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FUNDAMENTAL GROUPS OF LINKS OF ISOLATED SINGULARITIES

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Starting with Grothendieck's proof of the local version of the Lefschetz hyperplane theorems [Gro68], it has been understood that there are strong parallels between the topology of smooth projective varieties and the topology of links of isolated singularities. This relationship was formulated as one of the guiding principles in the monograph [GM88, p. 26]: "Philosophically, any statement about the projective variety or its embedding really comes from a statement about the singularity at the point of the cone. Theorems about projective varieties should be consequences of more general theorems about singularities which are no longer required to be conical."

The aim of this note is to prove the following, which we consider to be a strong exception to this principle.

Theorem 1. *For every finitely presented group G there is an isolated, 3-dimensional, complex singularity $\big(0 \in X_G\big)$ with link L_G such that $\pi_1\big(L_G\big) \cong G$.*

By contrast, the fundamental groups of smooth projective varieties are rather special; see [ABC$^+$96] for a survey. Even the fundamental groups of smooth quasi-projective varieties are quite restricted [Mor78, KM98a, CS08, DPS09]. This shows that germs of singularities can also be quite different from quasi-projective varieties.

We think of a complex singularity $(0 \in X)$ as a contractible Stein space sitting in some $\mathbb{C}^N$. Then its *link* is $\operatorname{link}(X) := X \cap \mathbb{S}^{2N-1}_\epsilon$, an intersection of X with a small $(2N-1)$-sphere centered at $0 \in X$. Thus $\operatorname{link}(X)$ is a deformation retract of $X \setminus \{0\}$.

There are at least three natural ways to attach a fundamental group to an isolated singularity $(0 \in X)$. Let $p : Y \to X$ be a resolution of the singularity with simple normal crossing exceptional divisor $E \subset Y$. (That is, the irreducible components of E are smooth and they intersect transversally.) We may assume that $Y \setminus E \cong X \setminus \{0\}$. The following 3 groups are all independent of the resolution:

- $\pi_1\big(\operatorname{link}(X)\big) = \pi_1\big(X \setminus \{0\}\big) = \pi_1\big(Y \setminus E\big)$.
- $\pi_1(Y) = \pi_1(E)$; we denote it by $\pi_1\big(\mathcal{R}(X)\big)$ to emphasize its independence of Y. These groups were first studied in [Kol93, Tak03].
- $\pi_1\big(D(E)\big)$ where $D(E)$ denotes the dual simplicial complex of E. (That is, the vertices of $D(E)$ are the irreducible components $\{E_i : i \in I\}$ of E

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and for $J \subset I$ we attach a $|J|$-simplex for every irreducible component of $\cap_{i\in J} E_i$; see Definition 41 for details.) We denote this group by $\pi_1\bigl(\mathcal{DR}(X)\bigr)$; it is actually the main object of our interest.

There are natural surjections between these groups:

$$\pi_1\bigl(\operatorname{link}(X)\bigr) = \pi_1\bigl(Y\backslash E\bigr) \twoheadrightarrow \pi_1(Y) = \pi_1\bigl(E\bigr) \twoheadrightarrow \pi_1\bigl(D(E)\bigr).$$

Usually neither of these maps is an isomorphism. The kernel of $\pi_1\bigl(Y\backslash E\bigr) \twoheadrightarrow \pi_1(Y)$ is generated by loops around the irreducible components of E, but the relations between these loops are not well understood; see [Mum61] for some computations in the 2-dimensional case.

The kernel of $\pi_1(E) \twoheadrightarrow \pi_1\bigl(D(E)\bigr)$ is generated by the images of $\pi_1(E_i) \to \pi_1(E)$. In all our examples the E_i are simply connected; thus $\pi_1(E) = \pi_1\bigl(D(E)\bigr)$. We do not investigate the difference between these two groups in general.

Our proof of Theorem 1 proceeds in two distinct steps.

Simpson showed in [Sim11, Theorem 12.1] that every finitely presented group G is the fundamental group of an irreducible, singular, projective variety. He posed the question if this irreducible variety can be chosen to have normal crossing singularities only. Our first result shows that a closely related result is true.

Theorem 2. *For every finitely presented group G there is a reducible, complex, projective surface S_G with simple normal crossing singularities only such that $\pi_1(S_G) \cong G$.*

Then we take a cone over S_G to get an affine variety $C(S_G)$ such that, essentially, $\pi_1\bigl(\mathcal{R}\bigl(C(S_G)\bigr)\bigr) \cong G$. (These cones have very non-isolated singularities and therefore it is not clear that $\pi_1\bigl(\mathcal{R}\bigl(C(S_G)\bigr)\bigr)$ really makes sense for them.) Then we use the method of [Kol11] to construct a normal singularity $(x \in X_G)$ whose tangent cone at x is $C(S_G)$. (X_G has an isolated singularity only in low dimensions; we get codimension 5 singularities in general.) In low dimensions we can also assure that $\pi_1\bigl(\mathcal{R}\bigl(X_G\bigr)\bigr) \cong G$. For some choices of S_G one can control the other two groups as well, completing the proof of Theorem 1.

It is interesting to study the relationship between the algebro-geometric properties of a singularity $(0 \in X)$ and the fundamental group of $\operatorname{link}(X)$. We prove the following results for rational singularities in Section 7:

- Let $(0 \in X)$ be a rational singularity (40). Then $\pi_1\bigl(\mathcal{DR}(X)\bigr)$ is $\mathbb{Q}$-superperfect; that is, $H_i\bigl(\pi_1\bigl(\mathcal{DR}(X)\bigr), \mathbb{Q}\bigr) = 0$ for $i = 1, 2$.
- Conversely, for every finitely presented, $\mathbb{Q}$-superperfect group G there is a 6-dimensional rational singularity $(0 \in X)$ such that $\pi_1\bigl(\mathcal{DR}(X)\bigr) = \pi_1\bigl(\mathcal{R}(X)\bigr) = \pi_1\bigl(\operatorname{link}(X)\bigr) \cong G$.
- Not every finite group G occurs as $\pi_1\bigl(\mathcal{DR}(X)\bigr)$ for a 3-dimensional rational singularity. (We do not know what happens in dimensions 4 and 5.)

3 (Open problems)**.** Theorem 1 and its proof raise many questions; here are just a few of them.

(3.1) Our examples show that links of isolated singularities are more complicated than *smooth* projective varieties. It would be interesting to explore the difference in greater detail.

(3.2) Several steps of the proof use general position arguments and it is probably impractical to follow it to get concrete examples. It would be nice to work out simpler versions in some key examples, for instance for Higman's group (as in Example 44) and to understand the geometry of the resulting singularities completely.

(3.3) We have been focusing only on the fundamental group of the dual simplicial complex $\pi_1\big(\mathcal{DR}(X)\big)$ associated to an isolated singularity $(0 \in X)$. In a subsequent paper we show that for every finite simplicial complex $\mathcal{C}$ there is a normal singularity $(0 \in X)$ such that $\mathcal{DR}(0 \in X)$ is homotopy equivalent to $\mathcal{C}$.

(3.4) Given an n-dimensional manifold (possibly with boundary) M, our constructions give a $(2n+1)$-dimensional singularity $(0 \in X)$ such that $\mathcal{DR}(X)$ is homotopy-equivalent to M. It is reasonable to expect that there is an $(n+1)$-dimensional singularity $(0 \in Z)$ such that $\mathcal{DR}(Z)$ is homeomorphic to M.

(3.5) For a complex algebraic variety X, its algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ is the profinite completion of its topological fundamental group $\pi_1(X)$. There are examples where the natural map $\pi_1(X) \to \pi_1^{\mathrm{alg}}(X)$ is not injective [Tol93, 92], but in all such known cases the image of $\pi_1(X) \to \pi_1^{\mathrm{alg}}(X)$ is infinite and very large. We now have (non-explicit) examples of isolated rational singularities such that $\pi_1(\mathrm{link}(X))$ is infinite yet $\pi_1^{\mathrm{alg}}(\mathrm{link}(X))$ is the trivial group; see Corollary 51.

(3.6) All the examples in Theorem 1 can be realized on varieties defined over $\mathbb{Q}$. Thus they have an algebraic fundamental group $\pi_1^{\mathrm{alg}}\big(\mathrm{link}(X_{\mathbb{Q}})\big)$ which is an extension of the above $\pi_1^{\mathrm{alg}}\big(\mathrm{link}(X)\big)$ and of the absolute Galois group $\mathrm{Gal}\big(\bar{\mathbb{Q}}/\mathbb{Q}\big)$. We did not investigate this extension; thus we do not have a complete description of all possible groups $\pi_1^{\mathrm{alg}}\big(\mathrm{link}(X_{\mathbb{Q}})\big)$.

1. Polyhedral complexes

A (convex) *Euclidean polyhedron* is a subset P of $\mathbb{R}^n$ given by a finite collection of linear inequalities (some of which may be strict and some not). The polyhedron P is *rational* if it can be given by linear inequalities with rational coefficients. The *dimension* of P is its topological dimension, which is the same as the dimension of its affine span $\mathrm{Span}(P)$. (Recall that the empty set has dimension -1.) Note that we allow polyhedra which are unbounded and neither open nor closed. A *face* of P is a subset of P which is given by converting some of these non-strict inequalities to equalities. Define the set $\mathrm{Faces}(P)$ to be the set of faces of P. The *interior* $\mathrm{Int}(P)$ of P is the topological interior of P in $\mathrm{Span}(P)$. Again, $\mathrm{Int}(P)$ is a Euclidean polyhedron. We will refer to $\mathrm{Int}(P)$ as an *open polyhedron*.

An (isometric) *morphism* of two polyhedra is an isometric map $f : P \to Q$ so that $f(P)$ is a face of Q. A morphism is *rational* if it is the restriction of a rational affine map.

The following definition is a variation on the one given in [Dav08], Definition A.2.12.

Definition 4. *A* (Euclidean) polyhedral complex *is a small category $\mathcal{C}$ whose objects are convex polyhedra and morphisms are their isometric morphisms satisfying the following axioms:*

(1) *For every $c_1 \in \mathrm{Ob}(\mathcal{C})$ and every face c_2 of c_1, $c_2 \in \mathrm{Ob}(\mathcal{C})$, the inclusion map $\iota : c_1 \to c_2$ is a morphism of $\mathcal{C}$.*

(2) *For every* $c_1, c_2 \in \mathrm{Ob}(\mathcal{C})$ *there exists at most one morphism* $f = f_{c_2,c_1} \in \mathrm{Mor}(\mathcal{C})$ *so that* $f(c_1) \subset c_2$. *Thus, for every* $c \in \mathrm{Ob}(\mathcal{C})$ *one defines* $c_{\leq} = \{c' \in \mathrm{Ob}(\mathcal{C}) : \exists f : c' \to c\}$.

(3) *Given* $a, b \in \mathrm{Ob}(\mathcal{C})$ *either* $a_{\leq} \cap b_{\leq} = \emptyset$ *or there exists* $c \in a_{\leq} \cap b_{\leq}$ *so that for every* $d \in a_{\leq} \cap b_{\leq}$, *there exists a morphism* $f : d \to c$.

