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COHOMOLOGY OF ARITHMETIC FAMILIES OF (φ, Γ)-MODULES

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1. Introduction

One of the most powerful tools available for the study of p-adic Galois representations is the theory of (φ, Γ)-modules, which provides an equivalence of categories between Galois representations and modules over a certain somewhat simpler group algebra. To name just one example, (φ, Γ)-modules form a key intermediate step in the p-adic local Langlands correspondence for the group $\mathrm{GL}_2(\mathbb{Q}_p)$, as discovered

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by Colmez [20]. One key feature of the theory, which plays a crucial role in the previous example, is that the full category of (φ, Γ)-modules is strictly larger than the subcategory on which the equivalence to Galois representations takes place (namely the subcategory of *étale* (φ, Γ)-modules); this makes it possible (and indeed quite frequent) for an *irreducible* p-adic Galois representation to become *reducible* when carried to the full category of (φ, Γ)-modules. This is explained largely using the theory of slope filtrations for φ-modules (as developed for instance in [33]) and has important applications in Iwasawa theory especially in cases of nonordinary reduction [42, 43].

One important feature of the theory of (φ, Γ)-modules is its compatibility with variation in analytic families. On the side of Galois representations, this means working with continuous actions of the Galois group of a finite extension of $\mathbb{Q}_p$ not on a finite-dimensional $\mathbb{Q}_p$-vector space, but on a finite projective module over a $\mathbb{Q}_p$-affinoid algebra (or more globally, a locally free coherent sheaf on a rigid analytic space over $\mathbb{Q}_p$). The work of Berger and Colmez [10] provides a functor from such Galois representations to a certain category of *relative* (φ, Γ)*-modules* (or more globally, to an *arithmetic family of* (φ, Γ)*-modules* in the language of [34]). In contrast to the usual theory, however, this functor is fully faithful but not essentially surjective, even onto the étale objects; this somewhat complicates the relative theory. Another complication is that the theory of slope filtrations does not extend in a completely satisfactory way to families; see [34] for a discussion of some of the difficulties.

This paper primarily concerns itself with the relative analogue of the theory of *Galois cohomology* for (φ, Γ)-modules. The mechanism for computing Galois cohomology of p-adic Galois representations on the side of (φ, Γ)-modules was introduced by Herr, and it was later shown by Liu [36] (using slope filtrations) that this extends in a satisfactory way to all (φ, Γ)-modules when the family is reduced to a point. This includes analogues of some basic results of Tate (finite dimensionality of cohomology, an Euler characteristic formula, and Tate local duality) and is important for such applications as [42]. This makes clear the need for an analogue of these results in families, but a straightforward imitation of [36] is infeasible for the sorts of reasons described in the previous paragraph. The arguments used form a combination of several different strategies, which are described in the discussion of the structure of the paper below.

In addition to results on relative Galois cohomology, we also obtain relative versions of some results on *Iwasawa cohomology*. Iwasawa cohomology, computed using the ψ operator (a left inverse of the Frobenius operator φ), is used in [43] to provide a structural link between (φ, Γ)-modules and Iwasawa theory, giving a way to clarify some previously mysterious phenomena on the latter side.

Arguments taken from [43] also show that our finiteness results imply compatibility with base change for both Galois cohomology and Iwasawa cohomology. As one might expect, these results must be stated at the level of derived categories except in the case of a *flat* base change.

We summarize the preceding results in the following theorem, referring to the body of the paper for the relevant terminology.

Theorem. *Let K be a finite extension of $\mathbb{Q}_p$, let A be a $\mathbb{Q}_p$-affinoid algebra, and let M be (the global sections of) a locally free coherent sheaf over the relative Robba ring $\mathcal{R}_A(\pi_K)$ with continuous semilinear (φ, Γ_K)-action, such that the induced linear* *map $\varphi^* M \to M$ is an isomorphism. Then M is finitely generated projective over $\mathcal{R}_A(\pi_K)$, its Galois cohomology $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ is quasi-isomorphic to $\mathrm{C}^\bullet_{\psi,\gamma_K}(M)$ and lies in $\mathbf{D}^\flat_{\mathrm{perf}}(A)$, and its Iwasawa cohomology $\mathrm{C}^\bullet_{\psi}(M)$ lies in $\mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}^\infty_A(\Gamma_K))$. Variants of Tate local duality and the Euler-Poincaré formula hold for $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ and $\mathrm{C}^\bullet_{\psi}(M)$.*

If $A \to B$ is a homomorphism of $\mathbb{Q}_p$-affinoid algebras, then the natural maps

$$\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \overset{\mathbf{L}}{\underset{A}{\otimes}} B \to \mathrm{C}^\bullet_{\varphi,\gamma_K}\left(M \underset{A}{\widehat{\otimes}} B\right) \quad \text{and} \quad \mathrm{C}^\bullet_{\psi}(M) \overset{\mathbf{L}}{\underset{\mathcal{R}^\infty_A(\Gamma_K)}{\otimes}} \mathcal{R}^\infty_B(\Gamma_K) \to \mathrm{C}^\bullet_{\psi}\left(M \underset{A}{\widehat{\otimes}} B\right)$$

are isomorphisms in the derived category.

While many applications of these results are expected, we limit ourselves in this paper to just one application: the problem of *global triangulations* of families of (φ, Γ)-modules, a triangulation being a useful notion due to Colmez [19]. As noted earlier, it occurs quite frequently that irreducible p-adic Galois representations become more reducible when pushed into the full category of (φ, Γ)-modules; our results show that this phenomenon occurs frequently in families as well.

One primary area of application is to the study of *eigenvarieties*, which parametrize p-adic analytic families of automorphic forms. In this case, we prove the existence of global triangulations after modifying the eigenvarieties along some proper birational morphisms. We also prove that the representation is trianguline at *every* point on the eigenvariety. In the special case of the Coleman-Mazur eigencurve, we prove the existence of a global triangulation after resolving the singularities; we prove that if a point corresponds to an overconvergent modular form which is in the image of the θ^{k-1}-operator of Coleman, then the global triangulation does *not* give a saturated triangulation at that point.

As explained in [42], the reducibility of trianguline families can be exploited to form analogues of Greenberg's families of Selmer groups for *nonordinary* families of Galois representations; the finiteness of cohomology also intervenes in a crucial way to guarantee their good behavior. Although we do not explain the following point in this paper, it should be noted that given our work on the eigencurve, Nekovář's methods then generalize immediately to show that the validity of the Parity Conjecture is constant along the irreducible components of the eigencurve; this reduces the conjecture immediately to the (presently unknown) claim that each irreducible component contains a classical weight two point of noncritical slope.

Some similar results for representations of $G_{\mathbb{Q}_p}$ have recently been announced by other authors. Using a significant technical improvement of Kisin's original method [35] of interpolating crystalline periods, Liu developed techniques for showing that refined families admit triangulations over Zariski-dense open sets, similar to our Example 6.4.3. As an application, he has recovered essentially our Proposition 6.4.5 in [37, Proposition 5.4.2] (in which the cases (1) and (2) treat nonordinary and ordinary points, respectively); see also [37, Theorem 5.4.3] for some information at singularities of the eigencurve. Also, Hellmann [29, Corollary 1.5] has obtained a result on the triangulation of eigenvarieties for definite unitary groups over imaginary quadratic fields, following a strategy suggested by the second author: to construct a universal family of (rigidified) trianguline (φ, Γ)-modules as in [15], then use automorphic data to construct a map from the eigenvariety to this moduli space.

Structure of the paper. In Section 2, we carry out some preliminary arguments relating sheaves on relative annuli to *coadmissible* modules in the sense of Schneider and Teitelbaum [46]. A key issue is to establish finite generation for certain such modules; for instance, we obtain (Proposition 2.2.7) an equivalence between the two flavors of relative (φ, Γ)-modules treated in [34] (one defined using modules over a relative Robba ring, the other defined using vector bundles over a relative annulus).

In Section 3, we make a careful study of the ψ operator, a one-sided inverse of the Frobenius operator φ. Some of these arguments (such as the finiteness of the cokernel of $\psi - 1$) are straightforward generalizations of arguments appearing in the usual (φ, Γ)-module theory.

In Section 4, we establish the formal framework for our results, and obtain relative versions of the results on Galois cohomology from [36] and the results of Iwasawa cohomology from [43], plus base change in both settings, modulo a key finiteness theorem for Iwasawa cohomology (Theorem 4.4.1). Given this result, most of the proofs proceed by reducing to the case of a point, for which we appeal to the previously known results; in particular, we do not give a new method for proving any results from [36] or [43].

In Section 5, we prove Theorem 4.4.1. After some initial simplifications, the argument consists of two main steps. The first step is to establish finiteness of Iwasawa cohomology in degree 1 assuming vanishing of outer (φ, Γ)-cohomology, using the existence of the duality pairing in general and its perfectness at points. The second step is to reduce to this case using a dévissage argument akin to those in [36], in which one carefully constructs an extension of the original (φ, Γ)-module (using slope filtrations) in order to kill off undesired cohomology in degree 2.

In Section 6, we give some applications of our results to triangulations in families of (φ, Γ)-modules for a finite extension K over $\mathbb{Q}_p$. We also derive some explicit consequences for eigenvarieties and in particular for the Coleman-Mazur eigencurve. In addition, we prove that all rank one arithmetic families of (φ, Γ)-modules are of character type (up to twisting by a line bundle on the base), answering in the affirmative a question of Bellaïche [2, Sec. 3, Question 1].

Notation. Throughout this paper, we fix a prime number p. Set $\omega = p^{-1/(p-1)}$.

For $H \subseteq G$ a subgroup of finite index and M a $\mathbb{Z}[H]$-module, we write $\mathrm{Ind}_H^G M$ for the induced $\mathbb{Z}[G]$-module; it is canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$.

We will exclusively work with affinoid and rigid analytic spaces in the sense of Tate, rather than Berkovich. The letter A will always denote a $\mathbb{Q}_p$-affinoid algebra; we use $\mathrm{Max}(A)$ to denote the associated rigid analytic space. For $z \in \mathrm{Max}(A)$, we use $\mathfrak{m}_z$ to denote the maximal ideal of A at z and put $\kappa_z = A/\mathfrak{m}_z$. For M an A-module, we write M_z to mean $M/\mathfrak{m}_z M$.

All Hom spaces consist of *continuous* homomorphisms for the relevant topologies (although we have no need to topologize the Hom spaces themselves, except for when specified). We will sometimes use the subscript "cont" to emphasize this point.

We normalize the theory of slope filtrations so that a nonzero pure submodule of an étale object has *negative* slope.

2. Families of (φ, Γ)-modules

In this section, we introduce the definition of a (φ, Γ)-module over the Robba ring with coefficients in the affinoid algebra A over $\mathbb{Q}_p$. Our definition is algebraic in nature, involving a finite projective module equipped with extra structures; however, we show that it agrees with the definition of an *arithmetic family of* (φ, Γ)*-modules* over the Robba ring in the sense of [34], which involves a vector bundle over a certain rigid analytic space. We also recall the relationship between (φ, Γ)-modules and families of Galois representations following Berger-Colmez [10] and Kedlaya-Liu [34], as well as the formalism of Galois cohomology for (φ, Γ)-modules following Herr and especially Liu [36].

Convention 2.0.1. Throughout this paper, all radii r and s are assumed to be rational numbers, except possibly $r = \infty$.

2.1. **Modules over relative discs and annuli.** In general, the module of global sections of a vector bundle over an open relative disc or a half-open relative annulus may not be finitely generated over the ring of analytic functions on the corresponding space. We establish a criterion for the module of sections to be finitely generated or finite projective.

Notation 2.1.1. For $r > 0$, define the r*-Gauss norm* on $\mathbb{Q}_p[T^{\pm 1}]$ by the formula $\left|\sum_i a_i T^i\right|_r = \max_{i \in \mathbb{Z}}\{|a_i| \omega^{ir}\}$, where $a_i \in \mathbb{Q}_p$. This is a multiplicative non-archimedean norm.

For $0 < s \leq r$, we write $\mathrm{A}^1[s, r]$ for the rigid analytic annulus in the variable T with radii $|T| \in [\omega^r, \omega^s]$; its ring of analytic functions, denoted by $\mathcal{R}^{[s,r]}$, is the completion of $\mathbb{Q}_p[T^{\pm 1}]$ with respect to the norm $|\cdot|_{[s,r]} = \max\{|\cdot|_r, |\cdot|_s\}$. We also allow r (but not s) to be ∞, in which case $\mathrm{A}^1[s, r]$ is interpreted as the rigid analytic disc in the variable T with radii $|T| \leq \omega^s$; then $\mathcal{R}^{[s,r]} = \mathcal{R}^{[s,\infty]}$ is the completion of $\mathbb{Q}_p[T]$ with respect to $|\cdot|_s$. We treat $[s, \infty]$ as a closed interval when referring to it.

Let $\mathcal{R}_A^{[s,r]}$ denote the ring of analytic functions on the relative annulus (or disc if $r = \infty$) $\mathrm{Max}(A) \times \mathrm{A}^1[s, r]$; its ring of analytic functions is $\mathcal{R}_A^{[s,r]} = \mathcal{R}^{[s,r]} \widehat{\otimes}_{\mathbb{Q}_p} A$. Put $\mathcal{R}_A^r = \bigcap_{0<s\leq r} \mathcal{R}_A^{[s,r]}$ and $\mathcal{R}_A = \bigcup_{0<r} \mathcal{R}_A^r$.

Hypothesis 2.1.2. In this subsection, we fix r_0 to be a positive rational number or ∞.

For any decreasing sequence of positive (rational) numbers $r_1, r_2, \ldots$ tending to zero with $r_0 > r_1$, we have:

(i) $\mathcal{R}_A^{[r_n,r_0]}$ and $\mathcal{R}_A^{[r_n,\infty]}$ are noetherian Banach A-algebras and
(ii) $\mathcal{R}_A^{[r_{n+1},r_0]} \to \mathcal{R}_A^{[r_n,r_0]}$ and $\mathcal{R}_A^{[r_{n+1},\infty]} \to \mathcal{R}_A^{[r_n,\infty]}$ are flat and have topologically dense images,

for any $n \geq 0$. This implies that $\mathcal{R}_A^{r_0}$ is a Fréchet-Stein algebra in the sense of [46, Section 3]. We recall the terminology therein as follows.

Definition 2.1.3. A *coherent sheaf* over $\mathcal{R}_A^{r_0}$ consists of one finite module $M^{[s,r]}$ over each ring $\mathcal{R}_A^{[s,r]}$ with $0 < s \leq r \leq r_0$, together with isomorphisms $M^{[s',r']} \cong M^{[s,r]} \otimes_{\mathcal{R}_A^{[s,r]}} \mathcal{R}_A^{[s',r']}$ for all $0 < s \leq s' \leq r' \leq r \leq r_0$ satisfying the obvious cocycle conditions. Its *module of global sections* is $M = \varprojlim_{s\to 0^+} M^{[s,r_0]}$. An $\mathcal{R}_A^{r_0}$-module is *coadmissible* if it occurs as the module of global sections of some coherent sheaf.

A *vector bundle* over $\mathcal{R}_A^{r_0}$ is a coherent sheaf $(M^{[s,r]})$ where each $M^{[s,r]}$ is flat over $\mathcal{R}_A^{[s,r]}$.

We list a few basic facts from [46, Section 3]. (Strictly speaking, [46, Section 3] only treats the case $r = r_0$, but one can deduce the results for general r using standard techniques from [11].)

Lemma 2.1.4. *Let $(M^{[s,r]})$ be a coherent sheaf over $\mathcal{R}_A^{r_0}$ with module of global sections M.*

(1) *For any $0 < s \leq r \leq r_0$, M (resp. $M[\frac{1}{T}]$ if $r_0 = \infty$ and $r \neq \infty$) is dense in $M^{[s,r]}$.*
(2) *We have $\mathbf{R}^i \varprojlim_{s \to 0^+} M^{[s,r_0]} = 0$ for $i \geq 1$.*
(3) *We have a natural isomorphism $M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{[s,r]} \cong M^{[s,r]}$ for any $0 < s \leq r \leq r_0$.*
(4) *The ring $\mathcal{R}_A^{[s,r]}$ is flat over $\mathcal{R}_A^{r_0}$ for any $0 < s \leq r \leq r_0$.*
(5) *The kernel and cokernel of an arbitrary $\mathcal{R}_A^{r_0}$-linear map between coadmissible $\mathcal{R}_A^{r_0}$-modules are coadmissible.*
(6) *Any finitely generated ideal of $\mathcal{R}_A^{r_0}$ is coadmissible, or more generally, any finitely generated $\mathcal{R}_A^{r_0}$-submodule of a coadmissible module is coadmissible.*
(7) *Any finitely presented $\mathcal{R}_A^{r_0}$-module is coadmissible.*

Corollary 2.1.5. *For any element $t \in \mathcal{R}_{\mathbb{Q}_p}^{r_0}$, the ring $\mathcal{R}_A^{r_0}/t$ is flat over A. In particular, $\mathcal{R}_A^{r_0}$, $\mathcal{R}_A/t$, and $\mathcal{R}_A$ are flat over A.*

Proof. Granted the claim for $\mathcal{R}_A^{r_0}/t$, the claim for $\mathcal{R}_A^{r_0}$ follows by taking $t = 0$, and the claims with $\mathcal{R}_A$ in place of $\mathcal{R}_A^{r_0}$ follow by taking the direct limit on r_0. So we prove the first statement.

Let $N \to N'$ be an injective morphism of finite A-modules. Since A is noetherian, $N \otimes_A \mathcal{R}_A^{r_0}/t$ and $N' \otimes_A \mathcal{R}_A^{r_0}/t$ are finitely presented and hence coadmissible by Lemma 2.1.4(7). To check the injectivity of $N \otimes_A \mathcal{R}_A^{r_0}/t \to N' \otimes_A \mathcal{R}_A^{r_0}/t$, by Lemma 2.1.4(2), it suffices to check the injectivity of $N \otimes_A \mathcal{R}_A^{[s,r]}/t \to N' \otimes_A \mathcal{R}_A^{[s,r]}/t$ for any $0 < s < r \leq r_0$. When $t \neq 0$, $\mathcal{R}_A^{[s,r]}/t$ is in fact finite and free over A, so the injectivity is obvious. When $t = 0$, we take a Schauder basis of $\mathcal{R}_{\mathbb{Q}_p}^{[s,r]}$ over $\mathbb{Q}_p$ to identify $\mathcal{R}_{\mathbb{Q}_p}^{[s,r]}$ with the completed direct sum $\widehat{\oplus}_{i \in I} \mathbb{Q}_p e_i$, where $(e_i)_{i \in I}$ form a potentially orthonormal basis. Under this identification, $N \otimes_A \mathcal{R}_A^{[s,r]} \cong \widehat{\oplus}_{i \in I} N$ and $N' \otimes_A \mathcal{R}_A^{[s,r]} \cong \widehat{\oplus}_{i \in I} N'$. Since $N \to N'$ is injective, so is $\widehat{\oplus}_{i \in I} N \to \widehat{\oplus}_{i \in I} N'$. The corollary follows. ☐

Lemma 2.1.6. *For a vector bundle $(M^{[s,r]})$ over $\mathcal{R}_A^{r_0}$, its module of global sections M is a finite projective $\mathcal{R}_A^{r_0}$-module if and only if M is finitely presented.*

Proof. A module over a commutative ring is finite projective if and only if it is finitely presented and flat [38, Corollary of Theorem 7.12]. It thus suffices to assume that M is finitely presented and prove that it is flat. We need to prove that, for any finitely generated ideal I of $\mathcal{R}_A^{r_0}$, the natural map $I \otimes_{\mathcal{R}_A^{r_0}} M \to M$ is injective.

We write M as the cokernel of an $\mathcal{R}_A^{r_0}$-linear homomorphism $f : (\mathcal{R}_A^{r_0})^{\oplus m} \to (\mathcal{R}_A^{r_0})^{\oplus n}$. We can then realize $I \otimes_{\mathcal{R}_A^{r_0}} M$ as the cokernel of the $\mathcal{R}_A^{r_0}$-linear map $I \otimes f : I^{\oplus m} \to I^{\oplus n}$. By Lemma 2.1.4(6), I is coadmissible, and hence so is $I \otimes_{\mathcal{R}_A^{r_0}} M$ by Lemma 2.1.4(5). By Lemma 2.1.4(2), to check the injectivity of the natural map $I \otimes_{\mathcal{R}_A^{r_0}} M \to M$, it suffices to check the injectivity of

$$(I \otimes_{\mathcal{R}_A^{r_0}} M) \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{[s,r_0]} \to M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{[s,r_0]}$$

for all $0 < s \leq r_0$. By the flatness in Lemma 2.1.4(4), the map above becomes

$$(I \cdot \mathcal{R}_A^{[s,r_0]}) \otimes_{\mathcal{R}_A^{[s,r_0]}} M^{[s,r_0]} \to M^{[s,r_0]}.$$

It is injective because $M^{[s,r_0]}$ is a flat $\mathcal{R}_A^{[s,r_0]}$-module. This finishes the proof. $\square$

Corollary 2.1.7. *Assume that A is reduced. Let M be a finitely presented $\mathcal{R}_A^{r_0}$-module such that for any $z \in \mathrm{Max}(A)\times \mathrm{A}^1(0,r_0]$, $M/\mathfrak{m}_z M$ has the same dimension. Then M is a finite projective $\mathcal{R}_A^{r_0}$-module.*

Proof. For any $0 < s \le r \le r_0$, $\mathcal{R}_A^{[s,r]}$ is noetherian, and hence $M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^{[s,r]}$ is finite and flat by simple commutative algebra (see Lemma 2.1.8(1) below). By Lemma 2.1.6, M is then a finite projective $\mathcal{R}_A^{r_0}$-module. $\square$

Lemma 2.1.8. (1) *Let B be a noetherian ring whose Jacobson radical is zero, or equivalently, a reduced noetherian ring whose maximal ideals form a dense subset of* $\mathrm{Spec}(B)$. *Let M be a finitely generated B-module. If $z \mapsto \dim_{\kappa_z} M/\mathfrak{m}_z M$ is a locally constant function on the set of maximal ideals z of B, then M is flat.*

(2) *Let A and B be $\mathbb{Q}_p$-affinoid algebras with B* reduced, *and N a coherent sheaf on* $\mathrm{Max}(A)\times\mathrm{Max}(B)$. *Assume that for every $y \in \mathrm{Max}(B)$, $N/\mathfrak{m}_y N$ is a finite projective $A\otimes_{\mathbb{Q}_p}\kappa_y$-module of rank depending only on the connected component of y in* $\mathrm{Spec}(B)$. *Then N is a finite flat $A\widehat{\otimes}_{\mathbb{Q}_p}B$-module.*

Proof. (1) Suppose $m_1,\dots,m_r \in M$ reduce mod $\mathfrak{m}_z$ to a basis, and let $f: B^r \to M$ denote the map associated to the m_i. It suffices to show that f is an isomorphism in a neighborhood of z in $\mathrm{Spec}(B)$. Consider the kernel K and cokernel K' of f. By Nakayama's lemma, the m_i generate $M_{\mathfrak{m}_z}$, so $K'_{\mathfrak{m}_z}=0$. But the support of K' in $\mathrm{Spec}(B)$ is a closed subset, so in a neighborhood of z we have that f is surjective. Replacing $\mathrm{Spec}(B)$ by a suitable affine neighborhood of z, we assume f to be surjective. On the other hand, for all closed points $z' \in \mathrm{Spec}(B)$ the surjectivity of $f_{z'}: (B/\mathfrak{m}_{z'})^r \to M/\mathfrak{m}_{z'}M$ implies, by comparing $\kappa_{z'}$-dimensions (and using the local constancy hypothesis), that $f_{z'}$ is also injective. Thus, elements of K have all entries in B^r belonging to every $\mathfrak{m}_{z'}$, and hence to the Jacobson radical of B, which was assumed to be zero.

(2) Since N is finite over $A\widehat{\otimes}_{\mathbb{Q}_p}B$, it is flat if and only if for all $z\in\mathrm{Max}(A\widehat{\otimes}_{\mathbb{Q}_p}B)$ one has $N_{\mathfrak{m}_z}$ flat over $(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_z}$. Given such z, let $y\in\mathrm{Max}(B)$ be such that $\mathfrak{m}_y=\mathfrak{m}_z\cap B$. Then since $B\to A\widehat{\otimes}_{\mathbb{Q}_p}B$ is flat, $B_{\mathfrak{m}_y}\to(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_y}\to(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_z}$ is flat, so [1, 0.10.2.5] shows that $N_{\mathfrak{m}_z}$ is flat over $(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_z}$ if and only if both $N_{\mathfrak{m}_z}/\mathfrak{m}_yN_{\mathfrak{m}_z}$ is flat over $(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_z}/\mathfrak{m}_y(A\widehat{\otimes}_{\mathbb{Q}_p}B)_{\mathfrak{m}_z}$ and $N_{\mathfrak{m}_z}$ is flat over $B_{\mathfrak{m}_y}$. The first condition is immediate, because by hypothesis $N/\mathfrak{m}_yN$ is flat over $(A\widehat{\otimes}_{\mathbb{Q}_p}B)/\mathfrak{m}_y(A\widehat{\otimes}_{\mathbb{Q}_p}B)$, and then we may localize at $\mathfrak{m}_z$. We next show that N is flat over B, so that $N_{\mathfrak{m}_y}$ is flat over $B_{\mathfrak{m}_y}$, from which it follows easily that $N_{\mathfrak{m}_z}$ is also flat over $B_{\mathfrak{m}_y}$.

One can find a decreasing sequence of ideals J_i of A such that $J_i/J_{i+1}\approx A/I_i$ for some radical ideal $I_i\subset A$, where, say, $J_0=A$ and $J_r=(0)$. We will show by induction on i that each $J_iN/J_{i+1}N$ is flat over B and, for all $y\in\mathrm{Max}(B)$, $(J_{i+1}N)\otimes_B\kappa_y\cong J_{i+1}(N\otimes_B\kappa_y)$; it follows from this that N is flat over B. There is nothing to prove when $i=0$. Suppose that we have proved the claim for $i-1$. Applying $\otimes_B\kappa_y$ to the exact sequence $0\to J_{i+1}N\to J_iN\to(J_iN/J_{i+1}N)\to 0$ and then using the inductive hypothesis gives

$$\begin{aligned}(J_iN/J_{i+1}N)\otimes_B\kappa_y &\cong (J_iN)\otimes_B\kappa_y/\,\mathrm{Image}((J_{i+1}N)\otimes_B\kappa_y)\\ &\cong J_i(N\otimes_B\kappa_y)/J_{i+1}(N\otimes_B\kappa_y)\cong(A/I_i)\otimes_A(N\otimes_B\kappa_y).\end{aligned}$$

The right-hand side is a finite projective module over A/I_i of constant rank; because A/I_i is reduced, the claim (1) then gives that $(J_iN/J_{i+1}N)$ is flat over $(A/I_i)\widehat{\otimes}_{\mathbb{Q}_p}B$ and hence flat over B. Moreover, this implies that the application of $\otimes_B k_y$ above was exact and hence $(J_{i+1}N)\otimes_B \kappa_y \cong J_{i+1}(N\otimes_B \kappa_y)$. □

Definition 2.1.9. For R a ring and M an R-module, we say that M is n-finitely generated (resp. (m,n)-*finitely presented*) if there exists an R-linear exact sequence $R^{\oplus n}\to M\to 0$ (resp. $R^{\oplus m}\to R^{\oplus n}\to M\to 0$).

By an *admissible cover* of $(0,r_0]$, we mean a cover of $(0,r_0]$ by closed intervals $\{[s_i,r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ with nonempty interiors, admitting a locally finite refinement. Given such a cover of $(0,r_0]$, the system of rigid annuli $\mathrm{A}^1[s_i,r_i]$ of respective valuations $[s_i,r_i]$ gives an admissible affinoid cover of the rigid annulus $\mathrm{A}^1(0,r_0]$. Thus, to define a coherent sheaf over $\mathcal{R}_A^{r_0}$ is equivalent to specify $M^{[s_i,r_i]}$ for each i together with the obvious compatibility data and conditions.

Let $(M^{[s,r]})$ be a coherent sheaf over $\mathcal{R}_A^{r_0}$. We say that the sheaf is *uniformly finitely generated* if there exist $n\in\mathbb{N}$ and an admissible cover $\{[s_i,r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ of $(0,r_0]$ such that each $M^{[s_i,r_i]}$ is n-finitely generated. We say that the sheaf is *uniformly finitely presented* if there exist $m,n\in\mathbb{N}$ and an admissible cover $\{[s_i,r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ of $(0,r_0]$ such that each $M^{[s_i,r_i]}$ is (m,n)-finitely presented.

Keeping track of the number of generators, a standard argument in homological algebra proves the following.

Lemma 2.1.10. *Let* $0\to(M'^{[s,r]})\to(M^{[s,r]})\to(M''^{[s,r]})\to 0$ *be a short exact sequence of coherent sheaves over* $\mathcal{R}_A^{r_0}$.

- *If* $(M'^{[s,r]})$ *and one of the other coherent sheaves are uniformly finitely presented, so is the third one.*
- *If both* $(M^{[s,r]})$ *and* $(M''^{[s,r]})$ *are uniformly finitely presented, then* $(M'^{[s,r]})$ *is uniformly finitely generated.*

Lemma 2.1.11. *Let* $(M^{[s,r]})$ *be a coherent sheaf over* $\mathcal{R}_A^{r_0}$ *with module of global sections* M. *Assume that there exist elements* $\mathbf{f}_1,\dots,\mathbf{f}_n\in M$ *and an admissible cover* $\{[s_i,r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ *of* $(0,r_0]$ *such that* $\mathbf{f}_1,\dots,\mathbf{f}_n$ *generate each* $M^{[s_i,r_i]}$ *as an* $\mathcal{R}_A^{[s_i,r_i]}$*-module. Then* $\mathbf{f}_1,\dots,\mathbf{f}_n$ *generate* M *as an* $\mathcal{R}_A^{r_0}$*-module.*

Proof. Consider the $\mathcal{R}_A^{r_0}$-linear map $f:(\mathcal{R}_A^{r_0})^{\oplus m}\to M$ given by $\mathbf{f}_1,\dots,\mathbf{f}_n$. The hypothesis of the lemma implies that $f\otimes\mathcal{R}_A^{[s_i,r_i]}:(\mathcal{R}_A^{[s_i,r_i]})^{\oplus n}\to M^{[s_i,r_i]}$ is surjective for any i. Hence f is surjective by Lemma 2.1.4(2), yielding the lemma. □

Lemma 2.1.12. *Let* N *be a finite Banach module over a Banach algebra* R *over* $\mathbb{Q}_p$ *with generators* $\mathbf{e}_1,\dots,\mathbf{e}_n$. *Then there exists* $\epsilon>0$ *such that, for any elements* $\mathbf{e}'_1,\dots,\mathbf{e}'_n\in N$ *with* $|\mathbf{e}_i-\mathbf{e}'_i|\leq\epsilon$ *for any* i, N *is generated by* $\mathbf{e}'_1,\dots,\mathbf{e}'_n$.

Proof. We have a continuous surjective morphism $f:R^{\oplus n}\to M$ of Banach R-modules given by $\mathbf{e}_1,\dots,\mathbf{e}_n$. By the Banach open mapping theorem ([12, Section I.3.3, Théorème 1] or [45, Proposition 8.6]), there exists $\epsilon>0$ such that for any $\mathbf{v}\in M$ with $|\mathbf{v}|\leq\epsilon$, we can write $\mathbf{v}=x_1\mathbf{e}_1+\cdots+x_n\mathbf{e}_n$ for some $x_i\in R$ with $|x_i|\leq\frac{1}{2}$. Given the data in the lemma, we write $\mathbf{e}'_i-\mathbf{e}_i=\sum_{j=1}^n x_{ji}\mathbf{e}_j$ for $x_{ji}\in R$ with $|x_{ji}|\leq\frac{1}{2}$. Set $X=(x_{ji})$. The transition matrix for the two sets of elements is $1+X$, which is invertible. Hence, N is generated by $\mathbf{e}'_1,\dots,\mathbf{e}'_n$. □

We are grateful to R. Bellovin for showing us a more elementary proof of the following result (see [4, Section 2.2]); we provide the following argument anyway, because its method is used to prove Lemma 2.1.15 below, to which her argument does not seem to apply.

Proposition 2.1.13. *Let $(M^{[s,r]})$ be a coherent sheaf over $\mathcal{R}_A^{r_0}$ with module of global sections M.*

(1) *The $\mathcal{R}_A^{r_0}$-module M is finitely generated if and only if $(M^{[s,r]})$ is uniformly finitely generated.*
(2) *The $\mathcal{R}_A^{r_0}$-module M is finitely presented if and only if $(M^{[s,r]})$ is uniformly finitely presented.*
(3) *The $\mathcal{R}_A^{r_0}$-module M is finite projective if and only if $(M^{[s,r]})$ is a uniformly finitely presented vector bundle.*

Proof. (3) follows from combining (2) with Lemma 2.1.6. (2) follows from applying (1) twice and using Lemma 2.1.10 to pass on the conditions. The sufficiency part of (1) is obvious.

We now prove the necessity part of (1). Assume that each $(M^{[s,r]})$ is n-finitely generated. We may refine and reorder the admissible cover $\{[s_i, r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ so that the subcollections $I_{\text{odd}} = \{[s_{2i+1}, r_{2i+1}]\}_{i\in\mathbb{Z}_{\geq 0}}$ and $I_{\text{even}} = \{[s_{2i}, r_{2i}]\}_{i\in\mathbb{Z}_{\geq 0}}$ consist of pairwise *disjoint* intervals in decreasing order. We claim that there exist global sections $\mathbf{f}_1^{\text{odd}}, \dots, \mathbf{f}_n^{\text{odd}} \in M$ (resp. $\mathbf{f}_1^{\text{even}}, \dots, \mathbf{f}_n^{\text{even}} \in M$) generating each $M^{[s_i,r_i]}$ where i is odd (resp. even). Then the proposition follows from Lemma 2.1.11 above. We will only prove the claim for the even subcollection, as the proof for the odd one proceeds the same way.

For each i, we use $|\cdot|_{M,[s_i,r_i]}$ to denote a fixed Banach norm on $M^{[s_i,r_i]}$. As in Definition 2.1.9, $M^{[s_{2i},r_{2i}]}$ is generated by n elements $\mathbf{f}_{2i,1}, \dots, \mathbf{f}_{2i,n} \in M^{[s_{2i},r_{2i}]}$. By the density of M (resp. $M[\frac{1}{T}]$ if $r_0 = \infty$ and $r_{2i} \neq \infty$) in $M^{[s_{2i},r_{2i}]}$ (Lemma 2.1.4(1)) and Lemma 2.1.12, we may assume that $\mathbf{f}_{2i,1}, \dots, \mathbf{f}_{2i,n} \in M$ are global sections. (When $r_0 = \infty$ and $r_{2i} \neq \infty$, we first get the generators in $M[\frac{1}{T}]$ and then we multiply them by some powers of T; they are still generators.)

By Lemma 2.1.12 again, there exists $\epsilon_{2i} > 0$ such that any elements $\mathbf{f}'_{2i,1}, \dots, \mathbf{f}'_{2i,n} \in M$ with $|\mathbf{f}'_{2i,j} - \mathbf{f}_{2i,j}|_{M,[s_{2i},r_{2i}]} \leq \epsilon_{2i}$ for all j also generate $M^{[s_{2i},r_{2i}]}$. For each j, we hope to define

$$\mathbf{f}_j^{\text{even}} = \alpha_{0,j} T^{\lambda_{0,j}} \mathbf{f}_{0,j} + \alpha_{2,j} T^{\lambda_{2,j}} \mathbf{f}_{2,j} + \alpha_{4,j} T^{\lambda_{4,j}} \mathbf{f}_{4,j} + \cdots,$$

where $\alpha_{2i,j} \in \mathbb{Q}_p$ and $\lambda_{2i,j} \in \mathbb{Z}_{\geq 0}$ are chosen inductively on i with $\lambda_{0,j} = 0, \alpha_{0,j} = 1$ so that $\lambda_{2i,j} \geq \lambda_{2i-2,j}$, and

$$\text{(2.1.13.1)} \qquad \text{for any } i' < i, \ \left| \frac{\alpha_{2i,j} T^{\lambda_{2i,j}}}{\alpha_{2i',j} T^{\lambda_{2i',j}}} \mathbf{f}_{2i,j} \right|_{M,[s_{2i'},r_{2i'}]} \leq \epsilon_{2i'},$$

$$\text{(2.1.13.2)} \qquad \text{for any } i' < i, \ \left| \frac{\alpha_{2i',j} T^{\lambda_{2i',j}}}{\alpha_{2i,j} T^{\lambda_{2i,j}}} \mathbf{f}_{2i',j} \right|_{M,[s_{2i},r_{2i}]} \leq \epsilon_{2i}, \text{ and}$$

(2.1.13.3)

$$\text{for any } i'' < 2i-1 \text{ (even or odd)}, \ \left| \alpha_{2i,j} T^{\lambda_{2i,j}} \mathbf{f}_{2i,j} \right|_{M,[s_{i''},r_{i''}]} \leq p^{-i}.$$

This would imply that the infinite sum defining $\mathbf{f}_j^{\mathrm{even}}$ converges and we have for any i,

$$\left|\mathbf{f}_{2i,j} - \frac{1}{\alpha_{2i,j}T^{\lambda_{2i,j}}}\mathbf{f}_j^{\mathrm{even}}\right|_{M,[s_{2i},r_{2i}]} \leq \epsilon_{2i}.$$

(This equation is also valid for $i = 0$ because $\alpha_{0,j} = 1$ and $\lambda_{0,j} = 0$.) These imply that $\mathbf{f}_1^{\mathrm{even}}, \dots, \mathbf{f}_n^{\mathrm{even}}$ generate $M^{[s_{2i},r_{2i}]}$ over $\mathcal{R}_A^{[s_{2i},r_{2i}]}$.

We now prove that we can choose $\alpha_{2i,j}$ and $\lambda_{2i,j}$ satisfying (2.1.13.1)–(2.1.13.3). It suffices to meet the following conditions:

$$\text{for any } i' < i,\ |\alpha_{2i,j}| \cdot \omega^{s_{2i'}\lambda_{2i,j}} \leq \omega^{s_{2i'}\lambda_{2i',j}} |\mathbf{f}_{2i,j}/\alpha_{2i',j}|^{-1}_{M,[s_{2i'},r_{2i'}]} \cdot \epsilon_{2i'},$$

$$\text{for any } i' < i,\ |\alpha_{2i,j}| \cdot \omega^{r_{2i}\lambda_{2i,j}} \geq \omega^{r_{2i}\lambda_{2i',j}} |\alpha_{2i',j}\mathbf{f}_{2i',j}|_{M,[s_{2i},r_{2i}]}/\epsilon_{2i}, \text{ and}$$

$$\text{for any } i'' < 2i-1,\ |\alpha_{2i,j}| \cdot \omega^{s_{i''}\lambda_{2i,j}} \leq |\mathbf{f}_{2i,j}|^{-1}_{M,[s_{i''},r_{2i''}]} \cdot p^{-i}.$$

Note that for any $i' < i$ and $i'' < 2i-1$, we have $r_{2i} < s_{2i'}$ and $r_{2i} < s_{i''}$ because the intervals in I_{even} are pairwise disjoint. Hence, by making $\lambda_{2i,j} - \lambda_{2i',j}$ sufficiently large, we may find a choice of $\alpha_{2i,j} \in \mathbb{Q}_p$ satisfying all conditions above. This finishes the proof. □

Remark 2.1.14. The condition in (3) may be weakened to only require $(M^{[s,r]})$ to be a uniformly finitely generated vector bundle. Indeed, $M^{[s,r]}$ is finite flat and hence finite projective, so if it is generated by n elements, it is automatically (n,n)-finitely presented. Hence, $(M^{[s,r]})$ is uniformly finitely presented.

We will later apply Proposition 2.1.13 in several cases: φ-bundles and φ-modules over $\mathcal{R}_A^{r_0}$ (Proposition 2.2.7 and Lemma 2.2.9, respectively), for a (φ,Γ)-module M over $\mathcal{R}_A^{r_0}$, the structure theorems for $M^{\psi=0}$ (Theorem 3.1.1) and $M^{\psi=1}$ (Proposition 5.2.10).

The following variant of Proposition 2.1.13 is tailored to the situation of Theorem 5.2.12.

Lemma 2.1.15. *Let M be a finite projective $\mathcal{R}_A^\infty$-module and N an $\mathcal{R}_A^\infty$-submodule of M such that $N \otimes_{\mathcal{R}_A^\infty} \mathcal{R}_A^{[s,\infty]} \xrightarrow{\sim} M \otimes_{\mathcal{R}_A^\infty} \mathcal{R}_A^{[s,\infty]}$ for any $s > 0$. Moreover, assume that N is complete for a Fréchet topology defined by a sequence of norms $|\cdot|_{N,q_1} \leq |\cdot|_{N,q_2} \leq \cdots$ indexed by $\mathbb{N}$, such that for each m, there exists $g_m \in \mathbb{N}$ such that the operator norm $|T^{g_m}|_{N,q_m} \leq \omega$. Then $N = M$.*

Proof. For $0 < s \leq r$, we write $M^{[s,r]}$ for $M \otimes_{\mathcal{R}_A^\infty} \mathcal{R}_A^{[s,r]}$. It suffices to exhibit a finite set of elements of N generating all $M^{[s_i,r_i]}$ over $\mathcal{R}_A^{[s_i,r_i]}$ for some admissible cover $\{[s_i,r_i]\}_{i\in\mathbb{Z}_{\geq 0}}$ of $(0,\infty]$, by Lemma 2.1.11.

We fix generators $\mathbf{e}_1, \dots, \mathbf{e}_n$ of M as an $\mathcal{R}_A^\infty$-module. We proceed as in the proof of Proposition 2.1.13 by separating the admissible cover into subcollections I_{odd} and I_{even} of pairwise disjoint decreasing intervals. We will prove that there exist $\mathbf{f}_1^{\mathrm{even}}, \dots, \mathbf{f}_n^{\mathrm{even}} \in N$ generating $M^{[s_{2i},r_{2i}]}$ for any i, and the corresponding statement for odd subcollection will follow by the same argument. For each i, we use $|\cdot|_{M,[s_{2i},r_{2i}]}$ to denote a fixed Banach norm on $M^{[s_{2i},r_{2i}]}$. Using the isomorphism $N \otimes_{\mathcal{R}_A^\infty} \mathcal{R}_A^{[s,\infty]} \cong M^{[s,\infty]}$, we may write each $\mathbf{e}_j$ in terms of a finite sum $\sum_\alpha m_\alpha \otimes f_\alpha(T)$ for $m_\alpha \in N$ and $f_\alpha(T) \in \mathcal{R}_A^{[s,\infty]}$. We may approximate each $f_\alpha(T)$ by elements in $\mathcal{R}_A^\infty[\frac{1}{T}]$ (resp. $\mathcal{R}_A^\infty$ if $i = 0$) and hence obtain $\mathbf{m}_{2i,j} \in N[\frac{1}{T}]$ (resp. $\mathbf{m}_{2i,j} \in N$ if $i = 0$) such that $|\mathbf{m}_{2i,j} - \mathbf{e}_j|_{M,[s_{2i},r_{2i}]}$ is sufficiently small so that $\mathbf{m}_{2i,1}, \dots, \mathbf{m}_{2i,n}$ form a basis of $M^{[s_{2i},r_{2i}]}$ by invoking Lemma 2.1.12. By multiplying some powers of T (when $i > 0$), we may take $\mathbf{f}_{2i,1}, \dots, \mathbf{f}_{2i,m} \in N$ generating $M^{[s_{2i},r_{2i}]}$.

