JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 4, October 2014, Pages 1147–1168
S 0894-0347(2014)00797-9
Article electronically published on June 6, 2014

SEMI-INFINITE SCHUBERT VARIETIES AND QUANTUM K-THEORY OF FLAG MANIFOLDS

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1. Introduction

1.1. **Spaces of quasi-maps.** Let G be a semi-simple, simply connected group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$; we will denote by $\check{\mathfrak{g}}$ the Langlands dual algebra of $\mathfrak{g}$. Also let $\mathcal{B}_\mathfrak{g}$ denote its flag variety. We have $H_2(\mathcal{B}_\mathfrak{g},\mathbb{Z})=\Lambda$, the coroot lattice of $\mathfrak{g}$. We will denote by Λ_+ the sub-semigroup of positive elements in Λ.

Let $\mathbf{C}\simeq\mathbb{P}^1$ denote a (fixed) smooth connected projective curve (over $\mathbb{C}$) of genus 0; we are going to fix a marked point $\infty\in\mathbf{C}$. For each $\alpha\in\Lambda_+$ we can consider the space $\mathcal{M}_\mathfrak{g}^\alpha$ of maps $\mathbf{C}\to\mathcal{B}_\mathfrak{g}$ of degree α. This is a smooth quasi-projective variety. It has a compactification $\mathcal{QM}_\mathfrak{g}^\alpha$ by means of the space of *quasi-maps* from $\mathbf{C}$ to $\mathcal{B}_\mathfrak{g}$ of degree α. Set-theoretically this compactification can be described as follows:

$$\mathcal{QM}_\mathfrak{g}^\alpha=\bigsqcup_{0\le\beta\le\alpha}\mathcal{M}_\mathfrak{g}^\beta\times\mathrm{Sym}^{\alpha-\beta}(\mathbf{C}),\tag{1.1}$$

where $\mathrm{Sym}^{\alpha-\beta}(\mathbf{C})$ stands for the space of "colored divisors" of the form $\sum\gamma_i x_i$ where $x_i\in\mathbf{C}$, $\gamma_i\in\Lambda_+$ and $\sum\gamma_i=\alpha-\beta$.

Let us fix a pair of opposite Borel subgroups $B,B_-\subset G$; then we can write $\mathcal{B}_\mathfrak{g}=G/B$. We can now consider the space $\overset{\circ}{Z}{}_\mathfrak{g}^\alpha$ of *based maps* $(\mathbf{C},\infty)\to(\mathcal{B}_\mathfrak{g}=G/B,e_-)$ (here $e_-\in G/B$ denotes the class of B_-, and a map $f:\mathbf{C}\to\mathcal{B}_\mathfrak{g}$ is called based if $f(\infty)=e_-$). This is a quasi-affine variety; the corresponding space $Z_\mathfrak{g}^\alpha$ of based quasi-maps (a.k.a. Zastava space in the terminology of [17] and [13]) is affine. It possesses a stratification similar to (1.1) but with $\mathbf{C}$ in the right-hand side of (1.1) replaced by $\mathbf{C}-\infty$.

The following theorem is the first main result of this article.

Theorem 1.2. (1) *For any $\mathfrak{g}$ and α the schemes $Z_\mathfrak{g}^\alpha$ and $\mathcal{QM}_\mathfrak{g}^\alpha$ are normal.*
(2) *Assume that $\mathfrak{g}$ is simply laced. Then $Z_\mathfrak{g}^\alpha$ (and $\mathcal{QM}_\mathfrak{g}^\alpha$) is Gorenstein (in particular, Cohen-Macaulay) and has canonical (hence rational) singularities.*

1.3. **Connection to quantum K-theory of $\mathcal{B}_\mathfrak{g}$.** In fact, we believe that $Z_\mathfrak{g}^\alpha$ must have rational singularities for all $\mathfrak{g}$ (not necessarily simply laced). Let us explain the importance of this assertion. Recall that a scheme Z has rational singularities, if for some (equivalently, for any) resolution $\pi:\widetilde{Z}\to Z$ we have $R\pi_*(\mathcal{O}_{\widetilde{Z}})=\mathcal{O}_Z$. The scheme $Z_\mathfrak{g}^\alpha$ has a resolution by means of the Kontsevich moduli space $M_\mathfrak{g}^\alpha$ of stable maps from a nodal curve C of genus 0 to $\mathcal{B}_\mathfrak{g}\times\mathbb{P}^1$ which have degree $(\alpha,1)$ and with some analog of the "based" condition (cf. Section 5 for more detail). The

Received by the editors December 6, 2011 and, in revised form, June 16, 2013 and September 6, 2013.
2010 *Mathematics Subject Classification.* Primary 17B37; Secondary 14N35, 19L47, 37J35.

space $M_{\mathfrak{g}}^\alpha$ is a smooth Deligne-Mumford stack which has a natural action of $T\times\mathbb{C}^*$, where $T\subset B\subset G$ is a maximal torus (here the action of T comes from the fact that it acts on $\mathcal{B}_{\mathfrak{g}}$ preserving e_- and the action of $\mathbb{C}^*$ comes from the action on $\mathbb{P}^1$ preserving ∞). Let $\mathfrak{J}_\alpha$ be the $T\times\mathbb{C}^*$-equivariant pushforward of $\mathcal{O}_{M_{\mathfrak{g}}^\alpha}$ to $\mathrm{Spec}(\mathbb{C})$ (i.e., the character of $[R\Gamma(M_{\mathfrak{g}}^\alpha,\mathcal{O}_{M_{\mathfrak{g}}^\alpha})]$ of $R\Gamma(M_{\mathfrak{g}}^\alpha,\mathcal{O}_{M_{\mathfrak{g}}^\alpha})$ with respect to $T\times\mathbb{C}^*$). This is a rational function on $T\times\mathbb{C}^*$ and we are going to write $\mathfrak{J}_\alpha=\mathfrak{J}_\alpha(z,q)$ where $z\in T, q\in\mathbb{C}^*$.

It is explained in [19] (cf. also the Appendix to [4] for the corresponding statement in cohomology (as opposed to K-theory)) that every $\mathfrak{J}_\alpha$ can be thought of as some generating function of *genus 0 K-theoretic Gromov-Witten invariants with gravitational descendants* of degree α. Moreover, it is shown in [24] that $\mathfrak{J}_\alpha$'s determine all genus zero K-theoretic Gromov-Witten invariants.[1]

Thus, computing $\mathfrak{J}_\alpha$ is an important problem. Theorem 1.2 implies that (for simply laced $\mathfrak{g}$) one can replace this equivariant pushforward with the character $[\mathcal{O}_{Z_{\mathfrak{g}}^\alpha}]$ of the ring of polynomial functions on $Z_{\mathfrak{g}}^\alpha$ with respect to the action of $T\times\mathbb{C}^*$.

It is often convenient to organize all $\mathfrak{J}_\alpha$ into a generating function:

$$J_{\mathfrak{g}}(z,x,q)=\sum_{\alpha\in\Lambda_+}x^\alpha\mathfrak{J}_\alpha, \tag{1.2}$$

where x lies in the dual torus $\check{T}$. This function is called the *equivariant K-theoretic J-function* of $\mathcal{B}_{\mathfrak{g}}$ (once again, it can be defined for any smooth projective variety X).

1.4. **Fermionic formula.** The function $J_{\mathfrak{g}}$ was studied in [19] for $G=\mathrm{SL}(N)$ and it was shown to be an eigen-function of the *quantum difference Toda integrable system* (cf. [11], [30]); this result was reproved in [7] using other methods. It was conjectured in [19] that the same result should hold for any $\mathfrak{g}$.

It is actually easy to see that verbatim this conjecture is false when $\mathfrak{g}$ is not simply laced. The main purpose of the second part of this article is to prove the above conjecture for any simply laced algebra $\mathfrak{g}$. More precisely, we are going to prove the following.

Theorem 1.5. *Assume that $\mathfrak{g}$ is simply laced. Then the functions $\mathfrak{J}_\alpha$ satisfy the following recursive relation:*

$$\mathfrak{J}_\alpha=\sum_{0\le\beta\le\alpha}\frac{q^{(\beta,\beta)/2}z^{\beta^*}}{(q)_{\alpha-\beta}}\mathfrak{J}_\beta. \tag{1.3}$$

Here $\beta\mapsto\beta^*$ stands for the natural isomorphism between the coroot lattice of $\mathfrak{g}$ and its root lattice.

The equation (1.3) appears in [14], where the authors show that (1.3) holds precisely if and only if the generating function of the $\mathfrak{J}_\alpha$'s is an eigen-function of the above-mentioned quantum difference Toda system. Thus, Theorem 1.5 and the main result of [14] imply the following.

[1] In fact, in [24] the authors work with an arbitrary smooth projective variety X instead of $\mathcal{B}_{\mathfrak{g}}$. In this case the definition of $\mathfrak{J}_\alpha$ is similar, however technically the pushforward must be taken with respect to certain *virtual fundamental cycle* in K-theory. In the case when X is a homogeneous space of a linear algebraic group, this reduces to the usual pushforward.

Corollary 1.6. *Let $\mathfrak{g}$ be simply laced. Then the equivariant K-theoretic J-function of $\mathcal{B}_{\mathfrak{g}}$ is an eigen-function of the quantum difference Toda integrable system associated with $\mathfrak{g}$.*

One may ask whether the assumption that $\mathfrak{g}$ is simply laced is really essential. As was mentioned above verbatim, Theorem 1.5 (and thus also Corollary 1.6) does not hold for non simply laced $\mathfrak{g}$ (the reason for this is explained in Section 7). On the other hand, in Section 7 we show how to modify the geometric problem a little (using the recent result of [31]) in order to make a correct statement for all $\mathfrak{g}$. It is worthwhile to note that the corresponding analog of $J_{\mathfrak{g}}$ in that case becomes an eigen-function of the quantum difference Toda system associated with $\check{\mathfrak{g}}$ (in the simply laced case we have $\mathfrak{g} = \check{\mathfrak{g}}$). The reader should compare this statement with the main result of [22] which deals with the "usual" (i.e., cohomological) J-function of $\mathcal{B}_{\mathfrak{g}}$.

1.7. **Representation-theoretic interpretation.** In this subsection we discuss possible interpretation of the above results in terms of geometric representation theory; this subsection will not be used in the future, so the uninterested reader may skip this discussion and go to Section 1.9.

Corollary 1.6 and the constructions of [30] and [14] also imply the following.

Corollary 1.8. *In the simply laced case the function $J_{\mathfrak{g}}$ is equal to the Whittaker matrix coefficient in the universal Verma of $U_q(\check{\mathfrak{g}})$.*

In [7] this result was proved directly for $G = \mathrm{SL}(N)$. Namely, in that case the space $Z^\alpha_{\mathfrak{g}}$ has a small resolution of singularities (usually called Laumon's resolution) which we will denote by $\mathcal{P}^\alpha$. In [7] we construct an action of the quantum group $U_q(\mathfrak{sl}(N))$ on $\mathcal{V} = \oplus_\alpha K_{T\times\mathbb{C}^*}(\mathcal{P}^\alpha)_{\mathrm{loc}}$ (here the subscript "loc" means "localized equivariant K-theory") and identify the corresponding $U_q(\mathfrak{sl}(N))$ module with the *universal Verma module*. Moreover, the natural pairing on $\mathcal{V}$ gets identified with the Shapovalov form on the Verma module. In addition, if we denote by $1_\alpha \in K_{T\times\mathbb{C}^*}(\mathcal{P}^\alpha)$ then the formal sum $\sum_\alpha 1_\alpha$ (lying in some completion of $\mathcal{V}$) is the Whittaker vector in $\mathcal{V}$ (i.e., an eigen-vector of the positive part of $U_q(\mathfrak{sl}(N))$). It is easy to see that these results imply Corollary 1.8 (we refer the reader to [7] for the details).

It would be very interesting to prove Corollary 1.8 along similar lines, however we don't know how to do this, since for general $\mathfrak{g}$ there is no resolution of $Z^\alpha_{\mathfrak{g}}$ similar to $\mathcal{P}^\alpha$. In addition we would like to mention that the notion of Whittaker vector for $\mathfrak{g}$ (or $\check{\mathfrak{g}}$), which is developed in [30] (cf. [11] for a closely related approach) depends on a choice of orientation of the Dynkin diagram of $\mathfrak{g}$; it would be very interesting to understand how it can be incorporated in the above constructions (for $\mathfrak{g} = \mathfrak{sl}(N)$ there is a natural choice of orientation).

1.9. **Idea of the proof of normality.** Let us now go back and explain the idea of the proof of Theorem 1.2(1), since in our opinion this proof is of independent interest.

Let $\mathrm{Gr}_G = G((t))/G[[t]]$ be the affine Grassmannian of G. It is well known that the orbits of $G[[t]]$ on Gr_G are in one-to-one correspondence with the elements of the dominant cone Λ^+; for each $\lambda \in \Lambda^+$ we will denote the corresponding orbit by Gr^λ_G. Its closure $\overline{\mathrm{Gr}}^\lambda_G$ is the union of all Gr^μ_G with $\mu \le \lambda$. It is well known (cf., e.g.,

[12]) that $\overline{\mathrm{Gr}}^\lambda_G$ is normal, Cohen-Macaulay and has rational singularities (in fact, it is also Gorenstein - cf. [9]).

