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KÄHLER-EINSTEIN METRICS ON FANO MANIFOLDS. II: LIMITS WITH CONE ANGLE LESS THAN 2π

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1. Introduction

This is the second of a series of three articles which provide proofs of results announced in [8]. We review the background briefly but refer the reader also to the Introduction in [9]. Let X be a Fano manifold of complex dimension n. Let $\lambda \geq 0$ be an integer and D be a smooth divisor in the linear system $|-\lambda K_X|$. (For purely expository purposes, we allow the case when $\lambda = 0$, in which case D is the empty set.) For $\beta \in (0,1]$ we consider a Kähler-Einstein metric ω with a cone angle $2\pi\beta$ along D, in the sense we defined in [9]. When $\beta = 1$ we mean that the metric is a smooth Kähler-Einstein metric in the usual sense, but in the present article we always consider $\beta < 1$. The Ricci curvature of such a Kähler-Einstein metric ω, with cone singularities, is $(1-\lambda(1-\beta))\omega$. As in [9], we will assume throughout this article that $\beta > 1-\lambda^{-1}$. But our arguments also apply in the general case; see Theorem 4. The point of this article is to study the convergence properties of sequences of such metrics. Thus, we consider a sequence of triples (X_i, D_i, ω_i) with fixed λ and fixed Hilbert polynomial $\chi(X, K_X^{-p})$ and a sequence of cone angles β_i with limit β_∞. (For our main application we can take $(X_i, D_i) = (X, D)$, so only the cone angle varies, but even in that case it becomes notationally easier to write (X_i, D_i).) We will also usually denote the positive line bundle K_X^{-1} by L, partly to streamline notation and partly because many of our results would apply to more general polarized manifolds. In this article, we consider the case when the limit β_∞ is strictly less than 1. In the sequel we will consider the case when $\beta_\infty = 1$ and also explain, in more detail than in [8], how our results lead to the main theorem announced there.

To state our main result we need to recall some theory of Kähler-Einstein metrics on singular varieties. A general reference for this is [12]. If W is a normal variety we write K_W for the canonical line bundle over the smooth part W_0 of W.

Definition 1. *A* **Q**-*Fano variety is a normal projective variety W which*

- *is* **Q**-*Gorenstein, so some power $K_W^{m_0}$ extends to a line bundle over W;*
- *for some multiple m of m_0, is embedded in projective space by sections of K_W^{-m}; and*
- *has Kawamata log terminal (KLT) singularities.*

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(For theoretical purposes we do not need to distinguish between m and m_0, but in reality one expects that m_0 can often be taken substantially smaller than m.)

Definition 2. *Let $W \subset \mathbf{CP}^N$ be a $\mathbf{Q}$-Fano variety as above, embedded by sections of K_W^{-m}. A Kähler metric on W in the class c_1 is a Kähler metric ω on the smooth part W_0 of W which has the form $m^{-1}\omega_{FS} + i\partial\overline{\partial}\phi$ where ω_{FS} is the restriction of the Fubini-Study metric and ϕ is a continuous function on W, smooth on W_0.*

Another way of expressing this uses metrics on line bundles. We get a reference metric on the line bundle K_W^{-m} from the identification with $\mathcal{O}(1)|_W$. This gives a reference metric h_0 on K_W^{-1} over W_0 and a Kähler form ω is defined by a metric $h = e^{-\phi}h_0$ on K_W^{-1}. Thus, we will often write $\omega = \omega_h$. Of course we could change h by a non-zero multiple without changing ω_h. We say that a metric h on the line bundle K_W^{-1} over W_0 is continuous on W if the function $h^m h_0^{-1}$ extends to a continuous function on W.

A metric h on K_W^{-1} defines a volume form Ω_h on W_0: if σ is an element of the fiber over a point with $|\sigma| = 1$ then the volume form at that point is $(\sigma \wedge \overline{\sigma})^{-1}$.

Definition 3. *Let W be a $\mathbf{Q}$-Fano variety and ω_h be a Kähler metric on W in the class c_1. The metric is Kähler-Einstein if*

$$\omega_h^n = \Omega_h \tag{1}$$

on the smooth set $W_0 \subset W$.

Now fix $\lambda \geq 0$ as above. Let s be a section of the line bundle $K_W^{-\lambda}$ over W_0 such that

- if $\lambda = 0$, the section s is a non-zero constant and
- if $\lambda > 0$, the power s^m extends to non-trivial holomorphic section of $K_W^{-m\lambda}$ over W.

Let Δ_0 be the divisor in W_0 defined by s (so $\Delta_0 = 0$ if $\lambda = 0$), and let Δ be the closure of Δ_0 in W. Thus, Δ is a Weil divisor in W. For $\beta \in (0, 1)$ we have an $\mathbf{R}$-divisor $(1 - \beta)\Delta$.

Definition 4. *The pair $(W, (1-\beta)\Delta)$ is KLT if for each point $w \in W$ there is a neighborhood U of w, a positive integer m, and a non-vanishing holomorphic section σ of K_W^{-m} over $U \cap W_0$ such that*

$$\int_{U\cap(W_0\setminus \mathrm{supp}\Delta_0)} (\sigma \wedge \overline{\sigma})^{-1/m}|s|_h^{2\beta-2} < \infty,$$

where h is any continuous metric on K_W^{-1}.

Definition 5. *Let W be a $\mathbf{Q}$-Fano variety and let Δ be a Weil divisor defined by a section s as above such that $(W, (1-\beta)\Delta)$ is KLT for some $\beta \in (0, 1)$. A weak conical Kähler-Einstein metric for the triple (W, Δ, β) is a metric h on K_W^{-1} which extends continuously to W, which is smooth on $W_0 \setminus \mathrm{supp}\Delta_0$ and which satisfies the equation*

$$\omega_h^n = \Omega_h |s|_h^{2(\beta-1)} \tag{2}$$

on $W_0 \setminus \mathrm{supp}\Delta_0$.

It would be more precise to say that (2) is an equation for the metric h determined by β and the section s; however, s is determined by Δ up to a non-zero constant multiple. Given Δ, we can normalize s and the metric h, taking non-zero multiples, by requiring that the L^2 norm of s (computed using h and the volume form $(n!)^{-1}\omega_h^n$) is 1.

With these definitions fixed we can state our main result.

Theorem 1. *Let X_i be a sequence of n-dimensional Fano manifolds with fixed Hilbert polynomial. Let $D_i \subset X_i$ be smooth divisors in $|-\lambda K_{X_i}|$, for fixed $\lambda \geq 0$. If $\lambda > 0$, let $\beta_i \in (0,1)$ be a sequence converging to a limit β_∞ with $1-\lambda^{-1} < \beta_\infty < 1$. Suppose that there are Kähler-Einstein metrics ω_i on X_i with cone angle $2\pi\beta_i$ along D_i. Then there is a $\mathbf{Q}$-Fano variety W and a Weil divisor Δ in W such that:*

1. *$(W, (1-\beta_\infty)\Delta)$ is KLT;*
2. *there is a weak conical Kähler-Einstein metric ω for the triple $(W, \Delta, \beta_\infty)$;*
3. *possibly after passing to a subsequence, there are embeddings $T_i : X_i \to \mathbf{CP}^N, T_\infty : W \to \mathbf{CP}^N$, defined by the complete linear systems $|-mK_{X_i}|$ and $|-mK_W|$ respectively (for a suitable m depending only on the dimension of X_i, λ, and a uniform positive lower bound of $\beta_i - (1-\lambda^{-1})$), such that $T_i(X_i)$ converge to $T_\infty(W)$ as projective varieties and $T_i(D_i)$ converge to Δ as algebraic cycles.*

The statement of Theorem 1 is designed to give exactly what we need for the main argument outlined in [8]: no more and no less. One would like to have more information about the singularities of the limiting metric at points of Δ. This seems quite complicated when the divisor has components of multiplicity greater than 1 and we leave a discussion for a later article. When there are no such components then, at smooth points of Δ, we do have complete results. In particular, we have the following.

Theorem 2. *With the same hypotheses as in Theorem* 1*, suppose also that W is smooth and $\Delta \subset W$ is a smooth divisor. Then the metric ω is a Kähler-Einstein metric with cone angle $2\pi\beta_\infty$ along Δ.*

We now outline the strategy of proof of Theorem 1. By the results of [9] we can approximate each metric ω_i arbitrarily closely in Gromov-Hausdorff distance by smooth Kähler metrics with Ricci curvature bounded below by fixed strictly positive number. Since the volume is fixed (as the leading term in the Hilbert polynomial) the metrics satisfy a fixed volume non-collapsing bound. This means that we can transfer all of the Cheeger-Colding theory of non-collapsed limits of metrics of positive Ricci curvature to our situation. So, perhaps taking a subsequence, we can suppose that the X_i have a Gromov-Hausdorff limit Z. (We usually just write X_i for (X_i, ω_i) or, more precisely, the metric space defined by ω_i.) The essential problem is to give Z a complex algebraic structure, in the sense of constructing a normal variety W and a homeomorphism $h : Z \to W$. Then we have to show that W is $\mathbf{Q}$-Fano, and that the Gromov-Hausdorff convergence can be mirrored by algebro-geometric convergence in the sense stated in Theorem 1. We also have to show that there is a limiting metric on W which satisfies the appropriate weak Kähler-Einstein equation. In the case when $\lambda = 0$, so the divisors do not enter the picture at all, this is what was done in [11] except for the last statement about the limiting metric (which is not difficult). So Theorem 1 is essentially established in [11] for the case when $\lambda = 0$. Of course the real case of interest for us now is when

$\lambda > 0$, and what we have to do is to adapt the arguments in [11] to take account of the divisors. The main work takes place in Section 2, establishing that all tangent cones to Z are "good". In Section 3.1 we put together the proof of Theorem 1. Theorem 2 is proved in Section 3.2.

As in [9], in this article when we say a "Kähler-Einstein metric with cone angle $2\pi\beta$ along a divisor D", we mean that the metric is defined by a $C^{2,\alpha,\beta}$ potential for some $\alpha \in (0, \min(1, \beta^{-1} - 1))$[1] and satisfies the Kähler-Einstein equation on $X \setminus D$. In the Appendix, we will recall this definition in more detail, and prove that once the potential is in $C^{2,\alpha,\beta}$ for one such α, then it is so for all such α. Therefore, the definition does not depend on the particular choice of α. Such a regularity seems to have an independent interest, but we should point out that for our main purpose it is not necessary.

2. Structure of Gromov-Hausdorff limits

2.1. **Preliminaries.** We collect some facts which apply to rather general length spaces and which will be important to us. We expect that these are entirely standard results for experts.

Let P be a p-dimensional length space which is either the based Gromov-Hausdorff limit of a non-collapsed sequence of manifolds with Ricci curvature bounded below, or the cross-section of a tangent cone of a limit of such a sequence. Let A be a compact subset of P. Define a subset $\mathcal{M}_A$ of $(0, \infty)$ as follows. A number C is in $\mathcal{M}_A$ if for all $\epsilon > 0$ there is a cover of A by at most $C\epsilon^{2-p}$ balls of radius ϵ centered at points of A. Define the *(codimension 2) Minkowski measure* $m(A)$ to the infimum of $\mathcal{M}_A$, with the understanding that $m(A) = \infty$ if $\mathcal{M}_A$ is empty. Next we say that A has *capacity zero* if for all $\eta > 0$ there is a Lipschitz function g, equal to 1 on a neighborhood of A, supported on the η-neighborhood of A and with $\|\nabla g\|_{L^2} \leq \eta$.

Remarks

- The terminology "Minkowski measure" and "capacity" are provisional since the definitions may not exactly match up with standard ones in the literature. Since we only ever use the codimension 2 case we just talk about Minkowski measure hereafter.
- In many of our applications the space P will be a smooth Riemannian manifold outside A and the meaning of $\|\nabla g\|_{L^2}$ is clear. In the general case this quantity can be defined by the theory in [7].

Proposition 1. *If $m(A)$ is finite then A has capacity zero.*

To prove this, recall first that in this setting the volumes of r-balls in P are bounded above and below by multiples of r^p. Thus, the definition of Minkoski measure implies that the volume of the r-neighborhood of A is bounded by Cr^2 for some fixed C. For $\delta > 0, Q > 1$ define a function $f_{\delta,Q}$ on $[0, \infty)$ by

- $f_{\delta,Q}(t) = 0$ if $0 \leq t \leq \delta$;
- $f_{\delta,Q}(t) = \log(t/\delta)$ if $\delta \leq t \leq Q\delta$; and
- $f_{\delta,Q}(t) = \log Q$ if $t \geq Q\delta$.

Then $|f'_{\delta,Q}(t)| \leq \min(t^{-1}, \delta^{-1})$. Let $D_A : P \to [0, \infty)$ be the distance to A and $f = f_{\delta,Q} \circ D_A$. So $|\nabla f| \leq \min(D_A^{-1}, \delta^{-1})$. Let S be the distribution function of

[1] Here the requirement on the range of α is needed in order that the linear estimate for the Laplacian operator (c.f. (38) in Section 3) holds in this setting with cone singularities.

$|\nabla f|^2$—so $S(\lambda)$ is the volume of the set where $|\nabla f|^2 \geq \lambda$. Thus, $S(\lambda)$ is bounded by the volume of the $\lambda^{-1/2}$-neighborhood of A and so by $C\lambda^{-1}$. Also, $S(\lambda)$ vanishes for $\lambda > \delta^{-2}$ and for all λ, $S(\lambda)$ is bounded by the volume of the $Q\delta$ neighborhood of A, and so by $CQ^2\delta^2$. By the co-area formula

$$\int_P |\nabla f|^2 = \int_0^\infty S(\lambda) d\lambda \ = \int_0^{\delta^{-2}} S(\lambda) d\lambda.$$

Now $S(\lambda) \leq \min(CQ^2\delta^2, C\lambda^{-1})$ and we get

$$\int_P |\nabla f|^2 \leq C(1 + 2\log Q).$$

Now set $g = (\log Q)^{-1}(\log Q - f)$. So $g = 1$ on the δ-neighborhood of A and vanishes outside the $Q\delta$-neighborhood. We have

$$\int_P |\nabla g|^2 \leq C(\log Q)^{-2}(1 + 2\log Q).$$

Given $\eta > 0$ we first make Q so large that the right hand side in this formula is less than η^2 and then choose δ so small that $Q\delta < \eta$. This gives the desired function.

We also record here the following.

Proposition 2. *Suppose that A is closed and has Hausdorff dimension strictly less than $p - 2$. Then A has capacity zero.*

This is proved in [11], Proposition 3.5. The relevant hypothesis there is that the volume ratio of any r-ball is bounded above and below uniformly for all $r \in (0, 1]$, and this hypothesis can be derived directly, in our context, from the Ricci curvature lower bound, and the non-collapsing assumption, by the Bishop-Gromov volume comparison theorem.

Beyond Proposition 1, another general reason for considering the notion of "Minkowski measure" is that it behaves well with respect to Gromov-Hausdorff limits. Suppose we have metric spaces X_i with Gromov-Hausdorff limit Z and suppose given any subsets $A_i \subset X_i$. We fix distance functions d_i on $X_i \sqcup Z$, as in the definition of Gromov-Hausdorff convergence. Then we can define a "limit" $A_\infty \subset Z$ as the set of points $z \in Z$ such that there are $a_i \in A_i$ with $d_i(a_i, z) \to 0$. However, without further information, we cannot say much about it. In particular, the notion does not behave well with respect to Hausdorff measure. For example, we could take A_i to be finite and such that, even if we pass to any subsequence, A_∞ is the whole of Z. On the other hand, if we know that A_i have bounded Minkowski measure, with a fixed bound, $m(A_i) \leq M$ then it is straightforward to prove that (after taking a subsequence) the same is true in the limit. In particular, the limit has Hausdorff codimension at least 2.