By Lemma 2.1.12 again, there exists $\epsilon_{2i} > 0$ such that any elements $\mathbf{f}'_{2i,1}, \dots, \mathbf{f}'_{2i,n} \in N$ with $|\mathbf{f}'_{2i,j} - \mathbf{f}_{2i,j}|_{M,[s_{2i},r_{2i}]} \leq \epsilon_{2i}$ for any j also generate $M^{[s_{2i},r_{2i}]}$. We define $\mathbf{f}_j^{\mathrm{even}}$ as in the proof of Proposition 2.1.13, but instead of (2.1.13.3), we require (inductively on i)

$$\text{for any } m \leq i \text{ with } r_{2i} < 1/g_m, \text{ we have } \left|\alpha_{2i,j} T^{\lambda_{2i,j}} \mathbf{f}_{2i,j}\right|_{N,q_m} \leq p^{-i},$$

which is implied by

$$\text{for any } m \leq i \text{ with } r_{2i} < 1/g_m, \text{ we have}$$
$$|\alpha_{2i,j}| \omega^{\lfloor \lambda_{2i,j}/g_m \rfloor} \leq |\mathbf{f}_{2i,j}|_{N,q_m}^{-1} \cdot p^{-i} \cdot \min\{1, |T|_{q_m}, \dots, |T^{g_m-1}|_{q_m}\}^{-1}.$$

By making $\lambda_{2i,j}$ sufficiently large, we may find a choice of $\alpha_{2i,j} \in \mathbb{Q}_p$ satisfying all required conditions, finishing the proof. □

Lemma 2.1.16. *Let $f : M \to N$ be a morphism of two $\mathcal{R}_A^{r_0}$-modules with finitely generated kernel (resp. cokernel). If $f \otimes \mathcal{R}_A : M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A \to N \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A$ is injective (resp. surjective), then there exists some $r \in (0, r_0]$ such that $f \otimes \mathcal{R}_A^r : M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^r \to N \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^r$ is injective (resp. surjective).*

Proof. Let Q denote the kernel (resp. cokernel) of f. We have $Q \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A = 0$. Since Q is finitely generated, there exists $r \in (0, r_0]$ such that $Q \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^r = 0$ and hence $f \otimes \mathcal{R}_A^r$ is injective (resp. surjective). □

Remark 2.1.17. In the preceding proposition, we only need the finite generation of N to test for surjectivity. To test for bijectivity, it is sufficient that M and N both be finite projective: after restricting to $r \in (0, r_0]$ we may assume $f \otimes \mathcal{R}_A^r$ is surjective, and the kernel is then a summand of M hence also finitely generated. (The issue we are working around here is that $\mathcal{R}_A^{r_0}$ is not noetherian.)

The following lemma will be needed later in Theorem 4.4.6.

Lemma 2.1.18. *If M is an $\mathcal{R}_A^{r_0}$-module which is finite over A, then there exists $s \in (0, r_0]$ such that $M \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^s = 0$.*

Proof. If A is a finite $\mathbb{Q}_p$-algebra then M is artinian, and so $M \to M^{[s_0,r_0]}$ must be an isomorphism for sufficiently small s_0. Any $s \in (0, s_0)$ then has the desired property.

In the general case, view M as a module over the subring $A\langle T\rangle \subseteq \mathcal{R}_A^{r_0}$; it is finitely generated over this ring, because it is finitely generated over A. By specializing modulo maximal ideals of A and invoking the preliminary case above, we see that the reduced closed affinoid subspace $\mathrm{Supp}(M)$ of $\mathrm{Max}(A\langle T\rangle)$ does not meet $\mathrm{Max}(A\langle T^{\pm 1}\rangle)$. By the maximum modulus principle, the spectral norm ω^{s_0} of T on $\mathrm{Supp}(M)$ therefore satisfies $\omega^{s_0} < 1$. Any $s \in (0, s_0)$ then has the desired property. □

We conclude this subsection with some constructions involving residues and duality of Robba rings. Denote by $\Omega_{\mathcal{R}_A} = \widehat{\Omega}^1_{\mathcal{R}_A/A}$ the module of continuous A-linear differentials of $\mathcal{R}_A$. The *residue* map $\mathrm{Res}_{\mathcal{R}_A} : \Omega_{\mathcal{R}_A} \to A$ sends $\sum_{n \in \mathbb{Z}} a_n T^n dT$ to a_{-1}.

Lemma 2.1.19. *The module $\Omega_{\mathcal{R}_A}$ is free of rank one over $\mathcal{R}_A$ with basis dT, and in particular $\mathrm{Res}_{\mathcal{R}_A}$ is well defined. Moreover, $\mathrm{Res}_{\mathcal{R}_A}$ depends only on $\mathcal{R}_A$ up to A-linear continuous isomorphism, and not on the choice of indeterminate T.*

The natural pairing $\mathcal{R}_A \times \mathcal{R}_A \to A$ sending $(f(T), g(T))$ to $\mathrm{Res}_{\mathcal{R}_A}(f(T)g(T)dT)$ induces two topological isomorphisms depending on T:

$$\mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A, A) \cong \mathcal{R}_A \quad \text{and} \quad \mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A/\mathcal{R}_A^\infty, A) \cong \mathcal{R}_A^\infty.$$

(The topology on the first Hom*-set is dual-to-LF and hence LF, and the topology on the second* Hom*-set is Fréchet.)*

Proof. The computation of $\Omega_{\mathcal{R}_A}$ reduces to the structure of $\widehat{\Omega}^1_{\mathcal{R}_A^{[r,s]}/A}$, and that the latter is free over A with basis dT is obvious. To see that $\mathrm{Res}_{\mathcal{R}_A}$ is independent of choices, it suffices to take $A = \mathbb{Q}_p$. The indeterminate T belongs to the bounded subring $\mathcal{E}^\dagger$ of $\mathcal{R}$, which is dense in $\mathcal{R}$ and is a Henselian discretely valued field with uniformizer p; call the completion of $\mathcal{E}^\dagger$ for the p-adic topology $\mathcal{E}$. Continuous automorphisms of $\mathcal{R}$ preserve $\mathcal{E}^\dagger$ and induce p-adically continuous automorphisms of $\mathcal{E}$. Thus, we may show the independence of the residue map for $\mathcal{E}$ instead of $\mathcal{R}$, and this is done in [25, A2 2.2.3].

The dualities are well known; they appear implicitly in [32, Section 8.5], and when A is a finite extension of $\mathbb{Q}_p$, they are proved in [23, Theorem 5.4]. For the convenience of the reader, we sketch a short proof of the bijection. The pairing obviously defines a map $\mathcal{R}_A \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A, A)$. Conversely, to any $\mu \in \mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A, A)$, we associate a power series $\sum_{n\in\mathbb{Z}} \mu(T^{-1-n})T^n$. One checks easily that the continuity of μ translates directly to the condition that this formal Laurent series lies in $\mathcal{R}_A$. This defines an inverse to the natural morphism $\mathcal{R}_A \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A, A)$, proving the first isomorphism. Having proved this, the second isomorphism follows immediately because the associated formal Laurent series lies in $\mathcal{R}_A^\infty$ if and only if $\mu(T^{-1-n}) = 0$ for any $n < 0$, or equivalently, $\mu(\mathcal{R}_A^\infty) = 0$. See [23, Proposition 5.5] for a discussion of the topologies in the case where A is a field, the general case being similar. □

We will also need a group-theoretic variant of the residue pairing.

Definition 2.1.20. Assume given the data (C, ℓ), where C is a multiplicative abelian topological group whose torsion subgroup C_{tors} is finite and C/C_{tors} is noncanonically isomorphic to $\mathbb{Z}_p$, and ℓ is a nonzero continuous homomorphism $C \to \mathbb{Q}_p$.

Let $0 < s \le r$ be given. In the case where C is torsion-free, we choose a topological generator c of C and we define $\mathcal{R}_A^{[s,r]}(C)$ to be the formal substitution of T by $c-1$ in $\mathcal{R}_A^{[s,r]}$. For any other choice c', the element $\frac{c'-1}{c-1}$ is a unit with all Gauss norms equal to 1, so this definition does not depend on c. Moreover, since one has $|f((1+T)^p-1)|_r = |f(T)|_{pr}$ for all $f(T) \in \mathbb{Q}_p[T^{\pm 1}]$ if $r < 1$ (recall that $|T|_r = \omega^r$), it follows that if either $r < 1$ or both $r = \infty$ and $s < 1$ then one has $\mathcal{R}_A^{[s,r]}(C) = \mathcal{R}_A^{[ps,pr]}(C^p) \otimes_{\mathbb{Z}[C^p]} \mathbb{Z}[C]$. Therefore, in the general case, if we choose a torsion-free open subgroup $D \subseteq C$ and set $\mathcal{R}_A^{[s,r]}(C) = \mathcal{R}_A^{[[C:D]s,[C:D]r]}(D) \otimes_{\mathbb{Z}[D]} \mathbb{Z}[C]$, then $\mathcal{R}_A^{[s,r]}(C)$ is independent of all choices if either $r < 1/\#C_{\mathrm{tors}}$ or both $r = \infty$ and $s < 1/\#C_{\mathrm{tors}}$. One defines as well $\mathcal{R}_A^r(C) = \bigcap_{0<s\le r} \mathcal{R}_A^{[s,r]}(C)$ and $\mathcal{R}_A(C) = \bigcup_{0<r} \mathcal{R}_A^r(C)$. For general open subgroups $D' \subseteq D$ of C, if either $r < 1/\#D_{\mathrm{tors}}$ or both $r=\infty$ and $s<1/\#D_{\mathrm{tors}}$ then one has $\mathcal{R}^?_A(D)=\mathcal{R}^{[D:D']?}_A(D')\otimes_{\mathbb{Z}[D']}\mathbb{Z}[D]$, where for $?=[s,r],r,\emptyset$ we write $[D:D']?=[[D:D']s,[D:D']r],[D:D']r,\emptyset$, respectively. In particular, in this case $\mathcal{R}^?_A(D)$ is a free $\mathcal{R}^{[D:D']?}_A(D')$-module of rank $[D:D']$.

If $a\in\mathbb{Z}$ and $c\in C$ is nontorsion, then $d\log(c^a)=\frac{d(c^a)}{c^a}=a\frac{dc}{c}=ad\log(c)$. It follows that the differential form $\omega_\ell=\ell(c)^{-1}d\log(c)=\ell(c)^{-1}\frac{dc}{c}\in\Omega_{\mathcal{R}_A(C)}$ is independent of the choice of nontorsion $c\in C$, and even independent of C, in the sense that for every open subgroup $D\subseteq C$ it belongs to $\Omega_{\mathcal{R}_A(D)}\subseteq\Omega_{\mathcal{R}_A(C)}$ and agrees with the differential form $\omega_{\ell|_D}$. One has that ω_ℓ is an $\mathcal{R}_A(C)$-basis of $\Omega_{\mathcal{R}_A(C)}$, and therefore it makes sense to define

$$\mathrm{Res}_C:\Omega_{\mathcal{R}_A(C)}\to A,\qquad f\omega_\ell\mapsto\mathrm{Res}_{\mathcal{R}_A(D)}(\mathrm{Tr}_{\mathcal{R}_A(C)/\mathcal{R}_A(D)}(f)\omega_\ell),$$

for $D\subseteq C$ a torsion-free open subgroup, where $\mathrm{Res}_{\mathcal{R}_A(D)}$ denotes the intrinsic residue map on the Robba ring $\mathcal{R}_A(D)$ as above. It is easy to check that Res_C for $\mathcal{R}_A(C)$ is independent of the auxilliary choices D and ℓ. As in the case of a Robba ring, the pairing

$$\{-,-\}_\ell:\mathcal{R}_A(C)\times\mathcal{R}_A(C)\to A,\qquad (f,g)\mapsto\mathrm{Res}_C(fg\omega_\ell),$$

depends only on ℓ, and is perfect and induces topological isomorphisms

(2.1.20.1)

$$\mathcal{R}_A(C)\xrightarrow{\sim}\mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A(C),A),\qquad \mathcal{R}^\infty_A(C)\xrightarrow{\sim}\mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A(C)/\mathcal{R}^\infty_A(C),A).$$

Remark 2.1.21. Continue with the data (C,ℓ) of Definition 2.1.20. Let c be an element of C whose image in C/C_{tors} is a topological generator. For X a $\mathbb{Q}_p$-vector space equipped with a continuous action C, the continuous cohomology $H^*(C,X)$ can be computed by the complex $[X^{C_{\mathrm{tors}}}\xrightarrow{c-1}X^{C_{\mathrm{tors}}}]$. In particular, $H^1(C,X)$ is isomorphic to $X^{C_{\mathrm{tors}}}/(c-1)\cong X_C$, the coinvariants of X. We wish to warn the reader that this isomorphism depends on the choice of the element c above, in the following sense. Given another choice c', the natural isomorphism of $[X^{C_{\mathrm{tors}}}\xrightarrow{c-1}X^{C_{\mathrm{tors}}}]$ with $[X^{C_{\mathrm{tors}}}\xrightarrow{c'-1}X^{C_{\mathrm{tors}}}]$ is given by $[1\quad\frac{c'-1}{c-1}]$. The two isomorphisms $H^1(C,X)\approx X_C$ differ thus by a factor of $\ell(c')/\ell(c)$; multiplying by $\ell(c)^{-1}$ the naïve identification $H^1(C,X)\xrightarrow{\sim}X_C$ arising from the choice c yields an isomorphism that is compatible with changing the choice of c, and depends only on ℓ.

The preceding discussion applies equally well when the p-torsion subgroup $C_{p\text{-tors}}$ is such that $C_{\mathrm{tors}}/C_{p\text{-tors}}$ is cyclic (the Chinese remainder theorem then implies $C/C_{p\text{-tors}}$ is procyclic), one replaces $X^{C_{\mathrm{tors}}}$ by $X^{C_{p\text{-tors}}}$, and one instead chooses $c\in C$ mapping to a topological generator of $C/C_{p\text{-tors}}$. These conditions are satisfied, in the notations to appear shortly, by $C=\Gamma_K$, $\ell=\log\circ\chi$, and $c=\gamma_K$. Then every occurrence of the term $\log\chi(\gamma_K)$ or its reciprocal in this paper can be explained as normalizing a passage between $H^1(\Gamma_K,X)$ and X_{Γ_K} to render it compatible with changing the choice of γ_K.

Remark 2.1.22. Except for Corollary 2.1.7 and Lemma 2.1.8, the results in this subsection remain valid even if A is only assumed to be a strongly noetherian Banach algebra over $\mathbb{Q}_p$, meaning that the Tate algebra over A in any number of variables is Noetherian.

2.2. (φ,Γ)**-modules and Galois representations.** In this subsection, we define (φ,Γ)-modules over the relative Robba ring and state their relationship with Galois representations. In particular, using the results from the previous subsection, we establish the equivalence of the two viewpoints of relative (φ,Γ)-modules treated in [34] (one defined using modules over the relative Robba ring, the other defined using vector bundles over a relative annulus).

Notation 2.2.1. Throughout the rest of the paper, fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and a finite extension K of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. Choose a system $\varepsilon = (\zeta_{p^n})_{n\geq 0}$, where each $\zeta_{p^n} \in \overline{\mathbb{Q}}_p$ is a primitive p^nth root of unity and $(\zeta_{p^{n+1}})^p = \zeta_{p^n}$.

Let μ_{p^∞} denote the set of p-power roots of unity. Set $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$, $H_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K(\mu_{p^\infty}))$, and $\Gamma_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$. The group Γ_K is naturally identified with a subgroup of $\Gamma_{\mathbb{Q}_p}$. The cyclotomic character $\chi : \Gamma_{\mathbb{Q}_p} = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ is given by $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for any $\zeta \in \mu_{p^\infty}$ and $\gamma \in \Gamma_{\mathbb{Q}_p}$.

We let k be the residue field of K, with $F = W(k)[1/p]$, and we let k' denote the residue field of $K(\mu_{p^\infty})$, with $F' = W(k')[1/p]$. We write $\tilde{e}_K = [K(\mu_{p^\infty}) : \mathbb{Q}_p(\mu_{p^\infty})]/[k' : k]$ for the ramification degree of $K(\mu_{p^\infty})/\mathbb{Q}_p(\mu_{p^\infty})$.

Definition 2.2.2. The theory of the field of norms allows a certain choice of indeterminate π_K, and gives rise to a constant $C(\pi_K) > 0$. (See [26, pp. 154–156] for details. We assume, in addition, that $C(\pi_K) < 1/\#(\Gamma_{\mathbb{Q}_p})_{\mathrm{tors}}$.) For $0 < s \leq r \leq C(\pi_K)$, we set $\mathcal{R}^{[s,r]}(\pi_K)$ to be the formal substitution of T by π_K in the ring $\mathcal{R}^{[s,r]}_{F'}$; we set $\mathcal{R}^{[s,r]}_A(\pi_K) = \mathcal{R}^{[s,r]}(\pi_K)\widehat{\otimes}_{\mathbb{Q}_p} A$. We similarly define $\mathcal{R}^r_A(\pi_K)$ and $\mathcal{R}_A(\pi_K)$; the latter is referred to as the *relative Robba ring* over A for K. If π_K and π'_K are two choices of indeterminates as above, then for $0 < s \leq r \leq \min\{C(\pi_K), C(\pi'_K)\}$ the respective rings $\mathcal{R}^{[s,r]}_A(\pi_K)$ and $\mathcal{R}^{[s,r]}_A(\pi'_K)$ are canonically isomorphic. There are commuting A-linear actions on $\mathcal{R}^{[s,r]}_A(\pi_K)$ of Γ_K and of an operator $\varphi : \mathcal{R}^{[s,r]}_A(\pi_K) \to \mathcal{R}^{[s/p,r/p]}_A(\pi_K)$. The actions on the coefficients F' are the natural ones, Γ_K through its quotient $\mathrm{Gal}(F'/F)$ and φ by Witt functoriality, but the actions on π_K depend on its choice; see [6] for more discussion of these actions. Moreover, φ makes $\mathcal{R}^{[s,r]}_A(\pi_K)$ into a free $\mathcal{R}^{[s/p,r/p]}_A(\pi_K)$-module of rank p, and we obtain a Γ_K-equivariant left inverse $\psi : \mathcal{R}^{[s/p,r/p]}_A(\pi_K) \to \mathcal{R}^{[s,r]}_A(\pi_K)$ by the formula $p^{-1}\varphi^{-1} \circ \mathrm{Tr}_{\mathcal{R}^{[s/p,r/p]}_A(\pi_K)/\varphi(\mathcal{R}^{[s,r]}_A(\pi_K))}$. The map ψ extends to functions $\mathcal{R}^{r_0/p}_A(\pi_K) \to \mathcal{R}^{r_0}_A(\pi_K)$ for $r_0 \leq C(\pi_K)$ and $\mathcal{R}_A(\pi_K) \to \mathcal{R}_A(\pi_K)$.

Notation 2.2.3. Since we will often be in the case $K = \mathbb{Q}_p$, we simplify our notations in this case, writing $\Gamma = \Gamma_{\mathbb{Q}_p}$ and $\mathcal{R}^?_A = \mathcal{R}^?_A(\pi_K)$ for $? = [s,r], r, \emptyset$ (the latter overloading our previous meaning for $\mathcal{R}^?_A$).

Moreover, in this case one has $F' = \mathbb{Q}_p$, and the choice of ε above gives rise to a preferred indeterminate $\pi = \pi_{\mathbb{Q}_p}$ with constant any number $C(\pi) < 1/(p-1)$. We can explicitly describe the (φ,Γ)-actions on this indeterminate by $\gamma(\pi) = (1+\pi)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma = \Gamma_{\mathbb{Q}_p}$ and $\varphi(\pi) = (1+\pi)^p - 1$. (Any other choice of ε is of the form $\gamma(\varepsilon)$, $\gamma \in \Gamma$, and leads to the indeterminate $\gamma(\pi)$.) The ψ-action is computed by knowing that, in this case, $\mathcal{R}^{[s,r]}_A = \bigoplus_{i=0}^{p-1}(1+\pi)^i\varphi\mathcal{R}^{[ps,pr]}_A$, so if $f = \sum_{i=0}^{p-1}(1+\pi)^i\varphi(f_i)$ then $\psi(f) = f_0$.

We use the following notation for subgroups of $\Gamma = \Gamma_{\mathbb{Q}_p}$. For $n \geq 1$ (resp. $n \geq 2$ if $p = 2$), write $\gamma_n \in \Gamma$ for the unique element with $\chi(\gamma_n) = 1 + p^n$, as well as $\gamma'_n = \gamma_1^{p^{n-1}}$ (resp. $\gamma'_n = \gamma_2^{p^{n-2}}$ if $p = 2$) so that $\chi(\gamma'_n) = (1+p)^{p^{n-1}}$ (resp. $\chi(\gamma'_n) = (1+p^2)^{p^{n-2}}$ if $p = 2$). Each of these elements topologically generates the subgroup Γ_n of Γ mapped isomorphically by χ onto $1+p^n\mathbb{Z}_p$, so that $\Gamma = \Gamma_1 \times \mu_{p-1}$ (resp. $\Gamma = \Gamma_2 \times \mu_2$ if $p = 2$).

Remark 2.2.4. For L a finite extension of K and for $0 < r \leq \min\{C(\pi_K), C(\pi_L)\tilde{e}_L/\tilde{e}_K\}$, $\mathcal{R}_A^{r\tilde{e}_K/\tilde{e}_L}(\pi_L)$ is naturally a finite étale $\mathcal{R}_A^r(\pi_K)$-algebra, free of rank $[L(\mu_{p^\infty}) : K(\mu_{p^\infty})]$; when L/K is Galois, it has Galois group H_K/H_L. In general, the inclusion is equivariant for the actions of Γ_L (as a subgroup of Γ_K) and φ.

Notation 2.2.5. For M^{r_0} a module over $\mathcal{R}_A^{r_0}(\pi_K)$ and for $0 < s \leq r \leq r_0$, we set $M^? = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A^?(\pi_K)$ where $? = [s,r], s, \emptyset$. For example, $M = M^{r_0} \otimes_{\mathcal{R}_A^{r_0}(\pi_K)} \mathcal{R}_A(\pi_K)$.

Definition 2.2.6. Let $r_0 \in (0, C(\pi_K)]$. A *φ-module* over $\mathcal{R}_A^{r_0}(\pi_K)$ is a finite projective $\mathcal{R}_A^{r_0}(\pi_K)$-module M^{r_0} equipped with an isomorphism $\varphi^*M^{r_0} \cong M^{r_0/p}$. A *$\varphi$-module* over $\mathcal{R}_A(\pi_K)$ is the base change to $\mathcal{R}_A(\pi_K)$ of a φ-module over some $\mathcal{R}_A^{r_0}(\pi_K)$. For M^{r_0} a φ-module over $\mathcal{R}_A^{r_0}(\pi_K)$, the isomorphism $\varphi^*M^{r_0} \cong M^{r_0/p}$ induces a continuous A-linear homomorphism

$$\psi : M^{r_0/p} \cong \varphi(M^{r_0}) \otimes_{\varphi(\mathcal{R}_A^{r_0}(\pi_K))} \mathcal{R}_A^{r_0/p}(\pi_K) \to M^{r_0}$$

given by $\psi(\varphi(m) \otimes f) = m \otimes \psi(f)$ for $m \in M^{r_0}$ and $f \in \mathcal{R}_A^{r_0/p}(\pi_K)$.

A *φ-bundle* over $\mathcal{R}_A^{r_0}(\pi_K)$ is a vector bundle $(M^{[s,r]})$ over $\mathcal{R}_A^{r_0}(\pi_K)$ equipped with isomorphisms $\varphi^*M^{[s,r]} \cong M^{[s/p,r/p]}$ for all $0 < s \leq r \leq r_0$ satisfying the obvious compatibility conditions.

Proposition 2.2.7. *The natural functor from φ-modules over $\mathcal{R}_A^{r_0}(\pi_K)$ to φ-bundles over $\mathcal{R}_A^{r_0}(\pi_K)$ is an equivalence of categories.*

Proof. This natural functor is obviously fully faithful. To check essential surjectivity, by Proposition 2.1.13, it suffices to check that any φ-bundle $(M^{[s,r]})$ is uniformly finitely presented. Since $\mathcal{R}_A^{[r_0/p,r_0]}(\pi_K)$ is noetherian, $M^{[r_0/p,r_0]}$ is (m,n)-finitely presented for some $m,n \in \mathbb{N}$. Pulling back along φ^a for any $a \in \mathbb{N}$, we see that $M^{[r_0/p^{a+1},r_0/p^a]} \cong (\varphi^a)^*M^{[r_0/p,r_0]}$ is also (m,n)-finitely presented. This checks that the φ-bundle is uniformly finitely presented, and finishes the proof. ☐

Remark 2.2.8. Proposition 2.2.7 shows that the two types of objects considered in [34], (φ,Γ)-modules over $\mathcal{R}_A$ and families of (φ,Γ)-modules, are actually the same thing. We will no longer distinguish them from now on. But we remind the reader that these categories do depend on the choice of r_0.

The following two technical lemmas will be used in the proof of Theorem 6.3.9, the first of which is a strengthening of Lemma 2.1.16 making use of the φ-structure.

Lemma 2.2.9. *Let M^{r_0} and N^{r_0} be two φ-modules over $\mathcal{R}_A^{r_0}(\pi_K)$ and let $g : M \to N$ be a morphism of φ-modules over $\mathcal{R}_A(\pi_K)$. Then g is induced by some morphism $g^{r_0} : M^{r_0} \to N^{r_0}$ of φ-modules over $\mathcal{R}_A^{r_0}(\pi_K)$. Moreover, if g is injective (resp. surjective), so is g^{r_0}.*

Proof. The morphism g is the base change of some $g^r : M^r \to N^r$ for some $r \in (0, r_0]$. For $r' = \min\{pr, r_0\}$, g^r induces $g^{r'/p} : M^{r'/p} \to N^{r'/p}$. We define

$$g^{r'} : M^{r'} \xrightarrow{\varphi} M^{r'/p} \xrightarrow{g^{r'/p}} N^{r'/p} \xrightarrow{\psi} N^{r'}.$$

It is easy to check that $g^{r'}$ is a homomorphism of φ-modules over $\mathcal{R}_A^{r'}(\pi_K)$, which agrees with g^r upon base change to $\mathcal{R}_A^r(\pi_K)$. Iterating this construction gives the morphism $g^{r_0} : M^{r_0} \to N^{r_0}$. Now, suppose g is injective (resp. surjective). Let Q^{r_0} be the kernel (resp. cokernel) of g^{r_0}; then $\varphi^*(Q^r) \cong Q^{r/p}$ for $r \in (0, r_0]$. Using a similar argument as in the proof of Proposition 2.2.7 (by invoking Proposition 2.1.13(1)), we know that Q^{r_0} is finitely generated over $\mathcal{R}_A^{r_0}(\pi_K)$. By the injectivity (resp. surjectivity) of g and Lemma 2.1.16, we know that $Q^r = 0$ for some r. But then for $r' = \min\{pr, r_0\}$, $\varphi^*(Q^{r'}) = Q^{r'/p} = 0$ implies that $Q^{r'} = 0$. Iterating this proves the lemma. □

Notation 2.2.10. For any $f \in A$, we let $Z(f) \subseteq \mathrm{Max}(A)$ denote the closed subspace defined by f.

Lemma 2.2.11. *Let $f \in A$ be any element. Let $g : M \to N$ be a morphism of φ-modules over $\mathcal{R}_A$ (or $\mathcal{R}_A^{r_0}$ for some $r_0 > 0$). Assume that $g_z : M_z \to N_z$ is surjective for every $z \in \mathrm{Max}(A)\backslash Z(f)$. Then $g : M[\frac{1}{f}] \to N[\frac{1}{f}]$ is surjective.*

Proof. If both φ-modules are defined over $\mathcal{R}_A$, we may choose $r_0 > 0$ such that both modules and the morphism are the base change from $\mathcal{R}_A^{r_0}$; in this case, the surjectivity of the model $g_z^{r_0}$ of g_z over $\mathcal{R}_{\kappa_z}^{r_0}$ for $z \in \mathrm{Max}(A)\backslash Z(f)$ is still guaranteed by Lemma 2.2.9. Therefore, we need only to prove the lemma when the conditions are stated for φ-modules over $\mathcal{R}_A^{r_0}$. Let Q denote the cokernel of g; it is finitely presented. The assumption of the lemma implies that $Q_z = 0$ for any $z \in \mathrm{Max}(A)\backslash Z(f)$. Therefore, $Q^{[r_0/p, r_0]}$ is supported on $Z(f) \times \mathrm{A}^1[r_0/p, r_0]$, and is then killed by f^m for some $m \in \mathbb{N}$. Pulling back along powers of φ, we have $Q^{[r_0/p^{n+1}, r_0/p^n]} \cong (\varphi^n)^* Q^{[r_0/p, r_0]}$ is also killed by f^m for any $n \in \mathbb{N}$. Therefore, Q is killed by f^m, concluding the lemma. □

Definition 2.2.12. Let $r_0 \in (0, C(\pi_K)]$. A (φ, Γ_K)-*module* over $\mathcal{R}_A^{r_0}(\pi_K)$ is a φ-module M^{r_0} over $\mathcal{R}_A^{r_0}(\pi_K)$ equipped with a commuting semilinear continuous action of Γ_K. A (φ, Γ_K)-*module* over $\mathcal{R}_A(\pi_K)$ is the base change to $\mathcal{R}_A(\pi_K)$ of a (φ, Γ_K)-module over $\mathcal{R}_A^{r_0}(\pi_K)$ for some $r_0 \in (0, C(\pi_K)]$. We note that the map ψ commutes with the action of Γ_K.

If L is a finite extension of K and M is a (φ, Γ_L)-module over $\mathcal{R}_A(\pi_L)$, we define the *induced* (φ, Γ_K)-module to be $\mathrm{Ind}_L^K M = \mathrm{Ind}_{\Gamma_K}^{\Gamma_L} M$, viewed as an $\mathcal{R}_A(\pi_K)$-module, where the action of φ is inherited from M.

Remark 2.2.13. The continuity condition in the preceding definition means that, for any $m \in M^{[s,r]}$ with $0 < s \le r \le r_0$, the map $\Gamma_K \to M^{[s,r]}$, $\gamma \mapsto \gamma(m)$ is continuous for the profinite topology on Γ_K and the Banach topology on $M^{[s,r]}$. In the next proposition, we reformulate this condition more explicitly, and show that one in fact obtains an action of $\mathcal{R}_A^\infty(\Gamma_K)$ on M, the latter ring being defined in Definition 2.1.20.

The following claims are well known, but not well documented in the literature, so we provide a proof.

Proposition 2.2.14. *Let $r_0 \in (0, C(\pi_K)]$, and let M^{r_0} be a finitely generated $\mathcal{R}_A^{r_0}(\pi_K)$-module, say generated by $m_1, \dots, m_N$. For any $s \in (0, r_0]$ let $|\cdot|_{M^{[s,r_0]}}$ denote any fixed $\mathcal{R}_A^{[s,r_0]}(\pi_K)$-Banach module norm on $M^{[s,r_0]}$, and $\|\cdot\|_{M^{[s,r_0]}}$ the associated operator norm. Assume M^{r_0} is equipped with a semilinear action of Γ_K.*

(1) *In order that the action of Γ_K on M^{r_0} be continuous, it is necessary and sufficient that for each $i = 1, \dots, N$ one has $\lim_{\gamma\to 1} \gamma(m_i) = m_i$.*
(2) *If the action of Γ_K on M^{r_0} is continuous, then for each s one has $\lim_{\gamma\to 1} \|\gamma - 1\|_{M^{[s,r_0]}} = 0$.*
(3) *If the action of Γ_K on M^{r_0} is continuous, then the action of $A[\Gamma_K]$ extends by continuity to an action of $\mathcal{R}_A^\infty(\Gamma_K)$. More precisely, taking any $n \geq 1$ (resp. $n \geq 2$ if $p = 2$) such that $\Gamma_n \subseteq \Gamma_K$, for any $s \in (0, r_0]$ there exists $g_s \in \mathbb{N}$ such that $\|(\gamma_n' - 1)^{g_s}\|_{M^{[s,r_0]}} \leq \omega$.*

Proof. For (1) and (2), if the action is continuous then clearly the condition on the m_i holds. Conversely, assuming the condition on the m_i holds, we now show that the condition on the operator norms holds, which implies that the action of Γ_K is continuous, because we only need to check continuity at the origin. A general $m \in M^{[s,r_0]}$ is of the form $\sum_{i=1}^N f_i m_i$ with $f_i \in \mathcal{R}_A^{[s,r_0]}(\pi_K)$, and since there are only finitely many terms, it suffices to treat each $f_i m_i$ separately. Writing

$$\gamma(f_i m_i) - f_i m_i = (\gamma - 1)(f_i) m_i + \gamma(f_i)(\gamma - 1)(m_i),$$

it suffices to make $(\gamma - 1)(f_i)$ arbitrarily small, to keep $\gamma(f_i)$ bounded, and to make $(\gamma - 1)(m_i)$ arbitrarily small, with respect to $\|\cdot\|_{M^{[s,r_0]}}$ as $\gamma \to 1$. The third term can be made small by hypothesis. The second term is bounded by $\|f_i\|_{M^{[s,r_0]}}$ as soon as we can make the first term small. So we have reduced the general case to treating the free rank one module $M = \mathcal{R}_A^{r_0}(\pi_K)$. Since the module $\mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{R}_A^{r_0}(\pi_K)$ is noncanonically a direct sum of finitely many copies of $\mathcal{R}_A^{r_0}(\pi_K)$, the desired claim for $\mathcal{R}_A^{r_0}(\pi_K)$ follows from the case of the $\mathcal{R}_A^{\tilde{e}_K r_0}(\pi_{\mathbb{Q}_p})$-module $\mathrm{Ind}_K^{\mathbb{Q}_p} \mathcal{R}_A^{r_0}(\pi_K)$ with $\Gamma_{\mathbb{Q}_p}$-action. But, as above, a general $\mathcal{R}_A^{\tilde{e}_K r_0}(\pi_{\mathbb{Q}_p})$-module with $\Gamma_{\mathbb{Q}_p}$-action reduces to the case of the free rank one module. Henceforth, we replace $\tilde{e}_K r_0$ by r_0 for simplicity. We now take $\pi_{\mathbb{Q}_p} = \pi$ as in Notation 2.2.3. Because $|(1+\pi)-1|_{[s,r_0]} < 1$ we have $|(1+\pi)^a - 1|_{[s,r_0]} \leq C(s, r_0) < 1$ for some constant $C(s, r_0)$ independent of $a \in \mathbb{Z}_p$, and because taking pth powers improves congruences, this implies that for any $\epsilon > 0$ we can make $|(1+\pi)^{ap^n} - 1|_{[s,r_0]} < \epsilon$ independently of $a \in \mathbb{Z}_p$ by taking n sufficiently large. For $n \geq 1$ (resp. $n \geq 2$ if $p = 2$) and $\gamma \in \Gamma_n$, write $\chi(\gamma) = 1 + ap^n$ with $a \in \mathbb{Z}_p$. Given $f \in \mathcal{R}_A^{[s,r_0]}$, in order to bound $|\gamma(f) - f|_{[s,r_0]}$ it suffices to consider the positive and negative powers of π separately. Elementary algebra gives

$$\gamma(\pi) - \pi = (1+\pi)((1+\pi)^{ap^n} - 1), \tag{2.2.14.1}$$

$$\frac{1}{\gamma(\pi)} - \frac{1}{\pi} = -\frac{1+\pi}{\pi} \frac{(1+\pi)^{ap^n} - 1}{(1+\pi)((1+\pi)^{ap^n} - 1) + \pi}.$$

For any $\epsilon > 0$, by taking n large enough the first (resp. second) right-hand side can be made less than $\epsilon|\pi|_{[s,r_0]}$ (resp. $\epsilon|\pi^{-1}|_{[s,r_0]}$) independently of $a \in \mathbb{Z}_p$. This proves (1) and (2).

For (3), the second claim implies the first, because it provides M with an action of $\mathcal{R}_A^\infty(\Gamma_n)$ and one has $\mathcal{R}_A^\infty(\Gamma_K) = \mathcal{R}_A^\infty(\Gamma_n) \otimes_{\mathbb{Z}[\Gamma_n]} \mathbb{Z}[\Gamma_K]$. Thus, we concern ourselves with the second claim. Because $\mathrm{Ind}_K^{\mathbb{Q}_p} M$ is noncanonically a direct sum of finitely many Γ_K-stable copies of M, and $\Gamma_K \subseteq \Gamma_{\mathbb{Q}_p}$, the desired bound follows from the case of the $\mathcal{R}_A^{\tilde{e}_K r_0}(\pi_{\mathbb{Q}_p})$-module $\mathrm{Ind}_K^{\mathbb{Q}_p} M$ with $\Gamma_{\mathbb{Q}_p}$-action. So we assume from now on that $K = \mathbb{Q}_p$ and we replace $\tilde{e}_K r_0$ by r_0. Since for any $a \in \mathbb{Z}_p$ and $n \geq 1$ (resp. $n \geq 2$ if $p = 2$) one has $|(1+\pi)^{ap^n} - 1|_{[s,r_0]} < |\pi|_{[s,r_0]}$, it follows from the identities (2.2.14.1) that the norm $|\cdot|_{[s,r_0]}$ on $\mathcal{R}_A^{[s,r_0]}(\pi_{\mathbb{Q}_p})$ is invariant under precomposition with $\gamma \in \Gamma_n$. It follows from (2) that for some $n' \geq n$, the norm $|\cdot|_{M^{[s,r_0]}}$ is invariant under $\Gamma_{n'}$. Then $C = \max_{\gamma \in \Gamma_n/\Gamma_{n'}} \|\tilde{\gamma}\|_{M^{[s,r_0]}}$, where $\tilde{\gamma}$ denotes any choice of lift of γ to Γ_n, is equal to $\max_{\gamma \in \Gamma_n} \|\gamma\|_{M^{[s,r_0]}}$. Keeping in mind that $|\cdot|_{M^{[s,r_0]}}$ is invariant under Γ_n, it follows that the average

$$|\cdot|'_{M^{[s,r_0]}} = \frac{1}{[\Gamma_n : \Gamma_{n'}]} \sum_{\gamma \in \Gamma_n/\Gamma_{n'}} |\cdot|_{M^{[s,r_0]}} \circ \tilde{\gamma}$$

does not depend on the choice of n', and is a Γ_n-invariant norm on $M^{[s,r_0]}$ such that

$$C^{-1}|\cdot|'_{M^{[s,r_0]}} \leq |\cdot|_{M^{[s,r_0]}} \leq C|\cdot|'_{M^{[s,r_0]}}.$$

and hence

$$C^{-2}\,\|\cdot\|'_{M^{[s,r_0]}} \leq \|\cdot\|_{M^{[s,r_0]}} \leq C^2\,\|\cdot\|'_{M^{[s,r_0]}}.$$

Assuming we know the result for the norm $|\cdot|'_{M^{[s,r_0]}}$, there exists $g'_s \in \mathbb{N}$ such that $\|(\gamma'_n - 1)^{g'_s}\|'_{M^{[s,r_0]}} \leq \omega$. Choosing $h \in \mathbb{N}$ large enough so that $\omega^h C^2 \leq 1$, we have

$$\|(\gamma'_n - 1)^{(h+1)g'_s}\|_{M^{[s,r_0]}} \leq C^2 \|(\gamma'_n - 1)^{(h+1)g'_s}\|'_{M^{[s,r_0]}} \leq C^2 \omega^{h+1} \leq \omega,$$

whence the claim for the norm $|\cdot|_{M^{[s,r_0]}}$. By (2), there exists $n' \geq n$ such that $||\gamma'_{n'} - 1||_{M^{[s,r_0]}} \leq C^{-2}\omega$ so that $||\gamma'_{n'} - 1||'_{M^{[s,r_0]}} \leq \omega$. Now write $(\gamma'_n - 1)^{p^{n'-n}} = (\gamma'_{n'} - 1) + P(\gamma'_{n'})$, where $P(X) \in \mathbb{Z}[X]$ is divisible by p; the first term has operator norm under $|\cdot|'_{M^{[s,r_0]}}$ at most ω by the choice of n', and the latter term has operator norm less than or equal to $|p| \leq \omega$, so the operator norm $||(\gamma'_n - 1)^{p^{n'-n}}||'_{M^{[s,r_0]}} \leq \omega$, proving part (3) of the proposition. □

It would be useful for technical purposes to know that the following conjecture holds (see also [2, Section 3, Question 1] and [34, Remark 6.2]).

Conjecture 2.2.15. *For M a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, there exists a finite admissible cover $\{\mathrm{Max}(A_i)\}_{i=1,\dots,n}$ of $\mathrm{Max}(A)$ such that each $M\widehat{\otimes}_A A_i$ is a finite free $\mathcal{R}_{A_i}(\pi_K)$-module.*

Remark 2.2.16. We will prove the preceding conjecture for rank one (φ, Γ_K)-modules in Theorem 6.2.14, and the conjecture follows for trianguline (φ, Γ_K)-modules. The conjecture is also known for objects in the essential image of the functor $\mathbf{D}_{\mathrm{rig}}$ (defined in Theorem 2.2.17 below) after replacing K by a finite extension, although not for a general étale (φ, Γ_K)-module.

The importance of (φ, Γ)-modules rests upon the following result.

Theorem 2.2.17. *Let V be a finite projective A-module equipped with a continuous A-linear action of G_K. Then there is functorially associated to V a (φ, Γ_K)-module $\mathbf{D}_{\mathrm{rig}}(V)$ over $\mathcal{R}_A(\pi_K)$. The rule $V \mapsto \mathbf{D}_{\mathrm{rig}}(V)$ is fully faithful and exact, and it commutes with base change in A.*

Proof. See [34, Theorem 3.11], which generalizes [10, Théorème 4.2.9][1]. (It was also pointed out by Gaëtan Chenevier that some modification of [10, Théorème 4.2.9] can recover the theorem; see [34, Remark 3.13].) For further details, see [42, Section 2.1]. ☐

Lemma 2.2.18. *Let V be a finite projective A-module equipped with a continuous A-linear action of G_L. Denote the induced representation of G_K by $\mathrm{Ind}_{G_L}^{G_K} V$, and let $\mathrm{Ind}_L^K \mathbf{D}_{\mathrm{rig}}(V)$ be as in Definition* 2.2.12. *Then we have $\mathbf{D}_{\mathrm{rig}}(\mathrm{Ind}_{G_L}^{G_K} V) \cong \mathrm{Ind}_L^K \mathbf{D}_{\mathrm{rig}}(V)$.*

Proof. This can be proved in the same way as in [36, Proposition 3.1]. ☐

2.3. **Cohomology of (φ,Γ)-modules and duality.** Fontaine and Herr [30] defined a cohomology theory for (φ,Γ)-modules, compatible with the theory of Galois cohomology. When A is a finite extension of $\mathbb{Q}_p$, Liu [36] showed that this cohomology theory of (φ,Γ)-modules admits analogues of local Tate duality and the Euler characteristic formula for p-adic representations of G_K. In this subsection, we set up the basic formalism in the relative setting, including Tate duality pairing. In the special case $K=\mathbb{Q}_p$, we give an explicit computation of the pairing in terms of residues.

Notation 2.3.1. Let Δ_K denote the p-torsion subgroup of Γ_K, which is trivial if $p\neq 2$ and at largest cyclic of order two. When $K=\mathbb{Q}_p$, we write $\Delta=\Delta_{\mathbb{Q}_p}$. Choose $\gamma_K\in\Gamma_K$ whose image in Γ_K/Δ_K is a topological generator. (This choice is useful for explicit formulas, but if desired one can reformulate everything to eliminate this choice.)

Notation 2.3.2. We follow [41] for sign conventions throughout. Thus, the tensor product of complexes $X^\bullet\otimes Y^\bullet$ has differential defined by $d^{i+j}_{X\otimes Y}(x\otimes y)=d^i_X x\otimes y+(-1)^i x\otimes d^j_Y y$ if $x\in X^i$ and $y\in Y^j$. For a morphism of complexes $f^\bullet: X^\bullet\to Y^\bullet$, define its *mapping fiber* $\mathrm{Fib}(f)=\mathrm{Cone}(f)[-1]$, i.e., the complex with $\mathrm{Fib}(f)^i = X^i\oplus Y^{i-1}$ and $d^i_{\mathrm{Fib}(f)}=d^i_X-d^{i-1}_Y-f^i$. If $f:X\to Y$ is a morphism of objects, then $\mathrm{Fib}(f)$ is the complex $[X\xrightarrow{-f}Y]$ concentrated in degrees $0,1$.