The schemes $Z^\alpha_\mathfrak{g}$ were originally defined in [17] in order to give a model for the singularities of $\overline{\mathrm{Gr}}^\lambda_G$ at a point of Gr^μ_G when both λ and μ are very large and $\lambda-\mu=\alpha$. However, although this statement was used as a guiding principle in many works related to $Z^\alpha_\mathfrak{g}$ (cf. [3] for a review), it was never given any precise meaning.

The purpose of Section 2 is to formulate some version of the above principle precisely. This formulation immediately implies normality of $Z^\alpha_\mathfrak{g}$ but other parts of Theorem 1.2 still have to be proven by other means. Roughly speaking, we show the following. Given λ and μ as above one can construct certain transversal slice $\overline{\mathcal{W}}^\lambda_{G,\mu}$ to Gr^μ_G in $\overline{\mathrm{Gr}}^\lambda_G$. This transversal slice is also acted on by $T\times\mathbb{C}^*$. In Section 2 we construct a $T\times\mathbb{C}^*$-equivariant map $\overline{\mathcal{W}}^\lambda_{G,\lambda-\alpha}\to Z^\alpha_\mathfrak{g}$ and we show that this map induces an isomorphism on functions of given homogeneity degree with respect to $\mathbb{C}^*$ when λ is very large. This easily implies that $Z^\alpha_\mathfrak{g}$ is normal.

1.10. **Affine case.** The definition of the schemes $Z^\alpha_\mathfrak{g}$ was generalized in [6] to the case when $\mathfrak{g}$ is an untwisted affine algebra. We conjecture that Theorem 1.2 and Theorem 1.5 hold in this case; this should be useful for studying the Nekrasov partition function of 5-dimensional pure gauge theory compactified on S^1 (cf. [27]) in the spirit of [5]. In Section 3, we prove Theorem 1.2 for $\mathfrak{g}=\mathfrak{sl}(N)_{\mathrm{aff}}$; this easily implies Theorem 1.5 in this case, in view of the results of [7].

1.11. **Contents.** This article is organized as follows. In Section 2, we discuss the relation between $Z^\alpha_\mathfrak{g}$ and the transversal slices $\overline{\mathcal{W}}^\lambda_{G,\mu}$ in the affine Grassmannian and prove that the schemes $Z^\alpha_\mathfrak{g}$ are normal. In Section 3, we use a different method to show that the affine analogs of $Z^\alpha_\mathfrak{g}$ are normal, Gorenstein and have rational singularities for $G=\mathrm{SL}(N)$. In Section 4, we study the equation of the boundary of $Z^\alpha_\mathfrak{g}$; we use it in Section 5 in order to prove the second part of Theorem 1.2. Theorem 1.5 is proved in Section 6. Finally, in Section 7 we explain how to extend Theorem 1.2 and Theorem 1.5 to non simply laced case using the twisted affine Grassmannian studied in [31].

2. Normality of Zastava spaces via transversal slices in the affine Grassmannian

2.1. **Quasi-maps and Zastava spaces.** In this section we recall the definition of $\mathcal{QM}^\alpha_\mathfrak{g}$ and $Z^\alpha_\mathfrak{g}$, cf., e.g., [17, Section 3]. Choose a Borel subgroup $B\subset G$ with unipotent radical U. The quotient G/U is a quasi-affine variety and we denote by $\overline{G/U}$ its affine closure. The torus $T=B/U$ acts on G/U on the right and this action extends to $\overline{G/U}$.

Let $\mathcal{F}_T$ be a T-bundle over $\mathbf{C}$. For every weight $\check\lambda: T\to\mathbb{G}_m$ of T we may consider the associated line bundle $\mathcal{F}^{\check\lambda}_T$ on $\mathbf{C}$. We say that $\mathcal{F}_T$ has degree $\alpha\in\Lambda$ if for every $\check\lambda$ as above the bundle $\mathcal{F}^{\check\lambda}_T$ has degree $\langle\check\lambda,\alpha\rangle$. Then the scheme $\mathcal{QM}^\alpha_\mathfrak{g}$ parametrizes the following data:

a) T-bundle $\mathcal{F}_T$ on $\mathbf{C}$ of degree $-\alpha$;

b) a T-equivariant map $\kappa:\mathcal{F}_T\to\mathbf{C}\times\overline{G/U}$ of fibre bundles over $\mathbf{C}$ such that over the generic point of $\mathbf{C}$ this map goes to $\mathbf{C}\times G/U$.

More explicitly, giving κ is equivalent to specifying the following data: for every dominant $\check\lambda$ the map we need to specify an embedding of locally free sheaves $\kappa^{\check\lambda}:\mathcal{F}_T^{\check\lambda}\to\mathcal{O}_{\mathbf{C}}\otimes V(\check\lambda)$ (where $V(\check\lambda)$ denotes the corresponding irreducible representation of G). The maps κ^λ must satisfy certain *Plücker relations*; we refer the reader to [3] for the details. It is easy to make the above into a modular problem, which defines $\mathcal{QM}^\alpha_{\mathfrak{g}}$ as a scheme, which is reduced, irreducible, and projective of dimension $2|\alpha|+\dim(G/B)$ (here $|\alpha|=\langle\check\rho_{\mathfrak{g}},\alpha\rangle$, where $\check\rho_{\mathfrak{g}}$ denotes the half-sum of the positive roots of $\mathfrak{g}$).

Given $(\mathcal{F}_T,\kappa)$ as above let $\mathbf{U}\subset\mathbf{C}$ be the open subset of $\mathbf{C}$ over which κ lands in $\mathbf{C}\times G/U$. If $x\in\mathbf{U}$ then we will say that the quasi-map $(\mathcal{F}_T,\kappa)$ has no defect at x. It is clear that κ defines a map $f:\mathbf{U}\to G/B$. The (open dense) subset of $\mathcal{QM}^\alpha_{\mathfrak{g}}$ consisting of those quasi-maps for which $\mathbf{U}=\mathbf{C}$ is precisely the space $\mathcal{M}^\alpha_{\mathfrak{g}}$ of maps $f:\mathbf{C}\to\mathcal{B}_{\mathfrak{g}}=G/B$ of degree α.

Let us now fix another Borel subgroup B_- such that $B\cap B_-\simeq T$; then B_- defines a point $e_-\in\mathcal{B}_{\mathfrak{g}}=G/B$. The scheme $Z^\alpha_{\mathfrak{g}}$ is a locally closed subscheme of $\mathcal{QM}^\alpha_{\mathfrak{g}}$ which corresponds to those quasi-maps which have no defect at $\infty\in\mathbf{C}$ and such that $f(\infty)=e_-$ where f is as above. The scheme $Z^\alpha_{\mathfrak{g}}$ is an affine, reduced, and irreducible scheme of dimension $2|\alpha|$. The intersection $\overset{\circ}{Z}{}^\alpha_{\mathfrak{g}}=Z^\alpha_{\mathfrak{g}}\cap\mathcal{M}^\alpha_{\mathfrak{g}}$ is the space of *based maps* $f:(\mathbf{C},\infty)\to(\mathcal{B}_{\mathfrak{g}},e_-)$ (i.e., those maps which send ∞ to e_-).

The schemes $\mathcal{QM}^\alpha_{\mathfrak{g}}$ and $Z^\alpha_{\mathfrak{g}}$ possess the following stratification:

$$(2.1)\qquad \mathcal{QM}^\alpha_{\mathfrak{g}}=\bigsqcup_{0\le\beta\le\alpha}\mathcal{M}^\beta_{\mathfrak{g}}\times\mathrm{Sym}^{\alpha-\beta}(\mathbf{C});\quad Z^\alpha_{\mathfrak{g}}=\bigsqcup_{0\le\beta\le\alpha}\overset{\circ}{Z}{}^\beta_{\mathfrak{g}}\times\mathrm{Sym}^{\alpha-\beta}(\mathbf{C}-\infty).$$

Here for any curve X and $\gamma\in\Lambda_+$ we denote by $\mathrm{Sym}^\gamma(X)$ the scheme parametrizing "colored divisors" $\sum\gamma_ix_i$ where $x_i\in X$, $\gamma_i\in\Lambda_+$ and $\sum\gamma_i=\gamma$.

2.2. **The affine Grassmannian.** Let $\mathcal{K}=\mathbb{C}((t))$, $\mathcal{O}=\mathbb{C}[[t]]$. By the *affine Grassmannian* of G we will mean the quotient $\mathrm{Gr}_G=G(\mathcal{K})/G(\mathcal{O})$. It is known (cf. [1,25]) that Gr_G is the set of $\mathbb{C}$-points of an ind-scheme over $\mathbb{C}$, which we will denote by the same symbol.

Since G is simply connected, its coweight (=cocharacter) lattice coincides with the coroot lattice $\Lambda=\Lambda_G$. We will denote the cone of dominant coweights by $\Lambda^+\subset\Lambda$. Let $\Lambda^\vee$ denote the dual lattice (this is the weight lattice of G). We let $2\check\rho_G$ denote the sum of the positive roots of G.

The group-scheme $G(\mathcal{O})$ acts on Gr_G on the left and its orbits can be described as follows. One can identify the lattice Λ_G with the quotient $T(\mathcal{K})/T(\mathcal{O})$. Fix $\lambda\in\Lambda_G$ and let t^λ denote any lift of λ to $T(\mathcal{K})$. Let Gr^λ_G denote the $G(\mathcal{O})$-orbit of t^λ (this is clearly independent of the choice of t^λ). The following result is well known.

Lemma 2.3. (1)
$$\mathrm{Gr}_G=\bigcup_{\lambda\in\Lambda_G}\mathrm{Gr}^\lambda_G.$$

(2) *We have* $\mathrm{Gr}^\lambda_G=\mathrm{Gr}^\mu_G$ *if and only if* λ *and* μ *belong to the same* W*-orbit on* Λ_G *(here* W *is the Weyl group of* G*). In particular,*
$$\mathrm{Gr}_G=\bigsqcup_{\lambda\in\Lambda^+_G}\mathrm{Gr}^\lambda_G.$$

(3) *For every $\lambda \in \Lambda^+$ the orbit Gr_G^λ is finite-dimensional and its dimension is equal to $\langle \lambda, 2\check{\rho}_{\mathfrak{g}} \rangle$.*

Let $\overline{\mathrm{Gr}_G}^\lambda$ denote the closure of Gr_G^λ in Gr_G; this is an irreducible projective algebraic variety. One has $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G}^\lambda$ if and only if $\lambda - \mu$ is a sum of positive roots of $\check{G}$.

2.4. **Transversal slices.** Consider the group $G[t^{-1}] = G(\mathbb{C}[t^{-1}]) \subset G((t)) = G(\mathbb{C}((t)))$; let us denote by $G_1 = G[t^{-1}]_1$ the kernel of the natural ("evaluation at ∞") homomorphism $G[t^{-1}] \to G$. For any $\lambda \in \Lambda$ let $\mathrm{Gr}_{G,\lambda} = G[t^{-1}] \cdot t^\lambda$. Then it is easy to see that one has

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \mathrm{Gr}_{G,\lambda}.$$

Let also $\mathcal{W}_{G,\lambda}$ denote the G_1-orbit of t^λ. For any $\lambda, \mu \in \Lambda^+$, $\lambda \geq \mu$ set

$$\mathrm{Gr}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_{G,\mu}, \quad \overline{\mathrm{Gr}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathrm{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \mathrm{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \quad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\mathrm{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that $\overline{\mathcal{W}}_{G,\mu}^\lambda$ contains the point t^μ in it.

Let $\mathbb{G}_m = \mathbb{C}^*$ act on Gr_G by loop rotation.

Lemma 2.5. (1) *The point t^μ is the only $\mathbb{C}^*$-fixed point in $\overline{\mathcal{W}}_{G,\mu}^\lambda$. The action of $\mathbb{C}^*$ on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is "repelling", i.e., for any $w \in \overline{\mathcal{W}}_{G,\mu}^\lambda$ we have $\lim_{a\to\infty} a(w) = t^\mu$.*

(2) *The orbit $G \cdot t^\mu$ is a connected component of the $\mathbb{C}^*$-fixed point set $\mathrm{Gr}_G^{\mathbb{C}^*}$, isomorphic to a partial flag variety of G. The action of $\mathbb{C}^*$ on $\mathrm{Gr}_{G,\mu}$ is "repelling", i.e., for any $w \in \mathrm{Gr}_{G,\mu}$ we have $\lim_{a\to\infty} a(w) \in G \cdot t^\mu$.*

(3) *There exists an open subset $\mathcal{U}$ in Gr_G^μ and an open embedding $\mathcal{U} \times \overline{\mathcal{W}}_{G,\mu}^\lambda \hookrightarrow \overline{\mathrm{Gr}}_G^\lambda$ such that the diagram*

$$\begin{array}{ccc} \mathcal{U} \times \{t^\mu\} & \longrightarrow & \mathrm{Gr}_G^\mu \times \{t^\mu\} \\ \big\downarrow & & \big\downarrow \\ \mathcal{U} \times \overline{\mathcal{W}}_{G,\mu}^\lambda & \longrightarrow & \overline{\mathrm{Gr}}_G^\lambda \end{array}$$

is commutative. In other words, $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is a transversal slice to Gr_G^μ inside $\overline{\mathrm{Gr}}_G^\lambda$. □

Proof. The first two statements are obvious, and the third one follows from [21, Propositions 1.3.1 and 1.3.2]. □

2.6. **Functions on $\mathcal{W}_{G,\mu}$.** Let $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$ denote the ring of functions on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ and let

$$\mathbb{C}[\mathcal{W}_{G,\mu}] = \lim_{\leftarrow} \mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$$

be the ring of functions on the ind-scheme $\mathcal{W}_{G,\mu}$. The group $T \times \mathbb{C}^*$ acts on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ and $\mathcal{W}_{G,\mu}$ and thus it acts on the corresponding ring of functions.