A polyhedral complex is rational *if all its objects and morphisms are rational.*

Analogously, one defines spherical, hyperbolic, affine, projective, etc., polyhedral complexes, but we will not need these concepts. Thus, a *polyhedral complex* for us will always mean a Euclidean polyhedral complex.

Given a finite rational polyhedral complex $\mathcal{C}$, by scaling one obtains an *integral* polyhedral cell complex $\mathcal{C}'$, where the polyhedra and morphisms are integral.

Example 5. 1. *Every simplicial complex* Z *corresponds canonically to a Euclidean polyhedral complex* $\mathcal{Z}$: *Identify each* k*-simplex in* Z *with the standard Euclidean simplex in* $\mathbb{R}^{k+1}$.

2. *Conversely, if* $\mathcal{Z}$ *is a polyhedral complex where every face is a simplex, then* $\mathcal{Z}$ *is a simplicial complex (note Conditions* 2 *and* 3 *in Definition* 4*).*

Objects of a polyhedral complex $\mathcal{C}$ are called *faces* of $\mathcal{C}$ and the morphisms of $\mathcal{C}$ are called *incidence maps* of $\mathcal{C}$. A *facet* of $\mathcal{C}$ is a face P of $\mathcal{C}$ so that for every morphism $f : P \to Q$ in $\mathcal{C}$, $f(P) = Q$. A *vertex* of $\mathcal{C}$ is a zero-dimensional face. The *dimension* $\dim(\mathcal{C})$ of $\mathcal{C}$ is the supremum of dimensions of faces of $\mathcal{C}$. A polyhedral complex $\mathcal{C}$ is called *pure* if the dimension function is constant on the set of facets of $\mathcal{C}$; the constant value in this case is the dimension of $\mathcal{C}$. A *subcomplex* of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$. If c is a face of a complex $\mathcal{C}$, then $\mathrm{Res}_{\mathcal{C}}(c)$, the *residue* of c in $\mathcal{C}$, is the minimal subcomplex of $\mathcal{C}$ containing all faces c' such that there exists an incidence map $c \to c'$. For instance, if c is a vertex of $\mathcal{C}$, then its residue is the same as the *star* of c in $\mathcal{C}$; however, in general these are different concepts.

We generate the equivalence relation $\sim$ on a polyhedral complex $\mathcal{C}$ by declaring that $c \sim f(c)$, where $c \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Mor}(\mathcal{C})$. This equivalence relation also induces the equivalence relation $\sim$ on points of faces of $\mathcal{C}$.

If $\mathcal{C}$ is a polyhedral complex, its *poset* $\mathrm{Pos}(\mathcal{C})$ is the partially ordered set $\mathrm{Ob}(\mathcal{C})$ with the relation $c_1 \leq c_2$ iff $c_1 \sim c_0$ so that $\exists f \in \mathrm{Mor}(\mathcal{C}), f : c_0 \to c_2$.

We define the *underlying space* or *amalgamation* $|\mathcal{C}|$ of a polyhedral complex $\mathcal{C}$ as the topological space which is obtained from the disjoint union

$$\amalg_{c \in \mathrm{Ob}(\mathcal{C})}\ c$$

by identifying points using the equivalence relation: $\sim$. We equip $|\mathcal{C}|$ with the quotient topology.

Definition 6. *Let* $\mathcal{B}$ *be a subcomplex of a polyhedral complex* $\mathcal{C}$. *For* $c \in \mathrm{Ob}(\mathcal{C})$ *define the polyhedron*

$$c' := c \Big\backslash \bigcup_{b \leq c, b \in \mathcal{B}} f(b), \ \textit{where } f : b \to c, f \in \mathrm{Mor}(\mathcal{C}).$$

For a morphism $f \in \mathrm{Mor}(\mathcal{C})$, $f : c_1 \to c_2$, *we set* $f' : c_1' \to c_2'$ *as the restriction of* f. *We define the* difference complex $\mathcal{C} - \mathcal{B}$ *as the following polyhedral complex:*

$$\mathrm{Ob}(\mathcal{C} - \mathcal{B}) = \{c' : c \in \mathrm{Ob}(\mathcal{C})\},$$
$$\mathrm{Mor}(\mathcal{C} - \mathcal{B}) = \{f' : c_1' \to c_2', \textit{where } f \in \mathrm{Mor}(\mathcal{C}), f : c_1 \to c_2\}.$$

Note that $\mathcal{C} - \mathcal{B}$ need not be a subcomplex of $\mathcal{C}$.

We will be exclusively interested in finite Euclidean polyhedral complexes (i.e., complexes of finite cardinality), where the underlying space $|\mathcal{C}|$ is connected.

Example 7. *For us the most important Euclidean polyhedral complexes are obtained by subdividing (a domain in) $\mathbb{R}^N$ into convex polyhedra. Let $U \subset \mathbb{R}^N$ be an open subset and $\mathcal{V}$ be a partition of U in convex Euclidean polyhedra so that*

(1) *For every $c_1 \in \mathcal{V}$ and every face c_2 of c_1, $c_2 \in \mathcal{V}$.*

(2) *For every two polyhedra c_1, $c_2 \in \mathcal{V}$, $c_1 \cap c_2 \in \mathcal{V}$.*

Then $\mathcal{V}$ becomes the set of faces of a polyhedral complex (again denoted by $\mathcal{V}$ by abusing the notation), where inclusions of faces are the incidence maps.

Example 8. *Let Δ^m be the closed Euclidean m-simplex. The (simplicial) cell complex of faces of Δ^m will be denoted $\mathcal{C}(\Delta^m)$.*

Definition 9. *Let $\mathcal{C}$ be a pure n-dimensional polyhedral complex. The* nerve $\operatorname{Nerve}(\mathcal{C})$ *of $\mathcal{C}$ is the simplicial complex whose vertices are facets of $\mathcal{C}$ (the notation is $v = c^*$, where c is a facet of $\mathcal{C}$); distinct vertices $v_0 = c_0^*, \dots, v_k = c_k^*$ or* $\operatorname{Nerve}(\mathcal{C})$ *span a k-simplex if there exists a face c of $\mathcal{C}$ and incidence maps $c \to c_i$, $i = 0, \dots, k$. The simplex $\sigma = [v_0, \dots, v_k]$ then is said to be* dual *to the face c, provided that c is a maximal face satisfying the above property. We will use the notation $\sigma = c^*$.*

Example 10. *Suppose that $\mathcal{C}$ consists of three edges e_1, e_2, e_3 sharing a common vertex. Then the simplicial complex* $\operatorname{Nerve}(\mathcal{C})$ *is the face-complex $\mathcal{C}(\Delta^2)$ of the 2-dimensional simplex Δ^2: Vertices of Δ^2 are e_i^*, $i = 1, 2, 3$; edges of Δ^2 correspond to the (nonempty) intersections of pairs of distinct edges e_i, e_j; and the 2-face of Δ^2 corresponds to the triple intersection of the edges of $\mathcal{C}$.*

Lemma 11. *If $\mathcal{C}$ is finite, then $|\mathcal{C}|$ is homotopy-equivalent to* $|\operatorname{Nerve}(\mathcal{C})|$.

Proof. Notice that the image of each face of $\mathcal{C}$ in $|\mathcal{C}|$ is contractible. Therefore, one can thicken each facet c of $\mathcal{C}$ to an open contractible subset $U(c) \subset |\mathcal{C}|$ so that

(1) The collection of open sets $U(c)$ (c's are facets of $\mathcal{C}$) is a covering of $|\mathcal{C}|$.

(2) For every $k+1$-tuple of facets $c_0, \dots, c_k$, the intersection

$$U(c_0) \cap \cdots \cap U(c_k)$$

is nonempty iff $[c_0^*, \dots, c_k^*]$ is a simplex in $\operatorname{Nerve}(\mathcal{C})$.

(3) Each intersection $U(c_0) \cap \cdots \cap U(c_k)$ as above is contractible.

Now, the assertion becomes a standard fact of algebraic topology; see e.g. [Hat02]. □

Note that, in general, a face can have more than one, or no, dual simplices, except that each facet is dual to the unique vertex.

Definition 12. *A polyhedral complex $\mathcal{C}$ is* simple *if*

(1) *$\mathcal{C}$ is pure,* $\dim(\mathcal{C}) = n$,

(2) *For $k = 0, \dots, n$ and every k-face c of $\mathcal{C}$,* $\operatorname{Nerve}(\operatorname{Res}_{\mathcal{C}}(c))$ *is isomorphic to the complex $\mathcal{C}(\Delta^{n-k})$.*

It is easy to see that each face c of a simple n-dimensional complex $\mathcal{C}$ is dual to a unique simplex c^* in $\operatorname{Nerve}(\mathcal{C})$. Moreover, $\dim(c) + \dim(c^*) = n$.

Lemma 13. *If $\mathcal{A}$ is a simple polyhedral complex and $\mathcal{B}$ is its subcomplex, then the complex $\mathcal{C} := \mathcal{A} - \mathcal{B}$ is again simple.*

Proof. It is easy to see that $\mathcal{C}$ is purely of the same dimension as $\mathcal{A}$ and for each face c of $\mathcal{C}$, the poset of $\mathrm{Res}_{\mathcal{C}}(c)$ is isomorphic to the poset $\mathrm{Res}_{\mathcal{A}}(a)$, where $c = a'$ (see Definition 6). $\square$

2. Voronoi complexes in $\mathbb{R}^N$

In what follows, we will use the notation d for the Euclidean metric on $\mathbb{R}^N$.

Definition 14. Let $Y \subset \mathbb{R}^N$ be a finite subset. The *Voronoi tessellation* $\mathcal{V}(Y)$ of $\mathbb{R}^N$ associated with Y is defined by: For each $y \in Y$ take the *Voronoi cell*

$$V(y) := \{x \in \mathbb{R}^N : d(x,y) \le d(x,y'), \forall y' \in Y\}.$$

Thus, each cell $V(y)$ is given by the collection of non-strict linear inequalities $d(x,y) \le d(x,y')$, i.e.,

$$2(y'-y)\cdot x \le y'\cdot y' - y\cdot y.$$

Then each cell $V(y)$ is a closed (possibly unbounded) polyhedron in $\mathbb{R}^N$. Every $V(y)$ is rational provided that $Y \subset \mathbb{Q}^N$. The union of Voronoi cells is the entire $\mathbb{R}^N$. We thus obtain the polyhedral complex, called the *Voronoi complex*, $\mathcal{V}(Y)$ using the faces $V(y)$ as in Example 7.

Not every Voronoi complex is simple, but most of them are. In order to make this precise, we consider ordered finite subsets of $\mathbb{R}^N$; thus, every k-element subset becomes a point in $\mathbb{R}^{kN}$.