2.2. **Generalities on Gromov-Hausdorff limits.** As in Section 1, we consider a sequence of Kähler-Einstein metrics ω_i on X_i with cone angle $2\pi\beta_i$ along D_i where β_i tends to $\beta_\infty < 1$ and we suppose that these have a Gromov-Hausdorff limit Z (following [9], see the discussion in the paragraph following Theorem 2 above). For convenience fix $\beta_- > 0$ and $\beta_+ < 1$ so that $\beta_- \leq \beta_i \leq \beta_+$ for all i. We can apply the deeper results of Cheeger and Colding to the structure of the limit space Z. Thus, we have a notion of tangent cones at points of Z (not *a priori* unique) and we can write $Z = R \cup S$ where R (the regular set) is defined to be the set of points where all tangent cones are $\mathbf{C}^n$, and S is defined to be $Z \setminus R$. We stratify the singular set as

$$S = S_1 \supset S_2 \supset S_3 \dots,$$

where S_j is defined to be the set of points where no tangent cone splits off a factor $\mathbf{C}^{n-j+1}$. The Hausdorff dimension of S_j does not exceed $2(n-j)$. (Our labeling of the strata differs from what is standard in the literature, first because we use co-dimension rather than dimension and second because in this Kähler situation we can restrict to even dimensions.) We remark that, when we only have Ricci curvature bounded from below, S may be not equal to S_2, and more crucially, $\mathcal{R}$ may not be open in general and it does not usually carry a smooth structure. The fact that we have a smooth Kähler-Einstein metric outside the divisor D_i, helps us to overcome both difficulty. We can suppose that we have chosen fixed metrics d_i on $X_i \sqcup Z$ realising the Gromov-Hausdorff convergence.

Proposition 3. *The set $R \subset Z$ is open and the limiting metric induces a smooth Kähler-Einstein metric ω_∞ on R.*

Suppose that q is a point in $D_i \subset X_i$. Then the limiting volume ratio (as the radius tends to 0) at q is clearly $\beta_i \leq \beta_+ < 1$. By Bishop-Gromov monotonicity the volume ratio of any ball centered at q is less than β_+. Now let p be a point of $R \subset Z$. By Colding's theorem on volume convergence it follows that there is some $\epsilon > 0$ such that $d_i(q,p) \geq \epsilon$ for all such points q. In particular, there is a small $\epsilon' \in (0, \epsilon)$ such that the volume ratio of the ball $B_{\epsilon'}(q, d_i)$ is close enough to Euclidean volume ratio. By a theorem of Anderson, we have uniform control on local geometry inside the ball $B_{\frac{\epsilon'}{2}}(q, d_i)$. It follows that $B_{\frac{\epsilon'}{2}}(q, d) \subset \mathcal{R}$ and the convergence in this small ball is smooth. In other words, near p we reduce to the standard theory of smooth Kähler-Einstein metrics and the limit is some smooth Kähler Einstein metric locally.

The essential new feature of our case, as compared with the standard theory of smooth Kähler-Einstein metrics, is that $S \neq S_2$. We write $\mathcal{D} = S \setminus S_2$ so by definition a point is in $\mathcal{D}$ if it has a tangent cone which splits off $\mathbf{C}^{n-1}$ but not $\mathbf{C}^n$. Such a tangent cone must have the form $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ where $0 < \gamma < 1$ and $\mathbf{C}_\gamma$ denotes the standard flat cone of angle $2\pi\gamma$.

Proposition 4.
1. *Suppose a point $p \in \mathcal{D}$ has a tangent cone $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ and a tangent cone $\mathbf{C}_{\gamma'} \times \mathbf{C}^{n-1}$. Then $\gamma = \gamma'$.*
2. *There is a $\gamma_0 \in (0,1)$, depending only on the volume non-collapsing constant of (X_i, ω_i), such that if a point in $\mathcal{D}$ has tangent cone $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ then $\gamma \geq \gamma_0$.*

These follow immediately from volume monotonicity, somewhat similar to Proposition 3. (In the second item recall that, as discussed in the Introduction, the metrics do satisfy a fixed volume non-collapsing condition.)

All of the above discussion applies equally well to scaled limits. Thus, suppose we have a sequence $r_i \to \infty$ and base points $p_i \in X_i$. Then (taking a subsequence) we have a based Gromov-Hausdorff limit Z' of $(X_i, r_i^2\omega_i, p_i)$ and the structure of the singular set of Z' is just as described above. In particular, we can apply this to tangent cones and iterated tangent cones of Z. Such a tangent cone has the form $C(Y)$ where Y is a $(2n-1)$-dimensional length space which has a smooth Sasaki-Einstein structure on an open set R_Y: the complement of R_Y has (real) Hausdorff codimension at least 2 and can be written as a disjoint union $\mathcal{D}_Y \cup S_{2,Y}$ where $S_{2,Y}$ has (real) Hausdorff codimension at least 4 and points of $\mathcal{D}_Y$ have tangent cones $\mathbf{C}_\gamma \times \mathbf{R} \times \mathbf{C}^{n-2}$.

2.3. **Review.** In this subsection we assume knowledge of the main construction in [11], Section 3. Let Z be a Gromov-Hausdorff limit of (X_i, ω_i), as above, and let p be a point in Z. Let $C(Y)$ be a tangent cone of Z at p.

Definition 6. *We say that $C(Y)$ is a good tangent cone if the singular set $S_Y \subset Y$ has capacity zero.*

Now one of the main points in the argument of [11] is the following.

Proposition 5. *If $\mathcal{D}_Y$ is empty then $C(Y)$ is a good tangent cone.*

This follows from Proposition 2 since if $\mathcal{D}_Y$ is empty the singular set has Hausdorff codimension ≥ 4 and is closed since the regular set is open by Proposition 3. Of course by Proposition 1 we have the following.

Proposition 6. *If the singular set S_Y has finite Minkowski measure then $C(Y)$ is a good tangent cone.*

Now we have the following.

Proposition 7. *Let p be a point in a limit space Z as above and suppose that there is a good tangent cone of Z at p. Then there are $b(p), r(p) > 0$, and an integer $k(p)$ with the following effect. Suppose (X_i, D_i, ω_i) is a sequence of Kähler-Einstein metrics with cone singularities and cone angles $\beta_i \to \beta_\infty < 1$ then there is a $k \leq k(p)$ such that for sufficiently large i there is a holomorphic section s of $L^k \to X_i$, with L^2 norm 1 and with $|s(x)| \geq b(p)$ for all points $x \in X_i$ with $d_i(x,p) < r(p)$.*

This statement is chosen to be exactly parallel to that of Theorem 3.2 in [11] (except for trivial changes of language) and the proof is almost exactly the same. That is, the only facts from the convergence theory that are needed for that proof are that the convergence is in C^∞ on the regular set and that the singular set in Y has capacity zero. The "Hörmander argument", constructing holomorphic sections, can be carried through in a straightforward way by working with smooth approximating metrics as in [9]. (It would also be straightforward to extend the Hodge theory constructions directly to the singular case.) An important point to note is that the C^0 and C^1estimates for holomorphic sections (Proposition 2.1 in [11]) go through without change. This is because the Ricci curvature enters with a favorable sign in the relevant Bochner-Weitzenbock formulae.

This gives the background to state our main technical result of this section.

Theorem 3. *With Z as above, all tangent cones are good.*

Our proof of this will require some work and on the way we will establish a variety of other useful statements.

2.4. **Gaussian sections.** We now want to refine the statement of Proposition 7, which involves reviewing in more detail the constructions in [11] and modifying them slightly. When working with the line bundle $L^k \to X_i$ it is convenient to use the scaled metric $k\omega_i$. As in [11] we use the notation $L^{2,\sharp}$, etc. to denote quantities calculated using the rescaled metric. Likewise for the distance function $d_i^\sharp$ on $X_i \sqcup Z$.

Proposition 8. *Suppose that p is a point in Z with a good tangent cone. There is a sequence $k_\nu \to \infty$ such that the following holds. For any $\zeta > 0$ there is an $R_0 > 1$*[2]*, an integer $m \geq 1$, and for large ν integers $t_{i,\nu}$ with $1 \leq t_{i,\nu} \leq m$ such that if $k = t_{i,\nu} k_\nu$ then for large enough i there is a holomorphic section s of $L^k \to X_i$ with the following properties:*

- $\left|\, |s(x)| - \exp(-d_i^\sharp(x,p)^2/4) \,\right| \leq \zeta$, *if* $d_i^\sharp(x,p) \leq R_0$;
- $|s(x)| \leq \zeta$ *if* $d_i^\sharp(x,p) \geq R_0$; *and*
- $\|s\|_{L^{2,\sharp}} \leq (2\pi)^n + \zeta$.

We begin by recalling the basic strategy. First take a sequence of scalings $r_\nu \to \infty$ realizing the given good tangent cone $C(Y)$ at p. Changing r_ν by bounded factors does not change the tangent cone so we can suppose that $r_\nu = \sqrt{k_\nu}$ for integers k_ν. For a point z in the cone $C(Y)$ we use the notation $[z]$ to denote the distance to the vertex of the cone. There is a holomorphic section σ_0 of the trivial line bundle over the smooth part of the cone, with

$$|\sigma_0| = \exp(-[z]^2/4).$$

Thus, the L^2 norm of σ_0 is $(2\pi)^{n/2}\sqrt{\kappa_Y}$ where $\kappa_Y \leq 1$ is the volume ratio of the cone. For $\eta' > 0$, let $Y_{\eta'}$ be the interior of the complement of the η'-neighborhood of the singular set in Y and for $\delta \ll 1 \ll R$ let $U = U_{\eta',\delta,R}$ be the intersection of the cone on $Y_{\eta'}$ with the "annulus" defined by $\delta < [z] < R$. For suitable choice of parameters η', δ, R we construct an appropriate cut-off function χ of compact support on U and equal to 1 on a smaller set $U_0 \subset U$. Then $\sigma = \chi\sigma_0$ is an "approximately holomorphic section" in that

$$\|\overline{\partial}\sigma\|_{L^2} \leq \|(\overline{\partial}\chi)\ e^{-[z]|^2/4}\|_{L^2}, \tag{3}$$

and the right-hand side will be chosen small. (In [11] we denoted cut-off functions by β but that symbol is taken here for the cone angle.)

We introduce some terminology. For $\epsilon > 0$ we say that a map $\Gamma : U \to X_i$ is an *ϵ-Kähler embedding* if it is a diffeomorphism onto its image with the properties that

$$|\Gamma^*(J_i) - J| < \epsilon \quad , \quad |\Gamma^*(k_\nu \omega_i) - \omega| < \epsilon. \tag{4}$$

Here, J_i denotes the complex structure on X_i and J, ω are the standard structures on the smooth part of the cone. By the definition of the tangent cone (and the smooth convergence on the regular set) we can, for any given ϵ, choose ν sufficiently large and then $i \geq i(\nu)$ so that we have such a map. Further, for $r > 0$ let $B_{\nu,i}(r) \subset X_i$ be the set of points $x_i \in X_i$ with distance $d_i^\sharp(x_i, p) \leq r$. Then we can define distance functions $d_{\nu,i}$ on $B_{\nu,i}(r) \sqcup \{z : [z] \leq r\} \subset X_i \sqcup C(Y)$ realizing the Gromov-Hausdorff convergence in the sense that for any fixed r and for any $\delta > 0$ then for sufficiently large ν and for $i \geq i(\nu)$ each component is δ-dense. Then (taking $r = 2R$ say) we can also choose Γ above such that

$$d_{\nu,i}(z, \Gamma(z)) \leq \epsilon. \tag{5}$$

Returning to our outline of the basic strategy, we choose such an ϵ-Kähler embedding $\Gamma = \Gamma_i : U \to X_i$ for a suitable ϵ. A complication now arises involving the potential "topological obstruction" from the holonomy of the connections. This

[2]The referee points out that it is not necessary to introduce the parameter R_0 here. However, since we find it useful in our later discussion we prefer to include it.

was handled in [11] by the introduction of the factors $t_{i,\nu}$. We ignore this for the time being, so we suppose that Γ is covered by a map of line bundles $\tilde{\Gamma}$ such that

$$|\tilde{\Gamma}^*(A_i) - A| < \epsilon, \tag{6}$$

where A is the preferred connection on the trivial bundle over the regular part of $C(Y)$ and A_i is the connection on $L^k \to X_i$. With this done, we have a transplanted section $\tilde{\Gamma}_*(\sigma)$, which we denote by $\sigma^\sharp$, of $L^k \to X_i$ with $L^{2,\sharp}$ norm approximately the same as that of σ_0 and with the $L^{2\sharp}$ norm of $\overline{\partial}\sigma^\sharp$ small. The basic "Hörmander construction" is to pass to the holomorphic section $s = \sigma^\sharp - \tau$ with $\tau = \overline{\partial}^* \Delta^{-1} \overline{\partial}\sigma^\sharp$, so that $\|\tau\|_{L^{2\sharp}} \leq \|\overline{\partial}\sigma^\sharp\|_{L^{2,\sharp}}$ will be small. Notice here that the third condition in Proposition 8 is entirely straightforward.

Next we recall that we have universal bounds on the C^0 and C^1 norms of holomorphic sections. We take these in local form; for a point q in X_i and a holomorphic section s' of L^k defined over a ball B of fixed size in the re-scaled metric (say the unit ball) centered at q we have

$$|s'(q)| \leq K_0 \|s'\|_{L^{2,\sharp}(B)}, \tag{7}$$

and

$$|(\nabla s')(q)| \leq K_1 \|s'\|_{L^{2,\sharp}(B)}. \tag{8}$$

The point to emphasize here is that K_0, K_1 are "universal" constants, which could be computed explicitly in terms of β_i and the Hilbert polynomial. We emphasize that, following the remark after Theorem 1.2 in [9], we have a uniform Sobolev constant bound for the metrics we are studying.

With these preliminaries in place we can begin the proof of Proposition 8. The argument is entirely elementary, but a little complicated. If written out in full the argument involves a large number of computable constants. For clarity, we suppress these and often use a notation such as "$\ll K_0^{-1}$", where strictly we mean "$\leq$ some computable constant times K_0^{-1}". We emphasize that while we have outlined the general scheme of construction above we are keeping the parameters $R, \eta', \delta, \epsilon$ free and they will be chosen below.

Step 1. Given $h > 0$ there is some $\eta(h) \leq h$ such that for all y in Y there is a point $y' \in Y$ with $d(y, y') \leq h$ and such that y' does not lie in the $2\eta(h)$-neighborhood of S_Y.

This follows from an elementary argument, using the facts that S_Y is closed and has empty interior.

Step 2. Choose $R_0 \gg 1$ such that $\exp(-R_0^2/4) \ll K_0^{-1}\zeta$. We will take $R \geq 2R_0$.

Step 3. In the set up-as described above, with $U_0 \subset U$ the set where $\chi = 1$, it is clear that we have a fixed bound on $|\nabla\sigma^\sharp|$ over $\Gamma(U_0)$. Using (8), and writing $\tau = s - \sigma^\sharp$ we get a bound of the form

$$|\nabla\tau|_{\Gamma(U_0)} \leq K_2. \tag{9}$$

Now choose $\rho \ll \min(\zeta K_2^{-1}, \zeta K_1^{-1})$.

Step 4. Set $h = \rho/R_0$ and let

$$\eta_0 = \eta(h),$$

with $\eta(h)$ as in Step 1.

Step 5. Now we recall that the cut-off function χ will have the form $\chi = \chi_{\eta',\eta}\ \chi_\delta\ \chi_R$ where:

- $\chi_{\eta',\eta}$ is a function on Y (pulled back to the cone in the obvious sense), equal to 1 outside the η-neighborhood of the singular set and to 0 on the η'-neighborhood. Given η, use the definition of a good tangent cone to choose such a function so that the L^2 norm of its derivative is less than η (and with $\eta' = \eta'(\eta) < \eta$, but otherwise uncontrolled).
- χ_δ is a standard cut-off function of $[z]$, vanishing for $[z] \leq \delta$ and equal to 1 for $[z] \geq 2\delta$.
- χ_R is a standard cut-off function of $[z]$, vanishing for $[z] \geq R$ and equal to 1 for $[z] \leq R/2$.

This fixes χ, given parameters η, δ, R: for the moment we have not fixed those parameters, but we will take $\eta < \eta_0$.

Step 6. Write $B(R_0)$ for the set of $z \in C(Y)$ with $[z] \leq R_0$ and $V(R_0)$ for the intersection of the cone on Y_{η_0} with the the annulus $2\delta < [z] < R_0$. Thus, under our assumptions, $\chi = 1$ in $V(R_0)$. Now, using Step 4, we can fix δ_0 so that if $\delta \leq \delta_0$ then for all z in $B(R_0)$ we can find a z' in $V(R_0)$ such that:

- the distance between z, z' does not exceed ρ;
- $$|\exp(-[z]^2/4) - \exp(-[z']^2/4)| \ll \zeta; \text{ and}$$
- the $\eta_0\rho$-ball centered at z' lies in $V(R_0)$.