For any linear algebraic group H, we are going to denote by H_n the subgroup of $H[t^{-1}]$ consisting of those maps $h(t)$ which are equal to the identity $e \in H$ modulo t^{-n}; in particular, $H_0 = H[t^{-1}]$. Also, let $R_n = \mathbb{C}[t^{-1}]/t^{-n}$; for any scheme X over $\mathbb{C}$ we can consider the scheme of maps $\mathrm{Spec}(R_n) \to X$ which (abusing slightly the notation) we will denote by $X(R_n)$. Also, given a $\mathbb{C}$-point $x \in X$ we will denote by $X(R_n)_{\mathrm{based}}$ the closed sub-scheme of based maps $\mathrm{Spec}(R_n) \to X$ (i.e., those maps which send the unique $\mathbb{C}$-point of $\mathrm{Spec}(R_n)$ to x). In particular, if H is an algebraic group over $\mathbb{C}$ then $H_1/H_n = H(R_n)_{\mathrm{based}}$ (where the role of the point x is played by the identity $e \in H$).

Let $\mathrm{St}_\mu \subset G_1$ be the stabilizer of t^μ in G_1. Thus, $\mathcal{W}_\mu = G_1/\mathrm{St}_\mu$.

Lemma 2.7. (1) *Fix $n \in \mathbb{Z}_{>0}$ and let $\mu \in \Lambda^+$ satisfy the following condition:*

$$\langle \mu, \check{\alpha} \rangle \geq n \text{ for every positive root } \check{\alpha} \text{ of } \mathfrak{g}. \tag{2.2}$$

Then the image of St_μ in $G_1/G_n = G(R_n)_{\mathrm{based}}$ is equal to $U_-(R_n)_{\mathrm{based}}$. In particular, we have a natural map $\pi_{\mu,n} : \mathcal{W}_{G,\mu} \to G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}}$.

(2) *Assume that condition (2.2) is satisfied. Then for every $k < n$ the map $\pi^*_{\mu,n} : \mathbb{C}[G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}}] \to \mathbb{C}[\mathcal{W}_{G,\mu}]$ induces an isomorphism on functions of homogeneity degree k with respect to $\mathbb{C}^*$.*

Proof. (1) is obvious, so let us prove (2). First, let us discuss some preliminary facts about the algebra $\mathbb{C}[G_1]$. It is clear that any regular function $F : G \to \mathbb{C}$ defines a map of ind-schemes $F_1 : G_1 \to \mathbb{C}[t^{-1}]$ such that for any $g(t) \in G_1$ the constant term of $F_1(g(t))$ is equal to $F(e)$. Thus, for any $i > 0$ we can define the function $a_{F,i}$ on G_1 as the coefficient of t^{-i} in F_1. It is easy to see that the algebra $\mathbb{C}[G_1]$ is topologically generated by all the $a_{F,i}$. Since every $a_{F,i}$ has degree i with respect to $\mathbb{C}^*$, it follows that any function of homogeneity degree $< n$ lies in the subalgebra generated by $a_{F,i}$ with $i < n$. On the other hand, if $i < n$ then any $a_{F,i}$ is invariant under the (normal) subgroup G_n of G_1. Hence, any function on G_1 of homogeneity degree $< n$ is invariant under G_n.

Let f be a function on $\mathcal{W}_{G,\mu}$ of homogeneity degree k with respect to $\mathbb{C}^*$. Then we can think of f as a function on G_1 which is invariant on the right under St_μ. Then the above discussion shows that f is automatically (left and right) invariant under G_n. In addition, since f is invariant under St_μ it follows from (1) that (under the condition (2.2)) the function f comes from a function $\overline{f}$ on $G_1/G_n \cdot \mathrm{St}_\mu = G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}}$. □

Now we pass to the main technical result of this section. Let $\alpha \in \Lambda_+$. Then we have a natural map $Z^\alpha_{\mathfrak{g}} \to G_1/U_{-,1}$. This map is defined as follows: let $(\mathcal{F}_T, \kappa)$ be a quasi-map in $Z^\alpha_{\mathfrak{g}}$. The fiber of $\mathcal{F}_T$ at ∞ is automatically trivialized. This trivialization uniquely extends to $\mathbf{C} - \{0\}$ and thus we get a based map $\mathbf{C} - \{0\} \to \overline{G/U_-}$. Restricting this map to n-th infinitesimal neighborhood of ∞ in $\mathbf{C}$ we get a natural morphism $Z^\alpha_{\mathfrak{g}} \to G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}}$.

Theorem 2.8. (1) *Let $\lambda, \mu \in \Lambda^+$ such that $\lambda \geq \mu$ and let $\alpha = \lambda - \mu$. Then there exists a natural birational $T \times \mathbb{C}^*$-equivariant morphism $s^\lambda_\mu : \overline{\mathcal{W}}^\lambda_{G,\mu} \to$*

$Z^\alpha_{\mathfrak g}$, such that for any n satisfying (2.2), the following diagram is commutative:

$$\begin{tikzcd} \overline{\mathcal W}^\lambda_{G,\mu} \arrow[r, "s^\lambda_\mu"] \arrow[d, "\pi_{\mu,n}"'] & Z^\alpha_{\mathfrak g} \arrow[d] \\ G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}} \arrow[r, "\mathrm{id}"] & G(R_n)_{\mathrm{based}}/U_-(R_n)_{\mathrm{based}} \end{tikzcd} \tag{2.3}$$

(here we use the right vertical map described above).

(2) *Assume again that (2.2) is satisfied. Then the map $(s^\lambda_\mu)^* : \mathbb{C}[Z^\alpha_{\mathfrak g}] \to \mathbb{C}[\overline{\mathcal W}^\lambda_{G,\mu}]$ induces an isomorphism on functions of degree $< n$.*

Proof. First of all, we claim that (1) implies (2). Indeed, since s^λ_μ is birational, it follows that $(s^\lambda_\mu)^*$ is injective. On the other hand, it is surjective by Lemma 2.7(2) in view of (2.3).

Hence, it is enough to explain the construction of s^λ_μ. We start with a modular description of $\overline{\mathcal W}^\lambda_\mu$. Recall that Gr_G is the ind-scheme parametrizing a G-bundle $\mathcal F_G$ on $\mathbf C$ together with a trivialization on $\mathbf C - \{0\}$. Also, the isomorphism classes of G-bundles on $\mathbf C$ are in one-to-one correspondence with $\Lambda^+ = \Lambda/W$ (see [20] or [28]). This identification can be described as follows: it is obvious that T-bundles on $\mathbf C$ are in one-to-one correspondence with elements of Λ. On the other hand, it is well known that any G-bundle on $\mathbf C$ has a reduction to T. Thus, we get a surjective map from Λ to isomorphism classes of G-bundles on $\mathbf C$ and it is easy to see that two T-bundles on $\mathbf C$ give the isomorphic induced G-bundles if and only if one is obtained from the other by means of twist by some $w \in W$. The $G[t^{-1}]$-orbit $\mathrm{Gr}_{G,\mu} \subset \mathrm{Gr}_G$ (see Section 2.4) parametrizes the G-bundles of isomorphism type $W\mu$ equipped with a trivialization on $\mathbf C - \{0\}$. According to Lemma 2.5 we have a contraction $c :\ \mathrm{Gr}_{G,\mu} \to G \cdot t^\mu$, and $\mathcal W_{G,\mu} = c^{-1}(t^\mu)$. It remains to describe the contraction c to the partial flag variety $G \cdot t^\mu$ in modular terms.

Recall that the Harder-Narasimhan flag of a G-bundle $\mathcal F_G$ is a canonical reduction of $\mathcal F_G$ to a parabolic subgroup $P \subset G$, [29]. In case $\mathcal F_G$ is of isomorphism type $W\mu$, the corresponding parabolic subgroup is nothing but the stabilizer of t^μ in G. Let $HN(\mathcal F_G)$ be the Harder-Narasimhan flag of a G-bundle $\mathcal F_G \in \mathrm{Gr}_{G,\mu}$. Since $\mathcal F_G$ is trivialized off $0 \in \mathbf C$, the fiber of $HN(\mathcal F_G)$ at $\infty \in \mathbf C$ lies in the partial flag variety $G \cdot t^\mu$. So the value of $c(\mathcal F_G)$ is just the fiber of the Harder-Narasimhan flag of $\mathcal F_G$ at $\infty \in \mathbf C$. All in all, $\overline{\mathcal W}^\lambda_\mu \subset \mathrm{Gr}_G$ parametrizes G-bundles on $\mathbf C$ equipped with a trivialization off $0 \in \mathbf C$ with a pole of order $\le \lambda$ at 0, such that the isomorphism class of $\mathcal F_G$ is $W\mu$, and the fiber of the Harder-Narasimhan flag of $\mathcal F_G$ at $\infty \in \mathbf C$ is the base point $t^\mu \in G \cdot t^\mu$.

Now let us view the Harder-Narasimhan flag of $\mathcal F_G \in \overline{\mathcal W}^\lambda_{G,\mu}$ as a reduction $\mathcal F_P$ of $\mathcal F_G$ to a parabolic subgroup $P \subset G$ (the stabilizer of t^μ, containing B_-). Let L be the Levi quotient of P, and let L' be the quotient of L modulo center. Then $\mathrm{Ind}_P^{L'} \mathcal F_P$ is trivial. Hence, the standard reduction to the Borel B in the fiber of $\mathcal F_G$ at $\infty \in \mathbf C$ canonically extends to the reduction of $\mathrm{Ind}_P^{L'} \mathcal F_P$. Thus, any $\mathcal F_G \in \overline{\mathcal W}^\lambda_{G,\mu}$ is canonically equipped with a reduction $\varkappa$ to B with the standard fiber e_- at $\infty \in \mathbf C$.

Finally, we are ready for the construction of s^λ_μ. Given $\mathcal F_G \in \overline{\mathcal W}^\lambda_{G,\mu}$ equipped with an isomorphism $\sigma :\ \mathcal F_G \xrightarrow{\sim} \mathcal F_G^{\mathrm{triv}}$ defined off $0 \in \mathbf C$, we transfer the canonical reduction $\varkappa$ to $\mathcal{F}_G^{\mathrm{triv}}$ to obtain a B-structure $\sigma(\varkappa)$ on $\mathcal{F}_G^{\mathrm{triv}}$ with a singularity at $0 \in \mathbf{C}$. Twisting by $\lambda \cdot 0$ (i.e., in notations of Section 2.1 replacing $\mathcal{F}_T^{\check\lambda} \to \mathcal{O}_{\mathbf{C}} \otimes V(\check\lambda)$ by $\mathcal{F}_T^{\check\lambda}(-\langle\lambda,\check\lambda\rangle \cdot 0) \to \mathcal{O}_{\mathbf{C}} \otimes V(\check\lambda)$) we obtain a (regular) generalized B-structure κ on $\mathcal{F}_G^{\mathrm{triv}}$ with an *a priori* defect at $0 \in \mathbf{C}$ (cf. [17, Section 11]). Clearly, κ has no defect off $0 \in \mathbf{C}$, its value at $\infty \in \mathbf{C}$ is e_-, and the degree of κ equals $\alpha = \lambda - \mu$. We define $s_\mu^\lambda(\mathcal{F}_G, \sigma) = \kappa$. It follows from *loc. cit.* that s_μ^λ maps $\mathcal{S}^\lambda \cap \overline{\mathcal{W}}_{G,\mu}^\lambda$ isomorphically onto $\mathring{Z}_{\mathfrak{g}}^\alpha$. Here $\mathcal{S}^\lambda$ is the semi-infinite orbit $U((t)) \cdot t^\lambda$.

The theorem is proved. $\square$

Remark 2.9. Recall the notations of [17, Definition 11.7]: $\overline{\mathrm{Gr}}_G^\lambda \xleftarrow{\mathbf{p}} \mathcal{G}\mathcal{Z}_\lambda^{-\mu} \xrightarrow{\mathbf{q}} Z_G^\alpha$ is the restricted convolution diagram (as before, $\alpha = \lambda - \mu$). The locally closed subvariety $\overline{\mathcal{W}}_{G,\mu}^\lambda \subset \overline{\mathrm{Gr}}_G^\lambda$ is open dense in the image of $\mathbf{p}$, and the restriction of $\mathbf{p}$ to (the preimage of) $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is an isomorphism. The proof of the theorem shows that $s_\mu^\lambda = \mathbf{q}\mathbf{p}^{-1}|_{\overline{\mathcal{W}}_{G,\mu}^\lambda}$.