Lemma 15. *k-element subsets $Y \subset \mathbb{R}^N$ (resp. $Y \subset \mathbb{Q}^N$) whose Voronoi complex $\mathcal{V}(Y)$ is simple are open and dense in $\mathbb{R}^{kN}$ (resp. $\mathbb{Q}^{kN}$).*

Proof. For a subset $Y \subset \mathbb{R}^N$, failure of simplicity of $\mathcal{V}(Y)$ means that there exists an m-element subset $W \subset Y$ so that the set of affine hyperplanes

$$H_{y_i,y_j} = \{x | d(y_i,x) = d(y_j,x)\}, y_i, y_j \in W$$

has intersection of dimension $> m-1$; equivalently, the points of W lie on an affine subspace of dimension $< m-1$. The subset $\Sigma_{m,N} \subset \mathbb{R}^{mN}$ of such W's is closed and has an empty interior. The lemma follows from the density of rational k-element subsets. $\square$

Delaunay triangulations. Dually, one defines the *Delaunay simplicial complex*

$$\mathcal{D}(Y) = \mathrm{Nerve}(\mathcal{V}(Y)),$$

i.e., vertices of this complex are points of Y, vertices $y_0, \dots, v_k$ span a k-simplex in $\mathcal{D}(Y)$ iff $\cap_{i=0}^{k} D(y_i) \neq \emptyset$. We have the canonical affine map $\eta : \mathcal{D}(Y) \to \mathbb{R}^N$ which is the identity on Y.

The proof of the following theorem can be found in [For97, Thm. 2.1].

Theorem 16. 1. *If $\mathcal{V}(Y)$ is simple, then the map $\eta : |\mathcal{D}(Y)| \to \mathbb{R}^N$ injective.*
2. *The image of the latter map is the closed convex hull* $\mathrm{Hull}(Y)$ *of the set Y.*

The Euclidean simplicial complex $\eta(\mathcal{D}(Y))$ is called the *Delaunay triangulation* of $\mathrm{Hull}(Y)$ with the vertex set Y.

Voronoi complexes associated with smooth submanifolds in $\mathbb{R}^N$. Let M be a subset of $\mathbb{R}^N$ and $\epsilon > 0$. A set $Y \subset \mathbb{R}^N$ is said to be ϵ-dense in M if every point $x \in M$ is within distance $< \epsilon$ from a point of Y. (Note that Y need not be contained in M.) By compactness and Lemma 15, for every bounded subset of $\mathbb{R}^N$, $\epsilon > 0$, there exists a finite simple rational subset $Y \subset \mathbb{R}^N$ which is ϵ-dense in M.

Theorem 17 (S. S. Cairns [Cai61])**.** *Let M be a C^2-smooth closed submanifold of $\mathbb{R}^N$. Then there exists $\epsilon_M > 0$ so that for every $\epsilon \in (0, \epsilon_M]$ the following holds.*

Let Y be a finite subset of $\mathbb{R}^N$ which is ϵ-dense in M. For every face $c \in \mathcal{V}(Y)$ define the set $c_M := c \cap M$.

Then each c_M is a (topological) cell in M and the collection of cells c_M, $c \in \mathcal{V}(Y)$, is a cellulation of M.

Note that for a generic choice of Y the intersections c_M are transversal and, hence,

$$\dim(c_M) = \dim(c) + \dim(M) - N.$$

Examination of Cairns's proof of this theorem shows that it can be repeated verbatim to prove the following.

Theorem 18. *Let S be a compact codimension 0 submanifold of $\mathbb{R}^N$ with C^2-smooth boundary M. Then for $\epsilon_M > 0$ as above, and for every $\epsilon \in (0, \epsilon_M]$ the following holds.*

Let Y be a finite subset of $\mathbb{R}^N$ which is ϵ-dense in S. For every face $c \in \mathcal{V}(Y)$ define the set $c_S := c \cap S$. Then each c_S is a (topological) cell in S and the collection of cells c_S, $c \in \mathcal{V}(Y)$, is a regular cellulation $\mathcal{V}_S$ of S.

Again, for a generic choice of Y, $\dim(c_M) = \dim(c)$ unless $c_S = \emptyset$.

We observe that $\{c \in \mathcal{V}(Y) : c_S = \emptyset\}$ is the face set of a subcomplex $\mathcal{W}(Y)$ of $\mathcal{V}(Y)$. We then let $\mathcal{C}_S := \mathcal{V}(Y) - \mathcal{W}(Y)$.

Clearly, $\mathcal{C}_S$ is pure, N-dimensional and $\mathrm{Nerve}(\mathcal{C}_S)$ is isomorphic to $\mathrm{Nerve}(\mathcal{V}_S)$. Therefore, $|\,\mathrm{Nerve}(\mathcal{C}_S)|$ is homotopy-equivalent to $|\,\mathrm{Nerve}(\mathcal{V}_S)| \cong S$.

By Lemma 15, we can assume that Y is simple, rational and generic; then $\mathcal{C}_S$ is again simple, rational and $|\mathcal{C}_S|$ is still homotopy-equivalent to S.

Corollary 19. *Given $S \subset \mathbb{R}^N$ as above, there exists a simple rational polyhedral complex $\mathcal{C} = \mathcal{C}_S$ so that $|\mathcal{C}|$ is homotopy-equivalent to S.*

Faces of $\mathcal{C}_S$, in general, are not closed polyhedra.

3. Euclidean thickening of simplicial complexes

We are grateful to Frank Quinn for leading us to the following reference.

Theorem 20 (M. Hirsch [Hir62])**.** *Let Z be a finite simplicial complex in a smooth manifold X. Then there exists a codimension 0 compact submanifold $S \subset X$ with smooth boundary which is homotopy-equivalent to $|Z|$.*

Corollary 21. *For every n-dimensional finite simplicial complex Z there exists a codimension 0 compact submanifold $M \subset \mathbb{R}^{2n+1}$ with smooth boundary, homotopy-equivalent to $|Z|$.*

Proof. We give two proofs of this corollary.

1. By the Whitney embedding theorem in the context of simplicial complexes, Z embeds in $\mathbb{R}^{2n+1}$. (Start with a generic map $f : Z^{(0)} \to \mathbb{R}^{2n+1}$, and then use the affine extension of f to simplices of Z.) Now, the assertion immediately follows from Hirsch's theorem.

2. Below we give a simple and self-contained proof of this corollary communicated to us by Kevin Walker via Mike Freedman. We will think of Z as a cell complex

and will construct M by induction on skeleta of Z. Let B_v denote pairwise disjoint intervals in $\mathbb{R}$, where $v \in Z^{(0)}$, and set

$$S_0 := \amalg_v B_v.$$

Then the map $Z^{(0)} \to S_0$ sending each v to B_v is a homotopy-equivalence. Suppose that we constructed a codimension 0 submanifold $M_k \subset \mathbb{R}^{2k+1}$ with smooth boundary and a homotopy-equivalence $h : Z^{(k)} \to M_k$. Let f_i denote the the attaching maps $\partial B^{k+1} \to Z$ of $(k+1)$-cells of Z. The maps f_i define elements of $\pi_k(M_k)$. Since the dimension of the manifold M_k is $2k+1$, these homotopy classes can be realized by smoothly embedded k-sphere s_i with trivial normal bundle in $\mathbb{R}^{2k+1}$, see [KM63, §6]. *A priori*, the spheres s_i may not be even homotopic to spheres contained in the boundary of M_k. However, we replace M_k with $M_k' \subset \mathbb{R}^{2k+3}$, obtained from $M_k \times B^2 \subset \mathbb{R}^{2k+3}$ by "smoothing the corners." Then s_i can be chosen in the boundary of M_k' and, for the dimension reasons, it bounds a smoothly embedded $(k+1)$-disk in $\mathbb{R}^{2k+3} \backslash M_k'$. Then we attach the handle $H_i \cong D^{k+1} \times D^{k+2} \subset \mathbb{R}^{2k+3}$ to M_k' along s_i, so that this handle intersects M_k' only along a tubular neighborhood of s_i in $\partial M'$. Moreover, we can assume that distinct handles H_i are pairwise disjoint. We again smooth the corners after the handles are attached. Let $M_{k+1} \subset \mathbb{R}^{2k+1}$ be the codimension 0 submanifold with smooth boundary resulting from attaching these handles and smoothing the corners. Then, clearly, the homotopy-equivalence $Z^{(k)} \to M_k$ extends to a homotopy-equivalence $Z^{(k+1)} \to M_{k+1}$. ☐

By combining Hirsch's theorem with Corollary 19 we obtain the following.

Corollary 22. *Given a finite n-dimensional simplicial complex Z, there exists a finite simple $(2n+1)$-dimensional rational Euclidean polyhedral complex $\mathcal{C}$ so that $|\mathcal{C}|$ is homotopy-equivalent to $|Z|$.*

We note that the dimension of $\mathcal{C}$ in this corollary can be easily reduced to $N = 2n$. For instance, by a theorem of Stallings (see [DR93]) Z is homotopy-equivalent to a finite simplicial complex W which embeds in $\mathbb{R}^{2n}$. One can improve on this estimate even further as follows. Suppose that Z is a finite simplicial complex which admits an *immersion* $j : |Z| \to \mathbb{R}^N$. Then taking pull-back of an open regular neighborhood of $j(|Z|)$ via j one obtains an open smooth locally Euclidean N-dimensional manifold X which is homotopy-equivalent to $|Z|$.

Definition 23. *If Z is a simplicial complex, then a locally Euclidean Riemannian manifold X is called a* Euclidean thickening *of Z if there exists an embedding $|Z| \to X$ which is a homotopy-equivalence. We say that X is* rational*, resp.* integral*, if there exists a smooth atlas on X with transition maps that belong to $\mathbb{Q}^n \rtimes GL(n, \mathbb{Q})$, resp. $\mathbb{Z}^n \rtimes GL(n, \mathbb{Z})$, where $n = dim(X)$.*

Note that if X is an n-dimensional locally Euclidean manifold which admits an isometric immersion in $\mathbb{R}^n$, then X is integral.

Suppose that X is a Euclidean thickening of Z. Applying Hirsch's theorem above to the embedding $|Z| \subset X$, we obtain an N-dimensional smooth manifold with boundary $S \subset X$ homotopy-equivalent to $|Z|$. Even though such X is not isometric to $\mathbb{R}^N$ (it is typically incomplete and, moreover, need not embed in $\mathbb{R}^N$ isometrically), the arguments in the proof of Cairns's theorem 17 are local and go through if we replace $\mathbb{R}^N$ with X. We thus obtain the following.

Corollary 24. *Suppose that Z is a finite simplicial complex and X is an N-dimensional Euclidean thickening of Z. Then there exists a simple N-dimensional*

Voronoi complex $\mathcal{C}$ so that $|\mathcal{C}|$ is homotopy-equivalent to Z. Moreover, if X is rational, resp. integral, the complex $\mathcal{C}$ can be taken rational, resp. integral.

4. Complexification of Euclidean polyhedral complexes

Definition 25. Let $\mathbf{V}$ denote either the category of varieties (over a fixed field k) or the category of topological spaces.

Let $\mathcal{C}$ be a finite polyhedral complex. A $\mathbf{V}$-*complex based on* $\mathcal{C}$ is a functor Φ from $\mathcal{C}$ to $\mathbf{V}$ so that morphisms $c_i \to c_j$ go to closed embeddings $\phi_{ij} : \Phi(c_i) \hookrightarrow \Phi(c_j)$. By abuse of terminology, we will sometimes refer to the image category $im(\Phi)$ as a $\mathbf{V}$-complex based on $\mathcal{C}$.

