JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 4, October 2014, Pages 1169–1212
S 0894-0347(2014)00804-3
Article electronically published on May 21, 2014

PARITY SHEAVES

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1. Introduction

1.1. **Overview.** In view of applications in geometric representation theory in positive characteristic, we introduce parity sheaves, a class of constructible complexes of sheaves on stratified varieties whose strata satisfy a cohomological parity vanishing condition. We show the existence and uniqueness of parity sheaves on several spaces arising in representation theory, including generalised flag varieties, nilpotent cones (at least for GL_n) and toric varieties.

With sheaf coefficients in a field of characteristic zero, parity sheaves correspond to classical objects in geometric representation theory. When the coefficients are of positive characteristic, parity sheaves are important new objects. We show that parity sheaves, unlike intersection cohomology complexes, satisfy a form of the Decomposition Theorem, and explain the role played by intersection forms in determining the decomposition of their direct images. On flag varieties parity sheaves allow us to retrieve in a uniform way the Beilinson-Bezrukavnikov-Mirković tilting sheaves and the special sheaves of Soergel, used by Fiebig in his proof of Lusztig's conjecture.

1.2. **Outline.** In Section 2 we define parity sheaves and develop some of their basic properties. Our notation and assumptions appear in 2.1. The definition of parity sheaves appears as Definition 2.14 and depends on the preceding uniqueness result (Theorem 2.12). Section 2.3 begins to explore the question of existence and gives a criterion for existence. In Section 2.4, we introduce the notion of an even map (Definition 2.33) and show that the push forward functor along proper, even maps preserves the class of parity complexes (Proposition 2.34). This is our key tool for producing examples and serves as a weak analogue of the Decomposition Theorem. Section 2.5 is concerned with the behaviour of parity sheaves under modular reduction. Proposition 2.41 shows that when an **IC**-sheaf with $\mathbb{Q}$-coefficients is parity, the corresponding modular **IC**-sheaf is parity for all but finitely many characteristics. Sections 2.6 and 2.7 review respectively the notions of torsion primes and ind-varieties.

Section 3 extends an observation of de Cataldo and Migliorini [dCM02, dCM05] from their recent Hodge theoretic proof of the Decomposition Theorem. In their work a crucial role is played by the case of semi-small resolutions, and certain intersection forms attached to the strata of the target. Indeed, they show that for a semi-small morphism the direct image of the intersection cohomology sheaf splits

Received by the editors November 5, 2012 and, in revised form, October 21, 2013.
2010 *Mathematics Subject Classification.* Primary 55N33, 20C20.
The first author was supported by ANR Grant No. ANR-09-JCJC-0102-01.
The second author was supported by an NSF postdoctoral fellowship.

as a direct sum of intersection cohomology complexes if and only if these forms are non-degenerate.

In Section 3.1, we recall the definition of these intersection forms. In Section 3.2, we extend the observation of de Cataldo and Migliorini and prove (Theorem 3.7) that the non-degeneracy of the modular reduction of these intersection forms (which are defined over the integers) determine exactly when the decomposition theorem fails in positive characteristic for a semi-small resolution. Section 3.3 addresses the case of a proper and even (but not necessarily semi-small) morphism from a smooth source. Theorem 3.13 shows that the multiplicities of parity sheaves which occur in the direct image are given in terms of the ranks of these forms. These theorems allow one to reformulate questions in representation theory in terms of such intersection forms.

The remaining sections explore three classes of examples: Kac-Moody flag varieties (4.1), toric varieties (4.2) and nilpotent cones (4.3).

1.3. **Related work.** The usefulness of some form of parity vanishing in representation theory and intersection/equivariant cohomology has been noticed by many authors (e.g. [KL80], [Spr82], [CPS93], [GKM98] and [BJ01]). In the following we comment briefly on ideas that are particularly closely related to the current work:

1.3.1. *Soergel's category $\mathcal{K}$.* The idea of considering another class of objects as "replacements" for intersection cohomology complexes when using positive characteristic coefficients is due to Soergel in [Soe00]. He considers the full additive subcategory $\mathcal{K}$ of the derived category of sheaves of k-vector spaces on the flag variety which occur as direct summands of direct images of the constant sheaf on Bott-Samelson resolutions. Furthermore, he shows (using arguments from representation theory) that if the characteristic of k is larger than the Coxeter number, then the indecomposable objects in $\mathcal{K}$ are parametrised by the Schubert cells. In fact, the indecomposable objects in Soergel's category $\mathcal{K}$ are parity sheaves, and our arguments provide a geometric way of understanding and expanding his result.

1.3.2. *Tilting perverse sheaves.* Since their introduction by Ringel [Rin92], an important role in representation theory is played by tilting objects in the highest weight categories. There are several important examples of categories of perverse sheaves which are highest weight, and it is desirable to have a local (i.e. in terms of stalks and costalks) characterization of the tilting sheaves. In [BBM04] Beilinson, Bezrukavnikov and Mirković give such a description for the Schubert constructible perverse sheaves on the flag variety. It is immediate from their description that the tilting perverse sheaves can also be characterised as parity sheaves (for the "dimension pariversity," see Section 2.2).

Another important example of a highest weight category of perverse sheaves is the Satake category of $G[[t]]$-constructible perverse sheaves on the affine Grassmannian. In [JMW] the authors show that the parity sheaves on the affine Grassmannian correspond to the tilting sheaves, under certain explicit bounds on the characteristic of the coefficients. Thus, in two important and quite different examples—the finite flag variety (or more generally any stratified variety satisfying the conditions of [BBM04] and our parity conditions) and the affine Grassmannian—we see that the indecomposable tilting sheaves are parity sheaves. Thus one is led to suspect a relation between parity sheaves and tilting sheaves on any variety satisfying our parity conditions for which the corresponding category of perverse sheaves is highest weight. For example one may show that, in the above situation, if the parity sheaves for the dimension pariversity are perverse, then they are tilting sheaves. In [AM12], Achar and the second author start to explore this phenomenon in the case of nilpotent cones.

1.3.3. *Combinatorial models for intersection cohomology.* There exist combinatorial algorithms (due to Bernstein and Lunts [BL94a] and Barthel, Brasselet, Fieseler and Kaup [BBFK99]) for calculating the rational equivariant intersection cohomology of a toric variety using commutative algebra. Similarly, Braden and MacPherson [BM01] gave an analogous combinatorial algorithm for Schubert varieties. In both cases the calculation of intersection cohomology with modular coefficients is significantly more difficult, and no algorithm is known. In [FW], Fiebig and the third author show that, when performed with coefficients of positive characteristics, the Braden-MacPherson algorithm computes the stalks of the parity sheaves. It is likely that an analogous result is true for toric varieties.

1.3.4. *The p-canonical basis.* Parity sheaves on generalised flag varieties with coefficients in a field of characteristic $p \geq 0$ may be used to define a "p-canonical basis" for the Hecke algebra which enjoys remarkable positivity properties (when $p = 0$ one recovers the Kazhdan-Lusztig basis). In low rank examples, there are sufficiently many constraints to force the p-canonical and Kazhdan-Lusztig bases to coincide for all p and almost all elements of the Weyl group [WB12]. This indicates where to look for non-trivial torsion, which does indeed occur. Braden had previously found 2-torsion in some Schubert varieties of types A_7 and D_4 (see the appendix of [WB12]). More recently, Polo has found n-torsion in a Schubert variety of type A_{4n-1}. The examples in A_7 led to the discovery of a relation between non-trivial parity sheaves and the reducibility of characteristic varieties [VW].

There is a parallel story using Lusztig complexes on moduli spaces of quiver representations, where one recovers the p-canonical basis for the negative part of the quantised enveloping algebra [Gro99]. The relationship with parity sheaves for linear quivers has been explained by Maksimau [Mak13]. These results may be used to rephrase the James conjecture in terms of parity sheaves.

1.3.5. *Weights and parity sheaves.* Replacing our complex variety X by a variety X_o defined over a finite field $\mathbb{F}_q$, one can consider Deligne's theory of weights in the derived category $D^b_c(X_o, \mathbb{Q}_\ell)$ of $\mathbb{Q}_\ell$-sheaves (see [BBD82] for details and notation).

In all examples considered in this paper, one can proceed naively, and say that $\mathcal{F}_o \in D^b_c(X_o, \mathbb{Z}_\ell)$ (resp. $D^b_c(X_o, \mathbb{F}_\ell)$) is pure of weight 0 if $\mathcal{H}^i(\mathcal{F})$ and $\mathcal{H}^i(\mathbb{D}\mathcal{F})$ vanish for odd i and, for all $x \in X_o(\mathbb{F}_{q^n})$ the Frobenius $F^*_{q^n}$ acts on the stalks of $\mathcal{H}^{2i}(\mathcal{F})$ and $\mathcal{H}^{2i}(\mathbb{D}\mathcal{F})$ as multiplication by q^{ni} (the image of q^{ni} in $\mathbb{F}_\ell$ respectively). With this definition one can show that, in all examples considered in this paper, there exist analogues of parity sheaves which are pure of weight 0. Note that the modular analogue of Gabber's theorem is not true: if $\mathcal{F}_o$ in $D^b_c(X_o, \mathbb{Z}_\ell)$ or $D^b_c(X_o, \mathbb{F}_\ell)$ is pure of weight 0, then $\mathcal{F}$ is not necessarily semi-simple.

Nevertheless, such considerations have been used by Riche, Soergel and the third author to deduce that the dg-algebra of extensions of the direct sum of all parity sheaves on the flag variety is formal. From this they deduce a modular form of Koszul duality [RSW].

2. Definition and first properties

2.1. **Notation and assumptions.** Let $\mathbb{O}$ denote a complete discrete valuation ring of characteristic zero (e.g., a finite extension of $\mathbb{Z}_p$), $\mathbb{K}$ its field of fractions (e.g., a finite extension of $\mathbb{Q}_p$), and $\mathbb{F}$ its residue field (e.g., a finite field $\mathbb{F}_q$). Unless stated otherwise, k denotes a complete local principal ideal domain, which may be for example $\mathbb{K}$, $\mathbb{O}$ or $\mathbb{F}$, and all sheaves and cohomology groups are to be understood with coefficients in k.

In what follows all varieties will be considered over $\mathbb{C}$ and equipped with the classical topology. Throughout, X denotes either a variety or a G-variety for some connected linear algebraic group G. In Sections 2 and 3 we deal with these two situations simultaneously, bracketing the features which only apply in the equivariant situation. In the examples, we will specify the set-up in which we work.

We fix an algebraic stratification (in the sense of [CG97, Definition 3.2.23])

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

of X into smooth connected locally closed (G-stable) subsets. For each $\lambda \in \Lambda$ we denote by $i_\lambda : X_\lambda \to X$ the inclusion and by d_λ the complex dimension of X_λ.

We denote by $D(X)$, or $D(X; k)$ if we wish to emphasise the coefficients, the bounded (equivariant) constructible derived category of k-sheaves on X with respect to the given stratification (see [BL94b] for the definition and basic properties of the equivariant derived category). The category $D(X)$ is triangulated with shift functor [1]. We call objects of $D(X)$ complexes. For all $\lambda \in \Lambda$, let $\underline{k}_\lambda$ denote the (equivariant) constant sheaf on X_λ. Given $\mathcal{F}$ and $\mathcal{G}$ in $D(X)$ we set $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}_{D(X)}(\mathcal{F}, \mathcal{G})$ and $\mathrm{Hom}^n(\mathcal{F}, \mathcal{G}) := \mathrm{Hom}(\mathcal{F}, \mathcal{G}[n])$. We can form the graded k-module $\mathrm{Hom}^\bullet(\mathcal{F}, \mathcal{G}) := \oplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(\mathcal{F}, \mathcal{G})$.

Recall that an additive category is Krull-Remak-Schmidt if every object is isomorphic to a finite direct sum of objects, each of which has a local endomorphism ring. In a Krull-Remak-Schmidt category all idempotents split and any object admits a unique decomposition into indecomposable objects. Moreover, an object is indecomposable if and only if its endomorphism ring is local. By our assumptions on k, $D(X)$ is a Krull-Remak-Schmidt category (see [LC07]).

Remark 2.1. The category $D(X)$ is Krull-Remak-Schmidt as soon as the ring of coefficients k is Noetherian and complete local. The Krull-Remak-Schmidt property of $D(X)$ is fundamental to all arguments below. Above we make the stronger assumption that k is a complete local principal ideal domain (equivalently a field or complete discrete valuation ring). We use this stronger assumption in Sections 2.3 and 2.5. The results of Sections 2.2, 2.4 and 4 remain valid for coefficients in any Noetherian complete local ring k. In Section 3 we assume that k is a field.

For each λ, denote by $\mathrm{Loc}_{\mathrm{f}}(X_\lambda, k)$ or $\mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ the category of (equivariant) local systems of free finite rank k-modules on X_λ. We make the following assumptions on our variety X, which are in force throughout the paper except in Section 3.2 and 4.3.3. For each $\lambda \in \Lambda$ and all $\mathcal{L}, \mathcal{L}' \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ we assume:

$$\mathrm{Hom}^n(\mathcal{L}, \mathcal{L}') = 0 \text{ for } n \text{ odd} \tag{2.1}$$

and

$$\mathrm{Hom}^n(\mathcal{L}, \mathcal{L}') \text{ is a free } k\text{-module for all } n. \tag{2.2}$$

Remark 2.2.

(1) When k is a field, all finite dimensional k-modules are free, so the second assumption can be ignored.

(2) Given two local systems $\mathcal{L}, \mathcal{L}' \in \mathrm{Loc_f}(X_\lambda)$ we have isomorphisms:

$$\mathrm{Hom}^\bullet(\mathcal{L}, \mathcal{L}') \cong \mathrm{Hom}^\bullet(\underline{k}_\lambda, \mathcal{L}^\vee \otimes \mathcal{L}') \cong \mathbb{H}^\bullet(\mathcal{L}^\vee \otimes \mathcal{L}').$$

Hence (2.1) and (2.2) are equivalent to requiring that $\mathbb{H}^\bullet(\mathcal{L})$ is a free k-module and vanishes in odd degree, for all $\mathcal{L} \in \mathrm{Loc_f}(X_\lambda)$.

(3) The condition (2.1) implies that there are no extensions between objects of the category $\mathrm{Loc_f}(X_\lambda)$. In particular, if k is a field, then $\mathrm{Loc_f}(X_\lambda)$ is semi-simple.

Finally, for $\lambda \in \Lambda$ and $\mathcal{L} \in \mathrm{Loc_f}(X_\lambda)$, we denote by $\mathbf{IC}(\lambda, \mathcal{L})$, or simply $\mathbf{IC}(\lambda)$ if $\mathcal{L} = \underline{k}_\lambda$, the intersection cohomology complex on $\overline{X}_\lambda$ with coefficients in $\mathcal{L}$, shifted by d_λ so that it is perverse, and extended by zero on $X \backslash \overline{X}_\lambda$. We always use the middle perversity $p_{1/2}$, which is self-dual when k is a field. When k is a ring of integers, it is not stable by duality, so there is a dual **IC** for the dual t-structure $p^+_{1/2}$ [BBD82, §3.3]. In this paper we only need the standard **IC**.

2.2. **Definition and uniqueness.** In this section the notation and assumptions are as in Section 2.1.

Definition 2.3. A *pariversity* is a function $\dagger : \Lambda \to \mathbb{Z}/2$.[1]

We will mainly be interested in two special pariversities: the constant function $\natural$ defined by $\natural(\lambda) = \overline{0}$ for all λ and the dimension function $\Diamond$ defined by $\Diamond(\lambda) = \overline{d_\lambda}$. Notice that if the strata are all even dimensional, then $\natural = \Diamond$.

Definition 2.4. Fix a pariversity $\dagger$. In the following $? \in \{*, !\}$.

- A complex $\mathcal{F} \in D(X)$ is $(\dagger, ?)$**-even** (resp. $(\dagger, ?)$**-odd**) if, for all $\lambda \in \Lambda$ and $n \in \mathbb{Z}$, the cohomology sheaf $\mathcal{H}^n(i_\lambda^? \mathcal{F})$ belongs to $\mathrm{Loc_f}(X_\lambda)$ and vanishes for $n \notin \dagger(\lambda)$ (resp. $n \in \dagger(\lambda)$).
- A complex $\mathcal{F}$ is $(\dagger, ?)$**-parity** if it is either $(\dagger, ?)$-even or $(\dagger, ?)$-odd.
- A complex $\mathcal{F}$ is $\dagger$**-even** (resp. $\dagger$**-odd**) if it is both $(\dagger, *)$- and $(\dagger, !)$-even (resp. odd).
- A complex $\mathcal{F}$ is $\dagger$**-parity** if it splits as the direct sum of a $\dagger$-even complex and a $\dagger$-odd complex.

Remark 2.5.

(1) A complex is $(\dagger, *)$-even if and only if for every $\lambda \in \Lambda$ the stalks on X_λ are free and concentrated in degrees $\dagger(\lambda)$.

(2) By (2.1) and a standard dévissage argument, $\mathcal{F}$ is $(\dagger, ?)$-even (resp. odd) if and only if the $i_\lambda^? \mathcal{F}$ are isomorphic to *direct sums* of objects in $\mathrm{Loc_f}(X_\lambda)$ shifted by elements of $\dagger(\lambda)$ (resp. $\dagger(\lambda) + 1$).

(3) A complex $\mathcal{F}$ is $(\dagger, *)$-even (resp. odd) if and only if $\mathbb{D}\mathcal{F}$ is $(\dagger, !)$-even (resp. odd).

(4) An indecomposable $\dagger$-parity complex is either $\dagger$-even or $\dagger$-odd.

(5) A complex is $\dagger$-parity if and only if it is $\dagger'$-parity, where $\dagger'(\lambda) = \dagger(\lambda) + \overline{1}$ for all $\lambda \in \Lambda$.

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[1] We regard elements of $\mathbb{Z}/2$ as cosets and denote by $\bar{\cdot} : \mathbb{Z} \to \mathbb{Z}/2$ the non-trivial homomorphism; that is, $\overline{0} = \{n \in \mathbb{Z} \mid n \text{ is even}\}$ and $\overline{1} = \{n \in \mathbb{Z} \mid n \text{ is odd}\}$.

(6) If the pariversity function † is clear from the context, we may drop it from the notation.
(7) This definition is a geometric analogue of a notion introduced by Cline-Parshall-Scott in [CPS93]. For example, the notion of $*$-even corresponds to their $\mathcal{E}^L$.

For the rest of the section, we fix a pariversity function † and drop † from the notation.

Given a $*$-even $\mathcal{F} \in D(X)$ write $X' := \operatorname{supp} \mathcal{F}$ for the support[2] of $\mathcal{F}$ and choose an open stratum $X_\mu \subset X'$. We denote by i and j the inclusions:

$$X_\mu \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} X' \backslash X_\mu.$$

We have a distinguished triangle of $*$-even complexes

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \overset{[1]}{\to} \tag{2.3}$$

which is the extension by zero of the standard distinguished triangle on X'. (Note that $j^! \mathcal{F} = j^* \mathcal{F}$ because j factors as an open immersion into X' followed by the inclusion of X' into X.) Dually, if $\mathcal{G} \in D(X)$ is !-even and i, j are as above we have a distinguished triangle of !-even complexes

$$i_! i^! \mathcal{G} \to \mathcal{G} \to j_* j^* \mathcal{G} \overset{[1]}{\to} . \tag{2.4}$$

Proposition 2.6. *If $\mathcal{F}$ is $*$-parity and $\mathcal{G}$ is !-parity, then we have a (non-canonical) isomorphism of graded k-modules*

$$\operatorname{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \cong \bigoplus_{\lambda \in \Lambda} \operatorname{Hom}^\bullet(i_\lambda^* \mathcal{F}, i_\lambda^! \mathcal{G}).$$

Moreover, both sides are free k-modules.

Proof. We may assume that $\mathcal{F}$ and $\mathcal{G}$ are indecomposable and, by shifting if necessary, that $\mathcal{F}$ is $*$-even and that $\mathcal{G}$ is !-even. We proceed by induction on the number N of $\lambda \in \Lambda$ such that $i_\lambda^* \mathcal{F} \neq 0$. If $N = 1$, then $\mathcal{F} \cong i_{\mu !} i_\mu^* \mathcal{F}$ for some $\mu \in \Lambda$, and by adjunction

$$\operatorname{Hom}^\bullet(\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}^\bullet(i_{\mu !} i_\mu^* \mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}^\bullet(i_\mu^* \mathcal{F}, i_\mu^! \mathcal{G}).$$

As we assumed $\mathcal{F}$ to be $*$-even and $\mathcal{G}$ to be !-even, the complexes $i_\mu^* \mathcal{F}$ and $i_\mu^! \mathcal{G}$ are direct sums of shifts of elements of $\operatorname{Loc_f}(X_\mu)$ concentrated in degrees congruent to $\dagger(\lambda)$. By (2.1) and (2.2) we conclude that $\operatorname{Hom}^\bullet(i_\mu^* \mathcal{F}, i_\mu^! \mathcal{G})$ is free and concentrated in even degrees.

If $N > 1$, applying $\operatorname{Hom}(-, \mathcal{G})$ to (2.3) yields a long exact sequence

$$\cdots \leftarrow \operatorname{Hom}^n(j_! j^! \mathcal{F}, \mathcal{G}) \leftarrow \operatorname{Hom}^n(\mathcal{F}, \mathcal{G}) \leftarrow \operatorname{Hom}^n(i_* i^* \mathcal{F}, \mathcal{G}) \leftarrow \cdots$$

Now both $\operatorname{Hom}^n(i_* i^* \mathcal{F}, \mathcal{G})$ and $\operatorname{Hom}^n(j_! j^! \mathcal{F}, \mathcal{G})$ vanish for n odd and are free for n even, respectively by induction and by the base case; hence $\operatorname{Hom}^n(\mathcal{F}, \mathcal{G})$ also vanishes for n odd, and it is an extension of $\operatorname{Hom}^n(i_* i^* \mathcal{F}, \mathcal{G})$ by $\operatorname{Hom}^n(j_! j^! \mathcal{F}, \mathcal{G})$, and hence also free, for n even. ☐

[2]Contrary to the common usage, we call support of a sheaf (or of a complex) the *closure* of the set of points where its stalks are non-zero.