Corollary 2.10. *$Z_{\mathfrak{g}}^\alpha$ is normal.*

Proof. $\overline{\mathrm{Gr}}_G^\lambda$ is normal (see [12]); hence $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is normal by Lemma 2.5(3). Suppose a function $f \in \mathbb{C}(Z_{\mathfrak{g}}^\alpha)$ is a root of a unitary polynomial $f^r + a_{r-1}f^{r-1} + \cdots + a_0 = 0$ with coefficients in $\mathbb{C}[Z_{\mathfrak{g}}^\alpha]$. We choose n bigger than the degrees of the (homogeneous components) of the coefficients a_i, $1 \le i \le r$. Now we choose $\mu \in \Lambda^+$ satisfying (2.2) and such that $\lambda = \mu + \alpha \in \Lambda^+$. Then all the coefficients a_i lie in $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$. Hence, $f \in \mathbb{C}(Z_{\mathfrak{g}}^\alpha) = \mathbb{C}(\overline{\mathcal{W}}_{G,\mu}^\lambda)$ lies in $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$. Moreover, the degree of (the highest homogeneous component of) f is less than n. Hence, $f \in \mathbb{C}[Z_{\mathfrak{g}}^\alpha]$. $\square$

3. Normality of affine Zastava for $G = \mathrm{SL}(N)$

The purpose of this section is to give another prove of normality of Zastava for $G = \mathrm{SL}(N)$ which works also in the affine case.

3.1. **Notations.** We denote by I the set of simple coroots of the affine group $G_{\mathrm{aff}} = \mathrm{SL}(N)_{\mathrm{aff}}$. For $\alpha \in \mathbb{N}[I]$ we denote by Z^α the Drinfeld Zastava space. In [18] and [8] we have constructed a normal scheme $\mathfrak{Z}^\alpha$ together with a morphism $\eta : \mathfrak{Z}^\alpha \to Z^\alpha$ giving a bijection at the level of $\mathbb{C}$-points. In this section we prove that η is an isomorphism.

Recall that $\mathfrak{Z}^\alpha$ is defined as the categorical quotient $\mathsf{M}^\alpha /\!/ G_\alpha$ where M^α is the moduli scheme of representations of a certain chainsaw quiver with relations Q of dimension α. According to [8, 2.3–2.5], the stacky quotient $\mathsf{M}^\alpha / G_\alpha$ is the moduli stack $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty)$ of perverse coherent sheaves on the Deligne-Mumford stack $\mathcal{S}_N$ equipped with a framing at the divisor $\mathcal{D}_\infty \subset \mathcal{S}_N$. Let us denote by $\mathsf{g}_{\mathfrak{Z}} : \mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \to \mathfrak{Z}^\alpha$ the canonical map, and let us denote by $\mathsf{g}_Z : \mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \to Z^\alpha$ the composition of $\mathsf{g}_{\mathfrak{Z}}$ with η. Let us denote by $z_0^Z \in Z^\alpha$ (resp. $z_0^{\mathfrak{Z}} \in \mathfrak{Z}^\alpha$) the unique point fixed by the loop rotation action of $\mathbb{G}_m$. According to [6, 5.14], in order to prove that η is an isomorphism over the base field $\mathbb{C}$, it suffices to check that the inclusion $\mathsf{g}_{\mathfrak{Z}}^{-1}(z_0^{\mathfrak{Z}}) \hookrightarrow \mathsf{g}_Z^{-1}(z_0^Z)$ is an equality. We will do this mimicking the argument of [6, 5.16–5.17].

3.2. **Perverse sheaves framed off the origin.** We consider the following closed substack $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$: for a scheme S, its S-points are S-families of coherent perverse sheaves on $\mathcal{S}_N$ of degree α equipped with a framing on $\mathcal{S}_N - 0$, i.e., an isomorphism $\mathcal{F}|_{\mathcal{S}_N - 0} \simeq \mathcal{O}_{\mathcal{S}_N} \oplus \mathcal{O}_{\mathcal{S}_N}(-\mathcal{D}_0) \oplus \cdots \oplus \mathcal{O}_{\mathcal{S}_N}((1-N)\mathcal{D}_0)$, and satisfying the following condition: for any choice of presentation $h_c : \ \mathcal{S}_N = \mathbf{C} \times \mathcal{X}_N$ and the corresponding factorization map $\pi_c : \ Z^\alpha \to (\mathbf{C} - \infty_\mathbf{c})^\alpha$ the composition $S \to \mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \to Z^\alpha \to (\mathbf{C} - \infty_\mathbf{c})^\alpha$ sends S to the point $\alpha \cdot 0_\mathbf{C}$.

Note that the choices of h_c are parametrized by $c \in \mathbb{A}^1$ according to the choices of vertical directions $\mathbf{d}_v$ in [6, 5.14]. Similarly to [6, Proposition 5.16] we have the following.

Lemma 3.3. *The composition* $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0) \to \mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \to \mathfrak{Z}^\alpha$ *is the constant map to the point* $z_0^\mathfrak{Z} \in \mathfrak{Z}^\alpha$.

Proof. For a collection $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ representing a point of $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty)$, let us denote by T_{W_l} any endomorphism of the line W_l obtained by composing the maps A_k, B_i, p_j, q_r, and by T_{V_l} any similarly obtained endomorphism of V_l. It is well known that the ring of regular functions on $\mathfrak{Z}^\alpha$ is generated by all the possible T_{W_l}'s and the traces of all the possible T_{V_l}'s.

Let $\mathcal{F}$ be an S-point of $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$. For an integer m, let $\mathcal{F}'$ be the constant S-family of coherent perverse sheaves on $\mathcal{S}_N$ corresponding to the torsion-free sheaf $\mathfrak{m}_0^m \oplus \mathfrak{m}_0^m(-\mathcal{D}_0) \oplus \cdots \oplus \mathfrak{m}_0^m((1-N)\mathcal{D}_0)$ where $\mathfrak{m}_0$ is the maximal ideal of the point $0 \in \mathcal{S}_N$. Then, when m is large enough, we can find a map $\mathcal{F}' \to \mathcal{F}$ respecting the framings of both sheaves on $\mathcal{S}_N - 0$. The cone of this map is set-theoretically supported at $0 \in \mathcal{S}_N$ and has cohomology in degrees $0, 1$.

Let $(V_l, W_l, A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ (resp. $(V'_l, W'_l, A'_l, B'_l, p'_l, q'_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$) be the linear algebra data corresponding to $\mathcal{F}$ (resp. $\mathcal{F}'$). From the constructions of [8, 2.3–2.5] it follows that there are maps $V'_l \to V_l$, $W'_l \xrightarrow{\sim} W_l$ which commute with all the homomorphisms. Moreover, $q'_l \equiv 0$. From this we obtain that all the T_{W_l}'s vanish, and the only nonzero T_{V_l}'s are matrices of the form $A_{l_1}^{k_1} \circ B^{k_2} \circ A_{l_3}^{k_3} \circ \cdots \circ B^{k_{M-1}} \circ A_{l_M}^{k_M}$ where $l_1 = l_M = l$, and $B^{k_{2i}}$ stands for the composition of k_{2i} successive (composable) matrices of the form B_r, and $l_{2i+1} + k_{2i} = l_{2i-1}$. It remains to show that any such matrix is traceless.

As we already noted, for any matrix T'_{V_l} defined as in the previous paragraph but with certain B_r replaced by $p_{r+1}q_r$, the trace vanishes (due to the cyclic invariance of the trace of a product, being equal to the trace of the corresponding endomorphism $T'_{V'_l} = 0$). Using the relation $A_{r+1}B_r - B_rA_r + p_{r+1}q_r = 0$ repeatedly we see that $\mathrm{Tr}\, T_{V_l} = \mathrm{Tr}(A_l^k(B_{l-1} \circ B_{l-2} \circ \cdots \circ B_l)^{k'}) = \mathrm{Tr}((B_{l-1} \circ B_{l-2} \circ \cdots \circ B_l)^{k'} A_l^k)$ for certain k, k'. Therefore, it suffices to show that the characteristic polynomial of a matrix $A_l + cB_{l-1} \circ B_{l-2} \circ \cdots \circ B_l$ equals $t^{\dim V_l}$ for all $c \in \mathbb{C}$. However, this characteristic polynomial is nothing but the value at our point $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}} \in \mathsf{M}^\alpha$ of the l-th component of the factorization map $\mathsf{M}^\alpha \to \mathfrak{Z}^\alpha \to Z^\alpha \xrightarrow{\pi_c} \mathbb{A}^\alpha \to \mathbb{A}^{(a_l)}$ (here a_l is the l-the component of $\alpha \in \mathbb{N}[I]$). This completes the proof of the lemma. $\square$

Lemma 3.4. *For a scheme S, any S-point of the stack* $\mathsf{g}_Z^{-1}(z_0^Z)$ *factors through an S-point of* $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$.

Proof. Repeats the argument of [6, 5.17]. One has only to replace the word "trivialization" in *loc. cit.* by "framing", and $\mathbf{S}$ by $\mathcal{S}_N$. $\square$

Now an application of [6, Lemma 5.15] establishes the following.

Theorem 3.5. *Over the base field $\mathbb{C}$, the morphism $\eta : \mathfrak{Z}^\alpha \to Z^\alpha$ is an isomorphism.*

Corollary 3.6. *Over the base field $\mathbb{C}$, the Zastava scheme Z^α is reduced, normal, Gorenstein, and has rational singularities.*

Proof. $\mathfrak{Z}^\alpha$ is proved to be reduced and normal in [18, Theorem 2.7]. Recall the resolution $\varpi : \mathcal{P}^\alpha \to Z^\alpha$ by the affine Laumon space (see, e.g., *loc. cit.*). We will prove that $\mathcal{P}^\alpha$ is Calabi-Yau. Then it follows by the Grauert-Riemenschneider Theorem that Z^α has rational singularities and is Gorenstein. In order to prove that $\mathcal{P}^\alpha$ is Calabi-Yau, note that if the support of α is not the whole of $I = \mathbb{Z}/N\mathbb{Z}$, then we are in the finite (as opposed to affine) situation, and the Calabi-Yau property of Laumon resolution is proved in [19, Theorem 3] (cf. [15, Corollary 4.3]). If the support of α is full, recall the boundary divisor $\partial Z^\alpha \subset Z^\alpha$ introduced in [6, 11.8]. The proof of [6, Theorem 11.9] shows that for an integer $M \in \mathbb{N}$ the divisor $M\partial Z^\alpha$ (i.e., all the components $\partial_l Z^\alpha,\ l \in I$, of the boundary enter with the same multiplicity M) is a principal divisor. Let us denote $\eta^{-1}(\partial_l Z^\alpha)$ by $\partial_l \mathcal{P}^\alpha$. We see that $M \sum_{l\in I} \partial_l \mathcal{P}^\alpha$ is a principal divisor in $\mathcal{P}^\alpha$.

Now recall the meromorphic symplectic form Ω on $\mathcal{P}^\alpha$ (see [18, 3.1–3.2]). Let $\omega = \Lambda^{\mathrm{top}}\Omega$ be the corresponding meromorphic volume form on $\mathcal{P}^\alpha$. The calculation of [18, Proposition 3.5] shows that the divisor of poles of ω equals $\sum_{l\in I} \partial_l \mathcal{P}^\alpha$. We conclude that the canonical class of $\mathcal{P}^\alpha$ is torsion. However, the Picard group of $\mathcal{P}^\alpha$ has no torsion since $\mathcal{P}^\alpha$ is cellular. This completes the proof of the corollary. □

Corollary 3.7. *The resolution by the affine Laumon space $\varpi : \mathcal{P}^\alpha \to Z^\alpha$ induces an isomorphism $\varpi^* : \Gamma(Z^\alpha, \mathcal{O}_{Z^\alpha}) \xrightarrow{\sim} \Gamma(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha})$. The higher cohomology of the structure sheaf of the affine Laumon space vanishes: $H^k(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha}) = 0$ for $k > 0$.*

4. The boundary of Zastava

4.1. **The Cartier property.** In this section $\mathfrak{g}$ will be an arbitrary finite dimensional simple or untwisted affine Lie algebra with coroot lattice $\Lambda_\mathfrak{g}$. Let $\Lambda_\mathfrak{g}^+ \subset \Lambda_\mathfrak{g}$ denote the cone of positive linear combinations of positive simple coroots. Let T be the torus with the cocharacter lattice $\Lambda_\mathfrak{g}$. For $\alpha \in \Lambda_\mathfrak{g}^+$, the Zastava scheme $Z_\mathfrak{g}^\alpha$ is constructed in [6, Section 9] (under the name of $\mathfrak{U}_{G,B}^\alpha$). It is a certain closure of the (smooth) scheme $\overset{\circ}{Z}{}_\mathfrak{g}^\alpha$ of degree α based maps from $(\mathbb{P}^1, \infty)$ to the Kashiwara flag scheme $\mathcal{B}_\mathfrak{g}$ of $\mathfrak{g}$. In case $\mathfrak{g} = \mathfrak{sl}(N)_{\mathrm{aff}}$ we have $Z_\mathfrak{g}^\alpha = Z^\alpha$ of Section 3. The complement $\partial Z_\mathfrak{g}^\alpha := Z_\mathfrak{g}^\alpha - \overset{\circ}{Z}{}_\mathfrak{g}^\alpha$ (the boundary) is a quasi-effective Cartier divisor in $Z_\mathfrak{g}^\alpha$ according to [6, Theorem 11.9]. More precisely, there is a rational function F_α on $Z_\mathfrak{g}^\alpha$ whose lift to the normalization of $Z_\mathfrak{g}^\alpha$ is regular and has the preimage of $\partial Z_\mathfrak{g}^\alpha$ as the zero-divisor.