Step 7. Let F_0 be any function on $V(R_0)$ with $|\nabla F_0| \leq K_2$. (Here F_0 serves as a prototype for $|\tau|$.) If $z' \in V(R_0)$ is any point as in Step 6 we get, by an elementary estimate, that there is a K_3 depending on the already chosen numbers ζ, ρ, η_0 such that if $\|F_0\|_{L^2} \leq K_3^{-1}$ then $|F_0(z')| \leq \zeta/10$.

Remark: The argument here is slightly different from that of [11], in that we avoided the use of L^p theory so it appears to us this is more elementary. See also Remark 2.5 in [11].

Step 8. Let F_1 be any function on $B(R_0)$ with $|\nabla F_1| \leq K_1$. (Here F_1 serves as a prototype for $|s|$). Then for any point $z \in B(R_0)$ we can find z' as in Step 6 so we have

$$|F_1(z) - F_1(z')| \ll \zeta \tag{10}$$

and also

$$|\exp(-[z]^2/4) - \exp(-[z']^2/4)| \ll \zeta. \tag{11}$$

Step 9. We now finally fix our parameters δ, η, R. We choose $\delta < \delta_0, \eta < \eta_0, R > 2R_0$ but also we require that

$$\|(\nabla\chi)e^{-[z]^2/4}\|_{L^2} \ll \min\left(K_3^{-1}, K_0^{-1}\zeta\right). \tag{12}$$

It is elementary calculation that this is possible (that is, the right-hand side is a fixed number and we can make the left-hand side as small as we please by taking η and δ small and R large).

Step 10. We are now in much the same position as in the corresponding stage of [11], with the difference that we have set up the background to obtain more precise control of the final holomorphic section s. We choose an ϵ-Kähler embedding Γ of U in X_i satisfying (4), (5). Suppose first that we can find a lift $\tilde{\Gamma}$ as in (6). Then we define $\sigma^\sharp$ as described above. By making ϵ sufficiently small we can suppose that, following on from (12) we also have

$$\|\bar{\partial}\sigma^\sharp\|_{L^{2,\sharp}} \ll \min\left(K_3^{-1}, K_0^{-1}\zeta\right). \tag{13}$$

At this point we can fix the parameter m. This goes just as in [11]: we need to arrange that after dilating by a suitable factor t with $1 \leq t \leq m$ the potential topological obstruction arising from holonomy is a small perturbation, so we can construct a section satisfying the constraints (13). To simplify exposition, assume that in fact $t = 1$.

Step 11. For suitably large ν and $i \geq i(\nu)$ we can suppose that for any point $x \in X_i$ with $d^\sharp(x,p) \leq R_0$ there is a point x' in $\Gamma_i(V)$ with $d^\sharp(x,x') \leq 2\rho$ and such that the ball of radius $\eta_0\rho$ centered at x' lies in $\Gamma_i(V)$. Here $V = V(R_0)$. Now we apply the estimates, modelled on the prototypes in (10), (11), to deduce that $||s(x)| - \exp(-d_i^\sharp(x,p)^2/4)| \leq \zeta$.

Step 12. The remaining task is to obtain the estimate $|s(x)| \leq \zeta$ when $d_i^\sharp(x,p) \geq R_0$. For this we use the C^0 estimate (7). If $x \in X_i$ is such a point then the $L^{2,\sharp}$ norm of $\sigma^\sharp$ over the unit ball centered at x is small (compared with ζ) by Step 1. Likewise, for the $L^{2,\sharp}$ norm of τ by Step 9. Then we obtain the conclusion from the C^0 estimate.

Note. In our applications below we can take ζ to be some fixed small number, say 1/100. Then, given p, the number m is fixed. Using a diagonal argument and passing to a subsequence we may suppose that all the $t_{\nu,i}$ are equal and this effectively means that we can ignore this extra complication.

If we have suitable holomorphic functions on the cone we can construct more holomorphic sections.

Proposition 9. *Suppose that p is a point in Z with a good tangent cone and that f is a holomorphic function on the regular part of the cone such that for some $\alpha \geq 1$ and $C > 0$ we have the following.*

- *There are smooth functions $G_\pm$ of one positive real variable with $|G_\pm(t)| \leq Ct^\alpha, |G'_\pm(t)| \leq Ct^{\alpha-1}$ and*

$$G_-([z]) \leq |f(z)| \leq G_+([z]). \tag{14}$$

- $|\nabla f(z)| \leq C[z]^{\alpha-1}$.

Let k_ν be the sequence as in Proposition 8. For any $\zeta > 0$ we can choose $R_0, m, t_{i,\nu}$ as in Proposition 8 so that in addition there is a holomorphic section s_f of $L^k \to X_i$ such that, writing $d = d_i^\sharp(x,p)$

- $\exp(-d^2/4)G_-(d) - \zeta \leq |s_f(x)| \leq \exp(-d^2/4)G_+(d) + \zeta$ *if* $d \leq R_0$;
- $|s_f(x)| \leq \zeta$ *if* $d \geq R_0$; *and*
- $\|s_f\|_{L^{2,\sharp}} \leq N + \zeta$;

where N can be computed explicitly from C, α.

While the statement is a little complicated the proof is essentially identical to that of Proposition 8, starting with the section $\chi f \sigma_0$. The Gaussian decay of σ_0 dominates the polynomial growth of f and the condition $\alpha \geq 1$ means that there are no problems at the vertex of the cone.

For $r > 0$ we write $B^\sharp = B^\sharp(p,r) \subset X_i$ for the set of points x with $d^\sharp(x,p) < r$. (For our purposes below one could take $r = 3$, say.) Given any fixed r then clearly if ζ is chosen sufficiently small the radius R_0 in Proposition 8 will be much bigger than r and we will have a definite lower bound $|s| \geq c > 0$ for our section over $B^\sharp$. If then we have a function f as in Proposition 9, we can form the quotient $\tilde{f} = s_f/s$ as a holomorphic function over $B^\sharp$. This satisfies a fixed C^1 bound

$$|f|, |\nabla f| \leq M(C, \alpha, K_0, K_1, c) \tag{15}$$

for a computable function M.

We can say more.

Proposition 10. *Let W be any pre-compact open subset of the regular set in the ball $\{[z] < r\}$ in $C(Y)$. For any $\epsilon, \zeta > 0$, for ν sufficiently large and for $i \geq i(\nu)$ there is an ϵ-Kähler embedding $\Gamma : W \to X_i$ such that $|f - \tilde{f} \circ \Gamma| \leq \zeta$ on W.*

The proof is again essentially the same as Proposition 8.

Now we consider the extension of all these ideas to scaled limits. So suppose that we have a sequence $r_i \to \infty$, base points $q_i \in X_i$ and that Z' is the based limit of $(X_i, r_i^2 \omega_i)$. There is no loss of generality in supposing $r_i^2 = a_i$, for integers a_i. Thus, algebro-geometrically, the scaling corresponds to considering line bundles $L^{a_i} \to X_i$. Let d_i now denote a distance function for the re-scaled metrics, realizing the Gromov-Hausdorff convergence. Given k we write, as before, $d_i^\sharp = k^{1/2} d_i$, and in general as before we add a $\sharp$ to denote quantities calculated in the metric scaled by $\sqrt{k}$. The point is that this is *additional* to the scalings we have already made by $\sqrt{a_i}$.

Proposition 11. *Suppose that p is a point in Z' with a good tangent cone. There is a sequence $k_\nu \to \infty$ such that the following holds. For any $\zeta > 0$ there is an $R_0 > 1$, an integer $m \geq 1$, and integers $t_{i,\nu}$ with $1 \leq t_{i,\nu} \leq m$ such that if $k = t_{i,\nu} k_\nu$ then for large enough i there is a holomorphic section s of $L^{a_i k} \to X_i$ such that:*

- $|\ |s(x)| - \exp(-d_i^\sharp(x,p)^2/4)\ | \leq \zeta$ *if* $d_i^\sharp(x,p) \leq R_0$;
- $|s(x)| \leq \zeta$ *if* $d_i^\sharp(x,p) \geq R_0$;
- $\|s\|_{L^{2,\sharp}} \leq (2\pi)^n + \zeta$.

This is *precisely* the same statement as Proposition 8 except that we have Z' in place of Z and $L^{a_i k}$ in place of L^k. (So we can regard Proposition 8 as a special case when all $a_i = 1$.) The point is that the proof is *precisely* the same. (It may be pedantic to have written out the whole statement but that seems the best way to be clear.) Likewise the statements in Propositions 9 and 10 apply to scaled limits, introducing the extra powers a_i.

2.5. **Points in $\mathcal{D}$.** Next we focus attention on a point $p \in \mathcal{D} \subset Z$. By definition this means that there is some tangent cone $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$. This is clearly a good tangent cone so the results above apply. We take complex co-ordinates $(u, v_1, \dots, v_{n-1})$ on $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ with the metric

$$|u|^{2\gamma - 2} |du|^2 + \sum |dv_i|^2. \tag{16}$$

Then if $z = (u, v_1, \dots, v_{n-1})$ we have, in our notation above,

$$[z]^2 = |\gamma|^{-2} |u|^{2\gamma} + \sum |v_i|^2. \tag{17}$$

Each of the co-ordinate functions u, v_i satisfy the hypotheses of Proposition 9 so we get holomorphic functions $\tilde{u}, \tilde{v}_i$ on $B^\sharp$. Now we regard these as the components of a map

$$F : B^\sharp \to \mathbf{C}^n.$$

An easy extension of Proposition 9 shows that for any $\zeta > 0$ we can suppose that

$$|[F(x)] - d(x,p)| \leq \zeta, \tag{18}$$

where we regard [ ] as the function on $\mathbf{C}^n$ defined by (17). Proposition 10 implies that for any precompact subset W in the set

$$\{z = (u, v_1, \ldots, v_{n-1}) : [z] < r, u \neq 0\}$$

and any $\epsilon, \zeta > 0$ we can suppose that there is an ϵ-Kähler embedding $\Gamma : W \to B^\sharp$ such that

$$|F \circ \Gamma(z) - z| \leq \zeta,$$

for all z in W. (Here, of course, the meaning of "we can suppose that" is that we should take ν sufficiently large and $i \geq i(\nu)$.)

Now fix $r = \gamma^{-1} + 2$. For points $z \in \mathbf{C}^n$ with $|z|^2 \leq 1$ we have an elementary inequality $[z] < \gamma^{-1}$ so if $\zeta < 1/10$ we have $|F(x)| > 1$ on the boundary of $B^\sharp$. Let $\Omega = \Omega_i \subset X_i$ be the preimage of the open unit ball in $\mathbf{C}^n$:

$$\Omega = \{x \in B^\sharp \subset X_i : |F(x)| < 1\}.$$

The next result is one of the central points in this article. Later, we will develop it further to show that Z has an natural complex manifold structure around points of $\mathcal{D}$.

Proposition 12. *For ν sufficiently large and $i \geq i(\nu)$ the map F is a holomorphic equivalence from $\Omega_i \subset X_i$ to the unit ball $B^{2n} \subset \mathbf{C}^n$.*

By construction $F : \Omega \to B^{2n}$ is a proper map and so has a well-defined degree. The fibers of F are compact analytic subsets but we have constructed a non-vanishing section s of L^k over $\Omega \subset B^\sharp$ so the fibers must be finite. To establish the proposition it suffices to show that the degree is 1. By (15) we have a fixed bound on the derivative of F over Ω, say $|\nabla F| \leq K_4$.

Recall that we are making the identification $\mathbf{C}^n = \mathbf{C}_\gamma \times \mathbf{C}^{n-1}$. We write $d_\gamma(z, z')$ for the distance between points z, z' in the singular metric, not to be confused with Euclidean distance $|z - z'|$.

Let z_0 be the point $(1/2, 0, \ldots, 0) \in B^{2n} \subset \mathbf{C}^n$. The distance, in the singular metric, from z_0 to the boundary of B^{2n} is $\gamma^{-1}(1-(1/2)^\gamma) = d$, say. The distance in the singular metric from z_0 to the singular set $S = \{u = 0\}$ is $\gamma^{-1}(1/2)^\gamma > d$. Fix a small number $\rho > 0$ and a pre-compact open set $W \subset B^{2n} \setminus S$ with the following properties:

(1) W contains the Euclidean ball $B(z_0, 2\rho)$ with center z_0 and radius 2ρ;
(2) $d_\gamma(B^{2n} \setminus W, B(z_0, 2\rho)) > d/2$; and
(3) for any $z \in B^{2n}$ there is a z' in W with $d_\gamma(z, z') < \delta_0 = \min(\rho/(20K_4), d/4)$.

Now choose ϵ, ζ such that

- $\zeta < \rho/2$;
- $K_4(\delta_0 + 2\epsilon) + \zeta < \rho/10$; and
- $\delta_0 < d/2 - 4\epsilon$.

We can suppose that there is an ϵ-Kähler embedding $\Gamma : W \to X_i$ such that $|F \circ \Gamma(z) - z| \leq \zeta < \rho/2$ for $z \in W$. This implies that $F \circ \Gamma$ maps the boundary of $B(z_0, \rho)$ to $\mathbf{C}^n \setminus \{z_0\}$. This boundary map has a well defined degree (defined by the action on $(2n-1)$-dimensional homology) and the degree is 1. Now once ϵ is reasonably small it is clear from the definition that the derivative of Γ is invertible at each point and orientation preserving. Then it follows by basic differential topology and complex analysis that for any point z with $|z - z_0| \leq \rho/4$ there is a unique $\tilde{z}$ with $|\tilde{z} - z_0| \leq \rho$ such that $F \circ \Gamma(\tilde{z}) = z$. In particular, we could take $z = z_0$, so we get a $\tilde{z}_0$ with $|\tilde{z}_0 - z_0| \leq \rho$ and $F \circ \Gamma(\tilde{z}_0) = z_0$. What we have to show is that the only point $x \in \Omega$ with $F(x) = z_0$ is $\Gamma(\tilde{z}_0)$.

Suppose then that $x \in \Omega$ and $F(x) = z_0$. By (5) and item 3 above we can find an x' in $\Gamma(W)$ with $d^\sharp(x, x') \leq \delta_0 + 2\epsilon$. Thus, $|F(x) - F(x')| \leq K_4(\delta_0 + 2\epsilon)$. Writing $x' = \Gamma(z')$ we have $|z_0 - z'| \leq |z_0 - F \circ \Gamma(z')| + |z' - F \circ \Gamma(z')| \leq K_4(\delta + 2\epsilon) + \zeta < \rho/10$. Thus z' lies in $B(z_0, \rho)$ and so $d_\gamma(z', B^{2n} \setminus W) \geq d/2$ and by (5) again $d^\sharp(x', X_i \setminus \Gamma(W)) \geq d/2 - 2\epsilon$. Since $\delta_0 < d/2 - 4\epsilon$ we see that x itself lies in $\Gamma(W)$ so we can take $x = x'$ above and we have $x = \Gamma(z')$ with $F \circ \Gamma(z') = z_0$. Thus $|z' - z_0| \leq \zeta < \rho/2$ and by the uniqueness statement in the previous paragraph we must have $z' = \tilde{z}_0$.

We can use the map F to regard $k\omega_i$ as a metric $\tilde{\omega}_i$ on the unit ball $B^{2n} \subset \mathbf{C}^n$, with co-ordinates $(u, v_1, \ldots, v_{n-1})$. We also write $\tilde{D}_i$ for $F_i(\Omega_i \cap D_i)$. Let N_δ be the (Euclidean) δ-neighborhood of the hyperplane $\{u = 0\}$. Let ω_{Euc} be the Euclidean metric and $\omega_{(\gamma)}$ be the standard cone metric representing $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$. To sum up what we know: by making ν sufficiently large and then $i \geq i(\nu)$ we can arrange the following, for fixed C:

- $\tilde{\omega}_i = i\partial\overline{\partial}\phi_i$, where $0 \leq \phi_i \leq C$ (notice we can choose ϕ_i be to $-\log|s_i|^2$ adjusted by a suitable constant, where s_i is the Gaussian section constructed in Proposition 11);
- $\tilde{\omega}_i \geq C^{-1}\omega_{\mathrm{Euc}}$; and
- for any ζ, δ we can suppose $|\tilde{\omega}_i - \omega_{(\gamma)}| \leq \zeta$ outside N_δ (and likewise for any given number of derivatives).

2.6. **Consequences.** We will work in the co-ordinates $(u, v_1, \ldots, v_{n-1})$ on the unit ball in $\mathbf{C}^n$, as above. It is clear that the singular sets $\tilde{D}_i$ must lie in the tubular neighborhood N_δ. This means that there are well-defined homological intersection numbers $\mu = \mu_i$ with a disc transverse to the hyperplane $\{u = 0\}$. The singular set is given, in this co-ordinate chart by a Weierstrasse polynomial

$$u^\mu + \sum_{j=0}^{\mu-1} A_j u^j = 0, \tag{19}$$

where the A_j are holomorphic functions of $v_1, \ldots, v_{n-1}$. (Since the precise size of this domain is arbitrary we can always suppose that picture extends to a slightly larger region.)

