

## STRUCTURE OF MEASURES IN LIPSCHITZ DIFFERENTIABILITY SPACES

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### 1. INTRODUCTION

One of our goals is to show the equivalence of two seemingly very different ways of generalising Rademacher’s theorem on the almost everywhere differentiability of (real valued) Lipschitz functions defined on Euclidean spaces to metric measure spaces  $(X, d, \mu)$ . The first stems from the work of Cheeger [Che99] and further developed into the concept of a *Lipschitz differentiability space* following the work of several authors, most notably Keith [Kei04]. It is based upon the notion of differentiability with respect to suitable *charts*—Lipschitz functions  $\varphi: X \rightarrow \mathbb{R}^n$ . The second originates in some ideas of Alberti [Alb93] and is based on the immediate consequence of Lebesgue’s theorem that one may always differentiate a Lipschitz function almost everywhere along a Lipschitz curve. For measures represented by integration over such curves, this leads to a notion akin to partial differentiability  $\mu$  almost everywhere and, if  $\mu$  has more such representations, can lead to a notion of differentiability.

To describe our results in more detail, we now informally introduce the two main concepts studied in this paper.

An *n-dimensional Lipschitz differentiability space* may be thought of as a metric measure space  $(X, d, \mu)$  for which there is a Lipschitz  $\varphi: X \rightarrow \mathbb{R}^n$  (which is termed a *chart*) with respect to which every real valued Lipschitz function is differentiable  $\mu$  almost everywhere (a precise definition that allows a countable decomposition into such charts is given in Definition 3.1). Here differentiability of a function  $f$  at  $x_0$  requires the approximation of  $f(x) - f(x_0)$  by a linear form of  $\varphi(x) - \varphi(x_0)$  for  $x$  close to  $x_0$ . More precisely, it requires the existence of a linear  $L: \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$f(x) = f(x_0) + L(\varphi(x) - \varphi(x_0)) + o(d(x, x_0)).$$

Perhaps the best known non-trivial example of Lipschitz differentiability spaces is the Heisenberg group (see [Hei01]). In fact, by the result of [Che99], any doubling space satisfying the Poincaré inequality is a Lipschitz differentiability space. However the converse does not hold; for example subspaces of Lipschitz differentiability spaces are Lipschitz differentiability spaces, while an analogous statement fails for spaces satisfying the Poincaré inequality.

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An *Alberti representation* of a measure  $\mu$  (see Definition 2.2) is an integral combination of 1-rectifiable measures, each supported on a Lipschitz curve in  $X$  (whose domain is not necessarily connected), that calculates the  $\mu$  measure of any Borel subset of  $X$ . Further, we distinguish different Alberti representations by considering their *direction*; given a Lipschitz function  $\varphi: X \rightarrow \mathbb{R}^n$  we say that an Alberti representation is in the direction of a cone  $C \subset \mathbb{R}^n$  if, for almost every Lipschitz curve  $\gamma$  (with respect to the integrating measure) and almost every  $t_0$ ,  $(\varphi \circ \gamma)'(t_0) \in C$ . We then say that a collection of  $n$  Alberti representations are *independent* if there exist a Lipschitz  $\varphi: X \rightarrow \mathbb{R}^n$  and a collection of  $n$  independent cones in  $\mathbb{R}^n$  such that each Alberti representation is in the direction of its own cone (see Definition 2.10).

Fubini's theorem provides a simple example of an Alberti representation: it represents the  $n$ -dimensional Lebesgue measure as the integral of one-dimensional Lebesgue measures on lines parallel to the  $x$ -axis. In a general metric measure space, as for this example, if the measure  $\mu$  has an Alberti representation, it is possible to define the notion of a partial derivative of a Lipschitz function in the direction of this representation at almost every point. Therefore, if a measure has  $n$  independent Alberti representations, there exist  $n$  independent partial derivatives. This can lead to a new notion of the derivative of a Lipschitz function.

The approach to differentiability via representations of the underlying measure by integration over curves appears in [ACP] (see [ACP10] for an announcement) in which Alberti, Csörnyei and Preiss investigate measures in Euclidean space with respect to which Rademacher's theorem holds (and more generally, they use similar notions to give a geometric characterisation of those sets that are contained within the non-differentiability set of a Lipschitz function). However, in this generality we will not consider it as a separate notion, but rather as a description of the usual notion of Lipschitz differentiability spaces. One advantage of this is that it naturally leads to several descriptions of Lipschitz differentiability spaces, and emphasises the key point that such spaces possess a very rich structure of curves.

The research presented here began with an observation of Preiss that, from a theorem of Cheeger and Kleiner, one may obtain an Alberti representation of any doubling Lipschitz differentiability space that satisfies the Poincaré inequality (see Section 10 for details). We significantly strengthen this observation by showing that (even in an arbitrary Lipschitz differentiability space) there exist so many Alberti representations that they completely describe the derivative of a Lipschitz function. This gives a natural realisation of the possibility of describing such derivatives by partial derivatives, which was first pointed out in [CK09] (see Corollary 6.7).

We note that notions similar to Alberti representations have appeared within classical analysis and analysis on metric spaces. In particular, the superposition principle for transport measures of Young [You69] (see also Bernard [Ber08]); the decomposition of 1-dimensional currents into curves by Smirnov [Smi93] and its generalisation to metric spaces by Paolini and Stepanov [PS13]; the representation of a continuous time dependent family of measures on a metric space by a measure on curves by Lisini [Lis07]; the notion of a *test plan* by Ambrosio, Gigli and Savaré [AGS11] (and many other papers by the same authors).

In the first half of this paper we show that an  $n$ -dimensional Lipschitz differentiability space possesses  $n$  independent Alberti representations and that the derivative obtained from the partial derivatives agrees with the existing derivative. Further,

this implies that, for every Lipschitz function, at almost every point one of the partial derivatives must be comparable to the pointwise Lipschitz constant. That is, this collection of Alberti representations is *universal* (see Definition 7.1).

Since our only hypothesis on Lipschitz differentiability spaces is that Lipschitz functions are differentiable almost everywhere, we prove these results by first reducing the problem to showing that a certain class of sets must all have measure zero. Then, for each set  $S$  in this class, we construct a Lipschitz function that is differentiable almost nowhere on  $S$ , so that  $S$  has measure zero.

In the second half of the paper we characterise Lipschitz differentiability spaces via Alberti representations and related null sets. First we see that a collection of Alberti representations being universal is exactly the condition required for the partial derivatives to form a derivative and so obtain our first characterisation. We observe that many subsets of Lipschitz differentiability space must necessarily have measure zero, namely the set of non-differentiability points of any Lipschitz function. One may then ask if there exists a collection of subsets of a metric measure space such that, if each of these sets has measure zero, the space is a Lipschitz differentiability space. We answer this positively by giving several characterisations of Lipschitz differentiability spaces via null sets that arise naturally when constructing many independent Alberti representations of a measure.

One may wonder if, by restricting our attention to (for example) geodesic Lipschitz differentiability spaces, it is possible to only consider Alberti representations defined using curves that are the Lipschitz images of intervals and hence avoid some of the required technical constructions. Although such results may be possible using the theory presented here, it is not clear if such spaces possess a rich enough structure of curves to describe all of the directions of differentiability and so it is necessary to study the most general notion of an Alberti representation.

The existence of Alberti representations, and the ability to calculate the derivative from them, gives us a strong connection to the differentiability theory in Euclidean spaces. However, the theory of Lipschitz differentiability spaces remains more exotic than the Euclidean case as this theory cannot be improved beyond representations by 1-rectifiable measures. Indeed, it is known that the Heisenberg group is a Lipschitz differentiability space consisting of a single chart of dimension two. By our characterisations, we may therefore conclude that there exists a large collection of pairs of independent Alberti representations. However, it is also known that the Heisenberg group is 2-purely unrectifiable (see [AK00], Theorem 7.2) and so there cannot exist a representation of its measure by 2-rectifiable measures. We therefore see that, although there exists a very rich collection of 1-dimensional representations, they cannot be combined to produce a higher-dimensional structure.

The paper is organised as follows. In Section 2 we define an Alberti representation and deduce the elementary properties of measures with an Alberti representation. In Section 3 we recall the notion of a derivative with respect to a chart given in [Che99] and [Kei04]. We also give a characterisation, that first appeared in [BS13], describing the uniqueness of such a derivative as a property inherent to the chart. While this observation is simple, its consequences will be essential to our construction of a non-differentiable Lipschitz function.

We then proceed to prove the existence of  $n$  independent Alberti representations of any  $n$ -dimensional Lipschitz differentiability space (see Theorem 6.6). This proof

consists of the following three constructions which are valid for any chart in a metric measure space.

In Section 4 we present a general construction of a non-differentiable Lipschitz function (see Proposition 4.3). Given a set  $S$  and a sequence of 1-Lipschitz functions that behave, at some scale, like non-differentiable functions at points of  $S$ , we are able to produce a Lipschitz function that is not differentiable almost everywhere on  $S$ . More precisely, suppose that for each function  $f_m$  of this sequence and each  $x \in S$ , there exist  $y, z \in X$  with  $0 < d(y, x), d(z, x) < 1/m$  such that

$$\frac{|f_m(x) - f_m(y)|}{d(x, y)} \leq \frac{1}{m} \quad \text{and} \quad \frac{|f_m(x) - f_m(z)|}{d(x, z)} \geq \frac{1}{2}.$$

Then we modify and combine the functions of this sequence to construct a Lipschitz function that is not differentiable almost everywhere on  $S$ .

In Section 5 we define a class of subsets of a metric measure space and show that the measure has many independent Alberti representations if and only if each of these sets has measure zero (see Theorem 5.14). This class is a natural generalisation of the class of sets considered in [ACP]. In this paper, such sets will play a fundamental role in our description of Lipschitz differentiability spaces, via Alberti representations.

In Section 6, we provide a construction of a sequence of Lipschitz functions that satisfy the hypotheses of the construction from Section 4 for any set in the above class. In particular, there exists a Lipschitz function that is differentiable almost nowhere on such a set, establishing the existence of many independent Alberti representations of a Lipschitz differentiability space.

We use the existence of Alberti representations to give complete descriptions of Lipschitz differentiability spaces. In Section 7 we show how the partial derivatives obtained from a universal collection of Alberti representations form a derivative. For example, suppose that a Lipschitz function has all partial derivatives with respect to a universal collection of Alberti representations equal to zero. Then it must have derivative zero almost everywhere. By using the linearity of the derivative and investigating universal Alberti representations further, we are able to generalise this argument to show that the partial derivatives of any Lipschitz function form a derivative almost everywhere. By combining this with our previous results, we characterise Lipschitz differentiability spaces as those metric measure spaces with a universal collection of Alberti representations (see Theorem 7.8).

In Section 8 we show that, for any metric measure space with  $n$  independent Alberti representations, there exists a  $\delta > 0$  (depending on the properties of the Alberti representations) such that, for almost every  $x$  and every sufficiently small  $r$ , there exists a  $\delta r$  separated subset of  $B(x, r)$  of size  $n$ . In particular, if the metric measure space is doubling, there exists a bound on the total number of independent Alberti representations. When combined with the results from Section 5 that produce many independent Alberti representations, such a bound imposes a derivative of every Lipschitz function at almost every point. In particular, we characterise Lipschitz differentiability spaces as those doubling metric measure spaces in which certain subsets have measure zero (see Theorem 8.10).

In Section 9 we prove the existence of many additional Alberti representations of a Lipschitz differentiability space. In fact, for any chart  $\varphi$  in an  $n$ -dimensional differentiability space and any cone  $C \subset \mathbb{R}^n$ , we show the existence of an Alberti representation in the direction of  $C$ .

In Section 10 we relate our characterisations to the previous works of Cheeger and Keith.

Throughout this paper we will denote by  $(X, d)$  a complete, separable metric space and by  $(X, d, \mu)$  the metric measure space obtained by equipping  $(X, d)$  with a finite Borel regular measure  $\mu$ , so that  $\mu$  is Radon. However, by applying the results proved here to any compact subset, the main results hold for any metric space equipped with a Radon measure (that must necessarily be doubling). Further, since a derivative is not defined at an isolated point, we also suppose that the set of isolated points has measure zero.

For  $(X, d)$  a metric space,  $Y \subset X$ ,  $x \in X$  and  $r > 0$ , we denote by  $B(x, r)$  and  $\overline{B}(x, r)$  the open and closed balls centred at  $x$  of radius  $r$  and by  $B(Y, r)$  and  $\overline{B}(Y, r)$  the open and closed  $r$ -neighbourhood of  $Y$  of radius  $r$ . For a function  $f: X \rightarrow \mathbb{R}$  we write  $\text{Lip}(f, x)$  for the pointwise Lipschitz constant of  $f$  at  $x$  given by

$$\text{Lip}(f, x) := \limsup_{r \rightarrow 0} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : 0 < d(x, y) < r \right\}.$$

For two measures  $\mu$  and  $\nu$  on  $X$  we say that  $\mu \ll \nu$  if  $\mu$  is absolutely continuous with respect to  $\nu$ .

## 2. ALBERTI REPRESENTATIONS

This section first defines an Alberti representation and proves some of its basic properties. We then demonstrate how the existence of an Alberti representation provides a metric measure space with a notion of almost everywhere differentiability of Lipschitz functions.

**2.1. Basic definitions and properties.** Before defining an Alberti representation we define the set of curves that form a representation.

**Definition 2.1.** Let  $(X, d)$  be a metric space. We denote by  $\Gamma(X)$  the set of bi-Lipschitz functions

$$\gamma: \text{Dom } \gamma \rightarrow X$$

with  $\text{Dom } \gamma \subset \mathbb{R}$  non-empty and compact. For a function  $f: Y \rightarrow Z$  we denote

$$\text{Graph } f = \{(y, z) \in Y \times Z : f(y) = z\}.$$

Finally, for  $Y$  non-empty and compact and continuous  $f$ ,  $\text{Graph } f$  is non-empty and compact and so we may equip  $\Gamma(X)$  with the Hausdorff metric:

$$d(\gamma, \gamma') = \text{Hausdorff}(\text{Graph } \gamma, \text{Graph } \gamma'),$$

so that  $\Gamma(X)$  is a complete and separable metric space. We also define  $\Pi(X)$  to be the set of all  $\gamma \in \Gamma(X)$  with  $\text{Dom } \gamma$  a closed interval.

We will refer to elements of  $\Gamma(X)$  as *Lipschitz curves*. Note that, as in most metric spaces,  $\Pi(X)$  may only consist of the Lipschitz curves whose domain is a single point.

Suppose that  $(X, d)$  is a metric space,  $f: X \rightarrow \mathbb{R}^n$  is Lipschitz and  $\gamma \in \Gamma(X)$ . Then  $f \circ \gamma: \text{Dom } \gamma \subset \mathbb{R} \rightarrow \mathbb{R}^n$  is Lipschitz and so may be extended to a Lipschitz function  $F$  defined on an interval. Therefore, by Lebesgue's theorem,  $F$  is differentiable almost everywhere. However, at any non-isolated point  $t_0$  of  $\text{Dom } \gamma$ , if  $F'(t_0)$  exists, then its value is determined by the value of  $f \circ \gamma$  near to  $t_0$ , independently of the extension  $F$ . We may therefore define the derivative of  $f \circ \gamma$  almost everywhere in  $\text{Dom } \gamma$  to be equal to the derivative of any Lipschitz extension to an interval.

With these constructions, we define an Alberti representation as follows.

**Definition 2.2.** Let  $(X, d, \mu)$  be a metric measure space,  $\mathbb{P}$  a Borel probability measure on  $\Gamma(X)$  and, for each  $\gamma \in \Gamma(X)$ , let  $\mu_\gamma$  be a Borel measure on  $X$  that is absolutely continuous with respect to  $\mathcal{H}^1 \llcorner \text{Im } \gamma$ , the restriction of  $\mathcal{H}^1$  to  $\text{Im } \gamma$ . For measurable  $A \subset X$  we say that  $(\mathbb{P}, \{\mu_\gamma\})$  is an Alberti representation of  $\mu \llcorner A$  if, for every Borel  $Y \subset A$ ,  $\gamma \mapsto \mu_\gamma(Y)$  is Borel measurable and

$$\mu(Y) = \int_{\Gamma(X)} \mu_\gamma(Y) d\mathbb{P}(\gamma).$$

Given an Alberti representation  $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$  we will write “almost every  $\gamma \in \mathcal{A}$ ” to mean “ $\mathbb{P}$ -almost every  $\gamma \in \Gamma(X)$ ” and given a curve  $\gamma \in \Gamma(X)$  we write “almost every  $t \in \text{Dom } \gamma$ ” to mean “ $\gamma_{\#}^{-1}(\mu_\gamma)$ -almost every  $t \in \text{Dom } \gamma$ .”

First observe that, if  $\mu \llcorner A$  has an Alberti representation  $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$  and  $f: X \rightarrow \mathbb{R}$  is a positive simple function, then

$$\int_A f d\mu = \int_{\Gamma(X)} \int_{\text{Im } \gamma} f d\mu_\gamma d\mathbb{P}(\gamma).$$

Therefore, by the monotone convergence theorem, this formula holds for any positive Borel function  $f: X \rightarrow \mathbb{R}$ . Also note that, if  $N \subset X$  is  $\mu$ -null, then  $\mathcal{A}$  is also an Alberti representation of  $\mu \llcorner A \cup N$ . Finally, given a measurable set  $B \subset A$ , we may define  $\mathcal{A} \llcorner B$ , the restriction of  $\mathcal{A}$  to  $B$ , given by  $(\mathbb{P}, \{\mu_\gamma \llcorner B\})$ .

We may also form new Alberti representations from existing representations.

