

# KÄHLER-EINSTEIN METRICS ON FANO MANIFOLDS. I: APPROXIMATION OF METRICS WITH CONE SINGULARITIES

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## 1. INTRODUCTION

This is the first of a series of three articles which provide proofs of results announced in [9]. Let  $X$  be a Fano manifold of complex dimension  $n$ . Let  $\lambda > 0$  be an integer and  $D$  be a smooth divisor in the linear system  $|\lambda K_X|$ . For  $\beta \in (0, 1)$  there is now a well-established notion of a Kähler-Einstein metric with a cone singularity of cone angle  $2\pi\beta$  along  $D$ . (It is often called an “edge-cone” singularity). For brevity, we will just say that  $\omega$  has cone angle  $2\pi\beta$  along  $D$ . The Ricci curvature of such a metric  $\omega$  is  $(1 - \lambda(1 - \beta))\omega$ . Our primary concern is the case of positive Ricci curvature, so we suppose throughout most of the article that  $\beta \geq \beta_0 > 1 - \lambda^{-1}$ . However our arguments also apply to the case of non-positive Ricci curvature: see Remark 3.10.

**Theorem 1.1.** *If  $\omega$  is a Kähler-Einstein metric with cone angle  $2\pi\beta$  along  $D$  with  $\beta \geq \beta_0$ , then  $(X, \omega)$  is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics with positive Ricci curvature and with diameter bounded by a fixed number depending only on  $\beta_0, \lambda$ .*

This suffices for most of our applications but we also prove a sharper statement.

**Theorem 1.2.** *If  $\omega$  is a Kähler-Einstein metric with cone angle  $2\pi\beta$  along  $D$  then  $(X, \omega)$  is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics  $\omega_i$  with  $\text{Ric}(\omega_i) \geq (1 - \lambda(1 - \beta))\omega_i$ .*

One consequence of our approximation results is that the Sobolev constant of  $(X, \omega)$  is uniformly bounded above by a constant depending only on  $\beta_0$  and  $\lambda$ . This bound has also been obtained by Jeffres, Mazzeo and Rubinstein in [15].

A Kähler-Einstein metric with cone angle  $2\pi\beta > 0$  along the divisor  $D$ , satisfies the equation of currents:

$$(1.1) \quad \text{Ric}(\omega) = (1 - (1 - \beta)\lambda)\omega + 2\pi(1 - \beta)[D],$$

where  $[D]$  is the current of integration along  $D$ . To prove our theorems, we first approximate  $[D]$  by a sequence of smooth positive forms, and solve the corresponding complex Monge-Ampère equations; then we show that this sequence of solutions converges to the initial Kähler-Einstein metric as expected. We will make this more precise in Section 2.

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We will treat the case when  $\lambda = 1$ . The general case can be done in an identical way.

In this article we fix  $\omega_0$  to be a smooth Kähler form in  $2\pi c_1(X)$ . Set the space of smooth Kähler potentials to be

$$\mathcal{H} = \{\varphi \in C^\infty(X; \mathbb{R}) : \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ in } X\}.$$

We now define for our purpose the precise notion of a *Kähler metric on  $X$  with cone angle  $2\pi\beta$*  ( $\beta \in (0, 1)$ ) *along  $D$* , following [11]. We will make use of this definition in this series of articles. We begin by recalling the definitions of the appropriate Hölder spaces. Given any point in  $D$ , we choose local holomorphic coordinates  $(u, \sigma_1, \dots, \sigma_{n-1})$  so that  $D$  is locally given by  $\{u = 0\}$ . Denote by  $\omega_{(\beta)}$  the standard cone metric  $\sqrt{-1}\beta^2|u|^{2\beta-2}du \wedge d\bar{u} + \sum_a \sqrt{-1}d\sigma_a \wedge d\bar{\sigma}_a$ . We say a function  $\phi$  is in  $\mathcal{C}^{\alpha,\beta}$  if it is  $C^\alpha$  with respect to the metric  $\omega_{(\beta)}$ . Write  $u = r^{\frac{1}{\beta}}e^{i\theta}$  and  $\epsilon = dr + \sqrt{-1}\beta r d\theta$  (this is well-defined even though  $\theta$  is only locally defined.) Then on  $\{u \neq 0\}$  we have an orthonormal basis for  $(1, 1)$  forms given by

$$\epsilon \wedge \bar{\epsilon}, \epsilon \wedge d\bar{\sigma}_a, \bar{\epsilon} \wedge d\sigma_a, d\sigma_a \wedge d\bar{\sigma}_b.$$

We say a  $(1, 1)$  form is in  $\mathcal{C}^{\alpha,\beta}$  if it can be decomposed as

$$f\epsilon \wedge \bar{\epsilon} + \sum_a f_a \epsilon \wedge d\bar{\sigma}_a + \sum_a \bar{f}_a \bar{\epsilon} \wedge d\sigma_a + \sum_{a,b} f_{ab} d\sigma_a \wedge d\bar{\sigma}_b,$$

where  $f, f_a, f_{ab}$  are all in  $\mathcal{C}^{\alpha,\beta}$ , and  $f_a$  vanishes on  $\{u = 0\}$  for all  $a$ . We say a function  $\phi \in \mathcal{H}$  is in  $\mathcal{C}^{2,\alpha,\beta}$  if both  $\phi$  and  $i\partial\bar{\partial}\phi$  are in  $\mathcal{C}^{\alpha,\beta}$ .

**Definition 1.3.** *A Kähler metric  $\omega'$  on  $X$  with cone angle  $2\pi\beta$  along  $D$  is a current in  $2\pi c_1(X)$  such that:*

- (1)  $\omega'$  is a closed positive  $(1, 1)$  current on  $X$ , and is a smooth Kähler metric in  $X \setminus D$ ;
- (2) near each point in  $D$ ,  $\omega'$  is uniformly equivalent to  $\omega_{(\beta)}$ , i.e., there is a constant  $C > 0$  so that  $C^{-1}\omega_{(\beta)} \leq \omega' \leq C\omega_{(\beta)}$ ; and
- (3) near each point in  $D$ ,  $\omega' = \omega_{(\beta)} + \sqrt{-1}\partial\bar{\partial}\phi$ , where  $\phi \in \mathcal{C}^{2,\alpha,\beta}$  for some  $\alpha \in (0, \min(1, \beta^{-1} - 1))$ .

We say that  $\omega'$  is a *Kähler-Einstein metric on  $X$  with cone angle  $2\pi\beta$  along  $D$*  if in addition,  $\omega'$  is Einstein on  $X \setminus D$ .

It is easy to see this definition does not depend on the choice of local chart. For any  $\beta \in (0, 1]$ , let  $\hat{\mathcal{H}}_\beta$  be the space of all potentials  $\varphi$  such that  $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$  is a Kähler metric on  $X$  with cone angle  $2\pi\beta$  along  $D$ . It is well known that, for any  $\varphi \in \mathcal{H}$ , and for  $\epsilon$  small enough (which may depend on  $\varphi$ ), we have

$$\varphi + \epsilon|S|_h^{2\beta} \in \hat{\mathcal{H}}_\beta,$$

where  $h$  is a smooth Hermitian metric on  $-K_X$  with Ricci curvature  $\omega_0$  and  $S$  is the defining section of  $D$ .