Proposition 13. *We have the identity* $(1 - \gamma) = \mu_i(1 - \beta_\infty)$.

A first consequence of this is that μ does not depend on i. A second consequence is that there are only a finite number of possibilities for γ; that is $\gamma = 1 - \mu(1 - \beta_\infty)$ for $\mu = 1, \ldots, \mu_{\max}$ where $\mu_{\max}$ is the largest integer μ such that $1 - \mu(1 - \beta_\infty) > 0$. So we can write

$$\mathcal{D} = \bigcup_{\mu=1}^{\mu_{\max}} \mathcal{D}_\mu. \tag{20}$$

Discussion. The possible existence of points in $\mathcal{D}_\mu$ for $\mu > 1$ is the major difficulty in the limit theory for Kähler-Einstein metrics with cone singularities.

As we will see in (3.2), we understand the points in $\mathcal{D}_1$ essentially as well as the regular points. But one expects that there will be cases when the $\mathcal{D}_\mu$ for $\mu > 1$ arise. For $\epsilon > 0$ and $\beta > 1/2$, cut out two disjoint wedge-shaped regions from $\mathbf{C}$ with angles $2\pi(1-\beta)$ and vertices at the points $\pm\epsilon$. Then identify the boundaries of the resulting space to get a manifold with two cone singularities of angle $2\pi\beta$. The limit as ϵ tends to 0 has a single singularity with cone angle $2\pi(2\beta-1)$. This is the model for the simplest way in which points of $\mathcal{D}_2$ can appear, but one expects that more complicated phenomena arise. Similarly, one should probably be careful in understanding the structure of the sets $\mathcal{D}_\mu$. Let D be the cusp curve with equation $u^2 = v^3$ in $\mathbf{C}^2$. Then it seems likely that for $\beta > 1/2$ there is a Ricci flat Kähler metric on the complement of the curve with standard cone singularities of cone angle $2\pi\beta$ away from the origin, so the tangent cones at these points would be $\mathbf{C}_\beta \times \mathbf{C}$. It seems then likely (or, at least, hard to rule out) that the tangent cone at the origin is $\mathbf{C}_\gamma \times \mathbf{C}$ with $\gamma = (2\beta - 1)$, i.e., an isolated point of $\mathcal{D}_2$.

To prove Proposition 13, we consider a disc T defined by fixing v_j at some fixed values close to 0 and constraining $|u| \leq 1/2$ say. We can suppose that the disc meets $\tilde{D}_i$ transversally in μ points.

Let σ be the local trivialising section

$$\sigma = (dudv_1 \cdots dv_{n-1})^{-1}$$

of the anti-canonical bundle K^{-1}, furnished by the co-ordinates. Away from the singular set $\tilde{D}_i$ we have a connection form

$$\alpha_i = \sigma^{-1}\nabla\sigma,$$

and $d\alpha_i = \rho_i$ where ρ_i is the Ricci form of $\tilde{\omega}_i$. Define

$$Hol_i = \int_{\partial T} \alpha_i.$$

This is a lift, from $\mathbf{R}/\mathbf{Z}$ to $\mathbf{R}$, of the holonomy of the anti-canonical bunde around ∂T; a lift which is provided by the trivialising section and depends only on the homotopy class of the section over ∂T.

Lemma 1. *Let* $\mathrm{Hol}_i \in S^1 \subset \mathbf{C}$ *denote the holonomy of the anti-canonical bundle around the boundary of* T*. Then*

$$\mathrm{Hol}_i = 2\pi\sqrt{-1}\left(\mu(1-\beta_i) + \int_T \rho\right). \tag{21}$$

Choose local complex co-ordinates $z_1, \ldots, z_n$ in a small neighborhood of a point of $T \cap \tilde{D}_i$ so that $\tilde{D}_i$ is defined locally by the equation $z_1 = 0$ and T by $z_2 = \ldots z_n = 0$. Let $\mathrm{Hol}(r)$ denote the integral of the connection form α_i around the circle $|z_1|^\beta = r$ in T. By Stokes' formula, it suffices to prove that, for all such intersection points, $\mathrm{Hol}(r)$ tends to $2\pi(1-\beta_i)$ as r tends to zero. Let L be the logarithm of the ratio of the volume form of $\tilde{\omega}_i$ and $|z_1|^{-2(1-\beta)}$ times the Euclidean volume element, in these co-ordinates. The hypothesis that $\tilde{\omega}_i$ has a standard cone singularity implies that L is a bounded function. Take polar co-ordinates r, θ on D, with $r = |z_1|^\beta$ and θ the argument of z_1. Then

$$\mathrm{Hol}(r) = 2\pi\left((1-\beta_i) + I(r)\right)$$

where

$$I(r) = r\int \frac{\partial L}{\partial r} d\theta.$$

(Here we use the fact that the trivialisations σ and $dz_1 \cdots dz_n^{-1}$ of the anticanonical bundle are homotopic over the circle $|z_1|^\beta = r$ in T, since they both extend over the interior disc.) The fact that the Ricci curvature of ω_i is bounded implies, using the Gauss-Bonnet formula again, that

$$|I(r_1) - I(r_2)| \leq C(r_1^2 - r_2^2),$$

so $I(r)$ has a limit h as r tends to 0. Thus,

$$\frac{d}{dr} \int L d\theta = hr^{-1} + o(r^{-1}),$$

and if h is not zero this implies that $\int L d\theta$ is unbounded as $r \to 0$ which is a contradiction. This completes the proof of Lemma 1.

Note that the point in the proof of the Lemma is that we can work for a fixed i on an arbitrarily small neighborhood of an intersection point. We are not claiming that there is any uniformity in the size of this neighborhood as i varies.

Making ν and i large we can make Hol_i as close as we please to the corresponding holonomy for the metric $\omega_{(\gamma)}$, which is $2\pi(1-\gamma)$. On the other hand $\rho = k^{-1}(1 - \lambda(1-\beta_i))\tilde{\omega}_i$ where k can be made as large as we want by making ν large. So (using Lemma 1), to establish Proposition 13 it suffices to prove that the area of T, in the metric $\tilde{\omega}_i$, satisfies a fixed bound.

Now we use a version of the the well-known Chern-Levine-Nirenberg inequality. We fix a standard cut-off function G of $|u|$, equal to 1 when $|u| \leq 1/2$ and vanishing for $|u| \geq 3/4$. Then

$$\int_T \tilde{\omega}_i \leq \int_{2T} G \omega_i^\sharp = \int_{2T} i\partial\overline{\partial} G \phi_i \leq C \int_{2T} |i\partial\overline{\partial} G|$$

where $2T$ has the obvious meaning and we use the bound on the Kähler potential ϕ_i.

Remarks

(1) It may be possible to prove Proposition 13 by adapting the Cheeger-Colding "slicing" theory using harmonic functions. But we avoid that by using the "holomorphic slicing" as above.

(2) A similar argument, applied to the Chern connection on the line bundle L^k, shows that in fact the potential "topological obstruction" arising from holonomy does not occur. So for points of $\mathcal{D}$ we do not need to introduce the scale factors $t_{i,\nu}$.

In the next two propositions we study the volume of the divisor $\tilde{D}_i \subset B^{2n}$, with respect to the metric $\tilde{\omega}_i$

Proposition 14. *There is a universal constant $c_1 > 0$ such that*

$$\int_{\tilde{D}_i} \tilde{\omega}_i^{2(n-1)} \geq c_1.$$

Since the Euclidean metric is dominated by a multiple of $\tilde{\omega}_i$ it suffices to get a lower bound on the Euclidean volume. But this is clear. Recall that if A is any p-dimensional complex analytic variety in $B^{2n} \subset \mathbf{C}^n$ and if we write $V(r)$ for the volume of the intersection of A with the r ball then $r^{-2p}V(r)$ is an increasing function of r. If the origin lies in A then the limit of this ratio as r tends to 0 is an least the volume of the ball in B^{2p}, so we get a lower bound on $V(1)$. In our case the divisor $\tilde{D}_i$ need not pass exactly through the origin but it contain a point within the δ neighborhood of the origin (in the Euclidean metric) so the same argument applies.

Next let $\frac{1}{2}B^{2n}$ denote as usual the ball of radius 1/2.

Proposition 15. *There is a universal constant c_2 (depending on μ) so that*

$$\int_{\frac{1}{2}B^{2n}\cap\tilde{D}_i} \tilde{\omega}_i^{2(n-1)} \leq c_2.$$

(Of course, by adjusting our set-up slightly, we could replace $\frac{1}{2}B^{2n}$ by any given sub-domain.)

The proof uses another variant of the Chern-Levine-Nirenberg argument. We fix a sequence of cut-off functions $G_1, \dots, G_{n-1}$ depending on $|\underline{v}|$ (where $\underline{v} = (v_1, \dots, v_{n-1})$), such that G_{n-1} vanishes when $|\underline{v}| \geq 3/4$, $G_1 = 1$ when $|v| \leq 1/2$ and $G_{j+1} = 1$ on the support of G_j, for $j = 1, \dots, n-2$. Let

$$\Theta = \sqrt{-1}\sum_{j=1}^{n-1} dv_i d\overline{v}_i,$$

i.e., the Euclidean form pulled up from the $\mathbf{C}^{n-1}$ factor. Then we can suppose that $i\partial\overline{\partial}G_j \leq C\Theta$ for some fixed C (since the G_j depend only on $\underline{v}$). On the other hand, it follows from our set-up that, if δ is reasonably small, the restrictions of G_j to D_i are functions of compact support. Now we write

$$\int_{\frac{1}{2}B^{2n}\cap\tilde{D}_i} \tilde{\omega}_i^{n-1} \leq \int_{B^{2n}\cap\tilde{D}_i} G_1(i\partial\overline{\partial}\phi_i)^{n-1}.$$

Integrating by parts, the right-hand side is bounded by

$$C\int_{B^{2n}\cap\tilde{D}_i} \phi_i\Theta(i\partial\overline{\partial}\phi_i)^{n-2},$$

and we continue in the usual fashion to exchange factors of $i\partial\overline{\partial}\phi_i$ for factors of Θ. After $(n-1)$ steps we get a bound in terms of

$$\int_{\tilde{D}_i\cap\{|\underline{v}|\leq 3/4\}} \Theta^{n-1},$$

which is exactly μ times the volume of the 3/4-ball in $\mathbf{C}^{n-1}$.

All of the discussion above is local and applies just as well to scaled limits Z', in the manner described in (2.5) above.

2.7. **Lower bound on densities.** Let $|\ |_*$ denote the norm on $\mathbf{C}^q$

$$|(y_1, \dots, y_q)|_* = \max_j |y_j|.$$

Proposition 16. *Suppose that p is a point in a scaled limit space Z' which is a based limit of $(X_i, a_i\omega_i)$ for integers a_i. Let d_i be distance functions as usual. Suppose that there is a neighborhood $N \subset Z'$ of p such that for all p' in N there is at least one good tangent cone to Z' at p'. Then we can find C and $\rho_1 > \rho_2 > 0$ and q such that the following is true. For large enough i there is a holomorphic map $F : \Omega_i \to \mathbf{C}^q$ where $\Omega_i \subset X_i$ is an open set containing the points $x \in X_i$ with $d_i(p, x) \leq \rho_1$ and*

- *$|\nabla F_i| \leq C$ (where the norm is computed using the scaled metric $a_i\omega_i$);*
- *for points x with $d_i(p, x) = \rho_1$ we have $|F_i(x)|_* \geq 1/2$;*
- *for points x with $d_i(p, x) \leq \rho_2$ we have $|F_i(x)|_* \leq 1/100$.*

We apply Proposition 11 to find k, R and a holomorphic section s of $L^{ka_i} \to X_i$ (for large i) such that $||s(x)| - \exp(-d^2/4)| \leq 1/200$ if $d \leq R$ and $|s(x)| \leq 1/200$ if $d \geq R$. Here, $d = \sqrt{k} d_i(p, x)$. We can take k arbitrarily large so there is no loss in supposing that all points in Z' a distance less than $2k^{-1/2}$ from p have good tangent cones. There is also no loss of generality (multiplying by a factor slightly larger than 1 if necessary) in supposing that there is some fixed $t_0 > 0$ such that $|s| \geq 1$ at points where $d \leq t_0$.

Let t_+, t_- be the fixed numbers such that

$$\exp(-t_\pm^2/4) = 1/2 \mp 1/100,$$

so $t_+ > t_-$. Let $\mathcal{A} \subset Z'$ be the closed annulus of points distance between $k^{-\frac{1}{2}} t_-, k^{-\frac{1}{2}} t_+$ from p. Since $t_\pm < 2$, each point p' of $\mathcal{A}$ has some good tangent cone. Applying Proposition 11 we can find some integer $l(p') \geq 10$ and a number $R(p')$ such that for all sufficiently large i there is a holomorphic section $\sigma_{p'}$ of $L^{kl(p')a_i}$ which obeys just the same estimates as above but with $d = \sqrt{kl(p')} d_i$. (That is, having fixed k we can replace L by L^k and then apply Proposition 11.) By compactness of $\mathcal{A}$ we can find a finite collection $p'_j \in \mathcal{A}$ for $j = 1, \ldots, q$ such that the balls of radii $r_j = \frac{t_-}{2}(kl(p'_j))^{-\frac{1}{2}}$ with these centers cover $\mathcal{A}$. Now define F_i, mapping into $\mathbf{C}^q$, to be the function with components $\sigma_{p'_j}/s^{l(p'_j)}$. Let $\rho_1 = k^{-1/2} t_-$. Then for points x with $d_i(x, p) \leq \rho_1$ we have $|s(x)| \geq 1/2$ so F_i is well defined at such points and satisfies a C^1 bound. For sufficiently large i, for any point x with $d_i(x, p) = \rho_1$ there is some j such that $d_i(x, p'_j) \leq \frac{t_-}{2}(kl(p'_j))^{-\frac{1}{2}}$ and this means that $|\sigma_{p'_j}| \geq 1/2$. Thus for such a point

$$|F_i(x)|_* \geq 1/2,$$

since it is clear that $|s| \leq 1$ at such points. Let $\rho_2 = k^{-1/2} \min(t_0, \frac{t_-}{2})$ so at points x with $d(x, p) \leq \rho_2$ we have $|s(x)| \geq 1$. By our choice that $l_j \geq 10$ it is clear that $|\sigma_{p'_j}(x)| \leq 1/100$ for each j. Thus, at such points

$$|F_i(x)|_* \leq 1/100.$$

For a point $x \in X_i$ and $r > 0$ define the density function

$$V(x, r) = r^{2-2n} \mathrm{Vol}(D_i \cap B_r(x)), \tag{22}$$

where of course the volume is computed using ω_i.

Proposition 17. *There is a constant $c > 0$ such that for all i, all points $x \in D_i \subset X_i$ and all $r \leq 1$ we have $V(x, r) \geq c$.*

The proof is by contradiction, so suppose we have sequences $x_i \in D_i$ and $r_i \leq 1$ such that $V(x_i, r_i) \to 0$. Rescale by factors r_i^{-1} and take the based limit space (Z', p). We claim first that every tangent cone to Z' at a point z in Z' of distance strictly less than 1 from p is good. In fact, we claim that if $C(Y)$ is such a tangent cone at z then $\mathcal{D}(Y)$ is empty. Given this the assertion that the tangent cone is good follows from Proposition 5.

Suppose that there is some $y \in \mathcal{D}_Y$. The tangent cone $C(Y)$ is itself a scaled limit of the X_i, for a suitable choice of scalings and base points. Thus, we can apply Proposition 14, noticing that the discussion there extends to more general scaled limits, as pointed out in the end of Section 2.6. It follows that there is some definite lower bound on the volume of the singular set D_i in the unit ball centered at x_i (for the re-scaled metric), contradicting our assumption.