**Lemma 2.3.** *Let  $(X, d, \mu)$  be a metric measure space and  $A \subset X$  measurable. Suppose that there exist a finite measure  $\mathbb{M}$  on  $\Gamma(X)$  and for each  $\gamma \in \Gamma(X)$  a measure  $\nu_\gamma \ll \mathcal{H}^1 \llcorner \text{Im } \gamma$  such that  $\mu \llcorner A$  is absolutely continuous with respect to the Borel measure*

$$\nu(B) = \int_{\Gamma(X)} \nu_\gamma(B) d\mathbb{M}$$

*and such that  $\nu$  is finite. Then  $\mu \llcorner A$  has an Alberti representation.*

*Proof.* By dividing by  $\mathbb{M}(\Gamma(X))$  we may suppose that  $\mathbb{M}$  is a probability measure. Since  $\mu$  and  $\nu$  are both finite, by the Radon-Nikodym Theorem there exists a Borel measurable  $F: A \rightarrow \mathbb{R}$  such that, for each Borel  $B \subset A$ ,

$$\mu(B) = \int_B F d\nu = \int_{\Gamma(X)} \int_\gamma \mathbf{1}_B F d\nu_\gamma d\mathbb{M}.$$

Therefore, if we define  $d\mu_\gamma = F d\nu_\gamma$ ,

$$\mu(B) = \int_{\Gamma(X)} \mu_\gamma(B) d\mathbb{M}$$

and so  $\mu \llcorner A$  has an Alberti representation. □

**Lemma 2.4.** *Let  $(X, d, \mu)$  be a metric measure space and for each  $k \in \mathbb{N}$  let  $A_k \subset X$  be measurable such that  $\mu \llcorner A_k$  has an Alberti representation. Then  $\mu \llcorner \cup_k A_k$  has an Alberti representation.*

*Proof.* For each  $k \in \mathbb{N}$  let  $\mathcal{A}_k = (\mathbb{P}_k, \{\mu_{\gamma,k}\})$  be an Alberti representation of  $\mu \llcorner A_k$  and set  $\mathbb{P} = \sum_k 2^{-k} \mathbb{P}_k$ , a Borel probability measure on  $\Gamma(X)$ . For each  $\gamma \in \Gamma(X)$

and each  $k \in \mathbb{N}$  let  $F_{\gamma,k}$  be the Radon-Nikodym derivative of  $\gamma^{-1}(\mu_\gamma)$  with respect to the Lebesgue measure and set

$$S_\gamma = \{t \in \text{Dom } \gamma : \exists k \in \mathbb{N}, F_{\gamma,k}(t) > 0\}.$$

Then  $S_\gamma$  is a bounded subset of  $\mathbb{R}$  of positive measure and so we may define the measure

$$\nu_\gamma(Y) := \begin{cases} \mathcal{H}^1 \llcorner \gamma(S_\gamma) / \mathcal{H}^1(\gamma(S_\gamma)) & \text{if } \mathcal{H}^1(\gamma(S_\gamma)) > 0 \\ 0 & \text{otherwise} \end{cases}$$

on  $X$ . Further, we may define the finite measure  $\nu$  on  $X$  given by

$$\nu(Y) = \int_{\Gamma(X)} \nu_\gamma(Y) d\mathbb{P}$$

for each Borel  $Y \subset X$ .

Now let  $Y \subset X$  be Borel with  $\nu(Y) = 0$  and let  $k \in \mathbb{N}$ . Then for  $\mathbb{P}_k$ -a.e.  $\gamma \in \Gamma(X)$  we have  $\nu_\gamma(Y) = 0$ . However, by definition,  $\mu_{\gamma,k} \llcorner S_\gamma \ll \nu_\gamma$  for every  $\gamma \in \Gamma(X)$ . Therefore, for  $\mathbb{P}_k$ -a.e.  $\gamma \in \Gamma(X)$ ,  $\mu_{\gamma,k}(Y) = 0$ . Since  $\mu \llcorner A_k$  has the Alberti representation  $\mathcal{A}_k$ , we must have  $\mu(A_k \cap Y) = 0$ . In particular  $\mu(Y \cap \cup_k A_k) = 0$  and so  $\mu \llcorner \cup_k A_k$  is absolutely continuous with respect to  $\nu$ . By applying the previous lemma we obtain the required Alberti representation  $(\mathbb{P}, \{\mu_\gamma\})$  of  $\mu \llcorner \cup_k A_k$ .  $\square$

The previous two lemmas preserve many of the properties of the Alberti representations in their hypotheses. More precisely, suppose under the hypotheses of Lemma 2.4 that  $\tilde{\Gamma} \subset \Gamma(X)$  is a set of full  $\mathbb{P}_k$ -measure for each  $k \in \mathbb{N}$  and that, for each  $\gamma \in \tilde{\Gamma}$ ,  $\tilde{\gamma} \subset \text{Im } \gamma$  is a set of full  $\mu_{\gamma,k}$ -measure, for each  $k \in \mathbb{N}$ . Then  $\tilde{\Gamma}$  is a set of full  $\mathbb{P}$ -measure and for each  $\gamma \in \tilde{\Gamma}$ ,  $\tilde{\gamma}$  is a set of full  $\mu_\gamma$ -measure. We will make particular use of this fact in the following form. Suppose that  $f: X \rightarrow \mathbb{R}$  is Lipschitz such that, for each  $k \in \mathbb{N}$ , almost every  $\gamma \in \mathcal{A}_k$  and almost every  $t \in \text{Dom } \gamma$ ,  $(f \circ \gamma)'(t) > 0$ . Then for almost every  $\gamma \in \mathcal{A}$  and almost every  $t \in \text{Dom } \gamma$ ,  $(f \circ \gamma)'(t) > 0$ .

In the definition of an Alberti representation we only require the representation to hold for all Borel subsets of  $X$ . However, as we will now see, it is easy to extend this representation to all  $\mu$ -measurable subsets of  $X$ .

**Lemma 2.5.** *Let  $(X, d, \mu)$  be a metric measure space and  $A \subset X$  measurable such that  $\mu \llcorner A$  has an Alberti representation  $(\mathbb{P}, \{\mu_\gamma\})$ . Then for every  $\mu$ -measurable  $Y \subset A$ ,  $\gamma \mapsto \mu_\gamma(Y)$  is  $\mathbb{P}$ -measurable and*

$$\mu(Y) = \int_{\Gamma(X)} \mu_\gamma(Y) d\mathbb{P}.$$

*Proof.* Let  $Y \subset A$  be  $\mu$ -measurable and  $B \subset Y \subset C$  be Borel with  $\mu(B) = \mu(C)$ . Then for every  $\gamma \in \Gamma(X)$ ,

$$\mu_\gamma(B) \leq \mu_\gamma(Y) \leq \mu_\gamma(C).$$

However, since  $\mu(B)$  and  $\mu(C)$  are given by the Alberti representation, we know that  $\mu_\gamma(B) = \mu_\gamma(C)$  for almost every  $\gamma \in \mathcal{A}$ . Therefore  $\gamma \mapsto \mu_\gamma(Y)$  is  $\mathbb{P}$ -measurable

and

$$\begin{aligned} \mu(Y) = \mu(B) &= \int_{\Gamma(X)} \mu_\gamma(B) d\mathbb{P} \\ &\leq \int_{\Gamma(X)} \mu_\gamma(Y) d\mathbb{P} \\ &\leq \int_{\Gamma(X)} \mu_\gamma(C) d\mathbb{P} = \mu(C) = \mu(Y). \end{aligned}$$

Thus  $\mu(Y)$  is given by the Alberti representation. □

**2.2. Relationship with differentiability.** We now demonstrate how an Alberti representation provides a metric measure space with a notion of almost everywhere differentiability of Lipschitz functions.

**Definition 2.6.** For a metric space  $(X, d)$  we define  $H(X)$  to be the collection of non-empty compact subsets of  $\mathbb{R} \times X$  with the Hausdorff metric, so that  $H(X)$  is complete and separable. We also identify  $\Gamma(X)$  with its isometric image in  $H(X)$  via  $\gamma \mapsto \text{Graph } \gamma$  and set

$$A(X) = \{(x, \gamma) \in X \times \Gamma(X) : \exists t \in \text{Dom } \gamma, \gamma(t) = x\}.$$

Finally, for any  $K \subset X$ , we define the set

$$DP(K) := \{(x, \gamma) \in A(X) : \gamma^{-1}(x) \text{ is a density point of } \gamma^{-1}(K)\}.$$

**Lemma 2.7.** *Let  $(X, d)$  be a metric space and  $L > 0$ . Then the set of all  $L$ -bi-Lipschitz  $\gamma \in \Gamma(X)$  is a closed subset of  $H(X)$ , and  $\Gamma(X)$  is a Borel subset of  $H(X)$ . Further,  $A(X)$  is a Borel subset of  $X \times H(X)$  with the product metric. Finally, for any  $a_1 = (\gamma_1(t_1), \gamma_1)$  and  $a_2 = (\gamma_2(t_2), \gamma_2)$  in  $A(X)$ ,*

$$|t_1 - t_2| \leq \left(2 \min_{i=1,2} \text{biLip } \gamma_i + 1\right) d(a_1, a_2).$$

*Proof.* For any  $L > 0$  suppose that  $\gamma_m \in \Gamma(X)$  are  $L$ -bi-Lipschitz and  $\gamma_m \rightarrow \gamma$ , for some  $\gamma \in H(X)$ . We will show that  $\gamma$  is the graph of some  $L$ -bi-Lipschitz function.

Indeed, let  $(s, x)$  and  $(t, y)$  belong to  $\gamma$  and, for each  $m \in \mathbb{N}$ ,  $(s_m, x_m)$  and  $(t_m, y_m) \in \gamma_m$  with  $(s_m, x_m) \rightarrow (s, x)$  and  $(t_m, y_m) \rightarrow (t, y)$ . Then since each  $\gamma_m$  is  $L$ -bi-Lipschitz,

$$d(x, y) = \lim_{m \rightarrow \infty} d(\gamma_m(s_m), \gamma_m(t_m)) \leq \lim_{m \rightarrow \infty} L|s_m - t_m| = L|s - t|$$

and similarly

$$d(x, y) \geq |s - t|/L.$$

Therefore  $\gamma$  is the graph of some  $L$ -bi-Lipschitz function  $\gamma : \text{Dom } \gamma \rightarrow X$ . In particular, the set of  $L$ -bi-Lipschitz elements of  $\Gamma(X)$  is a closed subset of  $H(X)$ . By taking a union over  $L \in \mathbb{N}$  we conclude that  $\Gamma(X)$  is a Borel subset of  $H(X)$ .

Now suppose that  $(x, \gamma) \in X \times \Gamma(X)$  does not belong to  $A(X)$ . Then since  $\text{Im } \gamma$  is a compact set there exists a  $\delta > 0$  such that  $d(x, \text{Im } \gamma) > \delta$ . In particular  $B((x, \gamma), \delta/2)$  is disjoint from  $A(X)$  and so  $A(X)$  is a closed subset of  $X \times \Gamma(X)$ .

Finally, given  $a_1 = (\gamma_1(t_1), \gamma_1), a_2 = (\gamma_2(t_2), \gamma_2) \in A(X)$ , suppose that

$$L = \text{biLip } \gamma_1 \leq \text{biLip } \gamma_2.$$

There exists a  $t \in \text{Dom } \gamma_1$  with

$$\max\{|t_2 - t|, d(\gamma_1(t), \gamma_2(t_2))\} \leq d(a_1, a_2)$$



and so

$$\begin{aligned} |t_1 - t_2| &\leq |t_1 - t| + |t - t_2| \\ &\leq Ld(\gamma_1(t_1), \gamma_1(t)) + |t - t_2| \\ &\leq Ld(\gamma_1(t_1), \gamma_2(t_2)) + Ld(\gamma_2(t_2), \gamma_1(t_1)) + |t - t_2| \\ &\leq (2L + 1)d(a_1, a_2) \end{aligned}$$

as required. □

We will describe an Alberti representation using a Borel set  $B \subset A(X)$  whose projection into  $\Gamma(X)$  is a set of full measure. We now demonstrate the particular Borel sets that we will use.

**Lemma 2.8.** *Let  $(X, d, \mu)$  be a metric measure space.*

(1) *For  $g: X \rightarrow \mathbb{R}$  continuous, the function  $f: A(X) \rightarrow \mathbb{R} \cup \{\infty\}$  defined by*

$$f(x, \gamma) = \begin{cases} (g \circ \gamma)'(\gamma^{-1}(x)) & \text{if it exists} \\ \infty & \text{otherwise*} \end{cases}$$

*is Borel measurable.*

(2) *For any compact  $K \subset X$ ,  $DP(K)$  is a Borel subset of  $A(X)$ .*

*\*To be explicit, we include those  $x$  for which  $\gamma^{-1}(x)$  is isolated in  $\text{Dom } \gamma$  here.*

*Proof.* Let  $\delta, \epsilon > 0$  and  $q \in \mathbb{R}$ . Then the set of  $(\gamma(t_0), \gamma) \in A(X)$  with

$$|g(\gamma(t)) - g(\gamma(t_0)) - q(t - t_0)| \leq \epsilon|t - t_0|,$$

for all  $t \in \text{Dom } \gamma \cap B(t_0, \delta)$  and for which  $t_0$  is an accumulation point of  $\text{Dom } \gamma$ , is closed. Therefore, after taking suitable countable intersections and unions of such sets, the set where  $(g \circ \gamma)'(\gamma^{-1}(x))$  exists and belongs to some open subset of  $\mathbb{R}$  is Borel. Thus the set of points where the derivative does not exist is Borel as is the set of points where the derivative belongs to some given Borel set.

To prove (2), let  $\gamma_m \rightarrow \gamma$  in  $H(X)$ ,  $t \in \mathbb{R}$  and  $r > 0$ . Then  $\text{Dom } \gamma_m \rightarrow \text{Dom } \gamma$  in the Hausdorff metric. In particular, for every  $\epsilon > 0$ ,  $\text{Dom } \gamma_m \subset \overline{B}(\text{Dom } \gamma, \epsilon)$  for sufficiently large  $m$ . Therefore

$$\begin{aligned} \limsup_{m \rightarrow \infty} \mathcal{L}^1(\overline{B}(t, r) \cap \text{Dom } \gamma_m) &\leq \lim_{\epsilon \rightarrow 0} \mathcal{L}^1(\overline{B}(t, r) \cap \overline{B}(\text{Dom } \gamma, \epsilon)) \\ (2.1) \qquad \qquad \qquad &= \mathcal{L}^1(\overline{B}(t, r) \cap \text{Dom } \gamma) \end{aligned}$$

since  $\text{Dom } \gamma$  is compact.

Now let  $K \subset X$  be compact. We first show that

$$\gamma \mapsto \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on  $\Gamma(X)$ . For this let  $\gamma_m \rightarrow \gamma$  in  $H(X)$  such that each  $\gamma_m^{-1}(K)$  is non-empty and for every  $m \in \mathbb{N}$  define  $\text{Dom } \tilde{\gamma}_m = \gamma_m^{-1}(K)$  and  $\tilde{\gamma}_m$  to be the restriction of  $\gamma_m$  to  $\text{Dom } \tilde{\gamma}_m$ . Then since  $\gamma_m \rightarrow \gamma$  we know that all of the  $\text{Dom } \tilde{\gamma}_m$  belong to some compact subset  $A$  of  $\mathbb{R}$ . In particular each  $\tilde{\gamma}_m$  is contained within  $A \times K$ , a compact set, inside which the Hausdorff metric on its non-empty subsets is also compact. Therefore, for any sequence  $m(k) \rightarrow \infty$  there exists a subsequence  $m(k_i) \rightarrow \infty$  such that the  $\gamma_{m(k_i)}$  converge to some non-empty compact  $B \subset A \times K$ .

However, since  $\gamma_m \rightarrow \gamma$ , we know that  $B$  must be a subset of  $\gamma$  restricted to  $K$  and so, by equation (2.1),

$$\limsup_{i \rightarrow \infty} \mathcal{L}^1(\overline{B}(t, r) \cap \text{Dom } \tilde{\gamma}_{m(k_i)}) \leq \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K)).$$

This is true for any sequence  $m(k) \rightarrow \infty$  and so

$$\gamma \mapsto \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on  $\Gamma(X)$ .

Further, if  $(\gamma_m(t_m), \gamma_m) \rightarrow (\gamma(t), \gamma)$  in  $A(X)$ , then by Lemma 2.7 we know that  $|t_m - t| \rightarrow 0$  and so

$$\begin{aligned} \limsup_{m \rightarrow \infty} \mathcal{L}^1(\overline{B}(t_m, r) \cap \gamma_m^{-1}(K)) &\leq \limsup_{m \rightarrow \infty} \mathcal{L}^1(\overline{B}(t, r) \cap \gamma_m^{-1}(K)) + 2|t_m - t| \\ &\leq \mathcal{L}^1(\overline{B}(t, r) \cap \gamma^{-1}(K)). \end{aligned}$$

In particular

$$(x, \gamma) \mapsto \mathcal{L}^1(B(\gamma^{-1}(x), r) \cap \gamma^{-1}(K))$$

is upper semicontinuous on  $A(X)$ . By taking a suitable countable collection of intersections and unions we see that  $DP(K)$  is a Borel subset of  $A(X)$ .  $\square$

By combining these results we now show, for almost every point  $x$  in a metric measure space with an Alberti representation, the existence of a Lipschitz curve of which  $x$  is a density point. Moreover, such a Lipschitz curve may be taken from a set of full measure, with respect to the integrating measure of the Alberti representation.

**Proposition 2.9.** *Let  $(X, d, \mu)$  be a metric measure space and  $M \subset X$  measurable such that  $\mu \llcorner M$  has an Alberti representation  $\mathcal{A}$ . Suppose that  $(Y, \rho)$  is a complete, separable metric space,  $B \subset Y$  is Borel and  $f: A(X) \rightarrow Y$  is a Borel function such that, for almost every  $\gamma \in \mathcal{A}$  and almost every  $t \in \text{Dom } \gamma$ ,  $f(\gamma(t), \gamma) \in B$ . Then the set*

$$P(M) := \{x \in M : \exists \gamma \in \Gamma(X), (x, \gamma) \in DP(M), f(x, \gamma) \in B\}$$

*is a set of full measure in  $M$  and for each  $x \in P(M)$  there exists a  $\gamma^x \in \Gamma(X)$  that satisfies the conditions given in the definition of  $P(M)$  such that  $x \mapsto f(x, \gamma^x)$  is measurable.*

*Proof.* We first prove the statement under the additional hypotheses that  $M$  is compact. If so, by Lemma 2.8 (2), the set

$$G := \{(f(x, \gamma), x, \gamma) \in B \times A(X) : x \in M, (x, \gamma) \in DP(M)\}$$

is the graph of a Borel function restricted to a Borel set, and so is Borel. In particular, its projection onto  $M$  is a Suslin set and so is measurable. This projection equals  $P(M)$ . Further, the projection of  $G$  onto  $B \times M$  given by

$$\{(y, x) \in B \times M : \exists \gamma \in \Gamma(X), (x, \gamma) \in DP(M), y = f(x, \gamma) \in B\}$$

is also a Suslin set. This set is the graph of a function defined on  $M$  and so, by the Jankov-von Neumann Selection Theorem (see [Kec95], Theorem 18.1), for every  $x \in P(M)$  there exists a  $\gamma^x \in \Gamma(X)$  such that  $(x, \gamma^x) \in DP(M)$ ,  $f(x, \gamma^x) \in B$  and such that  $x \mapsto f(x, \gamma^x)$  is measurable.

Suppose that  $\mathcal{A} = (\mathbb{P}, \{\mu_\gamma\})$  and let us consider  $C = M \setminus P(M)$ , a measurable set. Then for any  $\gamma \in \Gamma(X)$  and  $t \in \text{Dom } \gamma$  with  $(\gamma(t), \gamma) \in C$ , since  $(\gamma(t), \gamma) \in A(X)$ ,

we must have either  $f(\gamma(t), \gamma) \notin B$  or  $t$  not a density point of  $\gamma^{-1}(C)$ . Therefore, from our hypotheses on  $\mathcal{A}$ , there exists a  $\mathbb{P}$ -null set  $N$  such that either  $\gamma \in N$  or  $\mathcal{L}^1(\gamma^{-1}(C)) = 0$ . By Lemma 2.5,

$$\mu(C) = \int_{\Gamma(X) \setminus N} \mu_\gamma(C) d\mathbb{P} + \int_N \mu_\gamma(C) d\mathbb{P} = 0$$

so that  $P(M)$  is a set of full measure in  $M$ .