There are other definitions of metrics with cone singularities. One variant is to only require the uniform bound (2) in the definition. For Kähler-Einstein metrics, we believe this is equivalent to Definition 1.3 (c.f. [15] for a discussion there). Higher regularity is also studied in [15], where they stated that such metrics have an asymptotic expansion about points on the divisor. For our purpose in this series of articles, regularity beyond  $\mathcal{C}^{2,\alpha,\beta}$  is not needed.

2. PROOF OF THEOREM 1.1

Suppose  $\omega_{\varphi_\beta} = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_\beta$  is a Kähler-Einstein metric on  $X$  with cone angle  $2\pi\beta$  along a smooth anti-canonical divisor  $D$ . Then  $\omega_{\varphi_\beta}$  satisfies the following Monge-Ampère equation

$$(2.1) \quad \omega_{\varphi_\beta}^n = e^{-\beta\varphi_\beta + h_{\omega_0}} \frac{\omega_0^n}{|S|_h^{2(1-\beta)}} \quad \text{on } X \setminus D,$$

where  $h_{\omega_0}$  is the Ricci potential of  $\omega_0$  and we have chosen the normalization of  $\varphi_\beta$  so that

$$\int_X e^{-\beta\varphi_\beta + h_{\omega_0}} \frac{\omega_0^n}{|S|_h^{2(1-\beta)}} = \int_X \omega_0^n.$$

To prove Theorem 1.1, we need to achieve the following three goals simultaneously:

- (1) approximate  $\omega_{\varphi_\beta}$  by smooth Kähler metrics on  $X$  locally smoothly away from  $D$ ;
- (2) the Ricci curvature of this sequence of metrics is positive and diameter is uniformly bounded from above; and
- (3) the Gromov-Hausdorff limit of this sequence of metrics is precisely the metric  $(X, \omega_{\varphi_\beta})$ .

To achieve the first goal, we want to smooth the volume form of  $\omega_{\varphi_\beta}$  first, and then use the Calabi-Yau theorem on  $X$  to smooth the potential  $\varphi_\beta$ . Fix  $p_0 \in (1, (1 - \beta_0)^{-1})$ . Note that the volume form of  $\omega_{\varphi_\beta}$  is bounded in  $L^{p_0}$ . We can find a family of smooth volume forms  $\eta_\epsilon (\epsilon \in (0, 1])$  with  $\int_X \eta_\epsilon = \int_X \omega_0^n$ , which converge to  $\omega_{\varphi_\beta}^n$  strongly in  $L^{p_0}$  and smoothly away from  $D$ . For each  $\eta_\epsilon$ , by the Calabi-Yau theorem we can find a smooth Kähler potential  $\varphi_\epsilon \in \mathcal{H}$  such that

$$\omega_{\varphi_\epsilon}^n = \eta_\epsilon.$$

Following [16], we obtain a uniform bound on  $\|\varphi_\epsilon\|_{C^\gamma(X)}$  for some  $\gamma \in (0, 1)$ . This bound and  $\gamma$  depend only on  $X, D, \omega_0$  and the  $L^{p_0}$  norm of  $\frac{\omega_{\varphi_\beta}^n}{\omega_0^n}$ . Furthermore,  $\{\varphi_\epsilon\}$  converges by sequence to  $\varphi_\beta$  in  $C^{\gamma'}(X)$  for some  $\gamma'$  slightly smaller than  $\gamma$ .

To achieve our second goal, we need to modify the volume form to secure positive Ricci curvature. Following Yau [25], we can solve the following equation for  $\epsilon \in (0, 1]$ :

$$(2.2) \quad \omega_{\psi_\epsilon}^n = e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}}.$$

Here we need to normalize  $\varphi_\epsilon$  so that

$$\int_X e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}} = \int_X \omega_0^n.$$

In Theorem 8 of [25], Yau treated a more general case with meromorphic right hand side. Note our initial approximation  $\varphi_\epsilon$  is smooth but does not have high regularity control outside  $D$ , and we will discuss this further after Proposition 2.3. The study of singular Kähler-Einstein metrics certainly goes back to Yau [25] and work of Yau with his collaborators, for instance, [10] and [24]. For more recent references on singular Kähler-Einstein metrics and generalizations, we refer to two recent works [6] and [13].

A direction calculation shows the following.

**Proposition 2.1.** *The Ricci form of  $\omega_{\psi_\epsilon}$  approximates  $\beta\omega_\beta + 2\pi(1 - \beta)[D]$  as  $\epsilon \rightarrow 0$ . Moreover,*

$$Ric(\omega_{\psi_\epsilon}) \geq \beta\omega_{\varphi_\epsilon} > 0, \quad \forall \epsilon \in (0, 1].$$

*Proof.* For any smooth function  $f > 0$ , we have (c.f. [25])

$$\begin{aligned} \sqrt{-1}\partial\bar{\partial}\log(f + \epsilon) &= \sqrt{-1}\partial\frac{\bar{\partial}f}{f + \epsilon} \\ &= \frac{\sqrt{-1}\partial\bar{\partial}f}{f + \epsilon} - \frac{\sqrt{-1}\partial f \wedge \bar{\partial}f}{(f + \epsilon)^2} \\ &= \frac{f}{f + \epsilon} \left( \frac{\sqrt{-1}\partial\bar{\partial}f}{f} - \frac{\sqrt{-1}\partial f \wedge \bar{\partial}f}{f^2} + \frac{\sqrt{-1}\partial f \wedge \bar{\partial}f}{f^2} \right) \\ &\quad - \frac{\sqrt{-1}\partial f \wedge \bar{\partial}f}{(f + \epsilon)^2} \\ &= \frac{f}{f + \epsilon} \sqrt{-1}\partial\bar{\partial}\log f + \epsilon \frac{\sqrt{-1}\partial f \wedge \bar{\partial}f}{f(f + \epsilon)^2} \\ &\geq \frac{f}{f + \epsilon} \sqrt{-1}\partial\bar{\partial}\log f. \end{aligned}$$

Using this, we can calculate the Ricci form on  $X \setminus D$ :

$$\begin{aligned} Ric(\omega_{\psi_\epsilon}) &= -\sqrt{-1}\partial\bar{\partial}h_{\omega_0} + (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log(|S|_h^2 + \epsilon) + Ric(\omega_0) + \beta\sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \\ &= \omega_0 + (1 - \beta)\sqrt{-1}\partial\bar{\partial}\log(|S|_h^2 + \epsilon) + \beta\sqrt{-1}\partial\bar{\partial}\varphi_\epsilon \\ &= \beta\omega_{\varphi_\epsilon} + (1 - \beta)(\omega_0 + \sqrt{-1}\partial\bar{\partial}\log(|S|_h^2 + \epsilon)) \\ &\geq \beta\omega_{\varphi_\epsilon} + (1 - \beta)\frac{\epsilon}{|S|_h^2 + \epsilon}\omega_0 \\ &\geq \beta\omega_{\varphi_\epsilon}. \end{aligned}$$

Since  $\psi_\epsilon$  is smooth this also holds on the whole  $X$ . □

For a later purpose we denote

$$(2.3) \quad (1 - \beta)\chi_\epsilon = Ric(\omega_{\psi_\epsilon}) - \beta\omega_{\varphi_\epsilon}.$$

By the previous calculation, this converges to  $2\pi(1 - \beta)[D]$  in the sense of currents.