Thus, we can apply Proposition 16 to construct holomorphic maps $F_i : B_i \to \mathbf{C}^q$ where $B_i \subset X_i$ is the set of points of (scaled) distance at most ρ_1 from p. Fix $h > 0$ such that the h-neighborhood of the set $\{|y|_* \leq 1/100\}$ in $\mathbf{C}^q$ lies in the set $\{|y|_* \leq 1/2\}$. Pick a point $x \in D_i$ with $d_i(x, p) < \rho_2$ (which is possible for large i by the definition of Gromov-Hausdorff convergence) and set $y = F_i(x)$ so $|y|_* \leq 1/100$. The fiber $F_i^{-1}(y)$ cannot meet the boundary of B_i and so is a compact analytic subset. Since, as in the proof of Proposition 12, the line bundle L^k is trivial over B_i we see that the fiber is a finite set. It follows that the image $F_i(D_i)$ is an $(n-1)$-dimensional analytic set. More precisely, the intersection with the ball of radius h centered at y is a closed $(n-1)$-dimensional analytic set and so its volume is bounded below, by the discussion in the proof of Proposition 14. The bound on the derivative of F_i gives a lower bound on the volume of $D_i \cap B_i$ and since $\rho_1 < 1$ this yields a lower bound on the volume of D_i in the unit ball centered at x_i, for large i. This gives our contradiction.

Discussion. The construction in Proposition 16 can be refined to show that maps like F_i can be chosen to be embeddings. It is also possible to pass to a limit as $i \to \infty$, and thus get a local complex analytic model for the scaled limit space Z'. It is interesting to understand the algebro-geometric meaning of these. But here we have just constructed what we need for our present purposes, and leave further developments for the future.

2.8. **Good tangent cones.** We consider a general scaled limit $Z' = \lim(X_i, a_i\omega_i)$ as above. We define a variant of the density function; for $z \in Z'$ and $\rho > 0$

$$V(i, z, \rho) = \rho^{2-2n}\mathrm{Vol}\{x \in D_i \subset X_i : d_i(x, z) \leq \rho\}, \tag{23}$$

where the volume is computed using the metric $a_i\omega_i$.

Proposition 18. *There is a $c' > 0$ such that for all $z \in \mathcal{D}(Z')$ and all $\rho > 0$ we have*

$$\liminf_i V(i, z, \rho) \geq c'.$$

This follows easily from Proposition 17.

Now in the other direction we have

Proposition 19. *For any z in Z' which is not in $S_2(Z')$ there is a $\rho > 0$ such that $V(i, z, \rho)$ is bounded.*

This follows immediately from Proposition 15.

The proof of Theorem 3 is a little complicated. For purposes of exposition we will first prove a different but very similar result.

Proposition 20. *Let Z' be a limit space as above and let K be a compact subset of Z'. Then $S(Z') \cap K$ has capacity zero.*

The proof has seven steps.

Step 1. By the Hausdorff dimension property we can find a countable collection B_μ of open balls of radius r_μ with $\sum r_\mu^{2n-3}$ arbitrarily small which cover $S_2 \cap K$. We can suppose that the (1/10)-sized balls with the same centers are disjoint. Let $U = \bigcup_\mu B_\mu$ so U is open.

Step 2 Let $J = K \cap (Z' \setminus U)$. Thus J is a compact set and no point of J lies in S_2. Thus, we can apply Proposition 19 at each point of J. Taking a finite covering we arrive at an open set W such that $J \subset W$ and a $\rho > 0$ such that the (2n-2)-volume of the intersection of D_i with the ρ-neighborhood of $\overline{W}$ is bounded by a fixed constant, for all i.

Step 3. The set $K \cap (Z' \setminus W)$ is compact and contained in $\bigcup_\mu B_\mu$. Thus, we can find a finite sub cover, say by the B_μ for $\mu \leq M$. Then we can use the method of [11] to construct a function g_1, equal to 1 on $\bigcup_{\mu \leq M} B_\mu$, supported in a slightly larger set and with L^2 norm of the derivative small. (Note that the point here is that the construction of [11] will not work for infinite covers.)

Step 4. Now let T be the intersection of the closure of $\mathcal{D}$ with $\overline{W} \cap K$. We want to show that T has finite Minkowski measure. So we need to show that there is an $M > 0$ such that for any $\epsilon > 0$ we can cover T by at most $M\epsilon^{2-2n}$ balls of radius ϵ. We may suppose that $\epsilon < \rho$ where ρ is as in Step 2 above. By a standard argument we find a cover by ϵ- balls such that the half-size balls are disjoint. Then the estimate on the number of balls follows from the upper and lower volume bounds for the D_i (Propositions 18 and 19).

Step 5. By Proposition 1 we can construct a cut-off g_2 with L^2 norm of the derivative small, equal to 1 on an open neighborhood N of T and supported in a somewhat larger (but arbitrarily small) neighborhood.

Step 6. Now $g_1 + g_2$ is ≥ 1 on the open set $N^+ = N \cup \bigcup_{\mu \leq M} B_\mu$. Let x be a point of $(S \setminus S_2) \cap K$. Then if $x \in \overline{W}$ we have $x \in T$ so $x \in N$. If x is not in $\overline{W}$ then x is in $\bigcup_{\mu \leq M} B_\mu$. So, either way, x is in N^+. Write Σ for the intersection of $S \cap K$ with $Z \setminus N^+$. Then Σ is compact and, by the previous sentence, is contained in S_2. Thus, Σ has Hausdorff codimension > 2. As before, we can construct a cut-off g_3 for Σ supported on a an arbitrarily small neighborhod of Σ and with derivative arbitrarily small in L^2.

Step 7. Now $g_1 + g_2 + g_3$ is ≥ 1 on all points of $S \cap K$. We choose a function $F(t)$ with $F(0) = 0$ and with $F(t) = 1$ if $t \geq 0.9$. Then set $g = F(g_1 + g_2 + g_3)$.

Now we give the proof of Theorem 3, which is a variant of that above. Fix small $\delta > 0$ and let I be the interval $[1-\delta, 1+\delta]$. Consider $Y \times I$ as embedded in the cone $C(Y)$ in the obvious way and identify Y with $Y \times \{1\}$. To begin with we work on Y so we cover $S(Y)$ by balls $B_\mu \subset Y$ as in Step 1 above and let $U = \bigcup_\mu B_\mu \subset Y$. As in Step 2, we find an open set W containing $Y \setminus U$ and a $\rho > 0$ such that the volume of D_i in the ρ-neighborhood of $\overline{W} \times I$ is bounded. Then we find a finite set of balls B_μ which cover $C(Y) \setminus W$. We construct a cut-off g_1 on Y using this finite set of balls as in Step 3. Let T be the intersection of the closure of $S(Y) \setminus S_2(Y)$ with $\overline{W}$. We want to show that the volume of the ϵ-neighborhood of T is $O(\epsilon^2)$. But in an obvious way this volume is controlled by the volume of the ϵ-neighborhood of $T \times I$ in $C(Y)$ for which we can argue as in Step 4 above. Then we construct a cut-off g_2 on Y and the last steps are just the same as above.

3. Proofs of Theorems 1 and 2

3.1. **Global structure.** We want to define a notion of the complex structure on Z being smooth in a neighborhood of a point. We adopt a rather complicated definition, which is tailored to our needs. Given $p \in Z$, say the complex structure near p is smooth if the following is true. There is a neighborhood Ω_∞ of p which is the limit of open sets $\Omega_i \subset X_i$ and for some $k = k(p)$ there are sections $\sigma_{i,0}, \ldots, \sigma_{i,n}$ of $L^k \to X_i$ with the properties below.

(1) The $\sigma_{i,j}$ satisfy a fixed L^∞ bound.
(2) $|\sigma_{i,0}| \geq c > 0$ on Ω_i for some fixed c.
(3) The holomorphic maps $F_i : \Omega_i \to \mathbf{C}^n$ given by $\sigma_{i,j}/\sigma_{i,0}$ are homeomorphisms from Ω_i onto $B^{2n} \subset \mathbf{C}^n$, satisfying a fixed Lipschitz bound.
(4) Some limit $F_\infty : \Omega_\infty \to B^{2n}$ is also a homeomorphism.
(5) There are holomorphic functions P_i on B^{2n} such that (in a smaller domain) the divisor $F_i(D_i)$ is defined by the equation $P_i = 0$ and the P_i converge to a function P_∞, not identically zero, as i tends to ∞.

Let $Z_0 \subset Z$ be the set of points around which the complex structure is smooth. By the nature of the definition this set is open in Z.

Proposition 21. *All points of $\mathcal{D} \cup R$ lie in Z_0.*

It follows that the complement $Z \setminus Z_0$ is a closed set of Hausdorff codimension at least 4. This implies in particular that Z_0 is connected.

The essential content of Proposition 21 is that $\mathcal{D}$ is contained in Z_0. The proof for R is similar but easier. So suppose $p \in \mathcal{D}$ and go back to the discussion in (2.5). We have holomorphic equivalences

$$F_i : \Omega_i \to B^{2n}$$

where $\Omega_i \subset X_i$ lies between $B^\sharp(p, 2)$ and $B^\sharp(p, 1/2)$ and these maps satisfy a fixed Lipschitz bound. It follows that by taking a subsequence we can pass to the limit to get a Lipschitz map F_∞ from a neighborhood Ω_∞ of p in Z to B^{2n}. The difficulty is that while we know the F_i are injective we do not yet know the same for F_∞.

Proposition 22. *The map $F_\infty : \Omega_\infty \to B^{2n}$ is a homeomorphism.*

The essential thing is to show that the map is injective. So suppose we have distinct $x, y \in \Omega_\infty$ with $F_\infty(x) = F_\infty(y)$. Choose sequences x_i, y_i in X_i converging to x, y. Recall that we have locally non-vanishing sections s of $L^k \to X_i$. Taking a subsequence, and possibly interchanging x, y there is no loss in assuming that $|s(x_i)| \leq |s(y_i)|$ for all i. Applying Proposition 8 we can find some large m such that there is for each large i a section τ of L^{mk} such that $|\tau(x_i)| = 1$ and $|\tau(y_i)| \leq 1/2$ (say). Now set

$$f_i = \frac{\tau}{s^m}.$$

Then by construction $|f_i(x_i)| \geq 2|f_i(y_i)|$ and $|f_i(x_i)| \geq c > 0$ for some fixed c (depending on m). Now consider the maps $(F_i, f_i) : \Omega_i \to B^{2n} \times \mathbf{C}$. On the one hand these satisfy a Lipschitz bound so we can pass to the limit and get a map (F_∞, f_∞) from Ω_∞ to $B^{2n} \times \mathbf{C}$ which separates x, y, by construction. On the other hand, for finite i the image of (F_i, f_i) is the graph of a holomorphic function h_i on B^{2n}. The h_i satisfy a fixed L^∞ bound so (taking a subsequence) we can suppose they converge to h_∞. Clearly, the image of (F_∞, g_∞) is the graph of h_∞, which contradicts the fact that (F_∞, f_∞) separates x, y.

Proposition 22 now follows easily. For each i the divisor is defined by a Weierstrass polynomial (19) and it is clear that (taking a subsequence) these have a nontrivial limit.

Remark. One can give a different proof of Proposition 22 using the Kähler-Einstein equations, similar to Proposition 2.5 in [9]. We will give a version of this argument in the sequel. The advantage of the proof given is that it does not use the Kähler-Einstein equations and potentially extends to other situations.

Given this background we can proceed, following the arguments in [11], to analyze the global structure of Z.

- The arguments in [11] go over without any change to show that there is some large m with the following effect. Use the L^2 norm to define a metric on $H^0(X_i, L^m)$ and pick an orthonormal basis. Then we get projective embeddings $T_i : X_i \to \mathbf{CP}^N$, canonically defined up to unitary transformations. The images converge to a normal variety $W \subset \mathbf{CP}^N$ and there is a Lipschitz homeomorphism $h : Z \to W$ compatible with the Gromov-Hausdorff convergence, in the sense made precise in [11].
- Likewise, the set $Z_0 \subset Z$ maps under h to the smooth part of W. This is clear from the definition of Z_0. (More precisely, that definition shows that a neighborhood of $p \in Z_0$ is embedded smoothly in projective space by sections of $L^{k(p)}$. But once we know that the images of Z stabilize for large k we can take a fixed power m.)
- It is also clear that $h(Z_0)$ is exactly equal to the smooth part of W.
- Taking a subsequence, we can suppose that the $T_i(D_i)$ converge to an $n-1$ dimensional algebraic cycle $\Delta = \sum_a \nu_a \Delta_a$ in $\mathbf{CP}^N$. The fact that W is normal implies that the intersection of each Δ_a with the singular set of W is a proper algebraic subset of Δ_a. Then Δ is a Weil divisor in W.
- Since the volume of the D_i is a fixed number the lower bound on densities (Proposition 17) implies a fixed bound on the Minkowski measure $m(D_i)$. Then, as in the discussion at the end of (2.1), we get (after perhaps passing to a subsequence) a limit set $D_\infty \subset Z$ which is a closed set with $m(D_\infty) \leq M$. Thus D_∞ has finite codimension 2 Hausdorff measure.
- We have $\mathcal{D} \subset D_\infty \subset S(Z)$. This is clear from the discussion in (2.2).
- The homeomorphism $h : Z \to W$ maps D_∞ onto suppΔ. This follows from the compatability between the algebraic and Gromov-Hausdorff convergence.
- The complement $S_2 = S \setminus \mathcal{D}$ has Hausdorff dimension at most $2n-4$. Thus, the same is true for $D_\infty \setminus \mathcal{D}$, in particular, $D_\infty = \overline{\mathcal{D}}$. The Lipschitz property of h implies that supp $\Delta \setminus h(\mathcal{D})$ has Hausdorff dimension at most $2n-4$. In particular, $h(\mathcal{D})$ is dense in suppΔ.

All of the above discussion would apply equally well (with suitable hypotheses) in a more general situation where L is not equal to the anti-canonical bundle. We will now develop results specific to the "Fano case", which is our concern in this article. For simplicity, we assume that $\lambda = 1$. (The general case would be exactly the same, since we can take k to run through multiples of the fixed number λ.)

For each i, the divisor $D_i \subset X_i$ is defined by a section S_i of K^{-1}. We have a hermitian metric h_i on $L = K^{-1}$. The Kähler-Einstein equation takes the form

$$\omega_i^n = (S_i \wedge \overline{S_i})^{-1} |S_i|_{h_i}^{2\beta_i}, \tag{24}$$

and we can normalize so that the L^2 norm of S_i (defined using the metric h_i) is 1. Then from (7) we get a fixed upper bound on the L^∞ norm of S_i.

Now consider the situation around a point of Z_0 in the charts Ω_i occurring in the definition. We have local co-ordinates z_a and a Kähler potential $\phi_i = k^{-1} \log |\sigma_{0,i}|^2$ which is bounded above and below. Let $\Theta = dz_1 \dots dz_n$. We can write $\sigma_{0,i} =$

$V_i^k\Theta^{-k}, S_i = U_i P_i \Theta^{-1}$ where U_i, V_i are non-vanishing holomorphic functions on the ball, and P_i is the given local defining function for D_i. Then the equation becomes (dropping the index i)

$$\det(\phi_{a\bar{b}}) = |UP|^{2\beta-2}|V|^{-2\beta}e^{-\beta\phi}. \tag{25}$$

Proposition 23. *In such co-ordinates there is C, independent of i, such that on a fixed interior ball $C^{-1} < |U_i|, |V_i| < C$.*

Consider $\mu > 0$, an open set G in B^{2n} and the integral

$$I_{\mu,G} = \int_G |S_i|^{2\mu}\omega_i^n.$$

The L^∞ bound on S_i gives an upper bound on $I_{\mu,G}$, independent of i. On the other hand we claim that for any μ, G, the integral $I_{\mu,G}$ has a strictly positive lower bound, independent of i. We prove this by a contradiction argument (it is possible to write down a more constructive proof); so suppose that we have a G, μ such that $I_{\mu,G}$ tends to zero as i tends to infinity. We adopt for the moment the point of view of [11] so we regard the anticanonical bundle as an abstract line bundle L over X_i. Then just as in [11] we can construct a limiting line bundle over the regular part of the limit space and we can suppose that the S_i converge to a non-trivial holomorphic section S_∞ of this limiting line bundle. By Proposition 22, we can regard a dense open subset of G as an open set G^* in the limit space, and the condition that $I_{\mu,G}$ tends to zero implies that S_∞ vanishes on G^*. Now it is clear from the structure theory above that the regular set in the limit is connected. By analytic continuation, the section S_∞ vanishes identically and this gives our contradiction.