In general, the boundary $\partial Z_\mathfrak{g}^\alpha$ is not irreducible; its irreducible components $\partial_{\alpha_i} Z_\mathfrak{g}^\alpha$ are numbered by the simple coroots α_i which enter α with a nonzero coefficient. The argument in [6, 11.5–11.7] gives the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_\mathfrak{g}^\alpha$. To formulate the answer we assume that *all* the simple coroots enter α with nonzero coefficients (i.e., α has full support); otherwise, the question reduces to the similar one for a Levi subalgebra of $\mathfrak{g}$.

Lemma 4.2. *If α_i is a short coroot, the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is 1; if α_i is a long coroot, the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is the square length ratio of a long and a short coroot (that is,* 1, 2 *or* 3*).*

Proof. First let $\mathfrak{g}$ be simply laced. Since the restriction of the adjoint representation to a basic $\mathfrak{sl}_2$-subalgebra $\mathfrak{sl}_2^{\alpha_i} \subset \mathfrak{g}$ is independent of α_i as an $\mathfrak{sl}_2$-module, the argument of [6, Proof of Theorem 11.6] shows that the order of vanishing of F_α along each boundary component $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is the same, namely 1.

If $\mathfrak{g}$ is not simply laced, we realize it as the folding of a simply laced $\tilde{\mathfrak{g}}$, i.e., invariants of a pinning-preserving automorphism $\sigma : \tilde{\mathfrak{g}} \xrightarrow{\sim} \tilde{\mathfrak{g}}$. We have $\Lambda_+(\mathfrak{g}) = \Lambda_+(\tilde{\mathfrak{g}})^\sigma$, and given $\alpha \in \Lambda_+(\mathfrak{g})$ we will denote the corresponding σ-invariant element of $\Lambda_+(\tilde{\mathfrak{g}})^\sigma$ by $\tilde{\alpha}$. Then $Z_{\mathfrak{g}}^\alpha \simeq (Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}})^\sigma$, and $F_\alpha = F_{\tilde{\alpha}}|_{Z_{\mathfrak{g}}^\alpha}$. Now if α_i is short, $\tilde{\alpha}_i$ is a σ-invariant simple coroot of $\tilde{\mathfrak{g}}$, say β_i, and generically $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is the transversal intersection of $\partial_{\beta_i} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}$ with $Z_{\mathfrak{g}}^\alpha$. If α_j is long, $\tilde{\alpha}_j$ is a sum of $\frac{(\alpha_j,\alpha_j)}{(\alpha_i,\alpha_i)}(= 2 \text{ or } 3)$ simple coroots, say $\beta_j, \beta_{j'}, \beta_{j''}$. They are all disjoint in the Dynkin diagram of $\tilde{\mathfrak{g}}$, and the intersection $\partial_{\beta_j} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} \cap \partial_{\beta_{j'}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} \cap \partial_{\beta_{j''}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}$ is generically transversal. Moreover, each of $\partial_{\beta_j} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}, \partial_{\beta_{j'}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}, \partial_{\beta_{j''}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}$ is generically transversal to $Z_{\mathfrak{g}}^\alpha \subset Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}$, and generically $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha = Z_{\mathfrak{g}}^\alpha \cap \partial_{\beta_j} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} = Z_{\mathfrak{g}}^\alpha \cap \partial_{\beta_{j'}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} = Z_{\mathfrak{g}}^\alpha \cap \partial_{\beta_{j''}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} = Z_{\mathfrak{g}}^\alpha \cap \partial_{\beta_j} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} \cap \partial_{\beta_{j'}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}} \cap \partial_{\beta_{j''}} Z_{\tilde{\mathfrak{g}}}^{\tilde{\alpha}}$. The lemma follows. □

4.3. **The degree of F_α.** The function F_α is an eigenfunction of the torus $T \times \mathbb{G}_m^*$. Here, T (the Cartan torus) acts on $Z_{\mathfrak{g}}^\alpha$ via the change of framing at infinity, and $\mathbb{G}_m^*$ (loop rotations) acts on the source $(\mathbb{P}^1, \infty)$, and hence on $Z_{\mathfrak{g}}^\alpha$ via the transport of structure. We denote the coordinates on $T \times \mathbb{G}_m^*$ by (z, q). We define an isomorphism $\alpha \mapsto \alpha^*$ from the coroot lattice of (G, T) to the root lattice of (G, T) in the basis of simple coroots as follows: $\alpha_i^* := \check{\alpha}_i$ (the corresponding simple root). For an element α of the coroot lattice of (G, T) we denote by z^{α^*} the corresponding character of T.

Proposition 4.4. *The eigencharacter of F_α is $q^{(\alpha,\alpha)/2} z^{\alpha^*}$.*

The proposition will be proved in Section 4.9.

4.5. **Deligne pairing.** In order to compute the eigencharacter of F_α we recall the construction of F_α following Faltings [12]. To this end recall that given a family $f : \mathfrak{X} \to S$ of smooth projective curves and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathfrak{X}$ Deligne defines a line bundle $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ on S. In terms of determinant bundles the definition is simply

$$\begin{aligned} \langle \mathcal{L}_1, \mathcal{L}_2 \rangle = \det Rf_*(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \det Rf_*(\mathcal{O}_{\mathfrak{X}}) \\ \otimes (\det Rf_*(\mathcal{L}_1) \otimes \det Rf_*(\mathcal{L}_2))^{-1}. \end{aligned} \tag{4.1}$$

Deligne shows that the resulting pairing $\operatorname{Pic}(\mathfrak{X}) \times \operatorname{Pic}(\mathfrak{X}) \to \operatorname{Pic}(S)$ is symmetric (obvious) and bilinear (not obvious).

4.6. **Determinant bundles on Bun_T.** Let T be a torus and for consistency let Λ (resp. $\Lambda^\vee$) denote its coweight (resp. weight) lattice. Let also $(\cdot, \cdot)$ be an even pairing on Λ.

Let also C be a smooth projective curve (say, over the field $\mathbb{C}$) and let Bun_T denote the moduli stack of T-bundles on C. Then to the above data one associates a line bundle $\mathcal{D}$ on Bun_T in the following way. Let $e_1,\dots,e_n$ be a basis of Λ and let $f_1,\dots,f_n$ be the dual basis. For every $i=1,\dots,n$ let $\mathcal{L}_i$ denote the line bundle on $\mathrm{Bun}_T\times C$ associated to the weight f_i. Let also $a_{ij}=(e_i,e_j)\in\mathbb{Z}$. Then we define

$$\mathcal{D}=\left(\bigotimes_{i=1}^{n}\langle\mathcal{L}_i,\mathcal{L}_i\rangle^{\otimes\frac{a_{ii}}{2}}\right)\otimes\left(\bigotimes_{1\le i<j\le n}\langle\mathcal{L}_i,\mathcal{L}_j\rangle^{\otimes a_{ij}}\right). \tag{4.2}$$

It is easy to see that $\mathcal{D}$ does not depend on the choice of the basis (here, of course, we have to use the statement that Deligne's pairing is bilinear).

4.7. **The case $C=\mathbf{C}$.** Let now $C=\mathbf{C}$ (the projective line). In this case let us denote by $'\mathrm{Bun}(T)$ the space of T-bundles trivialized at ∞. This a scheme isomorphic to $\Lambda\times\mathrm{Spec}(\mathbb{C})$. We shall denote the pull-back of $\mathcal{D}$ to $'\mathrm{Bun}_T$ also by $'\mathcal{D}$.

Recall that $\mathbb{G}_m$ acts on $\mathbf{C}$. If v stands for a coordinate on $\mathbb{G}_m$, and c stands for a coordinate on $\mathbf{C}=\mathbb{P}^1$, we need the action $v(c):=v^2c$. Note that for this action any line bundle on $\mathbf{C}$ can be equipped with a $\mathbb{G}_m$-equivariant structure. Comparing to the action of Section 4.3 we have $q=v^2$. The $\mathbb{G}_m$-action extends to an action on $'\mathrm{Bun}_T$ and Bun_T. Since the construction of $\mathcal{D}$ is completely natural, it follows that $'\mathcal{D}$ is $\mathbb{G}_m$-equivariant. Since every component of $'\mathrm{Bun}_T$ is a point, this equivariance is given by a character of $\mathbb{G}_m$ (i.e., an integer) for every $\gamma\in\Lambda$.

Lemma 4.8. *The above integer is equal to (γ,γ).*

Proof. The main observation is the following. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1$ of degree n. It is isomorphic to $\mathcal{O}(n)$ and therefore it has a unique $\mathbb{G}_m$-equivariant structure such that the action of $\mathbb{G}_m$ on the fiber at ∞ is trivial (this makes sense since ∞ is a fixed point of $\mathbb{G}_m$). Then we claim that with respect to this equivariant structure $\mathbb{G}_m$ acts on $\det R\Gamma(\mathcal{L})$ by the character $v\mapsto v^{n(n+1)}$. Indeed, if $n\ge 0$ then $H^1(\mathcal{L})=0$ and $H^0(\mathcal{L})$ has dimension $n+1$ with weights $0,2,\dots,2n$ and their sum is $n(n+1)$. If $n<0$ then $H^0(\mathcal{L})=0$ and $H^1(\mathcal{L})$ has dimension $-n-1$ with weights $-2,-4,\dots,-2-2(-n-2)$ and their sum is equal to $-2(-n-1)-(-n-1)(-n-2)=-(-n)(-n-1)=-n(n+1)$.

Let now $\mathcal{L}_1$ and $\mathcal{L}_2$ be two line bundles on $\mathbb{P}^1$ of degrees n_1 and n_2. Then the action of $\mathbb{G}_m$ on $\langle\mathcal{L}_1,\mathcal{L}_2\rangle$ with respect to the above $\mathbb{G}_m$-equivariant structure on $\mathcal{L}_1$ and $\mathcal{L}_2$ corresponds to the integer

$$(n_1+n_2+1)(n_1+n_2)-(n_1+1)n_1-(n_2+1)n_2=2n_1n_2. \tag{4.3}$$

Let now $\mathcal{F}$ be a T-bundle of degree γ and let $\mathcal{L}_i$ be the line bundle associated with $f_i\in\Lambda^\vee$. Then the degree of $\mathcal{L}_i$ is $n_i=f_i(\gamma)$. Note that $\gamma=\sum n_ie_i$ and hence

$$(\gamma,\gamma)=\sum_{i=1}^{n}a_{ii}n_i^2+\sum_{1\le i<j\le n}2a_{ij}n_in_j.$$

Therefore according to (4.3) the action of $\mathbb{G}_m$ on the fiber of $'\mathcal{D}$ at $\mathcal{F}$ is by the character $v\mapsto v^m$ where

$$m=\sum_{i=1}^{n}\frac{a_{ii}}{2}(2n_i^2)+\sum_{1\le i<j\le n}2a_{ij}n_in_j=(\gamma,\gamma).$$

□

4.9. **Determinant bundle on Zastava space.** In this section $\mathfrak{g}$ is an arbitrary symmetrizable Kac-Moody Lie algebra, and T is the torus with the cocharacter lattice $\Lambda = \Lambda_{\mathfrak{g}}$: the coroot lattice of $\mathfrak{g}$. The Kashiwara flag scheme of $\mathfrak{g}$ is denoted by $\mathcal{B}_{\mathfrak{g}}$, and $\overset{\circ}{Z}{}^\gamma$ denotes the space of based maps $(\mathbb{P}^1, \infty) \to \mathcal{B}_{\mathfrak{g}}$ of degree γ (see [6, Theorem 18.1]). We let $(\cdot,\cdot)$ be the minimal even W-invariant form on Λ. Then we have the natural maps $f_\gamma : \overset{\circ}{Z}{}^\gamma \to \mathrm{Bun}_T^\gamma$, $'f_\gamma : \overset{\circ}{Z}{}^\gamma \to {'\mathrm{Bun}}_T^\gamma$. Consider the line bundle $f_\gamma^*\mathcal{D} = {'f}_\gamma^*({'\mathcal{D}})$ on $\overset{\circ}{Z}{}^\gamma$.

This line bundle acquires two different trivializations for the following reasons.

1) Since ${'\mathrm{Bun}}_T^\gamma$ is just one point, the line bundle $'\mathcal{D}$ is trivial there and hence its pull-back is trivial as well.

2) According to Faltings [12, Section 7], there is a trivialization of the similar bundle on the space of all maps from any smooth projective curve C to $\mathcal{B}_{\mathfrak{g}}$ (Faltings proves this only for finite-dimensional $\mathfrak{g}$ but his proof works word by word for any $\mathfrak{g}$).