Now suppose that  $M \subset X$  is measurable and let  $N$  be a  $\mu$ -null set such that  $M \setminus N$  is a countable union of compact sets  $K_i$ . Then by the above,  $\cup_i P(K_i)$  is a set of full measure in  $M$ . Moreover, for every  $i \in \mathbb{N}$ ,  $K_i \subset M$  and so  $P(K_i) \subset P(M)$ . In particular  $\mu(M \setminus P(M)) = 0$  so that such a  $\gamma^x$  exists for almost every  $x \in M$ .  $\square$

In particular we may use this proposition to find a *partial derivative* of a Lipschitz function at almost every point of a metric measure space with an Alberti representation. In fact, we may find a *gradient* of partial derivatives, whenever a metric measure space has many *independent* Alberti representations, each distinguished in the following way.

**Definition 2.10.** For  $w \in \mathbb{S}^{n-1}$  and  $0 < \theta < 1$ , define the *cone of width  $\theta$  centred on  $w$*  to be the set

$$C(w, \theta) = \{v \in \mathbb{R}^n : v \cdot w \geq (1 - \theta)\|v\|\}$$

and the *open cone of width  $\theta$  centred on  $w$*  to be

$$C^\circ(w, \theta) = \{v \in \mathbb{R}^n : v \cdot w > (1 - \theta)\|v\|\}.$$

Also, for a cone  $C$  we denote the open cone with the same centre and width by  $C^\circ$ .

Now let  $(X, d, \mu)$  be a metric measure space and  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz. For a cone  $C \subset \mathbb{R}^n$  we say that a Lipschitz curve  $\gamma \in \Gamma(X)$  is in the  $\varphi$ -*direction* of  $C$  if, for almost every  $t \in \text{Dom } \gamma$ ,

$$(\varphi \circ \gamma)'(t) \in C \setminus \{0\}.$$

Further, we say that an Alberti representation  $\mathcal{A}$  is in the  $\varphi$ -*direction* of  $C$  if almost every  $\gamma \in \mathcal{A}$  is in the  $\varphi$ -direction of  $C$ .

Finally, we say that closed cones  $C_1, \dots, C_m \subset \mathbb{R}^n$  are *independent* if, for any choice of  $v_i \in C_i \setminus \{0\}$ , the  $v_i$  are linearly independent and that a collection  $\mathcal{A}_1, \dots, \mathcal{A}_m$  of Alberti representations is  $\varphi$ -*independent* if there exist independent cones  $C_1, \dots, C_m$  such that each  $\mathcal{A}_i$  is in the  $\varphi$ -direction of  $C_i$ .

**Definition 2.11.** Let  $(X, d, \mu)$  be a metric measure space,  $x_0 \in X$  and  $f: X \rightarrow \mathbb{R}$  and  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz. Suppose that  $\gamma_1, \dots, \gamma_n \in \Gamma(X)$  such that each  $\gamma_i^{-1}(x_0) = 0$  is a density point of  $\text{Dom } \gamma_i$  and that the  $(\varphi \circ \gamma_i)'(0)$  exist and form a linearly independent set. We define the *gradient of  $f$  at  $x_0$  with respect to  $\varphi$  and  $\gamma_1, \dots, \gamma_n$*  to be the unique  $\nabla f(x_0) \in \mathbb{R}^n$  such that

$$(f \circ \gamma_i)'(0) = \nabla f(x_0) \cdot (\varphi \circ \gamma_i)'(0).$$

Further, we say that  $\nabla f(x_0) \in \mathbb{R}^n$  is a *gradient of  $f$  at  $x_0$  with respect to  $\varphi$*  if there exist such  $\gamma_1, \dots, \gamma_n \in \Gamma(X)$  such that  $\nabla f(x_0)$  is the gradient of  $f$  at  $x_0$  with respect to  $\varphi$  and  $\gamma_1, \dots, \gamma_n$ .

*Remark 2.12.* A very easy construction of a Lipschitz function on the plane with both partial derivatives equal to zero and a non-zero directional derivative at the origin shows that a gradient with respect to a fixed  $\varphi$  need not be unique.

We obtain the following Corollary of Proposition 2.9.

**Corollary 2.13.** *Let  $(X, d, \mu)$  be a metric measure space and  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz such that  $\mu$  has  $n$   $\varphi$ -independent Alberti representations. Then for any Lipschitz  $f: X \rightarrow \mathbb{R}$  there exists a gradient  $\nabla f$  of  $f$  almost everywhere with respect to  $\varphi$ . Further, this gradient may be chosen such that  $x_0 \mapsto \nabla f(x_0)$  is measurable.*

### 3. A FIRST ANALYSIS OF LIPSCHITZ DIFFERENTIABILITY SPACES

We begin by recalling the notion of differentiability in metric spaces introduced by Cheeger [Che99] and Keith [Kei04].

**Definition 3.1.** Let  $(X, d)$  be a metric space and  $n \in \mathbb{N}$ . We say that a Borel set  $U \subset X$  and a Lipschitz function  $\varphi: X \rightarrow \mathbb{R}^n$  form a *chart of dimension  $n$* ,  $(U, \varphi)$  and that a function  $f: X \rightarrow \mathbb{R}$  is *differentiable at  $x_0 \in U$*  with respect to  $(U, \varphi)$  if there exists a unique  $Df(x_0) \in \mathbb{R}^n$  (the *derivative of  $f$  at  $x_0$* ) such that

$$\limsup_{X \ni x \rightarrow x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

Further, we say that a metric measure space  $(X, d, \mu)$  is a *Lipschitz differentiability space* if there exists a countable decomposition of  $X$  into charts such that any Lipschitz function  $f: X \rightarrow \mathbb{R}$  is differentiable at almost every point of every chart.

Whenever it will not cause confusion, we will say “a chart in a Lipschitz differentiability space” to mean a chart in a Lipschitz differentiability space with respect to which every real valued Lipschitz function is differentiable almost everywhere.

Note that the derivative of a function at a point depends on the chart that we are considering. However, we do not emphasise this in the notation as it will always be clear from the context which chart the derivative is taken with respect to.

For a survey on the existing theory of Lipschitz differentiability spaces, the reader is referred to the primer written by Kleiner and Mackay [KM11]. (Note that, in this paper and the papers of Cheeger and Keith, a Lipschitz differentiability space is referred to as a *metric measure space that admits a measurable differentiable structure*.)

*Remark 3.2.* One may wonder about the notion of a *zero-dimensional chart*, where every Lipschitz function has derivative zero almost everywhere. In Corollary 4.9 we see that such a chart must have measure zero.

We first give a characterisation of the uniqueness of derivatives in a chart that will lead us to necessary conditions for a function to be differentiable at a point. This characterisation first appeared in [BS13].

**Lemma 3.3** ([BS13], Lemma 2.1). *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a metric measure space  $(X, d, \mu)$  and  $x_0 \in U$ . The following are equivalent:*

- (1) *There exist a  $\lambda > 0$  and  $X \ni x_m \rightarrow x_0$  such that, for any  $v \in \mathbb{S}^{n-1}$ ,*

$$(3.1) \quad \liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \geq \lambda.$$

- (2) *There exists a  $\lambda > 0$  such that, for any  $v \in \mathbb{S}^{n-1}$ ,*

$$\limsup_{x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} \geq \lambda.$$

(3) For any  $f: X \rightarrow \mathbb{R}$ , if there exists a  $Df(x_0) \in \mathbb{R}^n$  such that

$$\limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0,$$

then it is unique.

In particular, the fact that such a  $Df(x_0)$  is unique depends only upon the chart and is independent of  $f$ .

*Proof.* Equation (1) implies (2). Now suppose that (2) holds and for a function  $f: X \rightarrow \mathbb{R}$  there exist  $Df(x_0)$  and  $Df'(x_0) \in \mathbb{R}^n$  that satisfy the hypotheses of (3). Then by the triangle inequality

$$\limsup_{x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot (D - D')|}{d(x, x_0)} = 0$$

and so  $\|D - D'\| = 0$ .

Finally suppose that (3) holds for some  $f: X \rightarrow \mathbb{R}$ . Then for any  $v_0 \in \mathbb{S}^{n-1}$ ,

$$\limsup_{x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v_0|}{d(x, x_0)} > 0.$$

Therefore, there exist  $x_m^0 \rightarrow x_0$  and  $w_0 \in \overline{B}(0, \text{Lip } \varphi)$  such that  $w_0 \cdot v_0 > 0$  and

$$\lim_{m \rightarrow \infty} \frac{\varphi(x_m^0) - \varphi(x_0)}{d(x_m^0, x_0)} = w_0.$$

For each  $1 \leq i < n$  inductively choose  $v_i \in \mathbb{S}^{n-1}$  such that  $v_i \cdot w_j = 0$  for each  $0 \leq j < i$  and let  $x_m^i \rightarrow x_0$  and  $w_i \in \overline{B}(0, \text{Lip } \varphi)$  such that  $w_i \cdot v_i > 0$  and

$$\lim_{m \rightarrow \infty} \frac{\varphi(x_m^i) - \varphi(x_0)}{d(x_m^i, x_0)} = w_i.$$

Then for any  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ , suppose that  $\alpha_0 w_0 + \dots + \alpha_{n-1} w_{n-1} = 0$ . By taking the inner product with  $w_{n-1}$  we see that  $\alpha_{n-1} = 0$ . Repeating we see that  $\alpha_i = 0$  for each  $0 \leq i < n$  and so the  $w_i$  form a basis of  $\mathbb{R}^n$ . In particular, there exists a  $\lambda > 0$  such that, for each  $v \in \mathbb{S}^{n-1}$ , there exists a  $0 \leq j < n$  with  $|w_j \cdot v| \geq \lambda$ .

Therefore, if we set  $x_{mn+i} = x_m^i$  for each  $m \in \mathbb{N}$  and  $0 \leq i < n$ , for any  $v \in \mathbb{S}^{n-1}$  there exists a  $0 \leq j < n$  such that

$$\liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \geq \lim_{m \rightarrow \infty} \frac{|(\varphi(x_m^j) - \varphi(x_0)) \cdot v|}{d(x_m^j, x_0)} \geq \lambda$$

as required. □

Using this we may give a necessary condition for a function to be differentiable at a point in a chart.

**Lemma 3.4.** *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a metric space  $(X, d)$ ,  $x_0 \in U$  and  $x_m \rightarrow x_0$  satisfying (3.1) for some  $\lambda > 0$ . Then for any  $f: X \rightarrow \mathbb{R}$  that is differentiable at  $x_0$  we have*

$$\|Df(x_0)\| \leq \text{Lip}(f, x_0)/\lambda$$

and

$$\frac{\lambda}{\text{Lip } \varphi} \text{Lip}(f, x_0) \leq \liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i}, x_0)}.$$

*Proof.* If  $f: X \rightarrow \mathbb{R}$  is differentiable at  $x_0 \in X$ , then by the triangle inequality

$$(3.2) \quad \liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i}, x_0)} \geq \liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|(\varphi(x_{mn+i}) - \varphi(x_0)) \cdot Df(x_0)|}{d(x_{mn+i}, x_0)} \geq \lambda \|Df(x_0)\|.$$

This proves the first inequality. Secondly, by another application of the triangle inequality,

$$\begin{aligned} \text{Lip}(f, x_0) &= \limsup_{x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{d(x, x_0)} \\ &\leq \|Df(x_0)\| \limsup_{x \rightarrow x_0} \frac{\|\varphi(x) - \varphi(x_0)\|}{d(x, x_0)} \\ &\leq \|Df(x_0)\| \text{Lip } \varphi. \end{aligned}$$

Combining this with inequality (3.2) gives the required result. □

A very easy application of this lemma gives the following corollary on singletons in a Lipschitz differentiability space. This will be used without any specific reference.

**Corollary 3.5.** *Any singleton in a Lipschitz differentiability space has measure zero.*

*Proof.* Let  $(U, \varphi)$  be any  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$ . For almost any  $x_0 \in U$  let  $x_m \rightarrow x_0$  and  $\lambda > 0$  be obtained from Lemma 3.3 and by taking a suitable subsequence if necessary, we may suppose that

$$\max_{0 \leq i < n} d(x_{mn+i}, x_0) \leq \min_{0 \leq i < n} d(x_{(m+1)n+i}, x_0)/2$$

for each  $m \in \mathbb{N}$ . Then the 1-Lipschitz function

$$f(x) = \sum_{m \in 2\mathbb{N}} \max_{0 \leq i < n} (d(x_{mn+i}, x_0)/4 - d(x_{mn+i}, x))^+$$

satisfies

$$\frac{f(x_{mn+i}) - f(x_0)}{d(x_{mn+i}, x_0)} = \begin{cases} 1/4 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

for each  $0 \leq i < n$ . In particular, the condition given in Lemma 3.4 does not hold at  $x_0$  and so  $f$  is not differentiable at  $x_0$ , as required. □

When constructing a non-differentiable Lipschitz function, we will ensure that

$$\liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|f(x_{mn+i}) - f(x_0)|}{d(x_{mn+i}, x_0)}$$

is sufficiently small by first bounding  $d(x_{mn+i}, x_0)$  from below, for a fixed  $m$ , all  $0 \leq i < n$  and all  $x_0$  in a certain set of large measure. We now do this uniformly across the chart to simplify the construction.

**Definition 3.6.** Let  $(X, d)$  be a metric space and  $\lambda > 0$ . We say that  $U \subset X$  and a Lipschitz function  $\varphi: X \rightarrow \mathbb{R}^n$  form a  $\lambda$ -structured chart of dimension  $n$  if  $U$  is compact and there exists a  $U' \subset U$  of full measure such that, for every

$R > 0$  there exist an  $r > 0$  and for every  $x_0 \in U'$  points  $x_1, \dots, x_n \in U$  with each  $r < d(x_i, x_0) < R$  and

$$\max_{1 \leq i \leq n} \frac{|(\varphi(x_i) - \varphi(x_0)) \cdot v|}{d(x_i, x_0)} \geq \lambda$$

for every  $v \in \mathbb{S}^{n-1}$ .

We say that  $(U, \varphi)$  is a structured chart if it is a  $\lambda$ -structured chart for some  $\lambda > 0$ .

Note that a structured chart is also a chart and so we may consider the derivative of a real valued function with respect to a structured chart. The following lemma shows that we may just consider structured charts when working with a Lipschitz differentiability space.

**Lemma 3.7.** *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a metric measure space  $(X, d, \mu)$  such that*

$$\text{Lip}(v \cdot \varphi, x_0) > 0$$

*for every  $v \in \mathbb{S}^{n-1}$  and almost every  $x_0 \in U$ . Then for any  $\epsilon > 0$  there exist a  $U' \subset U$  with  $\mu(U \setminus U') < \epsilon$  and a countable set  $N \subset X$  such that  $(U' \cup N, \varphi)$  is a structured chart.*

*In particular, we may decompose any Lipschitz differentiability space into a  $\mu$ -null set and a countable collection of structured charts with respect to which any real valued Lipschitz function is differentiable almost everywhere.*

*Proof.* For  $x_0, x_1, \dots, x_n \in X$  let us denote by  $P(x_0; x_1, \dots, x_n)$  the property

$$\max_{1 \leq i \leq n} \frac{|(\varphi(x_i) - \varphi(x_0)) \cdot v|}{d(x_i, x_0)} \geq \lambda \text{ for every } v \in \mathbb{S}^{n-1}.$$

Then for any  $R, \lambda, r > 0$ , the set of  $x_0 \in U$  for which there exist  $x_1, \dots, x_n \in X$  with  $r < d(x_i, x_0) < R$  for each  $1 \leq i \leq n$  and such that  $P(x_0; x_1, \dots, x_n)$  holds is an open set. Therefore, the set of those  $x_0 \in U$  that satisfy (3.1) for a given  $\lambda > 0$  is a Borel set. In particular, for any  $\epsilon > 0$  there exist a compact  $U' \subset U$  with  $\mu(U \setminus U') < \epsilon$ , so that  $(U', \varphi)$  is still a chart, and a  $\lambda > 0$  such that every point of  $U'$  satisfies (3.1) for  $\lambda$ .

Now let  $R > 0$  and  $\lambda$  be as found above. The function

$$x_0 \mapsto \sup \left\{ \min_{1 \leq i \leq n} d(x_i, x_0) : d(x_i, x_0) \leq R \text{ and } P(x_0; x_1, \dots, x_n) \right\}$$

is positive and lower semicontinuous on  $U'$ . Therefore, there exists an  $r > 0$  such that this function is bounded below by  $r$  on  $U'$ . Let  $y^1, \dots, y^M$  be a finite  $\lambda r/2$  Lip  $\varphi$ -net of  $U'$  and for each  $1 \leq j \leq M$  let  $x_1^j, \dots, x_n^j \in X$  with  $r \leq d(x_i^j, y^j) \leq R$  for each  $1 \leq i \leq n$  and such that  $P(y^j; x_1^j, \dots, x_n^j)$  for each  $1 \leq j \leq M$ . We set

$$N_R = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq M\}.$$

Then for any  $x_0 \in U'$  and  $v \in \mathbb{S}^{n-1}$ , there exist a  $y^j$  with  $d(x_0, y^j) < \lambda r/2 \text{Lip } \varphi$  and a  $1 \leq i \leq n$  such that

$$\begin{aligned} \max_{1 \leq i \leq n} |(\varphi(x_i^j) - \varphi(x_0)) \cdot v| &\geq |(\varphi(x_i^j) - \varphi(y^j)) \cdot v| - \|\varphi(y^j) - \varphi(x_0)\| \\ &\geq \lambda d(x_i^j, y^j) - \lambda d(x_i^j, y^j)/2 \\ &\geq \lambda d(x_i^j, x_0)/(2 + \lambda/\text{Lip } \varphi) \\ &:= \lambda' d(x_i^j, x_0). \end{aligned}$$

Therefore, if we define  $N = \cup_i N_{1/i}$  and  $V = U' \cup N$ , for any  $x_0 \in U'$  there exists a sequence  $V \ni x_m \rightarrow x_0$  that satisfies (3.1) for  $\lambda'$ . Moreover, since each  $N_{1/k}$  is a finite subset of  $B(U', \lambda \text{Lip } \varphi/2k)$ , for any sequence  $x_m \in V$  either there exists a constant subsequence or  $d(x_m, U') \rightarrow 0$ . In particular,  $V$  is compact and so  $(V, \varphi)$  is a structured chart.

Finally, in a Lipschitz differentiability space  $N$  is a  $\mu$ -null set and so  $(V, \varphi)$  is a chart with respect to which any Lipschitz function is differentiable almost everywhere in  $V$ . □

To conclude this section we highlight a key fact that will be used when investigating Alberti representations in Lipschitz differentiability spaces. This result should be compared to the concept of a gradient (see Definition 2.11). We will use this without any specific reference.