Definition 2.3.3. For M a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$, we define the complexes $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ and $\mathrm{C}^\bullet_{\psi,\gamma_K}(M)$ concentrated in degree $[0,2]$, and a morphism Ψ_M between them, as follows:

(2.3.3.1)

$$\begin{array}{ccccccc}
\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) & = & [M^{\Delta_K} & \xrightarrow{(\varphi-1,\gamma_K-1)} & M^{\Delta_K}\oplus M^{\Delta_K} & \xrightarrow{(\gamma_K-1)\oplus(1-\varphi)} & M^{\Delta_K}] \\
\downarrow{\scriptstyle\Psi_M} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle -\psi\oplus\mathrm{id}} & & \downarrow{\scriptstyle -\psi} \\
\mathrm{C}^\bullet_{\psi,\gamma_K}(M) & = & [M^{\Delta_K} & \xrightarrow{(\psi-1,\gamma_K-1)} & M^{\Delta_K}\oplus M^{\Delta_K} & \xrightarrow{(\gamma_K-1)\oplus(1-\psi)} & M^{\Delta_K}].
\end{array}$$

These complexes and the morphism are independent of the choice of γ_K up to canonical A-linear isomorphism: if f is one of φ or ψ then $\Gamma_{\gamma_K,\gamma'_K,M}:\mathrm{C}^\bullet_{f,\gamma_K}(M)\xrightarrow{\sim}\mathrm{C}^\bullet_{f,\gamma'_K}(M)$ is given by $[1 \quad 1\oplus\frac{\gamma'_K-1}{\gamma_K-1} \quad \frac{\gamma'_K-1}{\gamma_K-1}]$.

[1] R. Liu made us aware of a small gap in this reference; see [37, Theorem 1.1.4] for a solution.

The complex $C^\bullet_{\varphi,\gamma_K}(M)$ is called the *Herr complex* of M. Its cohomology group is denoted $H^*_{\varphi,\gamma_K}(M)$ and is called the (φ,Γ)-*cohomology*. We similarly define $H^*_{\psi,\gamma_K}(M)$ to be the cohomology of $C^\bullet_{\psi,\gamma_K}(M)$, called the (ψ,Γ)-*cohomology* of M. We remark that the same definition applies to any module N with a commuting action of φ (resp. ψ) and Γ_K; we denote the resulting cohomology by $H^*_{\varphi,\gamma_K}(N)$ (resp. $H^*_{\psi,\gamma_K}(N)$).

Remark 2.3.4. For f one of φ or ψ, we have

$$C^\bullet_{f,\gamma_K}(M) = \mathrm{Fib}(1-\gamma_K|\,\mathrm{Fib}(1-f|M^{\Delta_K})).$$

Moreover, from the definition we easily deduce a useful short exact sequence

$$0 \to M^{\Delta_K,\psi=1}/(\gamma_K-1) \to H^1_{\psi,\gamma_K}(M) \to (M/(\psi-1))^{\Gamma_K} \to 0,$$

where the nontrivial maps are induced by inclusion of the second (resp. projection onto the first) coordinate in $C^1_{\psi,\gamma_K}(M) = M^{\Delta_K} \oplus M^{\Delta_K}$.

Lemma 2.3.5. *Let L be a finite extension of K. Let M be a (φ,Γ_L)-module over $\mathcal{R}_A(\pi_L)$. We have a natural quasi-isomorphism $C^\bullet_{\varphi,\gamma_L}(M) \to C^\bullet_{\varphi,\gamma_K}(\mathrm{Ind}_L^K(M))$ (inducing isomorphisms $H^i_{\varphi,\gamma_L}(M) \cong H^i_{\varphi,\gamma_K}(\mathrm{Ind}_L^K M)$ for any i).*

Proof. As $C^\bullet_{\varphi,\gamma_K}(M)$ is equal, up to signs within the differentials, to the mapping fiber of $1-\varphi$ on the complex

$$C^\bullet_{\gamma_K}(M) = [M^{\Delta_K} \xrightarrow{\gamma_K-1} M^{\Delta_K}],$$

it suffices to treat the complex $C^\bullet_{\gamma_K}(M)$ in place of $C^\bullet_{\varphi,\gamma_K}(M)$. But the latter complex is functorially quasi-isomorphic to the continuous cochain group $C^\bullet_{\mathrm{cont}}(\Gamma_K, M)$. Thus the lemma follows from the usual formulation of Shapiro's lemma for $\Gamma_L \subseteq \Gamma_K$. □

Proposition 2.3.6. *For M a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$, the morphism $\Psi_M : C^\bullet_{\varphi,\gamma_K}(M) \to C^\bullet_{\psi,\gamma_K}(M)$ is a quasi-isomorphism. Hence, we have $H^*_{\varphi,\gamma_K}(M) \cong H^*_{\psi,\gamma_K}(M)$.*

Proof. The morphism Ψ_M is surjective at each degree. The kernel of Ψ_M is the complex $[M^{\Delta_K,\psi=0} \xrightarrow{\gamma_K-1} M^{\Delta_K,\psi=0}]$ concentrated in degrees 1 and 2. The bijectivity of $\gamma_K - 1$ on $M^{\Delta_K,\psi=0}$ will be proved in Theorem 3.1.1 in the next subsection (the reader can check that there is no circular reasoning), and the proposition follows from this fact. □

From now on, we will use the (φ,Γ)-cohomology and the (ψ,Γ)-cohomology interchangeably without notification.

Proposition 2.3.7. *Let V be a finite projective A-module equipped with a continuous A-linear action of G_K. If we use $\mathbf{R}\Gamma_{\mathrm{cont}}(G_K,V)$ to denote the cohomology of continuous G_K-cochains with values in V, we have a functorial isomorphism $\mathbf{R}\Gamma_{\mathrm{cont}}(G_K,V) \cong \mathbf{R}\Gamma_{\varphi,\gamma_K}(\mathbf{D}_{\mathrm{rig}}(V))$ compatible with base change, in the derived category of perfect complexes over A.*

Proof. This is [42, Theorem 2.8]. □

Definition 2.3.8. As is customary, write $\mathbb{Z}_p(1) = \varprojlim_n \mu_{p^n}$, and for any $\mathbb{Z}_p$-module V with G_K-action (possibly via Γ_K), write $V(1)$ for $V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)$ with the diagonal action.

We write $\mathcal{R}_A(\pi_K)(1) = \mathbf{D}_{\mathrm{rig}}(A(1))$. For M a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$, we write $M(1)$ for the twist $M\otimes_{\mathcal{R}(\pi_K)}\mathcal{R}(\pi_K)(1)$, and $M^\vee$ for the module dual $\mathrm{Hom}_{\mathcal{R}_A(\pi_K)}(M,\mathcal{R}_A(\pi_K))$, as well as $M^* = M^\vee(1)$ for the *Cartier dual*. For L a finite extension of K and M a (φ,Γ_L)-module over $\mathcal{R}_A(\pi_L)$, we have $(\mathrm{Ind}_L^K(M))^*\cong \mathrm{Ind}_L^K(M^*)$.

Since A is flat over $\mathbb{Q}_p$, one has $H^2(G_K,\mathbb{Q}_p(1))\otimes_{\mathbb{Q}_p}A\cong H^2(G_K,A(1))$. From local class field theory one has the trace isomorphism $H^2(G_K,\mathbb{Q}_p(1))\cong\mathbb{Q}_p$, hence also $H^2(G_K,A(1))\cong A$. Combining this with the case $V=A(1)$ in the preceding proposition, one gets the *Tate isomorphism* $\mathrm{Ta}_K : H^2_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K)(1))\cong H^2(G_K,A(1))\cong A$.

We record a general formalism of cup products, which is well known in the literature.

Lemma 2.3.9. *Let R be a ring. Let $f_i,g_i : \mathrm{C}_i^\bullet\to\mathrm{D}_i^\bullet$ for $i=1,2$ be morphisms of R-complexes. Then we have a natural morphism*

$$\mathrm{Fib}(\mathrm{C}_1^\bullet\xrightarrow{g_1-f_1}\mathrm{D}_1^\bullet)\otimes\mathrm{Fib}(\mathrm{C}_2^\bullet\xrightarrow{g_2-f_2}\mathrm{D}_2^\bullet)\longrightarrow\mathrm{Fib}(\mathrm{C}_1^\bullet\otimes\mathrm{C}_2^\bullet\xrightarrow{g_1\otimes g_2-f_1\otimes f_2}\mathrm{D}_1^\bullet\otimes\mathrm{D}_2^\bullet)$$

given by the downward arrows of the diagram

$$\begin{array}{ccccc}
\mathrm{C}_1^\bullet\otimes\mathrm{C}_2^\bullet & \xrightarrow{\big((f_1-g_1)\otimes 1,\,1\otimes(f_2-g_2)\big)} & \mathrm{D}_1^\bullet\otimes\mathrm{C}_2^\bullet\oplus\mathrm{C}_1^\bullet\otimes\mathrm{D}_2^\bullet & \xrightarrow{\big(1\otimes(g_2-f_2)\big)\oplus\big((f_1-g_1)\otimes 1\big)} & \mathrm{D}_1^\bullet\otimes\mathrm{D}_2^\bullet\\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 1\otimes g_2\oplus f_1\otimes 1} & & \\
\mathrm{C}_1^\bullet\otimes\mathrm{C}_2^\bullet & \xrightarrow{f_1\otimes f_2-g_1\otimes g_2} & \mathrm{D}_1^\bullet\otimes\mathrm{D}_2^\bullet & &
\end{array}$$

where the maps $1\otimes(f_2-g_2)$ in the first horizontal arrow and $f_1\otimes 1$ in the second vertical arrow get multiplied by -1 on summands with a tensor factor that is an odd-degree piece of $\mathrm{Fib}(\mathrm{C}_1^\bullet\xrightarrow{g_1-f_1}\mathrm{D}_1^\bullet)$.

Proof. This is a straightforward computation. ☐

Definition 2.3.10. For M_1,M_2 two (φ,Γ_K)-modules over $\mathcal{R}_A(\pi_K)$, we construct cup products as follows. Write $\mathrm{C}_\varphi^\bullet(M)=\mathrm{Fib}(1-\varphi|M^{\Delta_K})=[M^{\Delta_K}\xrightarrow{\varphi-1}M^{\Delta_K}]$ for brevity. First, apply Lemma 2.3.9 (with $R=A$) to the case $\mathrm{C}_i^\bullet=\mathrm{D}_i^\bullet=M_i^{\Delta_K}$, $f_i=\varphi$, and $g_i=\mathrm{id}$, to get the first arrow in the composite

$$\begin{aligned}
\cup_\varphi : \mathrm{C}_\varphi^\bullet(M_1)\otimes_A\mathrm{C}_\varphi^\bullet(M_2) &= [M_1^{\Delta_K}\xrightarrow{\varphi-1}M_1^{\Delta_K}]\otimes_A[M_2^{\Delta_K}\xrightarrow{\varphi-1}M_2^{\Delta_K}]\\
&\longrightarrow[(M_1\otimes_A M_2)^{\Delta_K}\xrightarrow{\varphi\otimes\varphi-1\otimes 1}(M_1\otimes_A M_2)^{\Delta_K}]\\
&\twoheadrightarrow[(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2)^{\Delta_K}\xrightarrow{\varphi-1}(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2)^{\Delta_K}]=\mathrm{C}_\varphi^\bullet(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2).
\end{aligned}$$

Then, one applies Lemma 2.3.9 (with $R=A$) to the case $\mathrm{C}_i^\bullet=\mathrm{D}_i^\bullet=\mathrm{C}_\varphi^\bullet(M_i)$, $f_i=\gamma_K$, and $g_i=\mathrm{id}$, to get the arrow $\cup_{\gamma_K}$ in the composite

$$\begin{aligned}
&\cup_{\varphi,\gamma_K} : \mathrm{C}_{\varphi,\gamma_K}^\bullet(M_1)\otimes_A\mathrm{C}_{\varphi,\gamma_K}^\bullet(M_2)=\mathrm{Fib}\Big(\mathrm{C}_\varphi^\bullet(M_1)\\
&\xrightarrow{1-\gamma_K}\mathrm{C}_\varphi^\bullet(M_1)\Big)\otimes_A\mathrm{Fib}\Big(\mathrm{C}_\varphi^\bullet(M_2)\xrightarrow{1-\gamma_K}\mathrm{C}_\varphi^\bullet(M_2)\Big)\xrightarrow{\cup_{\gamma_K}}\mathrm{Fib}\Big(\mathrm{C}_\varphi^\bullet(M_1)\otimes_A\mathrm{C}_\varphi^\bullet(M_2)\\
&\xrightarrow{1\otimes 1-\gamma_K\otimes\gamma_K}\mathrm{C}_\varphi^\bullet(M_1)\otimes_A\mathrm{C}_\varphi^\bullet(M_2)\Big)\xrightarrow{\cup_\varphi}\mathrm{Fib}\Big(\mathrm{C}_\varphi^\bullet(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2)\\
&\xrightarrow{1-\gamma_K}\mathrm{C}_\varphi^\bullet(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2)\Big)=\mathrm{C}_{\varphi,\gamma_K}^\bullet(M_1\otimes_{\mathcal{R}_A(\pi_K)}M_2).
\end{aligned}$$

This morphism induces an A-bilinear pairing $H^i_{\varphi,\gamma_K}(M) \times H^j_{\varphi,\gamma_K}(N) \to H^{i+j}_{\varphi,\gamma_K}(M \otimes_{\mathcal{R}_A(\pi_K)} N)$. This gives rise to the explicit formulas of [31, Section 4.2]; especially, when $i = j = 1$, one has

$$\overline{(x_1, x_2)} \otimes \overline{(y_1, y_2)} \mapsto \overline{x_2 \otimes \gamma(y_1) - x_1 \otimes \varphi(y_2)}.$$

Using the cup product, functoriality for the morphism $M \otimes_{\mathcal{R}_A(\pi_K)} M^* \to \mathcal{R}_A(\pi_K)(1)$, and the Tate map $\mathrm{Ta}_K : H^2_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K)(1)) \cong A$, one gets *Tate duality pairings*

$$\cup_{\mathrm{Ta}} : \mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \times \mathrm{C}^\bullet_{\varphi,\gamma_K}(M^*) \xrightarrow{\cup_{\varphi,\gamma_K}} \mathrm{C}^\bullet_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K)(1))$$

$$\to H^2_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K)(1))[-2] \xrightarrow{\mathrm{Ta}_K} A[-2], \tag{2.3.10.1}$$

$$\cup_{\mathrm{Ta}} : H^i_{\varphi,\gamma_K}(M) \times H^{2-i}_{\varphi,\gamma_K}(M^*) \to H^2_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K)(1)) \xrightarrow{\mathrm{Ta}_K} A. \tag{2.3.10.2}$$

The following extension of Tate local duality is due to Liu [36]. We will later generalize this result to the case of a general affinoid algebra A in Theorem 4.4.5.

Theorem 2.3.11 (Liu)**.** *Suppose that A is a finite extension of $\mathbb{Q}_p$. Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$.*

(1) *The A-vector spaces $H^i_{\varphi,\gamma_K}(M)$ are finite-dimensional for $i = 0, 1, 2$.*
(2) *The Euler characteristic $\chi(M) = \sum_{i=0}^2 (-1)^i \dim_A H^i_{\varphi,\gamma_K}(M)$ is equal to $-[K : \mathbb{Q}_p] \operatorname{rank} M$.*
(3) *The duality cup product* (2.3.10.2) *is a perfect pairing.*

We focus for the remainder of this subsection on the case $K = \mathbb{Q}_p$, and describe an explicit construction of the Tate isomorphism $\mathrm{Ta}_{\mathbb{Q}_p}$ (up to a fixed $\mathbb{Q}_p^\times$-multiple) and duality pairings $\cup_{\mathrm{Ta}}$.

Confusing additive and multiplicative notations, we may write $\mathbb{Q}_p(1) = \mathbb{Q}_p \otimes \varepsilon$ and $\mathcal{R}(1) = \mathcal{R} \otimes \varepsilon$ (with φ acting trivially on ε). Recall that to our choice of ε is associated an indeterminate π for $\mathcal{R}$. We define the *normalized residue maps* $\mathrm{Res}_{\mathbb{Q}_p}, \mathrm{Res}_{\gamma_{\mathbb{Q}_p}} : \mathcal{R}_A(1) = \mathcal{R}_A \otimes \varepsilon \to A$ by $\mathrm{Res}_{\mathbb{Q}_p}(f \otimes \varepsilon) = \mathrm{Res}_{\mathcal{R}_A}(f \frac{d\pi}{1+\pi})$ and $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}} = (\log \chi(\gamma_{\mathbb{Q}_p}))^{-1} \mathrm{Res}_{\mathbb{Q}_p}$. These maps are formally checked to be independent of the choice of ε, and invariant under precomposition with φ, ψ, and all $\gamma \in \Gamma$ (cf. [21, Section I.2]). In particular, they factor through maps $H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}_A(1)) \cong \mathcal{R}_A(1)_\Gamma/(\varphi - 1) \to A$ which we also call $\mathrm{Res}_{\mathbb{Q}_p}$ and $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}}$, respectively. By the reasoning of Remark 2.1.21, the maps $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}}$ are compatible for varying choices of $\gamma_{\mathbb{Q}_p}$. Since $H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}_A(1))$ is isomorphic to A as above, and the residue map is obviously surjective, it follows that $\mathrm{Res}_{\mathbb{Q}_p}$ and $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}}$ are isomorphisms. By base changing from the case $A = \mathbb{Q}_p$, we see that $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}} = C_p \cdot \mathrm{Ta}_{\mathbb{Q}_p}$ for some $C_p \in \mathbb{Q}_p^\times$ independent of A and $\gamma_{\mathbb{Q}_p}$.

Remark 2.3.12. Although the precise value of C_p is irrelevant in this paper, we remark that it is claimed in [5, Theorem 2.2.6] that $C_p = -(p-1)/p$.

Notation 2.3.13. Let $\langle -, - \rangle$ denote the tautological $\mathcal{R}_A$-bilinear pairing $M \times M^* \to \mathcal{R}_A(1)$. We define the *residue pairings* $\{-,-\}_{\mathbb{Q}_p}, \{-,-\}_{\gamma_{\mathbb{Q}_p}} : M \times M^* \to A$ by $\mathrm{Res}_{\mathbb{Q}_p} \circ \langle -, - \rangle$ and $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}} \circ \langle -, - \rangle$, respectively. (Our notation differs from the twisted pairing defined by Colmez [20].)

From the properties of $\langle -, - \rangle$ and $\mathrm{Res}_{\gamma_{\mathbb{Q}_p}}$ it follows that $\{\gamma(x), \gamma(y)\} = \{\varphi(x), \varphi(y)\} = \{x, y\}$, that $\{\varphi(x), y\} = \{x, \psi(y)\}$, and that $\{\psi(x), y\} = \{x, \varphi(y)\}$

for any $\gamma \in \Gamma$, $x \in M$ and $y \in M^*$. Then the scaled Tate duality pairing $C_p \cdot \cup_{\mathrm{Ta}}$ is computed, under the identification

$$\Psi_{M^*} \circ \Gamma_{\gamma_{\mathbb{Q}_p}, \gamma_{\mathbb{Q}_p}^{-1}, M^*} : \mathrm{C}^\bullet_{\varphi, \gamma_{\mathbb{Q}_p}}(M^*) \xrightarrow{\sim} \mathrm{C}^\bullet_{\psi, \gamma_{\mathbb{Q}_p}^{-1}}(M^*),$$

by the diagram

$$\begin{array}{ccccccccc}
\mathrm{C}^\bullet_{\varphi, \gamma_{\mathbb{Q}_p}}(M) & = & [\ M^{\Delta_K} & \longrightarrow & M^{\Delta_K} \oplus M^{\Delta_K} & \longrightarrow & M^{\Delta_K}\ ] & & \\
 & & \times & & \times & & \times & & \\
 & & [\ (M^*)^{\Delta_K} & \longleftarrow & (M^*)^{\Delta_K} \oplus (M^*)^{\Delta_K} & \longleftarrow & (M^*)^{\Delta_K}\ ] & = & \mathrm{C}^\bullet_{\psi, \gamma_{\mathbb{Q}_p}^{-1}}(M^*), \\
 & & \Big\downarrow {\scriptstyle \{-,-\}_{\gamma_{\mathbb{Q}_p}}} & & \Big\downarrow {\scriptstyle \{-,-\}_{\gamma_{\mathbb{Q}_p},1}} & & \Big\downarrow {\scriptstyle \{-,-\}_{\gamma_{\mathbb{Q}_p}}} & & \\
 & & A & & A & & A & &
\end{array}$$

where $\{(m,n),(k,l)\}_{\gamma_{\mathbb{Q}_p},1} = \{m,l\}_{\gamma_{\mathbb{Q}_p}} - \{n,k\}_{\gamma_{\mathbb{Q}_p}}$. (The diagram only commutes up to sign, but it is compatible with the sign convention for the tensor product of complexes.)

3. The ψ operator

We now focus attention more closely on the action of the operator ψ on (φ, Γ)-modules, and particularly the kernel of ψ and the kernel and cokernel of $\psi - 1$. The technical importance of the ψ-action is apparent in much of the prior work on the cohomology of (φ, Γ)-modules. Moreover, Fontaine observed that, in addition to being a useful intermediate step in the computation of (φ, Γ)-cohomology, ψ-cohomology is itself important because of its identification with Iwasawa cohomology; see [16]. We thus take a bit of extra care in our analysis of ψ-cohomology here, in order to later obtain conclusions about Iwasawa cohomology in arithmetic families.

Notation 3.0.1. Throughout this section, the notations $\mathcal{R}^?_A(C)$ of Definition 2.1.20 are in force with $C = \Gamma_K$ and $c = \gamma_K$.

Remark 3.0.2. Although we have assumed throughout A to be a $\mathbb{Q}_p$-affinoid algebra, the statements and proofs in this section apply to any strongly noetherian Banach algebra A over $\mathbb{Q}_p$.

3.1. **Γ-action on $M^{\psi=0}$.** The aim of this subsection is to prove the following theorem regarding the Γ_K-action on $M^{\psi=0}$, by removing an auxiliary boundedness condition from [15, Théorème 2.4].

Theorem 3.1.1. *For M any (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, there exists $r_1 \in (0, C(\pi_K))$ such that for any $0 < r \leq r_1$, $\gamma_K - 1$ is invertible on $(M^r)^{\psi=0}$, and the $A[\Gamma_K, (\gamma_K - 1)^{-1}]$-module structure on $(M^r)^{\psi=0}$ extends uniquely by continuity to an $\mathcal{R}_A^{\tilde{e}_K r}(\Gamma_K)$-module structure, for which $(M^r)^{\psi=0}$ is finite projective of rank $[K : \mathbb{Q}_p] \cdot \operatorname{rank} M$.*

Proof. We first observe that

$$(\mathrm{Ind}_K^{\mathbb{Q}_p} M)^{\psi=0} = \big(\mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma], M)\big)^{\psi=0} = \mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma], M^{\psi=0}).$$

Since $\mathbb{Z}[\Gamma]$ is a finite free $\mathbb{Z}[\Gamma_K]$-module, the statement of the theorem for M is equivalent to that for $\mathrm{Ind}_K^{\mathbb{Q}_p} M$. (Note our restriction on $C(\pi_K)$ in Definition 2.2.2

allows us to compare $\mathcal{R}_A^{\tilde{e}_K r}(\Gamma_K)$ and $\mathcal{R}_A^r(\Gamma)$.) Hence, we can and will assume that $K = \mathbb{Q}_p$ from now on.

We may assume that M is a (φ, Γ)-module over $\mathcal{R}_A^{r_0}$, where $r_0 < C(\pi_K)$ is small enough so that $|p| < |\pi^p|_{r_0}$, and that ψ is defined. For $0 < s \leq r \leq r_0$, because of the decomposition $\mathcal{R}_A^{[s/p^n, r/p^n]} = \bigoplus_{i \in \mathbb{Z}/p^n\mathbb{Z}} (1+\pi)^{\tilde{\imath}} \varphi^n \mathcal{R}_A^{[s,r]}$, where $\tilde{\imath}$ denotes any lift of i to $\mathbb{Z}$, we have a decomposition

$$(M^{[s/p^n, r/p^n]})^{\psi=0} = \bigoplus_{i \in (\mathbb{Z}/p^n\mathbb{Z})^\times} (1+\pi)^{\tilde{\imath}} \varphi^n M^{[s,r]} \cong (1+\pi)\varphi^n M^{[s,r]} \otimes_{\mathbb{Z}[\Gamma_n]} \mathbb{Z}[\Gamma]$$

of $\mathbb{Z}[\Gamma]$-modules. Since $\mathcal{R}_A^{[s/p^n, r/p^n]}(\Gamma) = \mathcal{R}_A^{[s,r]}(\Gamma_n) \otimes_{\mathbb{Z}[\Gamma_n]} \mathbb{Z}[\Gamma]$ for all $n \geq 0$ (using the restriction on $C(\pi_K)$ in Definition 2.2.2), we need only to prove the following: for *some* $n \geq 2$, the module $(1+\pi)\varphi^n M^{r_0}$ has invertible action of $\gamma_n - 1$, it admits a unique extension of the $A[\Gamma_n, (\gamma_n - 1)^{-1}]$-action by continuity to an $\mathcal{R}_A^{r_0}(\Gamma_n)$-module structure, and with respect to this structure it is finite projective of the same rank as M. Then the theorem holds for $r_1 = r_0/p^n$.

Since M is a finite projective $\mathcal{R}_A^{r_0}$-module, there exists a finite (projective) $\mathcal{R}_A^{r_0}$-module N such that $M \oplus N$ is free with basis $\mathbf{e}_1, \ldots, \mathbf{e}_m$ over $\mathcal{R}_A^{r_0}$. For $I = [r_0, r_0]$, $[r_0/p, r_0/p]$, or $[r_0/p, r_0]$, set $N^I = N \otimes_{\mathcal{R}_A^{r_0}} \mathcal{R}_A^I$. We equip M^I and N^I with any Banach module norms $|\cdot|_{M,I}$ and $|\cdot|_{N,I}$, and $M^I \oplus N^I$ with the supremum of these norms, denoted $|\cdot|_{M \oplus N, I}$.

By Proposition 2.2.14(2), there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operator norms of $\gamma_n - 1$ satisfy $\|\gamma_n - 1\|_{M,I} < |\pi|_I$ for any $I = [r_0, r_0]$, $[r_0/p, r_0/p]$, or $[r_0/p, r_0]$.

We will prove in Corollary 3.1.3 that, for any $l \in \mathbb{N}$, $(1+\pi)\varphi^{n_0} M^{[r_0/p^l, r_0/p^{l-1}]}$ has invertible action of $\gamma_{n_0} - 1$, it admits a unique extension of the $A[\Gamma_{n_0}, (\gamma_{n_0} - 1)^{-1}]$-action by continuity to an $\mathcal{R}_A^{[r_0/p^l, r_0/p^{l-1}]}(\Gamma_{n_0})$-module structure, and it is finite projective of the same rank as M, in fact generated by m elements (and hence (m, m)-finitely presented because of the projectivity), such that

$$\begin{aligned}(1+\pi)\varphi^{n_0} M^{[r_0/p^l, r_0/p^{l-1}]} &\otimes_{\mathcal{R}_A^{[r_0/p^l, r_0/p^{l-1}]}(\Gamma_{n_0})} \mathcal{R}_A^{[r_0/p^l, r_0/p^l]}(\Gamma_{n_0}) \\ \cong (1+\pi)\varphi^{n_0} M^{[r_0/p^{l+1}, r_0/p^l]} &\otimes_{\mathcal{R}_A^{[r_0/p^{l+1}, r_0/p^l]}(\Gamma_{n_0})} \mathcal{R}_A^{[r_0/p^l, r_0/p^l]}(\Gamma_{n_0})\end{aligned}$$

for any l. By Proposition 2.1.13(3), this implies that $(1+\pi)\varphi^{n_0} M^{r_0}$ is a finite projective $\mathcal{R}_A^{r_0}(\Gamma_{n_0})$-module of the same rank as M. □

Lemma 3.1.2. *Retain the notations of the preceding proof. Let* $I = [r_0/p, r_0]$, $[r_0/p, r_0/p]$, *or* $[r_0, r_0]$. *If* $n \geq n_0$, *then* $(1+\pi)\varphi^n M^I$ *has invertible action of* $\gamma_n - 1$, *it admits a unique extension of the* $A[\Gamma_n, (\gamma_n - 1)^{-1}]$*-action by continuity to an* $\mathcal{R}_A^I(\Gamma_n)$*-module structure, and it is finite projective of the same rank as* M *with (at most)* m *generators. Moreover, we have natural isomorphisms*

$$(1+\pi)\varphi^n M^{[r_0/p, r_0/p]} \cong (1+\pi)\varphi^n M^{[r_0/p, r_0]} \otimes_{\mathcal{R}_A^{[r_0/p, r_0]}(\Gamma_n)} \mathcal{R}_A^{[r_0/p, r_0/p]}(\Gamma_n), \text{ and}$$

$$(1+\pi)\varphi^n M^{[r_0, r_0]} \cong (1+\pi)\varphi^n M^{[r_0/p, r_0]} \otimes_{\mathcal{R}_A^{[r_0/p, r_0]}(\Gamma_n)} \mathcal{R}_A^{[r_0, r_0]}(\Gamma_n).$$

Proof. We observe that $\gamma_n - 1$ acts on $(1+\pi)\varphi^n M^I$ by sending $(1+\pi)\varphi^n(x)$ to

$$\begin{aligned}\gamma_n((1+\pi)\varphi^n(x)) - (1+\pi)\varphi^n(x) &= (1+\pi)(1+\pi)^{p^n}\varphi^n\gamma_n(x) - (1+\pi)\varphi^n(x)\\ &= (1+\pi)\varphi^n((1+\pi)\gamma_n(x) - x)\\ &= (1+\pi)\varphi^n(G_{\gamma_n}(x)),\end{aligned}$$

where we denote by G_{γ_n} the operator on M^I given by

$$G_{\gamma_n} : x \mapsto (1+\pi)\gamma_n(x) - x = \pi\cdot\left(1+\frac{(1+\pi)}{\pi}(\gamma_n - 1)\right)(x).$$

We observe that, by our choice of n_0, the series

$$\sum_{k\geq 0}\left(-\frac{(1+\pi)}{\pi}(\gamma_n-1)\right)^k\cdot\pi^{-1}$$

converges for the operator norm $\|\cdot\|_{M,I}$, and is inverse to G_{γ_n}. Thus, $\gamma_n - 1$ acts invertibly on $(1+\pi)\varphi^n M^I$. Moreover, we have $\|(G_{\gamma_n})^{\pm1} - (\pi)^{\pm1}\|_{M,I} < |\pi^{\pm1}|_I$. As a result, M^I admits a unique extension of the action of $A[G_{\gamma_n}^{\pm1}]$ by continuity an $\mathcal{R}_A^I(G_{\gamma_n})$-module structure, and by transport of structure $(1+\pi)\varphi^n M^I$ admits a unique extension of the action of $A[(\gamma_n-1)^{\pm1}]$ by continuity to an $\mathcal{R}_A^I(\Gamma_n)$-module structure.

We extend the action of G_{γ_n} to $M^I\oplus N^I$ by having G_{γ_n} act as multiplication by π on N^I. We still have

$$\|(G_{\gamma_n})^{\pm1} - (\pi)^{\pm1}\|_{M\oplus N,I} < |\pi^{\pm1}|_I. \tag{3.1.2.1}$$

Consider the following two maps

$$\begin{aligned}\Phi : \bigoplus_{i=1}^m \mathcal{R}_A^I(\Gamma_n)\mathbf{e}_i &\longrightarrow M^I\oplus N^I & \Phi' : \bigoplus_{i=1}^m \mathcal{R}_A^I(\Gamma_n)\mathbf{e}_i &\longrightarrow M^I\oplus N^I\\ \sum_{i=1}^m f_i(\gamma_n-1)\mathbf{e}_i &\longmapsto \sum_{i=1}^m f_i(\pi)\mathbf{e}_i & \sum_{i=1}^m f_i(\gamma_n-1)\mathbf{e}_i &\longmapsto \sum_{i=1}^m f_i(G_{\gamma_n})\mathbf{e}_i,\end{aligned}$$

where $f_i(G_{\gamma_n})$ is the formal substitution of G_{γ_n} into the variable of the formal Laurent series. If we provide $\bigoplus_{i=1}^m \mathcal{R}_A^I(\Gamma_n)\mathbf{e}_i$ with the norm defined by the $\mathbf{e}_i$, the map Φ is a topological isomorphism. The norm condition (3.1.2.1) implies that

$$|\Phi'\circ\Phi^{-1}(x) - x|_{M\oplus N,I} < |x|_{M\oplus N,I} \qquad \text{for any } x\in M^I\oplus N^I,$$

forcing Φ' to be an isomorphism too. Hence, if we let $\mathcal{R}_A^I(G_{\gamma_n})$ denote the formal substitution of G_{γ_n} into the indeterminate of $\mathcal{R}_A^I$, then $M^I\oplus N^I$ is a free $\mathcal{R}_A^I(G_{\gamma_n})$-module of rank m. In particular, this implies that $(1+\pi)\varphi^n M^I$ is finite projective over $\mathcal{R}_A^I(\Gamma_n)$, and is generated by m elements. Moreover, since the definition of Φ' is compatible with changing the interval I, we have the base change property asserted in the lemma.

By our construction, we have $\operatorname{rank}_{\mathcal{R}_A^I}(M^I\oplus N^I) = m = \operatorname{rank}_{\mathcal{R}_A^I(G_{\gamma_n})}(M^I\oplus N^I)$. Since the G_{γ_n}-action on N^I is given by multiplication by π, we see immediately that $\operatorname{rank}_{\mathcal{R}_A^I(\Gamma_n)}(1+\pi)\varphi^n M^I$ agrees with $\operatorname{rank}_{\mathcal{R}_A} M$. □

Corollary 3.1.3. *Retain the notations of the preceding proof. For any $l\geq 0$, let $I_l = [r_0/p^{l+1}, r_0/p^l], [r_0/p^{l+1}, r_0/p^{l+1}]$, or $[r_0/p^l, r_0/p^l]$. Then $(1+\pi)\varphi^{n_0}M^I$ is a* *finite projective module over $\mathcal{R}_A^I(\Gamma_{n_0})$ of the same rank as M with (at most) m generators. Moreover, we have*

$$\begin{aligned}&(1+\pi)\varphi^n M^{[r_0/p^{l+1}, r_0/p^{l+1}]}\\ &\quad\cong (1+\pi)\varphi^n M^{[r_0/p^{l+1}, r_0/p^l]} \otimes_{\mathcal{R}_A^{[r_0/p^{l+1}, r_0/p^l]}(\Gamma_n)} \mathcal{R}_A^{[r_0/p^{l+1}, r_0/p^{l+1}]}(\Gamma_n), \text{ and}\\ &(1+\pi)\varphi^n M^{[r_0/p^l, r_0/p^l]}\\ &\quad\cong (1+\pi)\varphi^n M^{[r_0/p^{l+1}, r_0/p^l]} \otimes_{\mathcal{R}_A^{[r_0/p^{l+1}, r_0/p^l]}(\Gamma_n)} \mathcal{R}_A^{[r_0/p^l, r_0/p^l]}(\Gamma_n).\end{aligned}$$

Proof. This follows from Lemma 3.1.2 and the fact that, for any $0 < s \leq r \leq r_0$,

$$\begin{aligned}(1+\pi)\varphi^n M^{[s/p^l, r/p^l]} &= (1+\pi)\varphi^{n+l} M^{[s,r]} \otimes_{\mathbb{Z}[\Gamma_{n+l}]} \mathbb{Z}[\Gamma_n]\\ &= (1+\pi)\varphi^{n+l} M^{[s,r]} \otimes_{\mathcal{R}_A^{[s,r]}(\Gamma_{n+l})} \mathcal{R}_A^{[s/p^l, r/p^l]}(\Gamma_n).\end{aligned}$$

□

3.2. **ψ-cohomology of M/tM.** An important ingredient of Liu's proof [36] of Theorem 2.3.11 is the careful study of the (φ, Γ)-cohomology of "torsion" modules M/tM. This allows one to freely shift the slopes of the modules. We generalize this result to the relative setting, and give more information concerning the Γ-action.

Hypothesis 3.2.1. In this subsection, we assume that $K = \mathbb{Q}_p$.

Notation 3.2.2. We define the special elements $q = \varphi(\pi)/\pi$, $q_n = \varphi^{n-1}(q)$ for $n \geq 1$, and $t = \log(1+\pi) = \pi \prod_{n \geq 1}(q_n/p)$ of $\mathcal{R}^\infty$. We have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for any $\gamma \in \Gamma$. For $0 < s \leq r$, we have $q_n \notin (\mathcal{R}^{[s,r]})^\times$ if and only if $s \leq 1/p^{n-1} \leq r$; in this case, $\mathcal{R}^{[s,r]}/q_n \cong \mathbb{Q}_p[\pi]/q_n$ may be identified with $\mathbb{Q}_p(\zeta_{p^n})$ by identifying $1+\pi$ with ζ_{p^n}. This identification is equivariant for Γ, and for φ in the sense that $\varphi : \mathcal{R}^{[s,r]}/q_n \to \mathcal{R}^{[s/p, r/p]}/q_{n+1}$ is identified with the natural embedding $\mathbb{Q}_p(\zeta_{p^n}) \to \mathbb{Q}_p(\zeta_{p^{n+1}})$.

Recall that $r_0 \leq C(\pi_K) < 1$. Put $n_0 = \lceil -\log_p r_0 \rceil$. We define similar special elements of $\mathcal{R}_A^\infty(\Gamma_{n_0})$, namely $q_{0,\gamma_{n_0}} = \gamma_{n_0} - 1$, $q_{n,\gamma_{n_0}} = p^{-1}(\gamma_{n_0}^{p^n} - 1)/(\gamma_{n_0}^{p^{n-1}} - 1)$ for $n \geq 1$, and $\ell_{\gamma_{n_0}} = \prod_{n \geq 0} q_{n,\gamma_{n_0}}$.

The following lemma is essentially [19, Proposition 2.16].

Lemma 3.2.3. *Let M^{r_0} be a (φ, Γ)-module over $\mathcal{R}_A^{r_0}$, and let $n_0 = \lceil -\log_p r_0 \rceil$. If we denote $M_n^{r_0} = M^{r_0}/q_n M^{r_0}$ for $n \geq n_0$, then we have the following:*

(1) *$M^{r_0}/tM^{r_0} \cong \prod_{n \geq n_0} M_n^{r_0}$;*
(2) *$\varphi^{n'-n} \otimes 1$ induces an isomorphism $M_n^{r_0} \otimes_{\mathbb{Q}_p(\zeta_{p^n})} \mathbb{Q}_p(\zeta_{p^{n'}}) \cong M_{n'}^{r_0}$ as $A[\Gamma]$-modules for any $n' \geq n \geq n_0$;*
(3) *under the product decomposition of (1), the map $\varphi : M^{r_0}/tM^{r_0} \to M^{r_0/p}/tM^{r_0/p}$ takes $(x_n)_n$ to $(x_{n-1})_n$; and*
(4) *$\psi : M^{r_0/p}/tM^{r_0/p} \to M^{r_0}/tM^{r_0}$ takes $(x_n)_n$ to $(p^{-1}\mathrm{Tr}_{M_n^{r_0}/M_{n-1}^{r_0}}(x_n))_n$, where $\mathrm{Tr}_{M_n^{r_0}/M_{n-1}^{r_0}} : M_n^{r_0} \cong M_{n-1}^{r_0} \otimes_{\mathbb{Q}_p(\zeta_{p^{n-1}})} \mathbb{Q}_p(\zeta_{p^n}) \to M_{n-1}^{r_0}$ is given by the trace on the second factor.*

Proof. (1) For $0 < s < r_0$, we have

$$\mathcal{R}_A^{[s,r_0]}/t\mathcal{R}_A^{[s,r_0]} \xrightarrow{\sim} \bigoplus_{n_0 \leq n \leq -\log_p s} A \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\zeta_{p^n}),$$

compatible with varying s and r. This implies that $M^{[s,r_0]}/tM^{[s,r_0]} \xrightarrow{\sim} \bigoplus_{n_0\leq n\leq -\log_p s} M_n^{r_0}$. Taking the inverse limit as $s\to 0^+$ and applying Lemma 2.1.4(2) proves (1).

The statement (2) follows from

$$M_n^{r_0}\otimes_{\mathbb{Q}_p(\zeta_{p^n})}\mathbb{Q}_p(\zeta_{p^{n+1}})\cong \varphi(M^{r_0}/q_nM^{r_0})\otimes_{\varphi(\mathcal{R}_A^{r_0}/q_n)}\mathcal{R}_A^{r_0/p}/q_{n+1}$$
$$\cong M^{r_0/p}/q_{n+1}M^{r_0/p}\cong M_{n+1}^{r_0/p}=M_{n+1}^{r_0}.$$

The statements (3) and (4) are straightforward. □

The following proposition is a naïve generalization of [36, Theorem 3.7] for families of (φ,Γ)-modules.

Proposition 3.2.4. *Let M be a (φ,Γ)-module over $\mathcal{R}_A$. The cohomology groups $H^*_{\varphi,\gamma_{\mathbb{Q}_p}}(M/tM)$ are finitely generated A-modules, and vanish outside degrees* 0, 1.

Proof. Assume that M is the base change of a (φ,Γ)-module M^{r_0} over $\mathcal{R}_A^{r_0}$. Denote $n_0=\lceil -\log_p r_0\rceil$. Then $M^{r_0}/tM^{r_0}\cong\prod_{n\geq n_0}M_n$. We first prove that $\varphi-1: M^{r_0}/tM^{r_0}\to M^{r_0/p}/tM^{r_0/p}$ is surjective. For $(x_n)_{n\geq n_0+1}\in M^{r_0/p}/tM^{r_0/p}$, we take $y_{n_0}=0$ and $y_n=-x_n+x_{n-1}-x_{n-2}+\cdots+(-1)^{n-n_0}x_{n_0+1}$ for $n\geq n_0+1$ and then $(\varphi-1)(y_n)_{n\geq n_0}=(x_n)_{n\geq n_0+1}$. This proves that the cohomology vanishes outside degrees 0, 1.

By the description of φ-action in Lemma 3.2.3(3), we have $(M^r/tM^r)^{\varphi=1}\cong M_{\lceil -\log_p r\rceil}$ for $0<r\leq r_0$. Taking the limit as $r\to 0^+$, we have $(M/tM)^{\varphi=1}\cong \varinjlim_n M_n$, and hence $H^*_{\varphi,\gamma_{\mathbb{Q}_p}}(M/tM)$ is the cohomology of the complex $[(\varinjlim_n M_n)^\Delta \xrightarrow{\gamma_{\mathbb{Q}_p}-1} (\varinjlim_n M_n)^\Delta]$. By the description of the Γ-action in Lemma 3.2.3(2) and the knowledge that $\mathbb{Q}_p(\mu_{p^n})$ is noncanonically the sum of all the irreducible $\mathbb{Q}_p$-representations of Γ/Γ_n, it suffices to know the following claim: for any A-valued Γ-representation W, the operator $\gamma_{\mathbb{Q}_p}-1$ is bijective on all twists $W(\psi)$ by finite order characters ψ of sufficiently large conductor. Indeed, it suffices to check bijectivity of γ_n-1 for some n. Choose n large enough that the operator norm $||\gamma_n-1||_W$ is less than $p^{-1/(p-1)}$ (using an argument similar to Proposition 2.2.14(2), but easier because the Γ_K-action is A-linear). Then for any character ψ such that $\psi|_{\Gamma_n}$ is nontrivial, it is easy to see that γ_n-1 is bijective on $W(\psi)$. □

Since we will later be studying Iwasawa cohomology, we need a version of Proposition 3.2.4 for ψ-cohomology rather than (φ,Γ)-cohomology, which gives more understanding of the $\mathcal{R}_A^\infty(\Gamma)$-module structure.