Moreover, the line bundle $f_\gamma^*\mathcal{D}$ is naturally $\mathbb{G}_m$-equivariant ($\mathbb{G}_m$ acts on everything); both trivializations 1) and 2) are compatible with this structure if in 1) we let $\mathbb{G}_m$ act on the trivial bundle via the character $v \mapsto v^{(\gamma,\gamma)}$ (this follows from Lemma 4.8) and in 2) we let $\mathbb{G}_m$ act trivially on the trivial bundle (this follows from the fact that Faltings' construction is natural with respect to everything). Thus, 1) and 2) together give us an invertible function F_γ on $\overset{\circ}{Z}{}^\gamma$ such that $F_\gamma(v\phi) = v^{(\gamma,\gamma)}F_\gamma(\phi)$, that is $F_\gamma(q\phi) = q^{(\gamma,\gamma)/2}F_\gamma(\phi)$.

This completes the proof of the main part of Proposition 4.4: we have found the eigencharacter of F_α with respect to the loop rotations. It remains to compute the eigencharacter of F_α with respect to the Cartan torus. We must check the following. Given a one-parametric subgroup $\beta :\ \mathbb{G}_m \to T$, and a general point $\phi \in Z_{\mathfrak{g}}^\alpha$, the action map $\mathbb{G}_m \to Z_{\mathfrak{g}}^\alpha,\ c \mapsto \beta(c)\cdot\phi$, extends to the map $a :\ \mathbb{G}_m \subset \mathbb{A}^1 \to Z_{\mathfrak{g}}^\alpha$. We must check that the function $F_\alpha \circ a$ on $\mathbb{A}^1$ has the order of vanishing $\langle \beta, \alpha^*\rangle$ at the origin. By the factorization property of $Z_{\mathfrak{g}}^\alpha$ the question reduces to the case $\mathfrak{g} = \mathfrak{sl}(2),\ \alpha = 1$, which is obvious. It follows in particular that the function F_α vanishes along the boundary of $Z_{\mathfrak{g}}^\alpha$. $\square$

5. Gorenstein property of Zastava

Suppose $\mathfrak{g}$ is a simply laced simple Lie algebra.

Proposition 5.1. *$Z_{\mathfrak{g}}^\alpha$ is a Gorenstein (hence, Cohen-Macaulay) scheme with canonical (hence rational) singularities.*

Proof. We are going to apply Elkik's criterion [10] in order to prove that $Z_{\mathfrak{g}}^\alpha$ has rational singularities. To this end we will use the Kontsevich resolution $\pi :\ M_{\mathfrak{g}}^\alpha \to Z_{\mathfrak{g}}^\alpha$ (see [13, Section 8]). We will show that the discrepancy of π is strictly positive, that is the singularities of $Z_{\mathfrak{g}}^\alpha$ are canonical, hence rational.

Recall that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1,\alpha))$ is the moduli space of stable maps from curves of genus zero without marked points of degree $(1,\alpha)$ to $\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}$. It is a smooth Deligne-Mumford stack equipped with a birational projection to the space of Drinfeld quasimaps from $\mathbb{P}^1$ to $\mathcal{B}_{\mathfrak{g}}$. If C is such a curve of genus 0, then it has a distinguished irreducible component C_h (h for horizontal) which maps isomorphically onto $\mathbb{P}^1$. Using this isomorphism to identify C_h with $\mathbb{P}^1$ we obtain the points $0, \infty \in C_h$. Now $M_{\mathfrak{g}}^\alpha \subset \overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1,\alpha))$ is the locally closed substack cut out by the open condition that $\infty \in C_h$ is a smooth point of C, and by the closed condition that the stable map $\phi:\ C \to \mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}$ takes $\infty \in C_h \subset C$ to the marked point $e_- \in \mathcal{B}_{\mathfrak{g}}$.

The open substack $\overset{\circ}{M}{}^\alpha_{\mathfrak{g}} \simeq \overset{\circ}{Z}{}^\alpha_{\mathfrak{g}}$ of genuine-based maps is formed by the pairs (C,ϕ) such that C is irreducible. The complement is a normal crossing divisor with irreducible components D_β numbered by all $\beta \leq \alpha$. The generic point of D_β parametrizes the pairs (C,ϕ) such that $C = C_h \cup C_v$ is a union of 2 irreducible components (v for vertical), and the degree of $\phi|_{C_v}$ equals $(0,\beta)$. If $\beta = \alpha_i$ is a simple root, then $\pi:\ D_{\alpha_i} \to Z^\alpha_{\mathfrak{g}}$ is a birational isomorphism onto the boundary component $\partial_{\alpha_i} Z^\alpha_{\mathfrak{g}} \subset Z^\alpha_{\mathfrak{g}}$. If β is not a simple root, then D_β is an exceptional divisor of the Kontsevich resolution $\pi:\ M^\alpha_{\mathfrak{g}} \to Z^\alpha_{\mathfrak{g}}$.

Recall the symplectic form Ω on $\overset{\circ}{Z}{}^\alpha_{\mathfrak{g}}$ constructed in [16]. Its top exterior power $\Lambda^{|\alpha|}\Omega$ is a nonvanishing section of the canonical line bundle on $\overset{\circ}{Z}{}^\alpha_{\mathfrak{g}}$. It is well known that the complement $\overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}} \subset Z^\alpha_{\mathfrak{g}}$ to the union of codimension at least 2 boundary components is smooth. Let us denote its canonical line bundle by $\overset{\bullet}{\omega}$. Then according to [16, Remark 3], $\Lambda^{|\alpha|}\Omega$ has poles of order 1 at all the boundary divisors $\partial_{\alpha_i} Z^\alpha_{\mathfrak{g}} \subset \overset{\bullet}{Z}{}^\alpha_{\mathfrak{g}}$. According to Lemma 4.2, the product $F_\alpha \Lambda^{|\alpha|}\Omega$ is a regular nowhere vanishing section of $\overset{\bullet}{\omega}$, hence $\overset{\bullet}{\omega}$ is a trivial line bundle. We conclude that $Z^\alpha_{\mathfrak{g}}$ is $\mathbb{Q}$-Gorenstein with trivial canonical class ω_Z, and the discrepancy of the Kontsevich resolution $\pi:\ M^\alpha_{\mathfrak{g}} \to Z^\alpha_{\mathfrak{g}}$ is isomorphic to its canonical class ω_M. We have $\omega_M \otimes \pi^*\omega_Z^{-1} = \sum_{\beta\leq\alpha} m_\beta D_\beta$. We know that for $\beta = \alpha_i$ a simple root the multiplicity m_{α_i} is 0, and we will compute the multiplicities for all the rest β, and show that they are all strictly positive. In fact, due to the factorization property of Zastava, it suffices to compute a single multiplicity m_β: for $\beta = \alpha$.

Lemma 5.2. $m_\alpha = |\alpha| + \frac{(\alpha,\alpha)}{2} - 2$.

Proof. The loop rotations group $\mathbb{G}_m$ (cf. Section 4.3) acts on $M^\alpha_{\mathfrak{g}}$ via its action on the target $\mathbb{P}^1$. The fixed point set $D_\alpha^{\mathbb{G}_m}$ contains all the pairs (C,ϕ) such that C consists of 2 irreducible components, C_h and C_v, intersecting at the point $0 \in C_h$. We will compute m_α via comparison of the $\mathbb{G}_m$-actions in the fibers of ω_M and the normal bundle $\mathcal{N}$ to D_α at such a fixed point $(C,\phi) \in D_\alpha^{\mathbb{G}_m}$.

The fiber of the normal bundle $\mathcal{N}_{(C,\phi)}$ equals the tensor product of the tangent spaces at 0 to C_h and C_v. Hence, $\mathbb{G}_m$ acts on $\mathcal{N}_{(C,\phi)}$ via the character q^{-1} (recall that in the normalization of Section 4.3 and Section 4.7 $\mathbb{G}_m$ acts on the coordinate function on $\mathbb{P}^1 \simeq C_h$ via the character q; hence, it acts on the tangent space at 0 via the character q^{-1}).

According to [23, 1.3], the tangent space $T_{(C,\phi)} D_\alpha$ is $H^1(C, \mathcal{F}^\bullet(-\infty))$ where $\mathcal{F}^\bullet = \mathcal{F}^0 \to \mathcal{F}^1$ is the following complex of sheaves in degrees 0,1: $\mathcal{F}^0$ is the sheaf of vector fields on C vanishing at 0; while $\mathcal{F}^1 = \phi^*\mathcal{T}_{\mathbb{P}^1\times\mathcal{B}_{\mathfrak{g}}}$ is the pullback of the tangent sheaf of the target. Moreover, according to *loc. cit.*, $H^0(C,\mathcal{F}^\bullet) = H^2(C,\mathcal{F}^\bullet) = 0$.

It is immediate to check that $H^\bullet(C,\mathcal{F}^0(-\infty)) = H^0(C,\mathcal{F}^0(-\infty))$ is a 3-dimensional vector space with $\mathbb{G}_m$-character $1+1+1$, hence it contributes to $\det H^1(C,\mathcal{F}^\bullet(-\infty))$ the $\mathbb{G}_m$-character 1. It is also clear that $H^\bullet(C,\phi^*\mathcal{T}_{\mathbb{P}^1\times\mathcal{B}_{\mathfrak{g}}}(-\infty)) = H^0(C,\phi^*\mathcal{T}_{\mathbb{P}^1\times\mathcal{B}_{\mathfrak{g}}}(-\infty))$ is a direct sum of a $2|\alpha|$-dimensional vector space with trivial $\mathbb{G}_m$-action, and a 2-dimensional vector space with $\mathbb{G}_m$-character $1+q^{-1}$. Hence, it contributes the $\mathbb{G}_m$-character q^{-1} to $\det H^1(C,\mathcal{F}^\bullet(-\infty))$. All in all, $\mathbb{G}_m$ acts on $\det T_{(C,\phi)}M^\alpha_\mathfrak{g}$ with the character q^{-2}, and on the fiber of the canonical bundle $\omega_{M,(C,\phi)}$ with the character q^2.

Now recall that the canonical class ω_Z of $Z^\alpha_\mathfrak{g}$ is trivialized by the section $F_\alpha\Lambda^{|\alpha|}\Omega$ which is an eigensection of $\mathbb{G}_m$ with the character $q^{\frac{(\alpha,\alpha)}{2}+|\alpha|}$; this follows from Proposition 4.4 and [16, Remark 3]. Hence, $\mathbb{G}_m$ acts on the fiber at (C,ϕ) of the discrepancy line bundle $\omega_M\otimes\pi^*\omega_Z^{-1}$ with the character $q^{2-\frac{(\alpha,\alpha)}{2}-|\alpha|}$. It coincides with the character of $\mathbb{G}_m$ in the fiber $\mathcal{N}_{(C,\phi)}$ raised to the power $\frac{(\alpha,\alpha)}{2}+|\alpha|-2$. Hence, $\mathcal{O}(D_\alpha)$ must enter $\omega_M\otimes\pi^*\omega_Z^{-1}$ with coefficient $\frac{(\alpha,\alpha)}{2}+|\alpha|-2$. This completes the proof of the lemma. □

We return to the proof of the proposition. Since m_α is positive for nonsimple α, the singularities of $Z^\alpha_\mathfrak{g}$ are canonical, hence rational, therefore Cohen-Macaulay. It remains to prove the Gorenstein property. Let us denote by j the open embedding of $\overset{\bullet}{Z}{}^\alpha_\mathfrak{g}$ into $Z^\alpha_\mathfrak{g}$. Let us denote by $\mathcal{D}_Z$ the dualizing sheaf of $Z^\alpha_\mathfrak{g}$. We have to check that the natural map $\psi:\ \mathcal{D}_Z\to j_*\overset{\bullet}{\omega}$ is an isomorphism (the RHS is a trivial line bundle on $Z^\alpha_\mathfrak{g}$). Let $\varpi:Z^\alpha_\mathfrak{g}\to Y$ be a finite map to a smooth affine scheme. Then $\mathcal{D}_Z$ is locally free over Y, and $\varpi_*\psi$ is an isomorphism, hence ψ is an isomorphism itself.