**Lemma 3.8.** *Let  $(U, \varphi)$  be a chart in a metric measure space  $(X, d, \mu)$  and  $\gamma \in \Gamma(X)$ . Suppose that for some non-isolated  $t_0 \in \text{Dom } \gamma$  and  $f: X \rightarrow \mathbb{R}$ ,  $f$  is differentiable with respect to  $(U, \varphi)$  at  $\gamma(t_0)$  and that  $(\varphi \circ \gamma)'(t_0)$  exists. Then*

$$(f \circ \gamma)'(t_0) = Df(x_0) \cdot (\varphi \circ \gamma)'(t_0).$$

*Proof.* If  $\gamma$  is  $L$  Lipschitz and  $\gamma(t_0) = x_0$ , use the triangle inequality and the fact that

$$\limsup_{\text{Dom } \gamma \ni t \rightarrow t_0} \frac{|P(\gamma(t)) - P(\gamma(t_0))|}{|t - t_0|} \leq L \limsup_{\text{Im } \gamma \ni x \rightarrow x_0} \frac{|P(x) - P(x_0)|}{d(x, x_0)}$$

for any  $P: X \rightarrow \mathbb{R}$ . □

#### 4. CONSTRUCTION OF A NON-DIFFERENTIABLE LIPSCHITZ FUNCTION

In the following section we will see that a metric measure space possesses many Alberti representations whenever a certain class of subsets have measure zero. Following this we will show that such subsets do have measure zero in a Lipschitz differentiability space by constructing a Lipschitz function that is differentiable almost nowhere on such a set.

In this section we present a general method for constructing a Lipschitz function that is not differentiable on a given subset of a structured chart with certain properties. Note that the construction does not use the fact that we work inside a Lipschitz differentiability space, just that we have a chart structure.

We will construct such a Lipschitz function from a given sequence of Lipschitz functions. The first step will be to modify each function in such a sequence in the following way.

**Lemma 4.1.** *Let  $(X, d, \mu)$  be a metric measure space,  $h > 4\epsilon > 0$  and  $L > 0$ . Then for any  $L$ -Lipschitz  $f: X \rightarrow \mathbb{R}$  and Borel  $S \subset X$  there exist an  $L$ -Lipschitz*



$\tilde{f}: X \rightarrow \mathbb{R}$  and Borel  $\tilde{S} \subset S$  with  $\mu(\tilde{S}) \geq (1 - 4\epsilon/h)\mu(S)$  such that

- (1) The support of  $\tilde{f}$  is contained within  $B(S, 2h/L)$ .
- (2)  $0 \leq \tilde{f} \leq h$ .
- (3) For every  $x, y \in B(S, h/L)$  with  $x \neq y$ ,

$$\frac{|\tilde{f}(x) - \tilde{f}(y)|}{d(x, y)} \leq \frac{|f(x) - f(y)|}{d(x, y)}.$$

- (4) For every  $x_0 \in \tilde{S}$  and  $x \in X$  with  $0 < d(x, x_0) \leq \epsilon/L$ ,

$$\frac{|\tilde{f}(x) - \tilde{f}(x_0)|}{d(x, x_0)} = \frac{|f(x) - f(x_0)|}{d(x, x_0)}.$$

*Proof.* Let  $M$  be the greatest integer less than  $h/4\epsilon$  and for each  $0 \leq k < M$  define  $Z_k(t) = d(t, 2\epsilon k + 2h\mathbb{Z})$  and  $\hat{f}_k = Z_k \circ f: X \rightarrow \mathbb{R}$ . Then each  $\hat{f}_k$  satisfies (2) and (3). Now define

$$F_k = \{x_0 \in S : d(f(x_0), 2\epsilon k + 2h\mathbb{Z}) < \epsilon\}.$$

Then for integer  $0 \leq k < M$ , the  $F_k$  are disjoint Borel subsets of  $S$ . Indeed, if  $x_0 \in F_k \cap F_{k'}$ , then there exists an  $n \in \mathbb{N}$  with  $|k - k' + hn/\epsilon| < 1$ . Since  $|k - k'| < h/4\epsilon$  and  $h/\epsilon > 4$ , this can only happen if  $n = 0$  and  $k = k'$  and so such  $F_k$  and  $F_{k'}$  are either disjoint or equal. Therefore, there exists an  $m$  with

$$\mu(F_m) \leq \mu(S)/M \leq 4\epsilon\mu(S)/h.$$

We set  $\tilde{S} = S \setminus F_m$  so that, for  $x_0 \in \tilde{S}$  and  $x \in X$  with  $d(x, x_0) \leq \epsilon/L$ ,

$$|\hat{f}_m(x) - \hat{f}_m(x_0)| < \epsilon \text{ and } d(f(x_0), 2\epsilon m + 2h\mathbb{Z}) \geq \epsilon.$$

In particular

$$|\hat{f}_m(x) - \hat{f}_m(x_0)| = |f(x) - f(x_0)|$$

and so (4) holds.

Finally we define

$$\begin{aligned} \tilde{f}: B(S, h/L) \cup X \setminus B(S, 2h/L) &\rightarrow \mathbb{R} \\ \tilde{f}(x) &= \begin{cases} \hat{f}_m & x \in B(S, h/L) \\ 0 & x \in X \setminus B(S, 2h/L). \end{cases} \end{aligned}$$

Then, since  $0 \leq \tilde{f} \leq h$ ,  $\tilde{f}$  is  $L$ -Lipschitz and so we may extend it to a function on the whole of  $X$  that satisfies the required properties. □

We now give the main lemma required for the construction of a non-differentiable Lipschitz function. It takes a set  $S$  and a sequence of Lipschitz functions that, in some neighbourhood of  $S$ , witness both large and small difference quotients at all points of  $S$  and combines them to construct a Lipschitz function that witnesses such difference quotients in all neighbourhoods of  $S$ . Later in this section we will see that such a function cannot be differentiable at any point of  $S$ .

**Lemma 4.2.** *Let  $(X, d, \mu)$  be a metric measure space,  $S \subset X$  Borel and  $L, \beta > 0$ . Suppose that there exists a sequence of  $L$ -Lipschitz functions  $f_m: X \rightarrow \mathbb{R}$  such that*

- (1) *For every  $x_0 \in S$  there exists an  $M \in \mathbb{N}$  such that for each  $m \geq M$  there exists an  $x \in X$  with  $0 < d(x, x_0) < 1/m$  and*

$$|f_m(x) - f_m(x_0)| \geq L\beta d(x, x_0).$$

- (2) For every  $m \in \mathbb{N}$  there exists a  $\rho_m > 0$  such that, for every  $x_0 \in S$  and any  $y, z \in B(x_0, \rho_m)$ ,

$$|f_m(y) - f_m(z)| \leq d(y, z)/m.$$

Then for any  $\alpha > 0$  and  $R_i > r_i > 0$  such that  $R_i, r_i \rightarrow 0$ , there exist  $i(k) \rightarrow \infty$  and a Lipschitz function  $F: X \rightarrow \mathbb{R}$  such that

- For almost every  $x_0 \in S$ ,  $\text{Lip}(F, x_0) \geq L\beta - \alpha$ .
- For every  $k \in \mathbb{N}$ ,  $x_0 \in S$  and  $x \in X$  with  $r_{i(k)} < d(x, x_0) < R_{i(k)}$ ,

$$\frac{|F(x) - F(x_0)|}{d(x, x_0)} \leq \alpha.$$

*Proof.* By dividing by  $L$  if necessary, we may suppose that  $L = 1$  and by taking a suitable subsequence we may suppose that  $R_i \leq r_{i-1}$  for each  $i \in \mathbb{N}$ . We define sequences  $m(k)$ ,  $i(k) \rightarrow \infty$  inductively as follows, choosing  $m(0) = i(0) = 1$ . Given  $m(k)$  and  $i(k)$  choose  $i(k+1) > i(k)$  such that

$$R_{i(k+1)} < \rho_{m(k)} \text{ and } r_{i(k+1)} \leq 2^{-(k+1)}\alpha r_{i(j)}$$

for every  $j \leq k$ . Then choose  $m(k+1)$  such that

$$\frac{1}{m(k+1)} \leq 2^{-(k+1)}r_{i(k+1)} \text{ and } m(k+1) \geq 2^{k+1}/\alpha.$$

Note that these conditions imply, for every  $j \in \mathbb{N}$ ,

$$(4.1) \quad \sum_{k>j} r_{i(k)} \leq \alpha r_{i(j)} \text{ and } \sum_{k \in \mathbb{N}} 1/m(k) \leq \alpha.$$

We define 1-Lipschitz functions  $g_k: X \rightarrow \mathbb{R}$  and a Borel set  $S_k \subset S$  by applying Lemma 4.1 to  $f_{m(k)}$  with  $h = r_{i(k)}/2$  and  $\epsilon = 1/m(k)$ . Then  $\mu(S_k) \geq (1 - 2^{3-k})\mu(S)$  and the  $g_k$  have the following properties:

- (1) The support of  $g_k$  is contained within  $B(S, r_{i(k)})$ .
- (2)  $0 \leq g_k \leq r_{i(k)}/2$ .
- (3) For every  $x_0 \in S_k$  there exists an  $x$  with  $0 < d(x, x_0) < 1/m(k)$  such that

$$\frac{|g_k(x) - g_k(x_0)|}{d(x, x_0)} \geq \beta.$$

- (4) For every  $x_0 \in S$  and any  $x, y \in B(x_0, R_{i(k)})$  with  $x \neq y$ ,

$$\frac{|g_k(x) - g_k(y)|}{d(x, y)} \leq \frac{1}{m(k)}.$$

Finally we define  $F = \sum_k g_k$  and

$$S' = \bigcap_{m \in \mathbb{N}} \bigcup_{k \geq m} S_k,$$

a set of full measure in  $S$ .

We first show that  $F$  is Lipschitz. Let  $x \neq y \in X$  with

$$r_{i(k+1)} \leq d(x, y) < r_{i(k)}$$

and suppose that, for some  $k' \leq k$ ,

$$|g_{k'}(x) - g_{k'}(y)| > d(x, y)/m_{k'}.$$

Then one of  $x$  or  $y$  must necessarily belong to  $B(S, r_{i(k)})$ . However,

$$d(x, y) < r_{i(k)} \leq r_{i(k')}$$

and so there exists an  $x_0 \in S$  such that

$$x, y \in B(x_0, 2r_{i(k')}) \subset B(x_0, R_{i(k'-1)}).$$

Since the  $R_{i(k)}$  strictly decrease we have, for all  $j < k'$ ,

$$|g_j(x) - g_j(y)| \leq d(x, y)/m(j).$$

In particular this can only happen for at most one value of  $k' \leq k$ . If such a  $k'$  does not exist, set  $k' = k$ . Then we have

$$\begin{aligned} |F(x) - F(y)| &\leq \sum_{k' \neq j \leq k} |g_j(x) - g_j(y)| + |g_{k'}(x) - g_{k'}(y)| + \sum_{j \geq k+1} |g_j(x) - g_j(y)| \\ &\leq \sum_{k' \neq j \leq k} d(x, y)/m(j) + 2d(x, y) + \sum_{j > k+1} r_{i(j)}/2 \\ &\leq (\alpha + 2)d(x, y) + \alpha r_{i(k+1)} \\ &\leq 2(\alpha + 1)d(x, y) \end{aligned}$$

and so  $F: X \rightarrow \mathbb{R}$  is Lipschitz. Here, for the penultimate inequality, we have used the inequalities in (4.1).

Now let  $x_0 \in S'$  and  $K \in \mathbb{N}$  such that  $x_0 \in S_k$  for each  $k \geq K$ . Then for each  $k \geq K$  there exists an  $x \in X$  with  $0 < d(x, x_0) < 1/m(k)$  such that

$$\frac{|g_k(x) - g_k(x_0)|}{d(x, x_0)} \geq \beta.$$

Note that we also have  $x \in B(S, R_{i(j)})$  for any  $j < k$  and so

$$\begin{aligned} |F(x) - F(x_0)| &\geq |g_k(x) - g_k(x_0)| - \sum_{j < k} |g_j(x) - g_j(x_0)| - \sum_{j > k} |g_j(x) - g_j(x_0)| \\ &\geq \beta d(x, x_0) - \sum_{j < k} d(x, x_0)/m(j) - \sum_{j > k} r_{i(j)}/2 \\ &\geq \beta d(x, x_0) - \alpha d(x, x_0) - \alpha r_{i(k)} \\ &\geq (\beta - 2\alpha)d(x, x_0). \end{aligned}$$

Such an  $x$  exists for each  $k \geq K$  and so  $\text{Lip}(F, x_0) \geq \beta - 2\alpha$ .

Now let  $x_0 \in S$  and for any  $k \in \mathbb{N}$  let  $x \in X$  with  $r_{i(k)} > d(x, x_0) < R_{i(k)}$ . Then

$$\begin{aligned} |F(x_i) - F(x_0)| &\leq \sum_{j \leq k} |g_j(x) - g_j(x_0)| + \sum_{j > k} |g_j(x) - g_j(x_0)| \\ &\leq \alpha d(x, x_0) + \sum_{j > k} r_{i(j)}/2 \\ &\leq 2\alpha d(x, x_0). \end{aligned}$$

Therefore,  $F$  satisfies the conclusion of the lemma for  $2\alpha$ . □

By combining this construction with the definition of a structured chart, we show that the constructed function is differentiable almost nowhere on such a set.

**Proposition 4.3.** *Let  $(U, \varphi)$  be a structured chart in a metric measure space  $(X, d, \mu)$ ,  $S \subset U$  be Borel and  $L, \beta > 0$ . Suppose that there exists a sequence of*

*L-Lipschitz functions*  $f_m: X \rightarrow \mathbb{R}$  such that

- (1) For every  $x_0 \in S$  there exists an  $M \in \mathbb{N}$  such that, for each  $m \geq M$ , there exists an  $x \in X$  with  $0 < d(x, x_0) < 1/m$  and

$$|f_m(x) - f_m(x_0)| \geq \beta L d(x, x_0).$$

- (2) There exists a  $\rho_m > 0$  such that, for every  $x_0 \in S$  and any  $y, z \in B(x_0, \rho_m)$ ,

$$|f_m(y) - f_m(z)| \leq d(y, z)/m.$$

Then there exists a Lipschitz function  $F: X \rightarrow \mathbb{R}$  that is differentiable  $\mu$ -almost nowhere on  $S$ . In particular, if  $X$  is a Lipschitz differentiability space, then  $S$  is  $\mu$ -null.

*Proof.* For  $\lambda > 0$  let  $(U, \varphi)$  be a  $\lambda$ -structured chart and choose  $\alpha < \lambda\beta/(\text{Lip } \varphi + 1)$ . Then by the definition of a structured chart, there exist positive  $R_i > r_i \rightarrow 0$  such that, for almost every  $x_0 \in U$  and every  $i \in \mathbb{N}$ , there exist  $x_1, \dots, x_n \in U$  with  $r_i < d(x_j, x_0) < R_i$  and

$$\max_{1 \leq j \leq n} \frac{|(\varphi(x_j) - \varphi(x_0)) \cdot v|}{d(x_j, x_0)} \geq \lambda$$

for every  $v \in \mathbb{S}^{n-1}$ . We apply Lemma 4.2 to obtain a Lipschitz function  $F: X \rightarrow \mathbb{R}$  and  $i(k) \rightarrow \infty$  such that, for almost every  $x_0 \in S$ ,  $\text{Lip}(F, x_0) \geq \beta - \alpha$  and for every  $k \in \mathbb{N}$  and  $x \in X$  with  $r_{i(k)} < d(x, x_0) < R_{i(k)}$ ,

$$\frac{|F(x) - F(x_0)|}{d(x, x_0)} \leq \alpha.$$

In particular, there exists a sequence  $x_m \rightarrow x_0$  satisfying (3.1) for  $\lambda$  such that

$$\liminf_{m \rightarrow \infty} \max_{0 \leq i < n} \frac{|(F(x_{mn+i}) - F(x_0)) \cdot v|}{d(x_{mn+i}, x_0)} \leq \alpha$$

for every  $v \in \mathbb{S}^{n-1}$ . Therefore, by Lemma 3.4 and our choice of  $\alpha$ ,  $F$  is not differentiable at almost every point of  $S$ . □

The following result first appeared in [BS13]; however we now give a short proof as this result will be required when establishing our characterisations of Lipschitz differentiability spaces. We first require some notation.

**Definition 4.4.** Let  $(X, d)$  be a metric space,  $S \subset X$  Borel and  $x_0 \in S$ . We say that  $S$  is *porous at*  $x_0$  if there exist an  $\eta > 0$  and  $X \ni x_m \rightarrow x_0$  such that

$$B(x_m, \eta d(x_m, x_0)) \cap S = \emptyset.$$

Further, we say that  $S$  is *porous* if it is porous at  $x_0$  for each  $x_0 \in S$ .

**Corollary 4.5** ([BS13], Theorem 2.4). *Porous sets in a Lipschitz differentiability space  $(X, d, \mu)$  have measure zero.*

*Proof.* For any porous set  $S \subset X$  there exist a countable Borel decomposition  $S = \cup_i S_i \cup N$  and a sequence  $\eta_i > 0$  such that  $\mu(N) = 0$ , each  $S_i$  is compact and contained within a structured chart and for each  $i \in \mathbb{N}$  and  $x_0 \in S_i$ , there exist  $x_m \rightarrow x_0$  such that

$$B(x_m, \eta_i d(x_m, x_0)) \cap S_i = \emptyset.$$

Then for any  $r > 0$  and  $i \in \mathbb{N}$ , the function

$$x_0 \mapsto \sup\{d(x, x_0) < r : d(x, S_i) > \eta_i d(x, x_0)/2\}$$

is well defined, strictly positive and lower semicontinuous on  $S_i$  and so there exists an  $\epsilon_{r,i} > 0$  such that it is bounded below by  $\epsilon_{r,i}$ . Then the functions

$$f_m(x) = \min\{d(x, S_i) - \epsilon_{1/m,i}, 0\}$$

satisfy the hypotheses of Proposition 4.3 for  $S_i$  and so  $S_i$  is  $\mu$ -null. In particular,  $S$  is also  $\mu$ -null. □

**4.1. Additional remarks on Lipschitz differentiability spaces.** We give several simple consequences of Proposition 4.3.