Now, using the Kähler-Einstein equation, we can write

$$I_{\mu,G} = \int_G |PU|^{2\beta+2\mu-2}|V|^{-2\beta-2\mu}e^{-(\mu+\beta)\phi}\Theta\wedge\overline{\Theta},$$

where again we temporarily drop the index i. First, take $\mu = 1-\beta$. Then the integral becomes

$$I = \int |V|^{-2}e^{-\phi}\Theta\wedge\overline{\Theta}.$$

Since ϕ is bounded above and below this is equivalent to the integral of $|V|^{-2}$, with respect to the Euclidean measure. Begin by taking G to be the whole unit ball. Thus we have an L^2 bound on the holomorphic function $f = V^{-1}$ over the ball which gives an L^∞ bound $|f| \leq C$ over an interior ball. We claim that on the other hand we must have some lower bound $|f| \geq C^{-1}$ over an interior ball, say $(1/4)B^{2n}$. To see this we argue by contradiction: if not there is a violating sequence f_j with points $z_j \in (1/4)B^{2n}$ such that $|f_j(z_j)| \to 0$. Taking a subsequence, we can suppose that the z_j have limit z and the f_j converge to a limit f_∞ in C^∞ on compact subset of B^{2n}. We must have $f_\infty(z_\infty) = 0$. If f_∞ is not identically zero then, by basic complex analysis, we get a nearby zero of f_j for large j in contradiction to our hypothesis. On the other hand, we know that the integral of $|f_j|^2$ over a small ball centered at z_∞ has a strictly positive lower bound, so the limit cannot be identically zero.

This argument gives the required upper and lower bounds on $|V|$. Now take $\mu = 2-\beta$. Then $I_{\mu,G}$ is equivalent to the integral over G of $|U|^2|P|^2$. Re-instate the index i so $P = P_i$ which have non-trivial limit P_∞. Away from the zero set of P_∞ we can apply the argument above to get upper and lower bounds on $|U_i|$, for large i. Then we can extend these over the zero set by applying the Cauchy integral formula to the restrictions of U_i, U_i^{-1} to suitable small discs.

Given Proposition 23, we can pass to the limit $i = \infty$ (working over a slightly smaller ball). The upper and lower bounds on $|U_i|, |V_i|$ mean that we can take limits and get holomorphic functions U_∞, V_∞ with the same bounds. Then $U_\infty P_\infty \Theta$ is a local section of K_W^{-1} defining the divisor Δ. While this is a local discussion the local sections obviously glue together to define a global section S_∞ of K_W^{-1} which defines Δ (in the sense that Δ is the closure of the zero set of S_∞ over the smooth part of W, as discussed in Section 1).

Now to see that $(W, (1-\beta_\infty)\Delta)$ is KLT we write

$$(S \wedge \overline{S})^{-1} = (\sigma \wedge \overline{\sigma})^{-1/m} |\sigma|^{2/m} |S|^{-2},$$

where σ is any local trivializing section of K_W^{-m} (Here we drop the subscript ∞.) So the Kähler-Einstein equation gives

$$(\sigma \wedge \overline{\sigma})^{-1/m} |S|^{2\beta-2} = \omega^n |\sigma|^{-2/m}.$$

The KLT condition is that the integral over the ball of $(\sigma \wedge \overline{\sigma})^{-1/m}|S|^{2\beta-2}$ is finite but by the identity above this is the integral of $\omega^n|\sigma|^{-2/m}$ which is clearly finite (since σ has no zeros) and the finiteness of the integral is clear (since $|\sigma|$ is bounded below). It also follows immediately from the set-up that the limiting metric is a weak conical Kähler-Einstein metric, as defined in Section 1. This completes the proof of Theorem 1.

3.2. **Smooth limits: proof of Theorem 2.** We now turn to Theorem 2. Of course what we prove is a local result: if p is a smooth point of W and of the divisor Δ then, near p the limiting metric is a metric with cone singularities along Δ and cone angle $2\pi\beta_\infty$. Recall from the Introduction that, for our purposes, this means in the sense of the class of metrics defined in [10]. Also see the Appendix for a study of improving Hölder regularity for such metrics. Now go back to the converging sequence of metrics ω_i on X_i, working in local co-ordinates $(u, v_1, \dots, v_{n-1})$. There is no loss of generality in supposing that, near p each D_i is given by $u = 0$ (since we can always adjust the co-ordinates to achieve this, on a small ball). For any $\beta \in (0,1)$, let $\omega_{(\beta)}$ be the standard cone metric on $\mathbf{C}^n = \mathbf{C}_\beta \times \mathbf{C}^{n-1}$, as above. We first establish a uniform bound.

Proposition 24. *There is a constant C such that*

$$C^{-1}\omega_{(\beta_i)} \leq \omega_i \leq C\omega_{(\beta_i)}.$$

To prove Proposition 24, drop the index i and write $\beta_i = \beta$ and $\tilde{\omega}_i = \omega$. We have $\omega = i\partial\overline{\partial}\phi$ where $0 \leq \phi \leq C_0$. In the set up of (3.1) the Kähler-Einstein equation is

$$\omega^n = |u|^{2\beta-2}|Q|^2 e^{-\beta\phi}\Theta \wedge \overline{\Theta},$$

where Q is a holomorphic function on the ball with $C_1^{-1} \leq |Q| \leq C_1$. (That is, $Q = U^{\beta-1}V^{-\beta}$, in the notation of (3.1)—we can take fractional powers since the ball is simply connected.) Here $\Theta = du dv_1 \dots dv_{n-1}$. However, changing ϕ by subtracting $\beta^{-1}\log|Q|^2$ we can suppose that in fact $Q = 1$ so our equation is

$$\omega^n = e^{-\beta\phi}\omega^n_{(\beta)}, \tag{26}$$

since $\omega_{(\beta)}^n = |u|^{2\beta-2}\Theta\wedge\overline{\Theta}$. We know that $\omega \geq C_3^{-1}\omega_{\rm Euc}$, where $\omega_{\rm Euc}$ is the standard Euclidean metric on $\mathbf{C}^n$ (by item (3) in the definition at the beginning of (3.1)). To establish Proposition 24 it suffices to show that

$$\omega \geq C_4^{-1}\omega_{(\beta)}, \tag{27}$$

for by (26) the ratio of the volume forms $\omega^n, \omega_{(\beta)}^n$ is bounded above and below and the lower bound (27) gives an upper bound.

To prove (27), let h be the square root of the trace of $\omega_{(\beta)}$ with respect to ω, so

$$h^2\omega^n = \omega_{(\beta)} \wedge \omega^{n-1}. \tag{28}$$

We claim that on a smaller ball, say $(1/2)B^{2n}$, we have an integral bound

$$\int_{(1/2)B^{2n}} h^2\omega^n \leq C_5. \tag{29}$$

This follows from the Chern-Levine-Nirenberg argument. Let χ be a standard function of compact support on B^{2n} equal to 1 on $(3/4)B^{2n}$, say. Then we have $i\partial\overline{\partial}\chi \leq C_6\omega_{\rm Euc} \leq C_3C_6\omega_{(\beta)}$. Then

$$\int_{(3/4)B^{2n}} \omega_\beta \wedge \omega^{n-1} \leq \int_{B^{2n}} \chi\omega_\beta \wedge \omega^{n-1}.$$

Integrating by parts, the right-hand side is

$$\int_{B^{2n}} \phi(i\partial\overline{\partial}\chi)\omega_\beta\omega^{n-2},$$

which is bounded by

$$C_0C_6\int_{B^{2n}} \omega_\beta^2\omega^{n-2}.$$

Just as in Proposition 15 above, we continue with a nested sequence of cut-off functions to interchange powers of ω_β, ω and arrive at a bound (29). The function h^2 can also be regarded as $|\nabla f|^2$ where f is the identify map from the ball with metric ω to the ball with metric $\omega_{(\beta)}$. The fact that ω has positive Ricci curvature and $\omega_{(\beta)}$ has zero Riemannian curvature imply that away from the singular set we have $\Delta h \geq 0$. In fact, we have

$$\Delta h^2 = \Delta(|\nabla f|^2) \geq 2|\nabla\nabla f|^2 \tag{30}$$

where $\nabla f \in \Gamma(T^* \otimes T)$ is the identity endomorphism of the tangent bundle and the covariant derivative is that formed using ω on one factor and $\omega_{(\beta)}$ on the other. This can be seen as an instance of the Chern-Lu inequality, or more simply as the Bochner formula, since away from the singular set we can take local Euclidean co-ordinates for $\omega_{(\beta)}$. Using the fact that $|\nabla\nabla f| \geq |\nabla|\nabla f||$ we get $\Delta h \geq 0$.

Lemma 2. *The inequality $\Delta h \geq 0$ holds in a weak sense across the singular set. That is, for any compactly supported smooth non-negative test function F we have*

$$\int (\nabla h, \nabla F) \leq 0. \tag{31}$$

It is here that we use the fact that the cone angles match up. This implies, by the definition of a cone singularity, that h is a bounded function on the complement of the divisor $\{u = 0\}$, say $h \leq M$. Thus, any given metric ω is bounded above and below by *some* multiples of $\omega_{(\beta)}$. Let σ be a non-negative function of compact support on the intersection of the ball with $\{u \neq 0\}$. Then

$$\nabla(\sigma^2 h).\nabla h = |\nabla(\sigma h)|^2 - h^2|\nabla\sigma|^2.$$

Since

$$\int \nabla(\sigma^2 h).\nabla h = -\int (\sigma^2 h \Delta h) \leq 0,$$

we have

$$\int |\nabla(\sigma h)|^2 \leq \int h^2 |\nabla\sigma|^2 \leq M^2 \int |\nabla\sigma|^2.$$

By a construction like that in (2.1) we can choose functions σ_j equal to 1 in intersection of the half-ball $\frac{1}{2}B^{2n}$ with $\{|u| \geq \epsilon_j\}$ where $\epsilon_j \to 0$ but with the L^2 norm of $\nabla\sigma_j$ bounded by a fixed constant. (It suffices to do this using the model metric $\omega_{(\beta)}$ since, as we have noted above, the given metric ω is bounded above and below by *some* multiples of $\omega_{(\beta)}$.) Taking the limit as $j \to \infty$ we see that

$$\int_{\frac{1}{2}B^{2n}} |\nabla h|^2 < \infty.$$

By obvious arguments, it suffices for our purposes to prove (31) for functions F supported in $\frac{1}{2}B^{2n}$. Thus, we can suppose that $F\nabla\sigma_j$ is supported in $\{u \leq \epsilon_j\}$ and that the L^2 norm of $F\nabla\sigma_j$ tends to 0 as $j \to \infty$. Now

$$\int \sigma_j \nabla F.\nabla h = -\int F\nabla\sigma_j.\nabla h - \int F\sigma_j \Delta h \leq \|F\nabla\sigma_j\|_{L^2}\|\nabla h\|_{L^2}.$$

Taking the limit as $j \to \infty$ we see that

$$\int \nabla F.\nabla h \leq 0.$$

With this Lemma in place we return to the proof of Proposition 24. We use the Moser iteration technique, starting with the bound on the integral of h^2 and the inequality $\Delta h \geq 0$ to obtain an L^∞ bound on h. Here, of course, we use the fact that there is a uniform Sobolev inequality for ω, which follows from the remark after Theorem 1.2 in [9]. The Euclidean ball $(1/2)B^{2n}$ in our co-ordinates contains the metric ball (in the metric ω of radius $(2C_3)^{-1}$ centered at p (since $\omega \geq C_3^{-1}\omega_{\mathrm{Euc}}$). Then Moser iteration gives a bound on h over the metric ball of radius $(4C_3)^{-1}$ say. Finally, this metric ball contains a Euclidean ball of a definite size. This can be seen from the argument of [9] Proposition 2.4. (In fact, that argument shows that the identity map from $(B^{2n}, \omega_{\mathrm{Euc}})$ to (B^{2n}, ω) satisfies a fixed Hölder bound with exponent β. Notice the argument in [9] is written for an approximating metric but can be easily adapted to derive the estimate for the metric ω.) This completes the proof of Proposition 24.

To deduce Theorem 2 from Proposition 24 we need some way of improving the estimates. In this general area, one approach is to try to estimate higher derivatives. For metrics with cone singularities, this was achieved by Brendle [2] assuming that $\beta < 1/2$. Another approach in this general area is furnished by the Evans-Krylov theory and an analog of that theory for metrics with cone singularities was developed by Calamai and Zheng [3], assuming that $\beta < 2/3$. A general result of Evans-Krylov type is stated in [13] where two independent proofs are given, but at the time of writing we have had difficulty in following the one of these proofs that we have, so far, studied in detail. Partly for this reason, and partly because it has its own interest, we give a different approach (to achieve what we need) below. Our approach is related to ideas of Anderson [1] in the standard theory, and we have other results, which apply under slightly different hypotheses, using a line of argument closer to Anderson's, and which we will give elsewhere.

Write $\omega = i\partial\overline{\partial}\phi$ for the limit of ω_i as i tends to infinity. This is a smooth Kähler-Einstein metric outside $\{u = 0\}$ and is a local representation of the metric on the Gromov-Hausdorff limit. Proposition 24 and the C^1 estimate (8) imply that ϕ satisfies a Lipschitz bound with respect to the standard cone metric $\omega_{(\beta)}$.

For $\epsilon > 0$ let $T_\epsilon : \mathbf{C}^n \to \mathbf{C}^n$ be the linear map $T_\epsilon(u, \underline{v}) = (\epsilon^{1/\beta}u, \epsilon\underline{v})$. Suppose ϵ_j is any sequence with $\epsilon_j \to 0$ and define a sequence $\omega^{(j)}$ of re-scalings by

$$\omega^{(j)} = \epsilon^{-2}T_\epsilon^*(\omega). \tag{32}$$

The scaling behavior of the model $\omega_{(\beta)}$ implies that we still have uniform upper and lower bounds $C^{-1}\omega_{(\beta)} \leq \omega^{(j)} \leq C\omega_{(\beta)}$ so (perhaps taking a subsequence) we get C^∞ convergence on compact subsets of $\{u \neq 0\}$ to a limit $\omega^{(\infty)}$. The Kähler-Einstein equation satisfied by ϕ and the fact that ϕ satisfies a Lipschitz bound with respect to the $\omega_{(\beta)}$ metric implies that $(\omega^{(\infty)})^n = e^{-\beta\phi(0)}\omega_{(\beta)}^n$, and without loss of generality we may assume $\phi(0) = 0$. On the other hand, the uniform bound of Proposition 24 implies that this limit $\omega^{(\infty)}$ represents the metric on a tangent cone to (Z, ω) at the given point p.

Proposition 25. *Suppose ω' is a Kähler metric defined on the complement of the divisor $\{u = 0\}$ in $\mathbf{C}^n$ such that:*

- *the volume form of ω' is the same as that of $\omega_{(\beta)}$;*
- *there are uniform bounds*

$$C^{-1}\omega_{(\beta)} \leq \omega' \leq C\omega_{(\beta)}; \textit{and}$$

- *$(\mathbf{C}^n, \omega')$ is a metric cone with vertex 0.*

Then there is a complex linear isomorphism $g : \mathbf{C}^n \to \mathbf{C}^n$ preserving the subspace $\{u = 0\}$ such that $\omega' = g^*(\omega_{(\beta)})$.

In our context this implies that any tangent cone at a point of Δ is $\mathbf{C}_\beta \times \mathbf{C}^{n-1}$, in other words all points of Δ lie in $\mathcal{D}_1$. Conversely, it is clear from the Weierstrasse representation (19) that at points of $\mathcal{D}_1$ the limiting divisor is smooth so we are in the situation considered here.

The hypothesis that $(\mathbf{C}^n, \omega')$ is a cone means that there is a radial vector field X, initially defined over $\{u \neq 0\}$. In the model case of $\omega_{(\beta)}$ the corresponding vector field is

$$X_{(\beta)} = \mathrm{Re}\left(\beta^{-1}u\partial_u + \sum_{i=1}^{n-1} v_i\partial_{v_i}\right).$$

We want to show that, after perhaps applying a linear transformation we have $X = X_{(\beta)}$. Write

$$X = \mathrm{Re}\left(a_0\partial_u + \sum_{i=1}^{n-1} a_i\partial_{v_i}\right),$$

for functions a_i. The Kähler condition means that the a_i are holomorphic. The length of X, computed in the metric ω' is equal to the distance to the origin. By the uniform upper and lower bounds we get

$$|u|^{2\beta-2}|a_0|^2 + \sum_{i=1}^{n-1} |a_i|^2 \leq C'(|u|^{2\beta} + \sum_{i=1}^{n-1} |v_i|^2), \tag{33}$$

for another constant C'. This implies that for each $i \geq 0$ we have

$$|a_i| \leq C''|(u, \underline{v})|^{1+\beta}.$$

First, by Riemann's removal singularities theorem this means that the a_i extend holomorphically over $u = 0$. Second, since $\beta < 1$ this means that the a_i are linear functions. Also, we see (taking $u = 0$ in (33)) that a_0 vanishes when $u = 0$, and (taking $u \to \infty$) that a_i is independent of u for $i \geq 1$. Thus, the vector field is defined by an endomorphism $A : \mathbf{C}^n \to \mathbf{C}^n$, preserving the subspace $\{u = 0\}$. Write $A_{(\beta)}$ for the endomorphism corresponding to $X_{(\beta)}$.