The proposition is proved. □

6. Fermionic formulas and the boundary of Zastava

6.1. **Structure of the boundary.** If $\alpha=\beta+\gamma$ for $\alpha,\beta,\gamma\in\Lambda^+_\mathfrak{g}$, then according to [6, Section 10], we have a finite morphism $\iota_{\beta,\gamma}:\ Z^\beta_\mathfrak{g}\times(\mathbf{C}-\infty)^\gamma\to Z^\alpha_\mathfrak{g}$. Its image (a closed reduced subscheme of $Z^\alpha_\mathfrak{g}$) will be denoted by $\partial_\gamma Z^\alpha_\mathfrak{g}$. We will denote $Z^\beta_\mathfrak{g}\times(\mathbf{C}-\infty)^\gamma$ (resp. $\partial Z^\beta_\mathfrak{g}\times(\mathbf{C}-\infty)^\gamma$) by $\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}$ (resp. $\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}$) for short. The image of $\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}$ in $\partial_\gamma Z^\alpha_\mathfrak{g}$ (a closed reduced subscheme of $\partial_\gamma Z^\alpha_\mathfrak{g}$) will be denoted by $\partial\partial_\gamma Z^\alpha_\mathfrak{g}$. The union of $\partial_\gamma Z^\alpha_\mathfrak{g}$ over all $\gamma\in\Lambda^+_\mathfrak{g}$ such that $\alpha-\gamma\in\Lambda^+_\mathfrak{g}$ and $|\gamma|=n$ (a closed reduced equidimensional subscheme of $Z^\alpha_\mathfrak{g}$ of codimension n) will be denoted $\partial_n Z^\alpha_\mathfrak{g}$. Here for $\gamma=\sum c_i\alpha_i$ we set $|\gamma|=\sum c_i$. The disjoint union of $\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}$ (resp. $\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}$) over all $\gamma\in\Lambda^+_\mathfrak{g}$ such that $\alpha-\gamma\in\Lambda^+_\mathfrak{g}$ and $|\gamma|=n$ will be denoted by $\widetilde{\partial}_n Z^\alpha_\mathfrak{g}$ (resp. $\partial\widetilde{\partial}_n Z^\alpha_\mathfrak{g}$). Thus we have a finite morphism $\iota_n:\ \widetilde{\partial}_n Z^\alpha_\mathfrak{g}\to\partial_n Z^\alpha_\mathfrak{g}$, and the reduced preimage of $\partial_{n+1}Z^\alpha_\mathfrak{g}\subset\partial_n Z^\alpha_\mathfrak{g}$ is $\partial\widetilde{\partial}_n Z^\alpha_\mathfrak{g}\subset\widetilde{\partial}_n Z^\alpha_\mathfrak{g}$. Hence, we have an embedding $\iota_n^*:\ \Gamma(\partial_n Z^\alpha_\mathfrak{g},\mathcal{O}_{\partial_n Z^\alpha_\mathfrak{g}}(-\partial_{n+1}Z^\alpha_\mathfrak{g}))\hookrightarrow\Gamma(\widetilde{\partial}_n Z^\alpha_\mathfrak{g},\mathcal{O}_{\widetilde{\partial}_n Z^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_n Z^\alpha_\mathfrak{g}))$.

6.2. **Equivariant K-theory of affine Laumon spaces.** This subsection deals with the case $\mathfrak{g}=\mathfrak{sl}(N)_{\mathrm{aff}}$. We recall some facts from [7] and [14]. We consider the equivariant K-theory of the affine Laumon spaces $\mathcal{P}^\alpha$ with respect to certain torus $\widehat{T}=\widetilde{T}\times\mathbb{C}^*\times\mathbb{C}^*$. Here $\widetilde{T}$ is a certain 2^{N-1}-fold cover of a Cartan torus $T\subset\mathrm{SL}(N)$ acting on $\mathcal{P}^\alpha$ via the change of framing at infinity, while $\mathbb{C}^*\times\mathbb{C}^*$ acts on $\mathcal{S}_N$ by dilations, and hence on $\mathcal{P}^\alpha$ by the transport of structure. The coordinates on $\widehat{T}$ are denoted by $(t_1,\ldots,t_N,u,v)$, $t_1\cdot\ldots\cdot t_N=1$. Certain natural correspondences between the affine Laumon spaces give rise to the action of the affine quantum group $\mathcal{U}$ of type $\widetilde{A}_{N-1}$ (a $\mathbb{C}[v^{\pm 1}]$-algebra) on $\mathcal{M} = \bigoplus_\alpha \mathcal{M}_\alpha := \bigoplus_\alpha K^{\widehat{T}}(\mathcal{P}^\alpha) \otimes_{K^{\widehat{T}}(pt)} \mathrm{Frac}(K^{\widehat{T}}(pt))$. The Cartan subalgebra of $\mathcal{U}$ is $\mathcal{U}_0 = \mathbb{C}[v^{\pm 1}, C^{\pm 1}, L_i^{\pm 1},\ 0 \le i \le N-1]$ in notations of [7, 3.5]. Now $\mathcal{M}$ contains the universal Verma module $\mathcal{M}'$ over $\mathcal{U}' = \mathcal{U} \otimes \mathrm{Frac}(\mathcal{U}_0)$; here C acts as uv^N, while for $\alpha = \sum_{i=0}^{N-1} d_i\alpha_i$ the action of L_i on $\mathcal{M}_\alpha$ is given by $t_1^{-1}\cdots t_i^{-1}v^{d_i+i(n-i)/2}$, see [7, 3.4,3.17]. A certain completion $\widehat{\mathcal{M}}'$ contains the Whittaker vector $\mathfrak{u} = \sum_\alpha \mathfrak{u}_\alpha$, and the dual Whittaker vector $\mathfrak{n}' = \sum_\alpha \mathfrak{n}'_\alpha$. According to [7, Corollary 3.21], the Shapovalov scalar product $(\mathfrak{n}'_\alpha, \mathfrak{u}_\alpha) \in \mathbb{C}(t_1, \dots, t_N, u, v)$ equals the class $\mathfrak{J}_\alpha := [R\Gamma(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha})] \in \mathrm{Frac}(K^{\widehat{T}}(pt))$ up to a monomial in t, u, v. Comparing with [14, Theorem 3.1] we see that the collection of rational functions $\mathfrak{J}_\alpha$ is uniquely characterized by the condition $\mathfrak{J}_0 = 1$, and the recursion relation

$$\mathfrak{J}_\alpha = \sum_{0 \le \beta \le \alpha} \frac{q^{(\beta,\beta)/2} z^{\beta^*}}{(q)_{\alpha-\beta}} \mathfrak{J}_\beta, \tag{6.1}$$

where $q = v^2$, and $(q)_\gamma := \prod_{i=0}^{N-1}\prod_{s=1}^{c_i}(1-q^s)$ for $\gamma = \sum_{i=0}^{N-1} c_i\alpha_i$; while $z^{\gamma^*} := \prod_{i=0}^{N-1} z_i^{c_i}$ and $z_i = t_{i+1}t_i^{-1}u^{\delta_{0,i}}$ corresponds to the highest weight of the standard Cartan generator $K_i = L_i^2 L_{i+1}^{-1} L_{i-1}^{-1} C^{\delta_{i,0}} \in \mathcal{U}_0$ (i is understood as a residue mod N in the latter formula). Now Corollary 3.7 yields the following.

Corollary 6.3. *The classes* $\mathfrak{J}_\alpha = [\Gamma(Z^\alpha, \mathcal{O}_{Z^\alpha})] \in \mathrm{Frac}(K^{\widehat{T}}(pt))$ *satisfy the recursion relation* (6.1).

Proposition 6.4. *Let* $\mathfrak{g}$ *be a simply laced finite or affine Lie algebra such that* $Z^\alpha_\mathfrak{g}$ *is normal for every* α. *The embedding* $\iota_n^* :\ \Gamma(\partial_n Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_n Z^\alpha_\mathfrak{g}}(-\partial\partial_n Z^\alpha_\mathfrak{g})) \hookrightarrow \Gamma(\widetilde{\partial}_n Z^\alpha_\mathfrak{g}, \mathcal{O}_{\widetilde{\partial}_n Z^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_n Z^\alpha_\mathfrak{g}))$ *is an isomorphism for any* n *(equivalently,* $\iota_{|\gamma|}^* :\ \Gamma(\partial_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_\gamma Z^\alpha_\mathfrak{g}}(-\partial\partial_\gamma Z^\alpha_\mathfrak{g})) \hookrightarrow \Gamma(\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}))$ *is an isomorphism for any* $\gamma \le \alpha$*), if and only if the fermionic recursion* (1.3) *holds for any* α. $\square$

Proof. We have $[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}})] = \mathfrak{J}^\mathfrak{g}_{\alpha-\gamma} \cdot \frac{1}{(q)_\gamma}$ where $\mathfrak{J}^\mathfrak{g}_\beta$ stands for the class of $[\Gamma(Z^\beta_\mathfrak{g}, \mathcal{O}_{Z^\beta_\mathfrak{g}})]$ in $\mathrm{Frac}(K^{T\times\mathbb{C}^*}(pt))$. Also, $[\Gamma(\partial Z^\beta_\mathfrak{g}, \mathcal{O}_{\partial Z^\beta_\mathfrak{g}})] = (1 - q^{(\beta,\beta)/2}z^{\beta^*})\mathfrak{J}^\mathfrak{g}_\beta$ since the (reduced) subscheme $\partial Z^\beta_\mathfrak{g} \subset Z^\beta_\mathfrak{g}$ is cut out by the equation F_β whose $T \times \mathbb{G}_m$-degree is given by Proposition 4.4. In effect, the zero-subscheme of F_β is generically reduced (at each irreducible component) by Lemma 4.2, and hence reduced due to normality of $Z^\beta_\mathfrak{g}$. In effect, we must check that any function $f \in \Gamma(Z^\beta_\mathfrak{g}, \mathcal{O}_{Z^\beta_\mathfrak{g}})$ vanishing at the boundary is divisible by F_β. The rational function f/F_β is regular at the generic points of all the components of the boundary due to Lemma 4.2, so it is regular due to normality of $Z^\beta_\mathfrak{g}$.

All in all we see that $[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}))]$ is equal to

$$[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}})] - [\Gamma(\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial\widetilde{\partial}_\gamma Z^\alpha_\mathfrak{g}})] = \frac{q^{(\alpha-\gamma,\alpha-\gamma)/2} z^{(\alpha-\gamma)^*}}{(q)_\gamma}\, \mathfrak{J}_{\alpha-\gamma}.$$

Furthermore, since $[\Gamma(\partial_n Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_n Z^\alpha_\mathfrak{g}}(-\partial_{n+1} Z^\alpha_\mathfrak{g}))] = [\Gamma(\partial_n Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_n Z^\alpha_\mathfrak{g}})] - [\Gamma(\partial_{n+1} Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_{n+1} Z^\alpha_\mathfrak{g}})]$ we have $[\Gamma(Z^\alpha_\mathfrak{g}, \mathcal{O}_{Z^\alpha_\mathfrak{g}})] = \sum_{n\ge 0}[\Gamma(\partial_n Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_n Z^\alpha_\mathfrak{g}}(-\partial_{n+1} Z^\alpha_\mathfrak{g}))] = \sum_{\gamma \le \alpha}[\Gamma(\partial_\gamma Z^\alpha_\mathfrak{g}, \mathcal{O}_{\partial_\gamma Z^\alpha_\mathfrak{g}}(-\partial\partial_\gamma Z^\alpha_\mathfrak{g}))]$. Let us view this equality as an equality of formal power series in q, z with *nonnegative* powers and with *nonnegative* integral coefficients. Note that $[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}))] \succeq [\Gamma(\partial_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\partial_\gamma Z^\alpha_{\mathfrak{g}}))]$ (meaning that the LHS series is termwise bigger than or equal to the RHS series) since $\Gamma(\partial_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\partial_\gamma Z^\alpha_{\mathfrak{g}})) \hookrightarrow \Gamma(\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}))$.

Comparing to the equality (1.3), in view of the equality $[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}))] = \frac{q^{(\alpha-\gamma,\alpha-\gamma)/2} z^{(\alpha-\gamma)^*}}{(q)_\gamma}\mathfrak{J}_{\alpha-\gamma}$, we must have $[\Gamma(\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\widetilde{\partial}_\gamma Z^\alpha_{\mathfrak{g}}))] = [\Gamma(\partial_\gamma Z^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_\gamma Z^\alpha_{\mathfrak{g}}}(-\partial\partial_\gamma Z^\alpha_{\mathfrak{g}}))]$ which completes the proof of the proposition. □

6.5. **Proof of Theorem 1.5.**

Lemma 6.6. *The factorization morphism* $\pi: Z^\alpha_{\mathfrak{g}} \to \mathbb{A}^\alpha = (\mathbf{C} - \infty)^\alpha$ *is flat.*

Proof. According to Proposition 5.1, $Z^\alpha_{\mathfrak{g}}$ is Cohen-Macaulay. Evidently, $\mathbb{A}^\alpha$ is regular. It is well-known that all the fibers of π have the same dimension $|\alpha|$. It remains to apply [26, Theorem 23.1]. □

Let $\mathcal{F}^\alpha_{\mathfrak{g}} = \pi^{-1}(\alpha \cdot 0)$ stand for the scheme-theoretic fiber of $\pi: Z^\alpha_{\mathfrak{g}} \to \mathbb{A}^\alpha$ over $\alpha \cdot 0 \in \mathbb{A}^\alpha$. Let $\mathfrak{I}_\alpha = [\Gamma(\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\mathcal{F}^\alpha_{\mathfrak{g}}})] \in \mathrm{Frac}(K^{T\times\mathbb{C}^*}(pt))$ be the character of the ring of regular functions on the central fiber of π.

Corollary 6.7. $\mathfrak{I}_\alpha = (q)_\alpha \mathfrak{J}_\alpha$.

