the fact that $M/T > 8$, Lemma 3.6 and the inequality (3.76) that
\[
\exp \left\{ \left[ (1+17a)^{1+\delta} - 1 \right] \alpha T R^{1+\delta} \right\} \int_0^{T/2} \int_{\Sigma_t \cap B_r \setminus B(1+17\alpha)R} w^2 d\mu_t dt \\
\lesssim r^2 \exp \left( 2^{2+\delta} \alpha T^{1+\delta} - \frac{M r^2}{2T} \right) + \exp \left\{ 2 \left[ (1+2a)^{1+\delta} - 1 \right] \alpha T R^{1+\delta} \right\} + 1.
\] (3.77)
Note that $(1+17a)^{1+\delta} - 1 \geq 17a$ and $(1+2a)^{1+\delta} - 1 \leq 8a$. Thus, letting $r \to \infty$,
\[
\int_0^{T/2} \int_{\Sigma_t \setminus B(1+17\alpha)R} w^2 d\mu_t dt \lesssim \exp \left( -\alpha T R^{1+\delta} \right).
\] (3.78)
Then, let $\alpha \to \infty$ and thus,
\[
\int_0^{T/2} \int_{\Sigma_t \setminus B(1+17\alpha)R} w^2 d\mu_t dt \leq 0.
\] (3.79)
By the arbitrariness of $a$, we conclude that $w \equiv 0$ on $Q_{R,T/2}$ and thus $v \equiv 0$ on $\Sigma \setminus B_{2R}/\sqrt{T}$.
Therefore, Theorem 1.3 follows immediately by applying the strong continuation theorem (cf. [17,18]) to $L_{\Sigma}$ inside the compact set, and an open and closed argument.

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