Proposition 3.2.5. *Let M be a (φ,Γ)-module over $\mathcal{R}_A$. The map $\psi-1: M/tM\to M/tM$ is surjective, and its kernel admits a resolution $0\to P\to Q\to(M/tM)^{\psi=1}\to 0$ by finite projective $\mathcal{R}_A^\infty(\Gamma)$-modules P and Q.*

Proof. Assume that M is the base change of a (φ,Γ)-module M^{r_0} over $\mathcal{R}_A^{r_0}$ with $r_0<1/p$. Set $n_0=\lceil -\log_p r_0\rceil\geq 2$, so that $M^{r_0}/tM^{r_0}\cong\prod_{n\geq n_0}M_n$ by Lemma 3.2.3(1). To see the surjectivity of $\psi-1$, given $(x_n)_{n\geq n_0}\in M^{r_0}/tM^{r_0}$, the tuple (y_n) defined by $y_{n_0}=0$ and $y_n=x_{n-1}+x_{n-2}+\cdots+x_{n_0}$ for $n\geq n_0+1$ satisfies $(\psi-1)(y_n)_{n\geq n_0}=(x_n)_{n\geq n_0}$.

We now discuss the $\mathcal{R}_A^\infty(\Gamma)$-module structure of $(M^{r_0}/tM^{r_0})^{\psi=1}$. By Lemma 3.2.6(1) below, it suffices to find a two-term resolution by finite projective $\mathcal{R}_A^\infty(\Gamma_{n_0})$-modules.

By the description of the ψ-operator in Lemma 3.2.3(4), $(M^{r_0}/tM^{r_0})^{\psi=1} \cong \varprojlim_{n\geq n_0} M_n$ with transition map ψ (which admits a right inverse). In particular, this implies that, in contrast to the (φ,Γ)-cohomology case (Proposition 3.2.4), $(M^{r_0}/tM^{r_0})^{\psi=1}$ does *not* depend on the choice of r_0, i.e., $(M/tM)^{\psi=1} = \varinjlim_r (M^r/tM^r)^{\psi=1} \cong \varprojlim_{n\geq n_0} M_n$. On the other hand, Lemma 3.2.3(2) gives Γ_{n_0}-equivariant isomorphisms $M_n \cong M_{n_0} \otimes_{\mathbb{Q}_p(\zeta_{p^{n_0}})} \mathbb{Q}_p(\zeta_{p^n})$, with ψ on the left corresponding to $1 \otimes \frac{1}{p}\mathrm{Tr}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})}$ on the right. Thus,

$$(M/tM)^{\psi=1} \cong \varprojlim_{n\geq n_0, 1\otimes\frac{1}{p}\mathrm{Tr}} M_{n_0} \otimes_{\mathbb{Q}_p(\zeta_{p^{n_0}})} \mathbb{Q}_p(\zeta_{p^n}).$$

The normal basis theorem allows us to choose a system of Γ_{n_0}-equivariant isomorphisms $\mathbb{Q}_p(\zeta_{p^n}) \cong \mathbb{Q}_p(\zeta_{p^{n_0}})[\Gamma_{n_0}/\Gamma_n]$ such that $\frac{1}{p}\mathrm{Tr}_{\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p(\zeta_{p^{n-1}})}$ on the left corresponds to the projection onto $\mathbb{Q}_p(\zeta_{p^{n_0}})[\Gamma_{n_0}/\Gamma_{n-1}]$ on the right. On the other hand, one has a Γ_{n_0}-equivariant identification $\mathbb{Q}_p(\zeta_{p^{n_0}})[\Gamma_{n_0}/\Gamma_n] \cong \mathcal{R}_{\mathbb{Q}_p(\zeta_{p^{n_0}})}(\Gamma_{n_0})/(\gamma_{n_0}^{p^{n-n_0}}-1)$. Thus,

$$\begin{aligned}(M/tM)^{\psi=1} &\cong \varprojlim_n M_{n_0} \otimes_{\mathbb{Q}_p(\zeta_{p^{n_0}})} \mathcal{R}_{\mathbb{Q}_p(\zeta_{p^{n_0}})}(\Gamma_{n_0})/(\gamma_{n_0}^{p^{n-n_0}}-1)\\ &\cong \varprojlim_n M_{n_0} \otimes_{\mathbb{Q}_p} \mathcal{R}(\Gamma_{n_0})/(\gamma_{n_0}^{p^{n-n_0}}-1)\\ &\cong \varprojlim_n M_{n_0} \otimes_A \mathcal{R}_A(\Gamma_{n_0})/(\gamma_{n_0}^{p^{n-n_0}}-1).\end{aligned}$$

But M_{n_0} is a finite A-module, and $\gamma_{n_0}^{p^{n-n_0}}-1 = p^{n-n_0}\prod_{i=0}^{n-n_0} q_{i,\gamma_{n_0}}$ (with p a unit in $\mathcal{R}(\Gamma_{n_0})$), so

$$(M/tM)^{\psi=1} \cong M_{n_0} \otimes_A \varprojlim_n \mathcal{R}_A(\Gamma_{n_0})/(\gamma_{n_0}^{p^{n-n_0}}-1) \cong M_{n_0} \otimes_A \mathcal{R}_A(\Gamma_{n_0})/\ell_{\gamma_{n_0}}.$$

We stress that Γ_{n_0} acts diagonally on the two tensor factors above.

Now we view $M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_{n_0})/\ell_{\gamma_{n_0}}$ as the cokernel of the natural map

$$M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_{n_0}) \xrightarrow{\mathrm{id}\otimes\cdot\ell_{\gamma_{n_0}}} M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_{n_0}). \tag{3.2.5.1}$$

We will prove that $M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_{n_0})$ is a finite projective $\mathcal{R}_A^\infty(\Gamma_{n_0})$-module (again for the diagonal Γ_{n_0}-action), which would conclude the proposition. For this, we use an argument similar to that of Lemma 3.1.2. By Lemma 3.2.6(2) below, it suffices to prove that $M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_{n_0}) \approx M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_m)^{\oplus p^{m-n_0}}$, and hence equivalently $M_{n_0} \otimes_A \mathcal{R}_A^\infty(\Gamma_m)$, is finite projective as an $\mathcal{R}_A^\infty(\Gamma_m)$-module for some $m \geq n_0$. Since M_{n_0} is finite flat over A, there exists an A-module N_{n_0} (with trivial Γ_{n_0}-action) such that $M_{n_0} \oplus N_{n_0}$ is a free A-module with basis $\mathbf{e}_1, \dots, \mathbf{e}_d$. We choose the Banach norm $|\cdot|_{M_{n_0}\oplus N_{n_0}}$ on $M_{n_0} \oplus N_{n_0}$ to be the one given by the basis $\mathbf{e}_1, \dots, \mathbf{e}_d$. By an argument similar to Proposition 2.2.14(2) (but easier, because the action is now linear), there exists $m \in \mathbb{N}$ such that the operator norm $\|\gamma_m - 1\|_{M_{n_0}\oplus N_{n_0}} < 1$; let $\epsilon > 0$ be so that this quantity is ω^ϵ. As in Lemma 3.1.2, we consider the following two maps:

$$\begin{aligned} \bigoplus_{i=1}^{d} \mathcal{R}_A^\infty(\Gamma_m)\mathbf{e}_i &\longrightarrow (M_{n_0}\oplus N_{n_0})\otimes_A \mathcal{R}_A^\infty(\Gamma_m) \\ \Phi:\ \sum_{i=1}^{d} f_i(\gamma_m-1)\mathbf{e}_i &\longmapsto \sum_{i=1}^{d} f_i(\gamma_m-1)\cdot(\mathbf{e}_i\otimes 1) \\ \Phi':\ \sum_{i=1}^{d} f_i(\gamma_m-1)\mathbf{e}_i &\longmapsto \sum_{i=1}^{d} \mathbf{e}_i\otimes f_i(\gamma_m-1). \end{aligned}$$

Here, Φ' is a topological isomorphism of A-modules (not necessarily respecting the Γ_m-action); and Φ is an $\mathcal{R}_A^\infty(\Gamma_m)$-homomorphism which we will now prove to be an isomorphism.

For each $r>0$, let $|\cdot|_{M_{n_0}\oplus N_{n_0},r}$ denote the tensor product norm on $(M_{n_0}\oplus N_{n_0})\otimes_A \mathcal{R}_A^\infty(\Gamma_m)$ given by the prescribed norm $|\cdot|_{M_{n_0}\oplus N_{n_0}}$ on $M_{n_0}\oplus N_{n_0}$ and the r-Gauss norm on $\mathcal{R}_A^\infty(\Gamma_m)$. The bound on the operator norm of γ_m-1 on $M_{n_0}\oplus N_{n_0}$ implies that for any $r\in(0,\epsilon)$, we have

$$|\Phi\circ\Phi'^{-1}(x)-x|_{M_{n_0}\oplus N_{n_0},r} < |x|_{M_{n_0}\oplus N_{n_0},r} \qquad \text{for any } x\in(M_{n_0}\oplus N_{n_0})\otimes_A\mathcal{R}_A^\infty(\Gamma_m).$$

This implies that Φ is a topological isomorphism. Hence, $M_{n_0}\otimes_A \mathcal{R}_A^\infty(\Gamma_m)$ is a finite projective $\mathcal{R}_A^\infty(\Gamma_m)$-module. This finishes the proof. □

The proof of the following lemma follows from elementary homological algebra, and is left to the reader.

Lemma 3.2.6. *(1) Let S be a finite Galois algebra over a ring R with Galois group G and let N be an S-module. Assume that N, viewed as an R-module, admits a d-term resolution $0\to P'^{-d+1}\to\cdots\to P'^{0}\to N\to 0$ by finite projective R-modules. Then N admits a d-term resolution by finite projective S-modules.*

(2) Let S be a finite flat algebra over a ring R, and let M be a finitely generated R-module. If $\mathrm{Hom}_R(S,M)$ is projective over S then M is projective over R.

Remark 3.2.7. With some mild generalization of Lemma 3.2.6, one can show that the $\mathcal{R}_A^\infty(\Gamma)$-modules P and Q in Proposition 3.2.5 have the same rank.

3.3. **Finiteness of $M/(\psi-1)$.** We now make a calculation to control the cokernel of $\psi-1$. Note that this argument (which makes no use of Γ) is similar to arguments used to control φ-cohomology in the development of slope filtration theory, as in [33, Proposition 2.1.5].

Hypothesis 3.3.1. In this subsection, we assume that $K=\mathbb{Q}_p$.

Proposition 3.3.2. *Let M be a (φ,Γ)-module over $\mathcal{R}_A$.*

1. *The A-module $M/(\psi-1)$ is finitely generated.*
2. *For all sufficiently large integers n, the map $\psi-1: t^{-n}M\to t^{-n}M$ is surjective.*

Proof. For a suitable choice of $r_0\in(0,1)$, we can realize M as the base extension of a φ-module M^{r_0} over $\mathcal{R}_A^{r_0}$. Put $r=r_0/p^2$, choose generators $\mathbf{e}'_1,\dots,\mathbf{e}'_n$ of M^{r_0}, and take $\mathbf{e}_i=\varphi(\mathbf{e}'_i)\in M^{pr}$. Let N^{pr} be the free module over $\mathcal{R}_A^{pr}$ on the generators $\mathbf{e}_1,\dots,\mathbf{e}_n$ and choose an $\mathcal{R}_A^{pr}$-linear splitting $N^{pr}\cong M^{pr}\oplus P^{pr}$ of the surjection $\mathrm{proj}: N^{pr}\to M^{pr}$. For $s\in(0,pr]$, let $|\cdot|_s$ denote the s-Gauss norm on N^{pr} for the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, and restrict this norm to M^{pr}. Let $||\mathrm{proj}||_s$ denote the operator norm of $\mathrm{proj} : N^{pr} \to M^{pr}$ with respect to the s-Gauss norm.

Choose a matrix F' with entries in $\mathcal{R}_A^{pr}$ such that $\mathbf{e}_j = \sum_i F'_{ij}\mathbf{e}'_i$ and put $F = \varphi(F')$, so that $\varphi(\mathbf{e}_j) = \sum_i F_{ij}\mathbf{e}_i$ and F has entries in $\mathcal{R}_A^r$. Choose also a matrix G with entries in $\mathcal{R}_A^{pr}$ such that $\mathbf{e}'_j = \sum_i G_{ij}\mathbf{e}_i$, so that $\mathbf{e}_j = \varphi(\sum_i G_{ij}\mathbf{e}_i)$. We then have

$$\varphi\left(\sum_j c_j \mathbf{e}_j\right) = \sum_i \left(\sum_j F_{ij}\varphi(c_j)\right)\mathbf{e}_i \qquad (c_1, \dots, c_n \in \mathcal{R}_A^r), \tag{3.3.2.1}$$

$$\psi\left(\sum_j c_j \mathbf{e}_j\right) = \sum_i \left(\sum_j G_{ij}\psi(c_j)\right)\mathbf{e}_i \qquad (c_1, \dots, c_n \in \mathcal{R}_A^{r/p}). \tag{3.3.2.2}$$

Choose $\epsilon \in (0,1)$. Note that for $a \in \mathbb{R}$ sufficiently small and $b \in \mathbb{R}$ sufficiently large (they need not be integers), one has

$$\omega^{a(p^{-1}-1)r}||\mathrm{proj}||_r \cdot \|G\|_r \leq \epsilon, \qquad \omega^{b(p-1)r}||\mathrm{proj}||_r \cdot \|F\|_r \leq \epsilon.$$

Choose some such a, b and define the A-linear map $\Pi : M^r \to M^r$ by inclusion into N^r followed by the formula

$$\Pi\left(\sum_{j=1}^n \sum_{i \in \mathbb{Z}} a_{ij}\pi^i \mathbf{e}_j\right) = \mathrm{proj}\ \sum_{j=1}^n \sum_{i \in [a,b]\cap\mathbb{Z}} a_{ij}\pi^i\mathbf{e}_j.$$

Note that Π projects M^r onto a finite A-submodule of M^r and that $|\Pi(\mathbf{v})|_r \leq ||\mathrm{proj}||_r \cdot |\mathbf{v}|_r$ for $\mathbf{v} \in M^r$. Define the map $\lambda : M^r \to M^r$ by inclusion into N^r followed by the formula

$$\lambda\left(\sum_{j=1}^n \sum_{i \in \mathbb{Z}} a_{ij}\pi^i \mathbf{e}_j\right) = \mathrm{proj}\ \sum_{j=1}^n \sum_{i<a} a_{ij}\pi^i\mathbf{e}_j - \mathrm{proj}\ \sum_{j=1}^n \sum_{i>b} \varphi(a_{ij}\pi^i\mathbf{e}_j). \tag{3.3.2.3}$$

Here we use (3.3.2.1) to write $\varphi(a_{ij}\pi^i\mathbf{e}_j)$ as $\sum_l F_{lj}\varphi(a_{ij}\pi^i)\mathbf{e}_l$; hence, the right-hand side of (3.3.2.3) lies in M^r. From the definition, we have the following formula for a given element $\mathbf{v} = \sum_{j=1}^n \sum_{i\in\mathbb{Z}} a_{ij}\pi^i\mathbf{e}_j$ in M^r:

$$\begin{aligned}\mathbf{v} - \Pi(\mathbf{v}) + (\psi - 1)(\lambda(\mathbf{v})) &= \mathrm{proj}\sum_{j=1}^n\sum_{i<a}\psi(a_{ij}\pi^i\mathbf{e}_j) - \mathrm{proj}\sum_{j=1}^n\sum_{i>b}\varphi(a_{ij}\pi^i\mathbf{e}_j)\\ &= \mathrm{proj}\sum_{j,l=1}^n\sum_{i<a}G_{lj}\psi(a_{ij}\pi^i)\mathbf{e}_j - \mathrm{proj}\sum_{j,l=1}^n\sum_{i>b}F_{lj}\varphi(a_{ij}\pi^i)\mathbf{e}_j.\end{aligned} \tag{3.3.2.4}$$

By our choice of a, b, we have

$$|\mathbf{v} - \Pi(\mathbf{v}) + (\psi - 1)(\lambda(\mathbf{v}))|_r \leq \epsilon|\mathbf{v}|_r.$$

Given $\mathbf{v} \in M^r$, put $\mathbf{v}_0 = \mathbf{v}$ and

$$\mathbf{v}_{l+1} = \mathbf{v}_l - \Pi(\mathbf{v}_l) + (\psi - 1)(\lambda(\mathbf{v}_l)). \tag{3.3.2.5}$$

Then $\mathbf{w} = \sum_{l=0}^\infty \mathbf{v}_l$, provided this sum converges, is an element of M^r satisfying

$$\mathbf{v} - \Pi(\mathbf{w}) + (\psi - 1)(\lambda(\mathbf{w})) = 0.$$

This sum converges for the r-Gauss norm by (3.3.2.5). We will check the convergence for s-Gauss norm when $s < r/p$; this is enough because knowing the convergence for smaller s gives it for larger s. For any $s \in (0, r/p)$, we again choose $a' \leq a$ sufficiently small and $b' \geq b$ sufficiently large so that

$$\omega^{a'(p^{-1}-1)s}||\mathrm{proj}||_s \cdot \|G\|_s \leq \epsilon \quad \text{and} \quad \omega^{b'(p-1)s}||\mathrm{proj}||_s \cdot \|F\|_s \leq \epsilon.$$

We separate the powers of π in the expression (3.3.2.4) defining $\mathbf{v}_l$ into the ranges $[a', a)$, $(b, b']$, and $(-\infty, a') \cup (b', \infty)$. By the same argument, the summation over all powers of π in $(-\infty, a') \cup (b', \infty)$ has s-Gauss norm less than or equal to $\epsilon|\mathbf{v}_l|_s$. The summation over powers of π in $[a', a)$ has s-Gauss norm less than or equal to $\omega^{a(p^{-1}s-r)}\|G\|_s|\mathbf{v}_l|_r$ (note that we are comparing it with $|\mathbf{v}_l|_r$ as opposed to $|\mathbf{v}_l|_s$). The summation over powers of π in $(b, b']$ has s-Gauss norm less than or equal to $\omega^{b'(sp-r)}\|F\|_s|\mathbf{v}_l|_r$. In summary, we have

$$|\mathbf{v}_{l+1}|_s \leq \max\{\omega^{b'(sp-r)}\|F\|_s|\mathbf{v}_l|_r,\ \omega^{a(p^{-1}s-r)}\|G\|_s|\mathbf{v}_l|_r,\ \epsilon|\mathbf{v}_l|_s\}.$$

From this, it follows that the series defining $\mathbf{w}$ converges also under $|\cdot|_s$; since s was an arbitrary element of $(0, r/p)$, we deduce that $\mathbf{w} \in M^r$.

Given any $\mathbf{v} \in M$, we can find $s \in (0, r]$ such that $\mathbf{v} \in M^s$. Choose a nonnegative integer m for which $p^m s \geq r$; then $\psi^m(\mathbf{v})$ is an element of M^r representing the same class in $M/(\psi - 1)M$ as $\mathbf{v}$. By the previous paragraph, however, this class is represented by an element in the image of Π; we thus deduce (1).

To prove (2), note that replacing M with $t^{-n}M$ has the effect of replacing F with $p^{-n}F$ and G with p^nG. It is thus sufficient to check that for n large, we can find a single value $a \in \mathbb{R}$ with

$$\omega^{a(p^{-1}-1)r}p^{-n}||\mathrm{proj}||_r \cdot \|G\|_r \leq \epsilon, \qquad \omega^{a(p-1)r}p^n||\mathrm{proj}||_r \cdot \|F\|_r \leq \epsilon,$$

as then the previous argument applies with an empty generating set. Namely, such a choice of a exists if and only if

$$p^{-n(p-1)}||\mathrm{proj}||_r^{p+1} \cdot \|G\|_r^p \cdot \|F\|_r \leq \epsilon^{p+1},$$

and it is clear that this holds for n large enough. □

4. Finiteness of cohomology

In this section, we give our main results on finiteness of (φ, Γ)-cohomology and Iwasawa cohomology for relative (φ, Γ)-modules. To clarify the presentation, we leave one key statement (Theorem 4.4.1) unproven for the moment, in order to illustrate the overall structure of the arguments; we take up the technical arguments needed to prove Theorem 4.4.1 in the next section.

4.1. **Formalism of derived categories.** We give a short discussion of the derived category of complexes of modules.

Hypothesis 4.1.1. In this subsection, we fix a commutative ring R.

Notation 4.1.2. Let $\mathbf{D}^{[a,b]}_{\mathrm{perf}}(R)$ (resp. $\mathbf{D}^{\flat}_{\mathrm{perf}}(R)$, $\mathbf{D}^{-}_{\mathrm{perf}}(R)$) denote the subcategory of the derived category of complexes of R-modules consisting of the complexes of R-modules which are quasi-isomorphic to complexes of finite projective R-modules concentrated in degrees in $[a, b]$ (resp. bounded degrees, degrees bounded above).

Lemma 4.1.3. *Let $P^\bullet$ be a complex of projective (resp. flat) R-modules concentrated in degree $[0, d]$ and $Q^\bullet$ a complex of projective (resp. flat) R-modules bounded above. Suppose that we have a quasi-isomorphism $P^\bullet \to Q^\bullet$ or $Q^\bullet \to P^\bullet$. Then* *the complex* $Q'^\bullet = [\operatorname{Coker} d_Q^{-1} \xrightarrow{d_Q^0} Q^1 \xrightarrow{d_Q^1} \cdots]$ *is quasi-isomorphic to* $Q^\bullet$ *and* $\operatorname{Coker} d_Q^{-1}$ *is a projective (resp. flat) R-module.*

Proof. Since $P^\bullet$ and $Q^\bullet$ are quasi-isomorphic, $H^i(Q^\bullet) = 0$ for $i < 0$. This implies that $Q'^\bullet$ is quasi-isomorphic to $Q^\bullet$. To see the projectivity (resp. flatness) of $\operatorname{Coker} d_Q^{-1}$, we observe that, under either condition, $\operatorname{Fib}(P^\bullet \to Q'^\bullet)$ or $\operatorname{Fib}(Q'^\bullet \to P^\bullet)$ is an *acyclic* complex of projective (resp. flat) R-modules, except possibly the term $\operatorname{Coker} d_Q^{-1}$ which appears either in the leftmost nonzero term, or as a direct summand of the second leftmost nonzero term. In either case, it is easy to deduce that $\operatorname{Coker} d_Q^{-1}$ is a projective (resp. flat) R-module. □

Lemma 4.1.4. *If a complex in* $\mathbf{D}^-_{\mathrm{perf}}(R)$ *has trivial cohomology in degrees strictly greater than b, then it is quasi-isomorphic to a complex of finite projective R-modules* $P^\bullet$ *with* $P^{b+1} = P^{b+2} = \cdots = 0$. *If the given complex lies in* $\mathbf{D}^\flat_{\mathrm{perf}}(R)$, *we may choose* $P^\bullet$ *such that, in addition, we have* $P^{-n} = 0$ *for n sufficiently large.*

Proof. The given complex is quasi-isomorphic to a complex $[\cdots \xrightarrow{d^{b'-2}} Q^{b'-1} \xrightarrow{d^{b'-1}} Q^{b'}]$ of finite projective R-modules in degrees bounded above, and also below if the given complex lies in $\mathbf{D}^\flat_{\mathrm{perf}}(R)$. Then the given complex is also quasi-isomorphic to the complex $[\cdots \xrightarrow{d^{b-2}} Q^{b-1} \xrightarrow{d^{b-1}} \operatorname{Ker}(d^b)]$. It suffices to prove that $\operatorname{Ker} d^b$ is finite projective. Indeed, the condition implies that $0 \to \operatorname{Ker} d^b \xrightarrow{d^b} Q^b \to Q^{b+1} \to \cdots \to Q^{b'} \to 0$ is a long exact sequence, in which all terms except $\operatorname{Ker} d^b$ are known to be finite projective. Therefore, $\operatorname{Ker} d^b$ is also finite projective. □

Lemma 4.1.5. *Let* $P^\bullet \to Q^\bullet$ *be a morphism of complexes in* $\mathbf{D}^-_{\mathrm{perf}}(R)$. *Then it is a quasi-isomorphism if and only if* $P^\bullet \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m} \to Q^\bullet \overset{\mathbf{L}}{\otimes}_R R/\mathfrak{m}$ *is a quasi-isomorphism for every maximal ideal* $\mathfrak{m}$ *of R.*

Proof. We may assume that both complexes are complexes of finite projective R-modules in degrees bounded above. It suffices to show that the mapping fiber complex $S^\bullet = \operatorname{Fib}(P^\bullet \to Q^\bullet)$ is acyclic. Suppose not. Let n be the maximal integer such that $H^n(S^\bullet)$ is nontrivial. By Lemma 4.1.4, $H^n(S^\bullet)$ is finitely generated. Hence, there exists a maximal ideal $\mathfrak{m}$ of R such that $H^n(S^\bullet) \otimes_R R/\mathfrak{m} \neq 0$. Now, the spectral sequence $\operatorname{Tor}^R_j(H^i(S^\bullet), R/\mathfrak{m}) \Rightarrow H^{i-j}(S^\bullet \otimes_R R/\mathfrak{m}) = 0$ would lead to a contradiction at degree n, which is nonzero according to the spectral sequence and is zero by assumption. □

Lemma 4.1.6. *If A is a finite extension of* $\mathbb{Q}_p$ *and M is an* $\mathcal{R}^{r_0}_A$*-module which is finite over A, we have* $M \in \mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}^{r_0}_A)$.

Proof. In this case, M is supported at a finite set of closed points on $\operatorname{Max}(A) \times \mathrm{A}^1(0, r_0]$, so the lemma is obvious. □

Definition 4.1.7. For P a finite projective R-module, we denote its rank by $\operatorname{rank}_R P$, viewed as a function on the set of irreducible components of $\operatorname{Spec} R$ (and when $\operatorname{Spec} R$ is irreducible we regard it simply as a nonnegative integer).

For $P^\bullet \in \mathbf{D}^{[a,b]}_{\mathrm{perf}}(R)$ a complex of finite projective R-modules concentrated in degrees in $[a, b]$, we define its *Euler characteristic* by $\chi_R(P^\bullet) = \sum_{i=a}^b (-1)^i \operatorname{rank}_R P^i$; it is invariant under quasi-isomorphisms, and is additive for distinguished triangles.

Definition 4.1.8. Let $\mathbf{R}\mathrm{Hom}_R(-,R) : \mathbf{D}^{[a,b]}_{\mathrm{perf}}(R) \to \mathbf{D}^{[-b,-a]}_{\mathrm{perf}}(R)$ denote the duality functor given by sending a bounded complex $P^\bullet$ of finite projective R-modules to $(\mathrm{Hom}_R(P^{-i},R))_{i\in\mathbb{Z}}$. The functor is well-defined and $\mathbf{R}\mathrm{Hom}_R(-,R)\circ\mathbf{R}\mathrm{Hom}_R(-,R)$ is the identity.

4.2. **Iwasawa cohomology of** (φ,Γ)**-modules.** Following the work of Fontaine (see [16]) on computing the Iwasawa cohomology of a Galois representation using (φ,Γ)-modules, the second author pointed out in [43] that the Iwasawa cohomology of general (φ,Γ)-modules has many number-theoretic applications. This motivates a systematic study of the Iwasawa cohomology of a (φ,Γ)-module.

Notation 4.2.1. Let $\mathrm{Max}(\mathcal{R}^\infty_A(\Gamma_K))$ denote the union of the $\mathrm{Max}(\mathcal{R}^{[s,\infty]}_A(\Gamma_K))$ for all $s>0$. In other words, it is the set of closed points of the quasi-Stein rigid analytic space associated to $\mathcal{R}^\infty_A(\Gamma_K)$, viewed as a disjoint union of relative discs. A closed point of $\mathrm{Max}(\mathcal{R}^\infty_A(\Gamma_K))$ is the same datum as an equivalence class of pairs (z,η), where $z\in\mathrm{Max}(A)$ and η is a character of Γ_K with values in some finite extension of $\mathbb{Q}_p$, and two such pairs are considered equivalent if they become equal upon embedding their target fields into some common finite extension of $\mathbb{Q}_p$. We let $\mathfrak{m}_{(z,\eta)}$ be the corresponding maximal ideal and $\kappa_{(z,\eta)}$ the residue field (which is finite over $\mathbb{Q}_p$), so that η is given by the composition $\Gamma_K \to \mathcal{R}^\infty_A(\Gamma_K)^\times \to (\mathcal{R}^\infty_A(\Gamma_K)/\mathfrak{m}_{(z,\eta)})^\times = \kappa^\times_{(z,\eta)}$. Let $\bar{\mathfrak{m}}_{(z,\eta)}$ denote the corresponding maximal ideal of $\mathcal{R}^\infty_{A/\mathfrak{m}_z}(\Gamma_K)$.

For M a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$, let $M_z\otimes\eta$ denote the (φ,Γ_K)-module over $\mathcal{R}_{\kappa_{(z,\eta)}}(\pi_K)$-module given by $M_z\otimes_{\kappa_z}\kappa_{(z,\eta)}$ with Γ_K acting on $\kappa_{(z,\eta)}$ through η, and φ acting on $\kappa_{(z,\eta)}$ trivially.

When A is a finite extension of $\mathbb{Q}_p$, we omit z from the notation by simply writing, for example, $M\otimes\eta$ and $\mathfrak{m}_\eta$.

Definition 4.2.2. Let M be a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$. We write $\mathrm{C}^\bullet_\psi(M)$ for the complex $[M \xrightarrow{\psi-1} M]$ concentrated in degrees 1 and 2. Its cohomology groups $H^*_\psi(M)$ are called the *Iwasawa cohomology* of M; using Proposition 2.2.14, we view them as modules over $\mathcal{R}^\infty_A(\Gamma_K)$.

The following base change result from Iwasawa cohomology to Galois cohomology follows immediately from the definitions, and will often be used implicitly in the remainder of this paper.

Proposition 4.2.3. *For $\eta\in\mathrm{Max}(\mathcal{R}^\infty(\Gamma_K))$ a continuous character of Γ_K with coefficients in $L=\mathcal{R}^\infty(\Gamma_K)/\mathfrak{m}_\eta$, we have a natural quasi-isomorphism*

$$\mathrm{C}^\bullet_\psi(M) \overset{\mathbf{L}}{\underset{\mathcal{R}^\infty_A(\Gamma_K)}{\otimes}} \mathcal{R}^\infty_A(\Gamma_K)/\mathfrak{m}_\eta\mathcal{R}^\infty_A(\Gamma_K) \longrightarrow \mathrm{C}^\bullet_{\psi,\gamma_K}(M\otimes_{\mathbb{Q}_p} L(\eta^{-1})), \tag{4.2.3.1}$$

where $L(\eta^{-1})$ is a one-dimensional L-vector space with the trivial φ-action and the Γ_K-action by η^{-1}.

Notation 4.2.4. The involution $\iota:\Gamma_K\to\Gamma_K$ given by $\iota(\gamma)=\gamma^{-1}$ for $\gamma\in\Gamma_K$ induces an involution $\iota:\mathcal{R}^?_A(\Gamma_K)\to\mathcal{R}^?_A(\Gamma_K)$, where $?=[s,r],r,\emptyset$. For an $\mathcal{R}^?_A(\Gamma_K)$-module N, we write N^ι for the $\mathcal{R}^?_A(\Gamma_K)$-module with the same underlying abelian group N but with module structure twisted through ι.

For the remainder of this subsection we specialize to the case $K=\mathbb{Q}_p$ and, in this case, describe an explicit construction of duality pairings for Iwasawa cohomology.

We use the notations of Definition 2.1.20 with $C = \Gamma$, $\ell = \log \circ \chi$, and $c = \gamma_{\mathbb{Q}_p}$. Recall that the residue map on $\mathcal{R}_A(\Gamma)$ is denoted Res_Γ.

Lemma 4.2.5. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$. There is a continuous isomorphism of $\mathcal{R}_A(\Gamma)$-modules*

$$(M^*)^{\psi=0} \cong \mathrm{Hom}_{\mathcal{R}_A(\Gamma)}(M^{\psi=0}, \mathcal{R}_A(\Gamma))^\iota \tag{4.2.5.1}$$

sending $y \in (M^*)^{\psi=0}$ to the unique $\mathcal{R}_A(\Gamma)$-linear homomorphism $\{-, y\}^0_{\mathrm{Iw}} : M^{\psi=0} \to \mathcal{R}_A(\Gamma)$ such that for all $x \in M^{\psi=0}$ one has $\mathrm{Res}_\Gamma(\{x, y\}^0_{\mathrm{Iw}} \omega_{\log \circ \chi}) = \{x, y\}_{\mathbb{Q}_p}$. In particular,

$$\{-, -\}^0_{\mathrm{Iw}} : M^{\psi=0} \times (M^*)^{\psi=0,\iota} \to \mathcal{R}_A(\Gamma)$$

is $\mathcal{R}_A(\Gamma)$-bilinear.

Proof. We first note that the pairing $\{-, -\}_{\mathbb{Q}_p} : M \times M^* \to A$ is perfect. This perfection does not involve any φ- or Γ-action, only that M is finite projective over $\mathcal{R}_A$; one reduces to the free case by adding a complementary projective module, then reduces to the free rank one case, which is Lemma 2.1.19.

Returning to the (φ, Γ)-action on M, we recall that for any $x \in M$ and $y \in M^*$ one has $\{\gamma x, \gamma y\}_{\mathbb{Q}_p} = \{x, y\}_{\mathbb{Q}_p}$ for all $\gamma \in \Gamma$, $\{\varphi x, y\}_{\mathbb{Q}_p} = \{x, \psi y\}_{\mathbb{Q}_p}$, and $\{\psi x, y\}_{\mathbb{Q}_p} = \{x, \varphi y\}_{\mathbb{Q}_p}$. From the identity for $\gamma \in \Gamma$ it follows that the duality isomorphism is $A[\Gamma]$-linear (with the shown involution ι), and hence by continuity of the action it is $\mathcal{R}^\infty_A(\Gamma)$-linear. From the identities for φ and ψ and the direct sum decompositions $M = \varphi(M) \oplus M^{\psi=0}$ and $M^* = \varphi(M^*) \oplus (M^*)^{\psi=0}$, it follows that $M^{\psi=0}$ and $\varphi(M^*)$ are exact orthogonal complements. Therefore, $\{-, -\}_{\mathbb{Q}_p}$ restricts to a perfect pairing between $M^{\psi=0}$ and $(M^*)^{\psi=0}$, which in turn gives by adjunction a topological isomorphism $(M^*)^{\psi=0} \xrightarrow{\sim} \mathrm{Hom}_{A,\mathrm{cont}}(M^{\psi=0}, A)^\iota$. This isomorphism is $\mathcal{R}^\infty_A(\Gamma)[(\gamma_{\mathbb{Q}_p} - 1)^{-1}]$-linear, and hence by continuity of the action it is $\mathcal{R}_A(\Gamma)$-linear.

Combining the preceding isomorphism with tensor-hom adjunction and residue duality (2.1.20.1) for $\mathcal{R}_A(\Gamma)$, we obtain

$$\begin{aligned}(M^*)^{\psi=0} &\cong \mathrm{Hom}_{A,\mathrm{cont}}(M^{\psi=0}, A)^\iota \cong \mathrm{Hom}_{A,\mathrm{cont}}(M^{\psi=0} \otimes_{\mathcal{R}_A(\Gamma)} \mathcal{R}_A(\Gamma), A)^\iota \\ &\cong \mathrm{Hom}_{\mathcal{R}_A(\Gamma)}(M^{\psi=0}, \mathrm{Hom}_{A,\mathrm{cont}}(\mathcal{R}_A(\Gamma), A))^\iota \\ &\cong \mathrm{Hom}_{\mathcal{R}_A(\Gamma)}(M^{\psi=0}, \mathcal{R}_A(\Gamma))^\iota.\end{aligned} \tag{4.2.5.2}$$

One verifies immediately from the definitions that this isomorphism has desired form. □

Corollary 4.2.6. *Let M be a (φ, Γ)-module over $\mathcal{R}_A$. Then there exists $r_0 > 0$ such that $(M^{*r_0})^{\psi=0} \cong \mathrm{Hom}_{\mathcal{R}^{r_0}_A(\Gamma)}((M^{r_0})^{\psi=0}, \mathcal{R}^{r_0}_A(\Gamma))^\iota$.*

Proof. It is easy to see that for sufficiently small $r_0 > 0$, the isomorphism in (4.2.5.1) sends the $\mathcal{R}^{r_0}_A(\Gamma)$-structure of the left-hand side into that of the right-hand side. By Theorem 3.1.1 and Remark 2.1.17, after perhaps shrinking r_0, the resulting morphism is an isomorphism. □

Lemma 4.2.7. *For $x \in M^{\psi=1}$ and $y \in (M^*)^{\psi=1}$, then $(\varphi - 1)x \in M^{\psi=0}$ and $(\varphi - 1)y \in (M^*)^{\psi=0}$, and we have $\{(\varphi - 1)x, (\varphi - 1)y\}^0_{\mathrm{Iw}} \in \mathcal{R}^\infty_A(\Gamma) \subset \mathcal{R}_A(\Gamma)$.*

Proof. The claim that $(\varphi - 1)x \in M^{\psi=0}$ and $(\varphi - 1)y \in M^{\psi=0}$ follows immediately from the identity $\psi \circ \varphi = \mathrm{id}$. Unwinding the last step of (4.2.5.2) through (2.1.20.1), the second claim is equivalent to showing that for all $r \in \mathcal{R}_A^\infty(\Gamma) \subseteq \mathcal{R}_A(\Gamma)$, one has $\{r\cdot(\varphi-1)x,(\varphi-1)y\}_{\mathbb{Q}_p} = 0$. Indeed, r commutes with φ and preserves $M^{\psi=1}$, so

$$\begin{aligned}\{r\cdot(\varphi-1)x,(\varphi-1)y\}_{\mathbb{Q}_p} &= \{(\varphi-1)(rx),(\varphi-1)y\}_{\mathbb{Q}_p}\\ &= \{rx,y\}_{\mathbb{Q}_p} - \{\varphi(rx),y\}_{\mathbb{Q}_p} - \{rx,\varphi(y)\}_{\mathbb{Q}_p} + \{\varphi(rx),\varphi(y)\}_{\mathbb{Q}_p}\\ &= \{rx,y\}_{\mathbb{Q}_p} - \{rx,\psi(y)\}_{\mathbb{Q}_p} - \{\psi(rx),y\}_{\mathbb{Q}_p} + \{rx,y\}_{\mathbb{Q}_p} = 0\end{aligned}$$

because x, rx, and y are fixed by ψ. □

Definition 4.2.8. By Lemma 4.2.7 above, the rule

$$\{-,-\}_{\mathrm{Iw}} : M^{\psi=1}\times(M^*)^{\psi=1,\iota} \to \mathcal{R}_A^\infty(\Gamma), \qquad (x,y)\mapsto \{(\varphi-1)x,(\varphi-1)y\}^0_{\mathrm{Iw}}$$

gives a continuous, $\mathcal{R}_A^\infty(\Gamma)$-bilinear pairing called the *Iwasawa pairing.* From the definition, if $x \in M^{\psi=1}$ and $y \in (M^*)^{\psi=1}$ then $\{x,y\}_{\mathrm{Iw}}$ is the unique element of $\mathcal{R}_A^\infty(\Gamma)$ such that for all $r \in \mathcal{R}_A(\Gamma)$ one has

$$\mathrm{Res}_\Gamma(r\{x,y\}_{\mathrm{Iw}}\omega_{\log\circ\chi}) = \{r\cdot(\varphi-1)x,\varphi-1)y\}_{\mathbb{Q}_p}.$$

Proposition 4.2.9. *For any point $(z,\eta) \in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))$, the Iwasawa pairing $\{-,-\}_{\mathrm{Iw}}$ at (z,η) is compatible with the Tate pairing for $M_z\otimes\eta^{-1}$, in the sense that the diagram*

$$\begin{array}{ccccc}
M^{\psi=1}/\mathfrak{m}_{(z,\eta)} & \times & (M^*)^{\psi=1}/\mathfrak{m}_{(z,\eta^{-1})} & \xrightarrow{\{-,-\}_{\mathrm{Iw}} \bmod \mathfrak{m}_{(z,\eta)}} & \mathcal{R}_A^\infty(\Gamma)/\mathfrak{m}_{(z,\eta)}\\
\Big\downarrow{\scriptstyle \log\chi(\gamma_{\mathbb{Q}_p})} & & \Big\downarrow{\scriptstyle \log\chi(\gamma_{\mathbb{Q}_p})} & & \Big\|\\
H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z\otimes\eta^{-1}) & \times & H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z^*\otimes\eta) & & \\
\Big\uparrow{\scriptstyle \Psi_{M_z\otimes\eta^{-1}}\,\cong} & & \Big\uparrow{\scriptstyle \cong\,\Gamma_{\gamma_{\mathbb{Q}_p}^{-1},\gamma_{\mathbb{Q}_p},M_z^*\otimes\eta}} & & \\
H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z\otimes\eta^{-1}) & \times & H^1_{\psi,\gamma_{\mathbb{Q}_p}^{-1}}(M_z^*\otimes\eta) & \xrightarrow{C_p\cdot\cup_{\mathrm{Ta}}} & \kappa_{(z,\eta)}
\end{array}$$

commutes. (Here, the upper horizontal arrow is obtained by identifying $(M^*)^{\psi=1,\iota}/\mathfrak{m}_{(z,\eta)} \cong (M^*)^{\psi=1}/\mathfrak{m}_{(z,\eta^{-1})}$, and the first vertical arrows are $\log\chi(\gamma_{\mathbb{Q}_p})$ *times the naïve maps induced by* (4.2.3.1)*, or by Remark* 2.3.4.*)*

Proof. We assume that Δ is trivial for simplicity, as the proof for the case of nontrivial Δ is similar. By twisting the action of Γ on M, we may assume that η is trivial.

Let $x \in M^{\psi=1}$ and $y \in (M^*)^{\psi=1}$. Their images in $H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z)$ and $H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z^*)$ are $\log\chi(\gamma_{\mathbb{Q}_p})\overline{(0,x)}$ and $\log\chi(\gamma_{\mathbb{Q}_p})\overline{(0,y)}$, respectively. Since $(\varphi-1)x \in M^{\psi=0}$, and $\gamma_{\mathbb{Q}_p}-1$ acts invertibly on $M^{\psi=0}$, the element $(\gamma_{\mathbb{Q}_p}-1)^{-1}(\varphi-1)x$ makes sense, and Ψ_{M_z} sends $\overline{((\gamma_{\mathbb{Q}_p}-1)^{-1}(\varphi-1)x,x)}$ to $\overline{(0,x)}$. By definition, $\Gamma_{\gamma_{\mathbb{Q}_p}^{-1},\gamma_{\mathbb{Q}_p},M_z^*}$ sends $\overline{(0,\frac{\gamma_{\mathbb{Q}_p}^{-1}-1}{\gamma_{\mathbb{Q}_p}-1}y)}$ to $\overline{(0,y)}$. We now compute that the bottom-left route around the diagram sends $(\log \chi(\gamma_{\mathbb{Q}_p}))^{-2}(x,y)$ to

$$\begin{aligned}
&\left\{((\gamma_{\mathbb{Q}_p}-1)^{-1}(\varphi-1)x, x), \left(0, \frac{\gamma_{\mathbb{Q}_p}^{-1}-1}{\gamma_{\mathbb{Q}_p}-1}y\right)\right\}_{\gamma_{\mathbb{Q}_p},1} \\
&\quad = \left\{(\gamma_{\mathbb{Q}_p}-1)^{-1}(\varphi-1)x, \frac{\gamma_{\mathbb{Q}_p}^{-1}-1}{\gamma_{\mathbb{Q}_p}-1}y\right\}_{\gamma_{\mathbb{Q}_p}} \\
&\quad = \{(\gamma_{\mathbb{Q}_p}^{-1}-1)^{-1}(\varphi-1)x, y\}_{\gamma_{\mathbb{Q}_p}} \\
&\quad = \{(\gamma_{\mathbb{Q}_p}^{-1}-1)^{-1}(\varphi-1)x, (1-\varphi)y\}_{\gamma_{\mathbb{Q}_p}} \\
&\quad = (\log \chi(\gamma_{\mathbb{Q}_p}))^{-1}\{(\gamma_{\mathbb{Q}_p}^{-1}-1)^{-1}(\varphi-1)x, (1-\varphi)y\}_{\mathbb{Q}_p} \\
&\quad = (\log \chi(\gamma_{\mathbb{Q}_p}))^{-1} \operatorname{Res}_\Gamma\left(\frac{-1}{\gamma_{\mathbb{Q}_p}^{-1}-1}\{x,y\}_{\mathrm{Iw},\gamma_{\mathbb{Q}_p}}\omega_{\log\circ\chi}\right),
\end{aligned}$$

where the first equality is by definition of the residue pairing, the second equality is because of the identity $\{x, \gamma y\}_{\gamma_{\mathbb{Q}_p}} = \{\gamma^{-1}x, y\}_{\gamma_{\mathbb{Q}_p}}$, the third equality is because $(\varphi-1)x$ belongs to the orthogonal complement $M^{\psi=0}$ of $\varphi(M^*)$, and the final two equalities are by the definitions of normalized trace and the Iwasawa pairings, respectively.