We call the functor Φ *strictly faithful* if the following holds.

If $x_i \in \Phi(c_i)$, $x_j \in \Phi(c_j)$ and $\phi_{ik}(x_i) = \phi_{jk}(x_j)$ for some k, then there is an ℓ and $x_\ell \in \Phi(c_\ell)$ such that $\phi_{\ell i}(x_\ell) = x_i$ and $\phi_{\ell j}(x_\ell) = x_j$.

The relation $x_i \sim \phi_{ij}(x_i)$ for every i, j and $x_i \in X_i$ generates an equivalence relation on the points of $\amalg_{i\in I}\Phi(c_i)$, also denoted by $\sim$.

In the category of topological spaces, the direct limit $\lim\Phi(\mathcal{C})$ of the diagram $\Phi(\mathcal{C})$ exists and its points are identified with $\big(\amalg_{i\in I}\Phi(c_i)\big)/\sim$.

For example, suppose that Φ_{taut} is the tautological functor which identifies each face of $\mathcal{C}$ with the corresponding underlying topological space. Then $\lim\Phi_{taut}(\mathcal{C})$ is nothing but $|\mathcal{C}|$.

In general, Proposition 3.1 in [Cor92] proves the following.

Lemma 26. *Suppose that Φ is strictly faithful and $\Phi(\mathcal{C})$ consists of cell complexes and cellular maps of such complexes. Then $\pi_1\big(\lim\Phi(\mathcal{C})\big) \cong \pi_1(|\mathcal{C}|)$ provided that each $\Phi(c)$, $c \in \mathrm{Ob}(\mathcal{C})$ is 1-connected.* □

In the category of varieties direct limits usually do not exist; we deal with this question in Section 5. Thus for now assume that $\Phi(\mathcal{C})$ has a direct limit $\lim\Phi(\mathcal{C})$ in the category of varieties. There is a natural surjection

$$\big(\amalg_{i\in I}\Phi(c_i)\big)/\sim \quad\to\quad \lim\Phi(\mathcal{C}).$$

If this map is a bijection, we say that $\lim\Phi(\mathcal{C})$ is an *algebraic realization* of $|\mathcal{C}|$.

Our next goal is to describe two constructions of complexes of varieties based on polyhedral complexes.

Projectivization. Suppose that $\mathcal{C}$ is a Euclidean polyhedral complex.

We first construct a *projectivization* $\mathcal{P} = \mathcal{P}(\mathcal{C})$ of the Euclidean polyhedral complex $\mathcal{C}$. We regard each face c of $\mathcal{C}$ as a polyhedron in $\mathbb{R}^N$. The complex affine span, $\mathrm{Span}(c)$, of c is a linear subspace of $\mathbb{C}^N$. Let P_c denote its projective completion in $\mathbb{P}^N$. Note that, technically speaking, different faces c can yield the same space P_c if their affine spans are the same. To avoid these issues, we set $\mathcal{P}_c := P_c \times \{c\}$. For every morphism of $\mathcal{C}$, $f_{c_2,c_1} : c_1 \to c_2$, we have a unique linear embedding $F_{c_2,c_1} : \mathcal{P}_{c_1} \to \mathcal{P}_{c_2}$. Thus, we obtain the functor

$$\mathcal{P} : \mathcal{C} \to \text{Varieties}$$

which sends each $c \in \mathrm{Ob}(\mathcal{C})$ to $\mathcal{P}_c$ and each morphism f_{c_2,c_1} to F_{c_2,c_1}.

We refer to $\mathcal{P}$ as a *complex of projective spaces* based on the complex $\mathcal{C}$.

Observe, however, that $im(\mathcal{P})$ does not (in general) accurately capture the combinatorics of the complex $\mathcal{C}$, i.e., the corresponding functor $\Psi_{var} = \mathcal{P}$ need not be strictly faithful. For instance, in complex projective space any two hyperplanes intersect but a polyhedral complex usually has disjoint codimension 1 faces. More generally, any intersection $\cap_i P_{c_i}$ that is not equal to the projective span P_c of some face $c \in \mathrm{Ob}(\mathcal{C})$ shows that $\Psi_{var} = \mathcal{P}$ is not strictly faithful.

Definition 27 (Parasitic intersections)**.** Let $\sigma := (c_1, c_2, \dots, c_k)$ be a tuple of faces incident to a face c. Consider the intersections

$$I_{c,\sigma} := \cap_{i=1}^{k} F_{c,c_i}(\mathcal{P}_{c_i}) \subset \mathcal{P}_c$$

such that there is no face c_0 such that $I_{c,\sigma} = F_{c,c_0}(\mathcal{P}_{c_0})$ and c_0 is incident to all the $c_1, c_2, \dots, c_k$. Then the subspace $I_{c,\sigma} \subset \mathcal{P}_c$ is called a *parasitic intersection* in $\mathcal{P}_c$.

Note, however, that this collection of parasitic intersections in spaces $\mathcal{P}_c, c \in Ob(\mathcal{C})$, is not stable under applying morphisms $F_{c',c}$ and taking preimages under these morphisms.

Remark 28. *This lack of stability could cause a problem later on, as we will be blowing up parasitic intersections. After such blow-up, the collection of varieties will fail to form a complex of varieties. This means that we have to modify our notion of parasitic intersections in order to ensure stability of images and preimages.*

We thus have to *saturate* the collection of parasitic intersections using the morphisms $F_{c,c'}$. This is done as follows. Let T denote the pushout of the category $im(\mathcal{P}_{top})$, where we regard each $\mathcal{P}_b, b \in Ob(\mathcal{C})$ as a topological space, so the pushout exists. Then for each $a \in Ob(\mathcal{C})$ we have the (injective) projection map $\rho_a : \mathcal{P}_a \to T$. For each parasitic intersection $I_{c,\sigma} \subset \mathcal{P}_c$, we define

$$I_{c,\sigma,a} := \rho_a^{-1}\rho_c(I_{c,\sigma}).$$

We call such $I_{c,\sigma,a}$ a *parasitic subspace* in $\mathcal{P}_a$. It is immediate that each parasitic subspace in $\mathcal{P}_a$ is a projective space linearly embedded in $\mathcal{P}_a$. With this definition, the collection of parasitic subspaces $I_{c,\sigma,a}$ is stable under taking images and preimages of the morphisms $F_{c,c'}$.

Note that the maps ρ_c induce an embedding $\rho : |\mathcal{C}| \to T$. Furthermore, convexity of the polyhedra $c \in Ob(\mathcal{C})$ implies that if $a, b \in \mathrm{Ob}(\mathcal{C})$ and $\rho_a(a) \subset \rho_b(\mathcal{P}_b)$, then $a \le b$.

Lemma 29. 1. *All parasitic subspaces have dimension $\le N - 2$.*

2. *If c is simple and the intersection $I_{c,\sigma}$ contains $\mathcal{P}_{c'}$ for some face c' of c, then $I_{c,\sigma}$ is not parasitic.*

3. *Suppose that the complex $\mathcal{C}$ is simple. Then no parasitic subspace $I_{c,\sigma,b} \subset \mathcal{P}_b$ contains a face a of b.*

Proof. (1) is clear.

(2) Without loss of generality we may assume that $\sigma = (c_1, \dots, c_k)$ is such that $c_1, c_2, \dots, c_k, c'$ are faces of c. Hence, $I_{c,\sigma}$ also contains $P_{c'}$ and for each $i = 1, \dots, k$, P_{c_i} contains the face c'. By convexity of c it follows that $c' \subset c_i$, $i = 1, \dots, k$. Simplicity of c then implies that for the face $c_{k+1} := c_1 \cap \dots \cap c_k$ of c, we have

$$\mathrm{Span}(c_{k+1}) = \mathrm{Span}(c_1) \cap \dots \cap \mathrm{Span}(c_k).$$

Thus, $I_{c,\sigma} = P_{c_{k+1}}$ and, hence, $I_{c,\sigma}$ is not parasitic.

(3) Let $I_{c,\sigma,b} = \rho_b^{-1}\rho_c(I_{c,\sigma})$ be parasitic in $\mathcal{P}_b$. Assume that a is a face of b such that $a \subset I_{c,\sigma,b}$. Then $\rho_a(a) \subset \rho_c(I_{c,\sigma}) \subset \rho_c(\mathcal{P}_c)$ and, hence, $a \le c$. Now the assertion follows from (2). □

We next define a certain blow-up $\mathcal{BP}$ of $\mathcal{P}$ above which it eliminates parasitic subspaces. In the process of the blow-up, the projective spaces $\mathcal{P}_c \in \mathrm{Ob}(\mathrm{im}\,\mathcal{P})$ will be replaced with smooth rational varieties $B\mathcal{P}_c$ so that the linear morphisms $F_{c_2,c_1} \in \mathrm{Mor}(\mathrm{im}\,\mathcal{P})$ correspond to embeddings $bF_{c_2,c_1} : B\mathcal{P}_{c_1} \to B\mathcal{P}_{c_2}$. The varieties $B\mathcal{P}$ are obtained by a sequence of blow-ups of parasitic subspaces.

Proposition 30. *Let $\mathcal{C}$ be a simple Euclidean complex. Then there exists a strictly faithful complex of varieties $\mathcal{A} : \mathcal{C} \to \mathcal{BP}(\mathcal{C})$ based on $\mathcal{C}$ so that*

(1) *The direct limit of $im(\mathcal{A}) = \mathcal{BP}(\mathcal{C})$ exists and is a projective variety X with simple normal crossing singularities.*
(2) *X is an algebraic realization of $|\mathcal{C}|$.*
(3) *$\pi_1(X) \cong \pi_1(|\mathcal{C}|)$.*
(4) *If $\mathcal{C}$ is rational, then X is also defined over $\mathbb{Q}$.*

Proof. The complex $\mathcal{A}$ is constructed by inductive blow-up of the complex $\mathcal{P}$ above.