Now we derive estimates on  $\omega_{\psi_\epsilon}$ . We make the convention that unless otherwise emphasized all constants appearing below are positive and depend only on  $X, D, \omega_0, \omega_{\varphi_\beta}$ . Also, the norms of the functions appearing in this article are always taken with respect to the background metric  $\omega_0$ .

**Theorem 2.2.** *There exists a uniform constant  $C_1 > 0$  such that for any  $\epsilon \in (0, 1]$ ,*

$$C_1^{-1}\omega_0 \leq \omega_{\psi_\epsilon} \leq \frac{C_1}{(\epsilon + |S|_h^2)^{1-\beta}} \cdot \omega_0.$$

*Proof.* First, by construction we have a constant  $c_1$  and  $p_0 > 1$  such that

$$\left\| \frac{\omega_{\psi_\epsilon}^n}{\omega_0^n} \right\|_{L^{p_0}} \leq c_1.$$

By a theorem of Kolodziej [16], this implies  $\|\psi_\epsilon\|_{C^\gamma(X)} \leq c_2$  for some  $\gamma \in (0, 1)$ . To derive  $C^2$  estimate, we view the identity map  $id : (X, \omega_{\psi_\epsilon}) \rightarrow (X, \omega_0)$  as a harmonic map with energy density

$$e(\psi_\epsilon) = \text{tr}_{\omega_{\psi_\epsilon}} \omega_0.$$

Then, using the fact that  $\text{Ric}(\omega_{\psi_\epsilon}) > 0$  and  $\text{Rm}(\omega_0) \leq c_3$ ,  $e(\psi_\epsilon)$  satisfies the following Chern-Lu differential inequality [21] (c.f. also [15]):

$$\Delta_{\psi_\epsilon}(\log e(\psi_\epsilon) - c_4\psi_\epsilon) \geq c_5e(\psi_\epsilon) - c_6.$$

Since  $|\psi_\epsilon|$  is uniformly bounded by  $c_2$ , by maximum principle, we have

$$e(\psi_\epsilon) \leq c_7$$

or

$$c_7^{-1}\omega_0 \leq \omega_{\psi_\epsilon}.$$

Plugging this into the Monge-Ampère equation (2.2), we obtain

$$(2.4) \quad C_1^{-1}\omega_0 \leq \omega_{\psi_\epsilon} \leq C_1 \cdot (|S|_h^2 + \epsilon)^{-(1-\beta)}\omega_0.$$

□

It follows that we have a uniform bound on  $\Delta_{\omega_0}\psi_\epsilon$  locally away from  $D$ , and a global uniform bound on  $\|\psi_\epsilon\|_{C^\gamma(X)}$ . So  $\{\psi_\epsilon\}$  by sequence converges to a limit potential  $\psi_0$  globally in  $C^{\gamma'}(X)$  and in  $C^{1,\alpha}$  locally away from  $D$ .

**Proposition 2.3.** *We have  $\psi_0 = \varphi_\beta + \text{constant}$ .*

*Proof.* This follows directly by the general uniqueness theorem for Monge-Ampère equation [1, 4, 17]. For the convenience of readers, we give a detailed account in our special case. Since

$$\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n$$

converges to 0 as  $\epsilon \rightarrow 0$  in  $L^p$  topology for some fixed  $p > 1$ , we have

$$\int_X (\varphi_\epsilon - \psi_\epsilon)(\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n) \rightarrow 0.$$

It follows that

$$\begin{aligned} & \int_X \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon) \wedge (\bar{\partial}\psi_\epsilon - \bar{\partial}\varphi_\epsilon) \wedge \sum_{k=0}^{n-1} \omega_{\psi_\epsilon}^k \wedge \omega_{\varphi_\epsilon}^{n-1-k} \\ &= - \int_X (\psi_\epsilon - \varphi_\epsilon)(\omega_{\psi_\epsilon}^n - \omega_{\varphi_\epsilon}^n) \rightarrow 0. \end{aligned}$$

Positivity of the integrand means that for any  $\delta > 0$ , we have

$$\int_{X \setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon) \wedge (\bar{\partial}\psi_\epsilon - \bar{\partial}\varphi_\epsilon) \wedge \sum_{k=0}^{n-1} \omega_{\psi_\epsilon}^k \wedge \omega_{\varphi_\epsilon}^{n-1-k} \rightarrow 0,$$

where  $D_\delta$  is the  $\delta$ -tubular neighborhood of  $D$ , defined by the metric  $\omega_0$ . Since every term is non-negative, we have

$$\int_{X \setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon) \wedge (\bar{\partial}\psi_\epsilon - \bar{\partial}\varphi_\epsilon) \wedge \omega_{\psi_\epsilon}^{n-1} \rightarrow 0.$$

By Theorem 2.2,  $\omega_{\psi_\epsilon} \geq C_1^{-1}\omega_0$ . It follows that

$$\int_{X \setminus D_\delta} \sqrt{-1}(\partial\psi_\epsilon - \partial\varphi_\epsilon) \wedge (\bar{\partial}\psi_\epsilon - \bar{\partial}\varphi_\epsilon) \wedge \omega_0^{n-1} \rightarrow 0.$$

In other words,

$$\partial\psi_0 - \partial\varphi_\beta = 0, \quad \text{in } X \setminus D_\delta.$$

So,

$$\psi_0 = \varphi_\beta + \text{constant}, \quad \text{in } X \setminus D_\delta.$$

Since  $\delta$  is arbitrary, this finishes the proof. □

To obtain more regularity control, we can substitute  $\varphi_\epsilon$  in Equation (2.2) by  $\psi_\epsilon$ . Then, the right-hand side of the equation will have a uniform  $C^{1,1}$  bound (i.e. bound on its Laplacian  $\Delta_{\omega_0}$ ) locally away from  $D$ . Then we can repeat the same procedure to obtain a new  $\psi'_\epsilon$  for  $\epsilon \in (0, 1]$ . As before, this new sequence  $\psi'_\epsilon$  converges to  $\varphi_\beta$  globally in  $C^{\gamma'}$  and they satisfy the same estimate as in Theorem 2.2. Now following standard theory of Evans-Krylov [12] [18] and bootstrapping, we can obtain an interior  $C^{3,\gamma}$  estimate on  $X \setminus D$  for some  $\gamma \in (0, 1)$ . It follows that  $\{\psi'_\epsilon\}$  by sequence converges in  $C^{3,\gamma'}$  to  $\varphi_\beta$  locally away from  $D$ . For simplicity, from now on we will denote by  $\psi_\epsilon$  this new sequence. If we keep running this procedure, we get even higher derivative control away from  $D$ .