The claim of the proposition reduces to showing that for any $f \in \mathcal{R}_A^\infty(\Gamma)$ the quantity

$$(\log \chi(\gamma_{\mathbb{Q}_p})) \operatorname{Res}_\Gamma\left(\frac{-1}{\gamma_{\mathbb{Q}_p}^{-1}-1} f\omega_{\log\circ\chi}\right) \in A$$

is equal to the image of f under the natural projection to $\mathcal{R}_A^\infty(\Gamma)/(\gamma_{\mathbb{Q}_p}-1) \cong A$. To see this in general, it suffices to take $f=1$, in which case

$$\begin{aligned}
(\log \chi(\gamma_{\mathbb{Q}_p})) \operatorname{Res}_\Gamma\left(\frac{-1}{\gamma_{\mathbb{Q}_p}^{-1}-1}\omega_{\log\circ\chi}\right) &= (\log \chi(\gamma_{\mathbb{Q}_p})) \operatorname{Res}_\Gamma\left(\frac{\gamma_{\mathbb{Q}_p}}{\gamma_{\mathbb{Q}_p}-1}\frac{d\gamma_{\mathbb{Q}_p}}{(\log \chi(\gamma_{\mathbb{Q}_p}))\gamma_{\mathbb{Q}_p}}\right) \\
&= \operatorname{Res}_\Gamma\left(\frac{d(\gamma_{\mathbb{Q}_p}-1)}{\gamma_{\mathbb{Q}_p}-1}\right) = 1,
\end{aligned}$$

as was desired. □

4.3. **Iwasawa cohomology over a field.** In this subsection, we study properties of Iwasawa cohomology for (φ,Γ)-modules over the standard Robba rings. Most of the results are already included in [43]; we reproduce them here for the convenience of the reader.

Hypothesis 4.3.1. In this subsection, assume that A is a finite extension of $\mathbb{Q}_p$ and M is a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$.

Notation 4.3.2. Since we will be making use of the exact sequence $0 \to M^{\varphi=1} \to M^{\psi=1} \xrightarrow{\varphi-1} M^{\psi=0}$, we set $\mathscr{C} = (\varphi-1)M^{\psi=1} \subseteq M^{\psi=0}$.

Remark 4.3.3. Under this hypothesis, the rings $R = \mathcal{R}_A^{r_0}(\pi_K)$ and $\mathcal{R}_A^\infty(\Gamma_K)$ are products of Bézout domains that are one-dimensional Fréchet-Stein algebras, and the rings $R = \mathcal{R}_A(\pi_K)$ and $\mathcal{R}_A(\Gamma_K)$ are direct limits of such rings. Therefore, any finitely generated or coadmissible, torsion-free R-module is automatically finite free on each connected component of $\operatorname{Spec} R$.

Lemma 4.3.4. *An $\mathcal{R}_A^\infty(\Gamma_K)$-module N is finitely generated and coadmissible if and only if it lies in $\mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma_K))$.*

Proof. The backward implication follows from Lemma 4.1.4. We now prove the forward implication. Assume that N is generated by n elements, giving rise to an exact sequence $0 \to \mathrm{Ker} \to \mathcal{R}_A^\infty(\Gamma_K)^{\oplus n} \to N \to 0$. By Lemma 2.1.4(5), Ker is coadmissible. By Remark 4.3.3, the torsion-free coadmissible module Ker is finite projective over $\mathcal{R}_A^\infty(\Gamma_K)$. This gives a finite resolution of N. □

Lemma 4.3.5. *The A-module $M^{\varphi=1}$ is finite.*

Note that although we assumed A is a field, the proof applies whenever A is a strongly noetherian Banach algebra over $\mathbb{Q}_p$.

Proof. Recall that the residue pairing $\{-,-\}_{\mathbb{Q}_p} : M \times M^* \to A$ given in Notation 2.3.13 is nondegenerate and thus induces an injective map $M^{\varphi=1} \to \mathrm{Hom}_A(M^*, A)$. However, this map factors through $\mathrm{Hom}_A(M^*/(\psi-1), A)$ due to the identity $\{\varphi(x), y\} = \{x, \psi(y)\}$. Since $M^*/(\psi-1)$ is finite over A by Proposition 3.3.2(1), so then is $M^{\varphi=1}$. □

Proposition 4.3.6. *The complex $\mathrm{C}^\bullet_\psi(M)$ lies in $\mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma_K))$.*

This argument uses some basic properties of slope filtrations for φ-modules, for which one should see [33, Theorem 1.7.1].

Proof. First, we have $\mathrm{C}^\bullet_\psi(\mathrm{Ind}_K^{\mathbb{Q}_p} M) = \mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma], \mathrm{C}^\bullet_\psi(M))$ and hence it suffices to prove the proposition for $K = \mathbb{Q}_p$. Since $M/(\psi-1)$ is a finite A-module by Proposition 3.3.2(1), it lies in $\mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$ by Lemma 4.1.6. Now, we prove $M^{\psi=1} \in \mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$. By dévissage and slope filtrations, one may assume that M is of pure slope. When M is étale, this is well known (e.g., [20, Théorème I.5.2 and Proposition V.1.18], plus the well-known identity $\mathbf{D}(\mathbf{V}(M))^{\varphi=1} = \mathbf{V}(M)^{H_K}$). For general integer slopes, it follows from comparing $\mathrm{C}^\bullet_\psi(M)$ with $\mathrm{C}^\bullet_\psi(t^n M)$ for $n \in \mathbb{Z}$ and applying Proposition 3.2.5.

When the slope of M is c/d with c, d integers, we fix $d \in \mathbb{N}$ and do induction on the residue $c \pmod d$ with known case $c = 0$. Assume the statement is known for some $c-1$ and we check it for c. By Propositions 3.2.5 and 3.3.2(2), we may replace M by $t^{-n}M$ for $n \gg 0$, so that $(t^i M)/(\psi-1) = 0$ for $i = 0, \pm 1$.

Recall that [36, Lemma 5.2] gives a (φ,Γ)-module E over $\mathcal{R}$ of pure slope $-1/d$ which is a successive extension of $d-1$ copies of $\mathcal{R}$ with a copy of $t\mathcal{R}$ as the subobject. Since $M \otimes_{\mathcal{R}} E$ is pure of slope equal to that of M minus $1/d$, the inductive hypothesis gives $(M \otimes_{\mathcal{R}} E)^{\psi=1} \in \mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$, and hence $(M \otimes_{\mathcal{R}} E)^{\psi=1}$ is a finitely generated $\mathcal{R}_A^\infty(\Gamma)$-module. This module, by the vanishing of cokernels of $\psi - 1$, is a successive extension of $M^{\psi=1}$ and $(tM)^{\psi=1}$ with $d-1$ copies of $M^{\psi=1}$ as the quotient object. In particular, this implies that $M^{\psi=1}$ is a finitely generated $\mathcal{R}_A^\infty(\Gamma)$-module. Running the same argument with $t^{-1}M$ in place of M shows that $M^{\psi=1}$, which is already known to be finitely generated, is an $\mathcal{R}_A^\infty(\Gamma)$-submodule of the coadmissible module $(t^{-1}M \otimes_{\mathcal{R}} E)^{\psi=1}$. By Lemma 2.1.4(6), $M^{\psi=1}$ is also coadmissible, and hence lies in $\mathbf{D}^\flat_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$ by Lemma 4.3.4. This finishes the induction and the proof. □

Corollary 4.3.7. *For M a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, the subspace $\mathscr{C} \subseteq M^{\psi=0}$ is a finite free $\mathcal{R}_A^\infty(\Gamma_K)$-module of rank equal to $[K : \mathbb{Q}_p]\,\mathrm{rank}\, M$. Moreover,* *the cohomology groups* $H^i_\psi(M)$ *for* $i = 0, 1, 2$ *are of respective generic ranks* 0, $[K : \mathbb{Q}_p]\operatorname{rank} M, 0$ *as* $\mathcal{R}^\infty_A(\Gamma_K)$*-modules.*

Proof. Again by induction it suffices to prove the proposition for $K = \mathbb{Q}_p$. By Proposition 4.3.6, $\mathscr{C}$ is finitely generated over $\mathcal{R}^\infty_A(\Gamma)$. Sitting inside $M^{\psi=0}$, a finite projective $\mathcal{R}_A(\Gamma)$-module, $\mathscr{C}$ is torsion-free and hence finite free over each connected component of $\operatorname{Spec}\mathcal{R}^\infty_A(\Gamma)$. We compute its rank as follows. By Proposition 3.3.2(1) and Lemma 4.3.5, the two modules $M/(\psi - 1)$ and $M^{\varphi=1}$ have isolated support. It follows that there exists $\eta \in \operatorname{Max}(\mathcal{R}^\infty_A(\Gamma))$ in each connected component of $\operatorname{Spec}\mathcal{R}^\infty_A(\Gamma)$ such that $H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(M \otimes \eta^{-1}) = H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(M \otimes \eta^{-1}) = 0$ and $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M \otimes \eta^{-1}) \cong M^{\psi=1}/\mathfrak{m}_\eta \simeq \mathscr{C}/\mathfrak{m}_\eta$. By Theorem 2.3.11(2), $\mathscr{C}$ has rank equal to $[K : \mathbb{Q}_p]\operatorname{rank} M$. The generic rank computation in degree 0 is obvious, and in degree 2 follows from the fact that $M/(\psi - 1)$ is a finite A-module as noted above. To see the correct rank in degree 1, it suffices to note that, in the exact sequence

$$0 \to M^{\varphi=1} \to M^{\psi=1} \to \mathscr{C} \to 0,$$

we have already shown $M^{\varphi=1}$ to be a finite A-module and $\mathscr{C}$ to have generic rank $[K : \mathbb{Q}_p]\operatorname{rank} M$. □

Proposition 4.3.8. *Assume* $K = \mathbb{Q}_p$*, and let* M *be a* (φ, Γ)*-module over* $\mathcal{R}_A$ *such that* $M/(\psi - 1) = M^*/(\psi - 1) = 0$.

(1) *The natural maps* $\varphi - 1 : M^{\psi=1} \to M^{\psi=0}$ *and* $\varphi - 1 : (M^*)^{\psi=1} \to (M^*)^{\psi=0}$ *are injective.*

(2) *The Iwasawa pairing* $\{-, -\}_{\mathrm{Iw}}$ *is perfect, inducing an isomorphism between two finite free* $\mathcal{R}^\infty_A(\Gamma)$*-modules:*

$$(M^*)^{\psi=1} \cong \operatorname{Hom}_{\mathcal{R}^\infty_A(\Gamma)}\big(M^{\psi=1}, \mathcal{R}^\infty_A(\Gamma)\big)^\iota.$$

(3) *We have an isomorphism of* $\mathcal{R}^{r_0}_A(\Gamma)$*-modules* $\mathscr{C} \otimes_{\mathcal{R}^\infty_A(\Gamma)} \mathcal{R}^{r_0}_A(\Gamma) \cong (M^{r_0})^{\psi=0}$ *with the precise* $r_0 > 0$ *for which Corollary 4.2.6 holds.*

Proof. (1) Vanishing of the cokernels of $\psi - 1$ implies that for any $\eta \in \operatorname{Max}(\mathcal{R}^\infty_A(\Gamma))$, $H^2_{\psi,\gamma_{\mathbb{Q}_p}}(M \otimes \eta^{-1}) = H^2_{\psi,\gamma_{\mathbb{Q}_p}}(M^* \otimes \eta^{-1}) = 0$. By Tate duality (Theorem 2.3.11(3)), $H^0_{\psi,\gamma_{\mathbb{Q}_p}}(M \otimes \eta^{-1}) = H^0_{\psi,\gamma_{\mathbb{Q}_p}}(M^* \otimes \eta^{-1}) = 0$ for any η. Note that, by Lemma 4.3.5, $M^{\varphi=1}$ and $(M^*)^{\varphi=1}$ are unions of finite A-modules with a Γ-action. So the vanishing of H^0 for all twists of M and M^* implies that $M^{\varphi=1} = 0$ and $(M^*)^{\varphi=1} = 0$, forcing $\varphi - 1$ to be injective.

(2) Vanishing of the cokernels of $\psi - 1$ implies that the two vertical injections in Proposition 4.2.9 are actually isomorphisms. This means that $M^{\psi=1}$ and $(M^*)^{\psi=1}$ are dual to each other when reduced modulo $\mathfrak{m}_\eta$ for each $\eta \in \operatorname{Max}(\mathcal{R}^\infty_A(\Gamma))$. Since both $M^{\psi=1}$ and $(M^*)^{\psi=1}$ are finite free over $\mathcal{R}^\infty_A(\Gamma)$ (Corollary 4.3.7), they are in perfect duality with one another.

(3) By (2) and Corollary 4.2.6, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathscr{C}^* \otimes_{\mathcal{R}^\infty_A(\Gamma)} \mathcal{R}^{r_0}_A(\Gamma) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{R}^{r_0}_A(\Gamma)}\big(\mathscr{C} \otimes_{\mathcal{R}^\infty_A(\Gamma)} \mathcal{R}^{r_0}_A(\Gamma), \mathcal{R}^{r_0}_A(\Gamma)\big)^\iota \\
\downarrow & & \uparrow \\
(M^{*r_0})^{\psi=0} & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{R}^{r_0}_A(\Gamma)}\big((M^{r_0})^{\psi=0}, \mathcal{R}^{r_0}_A(\Gamma)\big)^\iota
\end{array}$$

This implies that the $\mathcal{R}_A^{r_0}(\Gamma)$-module in the top line is a direct summand of the $\mathcal{R}_A^{r_0}(\Gamma)$-module in the bottom line. Since they are both free of same rank $[K:\mathbb{Q}_p]$ rank M by Corollary 4.3.7 and Theorem 3.1.1, they must be isomorphic. □

Remark 4.3.9. It will be important in the proof of Proposition 5.2.10 that we have a choice of r_0 in Proposition 4.3.8 that is uniform with respect to points $z \in \mathrm{Max}(A)$.

4.4. **Statement of main results.** The main result of this paper is the following.

Theorem 4.4.1. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$. We have $\mathrm{C}_\psi^\bullet(M) \in \mathbf{D}_{\mathrm{perf}}^-(\mathcal{R}_A^\infty(\Gamma_K))$.*

The proof of this theorem takes up Section 5; see specifically the very end of subsection 5.3. We first list a few useful corollaries of the theorem, including the compatibility with base change, and the comparison between the Iwasawa cohomology and the (φ, Γ)-cohomology of the cyclotomic deformation.

Keeping in mind Proposition 4.2.3, the following theorem is an immediate consequence of the main result.

Theorem 4.4.2. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$. Then we have $\mathrm{C}_{\varphi,\gamma_K}^\bullet(M) \in \mathbf{D}_{\mathrm{perf}}^-(A)$. In particular, the cohomology groups $H_{\varphi,\gamma_K}^i(M)$ are finite A-modules.*

The following theorem is essentially [42, Proposition 2.6]. But having proved Theorems 4.4.1 and 4.4.2, we can state it as an unconditional result, which we reproduce here for the convenience of the reader.

Theorem 4.4.3. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, and let $A \to B$ be a morphism of $\mathbb{Q}_p$-affinoid algebras.*

(1) *The canonical morphism*

$$\mathrm{C}_\psi^\bullet(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}_A^\infty(\Gamma_K)} \mathcal{R}_B^\infty(\Gamma_K) \longrightarrow \mathrm{C}_\psi^\bullet(M \widehat{\otimes}_A B)$$

is a quasi-isomorphism.

(2) *The canonical morphism $\mathrm{C}_{\varphi,\gamma_K}^\bullet(M) \overset{\mathbf{L}}{\otimes}_A B \to \mathrm{C}_{\varphi,\gamma_K}^\bullet(M \widehat{\otimes}_A B)$ is a quasi-isomorphism.*

In particular, if B is flat over A, we have

$$H_\psi^i(M) \otimes_{\mathcal{R}_A^\infty(\Gamma_K)} \mathcal{R}_B^\infty(\Gamma_K) \cong H_\psi^i(M \widehat{\otimes}_A B) \quad \text{and} \quad H_{\varphi,\gamma_K}^i(M) \otimes_A B \cong H_{\varphi,\gamma_K}^i(M \widehat{\otimes}_A B).$$

Proof. When $B = A/\mathfrak{m}_z$ for some $z \in \mathrm{Max}(A)$, $M \widehat{\otimes}_A B \cong M \otimes_A B$ and $\mathcal{R}_B^\infty(\Gamma_K) \cong \mathcal{R}_A^\infty(\Gamma_K) \otimes_A B$ as the latter objects are already complete. In this case, both (1) and (2) are tautologies because M is flat over A by Corollary 2.1.5.

We now consider the general case. In case (2), applying the discussion above to every closed point $y \in \mathrm{Max}(B)$, we have a quasi-isomorphism

$$\begin{aligned}\big(\mathrm{C}_{\varphi,\gamma_K}^\bullet(M) \overset{\mathbf{L}}{\otimes}_A B\big) \overset{\mathbf{L}}{\otimes}_B B/\mathfrak{m}_y \xrightarrow{\sim} \mathrm{C}_{\varphi,\gamma_K}^\bullet(M) \overset{\mathbf{L}}{\otimes}_A B/\mathfrak{m}_y \xrightarrow{\sim} \mathrm{C}_{\varphi,\gamma_K}^\bullet(M \otimes_A B/\mathfrak{m}_y) \\ \xrightarrow{\sim} \mathrm{C}_{\varphi,\gamma_K}^\bullet(M \widehat{\otimes}_A B) \overset{\mathbf{L}}{\otimes}_B B/\mathfrak{m}_y.\end{aligned}$$

Then (2) follows from the result on detecting quasi-isomorphisms pointwise (Lemma 4.1.5).

In case (1), we know both sides lie in $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_B(\Gamma_K))$ by Theorem 4.4.1 and hence all cohomology groups are coadmissible. By Lemma 2.1.4(2), it suffices to prove that, for any $s > 0$, the natural morphism

$$\mathrm{C}^\bullet_\psi(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_A(\Gamma_K)} \mathcal{R}^{[s,\infty]}_B(\Gamma_K) \longrightarrow \mathrm{C}^\bullet_\psi(M\widehat{\otimes}_A B) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_B(\Gamma_K)} \mathcal{R}^{[s,\infty]}_B(\Gamma_K) \tag{4.4.3.1}$$

is a quasi-isomorphism. Similar to (2), for each $(y,\eta) \in \mathrm{Max}(\mathcal{R}^\infty_B(\Gamma_K))$, the natural morphism

$$\begin{aligned}\mathrm{C}^\bullet_\psi(M) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_A(\Gamma_K)} \mathcal{R}^\infty_B(\Gamma_K) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_B(\Gamma_K)} \mathcal{R}^\infty_B(\Gamma_K)/\mathfrak{m}_{(y,\eta)} \\ \longrightarrow \mathrm{C}^\bullet_\psi(M\widehat{\otimes}_A B) \overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_B(\Gamma_K)} \mathcal{R}^\infty_B(\Gamma_K)/\mathfrak{m}_{(y,\eta)}\end{aligned}$$

is a quasi-isomorphism. Applying Lemma 4.1.5 to (4.4.3.1) proves (1). □

Remark 4.4.4. Using Theorem 4.4.3, as we restrict M over varying affinoid subdomains of $\mathrm{Max}(A)$, each cohomology group $H^*_{\varphi,\gamma_K}(M)$ yields a coherent sheaf on $\mathrm{Max}(A)$ by Kiehl's theorem. Similarly, each Iwasawa cohomology $H^*_\psi(M)$ yields a coherent sheaf on $\mathrm{Max}(\mathcal{R}^\infty_A(\Gamma_K))$.

With unconditional base change in hand, we may prove more precise versions of the preceding finiteness results.

Theorem 4.4.5. *Let M be a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$.*

(1) *We have $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \in \mathbf{D}^{[0,2]}_{\mathrm{perf}}(A)$.*
(2) *(Euler characteristic formula) We have $\chi_A\big(\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)\big) = -[K:\mathbb{Q}_p]\,\mathrm{rank}\, M$.*
(3) *(Tate duality) The Tate duality pairing* (2.3.10.1) *induces a quasi-isomorphism*

$$\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A(\mathrm{C}^\bullet_{\varphi,\gamma_K}(M^*),A)[-2]. \tag{4.4.5.1}$$

Proof. (1) By Theorem 4.4.2, we have $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \in \mathbf{D}^-_{\mathrm{perf}}(A)$. Since $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ is concentrated in degrees 0, 1, and 2, by Lemma 4.1.4, this implies that $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ is quasi-isomorphic to a complex $[\cdots \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2]$ of finite projective A-modules, and hence quasi-isomorphic to the complex $[\mathrm{Coker}(d^{-1}) \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2]$. By Corollary 2.1.5, the complex $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ consists of flat A-modules. This forces $\mathrm{Coker}(d^{-1})$ to be also flat (and hence projective) over A, by Lemma 4.1.3. Hence, $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \in \mathbf{D}^{[0,2]}_{\mathrm{perf}}(A)$, as claimed.

(2) To check the Euler characteristic formula, it suffices to check it at one closed point on each connected component. This follows from Theorem 2.3.11(2).

(3) It is clear that (2.3.10.1) induces the natural morphism (4.4.5.1). By Theorem 2.3.11(3), (4.4.5.1) is a quasi-isomorphism when tensored with $A/\mathfrak{m}_x$ for any $x \in \mathrm{Max}(A)$. Our statement then follows from Lemma 4.1.5. □

We may use Theorem 4.4.5 to strengthen Theorem 4.4.1.

Theorem 4.4.6. *Let M be a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$.*

(1) *We have $\mathrm{C}^\bullet_\psi(M) \in \mathbf{D}^{[0,2]}_{\mathrm{perf}}(\mathcal{R}^\infty_A(\Gamma_K))$.*
(2) *The $\mathcal{R}^\infty_A(\Gamma_K)$-module $M^{\psi=1}$ is finitely presented.*

(3) *(Euler characteristic formula) We have* $\chi_{\mathcal{R}_A^\infty(\Gamma_K)}(\mathrm{C}_\psi^\bullet(M)) = -[K:\mathbb{Q}_p]\operatorname{rank} M$. *Moreover,* $H_\psi^0(M)=0$*, and as* $\mathcal{R}_A^\infty(\Gamma_K)$*-modules* $H_\psi^2(M)$ *is torsion and* $H_\psi^1(M)$ *has generic rank* $[K:\mathbb{Q}_p]\operatorname{rank} M$*. (Note this does not force* $\mathrm{C}_\psi^\bullet(M)\in\mathbf{D}_{\mathrm{perf}}^{[1,2]}(\mathcal{R}_A^\infty(\Gamma_K))$ *because, for example, there may be torsion in* $H_\psi^1(M)$*.)*

Proof. (1) Since $\mathrm{C}_\psi^\bullet(M)$ is concentrated in degrees 1 and 2, Lemma 4.1.4 implies that it is quasi-isomorphic to a complex $[\cdots \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2]$ of finite projective $\mathcal{R}_A^\infty(\Gamma_K)$-modules, and hence to $[Q\to P^1\to P^2]$ with $Q=\operatorname{Coker} d^{-1}$. To prove the first statement, we need to show that Q is a projective $\mathcal{R}_A^\infty(\Gamma_K)$-module (note that $\operatorname{Coker} d^0$ may not be projective in general and hence $\mathrm{C}_\psi^\bullet(M)\notin\mathbf{D}_{\mathrm{perf}}^{[1,2]}(\mathcal{R}_A^\infty(\Gamma_K))$). By Lemma 2.1.4(5), Q is a coadmissible $\mathcal{R}_A^\infty(\Gamma_K)$-module. By Lemma 2.1.6, it suffices to prove that Q gives a vector bundle over $\mathcal{R}_A^\infty(\Gamma_K)$, or equivalently $Q\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^{[s,\infty]}(\Gamma_K)$ is a finite flat $\mathcal{R}_A^{[s,\infty]}(\Gamma_K)$-module for all $s>0$. For this, we may assume that A is geometrically connected.

For any $\eta\in\operatorname{Max}(\mathcal{R}^\infty(\Gamma_K))$, we have $\mathrm{C}_\psi^\bullet(M)\overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty(\Gamma_K)}\mathcal{R}^\infty(\Gamma_K)/\mathfrak{m}_\eta\cong\mathrm{C}_{\psi,\gamma_K}^\bullet(M\otimes\eta^{-1})$ by Proposition 4.2.3. Since $P^\bullet$ consists of finite projective $\mathcal{R}_A^\infty(\Gamma_K)$-modules, it follows that the complex $P^\bullet\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta$ represents $\mathrm{C}_\psi^\bullet(M)\overset{\mathbf{L}}{\otimes}_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta$. But $P^\bullet\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta$ consists of finite projective $\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta$-modules, so there exists a morphism of complexes $P^\bullet\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta\to\mathrm{C}_{\psi,\gamma_K}^\bullet(M\otimes\eta^{-1})$ inducing an isomorphism on cohomology. Lemma 4.1.3 and Theorem 4.4.5 then show $Q\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta$ to be finite projective over $\mathcal{R}_A^\infty(\Gamma_K)/\mathfrak{m}_\eta\cong A\otimes_{\mathbb{Q}_p}\kappa_\eta$. Moreover, by Theorem 4.4.5(2) and the flatness of P^1 and P^2 over $\mathcal{R}_A^\infty(\Gamma_K)$, we know that $\eta\mapsto\operatorname{rank}_{A\otimes_{\mathbb{Q}_p}\kappa_\eta}(Q/\mathfrak{m}_\eta)$ defines a locally constant function on $\operatorname{Max}(\mathcal{R}^\infty(\Gamma_K))$ as η varies. By Lemma 2.1.8(2), $Q\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^{[s,\infty]}(\Gamma_K)$ is a finite flat $\mathcal{R}_A^{[s,\infty]}(\Gamma_K)$-module for any $s>0$. (1) follows.

(2) As $M^{\psi=1}$ is a cohomology group of the complex $\mathrm{C}_\psi^\bullet(M)\in\mathbf{D}_{\mathrm{perf}}^{[0,2]}(\mathcal{R}_A^\infty(\Gamma_K))$, it is coadmissible by Lemma 2.1.4. By Proposition 3.3.2, $M/(\psi-1)$ is finite over A. By Lemma 2.1.18, there exists $r>0$ such that $M/(\psi-1)\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^r(\Gamma_K)=0$. By part (1) above, this means that $M^{\psi=1}\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^r(\Gamma_K)$ is the only nonzero cohomology group of a complex in $\mathbf{D}_{\mathrm{perf}}^{[0,2]}(\mathcal{R}_A^r(\Gamma_K))$. In particular, it is finitely presented as an $\mathcal{R}_A^r(\Gamma_K)$-module. By Lemma 2.1.4(3) the coherent sheaf on $\mathcal{R}_A^r(\Gamma_K)$ determined by $M^{\psi=1}\otimes_{\mathcal{R}_A^\infty(\Gamma_K)}\mathcal{R}_A^r(\Gamma_K)$ is then uniformly finitely presented, and combining this with the fact that $\mathcal{R}_A^{[r,\infty]}(\Gamma_K)$ is noetherian, the coherent sheaf on $\mathcal{R}_A^\infty(\Gamma_K)$ determined by $M^{\psi=1}$ is uniformly finitely presented over $\mathcal{R}_A^\infty(\Gamma_K)$. By Proposition 2.1.13(2), we deduce that $M^{\psi=1}$ is a finitely presented $\mathcal{R}_A^\infty(\Gamma_K)$-module.

(3) Since the Euler characteristic can be tested at a closed point on each connected component, the first claim follows from Theorem 2.3.11(2). For the second claim, one may base change to any sufficiently general closed point to assume that A is a finite field extension of $\mathbb{Q}_p$, in which case the claim is treated by Corollary 4.3.7. □

We now explain the relation between the Iwasawa cohomology of M and the (φ,Γ)-cohomology of the cyclotomic deformation of M, following [43].

Definition 4.4.7. For $n \geq 1$, let $\Lambda_n = \mathcal{R}^{[1/p^n,\infty]}(\Gamma_K)$ and let $X_n = \mathrm{Max}(\Lambda_n)$. We view $X = \cup_{n\geq 1} X_n$ as a quasi-Stein rigid analytic space, which is a disjoint union of open discs. For $n \geq 1$, consider the rank one (φ, Γ_K)-module $\mathbf{Dfm}_n = \Lambda_n \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}(\pi_K)\mathbf{e} = \mathcal{R}_{\Lambda_n}(\pi_K)\mathbf{e}$, where $\varphi(1 \otimes \mathbf{e}) = 1 \otimes \mathbf{e}$ and $\gamma(1\otimes \mathbf{e}) = \gamma^{-1} \otimes \mathbf{e}$ for $\gamma \in \Gamma_K$. We put $\mathbf{Dfm} = \varprojlim_n \mathbf{Dfm}_n$; this is a (φ, Γ_K)-module over the relative Robba ring over X. For a closed point $\eta \in \mathrm{Max}(\Lambda_n)$, we have $\mathbf{Dfm}/\mathfrak{m}_\eta \cong \mathbf{Dfm}_n/\mathfrak{m}_\eta \cong \mathcal{R}(\pi_K)(\eta^{-1})$, where $\mathcal{R}(\pi_K)(\eta^{-1})$ is the (φ, Γ_K)-module associated to η.

For M a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, we define the *cyclotomic deformation of* M to be $\mathbf{Dfm}(M) = M \widehat{\otimes}_{\mathcal{R}(\pi_K)} \mathbf{Dfm} = \varprojlim_n M \widehat{\otimes}_{\mathcal{R}(\pi_K)} \mathbf{Dfm}_n$; it is a (φ, Γ_K)-module over the relative Robba ring over $\mathrm{Max}(A) \times X$.

Theorem 4.4.8. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$. Then we have a natural quasi-isomorphism of complexes of $\mathcal{R}^\infty_A(\Gamma_K)$-modules*

$$\mathrm{C}^\bullet_{\psi,\gamma_K}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \mathrm{C}^\bullet_\psi(M), \tag{4.4.8.1}$$

where Γ_K acts on the left through the $\mathcal{R}_{\Lambda_n}(\pi_K)$-structures on the respective $\mathbf{Dfm}_n$, and on the right via the (φ, Γ_K)-actions on M. Consequently, we have a natural Iwasawa duality quasi-isomorphism

$$\mathrm{C}^\bullet_\psi(M^*) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathcal{R}^\infty_A(\Gamma_K)}\big(\mathrm{C}^\bullet_\psi(M), \mathcal{R}^\infty_A(\Gamma_K)\big)^\iota[-2]. \tag{4.4.8.2}$$

Proof. The isomorphisms $(\mathbf{Dfm}(M)^{\Delta_K})^{\gamma_K=1} = \mathbf{Dfm}(M)^{\Gamma_K} = 0$ and

$$\mathbf{Dfm}(M)^{\Delta_K}/(\gamma_K - 1) = (M \widehat{\otimes}_{\mathcal{R}(\pi_K)} \mathbf{Dfm})^{\Delta_K}/(\gamma_K \otimes \gamma_K - 1) \cong M$$

can be packaged into a quasi-isomorphism $[\mathbf{Dfm}(M)^{\Delta_K} \xrightarrow{\gamma_K - 1} \mathbf{Dfm}(M)^{\Delta_K}] \to M[-1]$. Taking the mapping fiber of $1 - \psi$ on both sides and adjusting signs and degrees, one obtains the quasi-isomorphism $\mathrm{C}^\bullet_{\psi,\gamma_K}(\mathbf{Dfm}(M)) \to \mathrm{C}^\bullet_\psi(M)$.

The second statement follows from combining the first statement with Theorem 4.4.5(3). $\square$

Remark 4.4.9. It would be interesting to know if one can define the Iwasawa duality morphism (4.4.8.2) directly on the level of complexes without comparing it to the cyclotomic deformation.

Remark 4.4.10. One can give a direct proof of the following corollary without referring to our main theorem, but it would require some setup on the construction of the functor $\mathbf{D}_{\mathrm{rig}}$. In the case $A = \mathbb{Q}_p$, the comparison of the Iwasawa cohomology of V to that of its cyclotomic deformation (over $\mathbb{Z}_p[\![\Gamma_K]\!]$, not $\mathcal{R}^\infty(\Gamma_K)$) is essentially a variant of Shapiro's lemma, and to our knowledge first occurs in Iwasawa theory in works of Greenberg [27]. It also occurs in works of Colmez [18], together with consideration of the cyclotomic deformation over the subring of $\mathcal{R}^\infty(\Gamma_K)$ of tempered distributions. The claim over the full ring $\mathcal{R}^\infty(\Gamma_K)$ appears in [43].

Corollary 4.4.11. *Let V be a finite projective A-module equipped with a continuous A-linear action of G_K. If we use $H^i_{\mathrm{Iw}}(G_K, V)$ to denote the inverse limit $[\varprojlim_{n\to\infty} H^i(G_{K(\mu_{p^n})}, T)] \otimes \mathbb{Q}$ (where $T \subseteq V$ is the unit ball for a Galois-invariant Banach module norm) of the cohomology groups under the corestriction maps for any i, we have a functorial isomorphism $H^i_{\mathrm{Iw}}(G_K, V) \widehat{\otimes}_{\mathbb{Z}_p[\![\Gamma_K]\!]} \mathcal{R}^\infty(\Gamma_K) \cong H^i_\psi(\mathbf{D}_{\mathrm{rig}}(V))$ of $\mathcal{R}^\infty_A(\Gamma_K)$-modules for any i compatible with base change.*

Proof. By Shapiro's lemma, $H^i_{\mathrm{Iw}}(G_K,V)\cong H^i(G_K,V\widehat{\otimes}_{\mathbb{Z}_p}\tilde\Lambda)$, where $\tilde\Lambda=\mathbb{Z}_p[\![\Gamma]\!]$ with the Galois action factoring through Γ and via ι. The corollary follows from a sequence of isomorphisms

$$H^i_{\mathrm{Iw}}(G_K,V)\otimes_{\Lambda(\Gamma_K)}\mathcal{R}^\infty(\Gamma_K)\xrightarrow{[42,\ \text{Theorem 1.6}]}H^i(G_K,V\widehat{\otimes}_{\mathbb{Q}_p}\varprojlim_n\Lambda_n)$$
$$\xrightarrow{\text{Theorem 2.3.7}}H^i_{\varphi,\gamma_K}(\mathbf{D}_{\mathrm{rig}}(V)\widehat{\otimes}_{\mathcal{R}(\pi_K)}\mathbf{Dfm})\xrightarrow{\text{Theorem 4.4.8}}H^i_\psi(\mathbf{D}_{\mathrm{rig}}(V)).$$

□

5. Proof of the main theorem

We now complete the arguments of the previous section by proving finiteness of Iwasawa cohomology (Theorem 4.4.1). The reader will note that, in order to avoid any vicious circles, we refrain from invoking any results from subsection 4.4, although the rest of Section 4 is allowed. The central argument is to use the duality pairing to play the Iwasawa H^1 groups of a given (φ,Γ)-module and its Cartier dual against each other, forcing them both to be finite. This argument only works if both Iwasawa H^2 groups vanish; the remainder of the argument is a dévissage to reduce to this case, in the spirit of some arguments from [36] but not quite identical to any of them.

5.1. **Preliminary reductions.** We make some preliminary reductions for the proof of Theorem 4.4.1.

Lemma 5.1.1. *If Theorem* 4.4.1 *is true for* $K=\mathbb{Q}_p$ *(and any* (φ,Γ)*-module over* $\mathcal{R}_A$*), then it is true for any finite extension* K *of* $\mathbb{Q}_p$*.*

Proof. If M is a (φ,Γ_K)-module over $\mathcal{R}_A(\pi_K)$, then we have

$$\mathrm{C}^\bullet_\psi(\mathrm{Ind}_K^{\mathbb{Q}_p}M)=\mathrm{C}^\bullet_\psi\big(\mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma],M)\big)\cong\mathrm{Hom}_{\mathbb{Z}[\Gamma_K]}(\mathbb{Z}[\Gamma],\mathrm{C}^\bullet_\psi(M)).$$

Hence, Theorem 4.4.1 for $\mathrm{Ind}_K^{\mathbb{Q}_p}M$ implies that for M. □

Hypothesis 5.1.2. For the rest of this section, we assume that $K=\mathbb{Q}_p$. We hence write $\mathcal{R}_A$ for $\mathcal{R}_A(\pi_K)$, $\mathcal{R}_A(\Gamma)$ for $\mathcal{R}_A(\Gamma_K)$, and $\mathcal{R}^\infty_A(\Gamma)$ for $\mathcal{R}^\infty_A(\Gamma_K)$.

Lemma 5.1.3. *If* I *is an ideal of* A*, then a complex of* $\mathcal{R}^\infty_{A/I}(\Gamma)$*-modules in* $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_{A/I}(\Gamma))$*, when viewed as a complex of* $\mathcal{R}^\infty_A(\Gamma)$*-modules, lies in* $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_A(\Gamma))$*.*

Proof. Any resolution of A/I: $\cdots\to A^{\oplus n_1}\to A^{\oplus n_0}\to A/I\to 0$ by finite free A-modules is strict and hence induces an exact sequence

$$\cdots\to\mathcal{R}^\infty_A(\Gamma)^{\oplus n_1}\to\mathcal{R}^\infty_A(\Gamma)^{\oplus n_0}\to\mathcal{R}^\infty_{A/I}(\Gamma)\to 0.$$

The lemma follows from this. □

Lemma 5.1.4. *Let* M *be a* (φ,Γ)*-module over* $\mathcal{R}_A$ *and let* B *be a* finite A*-algebra. Then we have a natural quasi-isomorphism*

$$\mathrm{C}^\bullet_\psi(M)\overset{\mathbf{L}}{\otimes}_{\mathcal{R}^\infty_A(\Gamma)}\mathcal{R}^\infty_B(\Gamma)\longrightarrow\mathrm{C}^\bullet_\psi(M\otimes_A B).\tag{5.1.4.1}$$

In particular, if Theorem 4.4.1 *holds for* M*, then it holds for* $M\otimes_A B\cong M\widehat{\otimes}_A B$*.*

Proof. Straightforward. □

Lemma 5.1.5. *Let A_{red} denote the reduced quotient of A. Let M be a (φ, Γ)-module over $\mathcal{R}_A$. If Theorem 4.4.1 holds for $M \otimes_A A_{\mathrm{red}}$ then it holds for M.*

Proof. By dévissage, we may write A as a finite successive extension of A-modules, each of which is isomorphic to A/I_i for some radical ideal I_i of A. Then $M_i = M \otimes_A A/I_i$ is a (φ, Γ)-module over $\mathcal{R}_{A/I_i}$. By Lemma 5.1.4 (noting that A/I_i is finite over A_{red}), Theorem 4.4.1 for $M \otimes_A A_{\mathrm{red}}$ implies Theorem 4.4.1 for each M_i. Therefore, each $\mathrm{C}^\bullet_\psi(M_i)$ belongs to $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_{A/I_i}(\Gamma))$ and thus belongs to $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_A(\Gamma))$ by Lemma 5.1.3. Now, $\mathrm{C}^\bullet_\psi(M)$, as a successive extension of complexes $\mathrm{C}^\bullet_\psi(M_i)$, must also lie in $\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}^\infty_A(\Gamma))$. ☐

Lemma 5.1.6. *For M a (φ, Γ)-module over $\mathcal{R}_A$ and any integer n, Theorem 4.4.1 holds for M if and only if it holds for $t^n M$.*

Proof. This follows from Proposition 3.2.5 by induction on n. ☐

5.2. **Main theorem in a special case.** Using the Iwasawa duality pairing we established in Definition 4.2.8 and its compatibility with the Tate local duality as discussed in Proposition 4.2.9, one may deduce the finite projectivity of $M^{\psi=1}$ over $\mathcal{R}^\infty_A(\Gamma)$ in the special case when $M/(\psi - 1) = M^*/(\psi - 1) = 0$. This is one of the key steps in proving Theorem 4.4.1. We will reduce the proof of the theorem to this special case in the next subsection.

We remind the reader that Hypothesis 5.1.2 is still in force.

Hypothesis 5.2.1. Throughout this subsection, we assume that A is a *reduced* $\mathbb{Q}_p$-affinoid algebra. We also take M to be a (φ, Γ)-module of constant rank d over $\mathcal{R}_A$ such that $M/(\psi - 1) = M^*/(\psi - 1) = 0$.

Lemma 5.2.2. *For any $(z, \eta) \in \mathrm{Max}(\mathcal{R}^\infty_A(\Gamma))$, we have $M^{\psi=1} \otimes_{\mathcal{R}^\infty_A(\Gamma)} \kappa_{(z,\eta)} \cong H^1_{\varphi, \gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}))$, and the same holds for M^* in place of M.*

Proof. Since $M/(\psi - 1) = 0$, the spectral sequence

$$E_2^{j,-i} = \mathrm{Tor}_i^{\mathcal{R}^\infty_A(\Gamma)}\left(H^j_\psi(M), \kappa_{(z,\eta)}\right) \Rightarrow H^{j-i}_{\psi, \gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1})$$

stabilizes at E_2, giving the isomorphism in the lemma. The same argument holds with M replaced by M^*. ☐

Lemma 5.2.3. *For any $(z, \eta) \in \mathrm{Max}(\mathcal{R}^\infty_A(\Gamma))$, we have $H^i_{\varphi, \gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) = H^i_{\varphi, \gamma_{\mathbb{Q}_p}}(M^*_z \otimes \eta) = 0$ for $i = 0, 2$. Hence, $\dim_{\kappa_{(z,\eta)}} H^1_{\varphi, \gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) = \dim_{\kappa_{(z,\eta)}} H^1_{\varphi, \gamma_{\mathbb{Q}_p}}(M^*_z \otimes \eta) = d$.*

Proof. For $i = 2$, this follows from the fact that $H^2_{\psi, \gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) = M/(\psi - 1, \mathfrak{m}_{(z,\eta)})$ and $H^2_{\psi, \gamma_{\mathbb{Q}_p}}(M^*_z \otimes \eta) = M^*/(\psi - 1, \mathfrak{m}_{(z,\eta^{-1})})$. The result for $H^0_{\varphi, \gamma_{\mathbb{Q}_p}}$ follows from Tate duality over $\kappa_{(z,\eta)}$ (Theorem 2.3.11(3)). The dimension of $H^1_{\varphi, \gamma_{\mathbb{Q}_p}}$ is computed using the Euler characteristic formula (Theorem 2.3.11(2)). ☐

Lemma 5.2.4. *For any $z \in \mathrm{Max}(A)$, we have $M_z/(\psi - 1) = M^*_z/(\psi - 1) = 0$. Hence $M_z^{\psi=1}$ and $(M^*_z)^{\psi=1,\iota}$ are free $\mathcal{R}^\infty_{\kappa_z}(\Gamma)$-modules of rank d and are perfect dual to each other.*

Proof. The first statement is obvious, while the second one follows from Proposition 4.3.8(2). ☐

Lemma 5.2.5. *We have injective homomorphisms*

$$M^{\psi=1} \hookrightarrow \prod_{(z,\eta)\in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) \quad \textit{and}$$
$$(M^*)^{\psi=1} \hookrightarrow \prod_{(z,\eta)\in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M^*_z \otimes \eta).$$

Proof. We indicate the proof of the first injection, and the second can be proved in the same way. We first observe that $\mathcal{R}_A \hookrightarrow \prod_{z\in\mathrm{Max}(A)} \mathcal{R}_{\kappa_z}$. It follows that $M \hookrightarrow \prod_{z\in\mathrm{Max}(A)} M_z$. As $\mathcal{R}_A^\infty(\Gamma)$-submodules, we have $M^{\psi=1} \hookrightarrow \prod_{z\in\mathrm{Max}(A)} M_z^{\psi=1}$. By Corollary 5.2.4, each $M_z^{\psi=1}$ is a finite free module over $\mathcal{R}^\infty_{\kappa_z}(\Gamma)$. Since $\mathcal{R}^\infty_{\kappa_z}(\Gamma) \hookrightarrow \prod_{\eta\in\mathrm{Max}(\mathcal{R}^\infty_{\kappa_z}(\Gamma))} \mathcal{R}^\infty_{\kappa_z}(\Gamma)/\mathfrak{m}_\eta$, we can continue the above inclusion for $M^{\psi=1}$ to get

$$M^{\psi=1} \hookrightarrow \prod_{(z,\eta)\in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} \left(M_z^{\psi=1}/\bar{\mathfrak{m}}_{(z,\eta)}\right) \cong \prod_{(z,\eta)\in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}).$$

□

Proposition 5.2.6. *Fix* $(z,\eta) \in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))$. *Suppose that* S *is a finite subset of* $M^{\psi=1}$ *generating* $M^{\psi=1}/\mathfrak{m}_{(z,\eta)} \cong H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z\otimes\eta^{-1})$. *Then there exists* $f \in \mathcal{R}_A^\infty(\Gamma)$ *which is nonzero modulo* $\mathfrak{m}_{(z,\eta)}$, *such that every element of* $fM^{\psi=1}$ *is equal to an* $\mathcal{R}_A^\infty(\Gamma)$*-linear combination of* S.