Proof. First, $[\Gamma(\mathbb{A}^\alpha, \mathcal{O}_{\mathbb{A}^\alpha})] = (q)_\alpha^{-1}$. Second, the flatness of π implies that $\pi_*\mathcal{O}_{Z^\alpha_{\mathfrak{g}}}$ is a direct sum of finite dimensional $T \times \mathbb{C}^*$-equivariant vector bundles $\mathcal{V}_\xi$ s.t. $T \times \mathbb{C}^*$ acts in the fiber of $\mathcal{V}_\xi$ over $\alpha \cdot 0$ via a character ξ of $T \times \mathbb{C}^*$. Finally, the fiber of $\pi_*\mathcal{O}_{Z^\alpha_{\mathfrak{g}}}$ over $\alpha \cdot 0$ is nothing but $\Gamma(\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\mathcal{F}^\alpha_{\mathfrak{g}}})$. □

Now the fermionic recursion (1.3) is equivalent to

$$\mathfrak{I}_\alpha = \sum_{\beta \leq \alpha} q^{(\beta,\beta)/2} z^{\beta^*} \binom{\alpha}{\beta}_q \mathfrak{I}_\beta, \tag{6.2}$$

where for $\alpha = \sum_{i\in I} a_i\alpha_i,\ \beta = \sum_{i\in I} b_i\alpha_i$ we set $\binom{\alpha}{\beta}_q := \prod_{i\in I}\prod_{s=1}^{a_i}(1-q^s)\prod_{s=1}^{b_i}(1-q^s)^{-1}\prod_{s=1}^{a_i-b_i}(1-q^s)^{-1}$. We will prove the fermionic recursion (1.3) in the equivalent formulation (6.2). To this end, we introduce the schemes (cf. notations of Section 6.1): $\partial_\gamma\mathcal{F}^\alpha_{\mathfrak{g}},\ \widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}},\ \partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}},\ \partial\partial_\gamma\mathcal{F}^\alpha_{\mathfrak{g}},\ \widetilde{\partial}_n\mathcal{F}^\alpha_{\mathfrak{g}},\ \partial\widetilde{\partial}_n\mathcal{F}^\alpha_{\mathfrak{g}}$ as the scheme-theoretic fibers over $\alpha \cdot 0 \in \mathbb{A}^\alpha$ of the corresponding morphisms $\partial_\gamma Z^\alpha_{\mathfrak{g}} \to \mathbb{A}^\alpha$ etc. ("the fiber of Z is $\mathcal{F}$").

As in the proof of Proposition 6.4, we see that $[\Gamma(\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}}(-\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}))]$ is equal to

$$[\Gamma(\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}})] - [\Gamma(\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}})] = q^{(\alpha-\gamma,\alpha-\gamma)/2} z^{(\alpha-\gamma)^*}\binom{\alpha}{\gamma}_q \mathfrak{I}_{\alpha-\gamma}.$$

In effect, $\widetilde{\partial}_\gamma\mathcal{F}^\alpha_{\mathfrak{g}}$ projects to the fiber of $\mathbb{A}^{\alpha-\gamma} \times \mathbb{A}^\gamma \to \mathbb{A}^\alpha$ over $\alpha \cdot 0$. The character of the ring of regular functions on this fiber equals $\binom{\alpha}{\gamma}_q$, the projection is flat, and its fiber over $(\alpha-\gamma)\cdot 0 \times \gamma \cdot 0$ is nothing but $\mathcal{F}^{\alpha-\gamma}_{\mathfrak{g}}$.

Furthermore, since $[\Gamma(\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}}(-\partial_{n+1}\mathcal{F}^\alpha_{\mathfrak{g}}))] = [\Gamma(\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}})] - [\Gamma(\partial_{n+1}\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_{n+1}\mathcal{F}^\alpha_{\mathfrak{g}}})]$ we have $[\Gamma(\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\mathcal{F}^\alpha_{\mathfrak{g}}})] = \sum_{n\geq 0}[\Gamma(\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}, \mathcal{O}_{\partial_n\mathcal{F}^\alpha_{\mathfrak{g}}}(-\partial_{n+1}\mathcal{F}^\alpha_{\mathfrak{g}}))] =$

$\sum_{\gamma\leq\alpha}[\Gamma(\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}))]$. Let us view this equality as an equality of formal power series in q,z with *nonnegative* powers and with *nonnegative* integral coefficients. Note that $[\Gamma(\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}))]\succeq[\Gamma(\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}))]$ (meaning that the LHS series is termwise bigger than or equal to the RHS series) since $\Gamma(\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}))\hookrightarrow\Gamma(\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}))$. We conclude that the recursion (6.2) holds iff $[\Gamma(\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\widetilde{\partial}_\gamma\mathcal{F}^\alpha_\mathfrak{g}))]=[\Gamma(\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}}(-\partial\partial_\gamma\mathcal{F}^\alpha_\mathfrak{g}))]$.

Note that both sides of the above equality are well defined at $q=1$, and since we already know the inequality LHS$\succeq$RHS, it suffices to check the equality at $q=1$. Let us denote $\mathfrak{I}_\alpha|_{q=1}$ by $\mathfrak{I}'_\alpha$. Then $\mathfrak{I}'_\alpha$ is the class of $[\Gamma(\,'\mathcal{F}^\alpha_\mathfrak{g},\mathcal{O}_{'\mathcal{F}^\alpha_\mathfrak{g}})]\in\mathrm{Frac}(K^T(pt))$ where $'\mathcal{F}^\alpha_\mathfrak{g}$ is an *arbitrary* fiber of $\pi: Z^\alpha_\mathfrak{g}\to\mathbb{A}^\alpha$. If $\underline{x}\in\mathbb{A}^\alpha$ is a general point (a configuration of distinct points), then by factorization $\pi^{-1}(\underline{x})$ is isomorphic to $\prod_{i\in I}\mathbb{A}^{a_i}$, and $\mathfrak{I}'_\alpha=\prod_{i\in I}(1-z_i)^{-a_i}$. The fermionic recursion (6.2) at $q=1$ becomes the evident equality $\mathfrak{I}'_\alpha=\sum_{\beta\leq\alpha}z^{\beta^*}\binom{\alpha}{\beta}\mathfrak{I}'_\beta$. This completes the proof of Theorem 1.5.

7. Non simply laced case

7.1. As mentioned before, Theorem 1.5 does not hold verbatim for non-simply laced $\mathfrak{g}$. It is reasonable to ask whether one can modify the statement so that an analog of Theorem 1.5 becomes true in the non-simply laced case. For this one should either modify the geometric problem (i.e., modify the definition of $\mathfrak{J}_\alpha$) or modify the equations that we want our J-function to satisfy. Apparently, both ways are possible, however the former is much easier than the latter. In this last section we explain how to modify the definition of $\mathfrak{J}_\alpha$ and sketch a proof of the corresponding analog of Theorem 1.5. Details (as well as other variants of Theorem 1.5 in the non-simply laced case) will appear in another publication.

7.2. **Fermionic recursion.** We recall the results of [14]. Let $\check{\mathfrak{g}}$ be a simple Lie algebra with the corresponding adjoint Lie group $\check{G}$. Let $\check{T}$ be a Cartan torus of $\check{G}$. We choose a Borel subgroup $\check{B}\supset\check{T}$. It defines the set of simple roots $\{\alpha_i,\ i\in I\}$. Let $G\supset T$ be the Langlands dual groups. We define an isomorphism $\alpha\mapsto\alpha^*$ from the root lattice of $(\check{G},\check{T})$ to the root lattice of (G,T) in the basis of simple roots as follows: $\alpha_i^*:=\check{\alpha}_i$ (the corresponding simple coroot). For two elements α,β of the root lattice of $(\check{G},\check{T})$ we say $\beta\leq\alpha$ if $\alpha-\beta$ is a nonnegative linear combination of $\{\alpha_i,\ i\in I\}$. For such α we denote by z^{α^*} the corresponding character of T. As usual, q stands for the identity character of $\mathbb{G}_m$. We set $d_i=\frac{(\alpha_i,\alpha_i)}{2}$, and $q_i=q^{d_i}$. For $\gamma=\sum_{i\in I}c_i\alpha_i$, we set $(q)_\gamma:=\prod_{i\in I}\prod_{s=1}^{c_i}(1-q_i^s)$. According to [14, Theorem 3.1], the recurrence relations

$$\mathfrak{Z}_\alpha=\sum_{0\leq\beta\leq\alpha}\frac{q^{(\beta,\beta)/2}z^{\beta^*}}{(q)_{\alpha-\beta}}\mathfrak{Z}_\beta$$

uniquely define a collection of rational functions $\mathfrak{Z}_\alpha,\ \alpha\geq 0$, on $T\times\mathbb{G}_m$, provided $\mathfrak{Z}_0=1$. Moreover, these functions are nothing but the Shapovalov scalar products of the weight components of the Whittaker vectors in the universal Verma module over the corresponding quantum group.

We can now explain why Theorem 1.5 doesn't literally hold for non-simply laced $\mathfrak{g}$. This can be seen, for example, in the following way. Let α_i be a simple coroot of $\mathfrak{g}$. Then it is easy to see that we have

$$\mathfrak{J}_{\alpha_i}(z,q)=\frac{1}{(1-z^{\alpha_i^*}q_i)(1-q_i)}.$$

However, the character of $\mathbb{C}[Z_{\mathfrak{g}}^{\alpha_i}]$ is simply $\frac{1}{(1-z^{\alpha_i*}q)(1-q)}$. Moreover, in this case $Z_{\mathfrak{g}}^{\alpha_i}$ is isomorphic to the affine plane $\mathbb{A}^2$, so it obviously has rational singularities. Therefore the character of $\mathbb{C}[Z_{\mathfrak{g}}^{\alpha_i}]$ is equal to $\mathfrak{I}_{\alpha_i}$. Hence, we see that if $q_i\neq q$ then $\mathfrak{I}_{\alpha_i}\neq\mathfrak{J}_{\alpha_i}$.

7.3. **Geometric interpretation of $\mathfrak{J}_\alpha$.** We are now going to introduce a scheme $\widehat{Z}_{\mathfrak{g}}^{\alpha}$ equipped with the action of $T\times\mathbb{G}_m$ such that the character of $\mathbb{C}[\widehat{Z}_{\mathfrak{g}}^{\alpha}]$ equals $\mathfrak{J}_\alpha$.

To this end we realize $\check{\mathfrak{g}}$ as a *folding* of a simple simply laced Lie algebra $\check{\mathfrak{g}}'$, i.e., as invariants of an outer automorphism σ of $\check{\mathfrak{g}}'$ preserving a Cartan subalgebra $\check{\mathfrak{t}}'\subset\check{\mathfrak{g}}'$ and acting on the root system of $(\check{\mathfrak{g}}',\check{\mathfrak{t}}')$. In particular, σ gives rise to the same named automorphism of the Langlands dual Lie algebras $\mathfrak{g}'\supset\mathfrak{t}'$. We denote by Ξ the finite cyclic group generated by σ. Let $G'\supset T'$ denote the corresponding simply connected Lie group and its Cartan torus. The *coinvariants* $X_*(T')_\sigma$ of σ on the coroot lattice $X_*(T')$ of $(\mathfrak{g}',\mathfrak{t}')$ coincide with the root lattice of $\check{\mathfrak{g}}$. We have an injective map $a:\ X_*(T')_\sigma\to X_*(T')^\sigma$ from coinvariants to invariants defined as follows: given a coinvariant $\bar{\alpha}$ with a representative $\alpha\in X_*(T')$ we set $a(\bar{\alpha}):=\sum_{\xi\in\Xi}\xi(\alpha)$. Given $\bar{\alpha}\geq 0$ in the root lattice of $\check{\mathfrak{g}}$, we define an automorphism ς of the based quasimaps' space $Z_{\mathfrak{g}'}^{a(\bar{\alpha})}$ as follows. It is the composition of two automorphisms: a) σ on the target and b) multiplication by ζ on the source $\mathbf{C}\cong\mathbb{P}^1$. Here, ζ is a primitive root of unity of the order equal to the order of σ. One can check that the fixed point set $(\overset{\circ}{Z}{}_{\mathfrak{g}'}^{a(\bar{\alpha})})^\varsigma$ is connected. We define $\widehat{Z}_{\mathfrak{g}}^{\bar{\alpha}}$ as the closure of $(\overset{\circ}{Z}{}_{\mathfrak{g}'}^{a(\bar{\alpha})})^\varsigma$ in $Z_{\mathfrak{g}'}^{a(\bar{\alpha})}$.

The equality $\mathfrak{J}_{\bar{\alpha}}=[\mathbb{C}[\widehat{Z}_{\mathfrak{g}}^{\bar{\alpha}}]]$ is proved along the lines of the argument of the previous sections. In particular, the role of the affine Grassmannian of G in the simply laced case is played by the ramified Grassmannian of (G',σ), see [31].

Acknowledgments

The idea of proving Corollary 1.6 using [14] came to us during a talk by E. Feigin (explaining the results of [14]) in Oberwolfach in March 2009. Also, the ideas of Section 2 came to us during a conversation with J. Kamnitzer in Jerusalem in December 2010. In addition, we would like to thank B. Feigin, A. Knutson, A. Kuznetsov, and X. Zhu for illuminating discussions. We are grateful to an anonymous referee whose remarks resulted in the correction and simplification of the proof of the main Theorem 1.5. Part of this work was completed when both authors were visiting CIRM (Luminy) and IESC (Cargese). The authors would like to thank those institutions for their hospitality. A. B. was partially supported by NSF grants DMS-0854760 and DMS-0901274. M.F. was partially supported by the RFBR grants 12-01-00944, 12-01-33101, 13-01-12401/13, the National Research University Higher School of Economics' Academic Fund award No.12-09-0062 and the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. This study comprises research findings from the "Representation Theory in Geometry and in

Mathematical Physics" carried out within The National Research University Higher School of Economics' Academic Fund Program in 2012, grant No 12-05-0014.

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