For $\tau \in \mathbf{C}$ consider the vector field $\mathrm{Re}(\tau X)$ and the one parameter group f_τ of holomorphic transformations so generated. Thus, f_τ takes a point of distance 1 from the origin to a point of distance $\exp(\mathrm{Re}(\tau))$ from the origin. Using the equivalence of metrics again, this translates into the statement that there is a constant C'' such that for all $z \in \mathbf{C}^n$ with $|z| = 1$ we have

$$(C'')^{-1} \exp(\mathrm{Re}(\tau)) \leq [\exp(\tau A)z] \leq C'' \exp(\mathrm{Re}(\tau)), \tag{34}$$

where $[(u, \underline{v})] = \left(u^{2\beta} + |\underline{v}|^2\right)^{1/2}$. Let A_0 be the restriction of A to $\{u = 0\}$ and apply this to an eigenvector v of A_0. We see that the eigenvalue must be 1. Suppose that A_0 is not the identity. Then we can write $A_0 = 1 + N$ where for some $k \geq 1$ we have $N^k \neq 0$ but $N^{k+1} = 0$. Then

$$\exp(\tau A_0) = \exp(\tau)(\tau^k N^k/k! + \cdots),$$

and if we take v so that $N^k v \neq 0$ we see that this violates (34) for large τ. So A_0 is the identity. Arguing in the same way we see that the there must be an eigenvalue β^{-1} of A. Arguing in the same way we see that the there must be an eigenvalue β^{-1} of A. Since $a_0 = cu$ for some constant c and a_i is independent of u for $i \geq 1$, this implies that the matrix A is block diagonal and thus $A = A_{(\beta)}$ and so $X = X_{(\beta)}$.

To complete the proof of Proposition 25, we consider the function h defined as the square root of the trace of $\omega_{(\beta)}$ with respect to ω', just as in the proof of Proposition 24. This is defined on the complement of the singular set $\{u = 0\}$ and is invariant under the real one-parameter subgroup generated by $X = X_{(\beta)}$. As before, we have $\Delta h \geq 0$. Let $\mathcal{A}$ be the "annulus" consisting of points $z \in \mathbf{C}^n$ with distance (in the metric ω') between 1 and 2 from the origin. Let χ be a non-negative function on a neighborhood of $\mathcal{A}$, equal to 1 outside a small neighborhood of $\{u = 0\}$ and vanishing near to $\{u = 0\}$. Then, much as in the proof of Lemma 2,

$$-\int_{\mathcal{A}} \chi^2 h \Delta h = \int_{\mathcal{A}} |\nabla(\chi h)|^2 - h^2 |\nabla \chi|^2 - \int_{\partial \mathcal{A}} \chi^2 h (\nabla h.n),$$

where n is the unit normal vector to the boundary. The fact that h is invariant under the flow of X implies that $(\nabla h.n) = 0$ so the boundary term vanishes and taking the limit over a sequence of functions χ we see that the integral of $|\nabla h|^2$ vanishes, so h is a constant. Going back to (30), we see that ω' is covariant constant with respect to $\omega_{(\beta)}$. This easily implies that ω' is isometric to $\omega_{(\beta)}$ by a complex linear transformation (We remark that from the proof we could replace the first hypothesis in the statement of Proposition 25 by a weaker hypothesis that the Ricci curvature of ω' is non-negative.)

An alternative approach to this last part of the proof is to apply the maximum principle to h. The only difficulty is that the maximum might be attained at a singular point $q = (u, 0)$ with $u \neq 0$. We can work by induction on n so we suppose we have proved Proposition 25 in lower dimensions. If we blow up at this point q, we arrive at a Kähler Ricci flat metric cone $(C(Y) \times \mathbf{C}^{n-k}, \omega_\phi)$ which is quasi isometric to $(\mathbf{C}_\beta \times \mathbf{C}^{n-1}, \omega_\beta)$. By Cheeger-Colding theorem, we know that $n - k \geq 1$. The splitting of C^{n-k} is represented by $n - k$ dimensional complex holomorphic vector field with constant length w.r.t. ω_ϕ. By quasi isometry, this set of holomorphic vector fields is bounded in $(\mathbf{C}_\beta \times \mathbf{C}^{n-1}, \omega_\beta)$. It follows these are constant holomorphic vector fields in $\mathbf{C}^n$. It follows that $(C(Y) \times \mathbf{C}^{n-k}, \omega_\phi)$ splits holomorphic and isometrically so that $(C(Y), \omega_\phi \mid_{C(Y)})$ is Ricci flat, quasi-isometry to $(\mathbf{C}_\beta \times \mathbf{C}^{k-1}, \omega_\beta)$ in $\mathbf{C}_\beta \times \mathbf{C}^{k-1}$. By induction, we know that $(C(Y), \omega_\phi \mid_{C(Y)})$ is equivalent to $(\mathbf{C}_\beta \times \mathbf{C}^{k-1}, \omega_\beta)$. It follows that $(C(Y) \times \mathbf{C}^{n-k}, \omega_\phi)$ is standard. Then (using also the arguments below) we can prove that ω' is $C^{,\alpha,\beta}$ near q. Once we know this it is straightforward to see that the maximum principle can be applied.

We now go back to the metrics ω_i, before taking the limit. For $\epsilon \in (0, 1)$ let

$$\omega_{i,\epsilon} = \epsilon^{-2} T_\epsilon^*(\omega_i).$$

We consider this as a metric defined on the unit ball B^{2n}.

Proposition 26. *Given any $\zeta > 0$ we can find $\epsilon(\zeta)$ such that for $\epsilon \leq \epsilon(\zeta)$ we can find a linear map g_ϵ preserving the subspace $\{u = 0\}$ such that for $i \geq i(\epsilon)$ we have*

$$(1 - \zeta) g_\epsilon^*(\omega_{(\beta_i)}) \leq \omega_{i,\epsilon} \leq (1 + \zeta) g_\epsilon^*(\omega_{(\beta_i)}), \tag{35}$$

throughout $\frac{1}{2} B^{2n}$.

The argument is similar to that in the proof of Proposition 24. Let $h_{i,\epsilon}$ be the square root of the trace of $g^*\omega_{(\beta_i)}$ with respect to $\omega_{i,\epsilon}$, where g is the linear isomorphism determined by Proposition 25. The uniform Lipschitz bound on the Kähler potential and the Kähler-Einstein equation (26) imply that the ratio of the volume forms $\omega_{i,\epsilon}^n, \omega_{(\beta_i)}^n$ is within $c\epsilon$ of 1, for a fixed constant c. Thus, if λ_a are the eigenvalues of $\omega_{(\beta_i)}$ with respect to $\omega_{i,\epsilon}$ we have

$$h_{i,\epsilon}^2 = \sum \lambda_a, \prod \lambda_a \geq (1 - c\epsilon).$$

Thus,

$$h_{i,\epsilon}^2 \geq n(1 - c\epsilon)^{1/n} \geq n - c'\epsilon,$$

say, for a suitable fixed c'. Set $f_{i,\epsilon} = h_{i,\epsilon}^2 - n + c'\epsilon$, so $f_{i,\epsilon} \geq 0$ and $\Delta f_{i,\epsilon} \geq 0$. Suppose we know that, for some θ, the L^2 norm of $f_{i,\epsilon}$ over B^{2n} is bounded by θ. Then Moser iteration implies that $f_{i,\epsilon} \leq K\theta$ over the half-sized ball, for a fixed computable K. Thus, $\sum \lambda_a \leq n + K\theta$ and $\prod \lambda_a \geq 1 - c'\epsilon$. It follows by elementary arguments that $1 - \zeta \leq \lambda_a \leq 1 + \zeta$ provided that θ, ϵ are sufficiently small compared with ζ.

The problem then is to show that for any θ we can make the L^2 norm of $f_{i,\epsilon}$ less than θ, by taking ϵ small and then $i \geq i(\epsilon)$. Of course to do this we may need to apply a linear transformation g_ϵ, as in the statement of Proposition 26. Clearly, it is the same to get the L^2 bound on $h_{i,\epsilon}^2 - n$. First we work with the limiting metric ω_∞, writing $h_{\infty,\epsilon}$ in the obvious way. Since $h_{\infty,\epsilon}$ satisfies a fixed L^∞ bound, it suffices to show that as ϵ tends to zero the functions $h^2_{\infty,\epsilon} - n$ converge to zero uniformly on compact subsets of $\{u \neq 0\}$. But this follows from Proposition 25 (after applying linear transformations). Now the convergence of ω_i to ω_∞, again uniformly on compact subsets of $\{u \neq 0\}$ gives our result.

We now move on to complete the proof of Theorem 2. Given any fixed $\zeta > 0$ we can without loss of generality suppose (by the result above) that over the unit ball B^{2n} our metrics ω_i satisfy

$$(1-\zeta)\omega_{(\beta_i)} \leq \omega_i \leq (1+\zeta)\omega_{(\beta_i)}. \tag{36}$$

For $q \in \mathbf{C}^{2n}$ and $\rho > 0$ write $B^{\beta_i}(q,\rho)$ for the ball of radius ρ and center q, in the metric $\omega_{(\beta_i)}$. We also fix a number $\alpha \in (0,1)$ with $\alpha < \beta^{-1} - 1$.

Lemma 3. *There are $C, \zeta_0 > 0$ and $R > 2$ such that, if ω_i satisfies (36) on B^{2n} with $\zeta = \zeta_0$, then if q is any point of B^{2n} and ρ is such that $B^{\beta_i}(q, R\rho) \subset B^{2n}$ then on $B^{\beta_i}(q, 2\rho)$ we can write $\omega_i = \omega_{(\beta_i)} + i\partial\overline{\partial}\psi$ where $[\psi]_\alpha \leq C\rho^{2-\alpha}$.*

Here $[\ ]_\alpha$ denotes the Hölder seminorm defined by the metric $\omega_{(\beta_i)}$, which is of course equivalent to that defined by the metric ω_i. This Lemma is straightforward to prove using the Hörmander technique to construct a suitable section of $L^k \to X_i$ for an appropriate power k. The power $\rho^{2-\alpha}$ comes from the scaling behavior: that is when the ball $B^{\beta_i}(q,\rho)$ is scaled to unit size and the metric is scaled the estimate for the rescaled potential $\tilde{\psi}$ becomes $[\tilde{\psi}]_\alpha \leq C$. On the other hand, the result is local and can probably also be proved by using the "weight function" version of the Hörmander technique (with any $R > 2$), or by other methods from complex analysis.

To simplify notation we drop the index i. We want to fix a suitable value of ζ. First, consider the function $F(M) = \det M - \mathrm{Tr} M$ on the space of $n \times n$ Hermitian matrices. Since the derivative at the identity vanishes we can for any $\eta > 0$ find a $\zeta(\eta)$ such that if M_1, M_2 satisfy $(1-\zeta) \leq M_i \leq (1+\zeta)$ then

$$|F(M_1) - F(M_2)| \leq \eta |M_1 - M_2|. \tag{37}$$

Next, we recall the Schauder estimate of [10]. This asserts that for any β and $\alpha \in (0,1)$ with $\alpha < \beta^{-1} - 1$ there is some $K = K(\alpha,\beta)$ such that for all $\psi \in \mathcal{C}^{2,\alpha,\beta}(B^\beta(0,2))$ we have

$$[i\partial\overline{\partial}\psi]_{\alpha, B^\beta(0,1)} \leq K\left([\Delta\psi]_{\alpha,B^\beta(0,2)} + [\psi]_{\alpha,B^\beta(0,2)}\right). \tag{38}$$

Here, Δ is the Laplacian of $\omega_{(\beta)}$. More generally, we can choose $K = K(\alpha,\beta)$ such that for *any point* $q \in \mathbf{C}^n$, radius $\rho > 0$ and for $\psi \in \mathcal{C}^{2,\alpha,\beta}(B^{(\beta)}(q,2\rho))$ we have

$$[i\partial\overline{\partial}\psi]_{\alpha, B^\beta(q,\rho)} \leq K\left([\Delta\psi]_{\alpha,B^\beta(q,2\rho)} + \rho^{-2}[\psi]_{\alpha,B^{(\beta)}(q,2\rho)}\right). \tag{39}$$

This follows by a straightforward argument, using (38) (after scaling and translation) at points near the singular set and the standard Schauder estimate away from the singular set. Furthermore, for any β and $\alpha \in (0,1)$ with $\alpha < \beta^{-1} - 1$ we can suppose that the inequality, with a fixed K, holds for α', β' sufficiently close to α, β. Now fix η with

$$(R-2)^{-\alpha} K\eta \leq R^{-\alpha}/4, \tag{40}$$

and fix $\zeta < \zeta_0$ and $\zeta < \zeta(\eta)$, where ζ_0, R are as in Lemma 3. For distinct points x, y in the Euclidean unit ball B, not in the singular set, define

$$Q(x,y) = d_\beta(x,y)^{-\alpha} |\omega(x) - \omega(y)| \left(\min(d_\beta(x,\partial B), d_\beta(y,\partial B)\right)^\alpha. \tag{41}$$

Here, d_β denotes the distance in the metric $\omega_{(\beta)}$. The difference $\omega(x) - \omega(y)$ is interpreted in the sense of [10], in that we take the matrix entries with respect to a fixed orthonormal frame for the (1,1) forms (in which $\omega_{(\beta)}$ is constant). Thus, by definition,

$$[\omega]_{\alpha,B} = \sup_{x,y} \left(d_\beta(x,y)^{-\alpha} |\omega(x) - \omega(y)| \right),$$

where the supremum is taken over distinct points x, y in B. By hypothesis this supremum is finite, so we have a finite number $M = \sup_{x,y} Q(x,y)$. We seek an *a priori* bound on M. It is not immediate that this supremum is attained, but we can certainly choose x, y so that $Q(x,y) \geq M/2$. Set $d = \min(d_\beta(x, \partial B), d_\beta(y, \partial B))$. If $d_\beta(x,y) \geq d/R$ then $M/2 \leq R^\alpha |\omega(x) - \omega(y)|$ and we are done since we have an L^∞ bound on ω. So we can suppose that $d_\beta(x,y) < d/R$. Let x be the point closest to the boundary and take it to be the center q in Lemma 3, with radius $\rho = d/R$. Thus we can apply Lemma 3 to write $\omega = \omega_{(\beta)} + i\partial\overline{\partial}\psi$ over $B^\beta(x, 2\rho)$, with $[\psi]_\alpha \leq C\rho^{2-\alpha}$. Thus, by construction

$$[i\partial\overline{\partial}\psi]_{\alpha, B^\beta(x,\rho)} \geq (M/2)d^{-\alpha} = (M/2)(R\rho)^{-\alpha}, \tag{42}$$

since y lies in $B^\beta(x, \rho)$. On the other hand, if z, w are distinct points in $B^\beta(x, 2\rho)$ the inequality $Q(z,w) \leq M$ implies that

$$d_\beta(z,w)|\omega(z) - \omega(w)| \leq ((R-2)\rho)^{-\alpha} M,$$

since the distance from z, w to the boundary is at least $(R-2)\rho$. So

$$[i\partial\overline{\partial}\psi]_{\alpha, B^\beta(x,2\rho)} \leq ((R-2)\rho)^{-\alpha} M. \tag{43}$$

Write det for the ratio of the volume forms $\omega^n, \omega^n_{(\beta)}$. Then (37) implies that

$$[\det - \Delta\psi]_{\alpha, B^\beta(x,2\rho)} \leq \eta [i\partial\overline{\partial}\psi]_{\alpha, B^\beta(x,2\rho)},$$

So

$$\begin{aligned} [\Delta\psi]_{\alpha, B^\beta(x,2\rho)} &\leq \eta [i\partial\overline{\partial}\psi]_{\alpha, B^\beta(x,2\rho)} + [\det]_{\alpha, B^\beta(x,2\rho)} \\ &\leq ((R-2)\rho)^{-\alpha} M\eta + [\det]_{\alpha, B^\beta(x,2\rho)}. \end{aligned}$$

Thus by (39) and (42)

$$\begin{aligned} R^{-\alpha}\rho^{-\alpha} M/2 \leq [i\partial\overline{\partial}\psi]_{\alpha, B^\beta(x,\rho)} &\leq ((R-2)\rho)^{-\alpha} KM\eta \\ &+ K[\det]_{\alpha, B^\beta(x,2\rho)} + K\rho^{-2}[\psi]_{\alpha, B^\beta(x,2\rho)}, \end{aligned}$$

and by our choice $(R-2)^{-\alpha} K\eta \leq R^{-\alpha}/4$ we can rearrange to get

$$R^{-\alpha}\rho^{-\alpha} M/4 \leq K[\det]_{\alpha, B^\beta(x,2\rho)} + K\rho^{-2}[\psi]_{\alpha, B^\beta(x,2\rho)}.$$

Now, putting in the bound from Lemma 3 and multiplying by ρ^α, we have

$$R^{-\alpha} M/4 \leq K\rho^\alpha [\det]_\alpha + CK,$$

which gives our bound on M.