We proceed by induction on dimension of parasitic intersections. First, for each face $c \in \mathcal{C}$ we blow up all parasitic subspaces in $\mathcal{P}_c$ which are points. We will use the notation $B_0\mathcal{P}_c$ for the resulting smooth rational varieties, $c \in \mathrm{Faces}(\mathcal{C})$. Observe that this blow-up is consistent with linear embeddings F_{c_2,c_1}, which, therefore, extend to embeddings $b_0F_{c_2,c_1} : B_0\mathcal{P}_{c_1} \to B_0\mathcal{P}_{c_2}$. We let $\mathcal{B}_0\mathcal{P}$ denote the functor $\mathcal{C} \to Varieties$,

$$c \mapsto B_0\mathcal{P}_c, \quad f_{c_2,c_1} \mapsto b_0F_{c_2,c_1}.$$

Observe also that after the zeroth blow-up, all 1-dimensional parasitic subspaces become pairwise disjoint (as we blew up their intersection points). We, thus, can now blow up each $B_0\mathcal{P}_c$ along every 1-dimensional blown-up parasitic subspace $B_0I_{c,\sigma}$. The result is a collection of smooth rational varieties $B_1\mathcal{P}_c$, $c \in \mathcal{C}$. Again, the projective embeddings $b_0F_{c_2,c_1}$ respect the blow-up, so we also get a collection of injective morphisms $b_1F_{c_2,c_1} : B_1\mathcal{P}_{c_1} \to B_1\mathcal{P}_{c_2}$. We, therefore, continue inductively on the dimension of parasitic subspaces. After at most $N-1$ steps we obtain a complex $\mathcal{A} = \mathcal{BP}$, whose image $\mathrm{im}(\mathcal{A})$ has blown-up projective spaces $B\mathcal{P}_c$ as objects and embeddings bF_{c_2,c_1} as morphisms. Observe that the subvarieties along which we do the blow-up have dimension $\le N-2$. Moreover, we never have to blow up the entire $\mathcal{P}_c$ for any $c \in Ob(\mathcal{C})$ (see Lemma 29). Now, by the construction, the functor $\mathcal{A}$ is strictly faithful.

It is easy to see that the conditions of Proposition 33 hold for $\mathcal{A}$; the key is that the complex $\mathcal{C}$ was simple and the normality condition was satisfied by the complex $\mathcal{P}$. Therefore, the complex variety X which is the direct limit of $im(\mathcal{A})$ exists and is an algebraic realization of $|\mathcal{C}|$. We check in 34 that the variety X is projective.

By the construction, since the complex $\mathcal{C}$ is simple, the variety X has only normal crossing singularities. Since each $B\mathcal{P}_c$ is simply connected, Lemma 26 implies that $\pi_1(X) \cong \pi_1(|\mathcal{C}|)$. Lastly, if we start with a rational complex $\mathcal{C}$, then all the blow-ups are defined over $\mathbb{Q}$ and so is the direct limit X. $\square$

Note that every face $c \in \mathcal{C}$ is naturally a subset of (the real points of) $\mathcal{P}_c$. However, usually there are parasitic subspaces that intersect this image; thus, because of the corresponding blow-ups, c does not map to $B\mathcal{P}_c$. In particular, we do not get a map of $|\mathcal{C}|$ to $X(\mathbb{R})$. It is possible that, by analyzing carefully which blow-ups are necessary, one could obtain algebraic realizations $X'(\mathcal{C})$ which come with a natural map $|\mathcal{C}| \to X'(\mathcal{C})(\mathbb{R})$.

Suppose now that W is a finite, connected, simplicial complex. By Corollary 22, there exists a rational finite simple polyhedral complex $\mathcal{C}$ so that $|\mathcal{C}|$ is homotopy-equivalent to $|W|$. Thus, we conclude the following.

Theorem 31. *There exists a complex projective variety $Z = Z_{\mathcal{C}}$ defined over $\mathbb{Q}$ whose only singularities are simple normal crossings, so that $\pi_1(Z) \cong \pi_1(|W|)$.*

5. Direct limits of complexes of varieties

Example 35 below shows that direct limits need not exist in the category of varieties, not even if all objects are smooth and all morphisms are closed embeddings. By analyzing the example, we see that problems arise if some of the images $\phi_{ik}(X_i) \subset X_k$ and $\phi_{jk}(X_j) \subset X_k$ are tangent to each other but not if they are all transversal. The right condition seems to be the seminormality of the images.

Definition 32. Recall that a complex space X is called *normal* if for every open subset $U \subset X$, every bounded meromorphic function is holomorphic.

As a slight weakening, a complex space X is called *seminormal* if for every open subset $U \subset X$, every continuous meromorphic function is holomorphic.

The following are some key examples: $(x^2 = y^3) \subset \mathbb{C}^2$ and $(x^3 = y^3) \subset \mathbb{C}^2$ are not seminormal (as shown by x/y and x^2/y) but $(x^2 = y^2) \subset \mathbb{C}^2$ is seminormal.

The key property that we use is the following.

Assume that X is seminormal. Let Y be any variety (or complex analytic space) and $p : Y \to X$ be any algebraic (or complex analytic morphism) that is a homeomorphism in the Euclidean topology. Then p is an isomorphism of varieties (or of complex analytic spaces).

Proposition 33. *Let $\mathbf{X} := \{X_i : i \in I, \phi_{ij} : X_i \to X_j : (i,j) \in M\}$ be a complex of varieties based on a finite polyhedral complex $\mathcal{C}$. Assume that for each k and each $J \subset I$ the subvariety $\cup_{j\in J} \operatorname{im}(\phi_{jk}) \subset X_k$ is seminormal. Then*

(1) *the direct limit $\mathbf{X}^\infty$ exists,*
(2) *the points of $\mathbf{X}^\infty$ are exactly the equivalence classes of points of $\amalg_{i\in I} X_i$, and in particular, $\mathbf{X}^\infty$ is an algebraic realization of $|\mathcal{C}|$, and*
(3) *$\cup_{j\in J} \operatorname{im}(\phi_{j\infty}) \subset \mathbf{X}^\infty$ is seminormal for every $J \subset I$.*

Proof. The proof is by induction on $|I|$.

If there is a unique final object X_j, then $\mathbf{X}^\infty = X_j$.

If not, let X_j be a final object. Removing X_j and all maps to X_j, we get a smaller complex $\mathbf{Y}_j$. By induction it has a direct limit $\mathbf{Y}_j^\infty$.

From $\mathbf{Y}_j$ take away all the X_k that do not map to X_j and all maps to such an X_k. Again we get a smaller complex $\mathbf{Z}_j$ whose direct limit is $\mathbf{Z}_j^\infty$.

There are maps $\mathbf{Z}_j^\infty \to X_j$ and $\mathbf{Z}_j^\infty \to \mathbf{Y}_j^\infty$. We claim that these are both closed embeddings.

Both maps are clearly injective and their image is seminormal. For the first this follows from our assumption and in the second case by induction and (3). As we noted in Definition 32, these imply that these maps are closed embeddings.

Now we claim that $\mathbf{X}^\infty$ is the universal pushout

$$\begin{array}{ccc} \mathbf{Z}_j^\infty & \to & X_j \\ \downarrow & & \downarrow \\ \mathbf{Y}_j^\infty & \to & \mathbf{X}^\infty. \end{array}$$

The existence of the pushout as an algebraic space is proved in [Art70, Thm. 3.1] and as a variety in [Fer03]; see also [Kol08, Cor. 48]. If a limit of a diagram of seminormal varieties exists, it is automatically seminormal.

Finally we need to check that (3) holds. Let $W^\infty \subset \mathbf{X}^\infty$ be the union of the images of $\phi_{i\infty}(X_i)$ for $i \in J$ for some $J \subset I$. Then $W_j := W^\infty \cap X_j$, $W_j^Z := W^\infty \cap \mathbf{Z}_j^\infty$ and $W_j^Y := W^\infty \cap \mathbf{Y}_j^\infty$ are all unions of images of some of the X_i; hence these are seminormal by induction. Note that $\mathbf{Z}_j^\infty \cup W_j$ and $\mathbf{Z}_j^\infty \cup W_j^Y$ are also unions of images of some of the X_i, and hence seminormal.

To check seminormality, we may assume that all varieties are affine. Let h be a continuous meromorphic function on W^∞. We claim that the restriction of h to W_j^Z is defined and is holomorphic. Indeed, let $\pi : \bar{W}^\infty \to W^\infty$ be the normalization and $\bar{W}_j^Z \subset \bar{W}^\infty$ the reduced preimage of W_j^Z. Then $h \circ \pi$ is holomorphic and thus its restriction $(h \circ \pi)|_{\bar{W}_j^Z}$ is a holomorphic function that is constant on the fibers of $\bar{W}_j^Z \to W_j^Z$. Since W_j^Z is seminormal, this implies that $(h \circ \pi)|_{\bar{W}_j^Z}$ descends to a holomorphic function on W_j^Z. We can then extend this restriction to a holomorphic function h^Z on $\mathbf{Z}_j^\infty$.

The restriction of h to W_j (resp. W_j^Y) is a continuous meromorphic function; hence it is holomorphic since W_j (resp. W_j^Y) is seminormal.

Thus h^Z and $h|_{W_j}$ define a continuous meromorphic function on $\mathbf{Z}_j^\infty \cup W_j$. It is thus holomorphic and extends to a holomorphic function h_j on X_j. Similarly, h^Z and $h|_{W_j^Y}$ extend to a holomorphic function h_j^Y on $\mathbf{Y}_j^\infty$.

Finally h_j and h_j^Y agree on $\mathbf{Z}_j^\infty$; thus, by the universality of the pushout, they define a holomorphic function h^∞ on $\mathbf{X}^\infty$. Its restriction to W is h; hence h is also holomorphic. □

34 (Projectivity)**.** As the examples (36) or [Kol08, Example 15] show, even if the X_i are all projective and the direct limit $\mathbf{X}^\infty$ exists, the latter need not be projective. The main difficulty is the following.

Let L^∞ be a line bundle on $\mathbf{X}^\infty$. By pullback we obtain line bundles L_i and isomorphisms $L_i \cong \phi_{ij}^* L_j$ with the expected compatibility conditions. L^∞ is ample iff each L_i is ample. Conversely, giving line bundles L_i and isomorphisms $L_i \cong \phi_{ij}^* L_j$ with the expected compatibility conditions determines a line bundle L^∞ on $\mathbf{X}^\infty$. The practical difficulty is that we need to specify actual isomorphisms $L_i \cong \phi_{ij}^* L_j$; it is not enough to assume that L_i and $\phi_{ij}^* L_j$ are isomorphic. For line bundles this is not natural to do since we usually specify them only up to the fiberwise $\mathbb{C}^*$-action.

However, once we work only with subsheaves of a fixed reference sheaf F^∞, the isomorphisms are easy to specify.

More generally, let $X = \cup_i X_i$ be a scheme with irreducible components X_i. Let F be a coherent sheaf on Y. Then specifying a coherent subsheaf $G \subset F$ is equivalent to specifying coherent subsheaves $G_i \subset F|_{X_i}$ such that $G_i|_{X_i \cap X_j} = G_j|_{X_i \cap X_j}$ for every i, j. Furthermore, by the Nakayama lemma, G is locally free iff every G_i is locally free.

In our case, we start with each irreducible component identified with a $\mathbb{P}^n$ and then we blow up parasitic subvarieties. Thus each irreducible component X_i comes with a natural morphism to $\mathbb{P}^n$. For F we choose the pullback of $\mathcal{O}_{\mathbb{P}^n}(m)$ for some $m \gg 1$. These pullbacks are not ample since they are trivial on the fibers of $p_i : X_i \to \mathbb{P}^n$.

In Section 4 we construct the X_i as follows:

(1) Fix a smooth projective variety $\mathbf{P}$ with an ample line bundle L (in our case in fact $\mathbf{P} \cong \mathbb{P}^n$ and $L \cong \mathcal{O}_{\mathbb{P}^n}(1)$). Set $X_i^0 := \mathbf{P}$.
(2) If X_i^j is already defined, we pick a smooth subvariety $Z_i^j \subset X_i^j$ of dimension j and let $\pi_i^j : X_i^{j+1} \to X_i^j$ denote the blow-up of Z_i^j with exceptional divisor E_i^{j+1}.
(3) Set $X_i := X_i^n$ with morphisms $\Pi_i^j : X_i^n \to X_i^j$.