Remark 2.7. The proof above shows that $\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})$ is a free k-module. If, moreover, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable, then the stalks (resp. costalks) of $\mathcal{F}$ (resp. $\mathcal{G}$) are concentrated in degrees congruent to a fixed parity a (resp. b) in $\mathbb{Z}/2$, and it follows that $\mathrm{Hom}^\bullet(\mathcal{F},\mathcal{G})$ is concentrated in degrees congruent to $a+b$ modulo 2.

Corollary 2.8. *If $\mathcal{F}$ is $*$-even and $\mathcal{G}$ is $!$-odd then*

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) = 0.$$

Corollary 2.9. *If $\mathcal{F}$ and $\mathcal{G}$ are indecomposable parity complexes of the same parity and $j : X_\mu \to X$ denotes the inclusion of a stratum which is open in the support of both $\mathcal{F}$ and $\mathcal{G}$, then the functor j^* gives a surjection:*

$$\mathrm{Hom}(\mathcal{F},\mathcal{G}) \twoheadrightarrow \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}).$$

Proof. Apply $\mathrm{Hom}(\mathcal{F},-)$ to (2.4) and use Corollary 2.8. ☐

The last corollary says that we can extend morphisms $j^*\mathcal{F} \to j^*\mathcal{G}$ to morphisms $\mathcal{F} \to \mathcal{G}$. Now we want to investigate how parity complexes behave when restricted to an open union of strata. Before stating the result, let us recall the following simple result from ring theory (whose proof is left as an exercise):

Lemma 2.10. *A quotient of a local ring is local.*

Proposition 2.11. *Let $j : U \to X$ denote the inclusion of an open union of strata. Then given an indecomposable parity complex $\mathcal{P}$ on X, its restriction to U is either zero or indecomposable.*

Proof. Suppose that $\mathcal{P}$ has non-zero restriction to U. As in the proof of Corollary 2.9, the functor $\mathrm{Hom}(\mathcal{P},-)$ applied to the appropriate adjunction triangle together with Corollary 2.8 shows that restriction yields a surjection

$$\mathrm{End}(\mathcal{P}) \twoheadrightarrow \mathrm{End}(\mathcal{P}_{|U}).$$

It follows by Lemma 2.10 that $\mathrm{End}(\mathcal{P}_{|U})$ is a local ring, and hence $\mathcal{P}_{|U}$ is indecomposable. ☐

Theorem 2.12. *Let $\mathcal{F}$ be an indecomposable parity complex. Then*

1. *the support of $\mathcal{F}$ is irreducible, and hence of the form $\overline{X}_\lambda$, for some $\lambda \in \Lambda$;*
2. *the restriction $i_\lambda^*\mathcal{F}$ is isomorphic to $\mathcal{L}[m]$, for some indecomposable object $\mathcal{L}$ in $\mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ and some integer m;*
3. *any indecomposable parity complex supported on $\overline{X}_\lambda$ and extending $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$.*

Proof. Suppose for contradiction that X_λ and X_μ are open in the support of $\mathcal{F}$, where λ and μ are two distinct elements of Λ. Let $U = X_\lambda \cup X_\mu$. Then $\mathcal{F}_{|U} \simeq \mathcal{F}_{X_\lambda} \oplus \mathcal{F}_{X_\mu}$, contradicting Proposition 2.11. This proves (1). The assertion (2) also follows from Proposition 2.11.

Now let $\mathcal{G}$ be an indecomposable parity complex supported on $\overline{X}_\lambda$ and such that $i_\lambda^*\mathcal{G} \simeq \mathcal{L}[m]$. By composition, we have inverse isomorphisms $\alpha : i_\lambda^*\mathcal{F} \xrightarrow{\sim} i_\lambda^*\mathcal{G}$ and $\beta : i_\lambda^*\mathcal{G} \xrightarrow{\sim} i_\lambda^*\mathcal{F}$. By Corollary 2.9, the restriction $\mathrm{Hom}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}(i_\lambda^*\mathcal{F}, i_\lambda^*\mathcal{G})$ is surjective. So we can lift α and β to morphisms $\tilde{\alpha} : \mathcal{F} \to \mathcal{G}$ and $\tilde{\beta} : \mathcal{G} \to \mathcal{F}$. By Corollary 2.9 again, the restriction $\mathrm{End}(\mathcal{F}) \to \mathrm{End}(i_\lambda^*\mathcal{F})$ is surjective. Since $\tilde{\beta} \circ \tilde{\alpha}$ restricts to $\beta \circ \alpha = \mathrm{Id}$, the locality of $\mathrm{End}(\mathcal{F})$ implies that $\tilde{\beta} \circ \tilde{\alpha}$ is invertible itself, and similarly for $\tilde{\alpha} \circ \tilde{\beta}$. This proves (3). ☐

Remark 2.13. If k is a field, one can replace "indecomposable" by "simple" in (2), due to our assumptions on X.

We now introduce the main character of our paper.

Definition 2.14. Let † be a pariversity. A **†-parity sheaf** is an indecomposable †-parity complex with X_λ open in its support and extending $\mathcal{L}[d_\lambda]$ for some indecomposable $\mathcal{L} \in \mathrm{Loc_f}(X_\lambda)$. When such a complex exists, we will denote it by $\mathcal{E}^\dagger(\lambda, \mathcal{L})$. We call $\mathcal{E}^\dagger(\lambda, \mathcal{L})$ the †-parity sheaf associated to the pair $(\lambda, \mathcal{L})$.

Remark 2.15.

(1) More generally, for $\mathcal{L}$ not indecomposable, we will let $\mathcal{E}^\dagger(\lambda, \mathcal{L})$ denote the direct sum of the parity sheaves associated to the direct summands of $\mathcal{L}$. We may also use the notation $\mathcal{E}^\dagger(\overline{X}_\lambda, \mathcal{L})$.
(2) If $\mathcal{L} = \underline{k}_{X_\lambda}$ is the constant local system, we may write $\mathcal{E}^\dagger(\lambda, k)$ (or even $\mathcal{E}^\dagger(\lambda)$ if the coefficient ring k is clear from the context).
(3) If the pariversity is clear from the context, we may also drop it from our notation.

Thus, any indecomposable parity complex is isomorphic to some shift of a parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$. The reason for the normalisation chosen in the last definition is explained by the following proposition:

Proposition 2.16. *For any pariversity †, λ in Λ and $\mathcal{L}$ in $\mathrm{Loc_f}(X_\lambda)$, we have*

$$\mathbb{D}\mathcal{E}^\dagger(\lambda, \mathcal{L}) \simeq \mathcal{E}^\dagger(\lambda, \mathcal{L}^\vee).$$

Proof. The definition of †-parity sheaf is clearly self-dual, so $\mathbb{D}\mathcal{E}^\dagger(\lambda, \mathcal{L})$ is a †-parity sheaf. Moreover, it is supported on $\overline{X}_\lambda$ and extends $\mathcal{L}^\vee[d_\lambda]$. By the uniqueness theorem, it is isomorphic to $\mathcal{E}^\dagger(\lambda, \mathcal{L}^\vee)$. □

Remark 2.17. We give many examples of parity sheaves below. In the next section we also give a few examples of situations in which a full set of parity sheaves does not exist, and, at the end of the paper, examples of parity sheaves that are not perverse (see Proposition 4.22).

Despite such examples, in many cases of interest, parity sheaves do exist and are perverse. For example, if X is a flag variety stratified by Schubert cells and k a field of characteristic zero, the ♮-parity sheaves are the intersection cohomology complexes, while the ♢-parity sheaves are the indecomposable tilting sheaves.

2.3. **Parity extensions and an existence criterion.** We have now explained that associated to each indecomposable local system on a stratum, there exists (up to isomorphism) at most one parity extension for a fixed pariversity. In this section, we begin to explore when such an extension does in fact exist.

Following the construction of the intersection cohomology sheaves in [BBD82], we present a criterion for the existence of parity sheaves and, when they exist, an explicit construction. This section concludes with some examples of spaces and pariversities for which parity extensions do not exist.

We begin with a lemma that characterises extensions of a complex from an open set. We will use it below to develop our existence criterion.

2.3.1. *Extensions from an open set.* Let $j : U \to X$ be an open embedding and $i : Z \to X$ be the closed embedding of the complement. Recall that an extension of a complex $\mathcal{F}_U \in D(U)$ is a pair $(\mathcal{F}, \alpha)$ where $\mathcal{F} \in D(X)$ is a complex and $\alpha : j^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}_U$ is an isomorphism. Extensions of $\mathcal{F}_U$ form a category in a natural way.

Lemma 2.18. *Fix $\mathcal{F}_U \in D(U)$. There is a natural bijection between isomorphism classes of extensions $(\mathcal{F}, \alpha)$ of $\mathcal{F}_U$ and isomorphism classes of distinguished triangles in $D(Z)$ of the form:*

$$A \to i^* j_* \mathcal{F}_U \to B \xrightarrow{[1]}$$

Under this bijection we have $i^*\mathcal{F} \cong A$ and $i^!\mathcal{F} \cong B[-1]$.

Proof. We describe the maps in both directions. We leave it to the reader to check that these maps do indeed provide a bijection on isomorphism classes.

Suppose first that we are given an extension $(\mathcal{F}, \alpha)$. Then we associate to $(\mathcal{F}, \alpha)$ the distinguished triangle

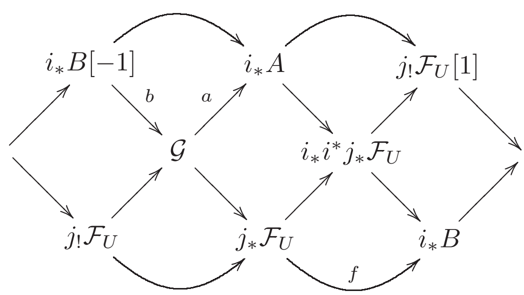
$$i^*\mathcal{F} \to i^* j_* \mathcal{F}_U \to i^!\mathcal{F}[1] \xrightarrow{[1]} \tag{2.5}$$

obtained by rotating the standard distinguished triangle $i^!\mathcal{F} \to i^*\mathcal{F} \to i^* j_* j^* \mathcal{F} \xrightarrow{[1]}$ [BBD82, 1.4.7.2] and using our isomorphism α. It is clear that the isomorphism class of the resulting triangle depends only on the isomorphism class of the extension $(\mathcal{F}, \alpha)$.

In the other direction, given a distinguished triangle:

$$A \to i^* j_* \mathcal{F}_U \to B \xrightarrow{[1]} \tag{2.6}$$

we can certainly build an octahedron:



To such a distinguished triangle we associate the extension $(\mathcal{G}, \beta)$, where β is obtained by adjunction from the map $\mathcal{G} \to j_*\mathcal{F}_U$.

The final statement of the lemma is clear by construction. □

Remark 2.19. Note that for $\mathcal{F}_U$ perverse, the perverse extensions ${}^p j_! \mathcal{F}_U$, ${}^p j_{!*} \mathcal{F}_U$ and ${}^p j_* \mathcal{F}_U$ correspond to the perverse truncation triangles

$${}^p\tau^{<n} i^* j_* \mathcal{F}_U \to i^* j_* \mathcal{F}_U \to {}^p\tau^{\geq n} i^* j_* \mathcal{F}_U \xrightarrow{[1]},$$

where $n = -1$, 0, 1 [BBD82, Proposition 1.4.23].

2.3.2. *Parity extensions.* Having considered the case of general extensions, we now turn our attention to the question of when parity extensions exist and how to construct them. For the rest of this section we use the notation and assumptions of Section 2.1.

We proceed by imitating Deligne's construction of the intersection cohomology sheaves. The method is a descending induction on the poset of strata. The following corollary describes the inductive step.

Corollary 2.20. *Let Z be a closed stratum. Fix a pariversity $\dagger$ and let $\mathcal{F}_U \in D(U)$ be a $\dagger$-even complex. There exists a $\dagger$-even extension of $\mathcal{F}_U$ if and only if there exists a distinguished triangle in $D(Z)$ of the form*

$$A \to i^* j_* \mathcal{F}_U \to B \xrightarrow{[1]}$$

where A is $\dagger$-even and B is $\dagger$-odd.

Proof. Immediate from the previous lemma. □

To identify when such extensions are indecomposable, we need the following lemma.

Lemma 2.21. *A complex $\mathcal{F} \in D(X)$ has no summands supported within a closed subset $i : Z \to X$ if and only the map $i^! \mathcal{F} \to i^* \mathcal{F}$ cannot be expressed as a direct sum $\mathrm{Id}_Q \oplus h$, where $i^! \mathcal{F} \cong Q \oplus \mathcal{G}$, $i^* \mathcal{F} \cong Q \oplus \mathcal{G}'$ and $h : \mathcal{G} \to \mathcal{G}'$, with $Q \neq 0$.*

Proof. Assume that $\mathcal{F} \cong i_* Q \oplus \mathcal{F}'$. Then $i^? \mathcal{F} \cong Q \oplus i^? \mathcal{F}'$ and the corestriction-to-restriction map decomposes as a direct sum of the corestriction-to-restriction map for $i_* Q$ (which is simply the identity map Id_Q) and that for $\mathcal{F}'$.

Conversely, suppose that the corestriction-to-restriction map for $\mathcal{F}$ can be expressed as $\mathrm{Id}_Q \oplus h$ as above. As the corestriction-to-restriction map factors through $\mathcal{F}$, we find that there is a commutative diagram:

$$\begin{array}{ccccccc}
 & & i_* i^! \mathcal{F} & \xrightarrow{\sim} & i_*(Q \oplus \mathcal{G}) & \longleftarrow & i_* Q \\
 & \swarrow & \downarrow & & \downarrow {\scriptstyle \mathrm{Id}_Q \oplus h} & & \downarrow {\scriptstyle =} \\
\mathcal{F} & \longrightarrow & i_* i^* \mathcal{F} & \xrightarrow{\sim} & i_*(Q \oplus \mathcal{G}') & \longrightarrow & i_* Q
\end{array}$$

It follows that $i_* Q$ is a direct summand of $\mathcal{F}$. □

We now return to the question of parity extensions and the notation of Corollary 2.20. As A and B are parity, (2.1) and (2.2) imply that $A \cong \oplus_n \mathcal{H}^n(A)[-n]$ and $B \cong \oplus_n \mathcal{H}^n(B)[-n]$. Let ϕ_n be the composition of the inclusion $\mathcal{H}^n(B)[-n] \hookrightarrow B$, the connecting map $B \to A[1]$, and the projection $A[1] \to \mathcal{H}^{n+1}(A)[-n]$.

Corollary 2.22. *The parity extension $\mathcal{F}$ is indecomposable if and only the image of each morphism ϕ_n defined above does not contain any non-zero direct-summand of $H^{n+1}(A)$. If k is a field, this condition is equivalent to $\phi_n = 0$ for all n.*

Proof. This statement is an application of the previous lemma together with the fact that $H^n(A)$ is in $\mathrm{Loc}_{\mathrm{f}}(Z)$ and therefore projective in the category of all local systems.

In this case, as mentioned in the remark above, the corestriction and restriction are both isomorphic to the direct sum of (the shifts of) their cohomology sheaves. Thus the condition of the lemma can be checked in each degree separately. Now for a single degree, the map ϕ_n maps to a projective object, so any direct summand contained in the image of the map is also a direct summand of the source. The result follows. $\square$

Proposition 2.23. *Assume again that Z consists of a single stratum and that $\mathcal{F}_U$ is a $\dagger$-even complex on U, where $\dagger$ is some pariversity on $X \backslash Z$. Then there exists a $\tilde{\dagger}$-even complex $\mathcal{F}$ on X extending $\mathcal{F}_U$ for both pariversities $\tilde{\dagger}$ extending $\dagger$ if and only if the complex $i^*j_*\mathcal{F}_U \in D(Z)$ is parity (i.e. isomorphic to a direct sum of shifts of elements of $\mathrm{Loc}_{\mathrm{f}}(Z)$).*

Proof. If $i^*j_*\mathcal{F}_U$ is isomorphic to a direct sum of shifts of objects in $\mathrm{Loc}_{\mathrm{f}}(Z)$, then it is clear that the required triangles exist and so both parity extensions exist.

Suppose that there exist parity extensions of each parity. Then there are triangles

$$A \to i^*j_*\mathcal{F}_U \to B \xrightarrow{[1]} \qquad \text{and} \qquad A' \to i^*j_*\mathcal{F}_U \to B' \xrightarrow{[1]}$$

with A, B' even and A', B odd. Without loss of generality, we may assume that the parity extensions have no non-zero summands with support contained in Z. By Corollary 2.22, this means that the images of the morphisms $\phi_{2n-1} : \mathcal{H}^{2n-1}(B) \to \mathcal{H}^{2n}(A)$ (resp. $\phi'_{2n} : \mathcal{H}^{2n}(B') \to \mathcal{H}^{2n+1}(A')$) do not contain a non-zero direct summand of $\mathcal{H}^{2n}(A)$ (resp. $\mathcal{H}^{2n+1}(A')$).

Consider the associated long exact sequences of cohomology sheaves for the above triangles. We find exact sequences

$$0 \to \mathrm{Im}(\phi_{2n-1}) \to \mathcal{H}^{2n}(A) \to \mathcal{H}^{2n}(i^*j_*\mathcal{F}_U) \to 0,$$
$$0 \to \mathrm{Im}(\phi'_{2n}) \to \mathcal{H}^{2n+1}(A') \to \mathcal{H}^{2n+1}(i^*j_*\mathcal{F}_U) \to 0,$$

as well as inclusions

$$0 \to \mathcal{H}^{2n}(i^*j_*\mathcal{F}_U) \to \mathcal{H}^{2n}(B'), \quad 0 \to \mathcal{H}^{2n+1}(i^*j_*\mathcal{F}_U) \to \mathcal{H}^{2n+1}(B).$$

As $\mathcal{H}^{2n}(i^*j_*\mathcal{F}_U)$ (resp. $\mathcal{H}^{2n+1}(i^*j_*\mathcal{F}_U)$) is a subobject of $\mathcal{H}^{2n}(B') \in \mathrm{Loc}_{\mathrm{f}}(Z)$ (resp. $\mathcal{H}^{2n+1}(B)$), it is an element of $\mathrm{Loc}_{\mathrm{f}}(Z)$ (remember that our ring of coefficients is a PID by assumption). In particular, it is a projective object in the category of all local systems on Z, by Remark 2.2(3). It follows that the short exact sequences above split and thus $\mathrm{Im}(\phi_{2n-1})$ and $\mathrm{Im}(\phi_{2n})$ are direct summands. By our assumption above, they must be trivial.

We conclude that the induced maps $\mathcal{H}^{2n}(A) \to \mathcal{H}^{2n}(i^*j_*\mathcal{F}_U)$ (resp. $\mathcal{H}^{2n+1}(A') \to \mathcal{H}^{2n+1}(i^*j_*\mathcal{F}_U)$) are isomorphisms. Thus the map $A \oplus A' \to X$ induces isomorphisms on cohomology sheaves and therefore is an isomorphism in $D(Z)$. The complex A is even and A' is odd, so $i^*j_*\mathcal{F}_U$ is a parity complex. $\square$

Remark 2.24. If the equivalent conditions of the above claim hold, and $\mathcal{F}$ is a parity extension with no summands supported on Z, then $i^*j_*\mathcal{F}_U$ is isomorphic to the direct sum $i^*\mathcal{F} \oplus i^!\mathcal{F}[1]$.

Remark 2.25. It is natural to ask whether the parity extension can be made into a functor. Then the assignment of A and B to $\mathcal{F}_U$ in the triangle

$$A \to i^*j_*\mathcal{F}_U \to B \xrightarrow{[1]} \tag{2.7}$$

should be functorial as well. Though there are some situations where this is the case, as we will see below (for example, when all strata are contractible), in general there is no reason why it should be true. By [BBD82, Proposition 1.1.9] a sufficient condition for the assignment of A and B to be functorial would be that

$\operatorname{Hom}(A[1], B) = 0$ for any two objects A (resp. B) appearing on the left (resp. right) of (2.7) for any $\mathcal{F}_U$. In the case of a parity extension, A is assumed to the †-even, and B †-odd; hence $\operatorname{Hom}(A[1], B) \neq 0$ in general, and this argument does not apply.

This is in contrast to the case of the ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ extensions which are obtained via a truncation triangle, which is functorial and satisfies $\operatorname{Hom}(A[1], B) = 0$. The three perverse extensions can then be constructed by either $j_!$ or j_* followed by a functor of truncation on the closed part, which is obtained by gluing a degenerate t-structure on the open part and a given t-structure on the closed part [BBD82, Proposition 1.4.23].

Lemma 2.21 gave a criterion for determining when a complex has summands supported on a fixed closed subset. The following proposition refines the lemma in a particular situation and will be useful when we come to discuss the Decomposition Theorem in Section 3.

Proposition 2.26. *Assume the ring of coefficients k is a field. Let $i : X_\lambda \hookrightarrow X$ be a closed stratum. Let $\mathcal{F} \in D(X)$ be a complex such that $i^!\mathcal{F}$ and $i^*\mathcal{F}$ are semi-simple.*[3] *Let D_λ denote the canonical morphism*

$$D_\lambda : i^!\mathcal{F} \to i^*\mathcal{F}$$

and set

$$D_\lambda^n := \mathcal{H}^n(D_\lambda) : \mathcal{H}^n(i^!\mathcal{F}) \to \mathcal{H}^n(i^*\mathcal{F}).$$

By assumption, both $\mathcal{H}^n(i^!\mathcal{F})$ and $\mathcal{H}^n(i^*\mathcal{F})$ are semi-simple and so we can (and do) choose splittings

$$\mathcal{H}^n(i^!\mathcal{F}) \cong \ker D_\lambda^n \oplus (\mathcal{H}^n(i^!\mathcal{F})/\ker D_\lambda^n), \tag{2.8}$$

$$\mathcal{H}^n(i^*\mathcal{F}) \cong \operatorname{Im} D_\lambda^n \oplus (\mathcal{H}^n(i^*\mathcal{F})/\operatorname{Im} D_\lambda^n). \tag{2.9}$$

Then there is an isomorphism

$$\mathcal{F} \cong \mathcal{G} \oplus \bigoplus_{n\in\mathbb{Z}} (i_*(\mathcal{H}^n(i^!\mathcal{F})/\ker D_\lambda^n)[-n])$$

where $\mathcal{G}$ is a complex having no direct summand supported on X_λ.