To establish Theorem 2 we fix $\alpha \in (0,1)$ with $\alpha < \beta_\infty^{-1} - 1$ and apply the above discussion to ω_i with $\beta = \beta_i$, so we can take a fixed constant K in the Schauder estimate. We see that the limiting metric, as $i \to \infty$, lies in $\mathcal{C}^{,\alpha,\beta_\infty}$.

Remark. To prove Theorem 2 we have only had to deal with the points in the divisor Δ, since the other points are covered by arguing as in [11], using a version of the Evans-Krylov theory. On the other hand, we can also handle these latter points by just the same argument used above, not invoking the Evans-Krylov theory (from PDE) but applying instead the Cheeger-Colding Theory (from Riemannian geometry).

In this article, we have been mainly focused on the case when the Ricci curvature is strictly positive, but it is easy to see that Theorems 1 and 2 also hold in the case of non-positive Ricci curvature, as long as we assume a volume non-collapsing condition. For our purpose in the sequel, we state the following convergence theorem.

Theorem 4. *Let X be a Fano manifold, and D be a smooth divisor in $|-\lambda K_X|$. Fix $\beta_0 \in (0, 1-\lambda^{-1})$, and let β_i be a sequence in $(\beta_0, 1-\lambda^{-1})$ converging to β_∞. Suppose there are C^{α,β_i} Kähler-Einstein metrics ω_i on X with cone angle $2\pi\beta_i$ along D for some $\alpha \in (0, \min(1, \beta_\infty^{-1}-1))$, then ω_i is uniformly bound in $C^{,\alpha,\beta_i}$, and it converges to a $C^{,\alpha,\beta_\infty}$ Kähler-Einstein metric ω_∞ with cone angle $2\pi\beta_\infty$ along D.*

The proof of this theorem is indeed easier than our main theorem. We choose a sequence of C^{α,β_i} Kähler metrics on X with cone angle $2\pi\beta_i$ along D, say ω_{β_i}, that converge in $C^{,\alpha,\beta_\infty}$ to a limit ω_{β_∞}. Then we can write $\omega_i = \omega_{\beta_i} + \sqrt{-1}\partial\bar\partial\phi_i$. with ϕ_i in C^{2,α,β_i}. The point here is that we are working on a *fixed* manifold, so by Theorem 3.11 in [9] we can directly obtain an L^∞ bound for the Kähler potential ϕ_i, uniform bound on diameter and the Sobolev constant. Moreover, Theorem 2.2 in [9] implies that the identity map $(X, \omega_i) \to (X, \omega_0)$ is uniformly bi-Hölder, where ω_0 is a fixed smooth metric on X. In particular, it follows that a ball of fixed size with respect to the metric ω_i is contained in ball of equivalent size with respect to ω_0, and vice versa. Then, Proposition 24 in this article implies that ω_i is uniformly equivalent to ω_{β_i} in any small ball centered at the divisor. The global equivalence follows since we already have uniform C^2 bound on ϕ_i away from divisor (again, by Theorem 2.2 in [9]). Alternatively, this global equivalence also follows from the general result in [13], given the L^∞ bound that is already at hand. Then it is straightforward to adapt the arguments in Section 3.2 to obtain a uniform C^{2,α,β_i} estimate on ϕ_i.

4. Appendix: Hölder regularity for Kähler-Einstein metrics with cone singularities

In this Appendix we prove that for Kähler-Einstein metrics with cone singularities, our definition using $C^{2,\alpha,\beta}$ potentials does not depend on the choice of $\alpha \in (0,\mu)$, where $\mu = \min(1, \beta^{-1}-1)$.[3] The result is purely local, so we focus on a unit ball B in $\mathbf{C}^n = \mathbf{C}_\beta \times \mathbf{C}^{n-1}$, with coordinates $(u, v_1, \dots v_{n-1})$. Denote by $S = B \cap \{u = 0\}$ the singular set. The standard cone metric is

$$\omega_0 = \beta^2 |u|^{2\beta-2} i du d\bar u + \sum_a i dv_a d\bar v_a.$$

Write $u = r^{1/\beta} e^{i\theta}$, and $\epsilon = dr + i\beta r d\theta$. Recall a $C^{,\alpha,\beta}$ Kähler metric is a $(1,1)$ current $\omega = \omega_0 + i\partial\bar\partial\phi$ that is uniformly equivalent to ω_0, and such that ϕ and $i\partial\bar\partial\phi$ are in $C^{,\alpha,\beta}$ with respect to ω_0. This means that under the decomposition

$$i\partial\bar\partial\phi = m\epsilon\bar\epsilon + \sum_a m_a \epsilon d\bar v_a + \sum_a \bar m_a dv_a \bar\epsilon + \sum_{a,b} m_{ab} dv_a d\bar v_b,$$

[3]After this article was written, Yuanqi Wang showed us a considerably simpler proof of this result which we hope will appear elsewhere.

the co-efficients m, m_a, m_{ab} are all in $C^{,\alpha,\beta}$ and m_a's vanishes on S. Then we have the following.

Proposition 27. *Suppose ω is a $C^{,\alpha,\beta}$ (for some $\alpha \in (0,\mu)$) Kähler metric on B and satisfies the Kähler-Einstein equation outside S, then ω is in $C^{,\bar{\alpha},\beta}(0.5B)$ for any $\bar{\alpha} < \mu$.*

First of all, we recall the linear estimates from [10]. Suppose a function ϕ is in $C^{,\alpha,\beta}(B)$, and $\Delta\phi$ is well-defined pointwise outside S and is in $C^{,\alpha,\beta}(B)$, then $i\partial\bar{\partial}\phi$ is in $C^{,\alpha,\beta}(B')$ for an interior ball B'. Furthermore, for any $\bar{\alpha} < \mu$,

$$[\phi]_{\bar{\alpha},B'} + [\nabla\phi]_{\bar{\alpha},B'} + [i\partial\bar{\partial}\phi]_{\alpha,B'} \leq K([\Delta\phi]_{\alpha,B} + Osc(\phi)), \tag{44}$$

where K depends only on $B', \alpha, \bar{\alpha}$. Here the operator and norms are taken with respect to the metric ω_0, but as remarked in [10], the same estimate holds when we use a metric whose difference from ω_0 has small $C^{,\alpha,\beta}$ norm.

Now we prove the Proposition. The local Kähler-Einstein equation takes the form

$$(\omega_0 + i\partial\bar{\partial}\phi)^n = e^{-\lambda\phi}\omega_0^n. \tag{45}$$

Here λ is a constant, and by scaling we may assume λ is sufficiently small so that (44) holds for the perturbed operator $\Delta + \lambda$. By assumption we have

$$[i\partial\bar{\partial}\phi]_{\alpha,B} \leq C.$$

By an easy iteration, it suffices to prove the following.

Lemma 4. *Let $\mu_1 = \frac{2\alpha\mu - \alpha^2}{\mu}$. For any $\alpha_1 \in (\alpha, \mu_1)$, there is a constant $C = C(\alpha_1) > 0$, so that for any $y', y'' \in 0.5B$, we have*

$$|i\partial\bar{\partial}\phi(y') - i\partial\bar{\partial}\phi(y'')| \leq C(\alpha_1) \cdot d(y', y'')^{\alpha_1}. \tag{46}$$

In the proof below, we fix an $\alpha_1 \in (\alpha, \mu_1)$. Write $y' = (u', v')$ and $y'' = (u'', v'')$. Clearly, we only need to prove the estimate assuming either $u' = u''$ or $v' = v''$. Now we divide our discussion into these two cases.

Case I: $u' = u''$. This is the easier case. By standard bootstrapping we know ϕ is smooth outside S. Denote by ϕ_a the derivative of ϕ with respect to ∂_{v_a}. Taking the derivative of both sides of (45), we obtain $\Delta_\omega \phi_a = -\lambda\phi_a$. By the above linear estimate, we know $\nabla\phi$ is in $C^{,\alpha,\beta}$. So $\Delta_\omega\phi_a$ is also in $C^{,\alpha,\beta}$ outside S. Then again by the linear estimates we obtain that $[i\partial\bar{\partial}\phi_a]_{\alpha,0.5B}$ is bounded. Then (46) follows easily.

Case II: $v' = v''$. Without loss of generality we assume $v' = v'' = 0$. By a linear transformation on u and v separately we may assume $\Theta(0,0) = 0$. By adjusting by a constant we may also assume $\phi(0,0) = 0$. We write the Kähler-Einstein equation as

$$(\Delta + \lambda)\phi = Q + e^{-\lambda\phi} + \lambda\phi,$$

where

$$Q = \sum_{i \geq 2} a_i \frac{\omega_0^{n-i}\Theta^i}{\omega_0^n}.$$

For any $\rho < 1/2$ we define the dilation L_ρ which sends (u, v) to $(\rho^{1/\beta}u, \rho v)$. Denote $\tilde{\phi} = L_\rho^*\phi$ and $\tilde{Q} = L_\rho^* Q$. Then $\Delta\tilde{\phi} = \rho^2(\tilde{Q} + e^{-\lambda\tilde{\phi}})$. Since $\Theta(0,0) = 0$, by assumption, we have $\Theta(y) \leq Cd(0,y)^\alpha\omega_0$. Since $[\Theta]_{\alpha,B} \leq C$, we have $[Q]_\alpha \leq C\rho^\alpha$ on ρB.

So $[\tilde{Q}]_\alpha \leq C\rho^{2\alpha}$ on B. Similarly, we have $[e^{-\lambda\tilde{\phi}} + \lambda\tilde{\phi}]_\alpha \leq C\rho^{2\alpha}$ and $[\tilde{\phi}]_\alpha \leq C\rho^\alpha$. On $0.8B$ we can solve the equation $(\Delta + \lambda\rho^2)\tilde{\psi} = \rho^2(\tilde{Q} + e^{-\lambda\tilde{\phi}} + \lambda\tilde{\phi})$ with estimates

$$[\tilde{\psi}]_\alpha + [i\partial\bar{\partial}\tilde{\psi}]_\alpha + |i\partial\bar{\partial}\tilde{\psi}| \leq C\rho^{2\alpha+2}.$$

Now let $\tilde{h} = \tilde{\phi} - \tilde{\psi}$; then on $0.8B$ we have

$$(\Delta + \lambda\rho^2)\tilde{h} = 0, \tag{47}$$

and

$$|i\partial\bar{\partial}\tilde{h}| \leq C\rho^{2+\alpha}, [\tilde{h}]_\alpha \leq C\rho^\alpha.$$

Following the usual proof of local $\partial\bar{\partial}$ lemma, we can write $i\partial\bar{\partial}\tilde{h} = i\partial\bar{\partial}F$ on $0.7B$, with

$$|F| \leq C\rho^{2+\alpha}.$$

Now since F satisfies the equation

$$\Delta F = -\lambda\rho^2\tilde{h}.$$

By the above recalled linear estimate, we have for any fixed $\bar{\alpha} < \mu$,

$$[i\partial\bar{\partial}\tilde{h}]_{\bar{\alpha},0.6B} = [i\partial\bar{\partial}F]_{\bar{\alpha},0.6B} \leq C([\lambda\rho^2\tilde{h}]_{\bar{\alpha},0.8B} + Osc(F)) \leq C\rho^{2+\alpha}.$$

Since $\tilde{\phi} = \tilde{h} + \tilde{\psi}$, rescale back we obtain for any $y_1, y_2 \in \frac{\rho}{2}B$ that

$$\begin{aligned} |i\partial\bar{\partial}\phi(y_1) - i\partial\bar{\partial}\phi(y_2)| &\leq C(\rho^\alpha d(y_1,y_2)^\alpha + \rho^{\alpha-\bar{\alpha}} d(y_1,y_2)^{\bar{\alpha}}) \\ &= Cd(y_1,y_2)^{\alpha_1}(\rho^\alpha d(y_1,y_2)^{\alpha-\alpha_1} + \rho^{\alpha-\bar{\alpha}} d_(y_1,y_2)^{\bar{\alpha}-\alpha_1}). \end{aligned}$$

Now we are ready to set the constants $\rho, \bar{\alpha}$ as follows. First set

$$\gamma = \frac{\alpha_1 - \alpha}{\alpha} < 1.$$

Then,

$$\frac{\bar{\alpha} - \alpha_1}{\bar{\alpha} - \alpha} = \gamma.$$

One can check that $\bar{\alpha} \in (\alpha, \mu)$. Finally, set $\rho = 10d(y_1, y_2)^\gamma$. Then

$$|i\partial\bar{\partial}\phi(y_1) - i\partial\bar{\partial}\phi(y_2)| \leq Cd(y_1,y_2)^{\alpha_1}.$$

Note we have assumed the condition that $y_1, y_2 \in \frac{\rho}{2}B$. Clearly, one can get the same estimate, with the origin replaced by any point $(0, v')$ in $S \cap 0.5B$. Since the linear change of coordinates is uniformly bounded for all such v', we obtain for $y', y'' \in \frac{\rho}{2}B$ with $v' = v''$ and $d(y', S) \leq 4d(y', y'')^\gamma$ a uniform estimate

$$|i\partial\bar{\partial}\phi(y') - i\partial\bar{\partial}\phi(y'')| \leq Cd(y', y'')^{\alpha_1}.$$

In particular, we can apply this estimate to the case when $y' \in S$, so we can write

$$\omega_0 + i\partial\bar{\partial}\phi = m\epsilon\bar{\epsilon} + \sum_a m_a \epsilon d\bar{v}_a + \sum_a \bar{m}_a dv_a \bar{\epsilon} + \sum_{a,b} m_{ab} dv_a d\bar{v}_b,$$

such that for all $y \in 0.5B$,

$$|m_a(y)| \leq Cd(y,S)^{\alpha_1}. \tag{48}$$

Now it remains to consider the case when $d(y', S) \geq 4d(y', y'')^\gamma$ and $d(y', S)$ is small. As before applying a linear transformation on u and v independently, we may assume $m(y') = 1$ and $m_{ab}(y') = \delta_{ab}$. The difference from previous situation is that the terms $m_a(y')$ are not automatically zero. For this we need to make a further linear transformation of the form $v_a \mapsto v_a + c_a d(y', S)^{\alpha_1+1-1/\beta} u$. Because of (48) we may choose c_a to be uniformly bounded. Denote by F the coordinate change, and $F^*(\omega_0 + i\partial\bar{\partial}\phi) = \omega_0 + i\partial\bar{\partial}\tilde{\phi}$, such that $i\partial\bar{\partial}\tilde{\phi}$ vanishes at y'. Now we focus on the ball $B(y', \rho)$ with $\rho = 2d(y', y'')^\gamma$. Here the distance is again measured using the old coordinates. But one can easily check that in the ball $B(y', \rho)$, the difference $F^*\omega_0 - \omega_0$ is bounded by $Cd(y', S)^{\alpha_1}$, and has uniformly bounded $C^{,\alpha_1,\beta}$ norm. Since we only need to consider the case when $d(y', S)$ is small, this means that in the end it does not matter if we work with either coordinate. Now we can perform the same scaling trick as before, and obtain

$$|i\partial\bar{\partial}\tilde{\phi}(y') - i\partial\bar{\partial}\tilde{\phi}(y'')| \leq Cd(y', y'')^{\alpha_1}. \tag{49}$$

Notice that here we only need to use the corresponding linear estimate for a smooth flat metric, since by assumption $B(y', \rho) \subset B(y', \frac{1}{2}d(y', S))$ does not intersect S. It follows we also obtain the same estimate for $i\partial\bar{\partial}\phi$, with a larger constant C.

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