Claim 34.4. For all $m_0 \gg m_1 \gg \cdots \gg m_n > 0$, the following line bundle is ample on X_i:

$$\Big(\big(\Pi_i^0\big)^* H^{m_0}\Big)\Big(-m_1\big(\Pi_i^1\big)^*(E_i^1) - \cdots - m_{n-1}\big(\Pi_i^1\big)^*(E_i^{n-1}) - m_n E_i^n\Big).$$

Proof. Let Y be a smooth variety and $Z \subset Y$ a smooth subvariety. Let $p_Y : B_Z Y \to Y$ denote the blow-up with exceptional divisor E_Y. Let H be an ample invertible sheaf on X. Then $p_Y^* H^a(-b \cdot E_Y)$ is ample on $B_Z Y$ for $a \gg b > 0$ (cf. [Har77, Prop. II.7.10]).

Applying this inductively to the blow-ups $\pi_i^j : X_i^{j+1} \to X_i^j$ we get our claim. □

For later use, also note the following. Assume that we have $Y_1 \subset Y$ smooth and $Z_1 := Z \cap Y_1$ also smooth. Let E_{Y_1} be the exceptional divisor of $p_{Y_1} : B_{Z_1} Y_1 \to Y_1$. Then there is an identity

$$\big(p_Y^* H^a(-b \cdot E_Y)\big)|_{B_{Z_1} Y_1} = p_{Y_1}^*\big(H|_{Y_1}\big)^a\big(-b \cdot E_{Y_1}\big).$$

(34.5) All the X_i map to $\mathbf{P}$ in a compatible manner; hence we have a fixed reference map $\Pi^\infty : X^\infty \to \mathbf{P}$. For $m_0 \gg 1$ we get our reference sheaf $F^\infty := \big(\Pi^\infty\big)^* H^{m_0}$.

(34.6) For each i we have $F_i := F^\infty|_{X_i} = \big(\Pi_i^0\big)^* H^{m_0}$. For fixed $m_0 \gg m_1 \gg \cdots \gg m_n > 0$ the formula (34.4) defines a subsheaf $G_i \subset F_i$ and G_i is an ample line bundle on X_i. As we noted above, all that remains is to prove that

$$F^\infty|_{X_i \cap X_j} \supset G_i|_{X_i \cap X_j} = G_j|_{X_i \cap X_j} \subset F^\infty|_{X_i \cap X_j} \quad \forall i, j.$$

This follows from the compatibility of blow-ups with restrictions noted after the proof of (34.4). □

The next example shows that direct limits need not exist in the category of varieties.

Example 35. Start with the polyhedral subcomplex of $\mathbb{R}^2$ whose objects are

$$(0,0), (x \le 0, 0), (x \ge 0, 0), (x, y \le 0), (x, y \ge 0).$$

We try to build an algebraic realization with objects

$$\mathbb{C}^1_x, \mathbb{C}^2_{x,y}, \mathbb{C}^2_{x,z}, \mathbb{C}^3_{\mathbf{u}}, \mathbb{C}^3_{\mathbf{v}} \tag{35.1}$$

where the maps are

$$\begin{aligned}
\mathbb{C}^1_x &\to \mathbb{C}^2_{x,y} : x \mapsto (x,0)\\
\mathbb{C}^1_x &\to \mathbb{C}^2_{x,z} : x \mapsto (x,0)\\
\mathbb{C}^2_{x,y} &\to \mathbb{C}^3_{\mathbf{u}} : (x,y) \mapsto (x,y,0)\\
\mathbb{C}^2_{x,z} &\to \mathbb{C}^3_{\mathbf{u}} : (x,z) \mapsto (x,z,z^2)\\
\mathbb{C}^2_{x,y} &\to \mathbb{C}^3_{\mathbf{v}} : (x,y) \mapsto (x,y,0)\\
\mathbb{C}^2_{x,z} &\to \mathbb{C}^3_{\mathbf{v}} : (x,z) \mapsto (x+z,z,z^2).
\end{aligned} \tag{35.2}$$

These are all embeddings (even scheme theoretically).

We claim that if X is any algebraic variety and $g^i_* : \mathbb{C}^i_* \to X$ are algebraic maps from the $\mathbb{C}^i_*$ in (35.1) to X with the expected compatibility properties, then $g^1_x : \mathbb{C}^1_x \to X$ is constant. In an algebraic realization all the maps $g^i_* : \mathbb{C}^i_* \to X$ should be injective; thus this example is not an algebraic realization. (A careful analysis of the proof shows that the direct limit does not exist in the category of varieties, or even in the category of schemes of finite type. The direct limit exists in the category of schemes but it is not Noetherian.)

So, take a regular function ϕ on X. We can pull it back to $\mathbb{C}^2_{x,y}$ and $\mathbb{C}^2_{x,z}$ to get polynomials

$$\sum_{ij} a(i,j)x^i y^j \quad \text{and} \quad \sum_{ij} b(i,j)x^i z^j.$$

Next we compute these 2 ways. First we pull ϕ back to $\mathbb{C}^3_{\mathbf{u}}$. We get a polynomial

$$f(u_1,u_2,u_3) = \sum_{ijk} c(i,j,k)u_1^i u_2^j u_3^k.$$

Pull it back to $\mathbb{C}^2_{x,y}$ and $\mathbb{C}^2_{x,z}$ to get that

$$a(i,j) = c(i,j,0) \quad \text{and} \quad b(i,j) = c(i,j,0) + c(i,j-2,1) + c(i,j-4,2) + \cdots .$$

Thus we obtain that

$$a(i,0) = b(i,0) \quad \text{and} \quad a(i,1) = b(i,1) \quad \forall\, i. \tag{35.3}$$

Next we pull ϕ back to $\mathbb{C}^3_{\mathbf{v}}$. We get a polynomial

$$g(v_1,v_2,v_3) = \sum_{ijk} d(i,j,k)v_1^i v_2^j v_3^k.$$

Pull it back to $\mathbb{C}^2_{x,y}$ to get that $a(i,j) = d(i,j,0)$. The pullback to $\mathbb{C}^2_{x,z}$ involves the binomial coefficients; we are interested in the first 2 terms only:

$$b(i,0) = d(i,0,0) \quad \text{and} \quad b(i,1) = d(i,1,0) + (i+1)d(i+1,0,0).$$

Thus we obtain that

$$a(i,0) = b(i,0) \quad \text{and} \quad a(i,1) = b(i,1) - (i+1)b(i+1,0) \quad \forall\, i. \tag{35.4}$$

Comparing (35.3) and (35.4) we see that $a(i+1,0) = b(i+1,0) = 0$ for $i \geq 0$; that is, ϕ is constant on the image of $\mathbb{C}^1_x$.

Note that the same argument holds if f, g are power series; thus the problem is analytically local everywhere along $\mathbb{C}^1_x$. In fact, the problem exists already if we work with C^2-functions (that is, if $X \subset \mathbb{R}^N$ and we require $\mathbb{C}^3_{\mathbf{u}} \to X \subset \mathbb{R}^N$ and $\mathbb{C}^3_{\mathbf{v}} \to X \subset \mathbb{R}^N$ to be at least C^2).

Example 36 (Triangular pillows)**.** Take 2 copies $\mathbb{P}^2_i := \mathbb{P}^2(x_i : y_i : z_i)$ of $\mathbb{CP}^2$ and the triangles $C_i := (x_i y_i z_i = 0) \subset \mathbb{P}^2_i$. Given $c_x, c_y, c_z \in \mathbb{C}^*$ define $\phi(c_x, c_y, c_z) : C_1 \to C_2$ by $(0 : y_1 : z_1) \mapsto (0 : y_1 : c_z z_1)$, $(x_1 : 0 : z_1) \mapsto (c_x x_1 : 0 : z_1)$ and $(x_1 : y_1 : 0) \mapsto (x_1 : c_y y_1 : 0)$ and glue the 2 copies of $\mathbb{P}^2$ using $\phi(c_x, c_y, c_z)$ to get the surface $S(c_x, c_y, c_z)$. (Note that the complex of varieties constructed here violates one of our axioms: Two 2-simplices glued along their boundaries do not form a polyhedral complex as Part 3 of Definition 4 fails.)

We claim that $S(c_x, c_y, c_z)$ is projective iff the product $c_x c_y c_z$ is a root of unity. To see this, note that $\operatorname{Pic}^0(C_i) \cong \mathbb{C}^*$ and $\operatorname{Pic}^r(C_i)$ is a principal homogeneous space under $\mathbb{C}^*$ for every $r \in \mathbb{Z}$. We can identify $\operatorname{Pic}^3(C_i)$ with $\mathbb{C}^*$ using the restriction of the ample generator L_i of $\operatorname{Pic}(\mathbb{P}^2_i) \cong \mathbb{Z}$ as the base point.

The key observation is that $\phi(c_x, c_y, c_z)^* : \operatorname{Pic}^3(C_2) \to \operatorname{Pic}^3(C_1)$ is the multiplication by $c_x c_y c_z$. Thus if $c_x c_y c_z$ is an rth root of unity, then L_1^r and L_2^r glue together to an ample line bundle but otherwise $S(c_x, c_y, c_z)$ carries only the trivial line bundle.

6. Proof of Theorem 1

So far, for every finitely presented group G we have constructed (Theorem 31) a complex projective variety Z with simple normal crossing singularities such that $\pi_1(Z) \cong G$. Using any such Z, we next construct a singularity. This relies on the following result which is mostly a combination of [Kol11, Thm. 8 and Prop. 10].

Theorem 37. *Let Z be an $(n \geq 2)$-dimensional projective variety with simple normal crossing singularities only and L an ample line bundle on Z. Then for $m \gg 1$ there are germs of normal singularities $\bigl(0 \in X = X(Z, L, m)\bigr)$ with a partial resolution*

$$\begin{array}{ccc} Z & \subset & Y \\ \downarrow & & \downarrow \pi \\ 0 & \in & X \end{array} \qquad \textit{where } Y \setminus Z \cong X \setminus \{0\}$$

such that

1. *Z is a Cartier divisor in Y,*
2. *the normal bundle of Z in Y is $K_Z \otimes L^{-m}$,*
3. *if $\dim Z \leq 4$, then $(0 \in X)$ is an isolated singular point,*
4. *$\pi_1\bigl(\mathcal{R}(X)\bigr) \cong \pi_1(Z)$ and*
5. *the map $\pi_1\bigl(\operatorname{link}(X)\bigr) \twoheadrightarrow \pi_1\bigl(\mathcal{R}(X)\bigr)$ is an isomorphism if the following holds: Every irreducible component $Z_i \subset Z$ contains two smooth rational curves C_i^1, C_i^2 such that Z is smooth along the C_i^j, $(L \cdot C_i^1) = (L \cdot C_i^2)$ and $(K_Z \cdot C_i^1) = (K_Z \cdot C_i^2) + 1$.*

Proof. The first 3 claims are explicitly stated in [Kol11, Thm. 8]. To see (4) note that Z is a deformation retract of Y; hence $\pi_1(Y) \cong \pi_1(Z)$. By [Kol11, 8.3] Y has terminal singularities; hence, by [Kol93, Tak03],

$$\pi_1\bigl(\mathcal{R}(X)\bigr) \cong \pi_1\bigl(\mathcal{R}(Y)\bigr) \cong \pi_1(Y) \cong \pi_1(Z).$$

Alternatively, if $\dim Z = 2$ (which is the only case that we need here), we have a complete description of the possible singularities of Y. By [Kol11, Claim 5.10] they are of the form $(x_1 x_2 = x_3 x_4) \subset \mathbb{C}^4$ with $x_3 = 0$ defining Z. There are 2 local irreducible components of Z given by $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$. We can resolve these singularities by a single blow-up. The exceptional divisor is simply connected; hence $\pi_1\big(\mathcal{R}(Y)\big) \cong \pi_1(Y)$.