Proof. We may choose $S = \{\alpha_1,\ldots,\alpha_d\}$ so that it maps to a basis $S_{(z,\eta)}$ of $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z\otimes\eta^{-1})$. By Theorem 2.3.11(3), $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z^*\otimes\eta)$ may be identified with the dual space of $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z\otimes\eta^{-1})$; let $S^*_{(z,\eta^{-1})}$ denote the dual basis to $S_{(z,\eta)}$. We may thus lift $S^*_{(z,\eta^{-1})}$ to a finite subset $S^* = \{\beta_1,\ldots,\beta_d\}$ of $(M^*)^{\psi=1}$ by Lemma 5.2.2.

The $n\times n$ matrix F over $\mathcal{R}_A^\infty(\Gamma)$ defined by $F_{ij} = \{\alpha_i,\beta_j\}_{\mathrm{Iw}}$ reduces to the identity modulo $\mathfrak{m}_{(z,\eta)}$, so $\det(F)$ is not zero modulo $\mathfrak{m}_{(z,\eta)}$. Then the adjugate matrix G of F satisfies $FG = GF = \det(F)\mathrm{Id}_n$, and $f = \det(F)^2$ has the desired property: given $\delta \in M^{\psi=1}$, the element

$$\det(F)\delta - \sum_{i,j} G_{ji}\{\delta,\beta_j\}_{\mathrm{Iw}}\alpha_i$$

pairs to zero with each β_j. Thanks to the duality isomorphism in Theorem 2.3.11(3) it vanishes in $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_{z'}\otimes\eta'^{-1})$ for any $(z',\eta') \in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))$ for which S^* reduces to a basis, namely on the nonvanishing locus of $\det(F)$. Multiplying the above displayed quantity through by $\det(F)$ gives an element vanishing in $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_{z'}\otimes\eta'^{-1})$ for all (z',η'), which must be zero by Lemma 5.2.5. The proposition follows. □

Notation 5.2.7. We set $N^s = M^{\psi=1}\otimes_{\mathcal{R}_A^\infty(\Gamma)}\mathcal{R}_A^{[s,\infty]}(\Gamma)$ and $N^{s*} = (M^*)^{\psi=1}\otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[s,\infty]}(\Gamma)$. Note that we simply took the algebraic tensor product, which means that neither space carries any topology *a priori*. This partly reflects the difficulties we are trying to work around.

Corollary 5.2.8. *For any* $s>0$, *the* $\mathcal{R}_A^{[s,\infty]}(\Gamma)$*-modules* N^s *and* N^{s*} *are finite and flat.*

Proof. All maximal ideals of $\mathcal{R}_A^{[s,\infty]}(\Gamma)$ are of the form $\mathfrak{m}_{(z,\eta)}$, so the ideal generated by all the f as in Proposition 5.2.6 is the unit ideal. Writing 1 in terms of these generators shows that only finitely many such f are in fact necessary. Hence N^s is generated by the corresponding lifts of local bases. The flatness follows from the dimension calculation in Lemma 5.2.3. □

Remark 5.2.9. Since our goal is to prove the finiteness theorem of Iwasawa cohomology, we will have to make some additional efforts to tackle the finite generation properties. If one is only interested in the finiteness of (φ, Γ)-cohomology, Corollary 5.2.8 together with the dévissage argument in the next subsection is enough to deduce Theorem 4.4.5.

Proposition 5.2.10. *The modules N^s form a uniformly finitely presented vector bundle over $\mathcal{R}_A^\infty(\Gamma)$. Moreover, if we use N to denote the module of global sections of the vector bundle (N^s), then N is a finite projective $\mathcal{R}_A^\infty(\Gamma)$-module.*

Proof. Let $r_0 > 0$ be the number provided by Corollary 4.2.6. For $0 < s \leq r_0$, consider the natural map

$$\varphi - 1 : N^s \otimes_{\mathcal{R}_A^{[s,\infty]}(\Gamma)} \mathcal{R}_A^{[s,r_0]}(\Gamma) \cong M^{\psi=1} \otimes_{\mathcal{R}_A^\infty(\Gamma)} \mathcal{R}_A^{[s,r_0]}(\Gamma) \to (M^{[s,r_0]})^{\psi=0}$$

of two finite flat $\mathcal{R}_A^{[s,r_0]}(\Gamma)$-modules (the latter by Corollary 4.4). By Proposition 4.3.8(3), it is an isomorphism modulo any $\mathfrak{m}_z$ for $z \in \mathrm{Max}(A)$. Therefore, it has to be an isomorphism itself. Since $(M^{[s,r_0]})^{\psi=0}$ as $s \to 0^+$ form a uniformly finitely presented vector bundle over $\mathcal{R}_A^{r_0}(\Gamma)$, we see that N^s also form a uniformly finitely presented vector bundle over $\mathcal{R}_A^\infty(\Gamma)$. The last statement now follows from Proposition 2.1.13(3). □

Lemma 5.2.11. *The natural map $M^{\psi=1} \to N$ is injective.*

Proof. Proposition 5.2.10 implies that N is a finite projective $\mathcal{R}_A^\infty(\Gamma)$-module. Since $\mathcal{R}_A^\infty(\Gamma) \hookrightarrow \prod_{(z,\eta)\in\mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} \kappa_{(z,\eta)}$ is injective, the module N embeds into the product

$$\begin{aligned}\prod_{(z,\eta)\in\mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} N/\mathfrak{m}_{(z,\eta)} &\cong \prod_{(z,\eta)\in\mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} M^{\psi=1}/\mathfrak{m}_{(z,\eta)} \\ &\cong \prod_{(z,\eta)\in\mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))} H^1_{\psi,\gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}).\end{aligned}$$

By Lemma 5.2.5, both $M^{\psi=1}$ and N embed into the same module. Hence, the natural map $M^{\psi=1} \to N$ is injective. □

Theorem 5.2.12. *Under Hypothesis* 5.2.1, *$M^{\psi=1}$ is a finite projective $\mathcal{R}_A^\infty(\Gamma)$-module.*

Proof. By Proposition 5.2.10, it suffices to prove that the natural map $M^{\psi=1} \to N$ in Lemma 5.2.11 is in fact an isomorphism. This follows from combining Proposition 5.2.10 and Lemma 5.2.11 with Lemma 2.1.15, where the continuity assumptions required in the second lemma are verified by Proposition 2.2.14(3). □

5.3. **Dévissage.** To complete the proof of the finiteness of Iwasawa cohomology of (φ,Γ)-modules, we must perform some dévissage in the style of [36] in order to arrive at a case in which we can control $M/(\psi-1)$ and $M^*/(\psi-1)$.

Hypothesis 5.3.1. We continue to assume that $K=\mathbb{Q}_p$, except in the proof of Theorem 4.4.1.

Notation 5.3.2. Let L be a finite extension of $\mathbb{Q}_p$ and assume A is an L-affinoid algebra. For $\delta:\mathbb{Q}_p^\times\to L^\times$ a continuous character and M a (φ,Γ)-module over $\mathcal{R}_A$, we equip the twist $M(\delta)=M\otimes_L L\mathbf{e}_\delta$ with the diagonal action of φ and Γ, where $\varphi(\mathbf{e}_\delta)=\delta(p)\mathbf{e}_\delta$ and $\gamma(\mathbf{e}_\delta)=\delta(\chi(\gamma))\mathbf{e}_\delta$. We define $w(\delta)=\log(\delta(a))/\log(a)$ for nontorsion $a\in\mathbb{Z}_p^\times$, called the *weight* of δ. We write x for the natural embedding $\mathbb{Q}_p^\times\to\mathbb{Q}_p^\times$. We have $\mathcal{R}(x)\cong t\mathcal{R}$ and it has weight 1 (from the action of $\mathbb{Z}_p^\times$) and slope -1 (from the action of p; note the sign convention). Similarly, $\mathcal{R}(x^k\delta)\cong t^k\mathcal{R}(\delta)$ for $k\in\mathbb{Z}$.

Remark 5.3.3. An easy application of local class field theory, as in [19, Proposition 3.1], shows that every rank one (φ,Γ)-module over $\mathcal{R}_L$ is of the form $\mathcal{R}(\delta)$ for some $\delta:\mathbb{Q}_p^\times\to L^\times$. In Section 6.2, we reprise and extend the preceding notation, replacing $\mathbb{Q}_p^\times$ by $K^\times$ for $K/\mathbb{Q}_p$ an arbitrary finite extension, and replacing the target by $\Gamma(X,\mathcal{O}_X)^\times$ for X any L-rigid analytic space, where L contains a full set of Galois conjugates of K. We then prove the corresponding generalization of the classification result.

Lemma 5.3.4. *Let $\delta:\mathbb{Q}_p^\times\to L^\times$ be a continuous character and let $\mathcal{R}(\delta)$ be the associated (φ,Γ)-module.*

(1) *We have $H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta))\neq 0$ if and only if $\delta=x^i$ for $i\in\mathbb{Z}_{\le 0}$. For $i\le 0$, we have $H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(x^i))=Lt^{-i}\mathbf{e}$.*
(2) *Any (φ,Γ)-submodule of $\mathcal{R}(\delta)$ is of the form $t^i\mathcal{R}(\delta)$ for some $i\in\mathbb{Z}_{\ge 0}$.*
(3) *If $\delta(p)\notin p^{\mathbb{Z}}$, $H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta))=H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta))=0$.*
(4) *If $w(\delta)\notin\mathbb{Z}$, the natural morphism $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(x^k\delta)\to H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(x^{k'}\delta)$ is an isomorphism for any $k\ge k'$.*

Proof. (1) is [19, Proposition 2.1]. From this, we have for any continuous character $\delta':\mathbb{Q}_p^\times\to L^\times$, $\mathrm{Hom}_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta'),\mathcal{R}(\delta))\cong H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta\delta'^{-1}))$ is nonzero if and only if $\delta'=\delta x^i$ for some $i\in\mathbb{Z}_{\ge 0}$, hence (2). (3) follows from (2) and the Tate duality (Theorem 2.3.11(3)). (4) is [19, Theorem 2.22]. □

Lemma 5.3.5. *If N is a (φ,Γ)-module over $\mathcal{R}_A$ such that $N/(\psi-1)=0$, then for any $z\in\mathrm{Max}(A)$, the map $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(N)\otimes_A A/\mathfrak{m}_z\to H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(N_z)$ is bijective.*

Proof. By Proposition 2.3.6, it suffices to prove it for the (ψ,Γ)-cohomology. Consider the spectral sequence $\mathrm{Tor}_i^A(H^j_{\psi,\gamma_{\mathbb{Q}_p}}(N),A/\mathfrak{m}_z)\Rightarrow H^{j-i}_{\psi,\gamma_{\mathbb{Q}_p}}(N_z)$ (this case of base change being known because $A\to A/\mathfrak{m}_z$ is finite). Since $N/(\psi-1)=0$, we have $H^2_{\psi,\gamma_{\mathbb{Q}_p}}(N)=0$. The lemma follows from an inspection of this spectral sequence. □

Lemma 5.3.6. *Let M be a (φ,Γ)-module over $\mathcal{R}_A$ such that $M^*/(\psi-1)=0$ and $M/(\psi-1)\neq 0$. Then there exists a short exact sequence $0\to M\to N\to P\to 0$ of (φ,Γ)-modules over $\mathcal{R}_A$ with the following properties.*

(a) *The* (φ, Γ)*-module* P *is of rank one and* $P/(\psi - 1) = P^*/(\psi - 1) = 0$.
(b) *The connecting homomorphism* $P^{\psi=1} \to M/(\psi - 1)$ *is nonzero.*

In particular, $N^*/(\psi - 1) = 0$ *and* $N/(\psi - 1)$ *is a quotient of* $M/(\psi - 1)$ *with nontrivial kernel.*

Proof. Since $M/(\psi - 1)$ is nonzero and is finite over A by Proposition 3.3.2(1), we may find (and fix) $(z, \eta) \in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))$ such that $H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) \neq 0$. Let $\delta : \mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ be any continuous character such that $\delta(p) \notin p^{\mathbb{Z}}$ and $w(\delta \otimes \eta) \notin \mathbb{Z}$, and take $P = \mathcal{R}_A(\delta)$. We will show that property (a) holds for any such δ, and that after replacing δ by $x^n\delta$ for $n \gg 0$ (which does not disturb the hypotheses on δ), we may choose $\alpha \in H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M(\delta^{-1}))$ corresponding to an extension N as above such that (b) holds.

Lemma 5.3.4(3) shows that for any $\eta' \in \mathrm{Max}(\mathcal{R}^\infty(\Gamma))$, we have $P/(\psi - 1, \mathfrak{m}_{\eta'}) \cong H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta) \otimes \eta'^{-1}) = 0$. Hence $P/(\psi - 1) = 0$ by its finiteness in Proposition 3.3.2(1). The same argument proves $P^*/(\psi - 1) = 0$. Thus (a) holds.

Recall our fixed $(z, \eta) \in \mathrm{Max}(\mathcal{R}_A^\infty(\Gamma))$ from above. We denote the map $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M(\delta^{-1})) \to H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z(\delta^{-1}))$ by $\alpha \mapsto \alpha_z$. To fulfill condition (b), we just need that the connecting map

$$P^{\psi=1}/\mathfrak{m}_{(z,\eta)} \cong H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta) \otimes \eta^{-1}) \xrightarrow{d^1_{\alpha_z \otimes \eta^{-1}}} H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z \otimes \eta^{-1}) = M/(\psi - 1, \mathfrak{m}_{(z,\eta)})$$

is nontrivial. In fact, dualizing the above map gives the boundary map

$$H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z^* \otimes \eta) \xrightarrow{d^0_{\alpha_z^* \otimes \eta}} H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta^{-1})(1) \otimes \eta),$$

where $\alpha_z^* \otimes \eta \in \mathrm{Ext}^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z^* \otimes \eta, \mathcal{R}(\delta^{-1})(1) \otimes \eta)$; fixing any nonzero $\beta_z \in H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z^* \otimes \eta)$, we will arrange a choice of α so that $d^0_{\alpha_z^* \otimes \eta}(\beta_z) \neq 0$.

As a first step, we compute $d^0_{\alpha_z^* \otimes \eta}(\beta_z)$. Considering β_z as a morphism $\mathcal{R}_{\kappa_{(z,\eta)}} \to M_z^* \otimes \eta$, and its twisted dual $\beta_z^* \otimes \delta^{-1}\eta$ as a map $M_z \otimes \delta^{-1} \to \mathcal{R}_{\kappa_{(z,\eta)}}(\delta^{-1})(1) \otimes \eta$, we have a commutative diagram (whose downward arrows are obtained by dualizing and twisting extensions)

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z^* \otimes \eta, \mathcal{R}(\delta^{-1})(1) \otimes \eta) & \xrightarrow{\mathrm{Ext}^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\beta_z, -)} & \mathrm{Ext}^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}_{\kappa_{(z,\eta)}}, \mathcal{R}(\delta^{-1})(1) \otimes \eta) \\
\downarrow \cong & & \downarrow \cong \\
H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z(\delta^{-1})) \otimes_{\kappa_z} \kappa_{(z,\eta)} & \xrightarrow{H^1(\beta_z^* \otimes \delta^{-1}\eta)} & H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}(\delta^{-1})(1) \otimes \eta).
\end{array}$$

It follows that

$$d^0_{\alpha_z^* \otimes \eta}(\beta_z) = \beta_z \cup (\alpha_z^* \otimes \eta) = \mathrm{Ext}^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\beta_z, -)(\alpha_z^* \otimes \eta) = H^1(\beta_z^* \otimes \delta^{-1}\eta)(\alpha_z \otimes 1).$$

Let E_z be the kernel of β_z^*, so that Lemma 5.3.4(2) gives a short exact sequence $0 \to E_z \to M_z \otimes \eta^{-1} \xrightarrow{\beta_z^*} t^r\mathcal{R}_{\kappa_{(z,\eta)}}(1) \to 0$ for some $r \geq 0$. By Proposition 3.3.2(2), we may replace δ by $x^n\delta$ for n sufficiently large so that $H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(E_z(\delta^{-1}) \otimes \eta) = 0$

and $M(\delta^{-1})/(\psi-1)=0$. We have the following diagram

(5.3.6.1)

$$\begin{array}{ccccc} H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z(\delta^{-1})) & \longrightarrow & H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(t^r\mathcal{R}_{\kappa_{(z,\eta)}}(\delta^{-1})(1)\otimes\eta) & \longrightarrow & H^2_{\varphi,\gamma_{\mathbb{Q}_p}}(E_z(\delta^{-1})\otimes\eta)=0 \\ & & \downarrow \cong & & \\ & & H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}_{\kappa_{(z,\eta)}}(\delta^{-1})(1)\otimes\eta). & & \end{array}$$

The top row is an exact sequence and the vertical arrow is an isomorphism by Lemma 5.3.4(4). By the Euler characteristic formula (Theorem 2.3.11(2)), $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(t^r\mathcal{R}_{\kappa_{(z,\eta)}}(\delta^{-1})(1)\otimes\eta)\neq 0$. To complete the proof, it suffices to choose an element $\alpha_z\in H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M_z(\delta^{-1}))$ with nontrivial image in $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(\mathcal{R}_{\kappa_{(z,\eta)}}(\delta^{-1})(1)\otimes\eta)$, and use Lemma 5.3.5 to lift this to an element α of $H^1_{\varphi,\gamma_{\mathbb{Q}_p}}(M(\delta^{-1}))$. □

We now can finish the proof of Theorem 4.4.1. For this, we return to the setting of its statement.

Proof of Theorem 4.4.1. First, by Lemmas 5.1.1, 5.1.5, and 5.1.6, we may assume that $K=\mathbb{Q}_p$, A is reduced, and $M^*/(\psi-1)=0$ (by Proposition 3.3.2(2)). We then apply Lemma 5.3.6 and noetherian induction to get a (φ,Γ)-module N over $\mathcal{R}_A$ which is a successive extension of M by finitely many rank one (φ,Γ)-modules P_i such that $P_i/(\psi-1)=P_i^*/(\psi-1)=0$, and $N/(\psi-1)=N^*/(\psi-1)=0$.

By Theorem 5.2.12, $N^{\psi=1}$ and $P_i^{\psi=1}$ are all finite projective $\mathcal{R}_A^\infty(\Gamma)$-modules. This implies that $\mathrm{C}^\bullet_\psi(M)\in\mathbf{D}^-_{\mathrm{perf}}(\mathcal{R}_A^\infty(\Gamma))$, finishing the proof of the main theorem. □

6. Triangulation

We conclude by giving an application of our finiteness results for cohomology of (φ,Γ)-modules to the study of global triangulations of (φ,Γ)-modules. Our main result in this area (Corollary 6.3.10) shows that the existence of triangulations at a small (Zariski dense) set of points implies the existence of a global triangulation over a much larger (Zariski open) subset. More precisely, we prove that the global triangulation exists after making a suitable blowup. This has direct applications to eigenvarieties; we get an especially precise result for the Coleman-Mazur eigencurve, and relate the error term in the global triangulation with the action of the Theta operator considered by Coleman [17]. Although our language and methods are different, this improves Kisin's result [35] on interpolating crystalline periods (needing no "Y-smallness" condition).

Hypothesis 6.0.1. Throughout this section, let K be a finite extension of $\mathbb{Q}_p$ of degree d, with ring of integers $\mathcal{O}_K$ and residue field k_K of cardinality p^f. Let K_0 denote the fraction field of the ring of Witt vectors $W(k_K)$.

The Artin map gives an isomorphism between the multiplicative group $K^\times$ and the maximal abelian quotient of the Weil group of K; we normalize it so that the image of a uniformizer of K is a *geometric* Frobenius. We hereafter identify the two groups.

We also let L denote a finite extension of $\mathbb{Q}_p$ (not necessarily contained in our chosen $\overline{\mathbb{Q}}_p$) such that the set Σ of $\mathbb{Q}_p$-algebra embeddings $K\hookrightarrow L$ has the maximal cardinality d. In the decomposition $K\otimes_{\mathbb{Q}_p}L=\prod_{\sigma\in\Sigma}L$, we write e_σ for the idempotent projecting onto the σ-factor, which satisfies $e_\sigma(x \otimes 1) = \sigma(x)$ for $x \in K$. Note that L may be replaced by a finite extension at any time without altering the hypothesis or Σ.

Notation 6.0.2. For M a finite projective module over a ring R, an R-submodule N of M is called *saturated* if M/N is a (finite) projective R-module. In particular, this condition implies that N itself is a finite projective R-module.

For M a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$, a (φ, Γ_K)-submodule N of M is called *saturated* if it is saturated as an $\mathcal{R}_A(\pi_K)$-submodule. In this case, M/N is a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$. When $A = L$ so that $\mathcal{R}_L(\pi_K)$ is a Bézout domain, the submodule N is saturated in M if and only if there is no $\mathcal{R}_L(\pi_K)$-submodule of M strictly containing N with the same rank.

6.1. **Moduli spaces of continuous characters.** We will show later (Construction 6.2.4 and Theorem 6.2.14) that (φ, Γ_K)-modules of rank one on a rigid L-analytic space X are essentially classified by continuous characters $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$. In preparation for this, we devote this subsection to the description of such characters in terms of a certain moduli space.

To begin with, we state a general result on moduli spaces of continuous characters.

Proposition 6.1.1. *Let G be a commutative p-adic Lie group that modulo some compact open subgroup becomes free abelian on finitely many generators. There exist a rigid analytic space $X^{\mathrm{an}}(G)$ over $\mathbb{Q}_p$ and a continuous character $\delta_G : G \to \Gamma(X^{\mathrm{an}}(G), \mathcal{O}_{X^{\mathrm{an}}(G)})^\times$ with the following property: for any rigid analytic space X, every continuous character $\delta : G \to \Gamma(X, \mathcal{O}_X)^\times$ arises by pullback from a unique morphism $X \to X^{\mathrm{an}}(G)$ of rigid analytic spaces. Moreover, $X^{\mathrm{an}}(G)$ is a smooth quasi-Stein space.*

Proof. Since this lemma is well known, we just give a sketch here. If G can be written as a product of groups satisfying the hypotheses of the lemma, then we may treat each factor separately, and assemble the product of the respective results. Thus, since G is (noncanonically) isomorphic to $G_0 \times \mathbb{Z}_p^r \times \mathbb{Z}^s$, where G_0 is a finite abelian group, it suffices to treat the cases of G_0, $\mathbb{Z}_p$, and $\mathbb{Z}$. Clearly one has $X^{\mathrm{an}}(G_0) = \mathrm{Max}(\mathbb{Q}_p[G_0])$ and $X^{\mathrm{an}}(\mathbb{Z}) = \mathbb{G}_m^{\mathrm{an}}$. One takes $X^{\mathrm{an}}(\mathbb{Z}_p)$ to be the generic fiber of the $\mathfrak{m}$-adic formal scheme $\mathrm{Spf}(\mathbb{Z}_p[\![\mathbb{Z}_p]\!])$, which via the Iwasawa isomorphism is the open unit disc in the variable $[\gamma_{\mathbb{Z}_p}] - 1$, where $\gamma_{\mathbb{Z}_p} \in \mathbb{Z}_p$ is a generator and the brackets denote a grouplike element in the completed group algebra. In other words, as one can readily verify, for any affinoid algebra A and any element $f \in A$ such that $f - 1$ is topologically nilpotent, there exists a unique continuous character $\mathbb{Z}_p \to A^\times$ sending $\gamma_{\mathbb{Z}_p}$ to f, and conversely every such character arises in this fashion. □

Remark 6.1.2. For G as in Proposition 6.1.1 and for any rigid L-analytic space X, the natural bijection between continuous characters $\delta : G \to \Gamma(X, \mathcal{O}_X)^\times$ and morphisms $X \to X^{\mathrm{an}}(G) \times_{\mathbb{Q}_p} L$ has the property that a morphism defines a character by pulling back the universal character δ_G. In turn, morphisms $X \to X^{\mathrm{an}}(G) \times_{\mathbb{Q}_p} L$ are in bijection with graphs of morphisms $X \to X^{\mathrm{an}}(G) \times_{\mathbb{Q}_p} L$, i.e., closed analytic subvarieties of $X \times_L (X^{\mathrm{an}}(G) \times_{\mathbb{Q}_p} L)$ which project isomorphically onto X.

We now specialize this construction to various cases of interest.

Example 6.1.3. For $G = \Gamma_K$, the space $X^{\mathrm{an}}(G)$ provided by Proposition 6.1.1 is the cyclotomic deformation space $\mathrm{Max}(\mathcal{R}^{\infty}_{\mathbb{Q}_p}(\Gamma_K))$ (see Notation 4.2.1).

Example 6.1.4. Take $G = K^\times$ in Proposition 6.1.1, then put $X^{\mathrm{an}}_L = X^{\mathrm{an}}(G) \times_{\mathbb{Q}_p} L$ and $\delta_L = \delta_G \otimes 1$. This example will control (φ,Γ_K)-modules of rank one, as described in Construction 6.2.4 and Theorem 6.2.14.

It will be useful to separate X^{an}_L into factors as follows.

Example 6.1.5. Choose a uniformizer ϖ_K of K and use it to split $K^\times$ as a product $\mathcal{O}_K^\times \times \varpi_K^{\mathbb{Z}}$. Then each continuous character δ on $K^\times$ factors uniquely as $\delta_1\delta_2$ where δ_1 is trivial on $\varpi_K^{\mathbb{Z}}$ and δ_2 is trivial on $\mathcal{O}_K^\times$. Correspondingly, we may factor X^{an}_L as a product $X^{\mathrm{an}}_{L,1} \times X^{\mathrm{an}}_{L,2}$ where $X^{\mathrm{an}}_{L,1} = X^{\mathrm{an}}(\mathcal{O}_K^\times) \times_{\mathbb{Q}_p} L$ and $X^{\mathrm{an}}_{L,2} = X^{\mathrm{an}}(\varpi^{\mathbb{Z}}) \times_{\mathbb{Q}_p} L$. In particular, there is a distinguished isomorphism $X^{\mathrm{an}}_{L,2} \cong \mathbb{G}^{\mathrm{an}}_{m,L}$ under which the coordinate on $\mathbb{G}^{\mathrm{an}}_{m,L}$ computes the evaluation of a character on ϖ_K.

It will also be helpful to further separate the factor $X^{\mathrm{an}}_{L,1}$ of X^{an}_L.

Definition 6.1.6. Let A be an L-affinoid algebra. A continuous character $\delta : K^\times \to A^\times$ is automatically locally $\mathbb{Q}_p$-analytic [14, Proposition 8.3] (but not locally K-analytic in general), so we may look at the action of the $\mathbb{Q}_p$-Lie algebra of $K^\times$. More precisely, the *weight* of δ is the image $\mathrm{wt}(\delta)$ under $\prod_{\sigma\in\Sigma} A \cong K \otimes_{\mathbb{Q}_p} A$ of the family of elements $(\mathrm{wt}_\sigma(\delta))_{\sigma\in\Sigma}$ such that

$$\lim_{\substack{a\to 0\\ a\in\mathcal{O}_K}} \frac{|\delta(1+a) - 1 - \sum_{\sigma\in\Sigma}\mathrm{wt}_\sigma(\delta)\sigma(a)|_A}{|a|_K} = 0$$

(for any Banach algebra norm $|\cdot|_A$ defining the topology on A). The existence of such a set of elements follows from the fact that δ is locally $\mathbb{Q}_p$-analytic, and the uniqueness is obvious.

Let X be a rigid L-analytic space. By globalizing the previous definition, we may attach to any continuous character $\delta : K^\times \to \Gamma(X,\mathcal{O}_X)^\times$ a *weight* $\mathrm{wt}(\delta) \in K \otimes_{\mathbb{Q}_p} \Gamma(X,\mathcal{O}_X)$.

Definition 6.1.7. Let $X^{\mathrm{an}}_{L,\mathrm{Sen}}$ be the rigid L-analytic variety obtained by forming $\mathbb{G}^{\mathrm{an}}_{a,K}$, taking the Weil restriction from K to $\mathbb{Q}_p$, then base extending from $\mathbb{Q}_p$ to L. From Definition 6.1.6, we obtain a homomorphism $X^{\mathrm{an}}_{L,1} \to X^{\mathrm{an}}_{L,\mathrm{Sen}}$ such that for any rigid L-analytic space X and any continuous character $\delta : K^\times \to \Gamma(X,\mathcal{O}_X)^\times$, the composition of the induced map $X \to X^{\mathrm{an}}_L$ with the projections $X^{\mathrm{an}}_L \to X^{\mathrm{an}}_{L,1} \to X^{\mathrm{an}}_{L,\mathrm{Sen}}$ is the map $X \to X^{\mathrm{an}}_{L,\mathrm{Sen}}$ associated to $\mathrm{wt}(\delta)$.

Let $X^{\mathrm{an}}_{L,\mathrm{fin}}$ denote the kernel of $X^{\mathrm{an}}_{L,1} \to X^{\mathrm{an}}_{L,\mathrm{Sen}}$. The space $X^{\mathrm{an}}_{L,\mathrm{fin}}$ classifies characters on $\mathcal{O}_K^\times$ of weight zero; since these characters are locally $\mathbb{Q}_p$-analytic, these are exactly the characters of finite order. It follows that $X^{\mathrm{an}}_{L,\mathrm{fin}}$ consists of an *infinite* discrete set of rigid analytic points.

6.2. (φ,Γ_K)**-modules of character type.** We next make a detailed study of (φ,Γ_K)-modules of rank one on rigid analytic spaces. In one direction, we show that continuous characters on $K^\times$ give rise to (φ,Γ_K)-modules, said to be of *character type*. The construction amounts to a translation of a construction of Nakamura [40] from Berger's language of $\mathbb{B}$-pairs into the language of (φ,Γ_K)-modules, which is better suited for arithmetic families. In the other direction, we show that every rank one (φ,Γ_K)-module on a rigid analytic space is of character type up to a twist by a line bundle on the base space. Along the way, we develop some tools which will be useful later for analyzing trianguline (φ, Γ_K)-modules, including a criterion for when a rank one submodule of a (φ, Γ_K)-module over $\mathcal{R}_L(\pi_K)$ is saturated (Lemma 6.2.10), and a definition of Sen weights in families.

In order to be able to work over rigid analytic spaces which are not necessarily affinoid, we introduce the following definition.

Definition 6.2.1. For X a rigid analytic space over $\mathbb{Q}_p$ and $r > 0$, we define $\mathcal{R}_X^r(\pi_K)$ to be the sheaf of analytic functions on $X \times \mathrm{Max}(\mathcal{R}^r_{\mathbb{Q}_p}(\pi_K))$. Let $\mathcal{R}_X(\pi_K)$ be the direct limit over $r > 0$ of the $\mathcal{R}_X^r(\pi_K)$. For $? = r, \emptyset$, a (φ, Γ_K)-module over $\mathcal{R}_X^?(\pi_K)$ is a compatible family of (φ, Γ_K)-modules over $\mathcal{R}_A^?(\pi_K)$ for each affinoid $\mathrm{Max}(A)$ of X. Note, in particular, that for X not quasicompact a (φ, Γ_K)-module over $\mathcal{R}_X(\pi_K)$ might not arise as the base change from a single $\mathcal{R}_X^r(\pi_K)$, as the required r may not be bounded away from zero as one ranges over the affinoids of X.

We begin by recalling a standard construction of $(\varphi, \Gamma_{\mathbb{Q}_p})$-modules over $\mathcal{R}_X$, generalizing Notation 5.3.2 from the case $X = \mathrm{Max}(\mathbb{Q}_p)$.

Notation 6.2.2. Let X be a rigid analytic space over $\mathbb{Q}_p$. For $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ a continuous character, we define $\mathcal{R}_X(\delta)$ to be the free rank one (φ, Γ)-module $\mathcal{R}_X \cdot \mathbf{e}$ with $\varphi(\mathbf{e}) = \delta(p)\mathbf{e}$ and $\gamma(\mathbf{e}) = \delta(\chi(\gamma))\mathbf{e}$ for $\gamma \in \Gamma$.

The above notion generalizes from $\mathbb{Q}_p$ to K, making use of the following variant of Hilbert's Theorem 90.

Lemma 6.2.3. *Let A be a K_0-algebra, and let $a \in A^\times$. Up to isomorphism, there exists a unique free rank one $K_0 \otimes_{\mathbb{Q}_p} A$-module $D_{f,a}$ equipped with a $\varphi \otimes 1$-semilinear operator φ satisfying $\varphi^f = 1 \otimes a$. One has $D_{f,ab} \simeq D_{f,a} \otimes_{K_0 \otimes_{\mathbb{Q}_p} A} D_{f,b}$ for all $a, b \in A^\times$.*

Proof. For existence, let $D_{f,a}$ be the free A-module on the basis $e_0, \dots, e_{f-1}$, extended to a $K_0 \otimes_{\mathbb{Q}_p} A$-module via the rule $(x \otimes 1)e_i = \varphi^i(x)e_i$. One checks directly that $D_{f,a}$ is free of rank one over $K_0 \otimes_{\mathbb{Q}_p} A$, and that the rule $\varphi : D_{f,a} \to D_{f,a}$ defined by

$$\varphi(e_0) = ae_{f-1}, \quad \varphi(e_1) = e_0, \quad \varphi(e_2) = e_1, \quad \dots, \quad \varphi(e_{f-1}) = e_{f-2}$$

and A-linearity is $(\varphi \otimes 1)$-linear with $\varphi^f = 1 \otimes a$.

For uniqueness, one reduces easily to the case $a = 1$, and must show that any candidate $D_{f,1}$ is isomorphic as $K_0 \otimes_{\mathbb{Q}_p} A$-module with φ-action to $K_0 \otimes_{\mathbb{Q}_p} A$ itself equipped with $\varphi \otimes 1$. Fix a basis e of $D_{f,1}$ and write $\varphi(e) = be$, where necessarily $b \in (K_0 \otimes_{\mathbb{Q}_p} A)^\times$ and $b \cdot (\varphi \otimes 1)(b) \cdots (\varphi \otimes 1)^{f-1}(b) = 1$. We must find $c \in (K_0 \otimes_{\mathbb{Q}_p} A)^\times$ such that $(\varphi \otimes 1)(c)/c = b$; given such c, the desired isomorphism sends $e \in D_{f,1}$ to $c \in K_0 \otimes_{\mathbb{Q}_p} A$. Writing, under the identification $K_0 \otimes_{\mathbb{Q}_p} A = \prod_{i=0}^{f-1} A$, the element b as the tuple $(b_0, \dots, b_{n-1})$, one easily checks that each b_i is a unit in A, and that the element c identified to the tuple $(1, b_0, b_0 b_1, \dots, b_0 b_1 \cdots b_{n-2})$ has the desired properties.

The identity $D_{f,ab} \simeq D_{f,a} \otimes_{K_0 \otimes_{\mathbb{Q}_p} A} D_{f,b}$ is straightforward to see. □

We now give the construction of (φ, Γ_K)-modules of *character type*.

Construction 6.2.4. (Compare with Nakamura's construction in the language of $\mathbb{B}$-pairs, in [39, Section 1.4].) To a continuous character $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$

we associate a (φ, Γ_K)-module of rank 1 over $\mathcal{R}_X(\pi_K)$, denoted $\mathcal{R}_A(\pi_K)(\delta)$, as follows. First, we may work locally, and assume that $X = \mathrm{Max}(A)$ is affinoid. Factor $\delta = \delta_1\delta_2$ as in Example 6.1.5; then δ_1, viewed as a character on $\mathcal{O}_K^\times$, extends to a character $\widehat{\delta}_1$ on G_K^{ab}. We let $\mathcal{R}_A(\pi_K)(\delta_1) = \mathbf{D}_{\mathrm{rig}}(\widehat{\delta}_1)$, and we let $\mathcal{R}_A(\pi_K)(\delta_2) = D_{f,\delta_2(\text{uniformizer})} \otimes_{(K_0 \otimes_{\mathbb{Q}_p} A)} \mathcal{R}_A(\pi_K)$ (in the notation of the preceding lemma) with the evident induced φ-action, and trivial Γ_K-action on the first factor. Then, we put $\mathcal{R}_A(\pi_K)(\delta) = \mathcal{R}_A(\pi_K)(\delta_1) \otimes_{\mathcal{R}_A(\pi_K)} \mathcal{R}_A(\pi_K)(\delta_2)$. Using the tensor-functoriality of $\mathbf{D}_{\mathrm{rig}}$ in Theorem 2.2.17 and $D_{f,a}$ in Lemma 6.2.3, this definition is easily checked to be independent of the choice of the uniformizer ϖ_K in Example 6.1.5 and also tensor-functorial.

For a (φ, Γ_K)-module M over $\mathcal{R}_X(\pi_K)$, we put $M(\delta) = M \otimes_{\mathcal{R}_X(\pi_K)} \mathcal{R}_X(\pi_K)(\delta)$.

We continue with some tools adapted to the case $A = L$, for use in treating the behavior of families at points of $\mathrm{Max}(A)$.

Definition 6.2.5. Let M be a (φ, Γ_K)-module over $\mathcal{R}_L(\pi_K)$ that is potentially semistable. Thus, $\mathbf{D}_{\mathrm{pst}}(M)$ is a filtered $(\varphi, N, \mathrm{Gal}(K'/K))$-module equipped with a $\mathbb{Q}_p$-linear action of L commuting with $(\varphi, N, \mathrm{Gal}(K'/K))$ and preserving the filtration; here K'/K is a finite Galois extension inside $\overline{\mathbb{Q}}_p$ over which M becomes semistable, and we write K_0' for its maximal absolutely unramified subfield. It is well known that $\mathbf{D}_{\mathrm{pst}}(M)$ is a free $K_0' \otimes_{\mathbb{Q}_p} L$-module, so that $\mathbf{D}_{\mathrm{dR}}(M) = (K' \otimes_{K_0'} D)^{G_K}$ is a free $K \otimes_{\mathbb{Q}_p} L$-module. We decompose $\mathbf{D}_{\mathrm{dR}}(M) = \bigoplus_{\sigma \in \Sigma} e_\sigma \mathbf{D}_{\mathrm{dR}}(M)$ as a filtered $K \otimes_{\mathbb{Q}_p} L$-module, and for $\sigma \in \Sigma$ we define the *σ-Hodge-Tate weights* of M to be the jumps in the filtration on $e_\sigma \mathbf{D}_{\mathrm{dR}}(M)$, counted with multiplicity given by the L-dimension of the respective graded pieces. (For details on the above constructions, see Berger's dictionary [7].)

Example 6.2.6. Choose a uniformizer ϖ_K of K.

(1) The unique homomorphism $\mathrm{LT}_{\varpi_K} : K^\times \to K^\times$ that is the identity on $\mathcal{O}_K^\times$ and satisfies $\mathrm{LT}_{\varpi_K}(\varpi_K) = 1$ extends to G_K^{ab}, whereupon it describes the action on the torsion in the Lubin-Tate group over $\mathcal{O}_K$ associated to ϖ_K. The structure of $\mathbf{D}_{\mathrm{cris}}(\mathrm{LT}_{\varpi_K})$ is well known. As a $K_0 \otimes_{\mathbb{Q}_p} K$-module with $\varphi \otimes 1$-linear operator, it is isomorphic to $D_{f,\varpi_K^{-1}}$ in the notation of Lemma 6.2.3, because the p^f-power Frobenius of the special fiber of the Lubin-Tate group is by construction equal to formal multiplication by ϖ_K. Choosing an embedding $\sigma_0 \in \Sigma$ and setting $\mathrm{LT}_{\varpi_K,\sigma_0} = \sigma_0 \circ \mathrm{LT}_{\varpi_K}$, the σ-Hodge-Tate weight of $\mathrm{LT}_{\varpi_K,\sigma_0}$ is -1 if $\sigma = \sigma_0$ and 0 if $\sigma \neq \sigma_0$; this is because the height and dimension of the Lubin-Tate group are d and 1, and the formal multiplication acts on its tangent space via the structure map.

(2) Fix an embedding $\sigma_0 \in \Sigma$. Use x_{σ_0} to denote the character $K^\times \to L^\times$ given by this embedding, and $\delta_{\sigma_0} : K^\times \to L^\times$ to denote the character given by $\delta_{\sigma_0}|_{\mathcal{O}_K^\times} = 1$ and $\delta_{\sigma_0}(\varpi_K) = \sigma_0(\varpi_K)$. One has $x_{\sigma_0} = \mathrm{LT}_{\varpi_K,\sigma_0} \cdot \delta_{\sigma_0}$, so that

$$\mathcal{R}_L(\pi_K)(x_{\sigma_0}) \simeq \mathcal{R}_L(\pi_K)(\mathrm{LT}_{\varpi_K,\sigma_0}) \otimes_{\mathcal{R}_L(\pi_K)} \mathcal{R}_L(\pi_K)(\delta_{\sigma_0}).$$

This allows us to compute, under Berger's dictionary [7, Théorème A], that $\mathbf{D}_{\mathrm{cris}}(\mathcal{R}_L(\pi_K)(x_{\sigma_0}))$ is isomorphic to $D_{f,1}$, and that its σ-Hodge-Tate weight is -1 if $\sigma = \sigma_0$ and 0 if $\sigma \neq \sigma_0$. In particular, $\mathcal{R}_L(\pi_K)(x_{\sigma_0})$ is a (φ, Γ_K)-submodule of the trivial one $\mathcal{R}_L(\pi_K)$.

(3) More generally, let $\delta : K^\times \to L^\times$ be a continuous character such that $\delta|_{\mathcal{O}_K^\times} = \prod_{\sigma \in \Sigma} x_\sigma^{k_\sigma}|_{\mathcal{O}_K^\times}$ for some integers k_σ. Then $\mathbf{D}_{\mathrm{cris}}(\mathcal{R}_L(\pi_K)(\delta))$ is isomorphic to $D_{f,\alpha}$

with $\alpha = \delta(\varpi_K) \cdot \prod_{\sigma\in\Sigma} \sigma(\varpi_K)^{-k_\sigma}$, and its σ-Hodge-Tate weight is $-k_\sigma$ for each $\sigma \in \Sigma$.

Notation 6.2.7. By Example 6.2.6(2), for each $\sigma \in \Sigma$, the module $\mathcal{R}_L(\pi_K)(x_\sigma)$ is a submodule of $\mathcal{R}_L(\pi_K)$. Since $\mathcal{R}_L(\pi_K)$ is a product of Bézout domains, there exists an element $t_\sigma \in \mathcal{R}_L(\pi_K)$ generating the submodule $\mathcal{R}_L(\pi_K)(x_\sigma)$. The element t_σ is not uniquely determined, but the ideal it generates is.

We remark that $\prod_{\sigma\in\Sigma} t_\sigma = tu$ for some unit $u \in \mathcal{R}_L(\pi_K)^\times$; this follows from the fact that $\prod_{\sigma\in\Sigma} x_\sigma = N_{K/\mathbb{Q}_p}$ as characters of $K^\times$. Also, the t_σ are strongly coprime, in that $(t_\sigma) + (t_\tau) = \mathcal{R}_L(\pi_K)$ for $\sigma \neq \tau$, which one sees by applying Berger's dictionary to $(t_\sigma) + (t_\tau)$ and considering its possible Hodge-Tate weights. In particular, one has

$$\mathcal{R}_L(\pi_K)/t \cong \bigoplus_{\sigma\in\Sigma} \mathcal{R}_L(\pi_K)/t_\sigma. \tag{6.2.7.1}$$

We now compute the cohomology of (φ, Γ_K)-modules of character type over L.