To achieve the second goal, we need to prove the following proposition.

**Proposition 2.4.** *For any  $\epsilon \in (0, 1]$ , the diameter of  $(X, \omega_{\psi_\epsilon})$  is uniformly bounded above by a constant  $C_2$ .*

*Proof.* Since  $D$  is smooth, there exists a small constant  $\delta > 0$  such that the restriction of the background metric  $\omega_0$  to the  $\delta$ -tubular neighborhood of  $D$ , denoted by  $D_\delta$ , is locally equivalent to the product metric on  $D \times \mathbb{B}$ , where  $\mathbb{B}$  is the standard disc of radius  $\delta$ . Following the estimate in Theorem 2.2, for every point in  $D_\delta$ , there is a curve connecting it to  $\partial D_\delta$  with length bounded by  $c_1\delta^\beta$ . On the other hand, the varying metric is bounded above by the metric  $c_2\delta^{2(\beta-1)}\omega_0$  in  $X \setminus D_\delta$ . Therefore, the diameter of  $(X, \omega_{\psi_\epsilon})$  is controlled above by  $c_3(\delta^\beta + \delta^{\beta-1})$ . □

Finally, to achieve the third goal, we need to study the Gromov-Hausdorff limit of a sequence of Riemannian manifolds with positive Ricci curvature.

**Proposition 2.5.**  *$(X, \omega_{\varphi_\beta})$  is the Gromov-Hausdorff limit of  $(X, \omega_{\psi_\epsilon})$  as  $\epsilon \rightarrow 0$ .*

*Proof.* First, we remark that, by definition,  $\omega_{\varphi_\beta}$  defines a metric on  $X$ , and makes it into a compact length space. This can also be considered as the metric completion of the incomplete Riemannian manifold  $(X \setminus D, \omega_{\varphi_\beta})$ . On the other hand, it is not immediately clear that  $(X \setminus D, \omega_{\varphi_\beta})$  is geodesically convex. By Theorem 2.2 and Proposition 2.4, we have the following:

- (1)  $\omega_{\psi_\epsilon}$  converges as a current to  $\omega_{\varphi_\beta}$ ;
- (2)  $(X \setminus D, \omega_{\psi_\epsilon})$  converges locally in  $C^{3,\gamma'}$  to  $(X \setminus D, \omega_{\varphi_\beta})$ ;
- (3) any fixed  $\delta$ -tubular neighborhood  $D_\delta$  of  $D$  with respect to the background metric  $\omega_0$  is contained in a  $\eta(\delta)$ -tubular neighborhood of  $D$  with respect to the varying metric  $\omega_{\psi_\epsilon}$ , where  $\eta(\delta)$  tends to zero as  $\delta$  tends to zero.

Let  $(Y, d_Y)$  be a sequential Gromov-Hausdorff limit of  $(X, \omega_{\psi_\epsilon})$ . By Proposition 2.4,  $Y$  is a compact length space. By item (2) and (3) we obtain a smooth dense open subset  $U$  of  $Y$  endowed with a Riemannian metric  $g_\infty$ , and a surjective local

isometry  $F_\infty : (X \setminus D, \omega_{\varphi_\beta}) \rightarrow (U, g_\infty)$  (as Riemannian manifolds). For any  $x, y \in U$ , clearly we have  $d_Y(x, y) \leq d_U(x, y)$ , where  $d_U$  is the metric on  $U$  induced by the Riemannian metric  $g_\infty$ . From this it is easy to see that  $F_\infty$  is a Lipschitz map, so  $F_\infty$  extends to a Lipschitz map from the metric completion  $(X, \omega_{\varphi_\beta})$  to  $Y$ , and the image is closed. Since  $U$  is dense, it follows that  $F_\infty$  is surjective. It is clear that  $Y \setminus U$  is contained in  $F_\infty(D)$ . Since  $D$  has zero codimension one Hausdorff measure with respect to  $\omega_{\varphi_\beta}$ , we see that  $Y \setminus U$  also has zero codimension one Hausdorff measure. Then by Theorem 3.7 in Cheeger-Colding [7], we know  $d_U(x, y) = d_Y(x, y)$  for any  $x, y \in U$ , and  $Y$  is the metric completion of  $(U, d_U)$ . It follows that  $F_\infty$  is an isometry.  $\square$

To finish the proof of Theorem 1.1, what is left to prove is the uniform diameter bound (depending only on  $\beta_0$ ). To see this, one notices that for a Kähler metric with cone singularities along  $D$ , by the Cheeger-Colding result [7] we can always find a minimizing geodesic in  $X \setminus D$ , with length as close as to the diameter of  $X$ . Then we can apply Myers' theorem to see that the diameter of  $\omega_\beta$  is uniformly bounded by  $\pi \sqrt{\frac{n-1}{\beta}} \leq \pi \sqrt{\frac{n-1}{\beta_0}}$ . Then for each  $\beta \geq \beta_0$ , we can apply Proposition 2.5 (choosing  $\epsilon$  sufficiently small) to obtain a sequence of smooth Kähler metrics of positive Ricci curvature with Gromov-Hausdorff limit  $(X, \omega_{\varphi_\beta})$  and the diameter of this sequence of metrics is uniformly bounded above by  $2\pi \sqrt{\frac{n-1}{\beta_0}}$ .

### 3. PROOF OF THEOREM 1.2

To prove Theorem 1.2, we need to achieve the fourth goal: to approximate the Kähler-Einstein metric  $\omega_{\varphi_\beta}$  by smooth Kähler metrics with Ricci curvature bounded from below by some uniform positive number. From the complex Monge-Ampère theory, this is very different from the first and second goal where we can obtain  $C^0$  estimate via Kolodziej's theorem [16] directly. What we will do is to use the metric constructed in the previous section as a starting point, and use continuity method to solve the twisted Kähler-Einstein equation up to  $t = \beta$ . To do this, we need to obtain a uniform  $C^0$  estimate. The key observation is that for the family of  $(1, 1)$  forms  $\chi_\epsilon$  (see Equation (2.3)) which converges to  $2\pi(1-\beta)[D]$ , the twisted K-energy  $E_{\epsilon, (1-\beta)D}$  dominates the K-energy  $E_{(1-\beta)D}$  from above (see formulas (3.1), (3.2) and Lemma 3.5 below for precise statements).

Following Szekelyhidi [23], we define

$$R(X) = \sup\{t : \exists \omega' \in 2\pi c_1(X) \text{ such that } Ric(\omega') > t\omega'\}.$$

Theorem 1.2 is a consequence of the following.