Proof. Fix isomorphisms $i^?\mathcal{F} \cong \bigoplus \mathcal{H}^n(i^?\mathcal{F})[-n]$. Composing these with D_λ we obtain a map

$$\bigoplus \mathcal{H}^n(i^!\mathcal{F})[-n] \xrightarrow{\sim} i^!\mathcal{F} \xrightarrow{D_\lambda} i^*\mathcal{F} \xrightarrow{\sim} \bigoplus \mathcal{H}^n(i^*\mathcal{F})[-n].$$

The splittings (2.8) and (2.9) then yield a map

$$c : \bigoplus (\mathcal{H}^n(i^!\mathcal{F})/\ker D_\lambda^n)[-n] \to i^!\mathcal{F} \xrightarrow{D_\lambda} i^*\mathcal{F} \to \bigoplus (\operatorname{Im} D_\lambda^n)[-n]. \tag{2.10}$$

Note that $\mathcal{H}^n(c)$ is an isomorphism for all n, and hence c is an isomorphism. Applying $i_! = i_*$ to (2.10) and using that the natural map $i_!i^!\mathcal{F} \to i_*i^*\mathcal{F}$ factors through $\mathcal{F}$ (as in the proof of Lemma 2.21) we get that $\bigoplus((\mathcal{H}^n(i^!\mathcal{F})/\ker D_\lambda^n)[-n])$ is a direct summand of $\mathcal{F}$. Hence we have an isomorphism

$$\mathcal{F} \cong \mathcal{G} \oplus \bigoplus i_*((\mathcal{H}^n(i^!\mathcal{F})/\ker D_\lambda^n)[-n]) \tag{2.11}$$

for some $\mathcal{G} \in D(X)$.

[3] We say a complex $Q \in D(F)$ is semi-simple if $Q \cong \bigoplus \mathcal{H}^n(Q)[-n]$ and $\mathcal{H}^n(Q)$ is semi-simple as an object of $\operatorname{Loc}_f(X_\lambda)$ for all n. (The second condition is automatic, by Remark 2.2(3).)

It remains to see that $\mathcal{G}$ does not have any direct summands supported on X_λ. To this end, fix $n \in \mathbb{Z}$ and let $\iota : \mathcal{G} \to \mathcal{F}$ and $\pi : \mathcal{F} \to i_*((\mathcal{H}^n(i^!\mathcal{F})/\ker D^n_\lambda)[-n])$ denote the inclusion and projections obtained from the isomorphism (2.11). If we apply $\mathcal{H}^n(i^!-)$ we see that $0 = \mathcal{H}^n(i^!(\pi \circ \iota)) : \mathcal{H}^n(i^!\mathcal{G}) \to \mathcal{H}^n(i^!\mathcal{F})/\ker D^n_\lambda$. Hence the image of $\mathcal{H}^n(i^!(\iota)) : \mathcal{H}^n(i^!\mathcal{G}) \to \mathcal{H}^n(i^*\mathcal{G})$ is contained in $\ker D^n_\lambda$. It follows that the map $\mathcal{H}^n(i^!\mathcal{G}) \to \mathcal{H}^n(i^*\mathcal{G})$ obtained by applying $\mathcal{H}^n(-)$ to the natural map $i^!\mathcal{G} \to i^*\mathcal{G}$ is zero.

We conclude that $\mathcal{H}^n(i^!\mathcal{G}) \to \mathcal{H}^n(i^*\mathcal{G})$ is zero for all n and hence that there is no direct summand of $i^!\mathcal{G}$ mapped isomorphically onto a direct summand of $i^*\mathcal{G}$. Hence $\mathcal{G}$ does not have any direct summands supported on X_λ by Lemma 2.21. This completes the proof of the proposition. □

2.3.3. *Constructible complexes with coefficients in a field for a stratification with contractible strata.* In this section we assume that we are in the non-equivariant setting and that k, our ring of coefficients, is a field. We use the construction from the previous paragraph to prove that if all strata are contractible, then parity sheaves exist on all strata and for all pariversities.

Lemma 2.27. *Suppose that Z is a contractible closed stratum of X and let $U := X \backslash Z$ denote the complement. Fix a pariversity † on U and let $\mathcal{F}_U \in D(U)$ denote a †-parity complex. Then there exists an extension $\mathcal{F}$ of $\mathcal{F}_U$ to X whose restriction and corestriction to Z is even. Similarly there exists an extension whose restriction and corestriction is odd.*

Proof. By Corollary 2.20 it suffices to show that there exists a triangle

$$A \to i^* j_* \mathcal{F}_U \to B \xrightarrow{[1]}$$

with $A, B \in D(Z)$ respectively even and odd. As Z is contractible and k a field, any object in $D(Z)$ is isomorphic to a direct sum of shifts of constant sheaves. Thus $i^* j_* \mathcal{F}_U \cong \mathcal{H}^{\mathrm{even}}(i^* j_* \mathcal{F}_U) \oplus \mathcal{H}^{\mathrm{odd}}(i^* j_* \mathcal{F}_U)$ and we have the split triangle:

$$\mathcal{H}^{\mathrm{even}}(i^* j_* \mathcal{F}_U) \to i^* j_* \mathcal{F}_U \to \mathcal{H}^{\mathrm{odd}}(i^* j_* \mathcal{F}_U) \xrightarrow{[1]} .$$

Under the bijection described in Corollary 2.20 this triangle corresponds to the desired extension. Entirely analogously, as the sequence splits, the maps can be directed in the other direction, producing an 'odd' extension. □

Corollary 2.28. *Let † be a pariversity. Let X be stratified by contractible strata and k be a field. Then for every stratum, there exists a parity sheaf $\mathcal{E}^\dagger(\lambda, k) \in D(X)$.*

2.3.4. *Non-examples.* By modifying the conditions required in the previous paragraph, one can find examples of spaces satisfying the parity conditions, for which some parity extensions do not exist. We conclude this section by mentioning three such examples.

Example 2.29. Let $k = \mathbb{Z}_2$, the ring of 2-adic integers. Consider the affine Grassmannian for GL_2, whose $GL_2(\mathbb{C}[[t]])$-orbits $\mathcal{G}r^\lambda$ are labelled by highest weights $(l_1, l_2) \in \mathbb{Z}^2$ for GL_2. Let X be the orbit closure of $\mathcal{G}r^\lambda$ for $\lambda = (1, -1)$. The strata will be $U = \mathcal{G}r^\lambda$ and $Z = \mathcal{G}r^0 \cong \mathrm{pt}$. The lower stratum Z is contractible and the larger stratum U has the cohomology of $\mathbb{P}^1$, and thus satisfies the parity conditions (note that in Proposition 2.27, only Z is assumed to be contractible, not U). Consider $i^* j_* \underline{k}_U$. It is quasi-isomorphic to the graded cohomology of the link, which is in turn homotopy equivalent to $\mathbb{RP}^3$. Now $H^*(\mathbb{RP}^3; \mathbb{Z}_2) = \mathbb{Z}_2[-3] \oplus \mathbb{Z}/2[-2] \oplus \mathbb{Z}_2$.

While there is a distinguished triangle

$$\mathbb{Z}_2[-2] \oplus \mathbb{Z}_2 \to \mathbb{Z}_2[-3] \oplus \mathbb{Z}/2[-2] \oplus \mathbb{Z}_2 \to \mathbb{Z}_2[-3] \oplus \mathbb{Z}_2[-1] \xrightarrow{[1]},$$

where the first term is even and the last term is odd (in particular, both are free over $\mathbb{Z}_2$), by Proposition 2.23 there is no such triangle in the other direction (as $H^*(\mathbb{RP}^3;\mathbb{Z}_2)$ is not a direct sum of free modules). We conclude that there exists a parity sheaf for the $\natural = \diamondsuit$ pariversity, but for not the pariversity $\dagger(\lambda) = \overline{0}$, $\dagger(0) = \overline{1}$.

Example 2.30. Let k be a field and X the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$ with two strata: the zero section E and its complement U. Let $\dagger$ be the pariversity that is even on U and odd on E. Similarly to the previous example, one finds that $\underline{k}_U$ has $\dagger$-parity extension if and only if the characteristic of k is equal to 2.

Example 2.31. Lastly, let k be a field and $X = \mathbb{P}^1$ viewed as a $\mathbb{C}^\times$-variety with the standard action ($z \cdot [x : y] = [x : zy]$ for $z \in \mathbb{C}^\times$). We fix the stratification indexed by $\Lambda = \{0, \infty\}$, where $X_0 = \{[1 : 0]\}$ is the north pole and X_∞ its complement.

Let $D(X) = D^b_{\mathbb{C}^\times,\Lambda}(\mathbb{P}^1, k)$ denote the bounded Λ-constructible equivariant derived category. The equivariant constant sheaf on X_∞ does not have a $\diamondsuit$-parity extension in the equivariant derived category $D(X)$. Note that in the non-equivariant derived category, such a $\diamondsuit$-parity extension does exist.

2.4. **Even resolutions and existence.** In this section we keep the notation and assumptions of Section 2.1.

In Theorem 2.12, we saw that, if we fix a stratum X_λ and a local system $\mathcal{L}$ on it, then there is at most one parity sheaf $\mathcal{F}$ such that $\operatorname{supp} \mathcal{F} = \overline{X}_\lambda$ and $i_\lambda^* \mathcal{F} \simeq \mathcal{L}[d_\lambda]$, up to isomorphism. Furthermore, in Corollary 2.28 we showed that, for an arbitrary pariversity $\dagger$ (in the non-equivariant set-up) if the strata are contractible and k is a field, then $\dagger$-parity sheaves exist, extending any local system. Unfortunately, the conditions of Corollary 2.28 are quite restrictive.

In this section, we will describe a very useful tool for constructing $\natural$-parity sheaves that are extensions of the trivial local system. Later, in Section 4, we will use this tool to provide large classes of examples.

Let us recall the following definition from [GM88, 1.6].

Definition 2.32. Let $X = \sqcup_{\lambda \in \Lambda_X} X_\lambda$ and $Y = \sqcup_{\mu \in \Lambda_Y} Y_\mu$ be stratified varieties. A morphism $\pi : X \to Y$ is **stratified** if

(1) for all $\mu \in \Lambda_Y$, the inverse image $\pi^{-1}(Y_\mu)$ is a union of strata;
(2) for each X_λ above Y_μ, the induced morphism $\pi_{\lambda,\mu} : X_\lambda \to Y_\mu$ is a submersion with smooth fibre $F_{\lambda,\mu} = \pi^{-1}_{\lambda,\mu}(y_\mu)$, where y_μ is some chosen base point in Y_μ.

Definition 2.33. A stratified morphism π is said to be **even** if for all λ, μ as above, and for any local system $\mathcal{L}$ in $\mathrm{Loc}_{\mathrm{f}}(X_\lambda)$, the cohomology of the fibre $F_{\lambda,\mu}$ with coefficients in $\mathcal{L}_{|F_{\lambda,\mu}}$ is torsion free and concentrated in even degrees.

A class of even morphisms, which are common in geometric representation theory and a motivation for the definition, are those whose stratifications induce "affine pavings" on the fibres—meaning that all of the $F_{\lambda,\mu}$ are affine spaces. Examples of such maps arise in the study of flag manifolds and will be discussed in Section 4.

Proposition 2.34. *The direct image of a $(\natural, ?)$-even (resp. odd) complex under a proper, even morphism is again $(\natural, ?)$-even (resp. odd). The direct image of a $\natural$-parity complex under such a map is $\natural$-parity.*

Proof. In this proof, all parity vanishing is with respect to the $\natural$-pariversity and so we will drop it from the notation. We use the notation of Definitions 2.32 and 2.33, and assume that our stratified even morphism π is proper.

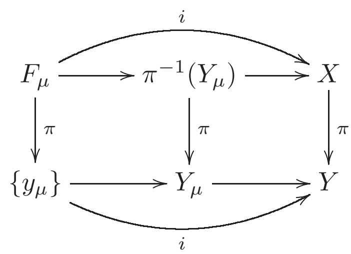
Note that if the statement is true for all !-even complexes, then it is true for all !-odd complexes (by shifting). It would then also be true for any $*$-parity complex because $\mathcal{F}$ is $*$-even (resp. odd) if and only if $\mathbb{D}\mathcal{F}$ is !-even (resp. odd) by Remark 2.5 and

$$\pi_* \mathbb{D}\mathcal{F} \cong \mathbb{D}\pi_! \mathcal{F} = \mathbb{D}\pi_* \mathcal{F}$$

as π is proper.

The second sentence of the theorem is an immediate corollary of the first. Thus it suffices to show the first statement for !-even sheaves.

Fix $\mu \in \Lambda_Y$ and let $\Lambda_X(\mu)$ denote the indices of strata in $\pi^{-1}(Y_\mu)$. Thus we have $\pi^{-1}(Y_\mu) = \bigsqcup_{\lambda \in \Lambda_X(\mu)} X_\lambda$. Moreover, let $F_\mu = \pi^{-1}(y_\mu) = \bigsqcup_{\lambda \in \Lambda_X(\mu)} F_{\lambda,\mu}$ where, as above, $F_{\lambda,\mu} = F_\mu \cap X_\lambda$. We have the following diagram with Cartesian squares:



We abuse notation and denote by i both inclusions $F_\mu \hookrightarrow X$ and $\{y_\mu\} \hookrightarrow Y$. Similarly, π denotes any vertical arrow in the above diagram.

Let $\mathcal{P} \in D(X)$ be a !-even complex. We wish to show that $\pi_*\mathcal{P}$ is !-parity. This is equivalent to $i^! \pi_* \mathcal{P}$ being !-parity for all μ. By the proper base change theorem,

$$i^! \pi_* \mathcal{P} \cong \pi_* i^! \mathcal{P} \cong \mathbb{H}^\bullet(F_\mu, i^! \mathcal{P}).$$

We will use the local-global spectral sequence to show that this latter cohomology group is parity.

Choose a filtration $F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_r = \emptyset$ of the fibre $\pi^{-1}(y_\mu)$ by closed subsets such that, for all p,

$$F_p \backslash F_{p+1} = F_{\lambda_p,\mu} \qquad \text{for some } \lambda_p \in \Lambda_X(\mu).$$

For all p, let $i_p : F_p \backslash F_{p+1} = F_{\lambda_p,\mu} \hookrightarrow X$ denote the inclusion. The local-global spectral sequence (see for example the proof of Proposition 3.4.4 in [Soe01]) has the form

$$E_1^{p,q} = \mathbb{H}^{p+q}(F_{\lambda_p,\mu}, i_p^! \mathcal{P}) \Rightarrow \mathbb{H}^{p+q}(F_\mu, i^! \mathcal{P}).$$

We may express i_p as the composition

$$F_{\lambda_p,\mu} \hookrightarrow X_{\lambda_p} \overset{i_{\lambda_p}}{\hookrightarrow} X$$

where $F_{\lambda_p,\mu}$ is a smooth subvariety of X_{λ_p}. It follows that, if d is the (complex) codimension of $F_{\lambda_p,\mu}$ in X_{λ_p}, we have

$$i_p^! \mathcal{P}[2d] \cong (i_{\lambda_p}^! \mathcal{P})_{|F_{\lambda_p,\mu}}.$$

(The isomorphism $i_p^![2d]\mathcal{F} \cong i_p^* \mathcal{F}$ is valid for any complex of sheaves on X_{λ_p} whose cohomology sheaves are local systems, as follows easily from the smoothness of $F_{\lambda_p,\mu}$ and X_{λ_p}.) As $\mathcal{P}$ is !-even by assumption, $i_p^! \mathcal{P}$ is isomorphic to a direct sum of local systems in even degrees, all obtained by restriction from torsion free local systems on X_{λ_p}.

By assumption, the cohomology of $F_{\lambda_p,\mu}$ with values in such local systems is free and concentrated in even degree and so the above spectral sequence degenerates for parity reasons, whence the claim. □

One practical application of the previous result is that the existence of parity sheaves follows from the existence of even resolutions:

Corollary 2.35. *Suppose that there exists an even, proper morphism*

$$\pi : \widetilde{X_\lambda} \to \overline{X}_\lambda \subset X$$

which is an isomorphism over X_λ.

Assume there exists a $\natural$*-parity complex* $\widetilde{\mathcal{P}}$ *on* $\widetilde{X_\lambda}$ *whose restriction to* $\pi^{-1}(X_\lambda) \cong X_\lambda$ *is isomorphic to a shifted indecomposable local system* $\mathcal{L}[d_\lambda]$ *on* X_λ. *Then there exists a* $\natural$*-parity sheaf* $\mathcal{P}$ *on* X *satisfying*

(1) $\operatorname{supp} \mathcal{P} = \overline{X}_\lambda$;
(2) $\mathcal{P}_{|X_\lambda} = \mathcal{L}[d_\lambda]$.

In particular, if π *is a resolution of singularities, then the above holds for* $\mathcal{L} = \underline{k}_\lambda$, *since in this case* $\underline{k}_{\widetilde{X_\lambda}}[d_\lambda]$ *is parity.*

Proof. By the previous proposition the direct image $\pi_*\widetilde{\mathcal{P}}$ is a parity complex. By proper base change we have that $(\pi_*\widetilde{\mathcal{P}})_{|X_\lambda} = \mathcal{L}[d_\lambda]$ and $\operatorname{supp} \pi_*\widetilde{\mathcal{P}} = \overline{X_\lambda}$. Now decompose $\pi_*\widetilde{\mathcal{P}}$ into indecomposable objects, and let $\mathcal{P}$ denote the (necessarily unique) direct summand whose restriction to X_λ is non-zero. Then $\mathcal{P}$ is a parity sheaf with $\mathcal{P}_{|X_\lambda} \cong \mathcal{L}[d_\lambda]$ and $\operatorname{supp} \mathcal{P} = \overline{X_\lambda}$ as claimed. □

2.5. **Modular reduction of parity sheaves.** Let $k \to k'$ be a ring homomorphism. In this section, we will consider the behaviour of parity sheaves under the extension of scalars functor, which we denote by

$$k'(-) := k' \overset{L}{\otimes}_k - : D(X;k) \to D(X;k').$$

Lemma 2.36. *Suppose that* $\mathcal{F} \in D(X,k)$ *is ?-even (resp. odd), then* $k'(\mathcal{F})$ *is ?-even (resp. odd). In particular, if* $\mathcal{F}$ *is a parity complex, then so is* $k'(\mathcal{F})$.

Proof. It suffices to prove the ?-even case. It is equivalent to show that the ?-restriction of $k'(\mathcal{F})$ to each point is even. For any complex $\mathcal{F} \in D(X;k)$ we have isomorphisms $i^?(k'(\mathcal{F})) \cong k'(i^?\mathcal{F})$ for i the inclusion of a point (this follows, for example, from Propositions 2.3.5 and 2.5.13 of [KS94]). By definition, if $\mathcal{F}$ is ?-even then the cohomology of $i^?\mathcal{F}$ vanishes in odd degrees and is free and therefore flat. Thus $k'(i^?_\lambda\mathcal{F}) = k' \otimes i^?_\lambda\mathcal{F}$ and $k'(\mathcal{F})$ is ?-even. □

We will now restrict our attention to the case when $k = \mathbb{O}$ and $k' = \mathbb{F}$ and $k \to k'$ is the residue map. Recall that $\mathbb{O}$ denotes a complete discrete valuation ring and $\mathbb{F}$ its residue field. We assume that (2.1) and (2.2) hold for $D(X,\mathbb{O})$.

In this case, $k'(-)$ is the modular reduction functor:

$$\mathbb{F}(-) := \mathbb{F} \overset{L}{\otimes}_{\mathbb{O}} - : D(X,\mathbb{O}) \to D(X,\mathbb{F}).$$

First we claim that in this situation, the implication of the previous theorem is in fact an equivalence.

Proposition 2.37. *A complex $\mathcal{F} \in D(X;\mathbb{O})$ is ?-even (resp, ?-odd or parity) if and only if $\mathbb{F}\mathcal{F}$ is.*

Remark 2.38. This proposition is analogous to [Ser67, Prop. 42(a)], which states, for a finite group G, an $\mathbb{O}[G]$-module which is free as an $\mathbb{O}$-module is projective if and only if its reduction is a projective $\mathbb{F}[G]$-module.

Proof. Having proved “only if” it remains to prove “if.”

Again, it suffices to check the ?-restrictions to points. As before, we have $i^?(\mathbb{F}\mathcal{F}) \cong \mathbb{F}(i^?\mathcal{F})$. This time, we wish to show that if $i^?(\mathbb{F}\mathcal{F})$ vanishes in odd degrees, then $i^?\mathcal{F}$ does too and is a free $\mathbb{O}$-module.