Proposition 6.2.8. *Let $\delta : K^\times \to L^\times$ be a continuous character.*

(1) *The cohomology $H^0_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta))$ is trivial unless $\delta = \prod_{\sigma\in\Sigma} x_\sigma^{k_\sigma}$ with all $k_\sigma \leq 0$, in which case the dimension is equal to one.*

(2) *The cohomology $H^2_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta))$ is trivial unless $\delta = |N_{K/\mathbb{Q}_p}|\cdot\prod_{\sigma\in\Sigma} x_\sigma^{k_\sigma}$ with all $k_\sigma \geq 1$, in which case the dimension is equal to one.*

(3) *The cohomology $H^1_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta))$ has dimension equal to $[K : \mathbb{Q}_p]$ unless either the zeroth or second cohomology does not vanish, in which case the dimension is equal to $[K : \mathbb{Q}_p] + 1$.*

Proof. This follows from Nakamura's computation [39, Section 2.3], together with the comparison of cohomology. For the convenience of the reader, we include a proof using the language of (φ, Γ_K)-modules.

By the Euler characteristic formula (Theorem 2.3.11(2)), the last statement follows from the first two. By Tate duality (Theorem 2.3.11(3)), the second statement follows from the first one. It then suffices to prove (1).

Suppose that $H^0_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta))$ is nontrivial for some continuous character $\delta : K^\times \to L^\times$. Then we have a nonzero morphism $\mathcal{R}_L(\pi_K) \to \mathcal{R}_L(\pi_K)(\delta)$ of (φ, Γ_K)-modules; this is equivalent to the existence of a morphism $j : \mathcal{R}_L(\pi_K)(\delta^{-1}) \to \mathcal{R}_L(\pi_K)$ of (φ, Γ_K)-modules. Since $\mathcal{R}_L(\pi_K)$ is a finite direct product of integral domains and φ acts transitively on these domains, this morphism j must be injective, realizing $\mathcal{R}_L(\pi_K)(\delta^{-1})$ as a subobject of $\mathcal{R}_L(\pi_K)$.

As a subobject of a crystalline object is crystalline, $\mathcal{R}_L(\pi_K)(\delta^{-1})$ corresponds to a nonzero subobject of the trivial filtered φ-module of rank one. In particular, it must be isomorphic to $K_0 \otimes_{\mathbb{Q}_p} L$ as a φ-module, and each σ-Hodge-Tate weight $-k_\sigma$ for $\sigma \in \Sigma$ is nonpositive. Note that this subobject is exactly the filtered φ-module associated to the character $\prod_{\sigma\in\Sigma} x_\sigma^{k_\sigma}$ in the computation in Example 6.2.6(3). It follows that $\delta = \prod_{\sigma\in\Sigma} x_\sigma^{-k_\sigma}$. As $\mathbf{D}_{\mathrm{cris}}$ is fully faithful on crystalline objects, and there is only one nonzero inclusion $\mathbf{D}_{\mathrm{cris}}(\mathcal{R}_L(\pi_K)(\delta^{-1})) \hookrightarrow \mathbf{D}_{\mathrm{cris}}(\mathcal{R}_L(\pi_K))$ up to scalar multiple, we see that $H^0_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta))$ is one-dimensional. ☐

Corollary 6.2.9. *Let $\delta : K^\times \to L^\times$ be a continuous character. Then a nonzero (φ, Γ_K)-submodule of $\mathcal{R}_L(\pi_K)(\delta)$ over $\mathcal{R}_L(\pi_K)$ is of the form $\left(\prod_{\sigma\in\Sigma} t_\sigma^{k_\sigma}\right) \mathcal{R}_L(\pi_K)(\delta)$ for some $k_\sigma \geq 0$.*

Proof. This follows immediately from Proposition 6.2.8(1). □

Lemma 6.2.10. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_L(\pi_K)$, and let λ be a nonzero element of $H^0_{\varphi,\gamma_K}(M) \cong \mathrm{Hom}_{\mathcal{R}_L(\pi_K)[\varphi,\Gamma_K]}(\mathcal{R}_L(\pi_K), M)$. Then $\lambda(\mathcal{R}_L(\pi_K))$ is saturated in M if and only if the image of λ in $H^0_{\varphi,\gamma_K}(M/t_\sigma)$ is nonzero for every $\sigma \in \Sigma$.*

Proof. For each embedding $\sigma \in \Sigma$, we have an exact sequence $0 \to H^0_{\varphi,\gamma_K}(M(x_\sigma)) \to H^0_{\varphi,\gamma_K}(M) \to H^0_{\varphi,\gamma_K}(M/t_\sigma)$. The image of λ in the last term is zero if and only if it is the image of some $\lambda' \in H^0_{\varphi,\gamma_K}(M(x_\sigma))$. This implies that if λ dies in $H^0_{\varphi,\gamma_K}(M/t_\sigma)$, then $\lambda(\mathcal{R}_L(\pi_K))$ embeds into $t_\sigma M$ and hence $t_\sigma^{-1}\lambda(\mathcal{R}_L(\pi_K))$ embeds into M, rendering the submodule $\lambda(\mathcal{R}_L(\pi_K))$ not saturated.

Conversely, if $\lambda(\mathcal{R}_L(\pi_K)) \hookrightarrow M$ is not saturated, then dualizing this inclusion gives a nonsurjective morphism $M^\vee \to \mathcal{R}_L(\pi_K)$ of (φ, Γ_K)-modules. By Corollary 6.2.9, the image of this map lies in $t_\sigma \mathcal{R}_L(\pi_K)$ for some $\sigma \in \Sigma$. Dualizing back this map and multiplying both the source and the target by t_σ gives a homomorphism $\lambda(\mathcal{R}_L(\pi_K)) \to t_\sigma M$. This proves that λ lies in the image of $H^0_{\varphi,\gamma_K}(t_\sigma M)$. □

We next discuss the notion of weights in families. Henceforth, A denotes an L-affinoid algebra.

Definition 6.2.11. Let M be a (φ, Γ_K)-module over $\mathcal{R}_A(\pi_K)$ of rank d. By Lemma 3.2.3, $(M/t)^{\varphi=1}$ is of the form

$$\mathbf{D}_{\mathrm{Sen},n}(M) \otimes_{K(\mu_{p^n}) \otimes_{\mathbb{Q}_p} A} (K(\mu_{p^\infty}) \otimes_{\mathbb{Q}_p} A)$$

for a sufficiently large $n \in \mathbb{N}$, where $\mathbf{D}_{\mathrm{Sen},n}(M)$ is a locally free module of rank d over $K(\mu_{p^n}) \otimes_{\mathbb{Q}_p} A$. Consider the *Sen operator* $\Theta_{\mathrm{Sen}} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$ on M for $\gamma \in \Gamma_K$ sufficiently close to 1. It induces a $K(\mu_{p^n}) \otimes_{\mathbb{Q}_p} A$*-linear* action on $\mathbf{D}_{\mathrm{Sen},n}(M)$. The characteristic polynomial of this linear action is monic with coefficients in $K(\mu_{p^n}) \otimes_{\mathbb{Q}_p} A$, but because it must be Γ_K-invariant it descends to $K \otimes_{\mathbb{Q}_p} A$. It is independent of n; we call it the *Sen polynomial* of M.

When the rank of M is one, the Sen operator acts by multiplication by an element of $K \otimes_{\mathbb{Q}_p} A$, which is the unique root of the Sen polynomial. We call this the *Sen weight* of M.

For a (φ, Γ_K)-module of character type, we may reconcile the Sen weight of the (φ, Γ_K)-module with the weight of the underlying character.

Lemma 6.2.12. *Let $\delta : K^\times \to A^\times$ be a continuous character. Then the Sen weight of $\mathcal{R}_A(\pi_K)(\delta)$ is $\mathrm{wt}(\delta)$, and for each $\sigma \in \Sigma$, and the operator Θ_{Sen} acts on $(\mathcal{R}_A(\pi_K)(\delta)/t_\sigma)^{\varphi=1}$ by multiplication by $\mathrm{wt}_\sigma(\delta)$.*

Proof. First suppose $\delta : K^\times \to L^\times$ is a continuous character such that $\delta|_{\mathcal{O}_K^\times} = \prod_{\sigma\in\Sigma} x_\sigma^{k_\sigma}|_{\mathcal{O}_K^\times}$ for some integers k_σ. Then $\mathrm{wt}_\sigma(\delta) = k_\sigma$ for each $\sigma \in \Sigma$ by the definition of weights, and so the first claim follows from the calculation of filtered φ-modules in Example 6.2.6(3).

We let $X_L^{\mathrm{an}} = X^{\mathrm{an}}(K^\times) \times_{\mathbb{Q}_p} L$ and $\delta_L = \delta_{K^\times} \otimes 1$ be as in Example 6.1.4, and we let S be the set of points of X_L^{an} corresponding to characters treated above. The weight and Sen weight of δ_L agree at all points of S, and since S is Zariski dense in X_L^{an} and X_L^{an} is reduced, this implies the weight and Sen weight must be equal in $\Gamma(X_L^{\mathrm{an}}, \mathcal{O}_{X_L^{\mathrm{an}}})$. Now let $f : \mathrm{Max}(A) \to X_L^{\mathrm{an}}$ be the unique morphism such that

$\delta = f^*\delta_L$. Pulling back along f the identity of the weight and Sen weight for δ_L, we deduce it for δ.

The second claim is just a restatement of the first, broken up according to the decomposition

$$\mathbf{D}_{\mathrm{Sen},n}(M) \cong \bigoplus_{\sigma\in\Sigma} \mathbf{D}_{\mathrm{Sen},n}(M) \otimes_{K\otimes_{\mathbb{Q}_p} A,\sigma\otimes 1} A \cong \bigoplus_{\sigma\in\Sigma} (M/t_\sigma)^{\varphi=1}$$

for n sufficiently large. □

The remainder of this section will be devoted to proving that Construction 6.2.4 is exhaustive up to twisting by line bundles on the base space. We begin with the case over an artinian ring. (Compare the following lemma with the corresponding claim in the language of $\mathbb{B}$-pairs, in [40, Proposition 2.15].)

Lemma 6.2.13. *Let A be an artinian L-algebra and let M be a (φ, Γ_K)-module of rank 1 over $\mathcal{R}_A(\pi_K)$. Then there exists a unique continuous character $\delta : K^\times \to A^\times$ such that $\mathcal{L} = H^0_{\varphi,\gamma_K}(M(\delta^{-1}))$ is free of rank 1 over A and the natural map $\mathcal{R}_A(\pi_K)(\delta) \otimes_A \mathcal{L} \to M$ is an isomorphism.*

Proof. First assume that A is a field, hence a finite extension of $\mathbb{Q}_p$. To see uniqueness, if $\mathcal{R}_A(\pi_K)(\delta) \simeq \mathcal{R}_A(\pi_K)(\delta')$ then $\mathcal{R}_A(\pi_K) \simeq \mathcal{R}_A(\pi_K)(\delta'\delta^{-1})$, and local class field theory forces $\delta'\delta^{-1} = 1$. We now show existence. If we view the (φ, Γ_K)-module M as a φ-module over $\mathcal{R}_A$ (of possibly larger rank), it has pure slope s. When it is étale, this is essentially local class field theory. Assume that A contains an element α of valuation s. Let $\delta : \mathbb{Q}_p^\times \to A^\times$ be the character trivial on $\mathbb{Z}_p^\times$ which sends p to α. Then it is clear that $M \otimes_{\mathcal{R}_A} \mathcal{R}_A(\delta^{-1})$ is étale and hence is of the form $\mathcal{R}_A(\pi_K)(\delta^*)$ for some character $\delta^* : K^\times \to A^\times$. Then M is isomorphic to $\mathcal{R}_A(\pi_K)(\delta^* \cdot \delta \circ N_{K/\mathbb{Q}_p})$. When the assumption that A contains an element of valuation s does not hold, choose a finite Galois extension A'/A with this property and apply the preceding argument to obtain $M \otimes_A A' \cong \mathcal{R}_{A'}(\pi_K)(\delta)$ for some A'-valued character δ. For each $\sigma \in \mathrm{Gal}(A'/A)$, one has $\mathcal{R}_{A'}(\pi_K)(\sigma\circ\delta) \cong M \otimes_A A' \otimes_{A',\sigma} A \cong M \otimes_A A' \cong \mathcal{R}_{A'}(\pi_K)(\delta)$, and therefore by uniqueness $\sigma \circ \delta = \delta$. It follows that δ takes values in A. To relate M to δ over A, note first that any nonzero element of $H^0_{\varphi,\gamma_K}((M \otimes_A A')(\delta^{-1}))$ is an isomorphism, since it is obtained by applying $\otimes\delta$ to a nonzero homomorphism from $\mathcal{R}_{A'}(\pi_K)$ into the trivial rank one (φ, Γ_K)-module $(M \otimes_A A')(\delta^{-1})$. Then note that

$$H^0_{\varphi,\gamma_K}(M(\delta^{-1})) \otimes_A A' \cong H^0_{\varphi,\gamma_K}((M \otimes_A A')(\delta^{-1})) \neq 0,$$

so we may choose a nonzero element $f : \mathcal{R}_A(\pi_K)(\delta) \to M$ of $H^0_{\varphi,\gamma_K}(M(\delta^{-1}))$. Since $A \to A'$ is faithfully flat, its image $f : \mathcal{R}_{A'}(\pi_K)(\delta) \to M \otimes_A A'$ in $H^0_{\varphi,\gamma_K}((M \otimes_A A')(\delta^{-1}))$ is nonzero, hence an isomorphism, and again by faithful flatness f is an isomorphism.

We now treat the case where A is an arbitrary artinian L-algebra, which we may freely assume is local. Let I be the maximal ideal of A; we induct on the nilpotency index e of I. If $e = 1$, then A is a field and the claim is treated in the preceding paragraph. Otherwise, using the induction hypothesis, we know that $M \otimes_A A/I^{e-1}$ is of the form $\mathcal{R}_{A/I^{e-1}}(\pi_K)(\delta_{e-1})$ for an A/I^{e-1}-valued character δ_{e-1}. Since the spaces constructed in Proposition 6.1.1 are smooth, we may lift δ_{e-1} to some character δ_e with values in $A/I^e = A$. Replacing M with $M(\delta_e^{-1})$, we may assume that $M \otimes_A A/I^{e-1}$ is trivial. In this case, the options for M are classified by $H^1_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K) \otimes_A I^{e-1})$. We may also assume that the composition $K^\times \xrightarrow{\delta} A^\times \to (A/I^{e-1})^\times$ is trivial, so that the options for δ are classified by $H^1(K, I^{e-1})$. The construction $\delta \mapsto \mathcal{R}_A(\pi_K)(\delta)$ defines a map $H^1(K, I^{e-1}) \to H^1_{\varphi,\gamma_K}(\mathcal{R}_A(\pi_K) \otimes_A I^{e-1})$ which coincides with the natural bijection from Proposition 2.3.7; this completes the induction. □

The corresponding statement over a general rigid L-analytic space is the following theorem. In the case $K = \mathbb{Q}_p$, it provides an affirmative answer to a question of Bellaïche [2, Section 3, Question 1].

Theorem 6.2.14. *Let X be a rigid L-analytic space and let M be a (φ, Γ_K)-module of rank* 1 *over $\mathcal{R}_X(\pi_K)$.*

(1) *There exist a continuous character $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and an invertible sheaf $\mathcal{L}$ on X, such that $M \simeq \mathcal{R}_X(\pi_K)(\delta) \otimes_{\mathcal{O}_X} \mathcal{L}$.*
(2) *The character δ corresponds to the morphism $X \to X_L^{\mathrm{an}}$ whose graph is the space Γ_M defined in Definition* 6.2.15*. In particular, it is unique.*
(3) *The sheaf $\mathcal{L}$ is unique up to isomorphism. Moreover, one can take $\mathcal{L} = H^0_{\varphi,\gamma_K}(M(\delta^{-1}))$, in which case the isomorphism is canonical.*

To complete the statement of the theorem, we exhibit the space Γ_M using the cohomology of (φ, Γ_K)-modules.

Definition 6.2.15. Form the (φ, Γ_K)-module $M(\delta_L^{-1})$ over $X \times_L X_L^{\mathrm{an}}$. By Theorems 4.4.2 and 4.4.3, we may view $N' = H^2_{\varphi,\gamma_K}(M(\delta_L^{-1})^*)$ and $N'' = H^2_{\varphi,\gamma_K}(M^\vee(\delta_L)^*)$ as coherent sheaves on $X \times_L X_L^{\mathrm{an}}$ whose formation commutes with arbitrary (not necessarily flat) base change on X. Let Γ'_M and Γ''_M be the closed analytic subspaces of $X \times_L X_L^{\mathrm{an}}$ cut out by the annihilator ideal sheaves of N' and N'', respectively. Put $\Gamma_M = \Gamma'_M \times_{X \times_L X_L^{\mathrm{an}}} \Gamma''_M$.

The proof of Theorem 6.2.14 occupies the remainder of this subsection. We first verify that the theorem is equivalent to the claim that Γ_M is indeed a graph.

Lemma 6.2.16. *Take X, M as in Theorem* 6.2.14*.*

(1) *If part (1) of Theorem* 6.2.14 *holds for M, then so does part (2); in particular, the map $\Gamma_M \to X$ is an isomorphism.*
(2) *If $\Gamma_M \to X$ is an isomorphism, then Theorem* 6.2.14 *holds in full for M.*
(3) *The formation of $\Gamma'_M, \Gamma''_M, \Gamma_M$ commutes with arbitrary base change on X. (We will use this property without comment in what follows.)*

Proof. We first check (1). By translating using the group structure on X_L^{an}, we may reduce the claim to the case where δ is the trivial character. Since the claim is local on X, we may also assume that M is itself a trivial (φ, Γ_k)-module. By Theorem 4.4.3, we may further reduce to the case $X = \mathrm{Max}(L)$. By Proposition 6.2.8(2), for $\delta' : K^\times \to L^\times$ a continuous character, in order for $H^2_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta')^*)$ to be nontrivial either δ' must be trivial or δ' must carry a uniformizer of K to an element of negative valuation. In particular, for both $H^2_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta')^*)$ and $H^2_{\varphi,\gamma_K}(\mathcal{R}_L(\pi_K)(\delta'^{-1})^*)$ to be nontrivial, the character δ' must be trivial. Consequently, Γ_M is supported at the identity point of X_L^{an}; it thus remains to check that Γ_M is reduced. Suppose the contrary; we can then choose a subspace $Y = \mathrm{Max}(A)$ of Γ_M which is local artinian of length 2. Let M_Y be the pullback of $M(\delta_L^{-1})$ to $\mathcal{R}_Y(\pi_K)$; then $\dim_L H^2_{\varphi,\gamma_K}(M_Y^*) > 1$. By restricting scalars from A to L

and then applying Theorem 4.4.5(3), we see that $\dim_L H^0_{\varphi,\gamma_K}(M_Y) > 1$; since $\dim_L H^0_{\varphi,\gamma_K}(\mathbf{m}_A M_Y) \leq 1$ and $M_Y/\mathbf{m}_A M_Y \cong \mathcal{R}_L(\pi_K)$ as a (φ, Γ_k)-module, the element $1 \in \mathcal{R}_L(\pi_K) \cong M_Y/\mathbf{m}_A M_Y$ must lift to some $\mathbf{e} \in H^0_{\varphi,\gamma_K}(M_Y)$. Since $\mathbf{e}$ maps to a free generator of $M_Y/\mathbf{m}_A M_Y$, by Nakayama's lemma it is also a free generator of M_Y. However, this means that we have a nontrivial character (namely the restriction of δ_L^{-1} to Y) whose corresponding (φ, Γ_K)-module is trivial, contradicting the uniqueness aspect of Lemma 6.2.13. Hence, Γ_M must be reduced, which completes the proof that Γ_M coincides with the graph of δ.

We next check (2). Let δ be the character corresponding to the morphism with graph Γ_M. To check the full conclusion of Theorem 6.2.14, again by translating on X_L^{an} we reduce to the case where δ is the trivial character. In case $X = \mathrm{Max}(A)$ for A an artinian local ring, we know by Lemma 6.2.13 that the conclusion of Theorem 6.2.14 holds, necessarily for the trivial character. In general, this implies that for each closed immersion $\mathrm{Max}(A) \to X$ with A an artinian local ring, $H^2_{\varphi,\gamma_K}(M_A) = 0$ and $H^0_{\varphi,\gamma_K}(M_A)$ is free of rank 1. By restricting scalars from A to L and applying Theorem 2.3.11, we also have

$$\dim_L H^1_{\varphi,\gamma_K}(M_A) = (1 + [K : \mathbb{Q}_p]) \dim_L A. \tag{6.2.16.1}$$

In particular, $H^2_{\varphi,\gamma_K}(M)$ is locally free of rank 0, so by Theorem 4.4.3(2), the formation of $H^1_{\varphi,\gamma_K}(M)$ also commutes with arbitrary base change. By (6.2.16.1), $H^1_{\varphi,\gamma_K}(M)$ must be locally free of rank $1 + [K : \mathbb{Q}_p]$, so by Theorem 4.4.3(2) again, the formation of $H^0_{\varphi,\gamma_K}(M)$ also commutes with arbitrary base change. Therefore, $H^0_{\varphi,\gamma_K}(M)$ is locally free of rank 1, so all parts of Theorem 6.2.14 hold.

We finally check (3). It suffices to check that the sheaves N' and N'' are locally monogenic; this follows from Proposition 6.2.8(2) and Nakayama's lemma. □

Lemma 6.2.16 allows us to parlay the artinian case of Theorem 6.2.14 into a partial result in the general case.

Lemma 6.2.17. *Take X, M as in Theorem* 6.2.14.

1. *The map $\Gamma_M \to X$ is an open immersion, is bijective on rigid analytic points, and is an isomorphism on affinoid subdomains of Γ_M.*
2. *For any positive integer m, the multiplication-by-m map on X_L^{an} induces an isomorphism $\Gamma_M \cong \Gamma_{M^{\otimes m}}$.*

Proof. For $X = \mathrm{Max}(A)$ with A an artinian ring, both parts are immediate from Lemma 6.2.13 and Lemma 6.2.16(1). To deduce (1), recall that an isomorphism of affinoid spaces may be detected on the level of completed local rings [11, Proposition 7.3.3/5], or equivalently on the level of artinian subspaces.

To deduce (2), note that since Γ_M and $\Gamma_{M^{\otimes m}}$ are closed analytic subspaces of $X \times_L X_L^{\mathrm{an}}$ and the multiplication-by-m map on X_L^{an} is finite, the induced map $\Gamma_M \to \Gamma_{M^{\otimes m}}$ is finite and hence quasicompact. Consequently, we may again use [11, Proposition 7.3.3/5] to reduce to the artinian case. □

Lemma 6.2.17(1) does not suffice to imply that $\Gamma_M \to X$ is an isomorphism (see Example 6.2.21); one must also know that $\Gamma_M \to X$ is quasicompact. To establish quasicompactness, we exploit the factorization of X_L^{an} and the relationship of $X_{\mathrm{Sen}}^{\mathrm{an}}$ to the Sen weight. In the following lemmas, take X, M as in Theorem 6.2.14 and assume also that X is affinoid.

Lemma 6.2.18. *The image of* $\Gamma_M \to X^{\mathrm{an}}_{L,2}$ *is contained in an affinoid subdomain of* $X^{\mathrm{an}}_{L,2}$.

Proof. Let T be the coordinate on $X^{\mathrm{an}}_{L,2} \cong \mathbb{G}^{\mathrm{an}}_{m,L}$ corresponding to evaluation at ϖ_K. By Proposition 3.3.2(2) applied to the restriction of scalars of M from $\mathcal{R}_X(\pi_K)$ to $\mathcal{R}_X$, the function $\log |T|$ is bounded below on Γ'_M and bounded above on Γ''_M, and hence bounded on Γ_M. This proves the claim. □

Corollary 6.2.19. *For any point* $\eta \in X^{\mathrm{an}}_{L,1}$, *the map* $\Gamma_M \times_{X^{\mathrm{an}}_{L,1}} \eta \to X$ *is a closed immersion.*

Proof. Identify $X^{\mathrm{an}}_{L,2}$ with $\mathbb{G}^{\mathrm{an}}_{m,L}$ and then embed the latter into $\mathbb{P}^{1,\mathrm{an}}_L$. Then $\Gamma_M \times_{X^{\mathrm{an}}_{L,1}} \eta \to X \times_L \eta \times_L X^{\mathrm{an}}_{L,2}$ is a closed immersion and hence proper. The composition with $X \times_L \eta \times_L X^{\mathrm{an}}_{L,2} \to X \times_L X^{\mathrm{an}}_{L,2}$ is also proper because the latter map is proper. The further composition with $X \times_L X^{\mathrm{an}}_{L,2} \to X \times_L \mathbb{P}^{1,\mathrm{an}}_L$ is also proper because the image of Γ_M in $X^{\mathrm{an}}_{L,2}$ is contained in an affinoid subdomain by Lemma 6.2.18. The further composition with $X \times_L \mathbb{P}^{1,\mathrm{an}}_L \to X$ is also proper because the latter map is proper. Consequently, the map $\Gamma_M \times_{X^{\mathrm{an}}_{L,1}} \eta \to X$ is both proper and a locally closed immersion (by Lemma 6.2.17), hence a closed immersion. □

Lemma 6.2.20. *There exist a positive integer* m *and a continuous character* $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$ *such that* $\Gamma_{M^{\otimes m}(\delta^{-1})}$ *is contained in* $X \times_L X^{\mathrm{an}}_{L,\mathrm{fin}} \times_L X^{\mathrm{an}}_{L,2}$.

Proof. Note that the Sen weight of $M^{\otimes m}$ equals m times the Sen weight of M. By choosing m to be a suitably large power of p, we may ensure that the Sen weight of $M^{\otimes m}$ belongs to the domain of convergence of the exponential map. In this case, it lifts to a continuous character $\delta : \mathcal{O}_K^\times \to \Gamma(X, \mathcal{O}_X)^\times$, which we may extend to $K^\times$ by decreeing that $\delta(\varpi_K) = 1$. To check that this choice of m and δ has the desired effect, we may reduce to the artinian case, then apply Lemmas 6.2.13 and 6.2.12. □

We can now establish that $\Gamma_M \to X$ is quasicompact and hence complete the proof of Theorem 6.2.14.

Proof of Theorem 6.2.14. By Lemma 6.2.16, it suffices to prove that $\Gamma_M \to X$ is an isomorphism. For this, we may assume X is affinoid and connected.

By Lemma 6.2.17(1), $\Gamma_M \to X$ is bijective on rigid analytic points and isomorphic on affinoid subdomains of Γ_M; it thus suffices to prove that $\Gamma_M \to X$ is quasicompact. By Lemma 6.2.17(2), it suffices to prove quasicompactness after replacing M with $M^{\otimes m}(\delta)$ for some positive integer m and some continuous character $\delta : K^\times \to \Gamma(X, \mathcal{O}_X)^\times$. By Lemma 6.2.20, we may thus reduce to the case where Γ_M is contained in $X \times_L X^{\mathrm{an}}_{L,1} \times_L X^{\mathrm{an}}_{L,\mathrm{fin}}$. Since the map $\Gamma_M \to X$ is surjective on points, there exists a point $\eta \in X^{\mathrm{an}}_{L,\mathrm{fin}}$ such that $\Gamma_M \times_{X^{\mathrm{an}}_{L,\mathrm{fin}}} \eta \neq \emptyset$. The map $\Gamma_M \times_{X^{\mathrm{an}}_{L,\mathrm{fin}}} \eta \to X$ is on one hand a closed immersion (by Corollary 6.2.19) and an open immersion (by Lemma 6.2.17(1) and the fact that $\eta \to X^{\mathrm{an}}_{L,\mathrm{fin}}$ is an open immersion). Since X is connected and $\Gamma_M \times_{X^{\mathrm{an}}_{L,\mathrm{fin}}} \eta$ is nonempty, $\Gamma_M \times_{X^{\mathrm{an}}_{L,\mathrm{fin}}} \eta \to X$ must be an isomorphism. In particular, the map $\Gamma_M \to X$, which is an open immersion by Lemma 6.2.17, must be an isomorphism and hence quasicompact. This completes the proof. □

We thank Michael Temkin for suggesting the following example.

Example 6.2.21. Put $X = \mathrm{A}^1[0,\infty]$ and $U = \mathrm{A}^1(0,\infty] \cup \mathrm{A}^1[0,0]$; that is, X is a closed unit disc and U is the union of the boundary circle and its complement. Then U is a quasi-Stein space and the natural map $U \to X$ is an open immersion which is a bijection on rigid analytic points but not an isomorphism. The corresponding map of Berkovich spaces is also a bijection, but the corresponding map of Huber spaces is not (it misses the type 5 point pointing into the open unit disc).

This example shows that Lemma 6.2.17(1) does not suffice for the proof of Theorem 6.2.14, thus necessitating the subsequent lemmas. It may be possible to give an alternate completion of the proof by proving directly that $\Gamma_M \to X$ is surjective on Huber points, but even the definition of the ring $\mathcal{R}_A(\pi_K)$ for A a field carrying a valuation of arbitrary rank is a bit delicate.

6.3. **Global triangulation.** The notion of a *trianguline* representation of $G_{\mathbb{Q}_p}$, one for which the associated (φ,Γ)-module is a successive extension of rank 1 (φ,Γ)-modules, was introduced by Colmez [19]. This notion was later generalized by Nakamura [39] to representations of G_K for K a finite extension of $\mathbb{Q}_p$, using Berger's language of $\mathbb{B}$-pairs. In this subsection, we redevelop the theory of trianguline representations of G_K using (φ,Γ)-modules. Although the two languages are equivalent, the approach using (φ,Γ)-modules is better suited for arithmetic families.

In this subsection, we use Construction 6.2.4 to define the notion of a trianguline (φ,Γ_K)-module. We then show that densely pointwise trianguline (φ,Γ)-modules admit triangulations on Zariski-dense open subsets.

Definition 6.3.1. Let $\delta_1,\dots,\delta_d : K^\times \to \Gamma(X,\mathcal{O}_X)^\times$ be continuous characters. A (φ,Γ_K)-module M of rank d over $\mathcal{R}_X(\pi_K)$ is *trianguline with ordered parameters* $\delta_1,\dots,\delta_d$ if, after perhaps enlarging L, there exists an increasing filtration $(\mathrm{fil}_i M)_{i=0,\dots,d}$ of M given by (φ,Γ_K)-submodules and line bundles $\mathcal{L}_1,\dots,\mathcal{L}_d$ on X such that each $\mathrm{gr}_i M \simeq \mathcal{R}_X(\pi_K)(\delta_i)\otimes_{\mathcal{O}_X}\mathcal{L}_i$. Such a filtration is called a *triangulation (with ordered parameters* $\delta_1,\dots,\delta_d$*)* of M. Since we proved in Theorem 6.2.14 that every rank one object is of the form $\mathcal{R}_X(\pi_K)(\delta)\otimes_{\mathcal{O}_X}\mathcal{L}$ for a uniquely determined δ and a uniquely determined (up to isomorphism) $\mathcal{L}$, it follows that M is trianguline if and only if after perhaps enlarging L it is a successive extension of rank one objects, and that given the filtration by saturated (φ,Γ_K)-stable submodules with rank one graded pieces, the ordered parameters and the required line bundles are well-defined (up to isomorphism).

In the case $X = \mathrm{Max}(L)$, we say that M is *strictly trianguline with ordered parameters* $\delta_1,\dots,\delta_d$ if, for each i, the submodule $\mathrm{fil}_{i+1}M$ is the unique way of enlarging $\mathrm{fil}_i M$ to a submodule of M with quotient isomorphic to $\mathcal{R}_X(\pi_K)(\delta_{i+1})$ (and in particular the trianguline filtration is the unique one with these ordered parameters). When M is already known to be trianguline with these ordered parameters, this is equivalent to $H^0_{\varphi,\gamma_K}((M/\mathrm{fil}_i M)(\delta_{i+1}^{-1}))$ being one-dimensional for all i, or alternatively $H^0_{\varphi,\gamma_K}((\mathrm{fil}_i M)^\vee(\delta_i))$ being one-dimensional for any i.

Definition 6.3.2. A subset Z of closed points of X is called *Zariski dense* if there exists an admissible affinoid cover $\{\mathrm{Max}(A_i)\}_{i\in I}$ of X such that $Z\cap\mathrm{Max}(A_i)$ is Zariski dense in $\mathrm{Max}(A_i)$ for each i. Note that this does not imply that Z is Zariski dense in all affinoid subdomains of X.

A (φ,Γ_K)-module M over $\mathcal{R}_X(\pi_K)$ is called *densely pointwise strictly trianguline* if there exist continuous characters $\delta_1,\dots,\delta_d : K^\times \to \Gamma(X,\mathcal{O}_X)^\times$ and a Zariski dense subset $X_{\mathrm{alg}} \subseteq X$ such that for each $z \in X_{\mathrm{alg}}$, M_z is strictly trianguline with ordered parameters $\delta_{1,z}, \ldots, \delta_{d,z}$.

Corollary 6.3.3. *Let M be a (φ, Γ_K)-module over $\mathcal{R}_X(\pi_K)$. Then $H^*_{\varphi,\gamma_K}(M)$ and $H^*_{\varphi,\gamma_K}(M/t_\sigma)$ for any $\sigma \in \Sigma$ are coherent sheaves over X. Moreover, locally on X they are the cohomology of complexes of locally free sheaves concentrated in degrees $[0, 2]$.*

Proof. The claims for $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M)$ follow from Theorem 4.4.5, so we show how to deduce the results for $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M/t_\sigma)$ from this. It suffices to assume that $X = \mathrm{Max}(A)$ is affinoid. Since one has a short exact sequence of complexes

$$0 \to \mathrm{C}^\bullet_{\varphi,\gamma_K}(t_\sigma M) \to \mathrm{C}^\bullet_{\varphi,\gamma_K}(M) \to \mathrm{C}^\bullet_{\varphi,\gamma_K}(M/t_\sigma) \to 0,$$

it is clear that $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M/t_\sigma) \in \mathbf{D}^{[-1,2]}_{\mathrm{perf}}(A)$. But Lemma 2.1.5 shows that $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M/t_\sigma)$ consists of flat A-modules, so Lemma 4.1.3 allows us to conclude. □

Before moving on to the main theorem, we briefly discuss a flattening technique using Fitting ideals. (This is inspired by the work of Raynaud and Gruson [44].)

Definition 6.3.4. Given a finitely generated module M over a noetherian ring R, write it as the cokernel of an R-linear map $\phi : R^m \to R^n$; let $\Phi \in \mathrm{M}_{n\times m}(R)$ be the matrix of ϕ. For $r \in \mathbb{Z}_{\geq 0}$, the *$r$th Fitting ideal* of M is defined to be the ideal $\mathrm{Fitt}_r(M)$ of R generated by the $(n - r)$-minors of Φ. This is independent of the choice of the presentation [24, Corollary-Definition 20.4]. Moreover, the construction of Fitting ideals commutes with arbitrary base change [24, Corollary 20.5].

Lemma 6.3.5. *Let R be a noetherian ring and let M be a finitely generated module given as the cokernel of an R-linear homomorphism $\phi : R^m \to R^n$.*

(1) If $M \cong N \oplus F$ is a direct sum of R-modules with F locally free of rank d, then $\mathrm{Fitt}_{r+d}(M) = \mathrm{Fitt}_r(N)$.

(2) For any $r \in \mathbb{Z}_{\geq 0}$, $\mathrm{Fitt}_r(M)$ *is a nil-ideal if and only if the rank of M at each generic point of* $\mathrm{Spec}\, R$ *is at least $r + 1$.*

(3) Assume that there exists $r \in \mathbb{N}$ such that $\mathrm{Fitt}_{r-1}(M) = (0)$ *and* $\mathrm{Fitt}_r(M)$ *is generated by $a \in R$ that is not a zero-divisor. Then the image of ϕ is flat over R of constant rank $n - r$. In particular, M has constant generic rank r and it has projective-dimension ≤ 1.*

Proof. (1) We may work Zariski locally on $\mathrm{Spec}\, R$, reducing to the case where F is free. In this case, the claim is straightforward from the definitions.

(2) The ideal $\mathrm{Fitt}_r(M)$ consists of nilpotent elements if and only if $\mathrm{Fitt}_r(M)$ is contained in all minimal primes $\mathfrak{p}$, if and only if, for each minimal prime $\mathfrak{p}$, $\mathrm{Fitt}_r(M \otimes_R \mathrm{Frac}(R/\mathfrak{p})) = (0)$. This reduces us to the case where R is a field, in which case the claim follows from elementary linear algebra.

(3) This follows from a variant of the proof of [44, Chapter 4, Proposition 1 and Lemma 1]. Let $\Phi = (\phi_{ij})_{1\leq i\leq n, 1\leq j\leq m}$ denote the matrix of ϕ for the standard bases $\mathbf{e}_1, \ldots, \mathbf{e}_m$ for R^m and $\mathbf{f}_1, \ldots, \mathbf{f}_n$ for R^n. Zariski locally on $\mathrm{Spec}\, R$, and after suitable permutation of the bases, and perhaps multiplying a by a unit in R, we may assume that $\det(\phi_{ij})_{1\leq i,j\leq n-r} = a$. Write Φ as a block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A is an $(n-r)\times(n-r)$-matrix with determinant a. Let $\mathrm{Adj}(A)$ denote the adjugate matrix of A.

We claim that we have the following identity

$$\begin{pmatrix} A \\ C \end{pmatrix} \operatorname{Adj}(A) B = a \begin{pmatrix} B \\ D \end{pmatrix}. \tag{6.3.5.1}$$

It suffices to check that $C \operatorname{Adj}(A)B = aD$, and for this it suffices to replace C by a single one of its rows, B by a single one of its columns, and D by the corresponding one of its entries (which is just an element of R). In this case, it is a simple linear algebra exercise to check that $a(aD - C\operatorname{Adj}(A)B) = a \cdot \det \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$; the right-hand side vanishes because it is an $(n-r+1)$-minor of Φ. Since a is not a zero-divisor, we have $aD = C \operatorname{Adj}(A)B$.

Note that each entry of $\operatorname{Adj}(A)B$ is an $(n-r)$-minor of Φ, and hence divisible by a. Since a is not a zero-divisor, there is a unique matrix E with coefficients in R such that $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right) E = \left(\begin{smallmatrix} B \\ D \end{smallmatrix}\right)$. This means that $\phi(\mathbf{e}_1), \dots, \phi(\mathbf{e}_{n-r})$ generate the image of ϕ. Since $\det A$ is not a zero-divisor, the image of ϕ is freely generated by $\phi(\mathbf{e}_1), \dots, \phi(\mathbf{e}_{n-r})$. The lemma is proved. □

Corollary 6.3.6. *Let X be a reduced rigid L-analytic space.*

(1) Let $\phi : C \to D$ be a homomorphism of locally free coherent sheaves on X. Assume that the generic rank r of the cokernel of ϕ is constant on X. Let $f : Y \to X$ be the blowup of X along the rth Fitting ideal of the cokernel of ϕ. Then the homomorphism $f^*\phi : f^*C \to f^*D$ has flat image. The formation of (Y, f) commutes with dominant base change in X. Moreover, for any morphism $g : Z \to Y$, the composite $f \circ g : Z \to X$ has the same property: $(f \circ g)^*\phi$ has flat image.

(2) Let $(C^\bullet, d^\bullet)$ be a bounded above complex of locally free coherent sheaves on X. Assume that the generic ranks of the $H^i(C^\bullet)$ and the $\operatorname{Coker}(d^i)$ are constant on X. Construct the sequence of morphisms

$$\cdots \to Y^i \xrightarrow{f_i} Y^{i+1} \xrightarrow{f_{i+1}} \cdots \to X$$

as follows. Let $Y^i = X$ for $i \gg 0$. Given $f_i, f_{i+1}, \dots$, put

$$g_i = \cdots \circ f_{i+1} \circ f_i : Y^i \longrightarrow X$$

and apply (1) to $g_i^* d^{i-1} : g_i^* C^{i-1} \to g_i^* C^i$ to obtain $f_{i-1} : Y^{i-1} \to Y^i$. Then the (Y_i, f_i) depend only on the quasi-isomorphism class of $(C^\bullet, d^\bullet)$, and their formation commutes with dominant base change in X.

Proof. We may assume that X is affinoid throughout. Also, we remark that generic ranks do not change under dominant base change.

Then the first claim of (1) is clear from Lemma 6.3.5. The second claim follows because the formations of Fitting ideals and blowups commute with base change. The third claim is obvious.

For (2), we prove that for each i the construction of f_{i-1} depends only on the quasi-isomorphism class of $(C^\bullet, d^\bullet)$. Note that our hypothesis on the generic ranks of the $H^i(C^\bullet)$ and the $\operatorname{Coker}(d^i)$ implies that the $\operatorname{Image}(d^i)$ and the C^i also have generic ranks that are constant on X. We have the short exact sequence

$$0 \to H^i(g_i^* C^\bullet) \to \operatorname{Coker}(g_i^* C^{i-1} \to g_i^* C^i) \to \operatorname{Image}(g_i^* C^i \to g_i^* C^{i+1}) \to 0,$$

in which the last term is a flat module by the construction of g_i, hence it is projective of constant rank s. In particular, the short exact sequence splits. Let r be the generic rank of $H^i(g_i^*C^\bullet)$, so that $\mathrm{Coker}(g_i^*C^{i-1}\to g_i^*C^i)$ has generic rank $r+s$. Then f_{i-1} is none other than the blowup of the ideal

$$\mathrm{Fitt}_{r+s}(\mathrm{Coker}(g_i^*C^{i-1}\to g_i^*C^i))=\mathrm{Fitt}_r(H^i(g_i^*C^\bullet))$$

on Y^i. As the complex $(C^\bullet,d^\bullet)$ consists of locally free coherent sheaves, it is clear that $H^i(g_i^*C^\bullet)$, and therefore also the above ideal, depends only on its quasi-isomorphism class. The commutation with base change follows from the corresponding property in (1). □

Lemma 6.3.7. *Let R be a noetherian ring. Let Q be the cokernel of an injective homomorphism $\phi: M\to N$ of finitely generated flat R-modules of constant ranks m and n, respectively. Let Z be the closed subscheme of* $\mathrm{Spec}\,R$ *defined by the $(n-m)$th Fitting ideal of Q. Then for any $z\in\mathrm{Spec}\,R$, one has $\mathrm{Tor}_1^R(Q,\kappa_z)\neq 0$ if and only if $z\in Z$.*

Proof. Tensoring ϕ with κ_z gives $0\to\mathrm{Tor}_1^R(Q,\kappa_z)\to M_z\xrightarrow{\phi_z}N_z\to Q_z\to 0$. So $\mathrm{Tor}_1^R(Q,\kappa_z)\neq 0$ if and only if ϕ_z is not injective, if and only if the rank of ϕ_z is not m, if and only if the $(n-m)$th Fitting ideal vanishes at z, if and only if $z\in Z$. □

Remark 6.3.8. In the following theorem and its corollary, we assume that X is a reduced rigid analytic space over L. The assumption on the reducedness of X is essential because we invoke arguments at the residue fields, which do not see non-reduced structure.

Theorem 6.3.9. *Let X be a reduced rigid L-analytic space. Let M be a (φ,Γ_K)-module over $\mathcal{R}_X(\pi_K)$ of rank d and let $\delta: K^\times\to\Gamma(X,\mathcal{O}_X)^\times$ be a continuous character. Suppose that there exists a Zariski dense subset X_{alg} of closed points of X such that for every $z\in X_{\mathrm{alg}}$, $H^0_{\varphi,\gamma_K}(M_z^\vee(\delta_z))$ is one-dimensional, and the image of $\mathcal{R}_{\kappa_z}(\pi_K)$ under any basis of this space is saturated in $M_z^\vee(\delta_z)$. Then there exist canonical data of*

- (a) *a proper birational morphism $f: X'\to X$ of reduced rigid analytic spaces and*
- (b) *a unique (up to $\mathcal{O}_{X'}^\times$) homomorphism $\lambda: f^*M\to\mathcal{R}_{X'}(\pi_K)(\delta)\otimes_{X'}\mathcal{L}$ of (φ,Γ_K)-modules over $\mathcal{R}_{X'}(\pi_K)$, where $\mathcal{L}$ is a line bundle over X' with trivial (φ,Γ_K)-actions,*

such that the following conditions are satisfied.