**Theorem 3.1.** *If there is a Kähler-Einstein metric  $\omega_{\varphi_\beta}$  with cone angle  $2\pi\beta$  along  $D$ , then  $(X, \omega_{\varphi_\beta})$  is the Gromov-Hausdorff limit of a sequence of Kähler metrics with Ricci curvature bounded below by  $\beta > 0$ . In particular,  $R(X) \geq \beta$ .*

This verifies one aspect of a conjecture by the second named author earlier [11]. We need to do some preparation first. Set

$$\chi = \sqrt{-1}\partial\bar{\partial} \log |S|_h^2 + \omega_0.$$

By the Poincaré-Lelong equation, this is the same as the current  $2\pi[D]$ . For  $\epsilon > 0$  sufficiently small, we define (c.f. Equation (2.3)):

$$\chi_\epsilon = \sqrt{-1}\partial\bar{\partial} \log(|S|_h^2 + \epsilon) + \omega_0 > 0.$$

Clearly as currents,

$$\chi_\epsilon \rightarrow \chi.$$

For any  $\varphi \in \mathcal{H}$ , we choose a smooth family of potentials  $\varphi(t) (t \in [0, 1])$  in  $\mathcal{H}$  with  $\varphi(0) = 0$  and  $\varphi(1) = \varphi$ . We denote  $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)$ . For any smooth function  $f(t, \cdot)$ , we write  $\dot{f}(t)$  for the time derivative  $\frac{\partial f}{\partial t}(t, \cdot)$ .

**Definition 3.2.** Define a functional  $J_{\chi_\epsilon}$  by

$$J_{\chi_\epsilon}(\varphi) = n \int_0^1 dt \int_X \dot{\varphi}(t)(\chi_\epsilon - \omega_t) \wedge \omega_t^{n-1},$$

and as  $\epsilon$  tends to zero, we define a functional  $J_\chi$  by

$$J_\chi(\varphi) = 2\pi n \int_0^1 dt \int_D \dot{\varphi}(t)\omega_t^{n-1} - n \int_0^1 dt \int_X \dot{\varphi}(t)\omega_t^n.$$

**Definition 3.3.** Define the K-energy functional  $E$  by

$$E(\varphi) = -n \int_0^1 dt \int_X \dot{\varphi}(t)(Ric(\omega_t) - \omega_t) \wedge \omega_t^{n-1}.$$

One can check these do not depend on the choice of the path and hence are well-defined functionals on  $\mathcal{H}$ . Then, we define the twisted K-energy

$$(3.1) \quad E_{\epsilon, (1-\beta)D}(\varphi) = E(\varphi) + (1 - \beta)J_{\chi_\epsilon}(\varphi),$$

and

$$(3.2) \quad E_{(1-\beta)D}(\varphi) = E(\varphi) + (1 - \beta)J_\chi(\varphi).$$

First, we give an explicit formula for  $J_{\chi_\epsilon}$  (c.f. [19]).

**Proposition 3.4.** We have

$$(3.3) \quad J_{\chi_\epsilon}(\varphi) = \int_X \log(|S|_h^2 + \epsilon) \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1}.$$

*Proof.* This is a direct calculation:

$$\begin{aligned} J_{\chi_\epsilon}(\varphi) &= n \int_0^1 dt \int_X \dot{\varphi}(t)\chi_\epsilon \wedge \omega_t^{n-1} - n \int_0^1 dt \int_X \dot{\varphi}(t)\omega_t^n \\ &= n \int_0^1 dt \int_X \dot{\varphi}(t)\sqrt{-1}\partial\bar{\partial}\log(|S|_h^2 + \epsilon) \wedge \omega_t^{n-1} \\ &\quad + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\ &= \int_0^1 dt \int_X \log(|S|_h^2 + \epsilon) \cdot \frac{\partial}{\partial t}\omega_t^n + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\ &= \int_X \log(|S|_h^2 + \epsilon) \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1}. \end{aligned}$$

□



Similarly, we have

$$J_X(\varphi) = \int_X \log |S|_h^2 \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1}.$$

Now we have the following observation

**Lemma 3.5.** *There exists a uniform constant  $C_3 = C_3(X, D, \omega_0)$  such that for any  $\epsilon \in (0, 1]$  and for any smooth Kähler potential  $\varphi$ , we have*

$$J_{X_\epsilon}(\varphi) \geq J_X(\varphi) - C_3.$$

As a consequence, we also have

$$E_{\epsilon, (1-\beta)D}(\varphi) \geq E_{(1-\beta)D}(\varphi) - C_3.$$

*Proof.* This follows from an elementary calculation:

$$\begin{aligned} J_{X_\epsilon}(\varphi) &= \int_X \log(|S|_h^2 + \epsilon) \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\ &= \int_X \log(|S|_h^2 + \epsilon) \cdot \omega_\varphi^n - \int_X \log(|S|_h^2 + \epsilon) \cdot \omega_0^n \\ &\quad + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\ &\geq \int_X \log |S|_h^2 \cdot \omega_\varphi^n - \int_X \log |S|_h^2 \cdot \omega_0^n - \int_X \log \frac{|S|_h^2 + \epsilon}{|S|_h^2} \cdot \omega_0^n \\ &\quad + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} \\ &\geq \int_X \log |S|_h^2 \cdot (\omega_\varphi^n - \omega_0^n) + n \int_0^1 dt \int_X \dot{\varphi}(t)(\omega_0 - \omega_t) \wedge \omega_t^{n-1} - C_3 \\ &= J_X(\varphi) - C_3. \end{aligned}$$

□

It is well known that the K-energy has an explicit expression (c.f. [8]):

$$E(\varphi) = \int_X \log \frac{\omega_\varphi^n}{\omega_0^n} \omega_0^n + I(\varphi) + Q(\varphi),$$

where

$$I(\varphi) = n \int_0^1 dt \int_X \dot{\varphi}(t) \omega_t^n,$$

and

$$Q(\varphi) = -n \int_0^1 dt \int_X \dot{\varphi}(t) Ric(\omega_0) \wedge \omega_t^{n-1}.$$

Following [23] in the smooth case and [19] in the case with cone singularities, we have the following.