The derived category over a point $D(\mathrm{pt};\mathbb{O})$ (resp. $D(\mathrm{pt};\mathbb{F})$) is equivalent to the bounded derived category of finitely generated $\mathbb{O}$-modules, $D(\mathrm{Mod}_{\mathbb{O}})$ (resp. finite dimensional $\mathbb{F}$-vector spaces, $D(\mathrm{Vect}_{\mathbb{F}})$). The ring $\mathbb{O}$ is hereditary, which implies that any object in $D(\mathrm{Mod}_{\mathbb{O}})$ is isomorphic to its cohomology. Using this it is easy to see that if the cohomology of $i^?\mathcal{F}$ had torsion, then $\mathbb{F}i^?\mathcal{F}$ would have non-trivial cohomology concentrated in two consecutive degrees. Hence $i^?\mathcal{F}$ is a free $\mathbb{O}$-module and is even. □

Proposition 2.39. *If $\mathcal{E} \in D(X,\mathbb{O})$ is a parity sheaf, then $\mathbb{F}\mathcal{E}$ is also a parity sheaf. In other words, for any $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda,\mathbb{O})$, we have*

$$\mathbb{F}\mathcal{E}(\lambda,\mathcal{L}) \cong \mathcal{E}(\lambda,\mathbb{F}\mathcal{L}).$$

Proof. For local systems $\mathcal{L},\mathcal{L}' \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda,\mathbb{O})$ on X_λ, we have

$$\mathbb{F} \otimes \mathrm{Hom}(\mathcal{L},\mathcal{L}') \cong \mathrm{Hom}(\mathbb{F}\mathcal{L},\mathbb{F}\mathcal{L}'). \tag{2.12}$$

Now consider $\mathcal{F}$ (resp. $\mathcal{G}$) in $D(X,\mathbb{O})$ which is $*$- (resp. !-) parity. Then using the proof of Proposition 2.6 and (2.12) above for $\mathcal{L} = i_\lambda^*\mathcal{F}, \mathcal{L}' = i_\lambda^!\mathcal{G}$, we see that the natural morphism yields an isomorphism:

$$\mathbb{F} \otimes \mathrm{Hom}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{F}\mathcal{F},\mathbb{F}\mathcal{G}). \tag{2.13}$$

Finally, let $\mathcal{F} = \mathcal{G} = \mathcal{E}(\lambda,\mathcal{L}) \in D(X;\mathbb{O})$ be a parity sheaf. It follows that we have a surjection

$$\mathrm{End}(\mathcal{E}(\lambda,\mathcal{L})) \twoheadrightarrow \mathrm{End}(\mathbb{F}\mathcal{E}(\lambda,\mathcal{L})).$$

It follows by Lemma 2.10 that $\mathrm{End}(\mathbb{F}\mathcal{E}(\lambda,\mathcal{L}))$ is a local ring, and hence $\mathbb{F}\mathcal{E}(\lambda,\mathcal{L})$ is indecomposable. We also know that $\mathbb{F}\mathcal{E}(\lambda,\mathcal{L})$ is a parity complex by Lemma 2.36. Hence we have an isomorphism $\mathcal{E}(\lambda,\mathbb{F}\mathcal{L}) \cong \mathbb{F}\mathcal{E}(\lambda,\mathcal{L})$ by Theorem 2.12. □

Remark 2.40. This is a partial analogue to [Ser67, Prop. 4.2(b)], which states that for each projective $\mathbb{F}[G]$-module F there exists a unique (up to isomorphism) projective $\mathbb{O}[G]$-module whose reduction is isomorphic to F.

Proposition 2.41. *If $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda,\mathbb{Z})$ is a local system on a stratum of X such that $\mathbf{IC}(\lambda,\mathbb{Q}\mathcal{L}) \cong \mathcal{E}(\lambda,\mathbb{Q}\mathcal{L})$ is a parity sheaf, then $\mathbf{IC}(\lambda,\mathbb{F}_p\mathcal{L}) \cong \mathcal{E}(\lambda,\mathbb{F}_p\mathcal{L})$ for all but finitely many primes p.*

Proof. Suppose there is no p-torsion in the cohomology of the stalks and costalks of $\mathbf{IC}(\lambda,\mathcal{L})$ for some prime p. Then the graded dimensions of the stalks and costalks of $\mathbb{Q}\mathbf{IC}(\lambda,\mathcal{L})(\cong \mathbf{IC}(\lambda,\mathbb{Q}\mathcal{L}))$ and $\mathbb{F}_p\mathbf{IC}(\lambda,\mathcal{L})$ coincide. It follows that $\mathbb{F}_p\mathbf{IC}(\lambda,\mathcal{L})$ is isomorphic to $\mathbf{IC}(\lambda,\mathbb{F}_p\mathcal{L})$ and, by the parity assumption on $\mathbf{IC}(\lambda,\mathbb{Q}\mathcal{L})$, is a parity sheaf.

It remains to show that the cohomology groups of the stalks and costalks of $\mathbf{IC}(\lambda, \mathcal{L})$ contain torsion for only finitely many prime numbers. This is true because they are finitely generated $\mathbb{Z}$-modules and are non-zero in finitely many degrees and on finitely many strata. □

Remark 2.42. In general, it is very difficult to determine for which p the conclusion of the previous proposition holds.

2.6. **Torsion primes.** Our assumptions (2.1) and (2.2) on the space X are quite strict. If we work in the equivariant setting, they might not even be satisfied when X is a single point. However, once we invert a set of prime numbers in k depending on the group G, called the torsion primes, the conditions are satisfied at least for a point. In this subsection, we recall from [Ste75] some facts about torsion primes.

Let G be a reductive group with associated root datum $(\mathbf{X}, \Phi, \mathbf{Y}, \Phi^\vee)$. A reductive subgroup of G is called regular if it contains a maximal torus T of G. Because our base field is $\mathbb{C}$, regular reductive subgroups containing T are in bijection with $\mathbb{Z}$-closed subsystems of Φ, i.e. $\Phi_1 \subset \Phi$ satisfying $\mathbb{Z}\Phi_1 \cap \Phi = \Phi_1$.

Definition 2.43. A prime p is a **torsion prime** for G if $\pi_1(H) = \mathbf{Y}/\mathbb{Z}\Phi_1^\vee$ has p-torsion, for some regular reductive subgroup H of G with root datum $(\mathbf{X}, \Phi_1, \mathbf{Y}, \Phi_1^\vee)$.

It follows from the definition that the torsion primes of any regular reductive subgroup H of G are among those of G. The reason that we are interested in torsion primes is the following theorem of Borel [Bor61, RS65, Dem73, Kac85].

Theorem 2.44. *The following conditions are equivalent:*

(1) *the prime p is not a torsion prime for G;*
(2) *the cohomology $H^*(G, \mathbb{Z})$ of G has no p-torsion;*
(3) *the cohomology $H^*(B_G, \mathbb{Z})$ of the classifying space of G has no p-torsion.*

Moreover, if k is a field whose characteristic either is zero or satisfies the above conditions, then $H^*(G, k)$ is an exterior algebra with generators of odd degrees, while $H^*(B_G, k)$ is a polynomial algebra on generators of one higher degree.

By the universal coefficient theorem, we can conclude that if k is a ring in which all torsion primes for G are invertible, then $H^*_G(\mathrm{pt}, k)$ is even and torsion-free, and the same is true for any regular reductive subgroup H of G.

We now recall how to determine the set of torsion primes. A reductive group has the same torsion primes as its derived subgroup. The torsion primes of a semi-simple group are those of its simply connected cover, together with the primes dividing the order of its fundamental group. The set of torsion primes of a semi-simple and simply connected group is the union of those of its simple factors.

Hence we are reduced to considering G simple and simply connected. Let us choose a system of simple roots Δ of Φ. Let then $\tilde{\alpha}$ denote the highest root of Φ, and let $\tilde{\alpha}^\vee = \sum_{\alpha \in \Delta} n_\alpha^\vee \alpha^\vee$ be the decomposition of the corresponding coroot into simple coroots. Finally, let $n^\vee$ denote the maximum of the $n_\alpha^\vee$.

Theorem 2.45. *If G is simple and simply connected and p is a prime, then the following conditions are equivalent:*

(1) *p is a torsion prime for G;*
(2) *$p \leq n^\vee$;*
(3) *p is one of the $n_\alpha^\vee$;*
(4) *p divides one of the $n_\alpha^\vee$.*

Thus the torsion primes of the simple, simply connected groups are given by the following table:

A_n, C_n	$B_n (n \geq 3), D_n, G_2$	E_6, E_7, F_4	E_8
none	2	2, 3	2, 3, 5

2.7. **Ind-varieties.** In this section we comment on how the results of this section generalise straightforwardly to the slightly more general setting of ind-varieties. Recall that an ind-variety X is a topological space, together with a filtration

$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

such that $X = \cup X_n$, each X_n is a complex algebraic variety, and the inclusions $X_n \hookrightarrow X_{n+1}$ are closed embeddings. We will always assume that each X_n carries the classical topology and equip X with the final topology with respect to all inclusions $X_n \hookrightarrow X$. By a stratification of X we mean a stratification of each X_n such that the inclusions preserve the strata. We will also consider the case where X is acted upon by a linear algebraic pro-group G, by which we mean that G acts on each X_n through a quotient isomorphic to a linear algebraic group, that each such action is algebraic, and that each of the inclusions $X_n \hookrightarrow X_{n+1}$ are G-equivariant. For the basic properties of ind-varieties and pro-groups we refer the reader to [Kum02, Chapter 4].

Now fix a complete local ring k and let

$$X = \bigsqcup_{\lambda \in \Lambda} X_\lambda$$

be a stratified ind-variety, or an ind-G-variety with G-stable stratification for some linear algebraic group G. We write $D(X)$ for the full subcategory of the bounded (equivariant) derived category of sheaves of k-vector spaces consisting of objects $\mathcal{F}$ such that:

(1) the support of $\mathcal{F}$ is contained in X_n for some n;
(2) the cohomology sheaves of $\mathcal{F}$ are constructible with respect to the stratification.

We assume that (2.1) holds for the strata of X_λ. The notion of parity still makes sense and it is immediate that the analogue of Theorem 2.12 applies. In particular, given any (equivariant) indecomposable local system $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ there is, up to isomorphism and shifts, at most one indecomposable parity sheaf $\mathcal{E}(\lambda) \in D(X)$ supported on $\overline{X}_\lambda$ and extending $\mathcal{L}[d_\lambda]$.

In what follows we will refer without comment to results which we have proved previously for varieties, but where an obvious analogue holds for ind-varieties.

3. The Decomposition Theorem and intersection forms

In their proof of the decomposition theorem for semi-small maps [dCM02], de Cataldo and Migliorini highlighted the crucial role played by intersection forms associated to the strata of the target: a certain splitting implied by the decomposition theorem is equivalent to these forms being non-degenerate. Then they prove the non-degeneracy using techniques from Hodge theory.

In our situation, where we consider modular coefficients, these forms may be degenerate. In this section we explain how the non-degeneracy of these forms, together with the semi-simplicity of certain local systems, provide necessary and sufficient conditions for the Decomposition Theorem to hold in positive characteristic. For this, we do not have to assume that X satisfies (2.1) or (2.2). If the map we consider is even (and not necessarily semi-small), then the direct image will be a direct sum of parity sheaves. Assuming now the parity conditions on the strata (ensuring the uniqueness of parity sheaves), we will see that even if the decomposition fails, one can still use intersection forms to determine the multiplicities of parity sheaves that occur in the direct image of the constant sheaf.

In Section 3.1 we recall the definition and basic properties of intersection forms on Borel-Moore homology. In Section 3.2 we relate these intersection forms to the failure of the Decomposition Theorem for semi-small maps in characteristic p. In Section 3.3 we examine the decomposition of the direct image into parity sheaves in the case where the morphism is even.

Aside from Section 3.1, we assume that k is a field (but see Remark 3.15). All cohomology groups are assumed to have values in k and, as always, dimension refers to the complex dimension unless otherwise stated.

3.1. **Borel-Moore homology and intersection forms.** In this subsection we recall some basic properties of Borel-Moore homology and intersection forms. For more details the reader is referred to [Ful93] or [CG97].

For any variety X we let $a_X : X \to \mathrm{pt}$ denote the projection to a point. The dualising sheaf on X is $\omega_X := a_X^! \underline{k}_{\mathrm{pt}}$. One may define the Borel-Moore homology of X to be

$$H_i^{BM}(X) = H^{-i}(a_{X*}a_X^!\underline{k}_{\mathrm{pt}}) = \mathrm{Hom}(\underline{k}_X, \omega_X[-i]).$$

A proper map $\pi : X \to Y$ induces a map $H_\bullet^{BM}(X) \to H_\bullet^{BM}(Y)$ which may be described as follows: given a class $\alpha : \underline{k}_X \to \omega_X[-i] \in H_i^{BM}(X)$ its image in $H_i^{BM}(Y)$ is the class

$$\underline{k}_Y \to \pi_*\pi^*\underline{k}_Y = \pi_*\underline{k}_X \xrightarrow{\pi_*\alpha} \pi_*\omega_X[-i] = \pi_!\pi^!\omega_Y[-i] \to \omega_Y[-i]$$

where the first and last arrows are the adjunction morphisms.

Let Y be a smooth and connected variety of dimension n. As Y has a canonical orientation (after choosing an orientation on $\mathbb{C}$) we have an isomorphism $\mu_Y : \underline{k}_Y \xrightarrow{\sim} \omega_Y[-2n]$. If we regard μ_Y as an element of $H_{2n}^{BM}(Y)$ it is called the **fundamental class**. Even if Y is singular of dimension n, $H_{2n}^{BM}(Y)$ is still freely generated by the fundamental classes of the irreducible components of Y of maximal dimension.

Now suppose that $F \overset{i}{\hookrightarrow} Y$ is a closed embedding of a variety F into a smooth variety of dimension n. For all m we have a canonical isomorphism

$$H_m^{BM}(F) \cong H^{2n-m}(Y, Y - F).$$

Recall that there exists a cup product on relative cohomology. We may use this to define an intersection form of F inside Y:

$$\begin{array}{ccccc} H_p^{BM}(F) & \otimes & H_q^{BM}(F) & \to & H_{p+q-2n}^{BM}(F) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ H^{2n-p}(Y, Y-F) & \otimes & H^{2n-q}(Y, Y-F) & \xrightarrow{\cup} & H^{4n-p-q}(Y, Y-F) \end{array}$$

Note that this product depends on the inclusion $F \hookrightarrow Y$.

Below we will need a slight variation of this intersection form in case $p + q = 2n$. If F is a proper subvariety of Y then we can compose the intersection form with the map $H_0^{BM}(F) \to H_0^{BM}(\mathrm{pt})$ induced from the proper map $F \to \mathrm{pt}$ to obtain a pairing

$$B_F^m : H_{n+m}^{BM}(F) \times H_{n-m}^{BM}(F) \to H_0^{BM}(\mathrm{pt}) = k.$$

Geometrically, this pairing corresponds to intersecting cycles but forgetting in which connected component of F the points live.

It is particularly interesting when F is proper and half-dimensional inside Y. In this case we obtain an intersection form

$$B_F^0 : H_{\mathrm{top}}^{BM}(F) \otimes H_{\mathrm{top}}^{BM}(F) \to k$$

where top denotes the real dimension of F. From the above comments, $H_{\mathrm{top}}^{BM}(F)$ has a basis given by the irreducible components of maximal dimension of F. It also follows that this intersection form over any ring is obtained by extension of scalars from the corresponding form over $\mathbb{Z}$.

The effect of forming the Cartesian product with a smooth and contractible space on Borel-Moore homology is easy to describe (and will be needed below). If U is an open contractible subset of $\mathbb{C}^m$, then for any $i \in \mathbb{Z}$, we have canonical isomorphisms

$$H_i^{BM}(X \times U) \cong H_{i-2m}^{BM}(X).$$

These isomorphisms are compatible with the intersection forms of $F \hookrightarrow Y$ and $F \times U \hookrightarrow Y \times U$.

3.2. **Multiplicities and intersection forms.** In this section, we do not assume that the stratified variety X satisfies the conditions (2.1) or (2.2). Moreover, in this section and the next, we assume that k is a field.

In Section 3.2.1 we explain how multiplicities of indecomposable objects in a k-linear Krull-Remak-Schmidt category are encoded in certain bilinear forms. In Section 3.2.2 we examine the splitting at the "most singular point" and relate it to an intersection form on the fibre. This is used in Section 3.2.3 to relate the Decomposition Theorem to the non-degeneracy of the intersection forms attached to each stratum.

3.2.1. *Bilinear forms and multiplicities in Krull-Remak-Schmidt categories.* Let H and H' be finite dimensional k-vector spaces and consider a bilinear map

$$B : H \times H' \to k.$$

We define

$$^{\perp}B := \{\alpha \in H \mid B(\alpha, \beta) = 0 \text{ for all } \beta \in H'\},$$
$$B^{\perp} := \{\beta \in H' \mid B(\alpha, \beta) = 0 \text{ for all } \alpha \in H\}.$$

Then B induces a perfect pairing

$$H/{}^{\perp}B \times H'/B^{\perp} \to k$$

and we have equalities

$$\dim(H/{}^{\perp}B) = \operatorname{rank} B = \dim(H'/B^{\perp}).$$

If $H = H'$ and B is a symmetric bilinear form then we write $\operatorname{rad} B$ instead of ${}^{\perp}B = B^{\perp}$.

Let $\mathcal{C}$ be a Krull-Remak-Schmidt k-linear category with finite dimensional Hom-spaces (see Section 2.1). Let $a \in \mathcal{C}$ denote an indecomposable object. Given any object $x \in \mathcal{C}$ we can write $x \simeq a^{\oplus m} \oplus y$ such that a is not a direct summand in y.

The integer m is called the **multiplicity** of a in x. This multiplicity is well defined because $\mathcal{C}$ is Krull-Remak-Schmidt.

Lemma 3.1. *Assume that* $\mathrm{End}(a) = k$. *Composition gives us a pairing:*

$$B : \mathrm{Hom}(a, x) \times \mathrm{Hom}(x, a) \longrightarrow \mathrm{End}(a) = k \qquad (3.1)$$
$$(\alpha, \beta) \longmapsto \beta \circ \alpha.$$

The multiplicity of a in x is equal to the rank of B.

Proof. Choose an isomorphism $\phi : x \xrightarrow{\sim} a^{\oplus m} \oplus y$. Inclusion and projection define subspaces $\mathrm{Hom}(a, y) \subset \mathrm{Hom}(a, x)$ and $\mathrm{Hom}(y, a) \subset \mathrm{Hom}(x, a)$. We will show that

$$\mathrm{Hom}(a, y) = {}^{\perp}B \qquad \text{and} \qquad \mathrm{Hom}(y, a) = B^{\perp}. \qquad (3.2)$$

Thus these subspaces do not depend on the choice of ϕ, and this will show that the multiplicity of a in x is equal to the rank of B.

The isomorphism ϕ induces isomorphisms

$$\mathrm{Hom}(a, x) \simeq \mathrm{Hom}(a, a^{\oplus m}) \oplus \mathrm{Hom}(a, y),$$
$$\mathrm{Hom}(x, a) \simeq \mathrm{Hom}(a^{\oplus m}, a) \oplus \mathrm{Hom}(y, a).$$

For α in $\mathrm{Hom}(a, x)$ and β in $\mathrm{Hom}(x, a)$, we write $\alpha = \alpha_1 \oplus \alpha_2$ and $\beta = \beta_1 \oplus \beta_2$ for the corresponding decompositions. Thus we have $B(\alpha, \beta) = \beta_1\alpha_1 + \beta_2\alpha_2$.

We actually have

$$\beta_2\alpha_2 = 0. \qquad (3.3)$$

Otherwise, since $\mathrm{End}(a) = k$, we could assume that $\beta_2\alpha_2 = \mathrm{Id}$, in which case $e := \alpha_2\beta_2$ would be an idempotent in $\mathrm{End}(y)$, and a would be a direct summand of y, contrary to our assumption.

Thus we have $B(\alpha, \beta) = \beta_1\alpha_1$ for all α and β. Hence we have inclusions

$$\mathrm{Hom}(a, y) \subset {}^{\perp}B \qquad \text{and} \qquad \mathrm{Hom}(y, a) \subset B^{\perp}, \qquad (3.4)$$

and B induces a bilinear form

$$\widetilde{B} : \mathrm{Hom}(a, a^{\oplus m}) \times \mathrm{Hom}(a^{\oplus m}, a) \longrightarrow \mathrm{End}(a) = k.$$

Now $\mathrm{Hom}(a, a^{\oplus m}) \simeq \mathrm{End}(a)^{\oplus m} \simeq k^{\oplus m}$ and similarly $\mathrm{Hom}(a^{\oplus m}, a) \simeq k^{\oplus m}$. With these identifications $\widetilde{B}$ is just the standard bilinear form on $k^{\oplus m}$; hence it is non-degenerate, and we have equalities in (3.4). □

Let us assume further that $\mathcal{C}$ is equipped with a duality[4] D and we have isomorphisms $a \xrightarrow{\sim} Da$ and $x \xrightarrow{\sim} Dx$. Then, using these isomorphisms, we may identify $\mathrm{Hom}(a, x)$ and $\mathrm{Hom}(x, a)$. In which case the composition (3.1) is given by a bilinear form on $H = \mathrm{Hom}(a, x) = \mathrm{Hom}(x, a)$ and the multiplicity of a in x is equal to the rank of this form.

[4]A duality is a contravariant equivalence $D : \mathcal{C} \xrightarrow{\sim} \mathcal{C}^{op}$ together with a natural isomorphism $D^2 \cong Id_{\mathcal{C}}$, where $Id_{\mathcal{C}}$ denotes the identity functor on $\mathcal{C}$.

3.2.2. *Splitting at the most singular point.* Consider a proper surjective morphism

$$\pi : \widetilde{X} \to X$$

with $\widetilde{X}$ smooth of dimension n. Fix a point $s \in X$ and form the Cartesian diagram:

$$\begin{array}{ccc} F & \xrightarrow{\tilde{i}} & \widetilde{X} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \{s\} & \xrightarrow{i} & X \end{array}$$

Let B_F^m denote the intersection form

$$B_F^m : H_{n+m}^{BM}(F) \times H_{n-m}^{BM}(F) \to H_0^{BM}(\mathrm{pt}) = k$$

associated to the inclusion $F \hookrightarrow \widetilde{X}$.[5]

Proposition 3.2. *The multiplicity of $i_* \underline{k}_s[m]$ as a direct summand of $\pi_* \underline{k}_{\widetilde{X}}[n]$ is equal to the rank of B_F^m.*

Remark 3.3. Because $\pi_* \underline{k}_{\widetilde{X}}[n]$ is self-dual, $i_* \underline{k}_s[m]$ and $i_* \underline{k}_s[-m]$ occur with equal multiplicities as direct summands of $\pi_* \underline{k}_{\widetilde{X}}[n]$. This also follows from the above proposition: B_F^m and B_F^{-m} are transpose, and hence have the same rank.