Note also that under any such blow-up the dual simplicial complex changes by getting a new vertex on the edge connecting the two local irreducible components. Thus its homeomorphism type is unchanged.

In order to prove (5) note that the kernel of $\pi_1\big(\operatorname{link}(X)\big) \twoheadrightarrow \pi_1\big(\mathcal{R}(X)\big)$ is generated by the loops around the irreducible components of Z. We use the curves C_i^j to show that these loops are trivial in $\pi_1\big(\operatorname{link}(X)\big)$.

Let N be the normal bundle of $Z \subset Y$. From (37.2) and the assumptions in (37.5) we conclude that

$$c_1(N) \cap [C_i^1] - c_1(N) \cap [C_i^2] = 1;$$

hence $c_1(N) \cap [C_i^1]$ and $c_1(N) \cap [C_i^2]$ are relatively prime.

Since Z and Y are both smooth along the C_i^j, the boundary of the normal disc bundle restricted to C_i^j is a lens space L_i^j with $|\pi_1(L_i^j)| = c_1(N) \cap [C_i^j]$. Thus, if γ_i denotes a small circle in Y around Z_i, then the order of γ_i in $\pi_1\big(\operatorname{link}(X)\big)$ divides both $c_1(N) \cap [C_i^1]$ and $c_1(N) \cap [C_i^2]$. Hence the γ_i is trivial in $\pi_1\big(\operatorname{link}(X)\big)$. $\square$

38 (Proof of Theorem 1)**.** In order to apply these results to prove Theorem 1 we start with the variety Z obtained in Theorem 31. Before we can apply Theorem 37, there are 2 issues to deal with.

First, in order to obtain an isolated singular point, we need a low dimensional variety Z. This is not a problem since, by (39), we can lower the dimension of Z to 2 without changing the fundamental group.

Second, we need to make sure that Z satisfies the assumptions in (37.5). It is easier to check this after some additional blow-ups; these again do not change the fundamental group.

For each irreducible component $Z_i \subset Z$ pick 2 points $p_i^j \in Z_i$ that are smooth on Z. We first blow up these points to get exceptional divisors E_i^j and then blow up a hyperplane $H_i \subset E_i^2$ to get an exceptional divisor F_i. Let C_i^1 be a line in E_i^1 and C_i^2 the (birational) transform of a line in E_i^2 not contained in H_i. We get a new variety $\tau : \tilde{Z} \to Z$ with irreducible components $\tilde{Z}_i \subset \tilde{Z}$.

By explicit computation we see that $(K_{\tilde{Z}} \cdot C_i^1) = (K_{\tilde{Z}} \cdot C_i^2) + 1$. Finally we take

$$\tilde{L} := \tau^* L^r \left(-\sum_i E_i^1 + 2E_i^2 + 3F_i \right).$$

We see that $\tilde{L}$ is ample for $r \gg 1$ and $(\tilde{L} \cdot C_i^1) = (\tilde{L} \cdot C_i^2)$. This completes the proof of Theorem 1. $\square$

The following is a singular version of the Lefschetz hyperplane theorem; see [GM88, Sec. II.1.2] for a stronger result and references.

Theorem 39. *Let X be a projective variety of dimension ≥ 3 with local complete intersection singularities and $H \subset X$ a general hyperplane section. Then $\pi_1(H) \cong \pi_1(X)$.*

7. Rational singularities and superperfect groups

Definition 40. A quasi-projective variety X has *rational singularities* if for one (equivalently every) resolution of singularities $p : Y \to X$ and for every algebraic (or holomorphic) vector bundle F on X, the natural maps $H^i(X,F)\to H^i(Y,p^*F)$ are isomorphisms. That is, for purposes of computing cohomology of vector bundles, X behaves like a smooth variety. See [KM98b, Sec. 5.1] for details.

Definition 41. Let $(0\in X)$ be a singularity that is not necessarily isolated and choose a resolution of singularities $p:Y\to X$ such that $E:=p^{-1}(0)$ is a simple normal crossing divisor. Let $\{E_i\subset E : i\in I\}$ be the irreducible components and for $J\subset I$ set $E_J:=\cap_{i\in J}E_i$.

The *dual simplicial complex* of E has vertices $\{v_i : i\in I\}$ indexed by the irreducible components of E. For $J\subset I$ we attach a $|J|$-simplex for every irreducible component of $\cap_{i\in J}E_i$. Thus $D(E)$ is a simplicial complex of dimension $\le \dim X-1$.

The dual simplicial complex of a singularity seems to have been known to several people but not explicitly studied until recently. The dual graph of a normal surface singularity has a long history. Higher dimensional versions appear in [Kul77, Per77, Gor80, 83] but systematic investigations were started only recently; see [Thu07, Ste08, Pay09, Pay11].

It is proved in [Thu07, Ste08] that the homotopy type of $D(E)$ is independent of the resolution $Y\to X$. As before, we denote it by $\mathcal{DR}(0\in X)$.

A possible argument runs as follows. Let $F\subset E_J$ be an irreducible component. If we blow up F, the dual simplicial complex changes by a barycentric subdivision of the $|J|$-simplex corresponding to F. If we blow up a smooth subvariety $Z\subset F$ that is not contained in any smaller $E_{J'}$, then the dual simplicial complex changes by attaching the cone over the star of the $|J|$-simplex corresponding to F. Thus in both cases, the homotopy type of $D(E)$ is unchanged and by [Wło03] this implies the general case.

If X has rational singularities, then $H^i\bigl(E,\mathcal{O}_E\bigr)=0$ for $i>0$ by [Ste83, 2.14]. By Part 1 of Lemma 42 below we conclude that $H^i\bigl(\mathcal{DR}(0\in X),\mathbb{Q}\bigr)=0$ for $i>0$, That is, $\mathcal{DR}(0\in X)$ is $\mathbb{Q}$-acyclic, as was observed in [ABW09].

Lemma 42. *Let X be a simple normal crossing variety over $\mathbb{C}$ with irreducible components $\{X_i : i\in I\}$. Let $T=D(X)$ be the dual simplicial complex of X. Then*

(1) *There are natural injections $H^r\bigl(T,\mathbb{C}\bigr)\hookrightarrow H^r\bigl(X,\mathcal{O}_X\bigr)$ for every r.*
(2) *For $J\subset I$ set $X_J:=\cap_{i\in J}X_i$ and assume that $H^r\bigl(X_J,\mathcal{O}_{X_J}\bigr)=0$ for every $r>0$ and for every $J\subset I$. Then $H^r\bigl(X,\mathcal{O}_X\bigr)=H^r\bigl(T,\mathbb{C}\bigr)$ for every r.*

Proof. The proof is essentially in [GS75, pp. 68–72]. More explicit versions can be found in [83, pp. 26–27] and [Ish85, ABW09].

Fix an ordering of I. It is not hard to check that there is an exact complex

$$0\to \mathbb{C}_X\to \sum_i \mathbb{C}_{X_i}\to \sum_{i<j}\mathbb{C}_{X_{ij}}\to\cdots$$

where the kth term is $\sum_{|J|=k}\mathbb{C}_{X_J}$ and $\mathbb{C}_{X_J}$ is the constant sheaf with support X_J. If $i\in J$, then the map $\mathbb{C}_{X_{J\setminus i}}\to\mathbb{C}_{X_J}$ is the natural restriction with a plus (resp. minus) sign if i is in odd (resp. even) position in the ordering of J.

Thus the cohomology of $\mathbb{C}_X$ is also the hypercohomology of the rest of the complex $\sum_i\mathbb{C}_{X_i}\to\sum_{i<j}\mathbb{C}_{X_{ij}}\to\cdots$. This is computed by a spectral sequence whose E_1 term is

$$\sum_{|J|=q} H^p\bigl(X_J,\mathbb{C}\bigr)\Rightarrow H^{p+q}(X,\,). \tag{42.3}$$

The key observation is that this spectral sequence degenerates at E_2. The reason is that $H^p\big(X_J,\mathbb{C}\big)$ carries a Hodge structure of weight p and there are no maps between Hodge structures of different weights.

Note also that the bottom (that is $p=0$) row of (42.3) is

$$0 \to \sum_i H^0\big(T_i,\mathbb{C}\big) \to \sum_{i<j} H^0\big(T_{ij},\mathbb{C}\big) \to \cdots$$

where $T_i \subset T$ denotes the open star of the vertex corresponding to $i \in I$ and $T_J = \cap_{i\in J} T_i$. The homology groups of this complex are exactly the $H^j\big(T,\mathbb{C}\big)$. Thus we have injections

$$H^j\big(T,\mathbb{C}\big) \hookrightarrow H^j\big(X,\mathbb{C}_X\big). \tag{42.4}$$

Similarly, there is an exact complex

$$0 \to \mathcal{O}_X \to \sum_i \mathcal{O}_{X_i} \to \sum_{i<j} \mathcal{O}_{X_{ij}} \to \cdots$$

which gives a spectral sequence whose E_1 term is

$$\sum_{|J|=q} H^p\big(X_J,\mathcal{O}_{X_J}\big) \Rightarrow H^{p+q}(X,\mathcal{O}_X). \tag{42.5}$$

By Hodge theory, the natural map from the spectral sequence (42.5) to the spectral sequence (42.3) is a split surjection; hence (42.5) also degenerates at E_2 and so

$$H^j\big(T,\mathbb{C}\big) \hookrightarrow H^j\big(X,\mathcal{O}_X\big) \tag{42.6}$$

is an injection. Under the assumptions of (2) only the bottom row of (42.5) is nonzero; hence, in this case, $H^j\big(T,\mathbb{C}\big) = H^j\big(X,\mathcal{O}_X\big)$. $\square$

In order to understand fundamental groups of links of rational singularities we need the following definition.