**Proposition 3.6.** *If there exists a Kähler-Einstein metric  $\omega_{\varphi_\beta}$  with cone angle  $2\pi\beta$  along  $D$ , then the twisted K-energy  $E_{(1-\beta)D}$  is proper on  $\mathcal{H}$ . In other words, there are constants  $C_4, C_5$  depending on  $X, D, \omega_0$ , and  $\beta$  such that for any smooth Kähler potential  $\varphi$ , we have*

$$E_{(1-\beta)D}(\varphi) \geq C_4 \cdot J_0(\varphi) - C_5,$$

where

$$J_0(\varphi) = \int_X \varphi(\omega_0^n - \omega_\varphi^n).$$

*Proof.* By [22] there is no non-trivial holomorphic vector field on  $X$  that is tangential to  $D$ . So by the openness theorem [11] for a slightly larger  $\beta' > \beta$  there exists a Kähler-Einstein metric on  $X$  with cone angle  $2\pi\beta'$  along  $D$ . By [3] and [2], the twisted Ding functional is bounded from below on  $\mathcal{H}$ . By [20], the infimum of the Ding functional and the infimum of the K-energy are the same in the anti-canonical class. This is generalized in [2] to the case with cone singularities. It follows that the twisted K-energy  $E_{(1-\beta')D}$  is bounded from below on  $\mathcal{H}$ . On the other hand, for  $\beta'' > 0$  sufficiently small  $E_{(1-\beta'')D}$  is proper on  $\mathcal{H}$ . This is easy when  $\lambda > 1$ , and in the case when  $\lambda = 1$  this follows from [2]. Since the twisted K-energy is linear in  $\beta$  [19], we see that  $E_{(1-\beta)D}$  is proper on  $\mathcal{H}$ .  $\square$

To prove Theorem 3.1, we need to set up a continuity path. For any  $\epsilon > 0$ , we consider the following equation for  $\phi_\epsilon(t) \in \mathcal{H}(t \in [0, \beta])$ :

$$(3.4) \quad Ric(\omega_{\phi_\epsilon(t)}) = t\omega_{\phi_\epsilon(t)} + (\beta - t)\omega_{\varphi_\epsilon} + (1 - \beta)\chi_\epsilon.$$

This is equivalent to the complex Monge-Ampère equation:

$$(3.5) \quad \begin{cases} \omega_{\phi_\epsilon(t, \cdot)}^n = e^{-t\phi_\epsilon(t, \cdot) - (\beta - t)\varphi_\epsilon + h\omega_0} \frac{1}{(|S|_h^2 + \epsilon)^{1-\beta}} \omega_0^n, \\ \phi_\epsilon(0, \cdot) = \psi_\epsilon. \end{cases}$$

Here  $\psi_\epsilon$  is the solution in Equation (2.2); in other words, we have

$$\omega_{\psi_\epsilon}^n = e^{-\beta\varphi_\epsilon + h\omega_0} \frac{1}{(|S|_h^2 + \epsilon)^{1-\beta}} \omega_0^n.$$

If we can solve Equation (3.5) up to  $t = \beta$ , then we have:

$$Ric(\omega_{\phi_\epsilon(\beta, \cdot)}) = \beta\omega_{\phi_\epsilon(\beta, \cdot)} + (1 - \beta)\chi_\epsilon \geq \beta\omega_{\phi_\epsilon(\beta, \cdot)}.$$

Although not needed for our main purpose in this article, we give here a uniform a priori estimate on the  $L^\infty$  norm of solutions to the above Monge-Ampère equation for small time.

**Lemma 3.7.** *There are constants  $t_0 \in (0, \beta]$  and  $K > 0$  independent of  $\epsilon \in (0, 1]$  such that if  $\phi_\epsilon(t, \cdot)$  is a solution to the Equation (3.5) for some  $t \in [0, t_0]$  and  $\epsilon \in (0, 1]$ , then we have  $|\phi_\epsilon(t, \cdot)|_{L^\infty} \leq K$ .*

*Proof.* Integrate Equation (3.5) we obtain that

$$\int_X \omega_{\phi_\epsilon(t, \cdot)}^n \geq e^{-t \sup \phi_\epsilon(t, \cdot)} \int_X e^{-(\beta - t)\varphi_\epsilon + h\omega_0} \frac{1}{(|S|_h^2 + \epsilon)^{1-\beta}} \omega_0^n.$$

Since  $\varphi_\epsilon$  is uniformly bounded independent of  $\epsilon$ , there is a constant  $B_1 > 0$ , such that  $e^{-t \sup \phi_\epsilon(t, \cdot)} \leq B_1$ . Fix a positive number  $\alpha_X$  slightly smaller than the  $\alpha$ -invariant of  $X$ , then there is a constant  $B_2 > 0$  depending only on  $X$  and  $\omega_0$  such that for any  $\phi \in \mathcal{H}$  we have

$$\int_X e^{-\alpha_X(\phi - \sup \phi)} \omega_0^n \leq B_2.$$

Writing

$$\omega_{\phi_\epsilon(t, \cdot)}^n = e^{-t \sup \phi_\epsilon(t, \cdot)} \cdot e^{-t(\phi_\epsilon(t, \cdot) - \sup \phi_\epsilon(t, \cdot))} \cdot \frac{e^{-(\beta-t)\varphi_\epsilon + h\omega_0}}{(|S|_h^2 + \epsilon)^{1-\beta}} \omega_0^n.$$

By standard Hölder inequality, we have for any  $p > 1$ ,  $q > 1$ ,  $\frac{1}{q} + \frac{1}{q^*} = 1$ ,

$$\begin{aligned} \int_X \left( \frac{\omega_{\phi_\epsilon(t, \cdot)}^n}{\omega_0^n} \right)^p \omega_0^n &\leq B_1^p \left( \int_X e^{-tpq(\phi_\epsilon(t, \cdot) - \sup \phi_\epsilon(t, \cdot))} \omega_0^n \right)^{\frac{1}{q}} \\ &\cdot \left( \int_X \left( \frac{e^{-(\beta-t)\varphi_\epsilon + h\omega_0}}{(|S|_h^2 + \epsilon)^{1-\beta}} \right)^{pq^*} \omega_0^n \right)^{\frac{1}{q^*}}. \end{aligned}$$

We first fix  $p > 1$  and  $q^* > 1$  so that  $pq^* < \frac{1}{1-\beta}$ , then the last term is uniformly bounded by a constant  $B_3$ . Then we let  $q = \frac{q^*}{q^*-1}$ , and  $t_0 = \min(p^{-1}q^{-1}\alpha_X, \beta) > 0$ , so  $\|\frac{\omega_{\phi_\epsilon(t, \cdot)}^n}{\omega_0^n}\|_{L^p}^p \leq B_1^p B_2^{\frac{1}{q}} B_3$ . As before we obtain the desired  $L^\infty$  estimate from Kolodziej's theorem.  $\square$

We now derive some energy estimates in order to prove Theorem 3.1.