Proof. By the discussion of the previous section, if B denotes the pairing given by composition

$$B : \mathrm{Hom}(i_* \underline{k}_s[m], \pi_* \underline{k}_{\widetilde{X}}[n]) \times \mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[n], i_* \underline{k}_s[m]) \to k,$$

then the multiplicity of $i_* \underline{k}_s[m]$ in $\pi_* \underline{k}_{\widetilde{X}}[n]$ is given by the rank of B. A string of adjunctions gives canonical identifications:

$$\mathrm{Hom}(i_* \underline{k}_s[m], \pi_* \underline{k}_{\widetilde{X}}[n]) = H_{n+m}^{BM}(F),$$
$$\mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[n], i_* \underline{k}_s[m]) = H_{n-m}^{BM}(F).$$

Hence we are interested in a pairing

$$B : H_{n+m}^{BM}(F) \times H_{n-m}^{BM}(F) \to k. \tag{3.5}$$

By the lemma below, this is the intersection form B_F^m. The proposition then follows. □

Lemma 3.4. *The pairing B in* (3.5) *is the intersection form B_F^m.*

Proof. In the course of the following proof all morphisms which are not described explicitly are either adjunction morphisms or the canonical morphisms $i^! \to i^*$ and $\tilde{i}^! \to \tilde{i}^*$. Also, every time we say "identify" we mean "canonically identify."

Recall that the intersection form is defined via the cup product on relative cohomology. Let γ_1 and γ_2 be classes in $H_{n+m}^{BM}(F) = H^{n-m}(\widetilde{X}, \widetilde{X} - F)$ and $H_{n-m}^{BM}(F) = H^{n+m}(\widetilde{X}, \widetilde{X} - F)$ respectively. If we represent them as morphisms $\gamma_1 : \underline{k}_{\widetilde{X}} \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[n-m]$ and $\gamma_2 : \underline{k}_{\widetilde{X}} \to \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[n+m]$, their cup product is the morphism

$$\gamma_1 \cup \gamma_2 : \underline{k}_{\widetilde{X}} \xrightarrow{\gamma_1} \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[n-m] \to \underline{k}_{\widetilde{X}}[n-m] \xrightarrow{\gamma_2[n-m]} \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[2n].$$

[5]See Section 3.1 for the definition of the intersection form. Note in particular that the image of B_F^m is $H_0^{BM}(\mathrm{pt})$ and not $H_0^{BM}(F)$.

Because $\widetilde{X}$ is a smooth variety the fundamental class gives an isomorphism $\mu_{\widetilde{X}} : \underline{k}_{\widetilde{X}} \xrightarrow{\sim} \omega_{\widetilde{X}}[-2n]$ (see 3.1). Using $i^! \omega_{\widetilde{X}} = \omega_F$ and adjunction we have an identification

$$\mathrm{Hom}(\underline{k}_{\widetilde{X}}, \tilde{i}_! \tilde{i}^! \underline{k}_{\widetilde{X}}[n-m]) = \mathrm{Hom}(\underline{k}_F, \omega_F[-n-m]).$$

One can now check that the intersection form

$$H^{BM}_{n+m}(F) \times H^{BM}_{n-m}(F) \to H^{BM}_0(F)$$

can be described as follows: given classes $\alpha : \underline{k}_F \to \omega_F[-n-m]$ and $\beta : \underline{k}_F \to \omega_F[-n+m]$, their pairing under the intersection form is the class

$$\underline{k}_F \to \tilde{i}^! \omega_{\widetilde{X}}[-n-m] = \tilde{i}^! \underline{k}_{\widetilde{X}}[n-m] \to \tilde{i}^* \underline{k}_{\widetilde{X}}[n-m] = \underline{k}_F[n-m] \to \omega_F.$$

Now consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(\tilde{\pi}^* \underline{k}_{\{s\}}[m], \tilde{i}^! \underline{k}_{\widetilde{X}}[n]) \otimes \mathrm{Hom}(\tilde{i}^* \underline{k}_{\widetilde{X}}[n], \tilde{\pi}^! \underline{k}_{\{s\}}[m]) & \xrightarrow{B_1} & H \\
\big\downarrow \mathrm{a} & & \big\downarrow \phi \\
\mathrm{Hom}(\underline{k}_{\{s\}}[m], \tilde{\pi}_* \tilde{i}^! \underline{k}_{\widetilde{X}}[n]) \otimes \mathrm{Hom}(\tilde{\pi}_* \tilde{i}^* \underline{k}_{\widetilde{X}}[n], \underline{k}_{\{s\}}[m]) & \xrightarrow{B_2} & \mathrm{End}(\underline{k}_{\{s\}}) \\
\big\downarrow \mathrm{pbc} & & \big\downarrow = \\
\mathrm{Hom}(\underline{k}_{\{s\}}[m], i^! \pi_* \underline{k}_{\widetilde{X}}[n]) \otimes \mathrm{Hom}(i^* \pi_* \underline{k}_{\widetilde{X}}[n], \underline{k}_{\{s\}}[m]) & \xrightarrow{B_3} & \mathrm{End}(\underline{k}_{\{s\}}) \\
\big\downarrow \mathrm{a} & & \big\downarrow i_* \\
\mathrm{Hom}(i_* \underline{k}_{\{s\}}[m], \pi_* \underline{k}_{\widetilde{X}}[n]) \otimes \mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[n], i_* \underline{k}_{\{s\}}[m]) & \xrightarrow{B_4} & \mathrm{End}(i_* \underline{k}_{\{s\}})
\end{array}$$

where

(1) $H := \mathrm{Hom}(\tilde{\pi}^* \underline{k}_{\{s\}}[m], \tilde{\pi}^! \underline{k}_{\{s\}}[m]) = \mathrm{Hom}(\underline{k}_F[m], \omega_F[m])$,
(2) each pairing B_i is canonical,
(3) the arrows labelled "a" are obtained via adjunction,
(4) the arrow labelled "pbc" is obtained via proper base change,
(5) the morphism ϕ is the map which sends $\gamma : \tilde{\pi}^* \underline{k}_{\{s\}}[m] \to \tilde{\pi}^! \underline{k}_{\{s\}}[m]$ to the morphism

$$\phi(\gamma) : \underline{k}_{\{s\}}[m] \to \tilde{\pi}_* \tilde{\pi}^* \underline{k}_{\{s\}}[m] \xrightarrow{\tilde{\pi}_* \gamma} \tilde{\pi}_* \tilde{\pi}^! \underline{k}_{\{s\}}[m] \to \underline{k}_{\{s\}}[m].$$

It is straightforward to check that this diagram is commutative.[6]

By the above discussion, B_1 may be identified with the intersection form

$$H^{BM}_{n+m}(F) \times H^{BM}_{n-m}(F) \to H = H^{BM}_0(F).$$

Also, if we identify $H = H^{BM}_0(F)$ and $\mathrm{End}(\underline{k}_{\{s\}}) = H^{BM}_0(\mathrm{pt})$, then ϕ is the map $H^{BM}_0(F) \to H^{BM}_0(\mathrm{pt})$ induced from the projection $F \to \mathrm{pt}$.

It follows that if we identify

$$\mathrm{Hom}(i_* \underline{k}_{\{s\}}[m], \pi_* \underline{k}_{\widetilde{X}}[2n]) = H^{BM}_{n+m}(F),$$
$$\mathrm{Hom}(\pi_* \underline{k}_{\widetilde{X}}[2n], i_* \underline{k}_{\{s\}}[m]) = H^{BM}_{n-m}(F),$$

[6]One needs that the following diagram of functors be commutative (where the vertical arrows are the base-change isomorphisms):

$$\begin{array}{ccc}
i^! \pi_* & \longrightarrow & i^* \pi_* \\
\big\downarrow & & \big\downarrow \\
\tilde{\pi}_* \tilde{i}^! & \longrightarrow & \tilde{\pi}_* \tilde{i}^*
\end{array}$$

This can be checked directly, by first checking the statement on sheaves.

then we may identify B_4 with the composition

$$H^{BM}_{n+m}(F) \otimes H^{BM}_{n-m}(F) \to H^{BM}_0(F) \to H^{BM}_0(\mathrm{pt})$$

which by definition is B^m_F. □

The proof of Proposition 3.2 has the following corollary:

Corollary 3.5. *The natural morphism*

$$H^m(i^!\pi_*\underline{k}_{\widetilde{X}}[n]) \to H^m(i^*\pi_*\underline{k}_{\widetilde{X}}[n])$$

may be canonically identified with the morphism

$$H^{BM}_{n-m}(F) \to H^{BM}_{n+m}(F)^*$$

induced by the intersection form.

3.2.3. *A criterion for the Decomposition Theorem.* Let X be a connected, equidimensional variety equipped with an algebraic stratification into connected strata

$$X = \bigsqcup_{\lambda\in\Lambda} X_\lambda.$$

In this subsection we do not make any parity assumptions on our stratification. We write d_X for the dimension of X and, as usual, write d_λ and i_λ for the dimension and inclusion of X_λ respectively. We fix a smooth variety $\widetilde{X}$ and a stratified, proper, surjective, semi-small morphism

$$f : \widetilde{X} \to X.$$

We want to understand when the perverse sheaf $f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ decomposes as a direct sum of intersection cohomology sheaves. This may be thought of as a global version of the previous section.

As we have assumed that the stratification of X is algebraic [CG97, 3.2.23], at each point $x \in X_\lambda$, we can choose a stratified slice N_λ to X_λ in X. We obtain a Cartesian diagram with $\widetilde{N_\lambda}$ smooth:

$$\begin{array}{ccc} F_\lambda & \xrightarrow{\tilde{i}} & \widetilde{N_\lambda} \\ \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \pi} \\ \{x\} & \xrightarrow{i} & N_\lambda \end{array} \tag{3.6}$$

Note that, as f is semi-small, the dimension of F is less than or equal to $\frac{1}{2}(d_X - d_\lambda)$. If equality holds we say that X_λ is **relevant** (see [BM83]).

Definition 3.6. The **intersection form associated to** X_λ is the intersection form on $H^{BM}_{d_X-d_\lambda}(F_\lambda)$ given by the inclusion $F_\lambda \hookrightarrow \widetilde{N_\lambda}$.

Note that $H^{BM}_{d_X-d_\lambda}(F_\lambda)$ is non-zero if and only if X_λ is relevant. Using the discussion at the end of Section 3.1 it is straightforward to see that the above intersection form does not depend on the choice of x. For each λ, we define

$$\mathcal{L}_\lambda := \mathcal{H}^{-d_\lambda}(i^!_\lambda f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]). \tag{3.7}$$

Note that $\mathcal{L}_\lambda$ is a local system on X_λ which is non-zero if and only if X_λ is relevant. The aim of this subsection is to show:

Theorem 3.7. *Suppose that the intersection forms associated to all strata are non-degenerate. Then one has an isomorphism:*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda} \mathbf{IC}(\mathcal{L}_\lambda).$$

In that case, the full Decomposition Theorem holds if and only if each local system $\mathcal{L}_\lambda$ is semi-simple.

Remark 3.8. It is a deep result of de Cataldo and Migliorini [dCM02] that in fact, these intersection forms are definite (and hence non-degenerate) over $\mathbb{Q}$. The semi-simplicity of each $\mathcal{L}_\lambda$ over a field of characteristic zero is much more straightforward (and is also pointed out in [dCM02]): For relevant strata X_λ the stalks of $\mathcal{L}_\lambda$ have a basis at any point $x \in X_\lambda$ consisting of the irreducible components of maximal dimension in the fibre $f^{-1}(s)$. The monodromy action permutes these components. Hence each local system factors through a representation of a finite group, and hence is semi-simple.

Remember that here, we do not assume condition (2.1). Otherwise, the semi-simplicity of the local systems would be automatic.

Assume that X_λ is a closed stratum and set $\mathcal{F} := f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$. For any $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda)$ we are interested in the pairing

$$\mathrm{Hom}(i_{\lambda *}\mathcal{L}[d_\lambda], \mathcal{F}) \times \mathrm{Hom}(\mathcal{F}, i_{\lambda *}\mathcal{L}[d_\lambda]) \to \mathrm{End}(i_{\lambda *}\mathcal{L}[d_\lambda]).$$

Applying two adjunctions on each side, this is equivalent to determining the pairing

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}_\lambda) \times \mathrm{Hom}(\mathcal{L}_\lambda^\vee, \mathcal{L}) \to \mathrm{End}(\mathcal{L})$$

where, given morphisms

$$\mathcal{L} \xrightarrow{f} \mathcal{L}_\lambda \text{ and } \mathcal{L}_\lambda^\vee \xrightarrow{g} \mathcal{L}$$

their pairing is given by the composition

$$\mathcal{L} \xrightarrow{f} \mathcal{L}_\lambda \to i_\lambda^! \mathcal{F}[-d_\lambda] \to i_\lambda^* \mathcal{F}[-d_\lambda] \to \mathcal{L}_\lambda^\vee \xrightarrow{g} \mathcal{L}$$

where all morphisms except f and g are canonical. Hence it is important to understand the morphism

$$D_\lambda : \mathcal{L}_\lambda \to \mathcal{L}_\lambda^\vee. \tag{3.8}$$

In the following lemma (and its proof), we use the notations in the diagram (3.6).

Lemma 3.9. *Given $x \in X_\lambda$ as above, the stalk of D_λ at x may be canonically identified with the morphism*

$$H^{BM}_{d_X - d_\lambda}(F_\lambda) \to H^{BM}_{d_X - d_\lambda}(F_\lambda)^*$$

induced by the intersection form associated to X_λ.

Proof. Without loss of generality we may assume that $X_\lambda = U$, $X = N_\lambda \times U$ and $\widetilde{X} = \widetilde{N_\lambda} \times U$, for some contractible open subset $U \subset \mathbb{C}^{d_\lambda}$. It follows that the stalk of D_λ may be identified with the morphism

$$\mathcal{H}^0(i^! \pi_* \underline{k}_{\widetilde{N_\lambda}}[d_{\widetilde{X}} - d_\lambda]) \to \mathcal{H}^0(i^* \pi_* \underline{k}_{\widetilde{N_\lambda}}[d_{\widetilde{X}} - d_\lambda])$$

in which case the result follows from Corollary 3.5. □

Applying the adjunction $(- \otimes \mathcal{L}_\lambda, - \otimes \mathcal{L}_\lambda^\vee)$ to D_λ we obtain a morphism

$$B_\lambda : \mathcal{L}_\lambda \otimes \mathcal{L}_\lambda \to \underline{k}_\lambda$$

and it follows from the above lemma that the stalk of this morphism at each point $x \in X_\lambda$ is given by the intersection form on $H^{BM}_{d_X - d_\lambda}(\pi^{-1}(x))$.

Let j denote the open inclusion of the complement of X_λ. We are now in a position to prove:

Proposition 3.10. *We have that $i_{\lambda *}\mathcal{L}_\lambda[d_\lambda]$ is a direct summand of $f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ if and only if the intersection form associated to X_λ is non-degenerate. If this is the case we have an isomorphism*

$$f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \simeq i_{\lambda *}\mathcal{L}_\lambda[d_\lambda] \oplus j_{!*}j^* f_*\underline{k}_{\widetilde{X}}[d_X].$$

Note that Theorem 3.7 now follows by a simple induction over the stratification.

Proof. The above discussion shows that $i_{\lambda *}\mathcal{L}_\lambda[d_\lambda]$ is a direct summand of $f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \iff D_\lambda$ is an isomorphism $\iff B_\lambda$ is non-degenerate $\iff$ the intersection form associated to X_λ is non-degenerate.

Now assume that $i_{\lambda *}\mathcal{L}_\lambda[d_\lambda]$ is a direct summand of $f_*\underline{k}_{\widetilde{X}}[d_X]$ and write $f_*\underline{k}_{\widetilde{X}}[d_X] \simeq i_{\lambda *}\mathcal{L}_\lambda[d_\lambda] \oplus \mathcal{F}$ for some perverse sheaf $\mathcal{F}$. Then $\mathcal{F}$ is necessarily self-dual because $f_*\underline{k}_{\widetilde{X}}[d_X]$ and $i_{\lambda *}\mathcal{L}_\lambda[d_\lambda]$ are. Also $\mathcal{H}^m(i_\lambda^*\mathcal{F}) = 0$ for $m \geq -d_\lambda$. Hence $\mathcal{F} \cong j_{!*}j^*f_*\underline{k}_{\widetilde{X}}[d_X]$ by the characterisation of $j_{!*}$ given in [BBD82, Proposition 2.1.9]. ☐

3.3. **Decomposing parity sheaves.** In this section we keep the notation from the previous section, but remove the semi-small assumption on f and assume instead that our stratified variety X satisfies (2.1) and that f is even. Recall that the parity assumption implies that, if it exists, the parity sheaf $\mathcal{E}(\lambda, \mathcal{L})$ corresponding to an irreducible local system $\mathcal{L} \in \mathrm{Loc_f}(X_\lambda)$ is well defined up to isomorphism.

Let f be a proper, surjective, even morphism:

$$f : \widetilde{X} \to X.$$

It follows that $f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ is a parity complex, and hence may be decomposed into a direct sum of parity sheaves:

$$f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus \mathcal{E}(\lambda, \mathcal{L})[-n]^{\oplus m_n(\lambda,\mathcal{L})}.$$

In this section we consider the problem of determining $m_n(\lambda, \mathcal{L}) \in \mathbb{N}$.

Remark 3.11. While we do not assume that $\mathcal{E}(\lambda, \mathcal{L})$ exists for all pairs $(\lambda, \mathcal{L})$, all the indecomposable summands of the direct image are parity sheaves and thus existence will follow for any pair occurring with non-zero multiplicity. Moreover, if f is semi-small, the multiplicities $m_n(\lambda, \mathcal{L}) = 0$ for $n \neq 0$ and any $\mathcal{E}(\lambda, \mathcal{L})$ which occurs in the direct image is perverse.

Define[7]

$$\mathcal{L}_\lambda^\bullet := i_\lambda^! f_*\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}].$$

Because f is even, if we define

$$\mathcal{L}_\lambda^n := \mathcal{H}^n(\mathcal{L}_\lambda^\bullet),$$

[7]We use the notation $\mathcal{L}_\lambda^\bullet$ to emphasise the fact that $\mathcal{L}_\lambda^\bullet$ is not necessarily concentrated in one degree, as was the case in the previous section.

then $\mathcal{L}^n_\lambda$ is zero if $n \not\equiv d_{\widetilde{X}} (\mathrm{mod}\ 2)$, and we have an isomorphism (which we fix)

$$\mathcal{L}^\bullet_\lambda \cong \bigoplus \mathcal{L}^n_\lambda[-n]. \tag{3.9}$$

Note that $i^*_\lambda f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \mathbb{D} i^!_\lambda f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ because f is proper and $\widetilde{X}$ is smooth (and so $\underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ is self-dual). Hence

$$i^*_\lambda f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \mathbb{D}\mathcal{L}^\bullet \cong \bigoplus (\mathcal{L}^n_\lambda)^\vee [2d_\lambda + n]. \tag{3.10}$$

As in the previous section we are interested in the canonical morphism

$$\mathcal{L}^\bullet_\lambda = i^! f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \xrightarrow{D_\lambda} i^* f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \mathbb{D}\mathcal{L}^\bullet_\lambda.$$

Taking cohomology and using the decompositions (3.9) and (3.10), we get maps

$$D^n_\lambda := \mathcal{H}^n(D_\lambda) : \mathcal{L}^n_\lambda \to (\mathcal{L}^{-2d_\lambda - n}_\lambda)^\vee.$$

Remark 3.12. One may interpret the maps D^n_λ in terms of an intersection form as follows. Fix a point $x \in X_\lambda$, and let F_λ denote the fibre of f over $x \in X$. Then, one may identify the stalk of $\mathcal{L}^n_\lambda$ (resp. of $(\mathcal{L}^{-2d_\lambda - n}_\lambda)^\vee$) at x with $H^{BM}_{d_{\widetilde{X}} - n - 2d_\lambda}(F_\lambda)$ (resp. $H^{BM}_{d_{\widetilde{X}} + n}(F_\lambda)^*$). Then, using similar arguments to the proof of Lemma 3.9 one may identify D^n_λ with the map

$$H^{BM}_{d_{\widetilde{X}} - n - 2d_\lambda}(F_\lambda) \to H^{BM}_{d_{\widetilde{X}} + n}(F_\lambda)^*$$

induced by the intersection form $H^{BM}_{d_{\widetilde{X}} - n - 2d_\lambda}(F_\lambda) \times H^{BM}_{d_{\widetilde{X}} + n}(F_\lambda) \to k$.

The following theorem shows that knowledge of these intersection forms allows one to decompose $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ into parity sheaves.

Theorem 3.13. *We have an isomorphism*

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda; n \in \mathbb{Z}} \mathcal{E}(\lambda, \mathcal{L}^n_\lambda / \ker D^n_\lambda)[-n - d_\lambda].$$

In particular, the multiplicity $m_n(\lambda, \mathcal{L})$ of an indecomposable parity sheaf $\mathcal{E}(\lambda, \mathcal{L})[-n]$ as a direct summand of $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ is equal to the multiplicity of $\mathcal{L}$ in $\mathcal{L}^{n-d_\lambda}_\lambda / \ker D^{n-d_\lambda}_\lambda$.

Remark 3.14. If f is semi-small then $\mathcal{L}^n_\lambda$ is zero for $n > -d_\lambda$ and hence $D^n_\lambda = 0$ unless $n = -d_\lambda$. In this case Theorem 3.13 gives an isomorphism

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus_{\lambda \in \Lambda} \mathcal{E}(\lambda, \mathcal{L}^{-d_\lambda}_\lambda / \ker D^{-d_\lambda}_\lambda).$$

Assumption (2.1) guarantees that each local system $\mathcal{L}^{-d_\lambda}_\lambda / \ker D^{-d_\lambda}_\lambda$ is semi-simple. Hence the Decomposition Theorem is true if and only if each $D^{-d_\lambda}_\lambda$ is an isomorphism, which is the case if and only if each intersection form is non-degenerate (by Lemma 3.9). We have already seen this result in Theorem 3.7 in the context of an arbitrary semi-small map. The advantage of the above theorem is that it explains (in the restricted context of varieties satisfying the parity assumptions) how to decompose $f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}]$ when the Decomposition Theorem fails.