**Corollary 4.6** ([BS13], Corollary 2.7). *For any measurable subset  $Y$  of a Lipschitz differentiability space  $(X, d, \mu)$ ,  $(Y, d, \mu)$  is a Lipschitz differentiability space with respect to the same chart structure.*

*Moreover, the derivative of a real valued Lipschitz function in  $Y$  agrees with the derivative of any Lipschitz extension of  $f$  to  $X$ , almost everywhere.*

*Proof.* Let  $f : Y \rightarrow \mathbb{R}$  be Lipschitz and  $(U, \varphi)$  a  $\lambda$ -structured chart of dimension  $n$  in  $X$ . We may extend  $f$  to a Lipschitz function defined on  $X$  and so, for almost every  $x_0 \in Y \cap U$ , there exists a unique  $Df(x_0)$  such that

$$\limsup_{X \ni x \rightarrow x_0} \frac{|f(x) - f(x_0) - Df(x_0) \cdot (\varphi(x) - \varphi(x_0))|}{d(x, x_0)} = 0.$$

This  $Df(x_0)$  will also be a derivative for the metric measure space  $(Y, d, \mu)$ , provided it is unique. As seen in Lemma 3.3 such a  $Df(x_0)$  is not unique if and only if there exists a  $v \in \mathbb{S}^{n-1}$  such that

$$\limsup_{Y \ni x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} = 0.$$

However, since  $(U, \varphi)$  is a  $\lambda$ -structured chart in  $X$ , there exist  $X \ni x_m \rightarrow x_0$  with

$$\lim_{m \rightarrow \infty} \frac{|(\varphi(x_m) - \varphi(x_0)) \cdot v|}{d(x_m, x_0)} \geq \lambda.$$

By combining these two relations and using the fact that  $\varphi$  is Lipschitz, we see that

$$B(x_m, \lambda d(x_m, x_0)/2 \text{Lip } \varphi) \cap Y = \emptyset$$

for sufficiently large  $m$ . In particular, for each  $v \in \mathbb{S}^{n-1}$ , the set of such  $x_0$  is a porous set in  $X$ . By taking the union of such  $x_0$  over a countable dense subset of  $\mathbb{S}^{n-1}$ , we see that the set of those  $x_0$  where  $Df(x_0)$  is not unique is  $\mu$ -null. Therefore the derivative of  $f$  in  $X$  is also the derivative of  $f$  in  $Y$ , almost everywhere. □

We also show how we may apply the construction of a non-differentiable Lipschitz function when we only have control over the infinitesimal behaviour of such a sequence of Lipschitz functions.

**Corollary 4.7.** *Let  $(X, d, \mu)$  be a Lipschitz differentiability space,  $S \subset X$  Borel and  $L, \beta > 0$ . Suppose that there exists a sequence of  $L$ -Lipschitz functions  $f_m : X \rightarrow \mathbb{R}$  such that, for almost every  $x_0 \in S$ ,*

- *There exists an  $M \in \mathbb{N}$  such that, for each  $m \geq M$ , there exists an  $x \in X$  with  $0 < d(x, x_0) < 1/m$  and*

$$|f_m(x) - f_m(x_0)| \geq \beta d(x, x_0).$$

- For each  $m \in \mathbb{N}$

$$\text{Lip}(f_m, x_0) \leq 1/m.$$

Then  $S$  is  $\mu$ -null.

*Proof.* By Lemma 3.7, it suffices to prove the result whenever  $S$  is contained within a structured chart  $(U, \varphi)$ . For this, we will apply Proposition 4.3 to a suitable subset of  $S$ .

For any  $\epsilon > 0$  there exist a Borel  $S' \subset S$  with  $\mu(S') \geq \mu(S) - \epsilon$ , an  $M \in \mathbb{N}$  and a sequence  $\rho_m > 0$  such that

- For every  $m \geq M$ ,  $x_0 \in S'$  and  $y \in X$  with  $0 < d(y, x_0) < \rho_m$ ,

$$\frac{|f_m(y) - f_m(x_0)|}{d(y, x_0)} \leq 2/m.$$

- For every  $m \geq M$  and  $x_0 \in S'$  there exists an  $x \in X$  with  $0 < d(x, x_0) < 1/m$  and

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \geq \beta.$$

In particular, for any  $x_0 \in S'$  and  $y, z \in B(x_0, \rho_m) \cap S'$ ,

$$\frac{|f_m(y) - f_m(z)|}{d(y, z)} \leq 2/m.$$

Also observe that, for any  $x_0 \in S'$ , there either exists an  $M' \geq M$  and for each  $m \geq M'$  an  $x \in S'$  with

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \geq \beta/2$$

or there exist  $X \ni x_j \rightarrow x_0$  such that  $B(x_j, \beta d(x_j, x_0)/2L) \cap S' = \emptyset$ . Therefore, there exists a Borel decomposition  $S' = S'' \cup P$  where  $P$  is a porous subset of  $X$  and  $S''$  satisfies the hypotheses of Proposition 4.3 for the Lipschitz differentiability space  $(S', d, \mu)$ . Therefore  $S''$  and so  $S'$  are  $\mu$ -null. Finally, since  $\epsilon > 0$  was arbitrary,  $S$  is also  $\mu$ -null.  $\square$

Using the previous corollary, we give an example of metric spaces that can only be Lipschitz differentiability spaces when they have zero measure.

**Corollary 4.8.** *Let  $(X, d, \mu)$  be a Lipschitz differentiability space,  $Y \subset X$  Borel and for each  $m \in \mathbb{N}$  let  $N_m$  be a  $1/m$ -net of  $Y$ . Suppose that, for each  $m \in \mathbb{N}$  and  $q \in N_m$ ,*

$$\limsup_{X \ni x \rightarrow x_0} \frac{|d(x, q) - d(x_0, q)|}{d(x, x_0)} = 0$$

for almost every  $x_0 \in Y \setminus \cup_m N_m$ . Then  $\mu(Y) = 0$ .

*In particular, for any metric measure space  $(X, d, \mu)$  and  $0 < \delta < 1$ ,  $(X, d^\delta, \mu)$  is a Lipschitz differentiability space if and only if  $\mu(X) = 0$ .*

*Proof.* Suppose that  $(U, \varphi)$  is a structured chart in  $X$  and  $S \subset Y \cap U \setminus \cup_m N_m$  is compact. Then for each  $m \in \mathbb{N}$  there exists a finite  $N'_m \subset N_m$  such that  $B(N'_m, 1/m) \supset S$ . We define the 1-Lipschitz function

$$f_m(x) = d(x, N'_m) = \min\{d(x, q) : q \in N'_m\}.$$

Then by the hypothesis on the  $N_m$ ,  $\text{Lip}(f_m, x_0) = 0$  for almost every  $x_0 \in S$ .

Further, for any  $x_0 \in S$  let  $q \in N'_m$  with  $d(x_0, q)$  minimal. Then  $d(x_0, q) < 1/m$  and

$$\frac{|f_m(x_0) - f_m(q)|}{d(x, q)} = 1.$$

Therefore the hypotheses of Corollary 4.7 are satisfied for  $S'$ , so that  $S'$  and hence  $S$  are  $\mu$ -null. Since  $S \subset U \setminus \cup_m N_m$  was an arbitrary compact set and each  $N_m$  is separated,  $Y \cap U$  must also be  $\mu$ -null.

Now let  $(X, d, \mu)$  be a metric measure space and  $0 < \delta < 1$ . For any  $a > 0$ ,  $|(a + r)^\delta - a^\delta|/r^\delta \rightarrow 0$  as  $r \rightarrow 0$ . In particular, for any  $x_0, z \in (X, d^\delta)$  with  $x_0 \neq z$ , we may use the triangle inequality in  $(X, d)$  to deduce

$$\limsup_{x \rightarrow x_0} \frac{|d^\delta(x, z) - d^\delta(x_0, z)|}{d^\delta(x, x_0)} \leq \frac{|(d(x_0, z) + d(x, x_0))^\delta - d^\delta(x_0, z)|}{d^\delta(x, x_0)} = 0.$$

Therefore, by choosing  $N_m$  to be any  $1/m$ -net of  $X$ , if  $(X, d^\delta, \mu)$  is a Lipschitz differentiability space we must have  $\mu(X) = 0$ . □

Finally, we address the notion of a zero-dimensional chart in a Lipschitz differentiability space.

**Corollary 4.9.** *Let  $(X, d, \mu)$  be a metric measure space and  $U \subset X$  Borel. Suppose that, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ ,*

$$\limsup_{X \ni x \rightarrow x_0} \frac{|f(x) - f(x_0)|}{d(x, x_0)} = 0$$

*for almost every  $x_0 \in U$ . Then  $\mu(U) = 0$ .*

*Proof.* First observe that, for any  $x_0 \in X$ , the function  $x \mapsto d(x, x_0)$  satisfies  $\text{Lip}(f, x_0) > 0$  and so singletons must have measure zero in  $U$ . In particular, almost every point of  $U$  is a limit point of  $U$ .

Now, for any  $m \in \mathbb{N}$  let  $N_m$  be a  $1/m$ -net of  $U$ ,  $S \subset U \setminus N_m$  be compact and  $N'_m \subset N_m$  be finite with  $B(N'_m, 1/m) \supset S$ . We define

$$f_m(x) = d(x, N'_m) = \min\{d(x, q) : q \in N'_m\}$$

and let  $S' \subset S$  be a compact and  $\rho > 0$  such that

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} \leq 1/m$$

for each  $x_0 \in S'$  and  $0 < d(x, x_0) < \rho$ . In particular, for any  $x_0 \in S'$  and  $y, z \in B(x_0, \rho) \cap S'$ ,

$$\frac{|f_m(y) - f_m(z)|}{d(y, z)} \leq 1/m.$$

Moreover, for any  $x_0 \in S'$  there exists a  $x \in U$  with  $0 < d(x, x_0) < 1/m$  and

$$\frac{|f_m(x) - f_m(x_0)|}{d(x, x_0)} = 1.$$

We apply Lemma 4.2 to obtain a Lipschitz function  $F: X \rightarrow \mathbb{R}$  with  $\text{Lip}(F, x_0) > 0$  for almost every  $x_0 \in S'$ . Therefore  $S'$  and hence  $S$  and  $X$  are  $\mu$ -null. □

*Remark 4.10.* Suppose we choose to define the lim sup of a function at an isolated point to equal zero (as is the case in [Kei04]). Then this corollary shows that a set in a Lipschitz differentiability space is a chart of dimension zero if and only if almost every point is isolated. This answers the question raised in Remark 2.1.3 of [Kei04] on the nature of charts of dimension zero.

5. THE EXISTENCE OF ALBERTI REPRESENTATIONS

**5.1. A single representation.** We now give a characterisation of the metric measure spaces with many independent Alberti representations (see Definition 2.10). Our method relies upon the Gliksberg-König-Seever Decomposition Theorem, which we state first. Recall that a measure  $\mu$  is *regular* if, for every measurable set  $A$ ,  $\mu(A) = \inf\{\mu(U) : U \supset A\}$ .

**Theorem 5.1** ([Rud08], Theorem 9.4.4). *Let  $(X, d, \mu)$  be a compact metric measure space and  $K$  a weak\*-compact, convex and non-empty set of regular Borel probability measures on  $X$ . Then there exists a Borel decomposition  $X = A \cup S$  such that  $\mu \llcorner A \ll \nu$  for some  $\nu \in K$  and  $k(S) = 0$  for every  $k \in K$ .*

We would like to apply this theorem with  $K$  a set of normalised  $\mathcal{H}^1$  measures supported on elements of a compact subset of  $\Gamma(X)$  (see Definition 2.1). However, this set need not be compact. Instead, in the following lemma, we apply it with  $K$  a set of normalised  $\mathcal{H}^1$  measures supported on elements of a compact subset of  $\Pi(X)$  (see Definition 2.1). Following this, we will use this lemma to obtain a conclusion in terms of  $\Gamma(X)$ .

**Lemma 5.2.** *Let  $(X, d, \mu)$  be a compact metric measure space and  $J \subset \Pi(X)$  closed. Then there exist a Borel decomposition  $X = A \cup S$  and an Alberti representation  $\mathcal{A}$  of  $\mu \llcorner A$  such that*

- (1) *Almost every  $\gamma \in \mathcal{A}$  belongs to  $J$ .*
- (2) *For every  $\gamma \in J$ ,  $\mathcal{H}^1(\gamma \cap S) = 0$ .*

*Remark 5.3.* If  $\mu$  gives measure zero to any set  $S$  that satisfies (2), then  $\mathcal{A}$  is an Alberti representation of  $\mu$ . Observe that, from the definition of an Alberti representation, this condition is also necessary for a representation of  $\mu$  satisfying (1) to exist. Therefore, this lemma gives a characterisation of the existence of such an Alberti representation.

*Proof.* If  $J$  is empty, then setting  $A = \emptyset$  and  $\mathcal{A} = (0, \{\mathcal{H}^1 \llcorner \gamma\})$  completes the proof. Otherwise, for each  $k \in \mathbb{N}$  let  $J_k$  be the set of  $k$ -bi-Lipschitz  $\gamma \in J$  with  $\text{Dom } \gamma \subset [-k, k]$  and  $\mathcal{L}^1(\text{Dom } \gamma) \geq 1/k$ . Then by Lemma 2.7, since  $J \subset \Pi(X)$  is closed, each  $J_k$  is isometrically equivalent to a closed subset of the compact subsets of  $[-k, k] \times X$  with the Hausdorff metric, and so is compact.

Now fix  $k \in \mathbb{N}$  and for each  $\gamma \in J_k$  let  $\mathcal{H}_\gamma$  be the pushforward of the Lebesgue measure on  $\text{Dom } \gamma$  under  $\gamma$ , multiplied by a suitable scalar such that  $\mathcal{H}_\gamma$  is a probability measure. Then since  $J_k$  is compact, the set of such measures is weak\*-compact and so  $K$ , its weak\*-closed convex hull, satisfies the hypothesis of Theorem 5.1. Moreover, probability measures on  $J_k$  are weak\*-compact and so any measure  $\nu \in K$  is of the form

$$\nu(A) = \int_{\Pi(X)} \mathcal{H}_\gamma(A) d\mathbb{P}'(\gamma),$$

for some Borel probability measure  $\mathbb{P}'$  on  $\Pi(X)$  with  $\mathbb{P}'(J_k) = 1$ .



Therefore, Theorem 5.1 gives a decomposition  $X = A_k \cup S_k$  where  $\mu \llcorner A_k$  is absolutely continuous with respect to some  $\nu \in K$  and  $\mathcal{H}_\gamma(S_k) = 0$  for every  $\gamma \in J_k$ . In particular, an application of Lemma 2.3 constructs an Alberti representation  $\mathcal{A}_k$  of  $\mu \llcorner A_k$  from  $\nu$  such that almost every  $\gamma \in \mathcal{A}_k$  belongs to  $J_k$  and  $\mathcal{H}^1(\gamma \cap S_k) = 0$  for every  $\gamma \in J_k$ . Finally, if we define  $A = \cup_k A_k$  and  $S = \cap_k S_k$ , then  $X = A \cup S$  is a Borel decomposition of  $X$  such that, by Lemma 2.4,  $\mu \llcorner A$  has an Alberti representation  $\mathcal{A}$  such that almost every  $\gamma \in \mathcal{A}$  belongs to  $J$ , and such that  $S$  satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in J$  with  $\mathcal{L}^1(\text{Dom } \gamma) > 0$ . Since  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in J$  with  $\mathcal{L}(\text{Dom } \gamma) = 0$ , this decomposition is of the required form.  $\square$

We summarise the exact setting in which we will use the above lemma. Not only does the resulting Alberti representation have a direction but we may also control the magnitude of the partial derivatives of a finite number of Lipschitz functions. Later (in Definition 5.7), this will be developed into the notion of the *speed* of an Alberti representation.

**Corollary 5.4.** *Let  $(X, d, \mu)$  be a compact metric measure space and, for some  $n \in \mathbb{N}$ , let  $\varphi, \psi: X \rightarrow \mathbb{R}^n$  be Lipschitz. Then for any cone  $C \subset \mathbb{R}^n$  and  $\delta_1, \dots, \delta_n > 0$ , there exists a Borel decomposition  $X = A \cup S$  such that*

- *There exists an Alberti representation  $\mathcal{A}$  of  $\mu \llcorner A$  in the  $\varphi$ -direction of  $C$  such that, for almost every  $\gamma \in \mathcal{A}$ ,  $\gamma$  belongs to  $\Pi(X)$  and*

$$\|(\psi_i \circ \gamma)'(t_0)\| \geq \delta_i \text{Lip}(\gamma, t_0)$$

*for almost every  $t_0 \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ .*

- *$\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Pi(X)$  in the  $\varphi$ -direction of  $C$  with*

$$(5.1) \quad \|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))\| \geq \delta_i \|\gamma(t) - \gamma(t')\|$$

*for every  $t, t' \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ .*

*Proof.* Let  $\gamma_m \in \Pi(X)$  be a sequence of Lipschitz curves in the  $\varphi$ -direction of  $C$  and, for some  $\gamma \in \Pi(X)$ , suppose that  $\gamma_m \rightarrow \gamma$ . Then for any  $t, t' \in \text{Dom } \gamma$  and for each  $m \in \mathbb{N}$  there exist  $t_m, t'_m \in \text{Dom } \gamma_m$  such that

$$\gamma_m(t_m) \rightarrow \gamma(t) \text{ and } \gamma_m(t'_m) \rightarrow \gamma(t').$$

Then

$$\begin{aligned} (\varphi(\gamma(t)) - \varphi(\gamma(t'))) \cdot w &= \lim_{m \rightarrow \infty} (\varphi(\gamma_m(t_m)) - \varphi(\gamma_m(t'_m))) \cdot w \\ &\geq \lim_{m \rightarrow \infty} (1 - \theta) \|\varphi(\gamma_m(t_m)) - \varphi(\gamma_m(t'_m))\| \\ &= (1 - \theta) \|\varphi(\gamma(t)) - \varphi(\gamma(t'))\| \end{aligned}$$

so that  $\gamma$  is in the  $\varphi$ -direction of  $C$ . Similarly, if each  $\gamma_m$  satisfies (5.1) for every  $t, t' \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ , then so does  $\gamma$ . Therefore, the set  $J$  of all  $\gamma \in \Pi(X)$  in the  $\varphi$ -direction of  $C$  that satisfy (5.1) for every  $1 \leq i \leq n$  and every  $t, t' \in \text{Dom } \gamma$  is closed. Moreover, for any  $\gamma \in J$ ,

$$\|(\psi_i \circ \gamma)'(t_0)\| \geq \delta_i \text{Lip}(\gamma, t_0)$$

for almost every  $t_0 \in \text{Dom } \gamma$  and each  $1 \leq i \leq n$ . By applying Lemma 5.2 to  $J$  we obtain a decomposition  $X = A \cup S$  of the required form.  $\square$

Observe that, if  $X$  is a metric space such that  $\Pi(X)$  only contains Lipschitz curves  $\gamma$  with  $\text{Dom } \gamma$  a single point, then the decomposition  $A = \emptyset, S = X$  satisfies the conclusion of the previous lemma. To obtain a meaningful result, we will instead apply the lemma to the closed, convex hull of an isometric copy of  $X$  within a Banach space  $\mathcal{B}$ . In this case  $\Pi(\mathcal{B})$  is very rich and so we obtain a much stronger conclusion.

So that we may describe such an Alberti representation purely in the language of the metric space, we now investigate Lipschitz curves in the singular set  $S$ .