Definition 43. Recall that a group G is called *perfect* if it has trivial abelianization or, equivalently, if $H_1(G,\mathbb{Z})=0$. Similarly, G is called *superperfect* (see [Ber02]) if $\tilde H_i(G,\mathbb{Z})=0$ for $i\le 2$. We generalize this notion to homology with coefficients in other commutative rings R: A group G is R-perfect if $H_1(G,R)=0$; G is R-superperfect if $\tilde H_i(G,R)=0$ for $i\le 2$. (We will be interested only in the cases $R=\mathbb{Z}$ and $R=\mathbb{Q}$.)

Let W be a cell complex. Recall that by a theorem of Hopf [Hop42] the natural homomorphism $H_2(W,R)\to H_2\big(\pi_1(W),R\big)$ is surjective and its kernel (in the case $R=\mathbb{Z}$) is the image of $\pi_2(W)$ under the Hurewicz homomorphism.

Therefore, if $\tilde H_i(W,R)=0$ for $i\le 2$, then $\tilde H_i\big(\pi_1(W),R\big)=0$ for $i\le 2$.

To see surjectivity in Hopf's theorem observe the following: For $G=\pi_1(|W|)$ we let $f: W\to V = K(G,1)$ be the map inducing the isomorphism of fundamental groups. Then there exists a map of the 2-skeleta $h: V^{(2)}\to W^{(2)}$ which is a homotopy-right inverse to f. Hence, $H_2(f): H_2(W,R)\to H_2(V,R)=H_2(G,R)$ is onto for every commutative ring R.

Example 44. Higman's group $G=\langle x_i | x_i[x_i,x_{i+1}], i\in \mathbb{Z}/4\mathbb{Z}\rangle$ is perfect and infinite and contains no proper finite index subgroups [Hig51]. If W is the (2-dimensional) presentation complex of G, then, clearly, $\chi(W)=1$. Thus, $\tilde H_i(W,\mathbb{Z})=0$, $i\le 2$. In particular, G is superperfect by Hopf's theorem. Moreover, W is $K(G,1)$; see e.g. [BG04]. Thus, $\tilde H_i(G,\mathbb{Z})=0$ for all i.

Theorem 45. *Let $(0 \in X)$ be a rational singularity. Then $\pi_1\big(\mathcal{DR}(X)\big)$ is $\mathbb{Q}$-superperfect and finitely presented. Conversely, for every finitely presented $\mathbb{Q}$-superperfect group G there is a 6-dimensional rational singularity $(0 \in X)$ such that*
$$\pi_1\big(\mathcal{DR}(X)\big) = \pi_1\big(\mathcal{R}(X)\big) = \pi_1\big(\operatorname{link}(X)\big) \cong G.$$

Remark 46. (1) The singularities constructed in Theorem 45 are not isolated. Their singular locus is 1-dimensional. Away from the origin it is the simplest possible non-isolated singularity, locally given by the equation
$$\big(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0\big) \subset \mathbb{C}^7.$$
We do not know if in Theorem 45 one can get isolated singularities or not.

(2) For an arbitrary rational singularity $(0 \in X)$, the three groups $\pi_1\big(\mathcal{DR}(X)\big)$, $\pi_1\big(\mathcal{R}(X)\big)$ and $\pi_1\big(\operatorname{link}(X)\big)$ need not be isomorphic. For example, if $\dim X = 2$, then $\pi_1\big(\mathcal{DR}(X)\big) = \pi_1\big(\mathcal{R}(X)\big) = 1$ yet $\pi_1\big(\operatorname{link}(X)\big)$ can be infinite [Mum61].

As another example, let S be a fake projective plane, that is, $H_i(S, \mathbb{Z}) \cong H_i(\mathbb{CP}^2, \mathbb{Z})$ for every i yet $\pi_1(S)$ is infinite. Such surfaces were classified in [PY07]. Let $\big(0 \in C(S)\big)$ denote a cone over S. Then $\pi_1\big(\mathcal{DR}(C(S))\big) = 1$ yet $\pi_1\big(\mathcal{R}(C(S))\big) = \pi_1(S)$ is infinite.

Proof of Theorem 45. The first claim of Theorem 45 follows from the above cited results of [Ste83] and [Hop42]. In order to see the converse, for every finitely presented $\mathbb{Q}$-superperfect group G we construct below (Theorem 49) a simple 5-dimensional, $\mathbb{Q}$-acyclic, Euclidean polyhedral complex $\mathcal{C}$ whose fundamental group is isomorphic to G. Once this is done, as in the proof of Theorem 1, we obtain a 5-dimensional projective variety Z with simple normal crossing singularities such that
$$G \cong \pi_1(Z) \cong \pi_1\big(\mathcal{DR}(X)\big) \cong \pi_1\big(\mathcal{R}(X)\big) \cong \pi_1\big(\operatorname{link}(X)\big).$$
Let T be the dual simplicial complex of Z. Then, by the construction of Z, $T \cong Nerve(\mathcal{C})$. Furthermore, for $i > 0$, $0 = H^i(\mathcal{C}, \mathbb{C}) \cong H^i(T, \mathbb{C}) \cong H^i(Z, \mathcal{O}_Z)$, the latter isomorphism follows from Lemma 42. (Note that the vanishing condition in Part 2 of Lemma 42 holds since intersections Z_J are blown-up projective spaces $B\mathcal{P}_c, c \in Ob(\mathcal{C})$, by the construction of the variety Z.)

We now apply Theorem 37 and the blow-up method of (38). The proof of [Kol11, Prop. 9.1] shows that for $m \gg 1$, the resulting X is a rational singularity. As we noted, X does not have isolated singularities but they are completely described by [Kol11, Claim. 5.10]. $\square$

Our next goal is to construct polyhedral complexes $\mathcal{C}$ used in the proof of Theorem 45. The following theorem was proven by Kervaire for $R = \mathbb{Z}$, but examination of the proofs in [Ker69] and [KM63] shows that they also apply to $R = \mathbb{Q}$.

Theorem 47. *Every finitely presented R-superperfect group is isomorphic to the fundamental group of a smooth R-homology k-sphere M^k for every $k \geq 5$; here $R = \mathbb{Z}$ or $R = \mathbb{Q}$.*

Corollary 48. *Let $R = \mathbb{Z}$ or $R = \mathbb{Q}$. Then a finitely presented group G is R-superperfect if and only if there exists a 5-dimensional finite simplicial complex W so that $\pi_1(|W|) \cong G$ and $|W|$ is R-acyclic; that is, $\tilde{H}_*(W, R) = 0$.*

Proof. One direction of this corollary follows from Hopf's result above. Suppose that G is R-superperfect and finitely presented. Take the 5-dimensional homology sphere M as in Theorem 47. Since M is smooth, we can assume that it is triangulated. Remove from M the interior of a closed simplex. The result is the desired simplicial complex W. $\square$

We now estimate the dimension of Euclidean thickening of Z in Corollary 48. A rough estimate is that Z immerses in $\mathbb{R}^{10}$, since Z is 5-dimensional. One can do much better as follows. Due to the results of [KM63, §6], the 5-dimensional manifold M^5 constructed in Theorem 47 can be chosen to be *almost parallelizable*; i.e., the complement to a point p in M^5 is parallelizable. Therefore, $M^5\setminus\{p\}$ admits an immersion in $\mathbb{R}^5$; see [Phi67]. Hence, W admits a 5-dimensional thickening Y; see Section 3. If $R = \mathbb{Z}$, then one can do even better and obtain a thickening Y of W which is an open subset of $\mathbb{R}^5$; see [Liv05].

In order to reduce the dimension of X from 6 to 5 in Theorem 45 (and, thus, obtain isolated singularities) we have to impose further restrictions on the fundamental group G. Recall that a finite presentation of a group is called *balanced* if it has equal number of generators and relators. A group G is called *balanced* if it admits a balanced presentation. Suppose that G is an R-superperfect group which is the fundamental group of a 2-dimensional R-acyclic cell complex W. Without loss of generality, W has exactly one vertex; i.e., W is a presentation complex of G. Then $H_i(W, R) \cong H_i(G, R) = 0$, $i = 1, 2$. In particular, $\chi(W) = 1$. It then follows that W has the same number of edges and 2-cells. Hence, G is balanced (with the balanced presentation complex W). Hausmann and Weinberger in [HW85] constructed examples of finite superperfect groups which are not balanced; see [CHRR04] for more examples and a survey. Examples of finite $\mathbb{Q}$-superperfect groups which are not balanced are easier to construct: Take, for instance, the k-fold direct product $A_{p,k} = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z}$ where $k \geq 2$. In particular, such groups do not admit $\mathbb{Q}$-acyclic presentation complexes and they do not occur as $\pi_1\big(\mathcal{DR}(X)\big)$ for a 3-dimensional rational singularity.

Suppose that G is balanced and R-superperfect ($R = \mathbb{Z}$ or $R = \mathbb{Q}$); then there exists a smooth 4-dimensional R-homology sphere M^4 with the fundamental group G; see [Ker69]. Moreover, in Kervaire's construction one can assume that M^4 is almost parallelizable (i.e., it is a 4-dimensional spin-manifold); see [Kap04]. Thus, for such G there is a 4-dimensional Euclidean thickening of its 2-dimensional (balanced) presentation complex W. More explicitly, in view of Stallings' theorem [DR93], we can assume that W embeds in $\mathbb{R}^4$. Since W is a balanced presentation complex of a perfect group, $\chi(W) = 0$, and, hence, $b_1(W) = b_2(W) = 0$. Thus we obtain a $\mathbb{Q}$-acyclic 4-dimensional Euclidean thickening of W.

Note that our methods cannot produce a 5-dimensional variety in Theorem 45 without the balancing condition. Specifically, given a $\mathbb{Q}$-superperfect group G we would need a 4-dimensional $\mathbb{Q}$-acyclic manifold with the fundamental group G. However, one can show, repeating the arguments of [HW85], that for all but finitely many finite groups G constructed in [HW85] such a 4-dimensional manifold does not exist.

Reducing the thickening dimension to 3 is, of course, very seldom possible since it amounts to assuming that G is a 3-manifold group, which is quite rare among finitely presented groups.

By combining these observations with Corollary 22, we conclude the following.

Theorem 49. *Let G be an R-superperfect finitely presented group ($R = \mathbb{Z}$ or $R = \mathbb{Q}$). Then there exists a finite simple 5-dimensional Euclidean polyhedral* *complex $\mathcal{C}$ so that $|\mathcal{C}|$ is R-acyclic and has fundamental group isomorphic to G. Moreover, if G admits a balanced presentation, then we can take such $\mathcal{C}$ to be 4-dimensional.*

Corollary 50. *Suppose that G is a finitely presented $\mathbb{Q}$-superperfect group which admits a balanced presentation. Then in Theorem 45 one can take X which is 5-dimensional and $(0 \in X)$ as an isolated singularity.*

Corollary 51. *There exists 5-dimensional, isolated, rational singularities $(0 \in X)$ so that the group $\pi_1^{\mathrm{alg}}(\mathrm{link}(X))$ is trivial yet $\pi_1(\mathrm{link}(X))$ is infinite.*

Proof. Take Higman's group G; see Example 44. Then G clearly has balanced presentation (its presentation has four generators and four relators); the group G is also superperfect, infinite and has no non-trivial finite quotients. Now, the assertion follows from Theorem 45 and Corollary 50. $\square$

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