Proof. By Proposition 2.11 it is enough to prove that, if $i_\lambda : X_\lambda \hookrightarrow X$ is the inclusion of a closed stratum, then

$$f_* \underline{k}_{\widetilde{X}}[d_{\widetilde{X}}] \cong \bigoplus (i_{\lambda *}(\mathcal{L}^n_\lambda / \ker D^n_\lambda)[-n]) \oplus \mathcal{G}$$

where $\mathcal{G}$ is a parity complex having no direct summands supported on X_λ. However this is an immediate consequence of Proposition 2.26. ☐

Remark 3.15. In this section we have only considered the case of field coefficients. However using (2.13) and idempotent lifting (e.g. [Fei82, Theorem 12.3]) one can show that if $\mathbb{O}$ is a complete discrete valuation ring with residue field $\mathbb{F}$ and $\mathcal{L} \in \mathrm{Loc}_{\mathrm{f}}(X_\lambda, \mathbb{O})$ (see Section 2.1), then the graded multiplicity of $\mathcal{E}(\lambda, \mathcal{L})$ in $f_* \underline{\mathbb{O}}_{\widetilde{X}}[d_{\widetilde{X}}]$ is equal to the graded multiplicity of $\mathcal{E}(\lambda, \mathbb{F}\mathcal{L})$ in $f_* \underline{\mathbb{F}}_{\widetilde{X}}[d_{\widetilde{X}}]$. Hence the results of this section also yield multiplicities with coefficients in $\mathbb{O}$.

4. Applications

4.1. **(Kac-Moody) flag varieties.** In this section we show the existence and uniqueness of parity sheaves on Kac-Moody flag varieties. Throughout we only work with the trivial pariversity $\dagger = \natural$. The reader unfamiliar with Kac-Moody flag varieties may keep the important case of a (finite) flag variety in mind. The standard reference for Kac-Moody Schubert varieties is [Kum02].

We begin by fixing some notation, which is identical to that of [Kum02]. Let A be a generalised Cartan matrix of size l and let $\mathfrak{g}(A)$ denote the corresponding Kac-Moody Lie algebra with Weyl group W, Bruhat order $\leq$, length function ℓ and simple reflections $S = \{s_i\}_{i=1,\dots l}$. To A one may also associate a Kac-Moody group $\mathcal{G}$ and subgroups N, $\mathcal{B}$ and $\mathcal{T}$ with $\mathcal{B} \supset \mathcal{T} \subset N$. Given any subset $I \subset \{1, \dots, l\}$ one has a standard parabolic subgroup $\mathcal{P}_I$ containing $\mathcal{B}$ and a canonical Levi subgroup $\mathcal{G}_I \subset \mathcal{P}_I$. The group $\mathcal{T}$ is a connected algebraic torus, $\mathcal{B}$, N, $\mathcal{P}_I$, $\mathcal{G}_I$ and $\mathcal{G}$ are all pro-algebraic groups and $(\mathcal{G}, \mathcal{B}, N, S)$ is a Tits system with Weyl group canonically isomorphic to W. The set $\mathcal{G}/\mathcal{P}_I$ may be given the structure of an ind-variety and is called a Kac-Moody flag variety.

Let $\mathfrak{h}_{\mathbb{Z}}$ denote the lattice of cocharacters of $\mathcal{T}$. Its dual $\mathfrak{h}_{\mathbb{Z}}^*$ may be identified with the lattice of characters of $\mathcal{T}$. In $\mathfrak{h}_{\mathbb{Z}}^*$ one has the set Δ of roots, together with a decomposition $\Delta = \Delta^+ \sqcup \Delta^-$ into the subsets of positive and negative roots. Let Δ_{re} denote the real roots and $\Delta_{\mathrm{re}}^+ = \Delta_{\mathrm{re}} \cap \Delta^+$ and $\Delta_{\mathrm{re}}^- = \Delta_{\mathrm{re}} \cap \Delta^-$ denote the positive and negative real roots respectively. Finally, given a subset $I \subset \{1, \dots, l\}$ we have have subsets Δ_I, $\Delta_{I,\mathrm{re}}$, $\Delta_{I,\mathrm{re}}^+$ etc. consisting of those (positive, real) roots in the span of simple roots indexed by I.

Example 4.1. If A is a Cartan matrix then $\mathfrak{g}(A)$ is a semi-simple finite-dimensional complex Lie algebra and $\mathcal{G}$ is the semi-simple and simply connected complex linear algebraic group with Lie algebra $\mathfrak{g}(A)$, $\mathcal{B}$ is a Borel subgroup, $\mathcal{T} \subset \mathcal{B}$ is a maximal torus, N is the normaliser of $\mathcal{T}$ in $\mathcal{G}$, $\mathcal{P}_I$ is a standard parabolic and $\mathcal{G}/\mathcal{P}_I$ is a partial flag variety.

Example 4.2. If A is now a Cartan matrix of size $l-1$ and $\mathfrak{g}(A)$ is the corresponding Lie algebra with semi-simple simply connected group G, one can obtain a generalised Cartan matrix $\tilde{A}$ by adding an l-th row and column with the values:

$$a_{l,l} = 2, a_{l,j} = -\alpha_j(\theta^\vee), a_{j,l} = -\theta(\alpha_j^\vee),$$

where $1 \leq j \leq l-1$, α_i are the simple roots of $\mathfrak{g}(A)$ and θ is the highest root. The corresponding Kac-Moody Lie algebra $\mathfrak{g}(\tilde{A})$ (resp. group $\mathcal{G}$) is the so-called (untwisted) affine Kac-Moody Lie algebra (resp. group) defined in [Kum02, Chapter 13]. It turns out that the associated Kac-Moody flag varieties have an alternative description as partial affine flag varieties. Let $\mathcal{K} = \mathbb{C}((t))$ denote the field of Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ the ring of Taylor series. Then, for example, the sets $G(\mathcal{K})/\mathcal{I}$ (the affine flag variety) and $G(\mathcal{K})/G(\mathcal{O})$ (the affine Grassmannian) may be given the structure of an ind-variety and are isomorphic to the Kac-Moody flag variety $\mathcal{G}/\mathcal{P}_I$ for $I = \emptyset$ and $I = \{1, \dots, l-1\}$ respectively. Here $G(\mathcal{K})$ (resp. $G(\mathcal{O})$) denotes the group of $\mathcal{K}$-points (resp. $\mathcal{O}$-points) of G and $\mathcal{I}$ denotes the Iwahori subgroup, defined as the inverse image of a Borel subgroup $B \subset G$ under the evaluation $G(\mathcal{O}) \to G$.

Given a subset $I \subset \{1, \dots, l\}$ we denote by A_I the submatrix of A consisting of those rows and columns indexed by I. For any such I, A_I is a generalised Cartan matrix. A subset $I \subset \{1, \dots, l\}$ is of **finite type** if A_I is a Cartan matrix. Equivalently, the subgroup $W_I \subset W$ generated by the simple reflections s_i for $i \in I$ is finite. Below we will mostly be concerned with subsets $I \subset \{1, \dots, l\}$ of finite type.[8]

For any two subsets $I, J \subset \{1, \dots, l\}$ of finite type we define

$$^I W^J := \{w \in W \mid s_i w > w \text{ and } w s_j > w \text{ for all } i \in I, j \in J\}.$$

The orbits of $\mathcal{P}_I$ on $\mathcal{G}/\mathcal{P}_J$ give rise to a Bruhat decomposition:

$$\mathcal{G}/\mathcal{P}_J = \bigsqcup_{w \in {}^I W^J} \mathcal{P}_I w \mathcal{P}_J/\mathcal{P}_J = \bigsqcup_{w \in {}^I W^J} {}^I X_w^J.$$

The Bruhat decomposition gives an algebraic stratification of $\mathcal{G}/\mathcal{P}_J$.

If $I = \emptyset$ each ${}^I X_w^J$ is isomorphic to an affine space of dimension $\ell(w)$. In general the decomposition of ${}^I X_w^J$ into orbits under $\mathcal{B}$ gives a cell decomposition

$$(4.1) \qquad {}^I X_w^J = \bigsqcup_{x \in W_I w W_J \cap {}^\emptyset W^J} \mathbb{C}^{\ell(x)}.$$

In the following proposition we analyse the strata ${}^I X_w^J$.

Proposition 4.3. *Let k be a ring.*

1. *The graded k-module $H^\bullet({}^I X_w^J, k)$ is torsion free and concentrated in even degrees.*
2. *The same is true of $H^\bullet_{\mathcal{P}_I}({}^I X_w^J, k)$ if all the torsion primes for A_I are invertible in k.*

Moreover, any local system or $\mathcal{P}_I$-equivariant local system on ${}^I X_w^J$ is constant.

We begin with the following lemma:

Lemma 4.4. *For any two subsets $I, J \subset \{1, \dots, l\}$ of finite type and $w \in {}^I W^J$, the variety ${}^I X_w^J$ is simply connected.*

Proof. For each subset $I \subset \{1, \dots, l\}$ of finite type, we can find a cocharacter $\lambda_I : \mathbb{C}^\times \to \mathcal{T}$ such that $\langle \lambda_I, \alpha \rangle = 0$ if $\alpha \in \Delta_{I,\mathrm{re}}$ and $\langle \lambda_I, \alpha \rangle > 0$ for all $\alpha \in \Delta_{\mathrm{re}}^+ \setminus \Delta_{I,\mathrm{re}}$. By working in suitable charts around each $\mathcal{T}$-fixed point on ${}^I X_w^J$ one may show that, for all $x \in {}^I X_w^J$, we have

$$\lim_{\mathbb{C}^\times \ni z \to 0} \lambda_I(z) \cdot x \in \mathcal{G}_I w \mathcal{P}_J/\mathcal{P}_J.$$

A similar argument shows that $\mathcal{G}_I w \mathcal{P}_J/\mathcal{P}_J$ is fixed by $\lambda_I(\mathbb{C}^\times) \subset \mathcal{T}$. It follows that $\mathcal{G}_I w \mathcal{P}_J/\mathcal{P}_J$ is a deformation retract of ${}^I X_w^J$.

[8] Much of the theory that we develop below is also valid $\mathcal{G}/\mathcal{P}_J$ even when J is not of finite type, but we will not make this explicit.

Now $\mathcal{G}_I w \mathcal{P}_J/\mathcal{P}_J$ is isomorphic to a (finite) partial flag variety for $\mathcal{G}_I$. It is standard that partial flag varieties are simply connected.[9] Hence

$$\pi_1({}^I X_w^J) = \pi_1(\mathcal{G}_I w \mathcal{P}_J/\mathcal{P}_J) = \{1\}$$

as claimed. □

Proof of Proposition 4.3. The first statement follows from the fact that (4.1) provides an affine paving of ${}^I X_w^J$. By Lemma 4.4 each ${}^I X_w^J$ is simply connected, and hence any local system on ${}^I X_w^J$ is constant. Now if H is the reductive part of the stabiliser of a point in ${}^I X_w^J$ then H is isomorphic to a regular reductive subgroup of a semi-simple connected and simply connected algebraic group with Lie algebra $\mathfrak{g}(A_I)$. It follows that any $\mathcal{P}_I$-equivariant local system on ${}^I X_w^J$ is constant. We also have

$$H^\bullet_{\mathcal{P}_I}({}^I X_w^J, \mathbb{Z}) \cong H^\bullet_H(\mathrm{pt}, \mathbb{Z}).$$

By Theorem 2.44 this has no p-torsion for p and no torsion prime for A_I, and the result follows. □

For the rest of this section we fix a complete local principal ideal domain k.

Fix $I, J \subset \{1, \ldots, l\}$ of finite type. We consider the following situations:

$$X = \mathcal{G}/\mathcal{P}_J, \text{ an ind-variety stratified by } \mathcal{P}_I\text{-orbits;} \tag{4.2}$$

$$X = \mathcal{G}/\mathcal{P}_J, \text{ an ind-}\mathcal{P}_I\text{-variety.} \tag{4.3}$$

If we are in situation (4.3), we make the following assumption:

$$\begin{array}{c}\text{for all subsets } K \subset \{1, \ldots, l\} \text{ of finite type} \\ \text{the torsion primes of } A_K \text{ are invertible in } k.\end{array} \tag{4.4}$$

Remark 4.5. Using the results of Section 2.6 the above assumption (4.4) may be easily read off the Cartan matrix A. For example, any complete local principal ideal domain k in which 2, 3 and 5 are invertible will always satisfy the above assumption.

In either case we let $D_I(\mathcal{G}/\mathcal{P}_J) := D(X, k) = D(X)$ be as in Section 2.1 (see also Section 2.7). Proposition 4.3 shows that the stratified ind-$(\mathcal{P}_I)$-variety $\mathcal{G}/\mathcal{P}_Y$ satisfies (2.1) and (2.2). By Theorem 2.12, it follows that there exists up to isomorphism at most one parity sheaf with support $\overline{{}^I X_w^J}$ for each $w \in {}^I W^J$.

The first aim of this section is to show the following:

Theorem 4.6. *Suppose that we are in situation* (4.2) *or* (4.3). *For each* $w \in {}^I W^J$, *there exists, up to isomorphism, one parity sheaf* $\mathcal{E}(w) \in D_I(\mathcal{G}/\mathcal{P}_Y)$ *with support* $\overline{{}^I X_w^J}$.

Remark 4.7. If $I = \emptyset$, then the $\mathcal{P}_I$-orbits on $\mathcal{G}/\mathcal{P}_J$ are isomorphic to affine spaces. In this case one can use the results of Section 2.3.3 to deduce the existence of †-parity sheaves $\mathcal{E}^\dagger(w)$ for all $w \in {}^I W^J$ and any pariversity in the setting (4.2).

[9] One possible proof: Every partial flag variety is isomorphic to the partial flag variety of a simply connected algebraic group. Now any homogeneous space with connected stabilisers for a simply connected group is simply connected, by the long exact sequence of homotopy groups for a fibration.

Recall that, if we are in the situation (4.3), then given any three subsets $I, J, K \subset \{1, \dots, l\}$ of finite type there exists a bifunctor

$$\begin{aligned} D_I(\mathcal{G}/\mathcal{P}_J) \times D_J(\mathcal{G}/\mathcal{P}_K) &\to D_I(\mathcal{G}/\mathcal{P}_K) \\ (\mathcal{F}, \mathcal{G}) &\mapsto \mathcal{F} * \mathcal{G} \end{aligned}$$

called convolution (see [Spr82, MV07]). It is defined using the convolution diagram (of topological spaces):

$$\mathcal{G}/\mathcal{P}_J \times \mathcal{G}/\mathcal{P}_K \overset{p}{\leftarrow} \mathcal{G} \times \mathcal{G}/\mathcal{P}_K \overset{q}{\to} \mathcal{G} \times_{\mathcal{P}_J} \mathcal{G}/\mathcal{P}_K \overset{m}{\to} \mathcal{G}/\mathcal{P}_K$$

where p is the natural projection, q is the quotient map and m is the map induced by multiplication. One sets

$$\mathcal{F} * \mathcal{G} := m_* \mathcal{K} \quad \text{where} \quad q^* \mathcal{K} \cong p^*(\mathcal{F} \boxtimes \mathcal{G}).$$

For the existence of $\mathcal{K}$ and how to make sense of $\mathcal{G} \times \mathcal{G}/\mathcal{P}_K$ algebraically, we refer the reader to [Nad05, Sections 2.2 and 3.3].

The second goal of this section is to show the following:

Theorem 4.8. *Suppose that we are in situation* (4.3)*. Then convolution preserves parity: If $\mathcal{F} \in D_I(\mathcal{G}/\mathcal{P}_J)$ and $\mathcal{G} \in D_J(\mathcal{G}/\mathcal{P}_K)$ are parity complexes, then so is $\mathcal{F} * \mathcal{G} \in D_I(\mathcal{G}/\mathcal{P}_K)$.*

Remark 4.9. The case of finite flag varieties was considered in [Spr82]. There Springer gives a new proof, due to MacPherson and communicated to him by Brylinski, of the fact that the characters of intersection cohomology complexes on the flag variety give the Kazhdan-Lusztig basis of the Hecke algebra. This uses parity considerations in an essential way. See also [Soe00].

Before turning to the proofs we prove some properties about the canonical quotient maps between Kac-Moody flag varieties and recall the definition of (generalised) Bott-Samelson varieties. Unless we state otherwise, in all statements below we assume that we are in either situation (4.2) or (4.3).

If $J \subset K$ are subsets of $\{1, \dots, l\}$ the canonical quotient map

$$\pi_K^J : \mathcal{G}/\mathcal{P}_J \to \mathcal{G}/\mathcal{P}_K$$

is a morphism of ind-varieties.

Proposition 4.10. *If K is of finite type then both $(\pi_K^J)_*$ and $(\pi_k^J)^*$ preserve parity.*

Proof. For the duration of the proof we abbreviate $\pi := \pi_K^J$. Because a complex is parity if and only if it is parity after applying the forgetful functor, it is clearly enough to deal with the non-equivariant case (i.e. that we are in situation (4.2)). Moreover, as the stratification of $\mathcal{G}/\mathcal{P}_K$ by $\mathcal{B}$-orbits refines the stratification by $\mathcal{P}_I$-orbits we may assume without loss of generality that $I = \emptyset$. It is known (see the discussion of [Kum02] around Proposition 7.1.5) that π is a stratified proper morphism between the stratified ind-varieties $\mathcal{G}/\mathcal{P}_J$ and $\mathcal{G}/\mathcal{P}_K$. Moreover, the same proposition shows that the restriction of π to a stratum in $\mathcal{G}/\mathcal{P}_K$ is simply a projection between affine spaces. It follows that π is even and hence π_* preserves parity complexes by Proposition 2.34.

We now prove that π^* preserves parity complexes. So assume that $\mathcal{F}$ is parity, or equivalently that $\mathcal{F}$ and $\mathbb{D}\mathcal{F}$ are $*$-parity. Then it is enough to show that $\pi^*\mathcal{F}$ and $\mathbb{D}\pi^*\mathcal{F} \cong \pi^!\mathbb{D}\mathcal{F}$ are $*$-parity. This is clear for $\pi^*\mathcal{F}$. For $\pi^!\mathbb{D}\mathcal{F}$ note that our assumptions on K guarantee that π is a smooth morphism with fibres of some (complex) dimension d. Hence $\pi^! \cong \pi^*[2d]$ and so $\pi^!\mathbb{D}\mathcal{F} \cong \pi^*\mathbb{D}\mathcal{F}[2d]$ is also $*$-parity. $\square$

Now, let $I_0 \subset J_1 \supset I_1 \subset J_2 \supset \cdots \subset J_n \supset I_n$ be finite type subsets of $\{1, \dots, l\}$. For $1 \le i \le k \le n$ consider the spaces

$$\mathrm{BS}(i, \dots, k) := \mathcal{P}_{J_i} \times^{\mathcal{P}_{I_i}} \mathcal{P}_{J_{i+1}} \times^{\mathcal{P}_{I_{i+1}}} \dots \mathcal{P}_{J_{k-2}} \times^{\mathcal{P}_{I_{k-1}}} \mathcal{P}_{J_k}/\mathcal{P}_{I_k},$$
$$Y(i, \dots, k) := \mathcal{G} \times^{\mathcal{P}_{I_i}} \mathcal{P}_{J_{i+1}} \times^{\mathcal{P}_{I_{i+1}}} \dots \mathcal{P}_{J_{k-2}} \times^{\mathcal{P}_{I_{k-1}}} \mathcal{P}_{J_k}/\mathcal{P}_{I_k}$$

defined as the quotient of $\mathcal{P}_{J_i} \times \mathcal{P}_{J_{i+1}} \times \cdots \times \mathcal{P}_{J_k}$ (resp. $\mathcal{G} \times \mathcal{P}_{J_{i+1}} \times \cdots \times \mathcal{P}_{J_k}$) by $\mathcal{P}_{I_i} \times \mathcal{P}_{I_{i+1}} \times \cdots \times \mathcal{P}_{I_k}$ where $(q_i, \dots, q_k)$ acts on $(p_i, \dots, p_k)$ by

$$(p_i q_i^{-1}, q_i p_{i+1} q_{i+1}^{-1}, \dots, q_{k-1} p_k q_k^{-1}).$$

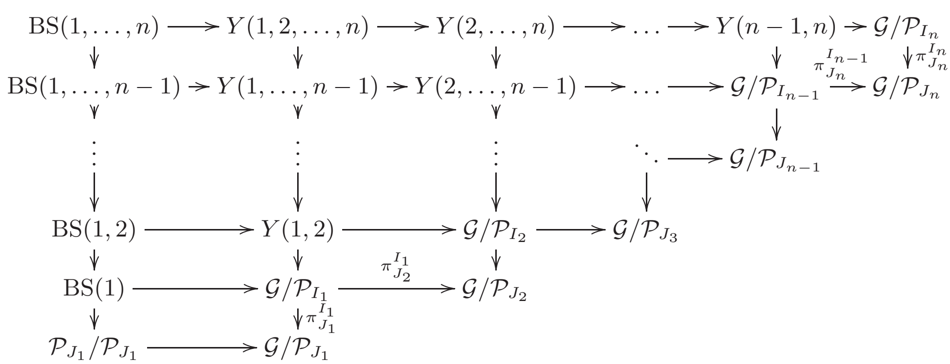
Then $Y(i, \dots, k)$ is a projective ind-$\mathcal{G}$-variety and $BS(i, \dots, k)$ is a closed subvariety. The space $BS(i, \dots, k)$ is called a generalised Bott-Samelson variety. (When $I_i = \emptyset$ and $|J_i| = 1$ for all i then $BS(i, \dots, k)$ is constructed in [Kum02, 7.1.3]. The construction of $BS(i, \dots, k)$ in general is discussed in [GL05]. The construction of $Y(i, \dots, k)$ is similar.) We will denote points in these varieties by $[p_i, \dots, p_k]$. For $i \le j \le k$ we have a morphism of ind-varieties $f_j : Y(i, \dots, k) \to \mathcal{G}/\mathcal{P}_{I_j} : [p_i, \dots, p_k] \mapsto p_i \dots p_j \mathcal{P}_{I_j}$. The map

$$Y(i, \dots, k) \to \mathcal{G}/\mathcal{P}_{I_i} \times \cdots \times \mathcal{G}/\mathcal{P}_{I_k}$$
$$p \mapsto (f_i(p), \dots, f_k(p))$$

is a closed embedding with image

$$\{(x_i, \dots, x_k) \in \mathcal{G}/\mathcal{P}_{I_i} \times \cdots \times \mathcal{G}/\mathcal{P}_{I_k} \mid \pi^{I_j}_{J_{j+1}}(x_j) = \pi^{I_{j+1}}_{J_{j+1}}(x_{j+1}),\ i \le j \le k-1\}. \tag{4.5}$$

The image of $BS(i, \dots, k) \subset Y(i, \dots, k)$ is the sublocus of such $(x_i, \dots, x_k)$ with $\pi^{I_i}_{J_i}(x_i) = \mathcal{P}_{J_i}/\mathcal{P}_{J_i} \in \mathcal{G}/\mathcal{P}_{J_i}$ (see [GL05, Section 7]). It follows that we have a diagram in which all squares are Cartesian (note that $Y(i) = \mathcal{G}/\mathcal{P}_{I_i}$):



Let $f : \mathrm{BS}(1, \dots, n) \to \mathcal{G}/\mathcal{P}_{I_n}$ denote the restriction of f_n to $\mathrm{BS}(1, \dots, n)$; it agrees with the map along the top of the above diagram.