**Lemma 3.8.** *There is a constant  $C_6 > 0$  such that*

$$|E(\epsilon, (1 - \beta)D)(\psi_\epsilon)| \leq C_6.$$

*Proof.* By Theorem 2.2 this is a direct verification using the explicit expressions of energy functionals described above.  $\square$

**Lemma 3.9.** *Along the continuity path,  $E_{\epsilon, (1-\beta)D}(\phi_\epsilon(t))$  decreases monotonically.*

*Proof.* Here we follow [23]. We take time derivative on both sides of Equation (3.5). Then,

$$\Delta_{\phi_\epsilon} \dot{\phi}_\epsilon = -t\dot{\phi}_\epsilon - (\phi_\epsilon - \varphi_\epsilon).$$

In this calculation we have omitted the parameter  $t$  for simplicity. A straightforward calculation then follows:

$$\begin{aligned} &\frac{d}{dt}(E + (1 - \beta)J_{X_\epsilon})(\phi_\epsilon(t)) \\ &= n \int_X \dot{\phi}_\epsilon(-Ric(\omega_{\phi_\epsilon}) + \omega_{\phi_\epsilon} + (1 - \beta)\chi_\epsilon - (1 - \beta)\omega_{\phi_\epsilon}) \wedge \omega_{\phi_\epsilon}^{n-1} \\ &= n \int_X \dot{\phi}_\epsilon(-t\omega_{\phi_\epsilon} - (\beta - t)\omega_{\varphi_\epsilon} - (1 - \beta)\chi_\epsilon + \beta\omega_{\phi_\epsilon} + (1 - \beta)\chi_\epsilon) \wedge \omega_{\phi_\epsilon}^{n-1} \\ &= n(\beta - t) \int_X \dot{\phi}_\epsilon(\omega_{\phi_\epsilon} - \omega_{\varphi_\epsilon}) \wedge \omega_{\phi_\epsilon}^{n-1} \\ &= (\beta - t) \int_X (\phi_\epsilon - \varphi_\epsilon) \cdot \Delta_{\phi_\epsilon} \dot{\phi}_\epsilon \omega_{\phi_\epsilon}^n \\ &= -(\beta - t) \int_X (\phi_\epsilon - \varphi_\epsilon) \cdot (t\dot{\phi}_\epsilon + (\phi_\epsilon - \varphi_\epsilon)) \omega_{\phi_\epsilon}^n \\ &= -(\beta - t) \int_X (\phi_\epsilon - \varphi_\epsilon)^2 \omega_{\phi_\epsilon}^n - nt(\beta - t) \int_X (\phi_\epsilon - \varphi_\epsilon) \cdot \dot{\phi}_\epsilon \omega_{\phi_\epsilon}^n \\ &\leq t(\beta - t) \int_X (\Delta_{\phi_\epsilon} \dot{\phi}_\epsilon + t\dot{\phi}_\epsilon) \cdot \dot{\phi}_\epsilon \omega_{\phi_\epsilon}^n \\ &\leq 0. \end{aligned}$$

The last inequality holds because  $Ric(\omega_{\phi_\epsilon(t,\cdot)}) > t\omega_{\phi_\epsilon(t,\cdot)}$  and  $\Delta_{\phi_\epsilon} + t$  is a negative operator.  $\square$

Now we are ready to prove Theorem 3.1.

*Proof.* According to Proposition 3.6, the twisted K-energy  $E_{(1-\beta)D}(\varphi)$  is proper on  $\mathcal{H}$ . Following Lemma 3.5,

$$E_{\epsilon,(1-\beta)D}(\varphi) \geq E_{(1-\beta)D}(\varphi) - C_3$$

is also proper on  $\mathcal{H}$ . By Lemma 3.9, we have

$$E_{\epsilon,(1-\beta)D}(\phi_\epsilon(t)) \leq E_{\epsilon,(1-\beta)D}(\phi_\epsilon(0)) = E_{\epsilon,(1-\beta)D}(\psi_\epsilon), \quad \forall t \in [0, \beta].$$

So by Lemma 3.8,

$$E_{\epsilon,(1-\beta)D}(\phi_\epsilon(t)) \leq C_6, \quad \forall t \in [0, \beta].$$

By definition of properness there is a constant  $C_7$  with

$$J_0(\phi_\epsilon(t)) \leq C_7.$$

For  $\epsilon > 0$  fixed, it follows from the standard argument that we can solve Equation (3.5) up to  $t = \beta$ . Now we claim that there is a constant  $C_8$  independent of  $\epsilon > 0$  such that

$$(3.6) \quad \sup_{\epsilon \in (0,1]} \max_{t \in [0,\beta]} \|\phi_\epsilon(t)\|_{L^\infty} \leq C_8.$$

To see this, notice that for any  $\delta \in (0, \beta]$ , and  $t \in [\delta, \beta]$ , we have

$$Ric(\omega_{\phi_\epsilon(t)}) \geq \delta\omega_{\phi_\epsilon(t)}.$$

It follows that the Sobolev constant of  $(M, \omega_{\phi_\epsilon(t)})$  is uniformly bounded (depending only on  $\delta$ ). By standard Moser iteration, this implies that there is a constant  $C_\delta > 0$  so that

$$\sup_{\epsilon \in (0,1]} \max_{t \in [\delta,\beta]} \|\phi_\epsilon(t)\|_{L^\infty} \leq C_\delta.$$

For our purpose of proving Theorem 3.1, we only need the convergence at  $t = \beta$ , so this is sufficient. However, for completeness of the argument, we would like a uniform  $L^\infty$  bound for small  $t$  as well. This indeed follows from Lemma 3.7, and we can simply take  $C_8 = \max(K, C_{t_0})$ . Note that we do not have a uniform positive lower bound for the Ricci curvature of the metrics  $\omega_{\phi_\epsilon(t,\cdot)}$  for all  $t$  and  $\epsilon$ , so it is not immediately clear if we can control Sobolev constant uniformly. This is different from the estimate required when we solve Equation (3.5) for a fixed  $\epsilon$ , where the solvability is automatic near  $t = 0$  by implicit function theorem, and then we have a uniform estimate along the rest of the continuity path.

As in the proof of Theorem 2.2, there is a constant  $C_9$  such that

$$C_9\omega_0 \leq \omega_{\phi_\epsilon(t)} \leq \frac{C_9}{(\epsilon + |S|_h^2)^{1-\beta}} \cdot \omega_0, \quad \forall t \in [0, \beta] \quad \epsilon \in (0, 1).$$

As before following [16] and Evans-Krylov theory to bootstrap regularity away from divisor, one can prove that  $\phi_\epsilon(t, \cdot)$  converges to  $\phi_0(t, \cdot)$  globally in  $C^{\gamma'}(X)$  and locally in  $C^{3,\gamma'}$  away from  $D$ . Moreover, in  $X \setminus D$ , it satisfies the equation

$$\omega_{\phi_0(t,\cdot)}^n = e^{-t\phi_0(t,\cdot) - (\beta-t)\varphi_\beta + h_{\omega_0}} \frac{1}{|S|_h^{2-2\beta}} \omega_0^n, \quad \forall t \in [0, \beta].$$

with

$$\phi_0(0, \cdot) = \varphi_\beta.$$

This can be written in a more concise form as

$$(3.7) \quad \omega_{\phi_0(t, \cdot)}^n = e^{-t(\phi_0(t, \cdot) - \varphi_\beta)} \omega_{\varphi_\beta}^n, \quad t \in [0, \beta].$$

Since  $\phi_\epsilon(t, \cdot)$  is uniformly bounded, we see  $\phi_0(\beta, \cdot)$  is in the weak sense (c.f. [2]) a Kähler-Einstein metric on  $X$  with cone angle  $2\pi\beta$  along  $D$ . By [22], there is no non-trivial holomorphic vector field on  $X$  which is tangential to  $D$ . So we can use the uniqueness theorem of Berndtsson [3] to obtain

$$\phi_0(\beta, \cdot) = \varphi_\beta(\cdot).$$

Then Proposition 2.5 implies that  $(X, \omega_{\phi_\epsilon(\beta, \cdot)})$  converges in the Gromov-Hausdorff topology to  $(X, \omega_{\varphi_\beta})$  as  $\epsilon \rightarrow 0$ . □

Here we give an alternative proof which makes use of the openness theorem proved by the second named author, bypassing Berndtsson’s theorem.