**Lemma 5.5.** *Let  $\mathcal{B}$  be a Banach space,  $X \subset \mathcal{B}$  closed and convex and let  $\varphi, \psi: \mathcal{B} \rightarrow \mathbb{R}^n$  be Lipschitz. For a cone  $C \subset \mathbb{R}^n$  and  $\delta_1, \dots, \delta_n > 0$  suppose that a Borel set  $S \subset \mathcal{B}$  satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for all  $\gamma \in \Pi(X)$  in the  $\varphi$ -direction of  $C$  with*

$$\|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))\| \geq \delta_i \|\gamma(t) - \gamma(t')\|$$

for every  $t, t' \in \text{Dom } \gamma$  and each  $1 \leq i \leq n$ . Then for any measurable  $D \subset \mathbb{R}$  and Lipschitz  $\gamma: D \rightarrow X$  the set

$$\{t_0 \in D : (\varphi \circ \gamma)'(t_0) \in C^\circ \text{ and } |(\psi_i \circ \gamma)'(t_0)| > \delta_i \text{Lip}(\gamma, t_0) \forall 1 \leq i \leq n\}$$

is Lebesgue null.

*Proof.* Suppose that the conclusion does not hold for some measurable  $D \subset \mathbb{R}$  of positive measure and  $\gamma: D \rightarrow X$ . We will construct a new function  $\tilde{\gamma} \in \Pi(X)$  that agrees with  $\gamma$  on a set of positive measure and satisfies the conditions given in the hypotheses of the lemma, producing a contradiction.

By standard measure theoretic techniques, there exist  $\Phi, \Psi \in \mathbb{R}^n, b, \xi > 0$  and  $\delta'_i > \delta_i$  with

$$\overline{B}(\Phi, \xi) \subset C \text{ and each } |\Psi_i| \geq \delta'_i b$$

such that, for every  $\epsilon > 0$ , there exists a bounded  $D' \subset D$  of positive measure with

$$\|(\varphi \circ \gamma)'(t_0) - \Phi\| < \epsilon, |(\psi_i \circ \gamma)'(t_0)| > |\Psi_i| \text{ and } \text{Lip}(\gamma, t_0) < b$$

for every  $1 \leq i \leq n$  and  $t_0 \in D'$ . Let  $D'$  be such a set for some choice of

$$0 < \epsilon < \min\{\|\Phi\|, \xi\}.$$

Further, let  $R > 0$  and  $D'' \subset D'$  have positive measure such that, for every  $t_0 \in D''$  and every  $t \in \text{Dom } \gamma$  with  $|t - t_0| \leq R$ ,

$$\|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - \Phi(t - t_0)\| \leq \epsilon|t - t_0|,$$

for each  $1 \leq i \leq n$

$$|\psi_i(\gamma(t)) - \psi_i(\gamma(t_0))| \geq |\Psi_i(t - t_0)|,$$

and

$$\|\gamma(t) - \gamma(t_0)\| \leq b|t - t_0|.$$

We set  $D_0$  to be the intersection of  $D''$  with an interval of diameter  $R$  chosen so that  $D_0$  has positive measure, and  $I$  to be the smallest closed interval containing  $D_0$ . Finally, we define  $\tilde{\gamma}$  to equal  $\gamma$  on  $D_0$  and extend  $\tilde{\gamma}$  to  $I$  whilst maintaining the Lipschitz constant by first extending to the closure of  $D_0$  and then linearly on the connected components of the complement.

First observe that, since  $X$  is closed and convex,  $\text{Im } \tilde{\gamma} \subset X$ . Further, since  $\gamma|_{D_0}$  is in the  $\varphi$ -direction of  $C$ ,  $\mathcal{H}^1(\gamma(D_0)) > 0$ . Now, for any connected component  $(c, d)$  of  $I \setminus \overline{D_0}$ , let  $c_m, d_m \in D_0$  with  $c_m \rightarrow c$  and  $d_m \rightarrow d$ . Then

$$\begin{aligned} \|\varphi(\tilde{\gamma}(c)) - \varphi(\tilde{\gamma}(d)) - \Phi(c - d)\| &= \lim_{m \rightarrow \infty} \|\varphi(\tilde{\gamma}(c_m)) - \varphi(\tilde{\gamma}(d_m)) - \Phi(c_m - d_m)\| \\ &\leq \epsilon |c - d|. \end{aligned}$$

Similarly

$$|\psi_i(\tilde{\gamma}(c)) - \psi_i(\tilde{\gamma}(d))| \geq |\Psi_i(c - d)|$$

for each  $1 \leq i \leq n$  and

$$\|\tilde{\gamma}(c) - \tilde{\gamma}(d)\| \leq b|c - d|.$$

In particular, since  $\tilde{\gamma}$  is extended linearly to  $(c, d)$ , for almost every  $t_0 \in I$ ,

$$\|(\varphi \circ \tilde{\gamma})'(t_0) - \Phi\| \leq \epsilon$$

so that  $\tilde{\gamma}$  is in the  $\varphi$ -direction of  $C$ . Similarly

$$|(\psi_i \circ \tilde{\gamma})'(t_0)| \geq |\Psi_i|$$

for each  $1 \leq i \leq n$  and, for every  $t, t' \in I$ ,

$$\|\tilde{\gamma}(t) - \tilde{\gamma}(t')\| \leq b|t - t'|.$$

Therefore, for any  $t \leq t' \in I$ ,

$$\begin{aligned} \|\tilde{\gamma}(t) - \tilde{\gamma}(t')\| &\geq \|\varphi(\tilde{\gamma}(t)) - \varphi(\tilde{\gamma}(t'))\| \\ &= \left\| \int_{[t, t']} (\varphi \circ \tilde{\gamma})'(t_0) \right\| \\ &\geq (\|\Phi\| - \epsilon)|t - t'| \end{aligned}$$

so that  $\tilde{\gamma}$  is bi-Lipschitz and so belongs to  $\Pi(X)$ . Similarly, for any  $1 \leq i \leq n$ ,

$$\begin{aligned} |\psi_i(\tilde{\gamma}(t)) - \psi_i(\tilde{\gamma}(t'))| &= \left| \int_{[t, t']} (\psi_i \circ \tilde{\gamma})'(t_0) \right| \\ &\geq \delta'_i b |t - t_0| \\ &\geq \delta_i \|\tilde{\gamma}(t) - \tilde{\gamma}(t')\|. \end{aligned}$$

Therefore  $\tilde{\gamma}$  satisfies the hypotheses of the lemma and so

$$0 = \mathcal{H}^1(\tilde{\gamma} \cap S) \geq \mathcal{H}^1(\tilde{\gamma}(D_0)) > 0,$$

a contradiction. □

To apply these results to a metric measure space, we now define an embedding into a Banach space that exposes additional structure given by some Lipschitz functions defined on the metric space.

**Definition 5.6.** For  $n \in \mathbb{N}$  we define  $\mathcal{B}$  to be the Banach space

$$\mathcal{B} = \mathbb{R}^n \times \mathbb{R}^n \times \ell_\infty$$

with norm

$$\|(u, v, s)\| = \max\{\|u\|_{\mathbb{R}^n}, \|v\|_{\mathbb{R}^n}, \|s\|_{\ell_\infty}\}.$$

Further, suppose that  $(X, d)$  is a metric space and  $\varphi, \psi: X \rightarrow \mathbb{R}^n$  Lipschitz. We fix a bi-Lipschitz embedding  $\iota: X \rightarrow \ell_\infty$  and define the bi-Lipschitz embedding  $\iota^*: X \rightarrow \mathcal{B}$  by

$$x \mapsto (\varphi(x), \psi(x), \iota(x)).$$

We identify  $X$  with its image in  $\mathcal{B}$  under  $\iota^*$  and note that the projection  $P_1$  onto the first factor (respectively  $P_2$  onto the second factor) in  $\mathcal{B}$  agrees with  $\varphi$  (respectively  $\psi$ ) on  $X$ . In particular, for any  $\gamma \in \Gamma(X)$ , the derivative  $(\varphi \circ \gamma)'$  (respectively  $(\psi \circ \gamma)'$ ) agrees with  $(P_1 \circ \gamma)'$  (respectively  $(P_2 \circ \gamma)'$ ) almost everywhere.

Finally, we define an additional property of an Alberti representation. The *speed* of an Alberti representation quantitatively describes how the partial derivative of a Lipschitz function with respect to the Alberti representation compares to the infinitesimal behaviour of such a function. This notion will have a particular importance when characterising Lipschitz differentiability spaces when we will be interested in Alberti representations with speed independent of the Lipschitz function.

**Definition 5.7.** Let  $(X, d)$  be a metric space,  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz and  $\delta > 0$ . We say that  $\gamma \in \Gamma(X)$  has *speed*  $\delta$  (or  $\varphi$ -*speed*  $\delta$  whenever the implied Lipschitz function is not clear) if for almost every  $t_0 \in \text{Dom } \gamma$ ,

$$\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \text{Lip}(\varphi, \gamma(t_0)) \text{Lip}(\gamma, t_0).$$

Further, we say that an Alberti representation  $\mathcal{A}$  has *speed*  $\delta$  (or  $\varphi$ -*speed*  $\delta$ ) if almost every  $\gamma \in \mathcal{A}$  has speed  $\delta$ .

**Corollary 5.8.** Let  $(X, d, \mu)$  be a metric measure space and  $\varphi, \psi: X \rightarrow \mathbb{R}^n$  Lipschitz such that  $\text{Lip}(\psi_i, x_0) > 0$  for each  $x_0 \in X$  and each  $1 \leq i \leq n$ . Then, for any cone  $C \subset \mathbb{R}^n$  and  $\delta_1, \dots, \delta_n > 0$ , there exists a Borel decomposition  $X = A \cup S$  such that

- There exists an Alberti representation of  $\mu \llcorner A$  in the  $\varphi$ -direction of  $C$  with  $\psi_i$ -speed  $\delta_i$  for each  $1 \leq i \leq n$ .
- $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C^\circ$  with  $\psi_i$ -speed strictly greater than  $\delta_i$ , for each  $1 \leq i \leq n$ .

In particular, for any cone  $C \subset \mathbb{R}^n$  there exists a Borel decomposition  $X = A \cup S$  where  $\mu \llcorner A$  has an Alberti representation in the  $\varphi$ -direction of  $C$  and  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C$ .

*Proof.* Fix  $\lambda > 1$ ,  $k_1, \dots, k_n \in \mathbb{Z}$  and let  $Y \subset X$  be compact with

$$\lambda^{-k_i} \geq \text{Lip}(\psi_i, x_0) \geq \lambda^{-k_i - 1}$$

for every  $x_0 \in Y$  and every  $1 \leq i \leq n$ . Since  $Y$  is compact, its closed convex hull  $\tilde{Y}$  is also compact. By applying Corollary 5.4 to  $(\tilde{Y}, d, \mu)$  we obtain a Borel decomposition  $Y = Y_a \cup Y_s$  such that

- There exists an Alberti representation  $\mathcal{A}$  of  $\mu \llcorner Y_a$  in the  $\varphi$ -direction of  $C$  such that

$$|(\psi_i \circ \gamma)'(t_0)| \geq \delta_i \lambda^{k_i} \text{Lip}(\gamma, t_0)$$

for almost every  $\gamma \in \mathcal{A}$ , almost every  $t_0 \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ .

- $\mathcal{H}^1(\gamma \cap Y_s) = 0$  for every  $\gamma \in \Pi(\tilde{Y})$  in the  $\varphi$ -direction of  $C$  with

$$|\psi_i(\gamma(t)) - \psi_i(\gamma(t'))| \geq \delta_i \lambda^{k_i} \|\gamma(t) - \gamma(t')\|$$

for each  $t, t' \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ .

By our choice of  $\lambda$ ,  $\mathcal{A}$  has  $\psi_i$ -speed  $\delta_i$  for each  $1 \leq i \leq n$ . Further, by Lemma 5.5, for any  $\gamma \in \Gamma(Y)$  in the  $\varphi$ -direction of  $C^\circ$  with each  $\psi_i$ -speed strictly greater than  $\delta_i \lambda$ ,  $\mathcal{H}^1(\gamma \cap S) = 0$ .

By varying the  $k_i \in \mathbb{Z}$ , there exists a countable cover of  $X$  by such  $Y$  except for a  $\mu$ -null set and so, after combining the respective representations using Lemma 2.4, for each  $\lambda > 1$  there exists a Borel decomposition  $X = A_\lambda \cup S_\lambda$  where  $A_\lambda$  has an Alberti representation in the  $\varphi$ -direction of  $C$  with  $\psi_i$ -speed  $\delta_i$  for each  $1 \leq i \leq n$  and such that  $S$  satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma$  in the  $\varphi$ -direction of  $C^\circ$  with each  $\psi_i$ -speed greater than  $\delta_i \lambda$ . Writing  $A = \cup_\lambda A_\lambda$  and  $S = \cap_\lambda S_\lambda$  (where the union and intersection are taken over  $\lambda \in \mathbb{Q}$ ,  $\lambda > 1$ ), Lemma 2.4 completes the proof.  $\square$

**5.2. Multiple representations.** We now improve upon Corollary 5.8 by producing multiple independent Alberti representations of a metric measure space  $(X, d, \mu)$ .

Suppose that  $\varphi: X \rightarrow \mathbb{R}^n$  is Lipschitz and that  $C_1, \dots, C_n$  is a collection of independent cones in  $\mathbb{R}^n$ . Then, by multiple applications of Corollary 5.8, there exists a Borel decomposition  $X = A \cup S$  such that  $\mu \llcorner A$  has  $n$   $\varphi$ -independent Alberti representations and  $S$  has a Borel decomposition  $S = S_1 \cup \dots \cup S_n$  such that, for each  $1 \leq i \leq n$ ,  $\mathcal{H}^1(\gamma \cap S_i) = 0$  for each  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C_i$ .

However, as we will see in the next section, the method of constructing a non-differentiable Lipschitz function on  $S$  requires, for any  $\epsilon > 0$ , a decomposition as above such that each  $C_i$  has width  $1 - \epsilon$ . Of course, for sufficiently small  $\epsilon$ , these cones will not be independent. So that we may produce independent Alberti representations and also satisfy this condition for  $S$ , we introduce a method that reduces the width of a cone that defines the direction of an Alberti representation, at the expense of a countable decomposition.

**Corollary 5.9.** *Let  $(X, d, \mu)$  be a metric measure space,  $A \subset X$  Borel and  $\varphi, \psi: X \rightarrow \mathbb{R}^n$  Lipschitz. Suppose that, for some cone  $C = C(w, \theta) \subset \mathbb{R}^n$ ,  $\mu \llcorner A$  has an Alberti representation  $\mathcal{A}$  in the  $\varphi$ -direction of  $C$  such that  $|(\psi_i \circ \gamma)'(t_0)| > 0$  for almost every  $\gamma \in \mathcal{A}$ , almost every  $t_0 \in \text{Dom } \gamma$  and every  $1 \leq i \leq n$ . Then, for any countable collection of cones  $C_m$  with*

$$\bigcup_{m \in \mathbb{N}} C_m^\circ \supset C \setminus \{0\},$$

*there exists a countable Borel decomposition  $A = \cup_k A_k$  such that each  $\mu \llcorner A_k$  has an Alberti representation  $\mathcal{A}_k$  in the  $\varphi$ -direction of some  $C_m$  with  $\psi_i$ -speed strictly greater than  $1/k$ , for each  $1 \leq i \leq n$ .*

*Moreover, if for some  $\delta_1, \dots, \delta_n > 0$ ,  $\mathcal{A}$  has  $\psi_i$ -speed strictly greater than  $\delta_i$  for each  $1 \leq i \leq n$ , then so does each  $\mathcal{A}_k$ .*

*Proof.* By applying Corollary 5.8 using each cone  $C_m$  in the hypotheses and each  $\delta_i = 1/k$ , we obtain a countable Borel decomposition  $A = \cup_k A_k$  such that each  $\mu \llcorner A_k$  has an Alberti representation of the required form and a set  $S$  that satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for each  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of some  $C_m^\circ$  and with positive  $\psi_i$ -speed for each  $1 \leq i \leq n$ . Therefore, since the  $C_m^\circ$  cover  $C \setminus \{0\}$ ,  $S$  satisfies this for all  $\gamma$  in the  $\varphi$ -direction of  $C$  with positive  $\psi_i$ -speed for each  $1 \leq i \leq n$ . Since  $\mu \llcorner A$  is represented by  $\mathcal{A}$ ,  $\mu(S) = 0$ .

Now suppose that  $\delta'_i > \delta_i$  and that  $\mathcal{A}$  has  $\psi_i$ -speed  $\delta'_i$  for each  $1 \leq i \leq n$ . Then if we repeat the same process but for each  $1 \leq i \leq n$  take each  $\mathcal{A}_k$  to have  $\psi_i$ -speed between  $\delta'_i$  and  $\delta_i$ , we obtain the same decomposition but with  $S$  satisfying

$\mathcal{H}^1(\gamma \cap S) = 0$  for each  $\gamma$  in the  $\varphi$ -direction of  $C$  with  $\psi_i$ -speed  $\delta'_i$  for each  $1 \leq i \leq n$ . Again, since  $\mu \llcorner A$  is represented by  $\mathcal{A}$ ,  $\mu(S) = 0$ .  $\square$

**Definition 5.10.** We will refer to the above process of obtaining new Alberti representations with the same properties as an existing representation but in the direction of thinner cones as *refining* an Alberti representation.

The requirement in the previous corollary that both  $\varphi$  and  $\psi$  take values in the same Euclidean space is not necessary. Indeed, suppose that  $(X, d, \mu)$  is a metric measure space satisfying the hypotheses of Corollary 5.9 for  $\varphi: X \rightarrow \mathbb{R}^n$  and  $\psi: X \rightarrow \mathbb{R}^m$  Lipschitz. If  $m < n$ , then by repeating any  $n - m + 1$  coordinate functions of  $\psi$ , we obtain a Lipschitz function  $\tilde{\psi}$  into  $\mathbb{R}^n$ . Similarly, if  $n < m$ , we may define a Lipschitz function  $\tilde{\varphi}$  into  $\mathbb{R}^m$  and  $\tilde{w} \in \mathbb{R}^m$  by  $\tilde{\varphi}_i = \varphi_i$  and  $\tilde{w}_i = w_i$  for  $1 \leq i \leq n$  and  $\tilde{\varphi}_i = \tilde{w}_i = 0$  for  $n < i \leq m$ . Then a Lipschitz curve is in the  $\varphi$ -direction of  $C(w, \theta)$  if and only if it is in the  $\tilde{\varphi}$ -direction of  $C(\tilde{w}, \theta)$ . In both cases, the original hypotheses of Corollary 5.9 are satisfied and its conclusion gives us Alberti representations of the required form.

We will require refinements of Alberti representations that maintain the speed of a vector valued function. Suppose an Alberti representation  $\mathcal{A}$  of  $(X, d, \mu)$  is in the  $\varphi$ -direction of a cone  $C$ , so that  $\|(\varphi \circ \gamma)'(t_0)\| > 0$  for almost every  $\gamma \in \mathcal{A}$  and almost every  $t_0 \in \gamma$ . Then we may refine  $\mathcal{A}$  into Alberti representations, each with positive  $\varphi$ -speed. Further, suppose that  $\mathcal{A}$  has  $\varphi$ -speed strictly greater than  $\delta$ , for some  $\delta > 0$ . Then there exist a countable Borel decomposition  $X = \cup_i X_i$  and for each  $i$ , rational  $\delta_1, \dots, \delta_n > 0$  with  $\delta_1^2 + \dots + \delta_n^2 > \delta^2$  such that  $\mathcal{A} \llcorner X_i$  has  $\varphi_i$ -speed strictly greater than  $\delta_i$ , for each  $1 \leq j \leq n$ . Therefore, we may take arbitrary refinements of  $\mathcal{A}$ , each with  $\varphi$ -speed strictly greater than  $\delta$ .