Proposition 4.11. *The sheaf $f_* \underline{k}_{\mathrm{BS}(1,\dots,n)} \in D_{I_0}(\mathcal{G}/\mathcal{P}_{I_n})$ is parity.*

Proof. (See [Soe00].) Repeated use of proper base change applied to the above diagram gives an isomorphism

$$f_* \underline{k}_{\mathrm{BS}(1,\dots,n)} \cong (\pi^{I_n}_{J_n})^* (\pi^{I_{n-1}}_{J_n})_* \dots (\pi^{I_2}_{J_2})^* (\pi^{I_1}_{J_2})_* (\pi^{I_1}_{J_1})^* \underline{k}_{\mathcal{P}_{J_1}}$$

where $\underline{k}_{\mathcal{P}_{J_1}}$ denotes the skyscraper sheaf on the point $\mathcal{P}_{J_1}/\mathcal{P}_{J_1} \in \mathcal{G}/\mathcal{P}_{J_1}$. However $\underline{k}_{\mathcal{P}_{J_1}}$ is certainly parity and the result follows from Proposition 4.10. □

We can now prove Theorems 4.6 and 4.8:

Proof. Fix subsets $I, J \subset \{1, \dots, l\}$ of finite type and choose $w \in {}^IW^J$. By Theorem 2.12 it is enough to show that there exists at least one parity sheaf $\mathcal{E}$ such that the support of $\mathcal{E}$ is $\overline{{}^IX_w^J}$.

One may show (see [Wil08, Proposition 1.3.4]) that there exists a sequence $I = I_0 \subset J_1 \supset I_1 \subset J_2 \supset \cdots J_n \supset I_n = J$ such that, if BS denotes the corresponding generalised Bott-Samelson variety, the morphism

$$f : \mathrm{BS} \to \mathcal{G}/\mathcal{P}_J$$

has image $\overline{{}^IX_w^J}$ and is an isomorphism over ${}^IX_w^J$.[10] Let d_{BS} denote the complex dimension of BS. Then $f_*\underline{k}_{\mathrm{BS}}[d_{\mathrm{BS}}]$ is self-dual (because f is proper and BS is smooth) and parity (by Proposition 4.11). Hence if we let $\mathcal{E}$ denote the unique indecomposable direct summand of $f_*\underline{k}_{\mathrm{BS}}[d_{\mathrm{BS}}]$ which is non-zero over ${}^IX_w^J$ then $\mathcal{E}$ is a parity sheaf with support $\overline{{}^IX_w^J}$. Theorem 4.6 then follows in either situation (4.2) and (4.3).

We now turn to Theorem 4.8 and assume we are in the situation (4.3). By the uniqueness of parity sheaves, and the above remarks, it is enough to show that if

$$I = I_0 \subset J_1 \supset I_1 \subset \cdots \subset J_n \supset I_n = J$$
$$J = I_n \subset J_{n+1} \supset I_{n+1} \subset \cdots \subset J_m \supset I_m = K$$

are two sequences of finite type subsets of $\{1, \dots, l\}$, BS_1 and BS_2 are the corresponding generalised Bott-Samelson varieties, and $f_1 : \mathrm{BS}_1 \to \mathcal{G}/\mathcal{P}_J$ and $f_2 : \mathrm{BS}_2 \to \mathcal{G}/\mathcal{P}_K$, then

$$f_{1*}\underline{k}_{\mathrm{BS}_1} * f_{2*}\underline{k}_{\mathrm{BS}_2} \in D_I(\mathcal{G}/\mathcal{P}_K)$$

is parity.

However, if BS denotes the Bott-Samelson variety associated to the concatenation $I = I_0 \subset J_1 \supset \cdots \supset I_n \subset \cdots \subset J_m \supset I_m = K$ and $f : \mathrm{BS} \to \mathcal{G}/\mathcal{P}_K$ is the multiplication morphism, then

$$f_{1*}\underline{k}_{\mathrm{BS}_1} * f_{2*}\underline{k}_{\mathrm{BS}_2} \cong f_*\underline{k}_{\mathrm{BS}}$$

and the result follows from the proposition above. □

Remark 4.12.

(1) Such theorems have been established for the finite flag varieties if k is a field of characteristic larger than the Coxeter number by Soergel in [Soe00].
(2) An important special case of the above is the affine Grassmannian. In this case, parity sheaves are closely related to tilting modules [JMW].

[10] Actually the condition that $f : \mathrm{BS} \to \mathcal{G}/\mathcal{P}_J$ be an isomorphism over ${}^IX_w^J$ is not necessary for the proof. One only needs that there exist a sequence $I = I_0 \subset J_1 \supset I_1 \subset J_2 \supset \cdots J_n \supset I_n = J$ such that the image of the corresponding generalised Bott-Samelson variety in $\mathcal{G}/\mathcal{P}_J$ is equal to $\overline{{}^IX_w^J}$. In this case any indecomposable summand of $f_*\underline{k}_{\mathrm{BS}}$ with support equal to $\overline{{}^IX_w^J}$ will give the desired parity sheaf (up to a shift).

4.2. **Toric varieties.** In this section we prove the existence and uniqueness of $\natural$-parity sheaves on toric varieties. As in the previous section, here parity sheaf means $\natural$-parity sheaf.

For notation, terminology, and basic properties of toric varieties we refer the reader to [Ful93] and [CLS11]. In this section T denotes a connected algebraic torus and $M = X^*(T)$ and $N = X_*(T)$ denote the character and cocharacter lattices respectively. If L is a lattice we set $L_{\mathbb{Q}} := L \otimes_{\mathbb{Z}} \mathbb{Q}$.

Recall that a fan in N is a collection Δ of polyhedral, convex cones in $N_{\mathbb{Q}}$ closed under taking faces and intersections. To a fan Δ in N one may associate a toric variety $X(\Delta)$ which is a connected normal T-variety. We write $X(\Delta, N)$ to specify the lattice if it is not clear from context.

There are finitely many orbits of T on $X(\Delta)$ and the decomposition into orbits gives a stratification

$$X(\Delta) = \bigsqcup_{\tau \in \Delta} O_\tau$$

indexed by the cones of Δ. For example the zero cone $\{0\}$ always belongs to Δ and $O_{\{0\}}$ is an open dense orbit, canonically identified with T.

In this section we fix a ring of coefficients k as in Section 2.1, take

$$X = X(\Delta) \text{ as a } T\text{-variety} \tag{4.6}$$

and let $D_T(X(\Delta)) = D(X)$ be as in Section 2.1. We use the notation of Section 2 without further comment.

Theorem 4.13. *For each orbit O_τ, there exists up to isomorphism one parity sheaf $\mathcal{E}(\tau) \in D_T(X(\Delta))$ with support $V(\tau) = \overline{O}_\tau$.*

Let $\tau \in \Delta$ and let N_τ denote the intersection of N with the linear span of τ. Then N_τ determines a connected subtorus $T_\tau \subset T$.

Lemma 4.14. *The stabiliser of a point $x \in O_\tau$ is T_τ and is therefore connected.*

Proof. This follows from the last exercise of Section 3.1 in [Ful93]. □

We now turn to the proof of the theorem.

Proof. By the quotient equivalence, the categories of T-equivariant local systems on O_τ and T_τ-equivariant local systems on a point are equivalent. Hence any torsion free equivariant local system on O_τ is isomorphic to a direct sum of copies of the trivial local system $\underline{k}_\tau$. We have

$$\mathrm{Hom}^\bullet(\underline{k}_\tau, \underline{k}_\tau) = H_T^\bullet(O_\tau) = H_{T_\tau}^\bullet(\mathrm{pt})$$

which is torsion free and vanishes in odd degrees. It follows that the T-variety $X(\Delta)$ satisfies (2.1) and (2.2). By Theorem 2.12, we conclude that for each $\tau \in \Delta$ there exists at most one parity sheaf $\mathcal{E}(\tau)$ supported on $V(\tau)$ and satisfying $i_\tau^* \mathcal{E}(\tau) \cong \underline{k}_\tau[d_\tau]$.

It remains to show existence. Recall the following properties of toric varieties:

(1) For $\tau \in \Delta$, $V(\tau)$ is a toric variety for T/T_τ ([Ful93, Section 3.1]).
(2) For any fan Δ there exists a refinement Δ' of Δ such that $X(\Delta')$ is quasi-projective and the induced T-equivariant morphism

$$\pi : X(\Delta') \to X(\Delta)$$

is a resolution of singularities ([Ful93, Section 2.6]).

(3) For all τ in Δ we have a Cartesian diagram (all morphisms are T-equivariant):

$$\begin{array}{ccccc} & & \overset{i'_\tau}{\frown} & & \\ O_\tau \times Z & \longrightarrow & O_\tau \times X(\Sigma, N_\tau) \cong X(\Sigma, N) & \longrightarrow & X(\Delta') \\ \downarrow{\scriptstyle \pi'} & & \downarrow & & \downarrow{\scriptstyle \pi} \\ O_\tau \times \{\gamma_\tau\} & \longrightarrow & O_\tau \times U_{\tau, N_\tau} \cong U_{\tau,N} & \longrightarrow & X(\Delta) \\ & & \underset{i_\tau}{\smile} & & \end{array}$$

Here the $U_{\tau,N}$ and U_{τ,N_τ} denote the affine toric varieties for the cone τ in N and N_τ, while Σ denotes the fan consisting of all cones in Δ' contained in τ. The square on the left is the product of O_τ with a fibre diagram.

By (1) it suffices to show the existence of $\mathcal{E}(\tau)$ when τ is the zero cone (corresponding to the open T-orbit). For this it suffices to show that $\pi_* \underline{k}_{X(\Delta')}$ is even. In fact, as $\underline{k}_{X(\Delta')}[d_\tau]$ is self-dual and π is proper, we need only show that $\pi_* \underline{k}_{X(\Delta')}$ is $*$-even.

By proper base change we have $i_\tau^* \pi_* \underline{k}_{X(\Delta')} \cong \pi'_* \underline{k}_{O_\tau \times Z}$. Under the quotient equivalence $D_T(O_\tau) \xrightarrow{\sim} D_{T_\tau}(\mathrm{pt})$, the sheaf $\pi'_* \underline{k}_{O_\tau \times Z}$ corresponds to $\tilde{\pi}_* \underline{k}_Z \in D_{T_\tau}(\mathrm{pt})$, where $\tilde{\pi} : Z \to \mathrm{pt}$ is the projection (of T_τ-varieties). We will see in the proposition below that $\tilde{\pi}_* \underline{k}_Z$ is always $*$-even. This proves the theorem. □

Proposition 4.15. *Let $\tau \subset N_{\mathbb{Q}}$ be a full-dimensional polyhedral convex cone, U_τ the corresponding affine toric variety, and Δ' a refinement of τ such that the corresponding toric variety $X(\Delta')$ is smooth and quasi-projective. Let x_τ denote the unique T-fixed point of U_τ. Consider the Cartesian diagram:*

$$\begin{array}{ccc} Z = \pi^{-1}(x_\tau) & \longrightarrow & X(\Delta') \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \{x_\tau\} & \longrightarrow & U_\tau \end{array}$$

Then $\pi_* \underline{k}_Z \in D_T(\mathrm{pt})$ is a direct sum of equivariant constant sheaves concentrated in even degree.

Proof. It is enough to show that the T-equivariant cohomology of Z with integral coefficients is free over $H_T^\bullet(\mathrm{pt}, \mathbb{Z})$ and concentrated in even degrees. We will show that the integral cohomology of Z is free, and generated by the classes of T-stable closed subvarieties. The result then follows by the Leray-Hirsch lemma (see [Bri, proof of Theorem 4]).

We claim in fact that Z has a T-stable affine paving, which implies the result by the long exact sequence of compactly supported cohomology. The argument is a straightforward adaption of [Dan78, 10.3–10.7] (which the reader may wish to consult for further details).

As $X(\Delta')$ is assumed to be quasi-projective we can find a piecewise linear function $g : N_{\mathbb{Q}} \to \mathbb{Q}$ which is strictly convex with respect to Δ'. In other words, g is continuous, convex and for each maximal cone $\sigma \in \Delta'$, g is given on σ by $m_\sigma \in M$. The function g allows us to order the maximal cones of σ as follows: We fix a generic point $x_0 \in N_{\mathbb{Q}}$ lying in a cone of Δ' and declare that $\sigma' > \sigma$ if $m_{\sigma'}(x_0) > m_\sigma(x_0)$. If σ' and σ satisfy $\sigma' > \sigma$ and intersect in codimension 1, then their intersection is said to be a positive wall of σ. Given a maximal cone σ we define $\gamma(\sigma)$ to be the intersection of σ with all its positive walls.

It is then easy to check (remembering that $X(\Delta')$ is assumed smooth) that if we set

$$C(\sigma) = \bigsqcup_{\gamma(\sigma)\subset\omega\subset\sigma} O_\omega$$

then $C(\sigma)$ is a locally closed subset of $X(\Delta')$ isomorphic to an affine space of dimension equal to the codimension of $\gamma(\sigma)$ in $N_{\mathbb{Q}}$. Lastly note that

$$Z = \bigsqcup O_\sigma$$

where the union takes place over those cones in Δ' which are not contained in any wall of τ. Hence the order on maximal cones yields a filtration of Z by T-stable closed subspaces $\cdots \subset F_{\sigma_{i+1}} \subset F_{\sigma_i} \subset \cdots$ such that $F_{\sigma_{i+1}} \backslash F_{\sigma_i}$ is isomorphic to an affine space for all i. The result then follows. □

Remark 4.16. With notation as above, $X(\Delta')$ retracts equivariantly onto Z. With this in mind, the above arguments (together with the reduction to the quasi-projective case in [Dan78]) can be used to establish the equivariant formality (over $\mathbb{Z}$) of convex smooth toric varieties. The elegant Mayer-Vietoris spectral sequence argument of [BZ03] may then be used to identify the equivariant cohomology ring with piecewise integral polynomials on the fan. This is probably well known to experts.

4.3. **Nilpotent cones.** Let $\mathcal{N}$ denote the nilpotent cone in the Lie algebra $\mathfrak{g}$ of a connected reductive group G. The group G acts on $\mathcal{N}$ by the adjoint action and has finitely many orbits [Ric67]. In this section we discuss the existence and uniqueness of $\natural$-parity sheaves on $\mathcal{N}$ stratified by the G-orbits, considered as a G-variety. Recall that the nilpotent orbits are even dimensional (e.g., [CM93, §1.4]), so $\natural = \Diamond$. All parity sheaves will be with respect to this pariversity. For $x \in \mathcal{N}$, let $A_G(x) = G_x/G_x^0$ and $C_x = (G_x^0)^{\mathrm{red}}$, the maximal reductive quotient of G_x^0. We fix a ring of coefficients k as in Section 2.1 and assume that for all $x \in \mathcal{N}$, the torsion primes of C_x (see Section 2.6) and the order of the group $A_G(x)$ are invertible in k.

4.3.1. *Uniqueness.*

Lemma 4.17. *The parity conditions* (2.1) *and* (2.2) *are satisfied.*

Proof. For any orbit $\mathcal{O} \subset \mathcal{N}$ and $x \in \mathcal{O}$, let $\tilde{\mathcal{O}} = G/G_x^0$ and $\pi : \tilde{\mathcal{O}} \to \mathcal{O}$ be the finite Galois cover given by $gG_x^0 \mapsto g \cdot x$ with Galois group $A_G(x)$. Note that $\mathrm{Loc}_{\mathrm{f},G}(\mathcal{O})$ is equivalent to the category of $k[A_G(x)]$-modules that are free over k. Using the assumption that $|A_G(x)|$ is invertible in k, one can show that any $k[A_G(x)]$-module free over k is projective (and hence a direct summand of a direct sum of copies of the regular representation). The regular representation corresponds to the pushforward $\pi_*\underline{k}_{\tilde{\mathcal{O}}}$. It thus suffices to show that the equivariant cohomology groups of $\pi_*\underline{k}_{\tilde{\mathcal{O}}}$ are free k-modules and vanish in odd degrees. We have

$$H_G^\bullet(\mathcal{O}, \pi_*\underline{k}_{\tilde{\mathcal{O}}}) = H_G^\bullet(\tilde{\mathcal{O}}) = H_{G_x^0}^\bullet(\mathrm{pt}) = H_{C_x}^\bullet(\mathrm{pt}).$$

Using the assumption that torsion primes for C_x are invertible, we apply Theorem 2.44 to conclude that the left hand side is a free k-module and vanishes in odd degrees. □

By Theorem 2.12 we conclude that for each pair $(\mathcal{O}, \mathcal{L})$ consisting of a nilpotent orbit together with an irreducible G-equivariant local system, there is at most one parity sheaf $\mathcal{E}(\overline{\mathcal{O}}, \mathcal{L})$ with support $\overline{\mathcal{O}}$ extending $\mathcal{L}[d_{\mathcal{O}}]$.

Remark 4.18. Our restriction on the ring of coefficients can be reformulated in terms of the root datum $(\mathbf{X}, \Phi, \mathbf{Y}, \Phi^\vee)$ of G. In [Her13], Herpel defines a notion of pretty good prime: a prime p is pretty good for G if the groups $\mathbf{X}/\mathbb{Z}\Phi_1$ and $\mathbf{Y}/\mathbb{Z}\Phi_1^\vee$ have no p-torsion for all subsets $\Phi_1 \subset \Phi$. One has a chain of implications: very good $\Longrightarrow$ pretty good $\Longrightarrow$ good. The class of reductive groups for which p is pretty good is characterised by the following properties (this is a variant of [Her13, Remark 5.4]):

(1) it contains all simple groups for which p is very good;
(2) it contains GL_n for all n;
(3) it is closed under taking products, replacing G by a p-separably isogenous group, and replacing $G = H \times S$ by H if S is a torus.

Using the tables of centralisers from [Car85] and the above characterisation, one can show that a prime p is pretty good for G if and only if for all $x \in \mathcal{N}$, p is not a torsion prime for C_x and does not divide the order of $A_G(x)$.

4.3.2. *Existence.* It is known [Lus86, V, Theorem 24.8] that the intersection cohomology complexes of nilpotent orbit closures, with coefficients in any irreducible G-equivariant local system in characteristic zero, are even. Thus a similar result holds for almost all characteristics (see Proposition 2.41). However, work still needs to be done to determine precise bounds on p for parity sheaves to exist, resp. to be perverse, resp. to be intersection cohomology sheaves. In what follows, we begin to address these questions.

Springer's resolution $\pi : \widetilde{\mathcal{N}} := G \times^B \mathfrak{u} \to \overline{\mathcal{O}}_{\mathrm{reg}} = \mathcal{N}$ is semi-small and even [DCLP88] (here B is a fixed Borel subgroup of G with unipotent radical U and $\mathfrak{u} = \mathrm{Lie}\, U$). Thus $\mathcal{E}(\overline{\mathcal{O}}_{\mathrm{reg}})$ exists and is perverse. By Remark 3.11, we also have existence of $\mathcal{E}(\overline{\mathcal{O}}, \mathcal{L})$ for all pairs appearing with non-zero multiplicity in the direct image $\pi_* \underline{k}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$. By semi-smallness, all of these are perverse. We remark that if $|W|$ is invertible in k, then those pairs are "the same as in characteristic zero."

In the case $G = GL_n$, every orbit $\mathcal{O}$ is equivariantly simply connected and there is a natural G-equivariant semi-small resolution of singularities of $\overline{\mathcal{O}}$ whose fibres admit affine pavings [BO11]. It follows that there exists a perverse parity sheaf $\mathcal{E}(\overline{\mathcal{O}})$ with support $\overline{\mathcal{O}}$ for any k as above.