*Proof.* Note that the expression  $e^{-t(\phi_0 - \varphi_\beta)}$  in Equation (3.7) is Hölder continuous on  $X$ , so it lies in the Hölder space  $C^{\gamma, \beta}$  for some  $\gamma < \frac{1}{\beta} - 1$ . Following [11], the Laplacian operator  $\Delta_{\varphi_\beta}$  defines a continuous and invertible map

$$\Delta_{\varphi_\beta} : C_0^{2, \gamma, \beta}(X, D) \rightarrow C^{\gamma, \beta}(X, D).$$

Here  $C_0^{2, \gamma, \beta}(X, D)$  consists of functions in  $C^{2, \gamma, \beta}(X, D)$  with zero average. It follows that, there exists a continuous family  $\psi(t, \cdot) \in C_0^{2, \gamma, \beta}(X, D)$  for small  $t$ , say  $t \in [0, \epsilon_0]$ , which solves

$$\omega_\psi^n = e^{-t(\phi_0 - \varphi_\beta)} \omega_{\varphi_\beta}^n$$

and  $\psi(0, \cdot) = \varphi_\beta$ . Then, either following the uniqueness in [17] or Proposition 2.3, we have

$$\psi(t, \cdot) = \phi_0(t, \cdot) + \text{constant}$$

for  $t \in [0, \epsilon_0]$ . It follows that  $\phi_0(t, \cdot) \in C^{2, \gamma, \beta}(X, D)$  for  $t \in [0, \epsilon_0]$ .

Now we define the constant family  $\varphi_\beta(t, \cdot) = \varphi_\beta(\cdot)$ , then it satisfies the same Equation (3.7) for  $t \in [0, \epsilon_0]$  :

$$(3.8) \quad \omega_{\varphi_\beta(t)}^n = e^{-t(\varphi_\beta(t) - \varphi_\beta)} \omega_{\varphi_\beta}^n, \quad t \in [0, \beta].$$

By [22], there is no non-trivial holomorphic vector fields on  $X$  which is tangential to  $D$ . It follows that the first eigenvalue of  $\omega_{\varphi_\beta} = \omega_{\phi_0(0, \cdot)}$  is strictly bigger than  $\beta > 0$ . Consequently, for  $t$  sufficiently small,  $\omega_{\phi_0(t, \cdot)}$  has eigenvalue strictly bigger than  $\frac{\beta}{2}$ . Now compare the two families of solutions to Equation (3.7), by implicit function theorem again we see the uniqueness holds. In other words, by making  $\epsilon_0$  even smaller we have

$$\phi_0(t, \cdot) = \varphi_\beta, \forall t \in [0, \epsilon_0].$$

Repeating the same procedure as we increase  $t \leq \beta$ , we see the same holds for all  $t \in [0, \beta]$ . Our theorem is then proved. □

**Remark 3.10.** *Finally, we remark that in the case when  $\lambda > 1$  and  $1 - (1 - \beta)\lambda \leq 0$  there is a complete existence theory [15], but the argument in this article also applies to prove the following.*

**Theorem 3.11.** *Let  $\lambda > 1$  and  $\beta_0 \in (0, 1 - \lambda^{-1}]$ . If  $\omega$  is a Kähler-Einstein metric with cone angle  $2\pi\beta$  along  $D \in |-\lambda K_X|$  with  $\beta \in [\beta_0, 1 - \lambda^{-1}]$ , then  $(X, \omega)$  is the Gromov-Hausdorff limit of a sequence of smooth Kähler metrics  $\omega_i$  with  $\text{Ric}(\omega_i) \geq c_\beta \omega_i$  where  $c_\beta = (1 - \lambda(1 - \beta)) \leq 0$ , and diameter bounded above by a uniform constant depending only on  $X, D$  and  $\beta_0$ .*

*Proof.* The main issue is that in our previous argument the diameter bound depends on the particular  $\beta$ , and in the case of positive Ricci curvature (as assumed before that  $\beta_0 > 1 - \lambda^{-1}$ ) we can apply Myers' theorem to show that the bound only depends on  $\beta_0$ . Under our assumptions, we are in the case of non-positive Ricci curvature. In general, the diameter can not have a uniform upper bound, if one varies the complex structure on  $(X, D)$ , even when  $X$  has complex dimension one. However for a fixed  $(X, D)$ , a closer look at the argument in the proof of Theorem 2.2 and Proposition 2.4 shows that the diameter bound really depends only on the lower bound on the Ricci curvature of  $\omega_{\varphi_\beta}$  and the  $L^\infty$  bound on  $\varphi_\beta$  (which in turn depends on the  $L^{p_0}$  bound on the volume form of  $\omega_{\varphi_\beta}$ ). Notice that we assume  $c_\beta \leq 0$ ; then the metric  $\omega_{\varphi_\beta}$  satisfies the equation

$$(3.9) \quad \omega_{\varphi_\beta}^n = e^{-c_\beta \varphi_\beta + h_{\omega_0}} \frac{1}{|S|_h^{2(1-\beta)}} \omega_0^n.$$

Similar to the arguments in Section 2, by Yau's theorem [25] for  $\epsilon \in (0, 1]$  one can solve the equation for  $\psi_\epsilon$ :

$$\omega_{\psi_\epsilon}^n = e^{-c_\beta \psi_\epsilon + h_{\omega_0}} \frac{1}{(|S|_h^2 + \epsilon)^{(1-\beta)}} \omega_0^n.$$

Direct calculation as before shows that  $\text{Ric}(\omega_{\psi_\epsilon}) \geq c_\beta \omega_{\psi_\epsilon}$ . Moreover, by the maximum principle we see that there are constants  $p_0 \in (1, \frac{1}{1-\beta_0})$ , and  $A > 0$  depending only on  $X, D, \omega_0$  and  $\beta_0$  such that for any  $\epsilon \in (0, 1]$ ,

$$\sup_X \psi_\epsilon + \left\| \frac{\omega_{\psi_\epsilon}^n}{\omega_0^n} \right\|_{L^{p_0}} \leq A.$$

Following the arguments in Section 2, one can show that as  $\epsilon \rightarrow 0$  the Gromov-Hausdorff limit of  $(X, \omega_{\psi_\epsilon})$  is exactly  $(X, \omega_{\varphi_\beta})$ . Moreover, there is a uniform diameter bound independent of  $\beta \in [\beta_0, 1 - \lambda^{-1}]$ .  $\square$

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