As in previous constructions, we will find multiple independent Alberti representations of a metric measure space  $(X, d, \mu)$  by decomposing  $X$  into two sets: one set on which  $\mu$  has many independent Alberti representations and another, singular set. We now define the general form that such a singular set takes. In fact, we will later see that such sets play a fundamental role when determining whether a metric measure space is a Lipschitz differentiability space. These sets, and the methods involving them in the following section, are a natural generalisation of those considered in [ACP] in which the authors investigate measures on Euclidean space with respect to which Rademacher’s Theorem holds.

**Definition 5.11.** Let  $(X, d)$  be a metric space,  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz and  $\delta, \theta, \lambda > 0$ . We define  $\tilde{A}(\varphi; \delta, \theta, \lambda)$  to be the set of  $S \subset X$  such that

- For every  $x_0 \in S$ ,
 
$$\text{Lip}(v \cdot \varphi, x_0) > \lambda \text{Lip}(\varphi, x_0) \quad \forall v \in \mathbb{S}^{n-1}.$$
- There exist a countable Borel decomposition  $S = \cup_k S_k$  and a countable collection of closed cones  $C_k \subset \mathbb{R}^n$  of width  $1 - \theta$  such that each  $S_k$  satisfies  $\mathcal{H}^1(\gamma \cap S_k) = 0$  for each  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C_k$  with  $\varphi$ -speed  $\delta$ .

Further, we define  $\tilde{A}(\delta, \theta, \lambda)$  to be the set of  $S \subset X$  for which there exist a countable Borel decomposition  $S = \cup_k S_k$  and a sequence of Lipschitz functions  $\varphi_k: X \rightarrow \mathbb{R}^{n_k}$  such that  $S_k \in \tilde{A}(\varphi_k; \delta, \theta, \lambda)$  for each  $k \in \mathbb{N}$ . Finally, we define  $\tilde{A}(\varphi)$ , respectively  $\tilde{A}$ , to be the set of  $S \subset X$  for which there exists a  $\lambda > 0$  such that  $S \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ , respectively  $S \in \tilde{A}(\delta, \theta, \lambda)$ , for every  $\delta, \theta > 0$ .

*Remark 5.12.* Observe that  $\tilde{A}(\varphi; \delta, \theta, \lambda) \subset \tilde{A}(\varphi; \delta', \theta', \lambda')$  whenever  $\delta \leq \delta'$ ,  $\theta \geq \theta'$  and  $\lambda \leq \lambda'$ , and hence  $\tilde{A}(\delta, \theta, \lambda) \subset \tilde{A}(\delta', \theta', \lambda')$ .

By combining the previous results of this section we may construct many independent Alberti representations of a measure, one by one.

**Proposition 5.13.** *Let  $(U, \varphi)$  be a  $\lambda$ -structured chart of dimension  $n$  in a metric measure space  $(X, d, \mu)$ . Then for any  $0 < \delta, \theta < 1$  there exists a countable Borel decomposition*

$$U = S \cup \bigcup_{i \in \mathbb{N}} U_i$$

such that each  $\mu \llcorner U_i$  has  $n$   $\varphi$ -independent Alberti representations with  $\varphi$ -speed strictly greater than  $\delta$  and  $S$  belongs to  $\tilde{A}(\varphi; \delta, \theta, \lambda / \text{Lip } \varphi)$ .

*Proof.* Observe that it suffices to prove the result for  $\text{Lip } \varphi = 1$ . In this case, by applying Corollary 5.8 using a cone of width  $1 - \theta$ , there exists a Borel decomposition  $U = U' \cup S$  such that  $\mu \llcorner U'$  has an Alberti representation with  $\varphi$ -speed strictly greater than  $\delta$  and  $S \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ .

Now suppose that, for some  $1 \leq m < n$ , there exists a countable Borel decomposition  $U = \cup_i U_i \cup S$  such that each  $\mu \llcorner U_i$  has  $m$  independent Alberti representations with  $\varphi$ -speed strictly greater than  $\delta$  and  $S \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ . Then, by refining if necessary, we may suppose that the representations of each  $\mu \llcorner U_i$  are in the  $\varphi$ -direction of cones of width  $\alpha$ , for some  $0 < \alpha < \sqrt{1 - \theta^2}$ .

Now fix an  $i \in \mathbb{N}$  and let  $w_1, \dots, w_m \in \mathbb{S}^{n-1}$  such that  $C(w_1, \alpha), \dots, C(w_m, \alpha)$  are independent and  $\mu \llcorner U_i$  has an Alberti representation in the  $\varphi$ -direction of  $C(w_i, \alpha)$ , for each  $1 \leq i \leq m$ . We choose  $w_{m+1} \in \mathbb{S}^{n-1}$  orthogonal to  $w_1, \dots, w_m$  so that, by our choice of  $\alpha$ ,  $C(w_1, \alpha), \dots, C(w_m, \alpha), C(w_{m+1}, \theta)$  are independent cones in  $\mathbb{R}^n$ . By applying Corollary 5.8 using  $C(w_{m+1}, \theta)$ , there exists a Borel decomposition  $U_i = U'_i \cup S_i$  such that  $\mu \llcorner U'_i$  has  $m + 1$  independent Alberti representations with  $\varphi$ -speed strictly greater than  $\delta$  and  $S_i \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ .

Since  $S \cup S_1 \cup S_2 \cup \dots \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ , we see that there exists a countable Borel decomposition  $X = \cup_i U'_i \cup S'$  such that each  $\mu \llcorner U'_i$  has  $m + 1$  independent Alberti representations and  $S' \in \tilde{A}(\varphi; \delta, \theta, \lambda)$ . By repeating this process  $n - 1$  times we obtain the required decomposition.  $\square$

**Theorem 5.14.** *Let  $(U, \varphi)$  be a  $\lambda$ -structured chart of dimension  $n$  in a metric measure space  $(X, d, \mu)$ . Then there exists a countable Borel decomposition  $U = \cup_i U_i$  such that each  $\mu \llcorner U_i$  has  $n$   $\varphi$ -independent Alberti representations if and only if, for every  $S \in \tilde{A}(\varphi)$  with  $S \subset U$ ,  $\mu(S) = 0$ .*

*Proof.* First suppose that, for every  $S \in \tilde{A}(\varphi)$  with  $S \subset U$ ,  $\mu(S) = 0$ . By the previous proposition, for each  $m \in \mathbb{N}$  there exists a Borel decomposition  $U = U_m \cup S_m$  such that  $U_m$  has a countable decomposition of the required form and  $S_m \in \tilde{A}(\varphi; 1/m, 1/m, \lambda)$ . This gives a Borel decomposition  $U = S \cup U_1 \cup U_2 \cup \dots$  where each  $U_i$  is of the required form and  $S = \cap_m S_m \in \tilde{A}(\varphi)$ , as required.

Conversely, suppose that such a decomposition exists. Then by refining each Alberti representation if necessary, for each  $U_j$  we may suppose that there exist a  $\delta > 0$  and independent cones  $C_1, \dots, C_n$  such that for each  $1 \leq i \leq n$ ,  $\mu \llcorner U_j$  has an Alberti representation in the  $\varphi$ -direction of  $C_i$  with  $\varphi$ -speed  $\delta$ . For such a  $U_j$  there exists an  $m \in \mathbb{N}$  such that  $1/m < \delta$  and such that any cone of width

$1 - 1/m$  must completely contain one of the  $C_i$ . Since  $\mu \ll U_j$  has the above Alberti representations, for any  $S \in \tilde{A}(\varphi; 1/m, 1/m, \lambda)$ ,  $\mu(S \cap U_j) = 0$ . In particular, for any  $S \in \tilde{A}(\varphi)$ ,  $\mu(S \cap U_j) = 0$  and hence  $\mu(S \cap U) = 0$ .  $\square$

6. FIRST ALBERTI REPRESENTATIONS IN DIFFERENTIABILITY SPACES

By combining our previous results, we now prove the existence of independent Alberti representations in Lipschitz differentiability spaces. We will apply Theorem 5.14 to construct independent representations and will show, for any chart  $(U, \varphi)$  and  $S \in \tilde{A}(\varphi)$ , that  $\mu(S \cap U) = 0$  by constructing a Lipschitz function that is differentiable almost nowhere in such a set, via Proposition 4.3.

**6.1. Constructions in Banach spaces.** We will again embed our metric space into a Banach space. This provides many Lipschitz curves that we may use to define the Lipschitz functions required for Proposition 4.3.

In this subsection we fix the following notation for simplicity.

**Notation 6.1.** For  $n \in \mathbb{N}$  we let  $\mathcal{B}$  be the Banach space defined in Definition 5.6, let  $\varphi: \mathcal{B} \rightarrow \mathbb{R}^n$  be the projection onto the first factor and fix  $w \in \mathbb{S}^{n-1}$ . We also fix  $X$  a compact subset of  $\mathbb{R}^n \times \{0\} \times \ell_\infty \subset \mathcal{B}$ ,  $0 < \delta, \theta < 1$  and  $S \subset X$  closed with  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w, \theta)$  with  $\varphi$ -speed  $\delta$ .

For each  $t \in \mathbb{R}$  we define  $\mathcal{P}_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$  to be the orthogonal projection onto

$$\{v \in \mathbb{R}^n : (v - tw) \cdot w = 0\}$$

and define  $P_t: \mathcal{B} \rightarrow \mathcal{B}$  by

$$P_t(u, v, s) = (\mathcal{P}_t(u), v, s).$$

We also define  $\mathcal{P} = \mathcal{P}_0$ ,  $\varphi_0 = \inf\{\varphi(x) \cdot w : x \in X\}$ ,  $P = P_{\varphi_0}$  and  $\Omega$  to be the closed, convex hull of  $P(X) \cup X$ , a compact, convex set.

Further, for any measurable  $I \subset \mathbb{R}$  and Lipschitz  $\gamma: I \rightarrow \Omega$ , we express

$$\gamma = (\varphi \circ \gamma, 0, \gamma^*)$$

and write  $\mathcal{V}_\gamma(t) = \text{Lip}(\gamma^*, t)$ , so that

$$\text{Lip}(\gamma, t) = \max\{\|(\varphi \circ \gamma)'(t)\|, \mathcal{V}_\gamma(t)\}$$

for almost every  $t \in I$ .

In addition, for any Borel  $V \subset \mathcal{B}$  and  $\epsilon > 0$ , we define

$$Q(V, \gamma, \epsilon) = \int_{\gamma^{-1}(\Omega \setminus V)} (\varphi \circ \gamma)' \cdot w + \int_{\gamma^{-1}(\Omega)} (K(\theta) \|\mathcal{P}((\varphi \circ \gamma)')\| + \delta \mathcal{V}_\gamma) + \epsilon \mathcal{H}^1(\gamma),$$

where  $K(\theta) = (1 - \theta) / \sqrt{\theta(2 - \theta)}$ .

**Lemma 6.2.** *For any  $\epsilon > 0$  there exists a  $\mathcal{B}$ -open set  $V \supset S$  such that, for any compact interval  $I$  and Lipschitz  $\gamma: I \rightarrow \Omega$  with  $(\varphi \circ \gamma)' \cdot w \geq 0$  almost everywhere,*

$$(6.1) \quad Q(V, \gamma, \epsilon) \geq (\varphi(\gamma_e) - \varphi(\gamma_s)) \cdot w - \epsilon,$$

where  $\gamma_s$  and  $\gamma_e$  are the endpoints of  $\gamma$ .



*Proof.* Suppose that the conclusion is false. Then there exist an  $\epsilon > 0$  and for each  $m \in \mathbb{N}$  a Lipschitz  $\gamma_m$  that violates (6.1) for the open set  $V_m = B(S, 1/m)$ . Since  $\Omega$  is compact, by replacing  $\epsilon$  by  $\epsilon/2$  if necessary, we may suppose that each  $\gamma_m$  has the same endpoints,  $\gamma_s$  and  $\gamma_e$ . In particular, for each  $m \in \mathbb{N}$ ,

$$\mathcal{H}^1(\gamma_m) \leq Q(V_m, \gamma_m, \epsilon)/\epsilon \leq (\varphi(\gamma_e) - \varphi(\gamma_s)) \cdot w/\epsilon.$$

Therefore, there exist an  $L > 0$  and a reparametrisation of each  $\gamma_m$  such that each is a 1-Lipschitz function defined on  $[0, L]$ . Further, by the Arzelà-Ascoli theorem and after possibly choosing a suitable subsequence, there exists a 1-Lipschitz  $\gamma: [0, L] \rightarrow \Omega$  such that  $\gamma_m \rightarrow \gamma$  uniformly. Moreover, since  $(\varphi \circ \gamma_m)' \geq 0$  almost everywhere for each  $m \in \mathbb{N}$ ,  $(\varphi \circ \gamma)' \geq 0$  almost everywhere. We will show that the image of  $\gamma$  intersects  $S$  in a set of positive measure from which we deduce a contradiction to our hypothesis.

Fix an  $m \in \mathbb{N}$  and  $\eta > 0$ . Since  $\beta = \gamma^{-1}(\Omega \setminus \overline{V_m})$  is an open subset of  $\mathbb{R}$  of finite measure, there exist disjoint compact intervals  $\beta_1, \dots, \beta_N \subset \beta$  such that  $\mathcal{L}^1(\beta \setminus \cup_i \beta_i) \leq \eta$ . Therefore,

$$\int_{\gamma^{-1}(\Omega \setminus \overline{V_m})} (\varphi \circ \gamma)' \cdot w \leq \int_{\cup_i \beta_i} (\varphi \circ \gamma)' \cdot w + \eta \text{Lip } \varphi.$$

Moreover, by the lower semicontinuity of the total variation of Lipschitz functions under uniform convergence,

$$\int_{\cup_i \beta_i} (\varphi \circ \gamma)' \cdot w \leq \liminf_{k \rightarrow \infty} \int_{\cup_i \beta_i} (\varphi \circ \gamma_k)' \cdot w.$$

Now, since  $\gamma_k \rightarrow \gamma$  uniformly and each  $\gamma(\beta_i)$  is compact, there exists a  $K \in \mathbb{N}$  such that, for all  $k \geq K$  and  $1 \leq i \leq N$ ,  $\gamma_k(\beta_i) \subset \Omega \setminus \overline{V_m}$ . Therefore, since  $(\varphi \circ \gamma_k)' \cdot w \geq 0$  almost everywhere for each  $k$ ,

$$\int_{\gamma^{-1}(\Omega \setminus \overline{V_m})} (\varphi \circ \gamma)' \cdot w \leq \liminf_{k \rightarrow \infty} \int_{\gamma_k^{-1}(\Omega \setminus \overline{V_m})} (\varphi \circ \gamma_k)' \cdot w + \eta \text{Lip } \varphi.$$

Further, by again using the lower semicontinuity of total variation and of  $\alpha \mapsto \mathcal{H}^1(\alpha)$ ,

$$Q(\overline{V_m}, \gamma, \epsilon) \leq \liminf_{k \rightarrow \infty} Q(\overline{V_m}, \gamma_k, \epsilon) + \eta \text{Lip } \varphi.$$

Finally, since  $(\varphi \circ \gamma)' \geq 0$  almost everywhere and  $V_k \subset V_m$  for every  $k \geq m$ ,

$$Q(\overline{V_m}, \gamma, \epsilon) \leq (\varphi(\gamma_e) - \varphi(\gamma_s)) \cdot w - \epsilon + \eta \text{Lip } \varphi.$$

This is true for all  $m \in \mathbb{N}$  and  $\eta > 0$ . Moreover, since  $S$  is closed,  $\cap_m \overline{V_m} = S$  and so

$$Q(S, \gamma, \epsilon) \leq (\varphi(\gamma_e) - \varphi(\gamma_s)) \cdot w - \epsilon.$$

In particular, by the fundamental theorem of calculus,

$$(6.2) \quad \epsilon \leq \int_{\gamma^{-1}(S)} (\varphi \circ \gamma)' \cdot w - \int_{\gamma^{-1}(\Omega)} (K(\theta) \|\mathcal{P}((\varphi \circ \gamma)')\| + \delta \mathcal{V}_\gamma) - \epsilon \mathcal{H}^1(\gamma).$$

On the other hand, when  $\gamma$  is restricted to

$$D := \{t_0 : (\varphi \circ \gamma)'(t_0) \cdot w > (1 - \theta) \|(\varphi \circ \gamma)'(t_0)\| \geq (1 - \theta) \delta \text{Lip}(\gamma, t_0)\},$$

it is a Lipschitz function in the  $\varphi$ -direction of  $C$  with  $\varphi$ -speed  $\delta$ . Moreover,  $\text{Lip}(\gamma, t_0) > 0$  for every  $t_0 \in D$  and so we may decompose  $D$  into a Lebesgue null set and a countable collection of compact sets  $K_i$  on which  $\gamma$  is bi-Lipschitz. In

particular, each  $\gamma|_{K_i} \in \Gamma(X)$  and so, since we are under the hypotheses of Notation 6.1,  $D \cap \gamma^{-1}(S)$  is Lebesgue null. Therefore, for almost every  $t_0 \in \gamma^{-1}(S)$ , either

$$(\varphi \circ \gamma)'(t_0) \leq (1 - \theta)\|(\varphi \circ \gamma)'(t_0)\| \leq K(\theta)\|\mathcal{P}(\varphi \circ \gamma)'(t_0)\|,$$

or

$$\|(\varphi \circ \gamma)'(t_0)\| \leq \delta \text{Lip}(\gamma, t_0) \text{ and so } \text{Lip}(\gamma, t_0) = \mathcal{V}_\gamma(t_0).$$

Thus

$$(6.3) \quad \int_{\gamma^{-1}(S)} (\varphi \circ \gamma)' \cdot w \leq \int_{\gamma^{-1}(S)} (K(\theta)\|\mathcal{P}((\varphi \circ \gamma)')\| + \delta \mathcal{V}_\gamma).$$

By combining equations (6.2) and (6.3) we obtain  $\epsilon \leq 0$ , giving the required contradiction.  $\square$

We use this result to construct a Lipschitz function on  $\Omega$  defined via Lipschitz curves connecting points in  $\Omega$ . We will see that functions of this form have the properties required to apply Proposition 4.3.