For a nilpotent orbit $\mathcal{O}$ in an arbitrary connected reductive group, let us recall how to construct a "standard" resolution of $\overline{\mathcal{O}}$ [Pan91]. Let x be an element of $\mathcal{O} \cap \mathfrak{u}$. By the Jacobson-Morozov theorem, there is an $\mathfrak{sl}_2$-triple (x, h, y) in $\mathfrak{g}$. The semi-simple element h induces a grading on $\mathfrak{g}$, and we can choose the triple so that all the simple root vectors have degree 0, 1 or 2. Let P be the standard parabolic subgroup of G corresponding to the set of simple roots with degree zero. Then there is a resolution of the form $\pi_{\mathcal{O}} : \widetilde{\mathcal{N}}_{\mathcal{O}} \to \overline{\mathcal{O}}$, where $\widetilde{\mathcal{N}}_{\mathcal{O}} = G \times^P \mathfrak{g}_{\geq 2}$ is a G-equivariant subbundle of $T^*(G/P) = G \times^P \mathfrak{u}_P$ (here $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical U_P of P), and $\pi_{\mathcal{O}}$ is the restriction of the moment map. To settle the question of existence for $\mathcal{E}(\overline{\mathcal{O}}, k)$ in general, one is led to the following problem.

Question 4.19. Is the resolution $\pi_{\mathcal{O}} : \widetilde{\mathcal{N}}_{\mathcal{O}} \to \overline{\mathcal{O}}$ even for any coefficients?

Given any parabolic subgroup P of G with Lie algebra $\mathfrak{p}$, and any P-stable ideal $\mathfrak{i} \subset \mathfrak{p}$, consider the natural morphism $\pi_{P,\mathfrak{i}} : G \times^P \mathfrak{i} \to \mathfrak{g}$. Fresse recently proved that if G is of classical type then $\pi_{P,\mathfrak{i}}$ is even [Fre13]. This answers our question positively in this case, taking P as above and $\mathfrak{i} = \mathfrak{g}_{\geq 2}$. In particular, it follows that there exists a parity extension for a constant local system on any nilpotent orbit of a classical group. In this way one actually constructs parity sheaves for a possibly larger set of local systems, but probably not more than those arising in the characteristic zero Springer correspondence [Som06, Conjecture 6.3].

4.3.3. *Minimal singularities.* Suppose that G is simple. Then there is a unique minimal (non-trivial) nilpotent orbit in $\mathfrak{g}$. We denote it by $\mathcal{O}_{\min}$. It is of dimension $d := 2h^\vee - 2$, where $h^\vee$ is the dual Coxeter number [Wan99]. We conclude by studying the singularity $\overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$. In this section we construct an indecomposable G-equivariant parity extension of the constant sheaf $\underline{k}[d]$ on $\mathcal{O}_{\min}$.

Consider the resolution of singularities

$$\pi : E := G \times^P \mathbb{C}x_{\min} \longrightarrow \overline{\mathcal{O}}_{\min} = \mathcal{O}_{\min} \cup \{0\}$$

where $x_{\min}$ is a highest weight vector of the adjoint representation and P is the parabolic subgroup of G stabilising the line $\mathbb{C}x_{\min}$. It is an isomorphism over $\mathcal{O}_{\min}$, and the fibre above 0 is the null section, isomorphic to G/P, which has even cohomology. Hence π is an even resolution, and so $\pi_*\underline{k}_E[d]$ is even.

Remark 4.20. The construction above works for any k (in fact for any commutative ring). However, the uniqueness theorem 2.12 does not apply unless we restrict to a k for which (2.1) and (2.2) hold. Rather than restrict to such k, we work here in the more general setting where parity sheaves may not be defined uniquely and so we can only discuss indecomposable parity complexes. One reason for doing this is that the singularities $\overline{\mathcal{O}}_{\min}$ arise in the affine Grassmannian where the parity conditions are satisfied for a larger class of coefficients.

We begin with a general lemma for isolated singularities.

Lemma 4.21. *Suppose $X = U \sqcup \{0\}$ is a stratified variety (thus 0 is the only singular point). We denote by $j : U \to X$ and $i : \{0\} \to X$ the inclusions.*

(1) *Let $\mathcal{P}$ be a $*$-even complex on X whose restriction to U is perverse. Then we have a short exact sequence*

$$0 \longrightarrow {}^p j_! j^* \mathcal{P} \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_* {}^p i^* \mathcal{P} \longrightarrow 0.$$

(2) *If $\mathcal{F}$ is any perverse sheaf on X whose composition factors are one copy of $\mathbf{IC}(X, \mathbb{F})$ and N copies of $\mathbf{IC}(0, \mathbb{F})$, then $\mathcal{H}^m(\mathcal{F})_0 \simeq \mathcal{H}^m(\mathbf{IC}(X, \mathbb{F}))_0$ for all $m \leq -2$.*

Proof. We have a distinguished triangle

$$j_! j^* \mathcal{P} \longrightarrow \mathcal{P} \longrightarrow i_* i^* \mathcal{P} \xrightarrow{[1]}$$

which gives rise to a long exact sequence of perverse cohomology sheaves, which ends with

$$i_* {}^p H^{-1} i^* \mathcal{P} \longrightarrow {}^p j_! j^* \mathcal{P} \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_* {}^p i^* \mathcal{P} \longrightarrow 0.$$

Now, ${}^p H^{-1} i^* \mathcal{P}$ is identified with $(\mathcal{H}^{-1}\mathcal{P})_0$ which is zero since $\mathcal{P}$ is $*$-even. This proves (1).

For (2), we proceed by induction on N. The result is trivial for $N = 0$. Now suppose $N > 1$. There is a perverse sheaf $\mathcal{G}$ such that we have a short exact sequence of one of the two following forms:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathbf{IC}(0, \mathbb{F}) \longrightarrow 0 \tag{4.7}$$

$$0 \longrightarrow \mathbf{IC}(0, \mathbb{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \tag{4.8}$$

and we can consider the corresponding long exact sequence for the cohomology of the stalk at zero. From $\mathcal{H}^m(\mathbf{IC}(0, \mathbb{F}))_0 = 0$ for $m \leq -1$, we deduce in both cases that $\mathcal{H}^m(\mathcal{F})_0$ is isomorphic to $\mathcal{H}^m(\mathcal{G})_0$ for $m \leq -2$ (at least). The result follows by induction. □

Proposition 4.22. *The following conditions are equivalent:*

1. *there exists a perverse, parity extension of* $\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]$*;*
2. *the standard sheaf* ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d])$ *is* $*$*-even;*
3. *the standard sheaf* ${}^p j_!(\underline{\mathbb{O}}_{\mathcal{O}_{\min}}[d])$ *has torsion free stalks;*
4. *for all* $m < d$*, the cohomology group* $H^m(\mathcal{O}_{\min}, \mathbb{Z})$ *has no* p*-torsion;*
5. *the characteristic of* $\mathbb{F}$ *is not one of the primes corresponding to the type of* G *in the following table:*

A_n	B_n, C_n, D_n, F_4	G_2	E_6, E_7	E_8
$-$	2	3	$2, 3$	$2, 3, 5$

Proof. First suppose that there exists a parity complex $\mathcal{E}$ extending $\underline{\mathbb{F}}_{\mathcal{O}_{\min}}$ that is also perverse. Then both $\mathcal{E}$ and ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d])$ are perverse sheaves whose composition factors are one copy of $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ and some number of copies of $\mathbf{IC}(0, \mathbb{F})$. By Lemma 4.21 (2), we have

$$\mathcal{H}^m({}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]))_0 \simeq \mathcal{H}^m(\mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F}))_0 \simeq \mathcal{H}^m(\mathcal{E})_0$$

for $m \leq -2$. Since $({}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]))_0$ is concentrated in degrees ≤ -2, this proves that ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d])$ is $*$-even. Thus (1) $\Longrightarrow$ (2).

Now assume that ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d])$ is $*$-even. Consider the parity complex $\mathcal{P} := \pi_* \underline{\mathbb{F}}_E[d]$ defined in the discussion proceeding Remark 4.20. By Lemma 4.21 (1), we have a short exact sequence

$$0 \longrightarrow {}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \longrightarrow {}^p H^0 \mathcal{P} \longrightarrow i_* {}^p i^* \mathcal{P} \longrightarrow 0.$$

Since the extreme terms are $*$-even, we deduce that ${}^p H^0 \mathcal{P}$ is $*$-even as well. But ${}^p H^0 \mathcal{P}$ is self-dual, because $\mathcal{P}$ is. Thus ${}^p H^0 \mathcal{P}$ is parity. The short exact sequence also shows that it is an extension of $\underline{\mathbb{F}}_{\mathcal{O}_{\min}}$. Thus (2) $\Longrightarrow$ (1).

The equivalences (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5) are proved in [Jut08a, Jut08b]. Briefly, the stalk ${}^p \mathcal{J}_!(\overline{\mathcal{O}}_{\min}, \mathbb{Z}_p)_0$ is given by a shift of $H^*(\mathcal{O}_{\min}, \mathbb{Z}_p)$ truncated in degrees $\leq d-2$, and $H^{d-1}(\mathcal{O}_{\min}, \mathbb{Z}) = 0$, so (3) $\Longleftrightarrow$ (4). Now, by a case-by-case calculation [Jut08a], one finds that (4) $\Longleftrightarrow$ (5).

The vanishing $H^{d-1}(\mathcal{O}_{\min}, \mathbb{O}) = 0$ implies that ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) = \mathbb{F} \otimes^L_{\mathbb{O}} {}^p j_!(\underline{\mathbb{O}}_{\mathcal{O}_{\min}}[d])$ by [Jut09]. Thus (2) $\Longleftrightarrow$ (3) by Proposition 2.37. □

Finally, let us recall from [Jut09] when the standard sheaf is equal to the intersection cohomology sheaf for a minimal singularity.

Proposition 4.23. *Let* Φ *denote the root system of* G*, with some choice of positive roots. Let* Φ' *denote the root subsystem of* Φ *generated by the long simple roots. Let*

H denote the fundamental group of Φ', that is, the quotient of its weight lattice by its root lattice. We have a short exact sequence

$$0 \longrightarrow i_*(\mathbb{F} \otimes_{\mathbb{Z}} H) \longrightarrow {}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \longrightarrow \mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F}) \longrightarrow 0.$$

Thus ${}^p j_!(\underline{\mathbb{F}}_{\mathcal{O}_{\min}}[d]) \simeq \mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ when the characteristic of $\mathbb{F}$ does not divide $|H|$.

Thus $\mathbf{IC}(\overline{\mathcal{O}}_{\min}, \mathbb{F})$ is an indecomposable parity complex if the characteristic of $\mathbb{F}$ does not belong to the list in Proposition 4.22 and does not divide $|H|$.

Acknowledgements

We would like to thank Alan Stapledon for help on the section about toric varieties. We would also like to thank David Ben-Zvi, Matthew Dyer, Peter Fiebig, Sebastian Herpel, Joel Kamnitzer, Frank Lübeck, David Nadler, Raphaël Rouquier, Olaf Schnürer, Eric Sommers, Tonny Springer, Catharina Stroppel, Ben Webster and Xinwen Zhu for useful discussions and comments. Finally, we thank Wilberd van der Kallen for pointing out some mistakes in a previous version and two anonymous referees for a thorough reading.

We also thank the Centre International de Rencontres Mathématiques where parts of this paper were written. The first and third authors would like to thank the Mathematical Sciences Research Institute, Berkeley and the Isaac Newton Institute, Cambridge for providing excellent research environments in which to pursue this project. The second author would also like to thank his advisor David Ben-Zvi, the geometry group at UT Austin, and David Saltman for travel support during his time as a graduate student.

References

[AM12] Pramod N. Achar, *Perverse coherent sheaves on the nilpotent cone in good characteristic*, Recent developments in Lie algebras, groups and representation theory, Proc. Sympos. Pure Math., vol. 86, Amer. Math. Soc., Providence, RI, 2012, pp. 1–23, DOI 10.1090/pspum/086/1409. MR2976995

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers* (French), Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR751966 (86g:32015)

[BBFK99] Gottfried Barthel, Jean-Paul Brasselet, Karl-Heinz Fieseler, and Ludger Kaup, *Equivariant intersection cohomology of toric varieties*, Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), Contemp. Math., vol. 241, Amer. Math. Soc., Providence, RI, 1999, pp. 45–68, DOI 10.1090/conm/241/03627. MR1718136 (2001i:14030)

[BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirković, *Tilting exercises* (English, with English and Russian summaries), Mosc. Math. J. **4** (2004), no. 3, 547–557, 782. MR2119139 (2006a:14022)

[BJ01] Michel Brion and Roy Joshua, *Vanishing of odd-dimensional intersection cohomology. II*, Math. Ann. **321** (2001), no. 2, 399–437, DOI 10.1007/s002080100235. MR1866494 (2003i:14022)

[BL94a] Arnaud Beauville and Yves Laszlo, *Conformal blocks and generalized theta functions*, Comm. Math. Phys. **164** (1994), no. 2, 385–419. MR1289330 (95k:14011)

[BL94b] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994. MR1299527 (95k:55012)

[BM83] Walter Borho and Robert MacPherson, *Partial resolutions of nilpotent varieties*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 23–74. MR737927 (85j:14087)

[BM01] Tom Braden and Robert MacPherson, *From moment graphs to intersection cohomology*, Math. Ann. **321** (2001), no. 3, 533–551, DOI 10.1007/s002080100232. MR1871967 (2003g:14030)

[BO11] Jonathan Brundan and Victor Ostrik, *Cohomology of Spaltenstein varieties*, Transform. Groups **16** (2011), no. 3, 619–648, DOI 10.1007/s00031-011-9149-2. MR2827037

[Bor61] Armand Borel, *Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes* (French), Tôhoku Math. J. (2) **13** (1961), 216–240. MR0147579 (26 #5094)

[Bri] *Geometry of toric varieties*, Séminaires et Congrès [Seminars and Congresses], vol. 6, Société Mathématique de France, Paris, 2002. Lectures from the Summer School held in Grenoble, June 19–July 7, 2000; Edited by Laurent Bonavero and Michel Brion. MR2072676 (2005a:14001)

[BZ03] Jean-Luc Brylinski and Bin Zhang, *Equivariant Todd classes for toric varieties* (2003), available at `arXiv/0311318`.

[Car85] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1985. Conjugacy classes and complex characters; A Wiley-Interscience Publication. MR794307 (87d:20060)

[CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR1433132 (98i:22021)

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR2810322 (2012g:14094)

[CM93] David H. Collingwood and William M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. MR1251060 (94j:17001)

[CPS93] Edward Cline, Brian Parshall, and Leonard Scott, *Abstract Kazhdan-Lusztig theories*, Tohoku Math. J. (2) **45** (1993), no. 4, 511–534, DOI 10.2748/tmj/1178225846. MR1245719 (94k:20079)

[Dan78] V. I. Danilov, *The geometry of toric varieties* (Russian), Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247. MR495499 (80g:14001)

[DCLP88] C. De Concini, G. Lusztig, and C. Procesi, *Homology of the zero-set of a nilpotent vector field on a flag manifold*, J. Amer. Math. Soc. **1** (1988), no. 1, 15–34, DOI 10.2307/1990965. MR924700 (89f:14052)

[dCM02] Mark Andrea A. de Cataldo and Luca Migliorini, *The hard Lefschetz theorem and the topology of semismall maps* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 5, 759–772, DOI 10.1016/S0012-9593(02)01108-4. MR1951443 (2003m:32026)

[dCM05] Mark Andrea A. de Cataldo and Luca Migliorini, *The Hodge theory of algebraic maps* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 5, 693–750, DOI 10.1016/j.ansens.2005.07.001. MR2195257 (2007a:14016)

[Dem73] Michel Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion* (French), Invent. Math. **21** (1973), 287–301. MR0342522 (49 #7268)

[Fei82] Walter Feit, *The representation theory of finite groups*, North-Holland Mathematical Library, vol. 25, North-Holland Publishing Co., Amsterdam, 1982. MR661045 (83g:20001)

[Fre13] Lucas Fresse, *Upper triangular parts of conjugacy classes of nilpotent matrices with finite number of B-orbits*, J. Math. Soc. Japan **65** (2013), no. 3, 967–992. MR3084986

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037 (94g:14028)

[FW] Peter Fiebig and Geordie Williamson, *Parity sheaves, moment graphs and the p-smooth locus of Schubert varieties*, available at `arXiv:1008.0719`. To appear in Ann. Inst. Fourier.

[GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. **131** (1998), no. 1, 25–83, DOI 10.1007/s002220050197. MR1489894 (99c:55009)

[GL05] S. Gaussent and P. Littelmann, *LS galleries, the path model, and MV cycles*, Duke Math. J. **127** (2005), no. 1, 35–88, DOI 10.1215/S0012-7094-04-12712-5. MR2126496 (2006c:20092)

[GM88] Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988. MR932724 (90d:57039)

[Gro99] Ian Grojnowski, *Affine $\mathfrak{sl}_p$ controls the representation theory of the symmetric groups and related Hecke algebras* (1999), available at `arXiv:math/9907129`.

[Her13] Sebastian Herpel, *On the smoothness of centralizers in reductive groups*, Trans. Amer. Math. Soc. **365** (2013), no. 7, 3753–3774, DOI 10.1090/S0002-9947-2012-05745-X. MR3042602

[JMW] Carl Mautner Daniel Juteau and Geordie Williamson, *Parity sheaves and tilting modules*. In preparation.

[Jut08a] Daniel Juteau, *Cohomology of the minimal nilpotent orbit*, Transform. Groups **13** (2008), no. 2, 355–387, DOI 10.1007/s00031-008-9009-x. MR2426135 (2009g:14058)

[Jut08b] Daniel Juteau, *Modular representations of reductive groups and geometry of affine Grassmannians* (2008), available at `arXiv:0804.2041`.

[Jut09] Daniel Juteau, *Decomposition numbers for perverse sheaves* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **59** (2009), no. 3, 1177–1229. MR2543666 (2011a:14028)

[Kac85] V. G. Kac, *Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups*, Invent. Math. **80** (1985), no. 1, 69–79, DOI 10.1007/BF01388548. MR784529 (86m:57041)

[KL80] David Kazhdan and George Lusztig, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203. MR573434 (84g:14054)

[KS94] Masaki Kashiwara and Pierre Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel. MR1074006 (92a:58132)

[Kum02] Shrawan Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002. MR1923198 (2003k:22022)

[LC07] Jue Le and Xiao-Wu Chen, *Karoubianness of a triangulated category*, J. Algebra **310** (2007), no. 1, 452–457, DOI 10.1016/j.jalgebra.2006.11.027. MR2307804 (2007k:18016)

[Lus86] George Lusztig, *Character sheaves. IV*, Adv. in Math. **59** (1986), no. 1, 1–63, DOI 10.1016/0001-8708(86)90036-8. MR825086 (87m:20118b)

[Mak13] Ruslan Maksimau, *Canonical basis, KLR-algebras and parity sheaves* (2013), available at `arXiv:1301.6261`.

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143, DOI 10.4007/annals.2007.166.95. MR2342692 (2008m:22027)

[Nad05] David Nadler, *Perverse sheaves on real loop Grassmannians*, Invent. Math. **159** (2005), no. 1, 1–73, DOI 10.1007/s00222-004-0382-3. MR2142332 (2006c:14021)

[Pan91] D. I. Panyushev, *Rationality of singularities and the Gorenstein property of nilpotent orbits* (Russian), Funktsional. Anal. i Prilozhen. **25** (1991), no. 3, 76–78, DOI 10.1007/BF01085494; English transl., Funct. Anal. Appl. **25** (1991), no. 3, 225–226 (1992). MR1139878 (92i:14047)

[Ric67] R. W. Richardson Jr., *Conjugacy classes in Lie algebras and algebraic groups*, Ann. of Math. (2) **86** (1967), 1–15. MR0217079 (36 #173)

[Rin92] Claus Michael Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. **208** (1991), no. 2, 209–223, DOI 10.1007/BF02571521. MR1128706 (93c:16010)

[RS65] M. Rothenberg and N. E. Steenrod, *The cohomology of classifying spaces of H-spaces*, Bull. Amer. Math. Soc. **71** (1965), 872–875. MR0208596 (34 #8405)

[RSW] Simon Riche, Wolfgang Soergel, and Geordie Williamson, *Modular Koszul duality*, available at `arXiv:1209.3760`. To appear in Compos. Math.

[Ser67] Jean-Pierre Serre, *Représentations linéaires des groupes finis* (French), Hermann, Paris, 1967. MR0232867 (38 #1190)

[Soe00] Wolfgang Soergel, *On the relation between intersection cohomology and representation theory in positive characteristic*, J. Pure Appl. Algebra **152** (2000), no. 1-3, 311–335, DOI 10.1016/S0022-4049(99)00138-3. Commutative algebra, homological algebra and representation theory (Catania/Genoa/Rome, 1998). MR1784005 (2001k:20098)

[Soe01] Wolfgang Soergel, *Langlands' philosophy and Koszul duality*, Algebra—representation theory (Constanta, 2000), vol. 28, Kluwer Acad. Publ., Dordrecht, 2001.MR1858045 (2002j:22019)

[Som06] Eric N. Sommers, *Equivalence classes of ideals in the nilradical of a Borel subalgebra*, Nagoya Math. J. **183** (2006), 161–185. MR2253889 (2007h:17010)

[Spr82] T. A. Springer, *Quelques applications de la cohomologie d'intersection* (French), Bourbaki Seminar, Vol. 1981/1982, Astérisque, vol. 92, Soc. Math. France, Paris, 1982, pp. 249–273. MR689533 (85i:32016b)

[Ste75] Robert Steinberg, *Torsion in reductive groups*, Advances in Math. **15** (1975), 63–92. MR0354892 (50 #7369)

[VW] Kari Vilonen and Geordie Williamson, *Characteristic cycles and decomposition numbers*, available at `arXiv:1208.1198`. To appear in Math. Res. Let.

[Wan99] Weiqiang Wang, *Dimension of a minimal nilpotent orbit*, Proc. Amer. Math. Soc. **127** (1999), no. 3, 935–936, DOI 10.1090/S0002-9939-99-04946-1. MR1610801 (99f:17010)

[WB12] Geordie Williamson and Tom Braden, *Modular intersection cohomology complexes on flag varieties*, Math. Z. **272** (2012), no. 3-4, 697–727, DOI 10.1007/s00209-011-0955-y. MR2995137

[Wil08] Geordie Williamson, *Singular soergel bimodules*. Ph.D. thesis, Albert-Ludwigs-Universität Freiburg, 2008, available at http://www.freidok.uni-freiburg.de.

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