1. *The set Z of closed points $z\in X'$ failing to have the following property is Zariski closed and disjoint from $f^{-1}(X_{\mathrm{alg}})$ (hence its complement is Zariski open and dense): the induced homomorphism $\lambda_z: M_z\to\mathcal{R}_{\kappa_z}(\pi_K)(\delta_z)$ is surjective and the corresponding element spans $H^0_{\varphi,\gamma_K}(M_z^\vee(\delta_z))$ (hence the latter is one-dimensional).*
2. *Locally on X', the cokernel of λ is killed by some power of t, and is supported over Z in the sense that for any analytic function g vanishes along Z, some power of g kills the cokernel of λ too.*
3. *The kernel of λ is a (φ,Γ_K)-module over $\mathcal{R}_{X'}(\pi_K)$ of rank $d-1$.*

Moreover, when X is a smooth rigid analytic curve, we can take $X'=X$.

Proof. If we prove the theorem for the normalization $p:\widetilde{X}\to X$ (with $p^*\delta$ and $p^{-1}(X_{\mathrm{alg}})$) to obtain $\widetilde{f}:\widetilde{X}'\to\widetilde{X}$, then the composite $f=p\circ\widetilde{f}: X'=\widetilde{X}'\to X$ satisfies the claims of the theorem, so we replace X by $\widetilde{X}$. We remark here that in the case where X is a smooth rigid analytic curve, there does not exist a nonisomorphic proper birational morphism into X, so we have $\widetilde{X} = X$ (and, later in the proof, $X_0' = X$ and $X' = X$) in this case. It also suffices to treat connected components separately. So we assume henceforth that X is normal and connected. Note that every connected affinoid domain $\mathrm{Max}(A)$ in X has $\mathrm{Spec}(A)$ irreducible: using [22, Lemma 2.1.1] and excellence to pass between completions, we see that $\mathrm{Max}(A)$ is normal, and since it is also connected it is irreducible by definition, so that [22, Lemma 2.1.4] shows that $\mathrm{Spec}(A)$ is irreducible. It follows that any coherent sheaf on X, or its pullback under any dominant morphism, has constant generic rank.

By Corollary 6.3.3, locally on X, say over the affinoid subdomain $W \subseteq X$, the complex $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M^\vee(\delta))$ (resp. $\mathrm{C}^\bullet_{\varphi,\gamma_K}(M^\vee(\delta)/t_\sigma)$) is quasi-isomorphic to some complex $A^\bullet$ of locally free sheaves concentrated in degrees $[0,2]$ (resp. $[-1,2]$), and by Theorem 4.4.3 for any morphism $j : W' \to W$ one has that $\mathrm{C}^\bullet_{\varphi,\gamma_K}(j^*M^\vee(\delta))$ (resp. $\mathrm{C}^\bullet_{\varphi,\gamma_K}(j^*M^\vee(\delta)/t_\sigma)$) is represented by the naïve pullback $j^*A^\bullet$. With this in mind, by Corollary 6.3.6(2), locally on X there exists a proper birational morphism $f_0 : X_0' \to X$ such that $N_0 = f_0^*(M^\vee(\delta))$ satisfies the conditions

(i) $H^0_{\varphi,\gamma_K}(N_0)$ is flat and $H^i_{\varphi,\gamma_K}(N_0)$ has Tor-dimension less than or equal to 1 for each $i = 1, 2$ and

(ii) for each $\sigma \in \Sigma$, $H^0_{\varphi,\gamma_K}(N_0/t_\sigma)$ is flat and $H^i_{\varphi,\gamma_K}(N_0/t_\sigma)$ has Tor-dimension less than or equal to 1 for each $i = 1, 2$.

By Theorem 4.4.3(2) and the invariance under base change in Corollary 6.3.6(2), the locally constructed morphisms glue to a proper birational morphism $f_0 : X_0' \to X$ locally satisfying the same conditions. Note that X_0' is connected; we may and will always take X_0' to be reduced and normal.

For a closed point $z \in X_0'$, the base change spectral sequence $E_2^{i,j} = \mathrm{Tor}_{-i}^{X_0'}(H^j_{\varphi,\gamma_K}(N_0), \kappa_z) \Rightarrow H^{i+j}_{\varphi,\gamma_K}(N_{0,z})$ gives a short exact sequence (using that $H^1_{\varphi,\gamma_K}(N_0)$ has Tor-dimension at most 1)

$$0 \to H^0_{\varphi,\gamma_K}(N_0) \otimes_{X_0'} \kappa_z \to H^0_{\varphi,\gamma_K}(N_{0,z}) \to \mathrm{Tor}_1^{X_0'}(H^1_{\varphi,\gamma_K}(N_0), \kappa_z) \to 0. \tag{6.3.9.1}$$

The condition (i) allows us to invoke Lemma 6.3.7, so that the set Z_0' of $z \in X_0'$ for which the last term of (6.3.9.1) does not vanish is Zariski closed, and $X_0' - Z_0'$ is open and dense. (Note that the formation of Z_0' is compatible with pullback because it is given by the zero locus of a Fitting ideal.) Thus for any point z in the Zariski dense subset $f_0^{-1}(X_{\mathrm{alg}}) \backslash Z_0'$ of X_0', $H^0_{\varphi,\gamma_K}(N_0) \otimes_{X_0'} \kappa_z \cong H^0_{\varphi,\gamma_K}(N_{0,z})$ is one-dimensional. Therefore, in view of condition (i), $H^0_{\varphi,\gamma_K}(N_0)$ is a locally free sheaf over X_0' of rank one. Let $\mathcal{L}_0$ denote the dual line bundle of $H^0_{\varphi,\gamma_K}(N_0)$. Dualizing the natural homomorphism $\mathcal{R}_{X_0'}(\pi_K) \otimes_{X_0'} \mathcal{L}_0^\vee \to N_0$ gives a homomorphism $\lambda_0 : f_0^*M \to \mathcal{R}_{X_0'}(\pi_K)(\delta) \otimes_{X_0'} \mathcal{L}_0$ of (φ, Γ_K)-modules.

The uniqueness of λ_0 follows from the construction, and that $H^0_{\varphi,\gamma_K}(N_0)$ is a line bundle. We next verify properties (1)–(2) for X_0'. Afterwards we will construct morphism $f : X' \to X_0' \to X$ by blowing up in X_0', which will preserve the properties (1)–(2) shown over X_0', and check (3) in this setting.

(1) Consider the locus Z_0 consisting of closed points $z \in X_0'$ that fail the condition that $\lambda_z : M_z \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_z)$ is surjective and the corresponding element spans $H^0_{\varphi,\gamma_K}(M_z^\vee(\delta_z))$. For each $\sigma \in \Sigma$ we have a natural map $\chi_\sigma : H^0_{\varphi,\gamma_K}(N_0) \to H^0_{\varphi,\gamma_K}(N_0/t_\sigma)$. By the discussion above, (6.3.9.1) implies that $H^0_{\varphi,\gamma_K}(N_{0,z})$ is one-dimensional if and only if $z \notin Z_0'$. In this case, Lemma 6.2.10 implies that

$\mathcal{R}_{\kappa_z}(\pi_K) \to M_z^\vee(\delta_z)$ is saturated, or equivalently $M_z \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_z)$ is surjective, if and only if for each $\sigma \in \Sigma$ the map

$$H^0_{\varphi,\gamma_K}(M_z^\vee(\delta_z)) \cong H^0_{\varphi,\gamma_K}(N_0)\otimes_{X_0'}\kappa_z \to H^0_{\varphi,\gamma_K}(N_0/t_\sigma)\otimes_{X_0'}\kappa_z \hookrightarrow H^0_{\varphi,\gamma_K}(M_z^\vee(\delta_z)/t_\sigma)$$

is nontrivial. Here, the injectivity of the last homomorphism above follows from the base change spectral sequence $E_2^{i,j} = \mathrm{Tor}_{-i}^{X'}(H^j_{\varphi,\gamma_K}(N_0/t_\sigma), \kappa_z) \Rightarrow H^{i+j}_{\varphi,\gamma_K}(N_{0,z}/t_\sigma)$ and the fact that $H^1_{\varphi,\gamma_K}(N_0/t_\sigma)$ has Tor-dimension at most 1. It follows that Z_0 is the union of Z_0' and the subspaces where the respective χ_σ vanish, hence is Zariski closed. Finally, $f_0^* X_{\mathrm{alg}}$ is disjoint from Z_0 because our hypotheses on each point of X_{alg} are simply a restatement of the negation of the condition on membership in Z_0.

(2) Let Q_0 denote the cokernel of λ_0. By the discussion in (1), $Q_{0,z} = 0$ if $z \notin Z_0$. By Lemma 2.2.11, this implies that Q_0 is supported on Z_0, in the sense that, locally on X_0', for any analytic function g vanishing along Z_0, some power of g kills Q_0.

By (6.3.9.1), $H^0_{\varphi,\gamma_K}(N_0)\otimes\kappa_z$ injects into $H^0_{\varphi,\gamma_K}(N_{0,z})$ and hence λ_z is nontrivial. Thus, the image of λ_z is a (φ,Γ_K)-submodule of $\mathcal{R}_{\kappa_z}(\pi_K)(\delta_z)$. By Corollary 6.2.9, $Q_{0,z}$ is killed by some power of t.

We claim that, locally on X_0', Q_0 itself is killed by some power of t. So assume that X_0' is affinoid. We first observe that λ_0 is the base change of a homomorphism $\lambda_0^r : f_0^* M^r \to \mathcal{R}^r_{X_0'}(\pi_K)(\delta) \otimes_{X_0'} \mathcal{L}_0$ for some $r \in (0, C(\pi_K)]$. Let Q_0^r denote the cokernel of λ_0^r. We first note that $Q_0^{[r/p,r]}$ is killed by some power of t, say t^n. Indeed, $Q_0^{[r/p,r]}$ is supported on the zero locus of t in $\mathrm{Max}(\mathcal{R}^{[r/p,r]}_{X_0'}(\pi_K))$, which is an affinoid rigid analytic space, finite over X_0'. This implies that $Q_0^{[r/p^m,r/p^{m-1}]} = (\varphi^m)^* Q_0^{[r/p,r]}$ is also killed by t^n for every $m \in \mathbb{N}$. Hence, Q_0^r itself is killed by t^n, proving that Q_0 is killed by t^n.

To obtain the morphism $f : X' \to X$, affinoid-locally in X_0' we apply Corollary 6.3.6(1) to any finite presentation of $Q_0^{[r/p,r]}$ (which is a finite module by the preceding paragraph), and we glue these local constructions globally over X_0' using the invariance under dominant base change provided by the corollary. Write $\mathcal{L}$, λ, Q, Q^r, and $Q^{[r,s]}$ to denote the respective pullbacks along $X' \to X_0'$ of $\mathcal{L}_0$, λ_0, Q_0, Q_0^r, and $Q_0^{[r,s]}$. By our construction of X' the module $Q^{[r/p,r]}$ has Tor-dimension at most 1; pulling back by φ shows that $Q^{[r/p^m,r/p^{m-1}]}$ also has Tor-dimension at most 1 for all $m \in \mathbb{N}$; gluing shows that Q^r has Tor-dimension at most 1.

(3) Recall that Q is the cokernel of λ, and let P denote the kernel of λ. For any closed point $z \in X'$, the Tor spectral sequence computing the cohomology of the complex $\left[M \xrightarrow{\lambda} \mathcal{R}_{X'}(\pi_K)(\delta) \otimes_{X'} \mathcal{L}\right] \overset{\mathbf{L}}{\otimes}_{X'} \kappa_z$ gives rise to the exact sequence

$$0 \to \mathrm{Tor}_2^{X'}(Q,\kappa_z) \to P_z \to \mathrm{Ker}\left(\lambda_z : M_z \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_z)\right) \to \mathrm{Tor}_1^{X'}(Q,\kappa_z) \to 0.$$

By our construction of X' the first term vanishes for all z, and (2) implies that the last term in this sequence is killed by a power of t. In particular, P_z is a (φ,Γ_K)-module over $\mathcal{R}_{\kappa_z}(\pi_K)$ of rank $d-1$. For each affinoid subdomain $\mathrm{Max}(A) \subseteq X$, choose a model M^r for M over $\mathcal{R}^r_A(\pi_K)$ for some r; by Lemma 2.2.9, λ arises from a map $\lambda^r : M^r \to \mathcal{R}^r_A(\pi_K)(\delta)$, giving rise to a model $P^r = \mathrm{Ker}(\lambda^r)$ for P over $\mathcal{R}^r_A(\pi_K)$. Each P^r_z is a coadmissible model for P_z over $\mathcal{R}^r_{\kappa_z}(\pi_K)$, and is finite projective of rank $d-1$ by φ-equivariance. Invoking Lemma 2.1.8(2), we see that

P^r is the global sections of a φ-bundle over $\mathcal{R}^r_A(\pi_K)$, hence by Proposition 2.2.7 is finite projective of rank $d-1$. □

Corollary 6.3.10. *Let X be a reduced rigid analytic space over L. Let M be a densely pointwise strictly trianguline (φ,Γ_K)-module over $\mathcal{R}_X(\pi_K)$ of rank d, with respect to the ordered parameters $\delta_1,\dots,\delta_d$ and the Zariski dense subset X_{alg}. Then there exist canonical data of*

(a) *a proper birational morphism $f: X' \to X$ of reduced rigid analytic spaces and*
(b) *a unique increasing filtration $(\mathrm{fil}_i(f^*M))_{i=1,\dots,d}$ on the pullback (φ,Γ_K)-module f^*M over $\mathcal{R}_{X'}(\pi_K)$ via (φ,Γ_K)-stable coherent $\mathcal{R}_{X'}(\pi_K)$-submodules,*

such that the following conditions are satisfied.

(1) *The set Z of closed points $z \in X'$ at which $(\mathrm{fil}_\bullet(f^*M))_z$ fails to be a strictly trianguline filtration on M_z with ordered parameters $\delta_{1,z},\dots,\delta_{d,z}$ is Zariski closed in X' and disjoint from $f^{-1}(X_{\mathrm{alg}})$ (hence the complement of Z is Zariski open and dense).*
(2) *Each $\mathrm{gr}_i(f^*M)$ embeds (φ,Γ_K)-equivariantly into $\mathcal{R}_{X'}(\pi_K)(\delta_i) \otimes_{X'} \mathcal{L}_i$ for some line bundle $\mathcal{L}_i$ over X', and the cokernel of the embedding is, locally on X', killed by some power of t and supported on Z.*
(2′) *The first graded piece $\mathrm{gr}_1(f^*M)$ is isomorphic to $\mathcal{R}_{X'}(\pi_K)(\delta_1) \otimes_{X'} \mathcal{L}_1$.*

Moreover, when X is a smooth rigid analytic curve, we can take X' equal to X.

Proof. The existence of data satisfying all properties (1) and (2) follows from Theorem 6.3.9 by induction, noting for (1) the equivalent description of a strict triangulation in Definition 6.3.1.

We now prove (2'). For this, we go into the proof of Theorem 6.3.9. The claim is local on X', so we replace X' by a member of an admissible affinoid covering such that $\mathcal{L}_1$ is trivial and we have a natural injective homomorphism $\lambda^r : \mathrm{gr}_1(f^*M^r) \to \mathcal{R}^r_{X'}(\pi_K)(\delta_1)$ of (φ,Γ_K)-modules over $\mathcal{R}^r_{X'}(\pi_K)$ for some $r \in (0, C(\pi_K)]$. Let $\lambda^{[s,r]}$ denote its base change to $\mathcal{R}^{[s,r](\pi_K)}_{X'}$ for any $s \in (0,r]$ and let $Q^{[s,r]}$ denote the cokernel of the latter; we must prove that the latter is zero. Since $Q^{[s,r]}$ is killed by some power of t, it is a finite $\mathcal{O}_{X'}$-module. Tensoring $\lambda^{[s,r]}$ with κ_z for $z \in X'$ gives an exact sequence

(6.3.10.1)
$$0 \to \mathrm{Tor}_1^{X'}(Q^{[s,r]},\kappa_z) \to \mathrm{gr}_1(f^*M^{[s,r]})_z \xrightarrow{\lambda_z} \mathcal{R}^r_{\kappa_z}(\pi_K)(\delta_{1,z}) \to Q_z^{[s,r]} \to 0,$$

where the first zero is by Corollary 2.1.5. Since λ_z is a homomorphism of finitely generated, torsion-free modules over a finite product of Bézout domains and is nontrivial generically, it must be injective, so that $\mathrm{Tor}_1^{X'}(Q^{[s,r]},\kappa_z) = 0$. On the other hand, since $Q^{[s,r]}$ is finitely generated over $\mathcal{O}_{X'}$, [24, Exercise 6.2] shows that $Q^{[s,r]}$ is flat over X'. However, this is only possible if $Q^{[s,r]}$ is zero, because we know that Q_z and hence $Q_z^{[s,r]}$ is trivial at $z \notin Z$. □

Remark 6.3.11. Although the proper birational morphism $X' \to X$ appearing in Theorem 6.3.9 is canonically constructed, it is not clear to us if there is a *universal* such X' satisfying the conditions of Corollary 6.3.10.

The "bad locus" Z is necessary in the theorem; we will see this in the example of the Coleman-Mazur eigencurve (Proposition 6.4.5).

Remark 6.3.12. If M is the base change of a (φ, Γ)-module over $\mathcal{R}_X^{r_0}(\pi_K)$, then the filtrations obtained in Corollary 6.3.10 can be taken to be base-changed from a filtration over $\mathcal{R}_{X'}^{r_0}(\pi_K)$, at least locally on X'. This is because we may start by working with $M \otimes_{\mathcal{R}_{X'}^{r_0}(\pi_K)} \mathcal{R}_{X'}(\pi_K)$ and invoking Lemma 2.2.9.

Despite the fact that the global triangulation only behaves well away from the bad locus Z, we can show that actually at each closed point $z \in X$, the (φ, Γ_K)-module is trianguline (with slightly different parameters). We thank Matthew Emerton for suggesting this application.

Theorem 6.3.13. *Let X be a rigid analytic space over L. Let M be a densely pointwise strictly trianguline (φ, Γ_K)-module over $\mathcal{R}_X(\pi_K)$ of rank d, with ordered parameters $\delta_1, \dots, \delta_d$ and the Zariski dense subset X_{alg}. Then for every $z \in X$, the specialization M_z is trianguline with parameters $\delta'_{1,z}, \dots, \delta'_{d,z}$, where $\delta'_{i,z} = \delta_{i,z} \prod_{\sigma \in \Sigma} x_\sigma^{n_{i,z,\sigma}}$ for some $n_{i,z,\sigma} \in \mathbb{Z}$.*

Proof. Fix $z \in X$. After replacing X by the reduced subspace of a connected component of the normalization, we may assume that X is reduced and irreducible. Applying Corollary 6.3.10 to M gives a proper birational morphism $f : X' \to X$ such that f^*M admits a filtration of (φ, Γ_K)-modules over $\mathcal{R}_{X'}(\pi_K)$ as stated therein. For any point $z' \in f^{-1}(z)$ one has $(f^*M)_{z'} \cong M_z \otimes_{\kappa_z} \kappa_{z'}$, so it suffices to replace (X, M, z) by (X', f^*M, z') in what follows.

Since the cokernel of the inclusion $\mathrm{gr}_i(M) \to \mathcal{R}_X(\pi_K)(\delta_i)$ is killed by some power of t, the cohomology groups of the complex

$$[0 \to \mathrm{fil}_{i-1}(M)_z \to \mathrm{fil}_i(M)_z \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_i) \to 0]$$

are killed by some power of t. The claim now follows from Corollary 6.2.9. □

Example 6.3.14. We work out the argument of Theorem 6.3.13 in detail for $\mathrm{fil}_2(M)$ when X is a smooth (and hence reduced with all connected components irreducible) rigid analytic curve. This method in principle can be extended to any $\mathrm{fil}_i(M)$, although the argument is more complicated. In this case, Corollary 6.3.10 implies that we have the following exact sequence

$$0 \to \mathcal{R}_X(\pi_K)(\delta_1) \otimes_X \mathcal{L}_1 \xrightarrow{\lambda} \mathrm{fil}_2(M) \xrightarrow{\mu} \mathcal{R}_X(\pi_K)(\delta_2) \otimes_X \mathcal{L}_2 \to Q \to 0, \tag{6.3.14.1}$$

where the cokernel Q is killed by some power of t and is supported on a subset $Z \subset X$ disjoint from X_{alg} meeting each affinoid at finitely many points. After shrinking X, we assume Q is supported at a unique point z of Z.

Tensoring (6.3.14.1) with κ_z, we obtain

$$\mathrm{fil}_2(M)_z \xrightarrow{\mu_z} \mathcal{R}_{\kappa_z}(\pi_K)(\delta_{2,z}) \to Q_z \to 0, \text{ and} \tag{6.3.14.2}$$

(6.3.14.3)

$$0 \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_{1,z}) \xrightarrow{\lambda_z} \mathrm{Ker}\left(\mu_z : \mathrm{fil}_2(M)_z \to \mathcal{R}_{\kappa_z}(\pi_K)(\delta_{2,z})\right) \to \mathrm{Tor}_1^X(Q, \kappa_z) \to 0.$$

Since the kernel of μ_z is torsion-free and has generic rank one, it is a (φ, Γ_K)-module over $\mathcal{R}_{\kappa_z}(\pi_K)$. Since it also contains $\mathcal{R}_{\kappa_z}(\pi_K)(\delta_{1,z})$ as a subobject, Corollary 6.2.9 implies that it is isomorphic to $\mathcal{R}_{\kappa_z}(\pi_K)(\delta'_{1,z})$, where $\delta'_{1,z} = \delta_{1,z} \prod_{\sigma \in \Sigma} x_\sigma^{-k_{z,\sigma}}$ for some $k_{z,\sigma} \in \mathbb{Z}_{\geq 0}$. In particular, this implies that $\mathrm{Tor}_1^X(Q, \kappa_z)$ is isomorphic to

$\mathcal{R}_{\kappa_z}(\pi_K)(\delta'_{1,z})/(\prod_{\sigma\in\Sigma} t_\sigma^{k_{z,\sigma}})$. After shrinking X, we may assume that it is affinoid and its global sections are a PID, which shows that Q_z must also be isomorphic to $\mathcal{R}_{\kappa_z}(\pi_K)(\delta'_{1,z})/(\prod_{\sigma\in\Sigma} t_\sigma^{k_{z,\sigma}})$, and the gradeds of Q for the $\mathfrak{m}_z$-adic filtration (where $\mathfrak{m}_z$ is the maximal ideal at z) must similarly have the form

$$\mathfrak{m}_z^n Q/\mathfrak{m}_z^{n+1} Q \cong \mathcal{R}_{\kappa_z}(\pi_K)(\delta'_{1,z}) \Big/ \left(\prod_{\sigma\in\Sigma} t_\sigma^{k_{z,\sigma,n}}\right)$$

where for each fixed $\sigma \in \Sigma$ the $k_{z,\sigma,n}$ form a nonincreasing sequence of nonnegative integers that eventually vanish.

In view of (6.3.14.2), Q_z is also isomorphic to $\mathcal{R}_{\kappa_z}(\pi_K)(\delta_{2,z})/\big(\prod_{\sigma\in\Sigma} t_\sigma^{k_{z,\sigma}}\big)$. In particular, for any $\sigma \in \Sigma$ such that $k_{z,\sigma} > 0$, we have a (φ, Γ)-equivariant isomorphism

$$\mathcal{R}_{\kappa_z}(\pi_K)(\delta'_{1,z})/(t_\sigma) \cong \mathcal{R}_{\kappa_z}(\pi_K)(\delta_{2,z})/(t_\sigma).$$

Note that these two objects are not genuine (φ, Γ_K)-modules, so in particular, the isomorphism does not provide us with any information about the φ-action. Still, it gives an isomorphism

$$\mathbf{D}_{\mathrm{Sen},m}(\delta'_{1,z}) \otimes_{K\otimes_{\mathbb{Q}_p}\kappa_z,\sigma} \kappa_z \cong \mathbf{D}_{\mathrm{Sen},m}(\delta_{2,z}) \otimes_{K\otimes_{\mathbb{Q}_p}\kappa_z,\sigma} \kappa_z,$$

for some m sufficiently large. Looking at the action of Θ_{Sen} on both sides of the isomorphism, Corollary 6.2.12 implies that

$$\mathrm{wt}_\sigma(\delta_{2,z}) = \mathrm{wt}_\sigma(\delta'_{1,z}) = \mathrm{wt}_\sigma(\delta_{1,z}) - k_{z,\sigma}.$$

In conclusion, for any $\sigma \in \Sigma$, either $k_{z,\sigma}$ is zero or it is strictly positive and equal to $\mathrm{wt}_\sigma(\delta_{1,z}) - \mathrm{wt}_\sigma(\delta_{2,z})$; the local triangulation at z is given by

(6.3.14.4)

$$0 \to \mathcal{R}_{\kappa_z}(\pi_K)\left(\delta_{1,z}\prod_{\sigma\in\Sigma} x_\sigma^{-k_{z,\sigma}}\right) \to \mathrm{fil}_2(M)_z \to \mathcal{R}_{\kappa_z}(\pi_K)\left(\delta_{2,z}\prod_{\sigma\in\Sigma} x_\sigma^{k_{z,\sigma}}\right) \to 0.$$

6.4. **Triangulation over eigenvarieties.** We apply the triangulation result Corollary 6.3.10 (resp. Theorem 6.3.13) to the case of eigenvarieties and obtain global (resp. pointwise) triangulation immediately. For simplicity, we restrict ourselves to the case when $K = \mathbb{Q}_p$, where $\Sigma = \{\mathrm{id}\}$, the id-Hodge-Tate weight is just the usual Hodge-Tate weight, and we can take $t_{\mathrm{id}} = t$.

Definition 6.4.1. Let V be a p-adic representation of $G_{\mathbb{Q}_p}$ over a finite extension L of $\mathbb{Q}_p$, and let $\delta_1, \dots, \delta_d : \mathbb{Q}_p^\times \to L^\times$ be potentially crystalline characters. We write $F_i = \delta_i(p)$, and let $\kappa_i \in \mathbb{Z}$ be the Hodge-Tate weight of δ_i, so that $(\delta_i \cdot x^{\kappa_i})|_{\mathbb{Z}_p^\times}$ has finite order, thus giving a character $\psi_i : G_{\mathbb{Q}_p} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^n})/\mathbb{Q}_p) \cong (\mathbb{Z}/p^n)^\times \to L^\times$ for some n. We say that V is *refined trianguline* with ordered parameters $\delta_1, \dots, \delta_d : \mathbb{Q}_p^\times \to L^\times$ if

(a) V becomes semistable over some $\mathbb{Q}_p(\mu_{p^n})$,
(b) V has Hodge-Tate weights $\kappa_1 \le \cdots \le \kappa_d$ and *distinct* Frobenius eigenvalues $p^{\kappa_1}F_1, \dots, p^{\kappa_d}F_d$,
(c) the action of $G_{\mathbb{Q}_p}$ on $\mathbf{D}_{\mathrm{pst}}(V)$ stabilizes each $p^{\kappa_i}F_i$-eigenspace and, on this line, is via ψ_i, and

(d) the complete flag on $\mathbf{D}_{\mathrm{pst}}(V)$ whose ith subobject is spanned by the eigenspaces for $p^{\kappa_1}F_1, \ldots, p^{\kappa_i}F_i$ is in general position with respect to the Hodge filtration, with weights $\kappa_1, \ldots, \kappa_i$, and each step is stable under the monodromy operator.

Lemma 6.4.2. *If V is refined trianguline with ordered parameters $\delta_1, \ldots, \delta_d$, then $\mathbf{D}_{\mathrm{rig}}(V)$ is strictly trianguline with ordered parameters $\delta_1, \ldots, \delta_d$.*

Proof. By Berger's correspondence between filtered $(\varphi, N, G_{\mathbb{Q}_p})$-modules and potentially semistable (φ, Γ)-modules described in [7, Théorème A], the complete flag described in Definition 6.4.1(d) gives a triangulation of $M = \mathbf{D}_{\mathrm{rig}}(V)$ with ordered parameters $\delta_1, \ldots, \delta_d$. Note that $M/\mathrm{fil}_i M(\delta_{i+1}^{-1})$ corresponds via Berger's correspondence to a filtered $(\varphi, N, G_{\mathbb{Q}_p})$-module with a unique φ-eigenvalue equal to 1. This shows that $H^0_{\varphi,\gamma_{\mathbb{Q}_p}}(M/\mathrm{fil}_i M(\delta_{i+1}^{-1}))$ is at most one-dimensional, and therefore one-dimensional, so that $\mathbf{D}_{\mathrm{rig}}(V)$ is strictly trianguline. □

Example 6.4.3. Let X be a rigid analytic space over $\mathbb{Q}_p$, let V be a family of $G_{\mathbb{Q}_p}$-representations over X, take $M = \mathbf{D}_{\mathrm{rig}}(V)$, and assume given continuous characters $\delta_1, \ldots, \delta_d : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and a Zariski-dense subset $X_{\mathrm{alg}} \subset X$ satisfying the following condition:

- For each $z \in X_{\mathrm{alg}}$ the fiber V_z is refined trianguline with parameters $\delta_{1,z}, \ldots, \delta_{d,z}$.

Putting $F_i = \delta_i(p)$ and $\kappa_i = -\mathrm{wt}(\delta_i)$, so that $\kappa_1, \ldots, \kappa_d, F_1, \ldots, F_d \in \Gamma(X, \mathcal{O}_X)$, such data are almost the same as those employed by Bellaïche [2, 3.2.1] (or Bellaïche-Chenevier [3, Chapter 4], where a family of pseudocharacters is treated), except these authors have a stronger denseness condition, and for simplicity they require the V_z to be crystalline. By Lemma 6.4.2, the fiber V_z for any $z \in X_{\mathrm{alg}}$ is strictly trianguline with ordered parameters $\delta_{1,z}, \ldots, \delta_{d,z}$. Then Corollary 6.3.10 (if X is reduced and its connected components are irreducible) and Theorem 6.3.13 apply to this situation.

Example 6.4.4. We conclude this subsection by specializing to the case where

- X is any disjoint union of irreducible subspaces of the Coleman-Mazur eigencurve and V its family of $G_{\mathbb{Q}_p}$-representations,
- X_{alg} is the set of classical points of weight $k \geq 2$ not in the image of the θ^{k-1}-map,
- $\delta_1|_{\mathbb{Z}_p^\times}$ is trivial and $\delta_2|_{\mathbb{Z}_p^\times}$ is the inverse of the "weight-nebentypus character" map,
- κ_1 is the constant function 0 and κ_2 measures one minus the weight, and
- F_1 is the U_p-map and F_2 is the inverse of the U_p-map,

and we obtain the following more precise result. If necessary, we may enlarge each irreducible subspace of X within the eigencurve so that it encompasses some point of classical weight. Then Coleman's classicality theorem implies that all nearby points of sufficiently large classical weight are classical, and X_{alg} is indeed dense.

Proposition 6.4.5. *With the preceding notations, there exists a triangulation over the entire desingularization X' of X. For any point $z \in X'$, the global triangulation restricts to a triangulation of $\mathbf{D}_{\mathrm{rig}}(V_z)$ if and only if the overconvergent modular form f corresponding to z is not in the image of the θ^{k-1}-map for some integer $k \geq 2$. If z corresponds to an overconvergent modular form of integer weight $k \geq 2$*

in the image of the θ^{k-1}-map, the triangulation extends to a submodule whose fiber at z has index t^{k-1} in its saturation.

Proof. The existence of global triangulation follows immediately from Corollary 6.3.10. We discuss the rest of the statement case by case.

If $z \in X'$ corresponds to an overconvergent modular form with a non-integer weight or with weight less than or equal to 1, then $\mathrm{wt}_{\mathrm{id}}(\delta_{1,z}) - \mathrm{wt}_{\mathrm{id}}(\delta_{2,z})$ is not a positive integer. By the calculation of Example 6.3.14, the global triangulation is saturated at the point z.

If $z \in X'$ corresponds to an overconvergent modular form with integer weight $k \geq 2$ that is not in the image of θ^{k-1}, [13, Proposition 5.4.3] implies that z corresponds to a classical modular form f of weight k and U_p-eigenvalue α_z, and [13, Théorème 1.1.3] implies that the representation V_z of $G_{\mathbb{Q}_p}$ attached to f is nonsplit. In the computations of Example 6.3.14 the integer $k_{z,\mathrm{id}}$ is either 0 or $k-1$, and it suffices to show that $k_{z,\mathrm{id}} = 0$. Weak admissibility of $\mathbf{D}_{\mathrm{pst}}(V_z)$ implies that if $k_{z,\mathrm{id}} = k-1$, then $\mathrm{ord}_p\, \alpha_z = k-1$, and it is easy to see that this forces $\mathbf{D}_{\mathrm{cris}}(V_z)$ to be a direct sum of two weakly admissible submodules, contradicting the fact that V_z is not split. It follows that one must have $k_{z,\mathrm{id}} = 0$.

Now suppose that $z \in X'$ corresponds to an overconvergent modular form f of weight $k \geq 2$ that is in the image of θ^{k-1}, i.e., there exists an overconvergent modular form f' of weight $2-k$ such that $(q\frac{d}{dq})^{k-1}(f'(q)) = f(q)$. If necessary, add to X an irreducible region of the eigencurve so that there exists $z' \in X'$ corresponding to f'. Let V_z (resp. $V_{z'}$) be the representation of $G_{\mathbb{Q}}$ attached to f (resp. f'); by looking at the eigenvalues of Hecke operators away from p, we know that $V_z = V_{z'} \otimes \chi^{1-k}$. It is also clear that $\delta_{1,z'} = \delta_{1,z}|x|^{k-1}$ and $\delta_{2,z'} = \delta_{2,z}x^{2k-2}|x|^{k-1}$. The computation in Example 6.3.14 gives us the two integers $k_{z,\mathrm{id}}$ and $k_{z',\mathrm{id}}$. Because the weight at z' is $2-k < 1$, the first case we treated above shows that the global triangulation is saturated at z', i.e., $k_{z',\mathrm{id}} = 0$. By the conclusion of Example 6.3.14, $k_{z,\mathrm{id}}$ is equal to 0 or $k-1$. We claim that $k_{z,\mathrm{id}} = k-1$, which would finish the proof of this proposition. Suppose not, i.e., $k_{z,\mathrm{id}} = 0$. Let κ be a finite extension of $\mathbb{Q}_p$ containing both κ_z and $\kappa_{z'}$. After base changing to κ, the exact sequence (6.3.14.4) for z is

$$0 \to \mathcal{R}_\kappa(\delta_{1,z}) \xrightarrow{\lambda_z} \mathbf{D}_{\mathrm{rig}}(V_z) \xrightarrow{\mu_z} \mathcal{R}_\kappa(\delta_{2,z}) \to 0, \tag{6.4.5.1}$$

whereas the sequence (6.3.14.4) for z', twisted by χ^{1-k}, is

$$0 \to t^{1-k}\mathcal{R}_\kappa(\delta_{1,z}) \xrightarrow{\lambda_{z'}} \mathbf{D}_{\mathrm{rig}}(V_z) \xrightarrow{\mu_{z'}} t^{k-1}\mathcal{R}_\kappa(\delta_{2,z}) \to 0. \tag{6.4.5.2}$$

Note that μ_z in (6.4.5.1) does not factor through $\mu_{z'}$ in (6.4.5.2). Thus, the composition $t^{1-k}\mathcal{R}_\kappa(\delta_{1,z}) \xrightarrow{\lambda_{z'}} \mathbf{D}_{\mathrm{rig}}(V_z) \xrightarrow{\mu_z} \mathcal{R}_\kappa(\delta_{2,z})$ is nonzero. By comparing weights, we conclude that $x^{1-k}\delta_{1,z} = \delta_{2,z}$. This implies that $\alpha_z^2 = p^{k-1}$, where α_z is the U_p-eigenvalue of f. This is impossible because, writing α'_z for the U_p-eigenvalue of f', one has $\alpha_z = p^{k-1}\alpha'_z$ and so $\mathrm{ord}_p\, \alpha_z \geq k-1$. ☐

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References

[1] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961), 167. MR0217085 (36 #177c)

[2] Joël Bellaïche, *Ranks of Selmer groups in an analytic family*, Trans. Amer. Math. Soc. **364** (2012), no. 9, 4735–4761, DOI 10.1090/S0002-9947-2012-05504-8. MR2922608

[3] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups* (English, with English and French summaries), Astérisque **324** (2009), xii+314. MR2656025 (2011m:11105)

[4] R. Bellovin, *p-adic Hodge theory in rigid analytic families*, preprint, `arXiv:1306.5685v1`.

[5] Denis Benois, *A generalization of Greenberg's $\mathcal{L}$-invariant*, Amer. J. Math. **133** (2011), no. 6, 1573–1632, DOI 10.1353/ajm.2011.0043. MR2863371

[6] Laurent Berger, *An introduction to the theory of p-adic representations* (English, with English and French summaries), Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 255–292. MR2023292 (2005h:11265)

[7] Laurent Berger, *Équations différentielles p-adiques et (ϕ, N)-modules filtrés* (French, with English and French summaries), Astérisque **319** (2008), 13–38. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ,Γ)-modules. MR2493215 (2010d:11056)

[8] Laurent Berger, Christophe Breuil, and Pierre Colmez (eds.) *Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ,Γ)-modules* (French), Société Mathématique de France, Paris, 2008. Astérisque No. 319 (2008). MR2482309

[9] Laurent Berger, Christophe Breuil, and Pierre Colmez (eds.), *Représentations p-adiques de groupes p-adiques I: Répresentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (φ,Γ)*-modules*, Astérisque **330** (2010).

[10] Laurent Berger and Pierre Colmez, *Familles de représentations de Rham et monodromie p-adique* (French, with English and French summaries), Astérisque **319** (2008), 303–337. *Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ,Γ)-modules.* MR2493221 (2010g:11091)

[11] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry. MR746961 (86b:32031)

[12] N. Bourbaki, *Espaces Vectoriels Topologiques*, reprint of the 1981 original, Springer, Berlin, 2007.

[13] Christophe Breuil and Matthew Emerton, *Représentations p-adiques ordinaires de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et compatibilité local-global* (French, with English and French summaries), Astérisque **331** (2010), 255–315. MR2667890 (2011m:22026)

[14] Kevin Buzzard, *Eigenvarieties, L-functions and Galois representations*, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 59–120, DOI 10.1017/CBO9780511721267.004. MR2392353 (2010g:11076)

[15] Gaëtan Chenevier, *Sur la densité des représentations cristallines de Gal*$(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ (French, with English summary), Math. Ann. **355** (2013), no. 4, 1469–1525, DOI 10.1007/s00208-012-0815-z. MR3037022

[16] Frédéric Cherbonnier and Pierre Colmez, *Théorie d'Iwasawa des représentations p-adiques d'un corps local* (French), J. Amer. Math. Soc. **12** (1999), no. 1, 241–268, DOI 10.1090/S0894-0347-99-00281-7. MR1626273 (99g:11141)

[17] Robert F. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), no. 1-3, 215–241, DOI 10.1007/s002220050051. MR1369416 (97d:11090a)

[18] Pierre Colmez, *Théorie d'Iwasawa des représentations de de Rham d'un corps local* (French), Ann. of Math. (2) **148** (1998), no. 2, 485–571, DOI 10.2307/121003. MR1668555 (2000f:11077)

[19] Pierre Colmez, *Représentations triangulines de dimension 2* (French, with English and French summaries), Astérisque **319** (2008), 213–258. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules. MR2493219 (2010f:11173)

[20] Pierre Colmez, *Représentations de* $\mathrm{GL}_2(\mathbf{Q}_p)$ *et* (ϕ, Γ)*-modules* (French, with English and French summaries), Astérisque **330** (2010), 281–509. MR2642409 (2011j:11224)

[21] Pierre Colmez, (ϕ, Γ)*-modules et représentations du mirabolique de* $\mathrm{GL}_2(\mathbf{Q}_p)$ (French, with English and French summaries), Astérisque **330** (2010), 61–153. MR2642405 (2011i:11170)

[22] Brian Conrad, *Irreducible components of rigid spaces* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541. MR1697371 (2001c:14045)

[23] Richard Crew, *Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 6, 717–763, DOI 10.1016/S0012-9593(99)80001-9. MR1664230 (2000a:14023)

[24] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. MR1322960 (97a:13001)

[25] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I* (French), The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 249–309. MR1106901 (92i:11125)

[26] J.-M. Fontaine and Y. Ouyang, *Theory of p-adic Galois representations*, book to appear, downloaded from `http://staff.ustc.edu.cn/~yiouyang/galoisrep.pdf` on 6 August 2013.

[27] Ralph Greenberg, *Iwasawa theory and p-adic deformations of motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 193–223. MR1265554 (95i:11053)

[28] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[29] E. Hellmann, *Families of trianguline representations and finite slope spaces*, preprint, `arXiv:1202.4408v1`.

[30] Laurent Herr, *Sur la cohomologie galoisienne des corps p-adiques* (French, with English and French summaries), Bull. Soc. Math. France **126** (1998), no. 4, 563–600. MR1693457 (2000m:11118)

[31] Laurent Herr, *Une approche nouvelle de la dualité locale de Tate* (French), Math. Ann. **320** (2001), no. 2, 307–337, DOI 10.1007/PL00004476. MR1839766 (2002d:11136)

[32] Kiran S. Kedlaya, *Finiteness of rigid cohomology with coefficients*, Duke Math. J. **134** (2006), no. 1, 15–97, DOI 10.1215/S0012-7094-06-13412-9. MR2239343 (2007m:14021)

[33] Kiran S. Kedlaya, *Slope filtrations for relative Frobenius* (English, with English and French summaries), Astérisque **319** (2008), 259–301. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules. MR2493220 (2010c:14024)

[34] Kiran Kedlaya and Ruochuan Liu, *On families of* ϕ*,* Γ*-modules*, Alg. Number Theor. **4** (2010), no. 7, 943–967, DOI 10.2140/ant.2010.4.943. MR2776879 (2012h:11081)

[35] Mark Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153** (2003), no. 2, 373–454, DOI 10.1007/s00222-003-0293-8. MR1992017 (2004f:11053)

[36] Ruochuan Liu, *Cohomology and duality for* (ϕ, Γ)*-modules over the Robba ring*, Int. Math. Res. Not. IMRN **3** (2008), Art. ID rnm150, 32, DOI 10.1093/imrn/rnm150. MR2416996 (2009e:11222)

[37] R. Liu, *Triangulation of refined families*, preprint, `arXiv:1202.2188v3`.

[38] Hideyuki Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461 (90i:13001)

[39] Kentaro Nakamura, *Classification of two-dimensional split trianguline representations of p-adic fields*, Compos. Math. **145** (2009), no. 4, 865–914, DOI 10.1112/S0010437X09004059. MR2521248 (2010f:11172)

[40] K. Nakamura, *Deformations of trianguline* $\mathbb{B}$*-pairs and Zariski density of two dimensional crystalline representations*, preprint. `arXiv:1006.4891v5`.

[41] Jan Nekovář, *Selmer complexes* (English, with English and French summaries), Astérisque **310** (2006), viii+559. MR2333680 (2009c:11176)

[42] Jonathan Pottharst, *Analytic families of finite-slope Selmer groups*, Alg. Number Theor. **7** (2013), no. 7, 1571–1612, DOI 10.2140/ant.2013.7.1571. MR3117501

[43] J. Pottharst, *Cyclotomic Iwasawa theory of motives*, preprint.

[44] M. Raynaud, *Flat modules in algebraic geometry*, Compos. Math. **24** (1972), 11–31. MR0302645 (46 #1789)

[45] Peter Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. MR1869547 (2003a:46106)

[46] Peter Schneider and Jeremy Teitelbaum, *Algebras of p-adic distributions and admissible representations*, Invent. Math. **153** (2003), no. 1, 145–196, DOI 10.1007/s00222-002-0284-1. MR1990669 (2004g:22015)

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