**Lemma 6.3.** *For any  $\epsilon > 0$  there exist a  $\mathcal{B}$ -open set  $V \supset S$  and a  $(1 + K(\theta) + \delta + \epsilon)$ -Lipschitz function  $f: \Omega \rightarrow \mathbb{R}$  such that*

- (1) *For every  $x, x_0 \in \Omega$  with  $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$ ,*  

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \epsilon.$$
- (2) *For every  $y, z$  contained in a ball  $B \subset V$ ,*  

$$|f(y) - f(z)| \leq K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)\|y - z\|.$$

*Proof.* For  $\epsilon > 0$  let  $V \supset S$  be the  $\Omega$ -open set obtained from an application of Lemma 6.2. By our definition of  $\Omega$ , for every  $x \in \Omega$  the straight line segment joining  $P(x)$  to  $x$  lies in  $\Omega$ . Therefore we may define a function  $f: \Omega \rightarrow \mathbb{R}$  by

$$f(x) = \inf Q(V, \gamma, \epsilon)$$

where the infimum is taken over all  $l \geq 0$  and all Lipschitz  $\gamma: [0, l] \rightarrow \Omega$  with  $(\varphi \circ \gamma)' \cdot w \geq 0$  almost everywhere such that  $\gamma(0) \in P(\Omega)$  and  $\gamma(l) = x$ . We will call such a curve *admissible for  $x$* .

We now use the conclusion of Lemma 6.2 to deduce the required properties of  $f$ . We first show that  $f$  is Lipschitz and satisfies (2). Indeed, let  $y, z \in \Omega$  with  $(\varphi(z) - \varphi(y)) \cdot w \geq 0$  and let  $\gamma: [0, l] \rightarrow \Omega$  be any admissible curve for  $y$ . Define  $\tilde{\gamma}: [0, l + 1] \rightarrow \mathcal{B}$  by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & t \in [0, l] \\ y + (t - l)(z - y) & t \in (l, l + 1]. \end{cases}$$

Then since  $(\varphi(z) - \varphi(y)) \cdot w \geq 0$ ,  $(\varphi \circ \tilde{\gamma})' \geq 0$  almost everywhere and so  $\tilde{\gamma}$  is admissible for  $z$ . Therefore

$$\begin{aligned} f(z) &\leq f(y) + Q(V, \tilde{\gamma}|_{[l, l+1]}) \\ &\leq f(y) + \mathcal{H}^1(\tilde{\gamma}([l, l + 1]) \setminus V) + K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| \\ &\quad + (\delta + \epsilon)\mathcal{H}^1(\tilde{\gamma}([l, l + 1])) \\ (6.4) \quad &\leq f(y) + K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)\|y - z\| \text{ if } y, z \in \text{Ball} \subset V \\ (6.5) \quad &\leq f(y) + K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (1 + \delta + \epsilon)\|y - z\| \text{ otherwise.} \end{aligned}$$

To bound  $f(y)$ , let  $\gamma: [0, l] \rightarrow \Omega$  be admissible for  $z$  and set

$$t_0 = \inf\{t \in I : \varphi(\gamma(t)) \cdot w \geq \varphi(y) \cdot w\}.$$

Then since  $(\varphi \circ \gamma)' \geq 0$  almost everywhere,  $(\varphi(\gamma(t)) - \varphi(y)) \cdot w \geq 0$  for all  $t \geq t_0$ . We define  $\tilde{\gamma}$  by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq t_0 \\ P_{\varphi(y) \cdot w}(\gamma(t)) & t_0 < t \leq l \\ \tilde{\gamma}(l) + (t - l)(y - \tilde{\gamma}(l)) & t \in (l, l + 1]. \end{cases}$$

Then  $\tilde{\gamma}(l + 1) = y$  and  $(\varphi \circ \tilde{\gamma})'(t) \cdot w = 0$  for almost every  $t \in [t_0, l + 1]$ , so that  $\tilde{\gamma}$  is admissible for  $y$ . Further, for almost every  $t \in [t_0, l]$ ,

$$\|\mathcal{P}((\varphi \circ \tilde{\gamma})'(t))\| \leq \|\mathcal{P}((\varphi \circ \gamma)'(t))\| \text{ and } \mathcal{V}_{\tilde{\gamma}}(t) = \mathcal{V}_{\gamma}(t).$$

Therefore

$$Q(V, \tilde{\gamma}|_{[t_0, l]}) \leq Q(V, \gamma|_{[t_0, l]})$$

and so

$$(6.6) \quad f(y) \leq f(z) + K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)\|y - z\|.$$

By combining equations (6.4), (6.5) and (6.6) we see that  $f$  is a  $(1 + K(\theta) + \delta + \epsilon)$ -Lipschitz function such that, for every  $y, z$  belonging to a ball contained in  $V$ ,

$$|f(y) - f(z)| \leq K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)\|y - z\|.$$

To prove (1) let  $x, x_0 \in \Omega$  with  $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$ . By the definition of  $\Omega$  and  $P$ , the straight line segment  $\gamma$  joining  $P(x_0)$  to  $x_0$  lies entirely within  $\Omega$  and satisfies  $(\varphi \circ \gamma)' \geq 0$  almost everywhere and so is admissible for  $x_0$ . Further,  $\mathcal{P}((\varphi \circ \gamma)'(t_0)) = 0 = \mathcal{V}_{\gamma}(t_0)$  for almost every  $t_0$ . Therefore,

$$f(x_0) \leq (\varphi(x_0) - \varphi(P(x_0))) \cdot w + \epsilon\|x_0 - P(x_0)\|.$$

Further, by the conclusion of Lemma 6.2,

$$f(x) \geq (\varphi(x) - \varphi(P(x))) \cdot w - \epsilon.$$

Therefore, since  $(\varphi(P(x)) - \varphi(P(x_0))) \cdot w = 0$ ,

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \epsilon(1 + \text{diam } X).$$

This proves the result for  $(1 + \text{diam } X)\epsilon$ , which suffices. □

**6.2. Application to Lipschitz differentiability spaces.** We first apply our previous construction to subsets of metric measure spaces.

**Lemma 6.4.** *Let  $(X, d, \mu)$  be a metric measure space,  $U \subset X$  compact and  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz. Suppose that, for some  $w \in \mathbb{S}^{n-1}$ ,  $0 < \theta < 1$  and  $\delta > 0$ ,  $S \subset U$  is closed and satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w, \theta)$  with  $\varphi$ -speed  $\delta$ . Then for any  $\epsilon > 0$  there exists a  $(1 + K(\theta) + \delta + \epsilon)$  Lip  $\varphi$ -Lipschitz function  $f: U \rightarrow \mathbb{R}$  and a  $\rho > 0$  such that*

- For every  $x_0 \in S$  and  $x \in U$  with  $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$ ,

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \epsilon$$

- For every  $x_0 \in S$  and  $y, z \in B(x_0, \rho)$ ,

$$|f(y) - f(z)| \leq K(\theta)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon) \text{Lip } \varphi d(y, z).$$

Here  $\mathcal{P}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the orthogonal projection onto the hyperplane orthogonal to  $w$  passing through the origin.

*Proof.* Let us identify  $U$  with its image in  $\mathcal{B}$  via the bi-Lipschitz isomorphism  $\iota^*$  defined in Definition 5.6 (for  $\psi = 0$ ). Note that  $\varphi$  agrees with the projection onto the first factor on  $U$  and so we may extend  $\varphi$  to all of  $\mathcal{B}$  by defining it to be this projection. Then  $S$  is also a subset of  $\mathcal{B}$  that satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Gamma(U)$  in the  $\varphi$ -direction of  $C(w, \theta)$  with  $\varphi$ -speed  $\delta$ . We denote by  $\|x - y\|$  the distance between  $x$  and  $y$  in  $\mathcal{B}$  and by  $d(x, y)$  the original distance between  $x$  and  $y$  in  $U$ , so that

$$d(x, y) \leq \|x - y\| \leq \text{Lip } \varphi d(x, y).$$

Define  $f$  to be the Lipschitz function and  $V$  the open set obtained from an application of Lemma 6.3. Since  $U \subset V$  is compact there exists a  $\rho > 0$  such that, for every  $x_0 \in S$ ,  $B(x_0, \rho) \subset V$ . This function has the required properties.  $\square$

We now combine the functions constructed in the previous lemma to satisfy the hypotheses of Proposition 4.3 for arbitrarily large subsets of any  $S \in \tilde{A}(\varphi)$ , for any structured chart  $(U, \varphi)$ .

**Lemma 6.5.** *Let  $(U, \varphi)$  be a structured chart in a metric measure space  $(X, d, \mu)$  and suppose that  $S \in \tilde{A}(\varphi)$  with  $S \subset U$ . Then there exist a  $\beta > 0$  and, for any  $\delta > 0$ , a 1-Lipschitz function  $f: X \rightarrow \mathbb{R}$ , a compact  $S' \subset S$  with  $\mu(S') \geq \mu(S) - \delta$  and a  $\rho > 0$  such that*

- For every  $x_0 \in S'$  there exists an  $x \in U$  with  $0 < d(x, x_0) < \delta$  and

$$|f(x) - f(x_0)| \geq \beta d(x, x_0).$$

- For every  $x_0 \in S$  and  $y, z \in B(x_0, \rho)$ ,

$$|f(y) - f(z)| \leq \delta d(y, z).$$

*Proof.* Note that it suffices to prove the result for any 1/2-Lipschitz  $\varphi$  and  $0 < \delta < 1/2$ . If so, let  $\lambda > 0$  and  $0 < \theta < 1$  such that  $(U, \varphi)$  is a  $\lambda$ -structured chart and  $K(\theta) \leq \delta/2$ . Then there exist a countable Borel decomposition  $S = \cup_i S_i$  and for each  $i \in \mathbb{N}$  a  $w_i \in \mathbb{S}^{n-1}$  such that  $\mathcal{H}^1(\gamma \cap S_i) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w_i, \theta)$  with  $\varphi$ -speed  $\delta$ .

For every  $i \in \mathbb{N}$ , let  $Q_i \subset S_i$  be disjoint and compact and let  $N \in \mathbb{N}$  such that

$$\mu(Q_1 \cup \dots \cup Q_N) > \mu(S) - \delta.$$

Then there exists a  $h > 0$  such that the  $B(Q_i, 2h)$  are disjoint. Further, let  $0 < 4R < \min\{h, \delta\}$  such that  $(1 - 4R/h)\mu(Q_1 \cup \dots \cup Q_N) > \mu(S) - \delta$ . Then, since  $(U, \varphi)$  is a structured chart, there exists an  $r > 0$  such that, for every  $x_0 \in U$ , there exists an  $x \in X$  with  $r < d(x, x_0) < R$  and

$$\frac{|(\varphi(x) - \varphi(x_0)) \cdot w_i|}{d(x, x_0)} \geq \lambda.$$

We also set  $\epsilon = \lambda r/2$  and, for each  $1 \leq i \leq N$ , define

$$g_i: B(Q_i, h) \rightarrow \mathbb{R}$$

to be the 1-Lipschitz function obtained from Lemma 6.4 restricted to  $B(Q_i, h)$  for this choice of  $\epsilon$ ,  $\theta$  and  $w_i$ . Further, we let  $f_i: B(Q_i, h) \rightarrow \mathbb{R}$  and  $P_i \subset Q_i$  be obtained from applying Lemma 4.1 to  $g_i$  and  $Q_i$  with the choice of  $\epsilon = R$ . We extend each  $f_i$  to a 1-Lipschitz function defined on  $X$  with value zero outside  $B(Q_i, 2h)$ .

Then for any  $x_0 \in P_i$  there exists an  $x \in X$  with  $r < d(x, x_0) < R < \delta$  and

$$\begin{aligned} |f_i(x) - f_i(x_0)| &= |g_i(x) - g_i(x_0)| \\ &\geq |(\varphi(x) - \varphi(x_0)) \cdot w_i| - \epsilon \\ &\geq \lambda d(x, x_0) - \lambda r/2 \\ &\geq \lambda d(x, x_0)/2. \end{aligned}$$

Further, we let  $\rho > 0$  such that, for any  $y, z \in \text{Ball} \subset B(Q_i, \rho)$ ,

$$\begin{aligned} |f_i(y) - f_i(z)| &\leq |g_i(y) - g_i(z)| \\ &\leq K(\theta) \|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\delta + \epsilon)d(y, z) \\ &\leq 2\delta d(y, z). \end{aligned}$$

Finally we set  $S' = P_1 \cup \dots \cup P_N$  so that

$$\mu(S') \geq (1 - 4R/h)\mu(Q_1 \cup \dots \cup Q_N) \geq \mu(S) - \delta$$

and

$$f = \sum_{1 \leq i \leq N} g_i.$$

Then since the  $g_i$  have disjoint support,  $f$  and  $S'$  satisfy the conclusion of the lemma for  $2\delta$  and  $\beta = \lambda/2$ .  $\square$

We now combine all of our previous results to obtain our first statement on the structure of measures in Lipschitz differentiability spaces.

**Theorem 6.6.** *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$ . Then any  $\tilde{A}(\varphi)$  subset of  $U$  is  $\mu$ -null. Therefore, there exists a countable Borel decomposition*

$$U = \bigcup_{k \in \mathbb{N}} U_k$$

such that, for each  $k \in \mathbb{N}$ ,  $\mu \llcorner U_k$  has  $n$   $\varphi$ -independent Alberti representations.

Moreover, for any Lipschitz  $\psi: X \rightarrow \mathbb{R}^m$  and  $U' \subset U$  of positive measure, if  $U'$  has  $m$   $\psi$ -independent Alberti representations, then  $m \leq n$ .

*Proof.* By Lemma 3.7 it suffices to prove that any  $\tilde{A}(\varphi)$  subset  $S$  of a structured chart  $(U, \varphi)$  is  $\mu$ -null. If  $S$  is such a set, for any  $\epsilon > 0$  and each  $m \in \mathbb{N}$  let  $0 < \delta_m < 1/m$  such that  $\sum_m \delta_m < \epsilon$  and let  $f_m: X \rightarrow \mathbb{R}$  and  $S_m \subset S$  be obtained by applying Lemma 6.5 with  $\delta_m$ . Then  $S' := \bigcap_m S_m$  satisfies  $\mu(S') \geq \mu(S) - \epsilon$  and the  $f_m$  satisfy the hypotheses of Proposition 4.3 for  $S'$ . Therefore there exists a Lipschitz function differentiable  $\mu$ -almost nowhere on  $S'$ . Thus  $S'$  and hence  $S$  are  $\mu$ -null.

In particular, any  $\tilde{A}(\varphi)$  subset of  $U$  is  $\mu$ -null and so, by Theorem 5.14, there exists the required collection of Alberti representations.

Now suppose that  $(U, \varphi)$  is any chart,  $\psi: X \rightarrow \mathbb{R}^m$  is Lipschitz and  $U' \subset U$  has  $m$   $\psi$ -independent Alberti representations. Then for almost every  $x_0 \in U'$  we have  $\text{Lip}(v \cdot \psi, x_0) > 0$  for each  $v \in \mathbb{S}^{m-1}$ . However, for almost every  $x_0 \in U$ , each  $D\psi_i(x_0)$  exists. Therefore, if  $m > n$ , there exists a  $v \in \mathbb{S}^{m-1}$  such that  $\sum_i v_i D\psi_i(x_0) = 0$ . In particular we have  $\text{Lip}(v \cdot \psi, x_0) = 0$  and so  $U'$  must be  $\mu$ -null.  $\square$

We now apply our theory of Alberti representations to charts, relating the behaviour of Lipschitz differentiability spaces to the existing differentiability theory of Euclidean spaces. Recall the notion of a gradient given in Definition 2.11.

**Corollary 6.7.** *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$ . Then for almost every  $x \in U$  there exist  $\gamma_1^x, \dots, \gamma_n^x \in \Gamma(X)$  such that each  $(\gamma_i^x)^{-1}(x) = 0$  is a density point of  $(\gamma_i^x)^{-1}(U)$  and the  $(\varphi \circ \gamma_i^x)'(0)$  are linearly independent.*

*Moreover, for any such  $\gamma_i^x$ , for any Lipschitz  $f: X \rightarrow \mathbb{R}$  and almost every  $x \in U$ , the gradient of  $f$  at  $x$  with respect to  $\varphi$  and  $\gamma_1^x, \dots, \gamma_n^x$  equals  $Df(x)$ .*

*Proof.* By the previous theorem there exists a countable Borel decomposition  $U = \cup_i U_i$  of  $U$  into sets with  $n$   $\varphi$ -independent Alberti representations. Therefore, by applying Proposition 2.9 to each representation, for almost every  $x$  in any  $U_i$  there exists such  $\gamma_1^x, \dots, \gamma_n^x$ .

Moreover, if  $Df(x)$  exists, then since

$$(f \circ \gamma_i^x)'(0) = Df(x) \cdot (\varphi \circ \gamma_i^x)'(0),$$

$Df(x)$  equals the gradient of  $f$  at  $x$  with respect to  $\varphi$  and  $\gamma_1^x, \dots, \gamma_n^x$ . □

The previous corollary should be compared to [CK09], Theorem 3.3. This theorem asserts that, for any  $n$ -dimensional chart in a doubling Lipschitz differentiability space that satisfies the Poincaré inequality, and for any collection  $f_1, f_2 \dots$  of Lipschitz functions, for almost every  $x \in U$  there exist  $\gamma_1^x, \dots, \gamma_n^x \in \Gamma(X)$  (whose domains are in fact intervals) such that, for each  $i \in \mathbb{N}$ , the gradient of  $f_i$  at  $x$  with respect to  $\varphi$  and  $\gamma_1^x, \dots, \gamma_n^x$  equals  $Df_i(x)$ .

We now use the derivative of a Lipschitz function to show that any  $\tilde{A}$  subset of a Lipschitz differentiability space has measure zero. For this, we first investigate how the direction of a Lipschitz curve varies with respect to different Lipschitz functions in a Lipschitz differentiability space.

**Lemma 6.8.** *Let  $(U, \varphi)$  be an  $n$ -dimensional  $\lambda$ -structured chart in a Lipschitz differentiability space  $(X, d, \mu)$ ,  $S \subset U$  Borel and  $\eta > 0$ . Suppose that  $\psi: X \rightarrow \mathbb{R}^m$  is Lipschitz and*

$$\text{Lip}(v \cdot \psi, x_0) > \eta \text{Lip}(\psi, x_0)$$

*for every  $x_0 \in S$  and  $v \in \mathbb{S}^{m-1}$ . Then for any  $\tilde{w} \in \mathbb{S}^{m-1}$  and  $0 < \epsilon < \theta < 1$ , there exists a countable Borel decomposition  $S = \cup_i S_i \cup N$  where  $\mu(N) = 0$  and, for each  $i \in \mathbb{N}$ , there exists a  $w_i \in \mathbb{S}^{m-1}$  such that, for any  $\delta > 0$ , any  $\gamma \in \Gamma(S_i)$  in the  $\varphi$ -direction of  $C(w_i, \theta - \epsilon)$  with  $\varphi$ -speed  $\delta$  is in the  $\psi$ -direction of  $C(\tilde{w}, 1 - (1 - \theta)\lambda\eta / \text{Lip } \varphi)$  with  $\psi$ -speed  $\delta\eta$ .*

*Proof.* We use the derivative of each  $\psi_i$  to transform the direction of such a Lipschitz curve. Indeed, for  $x_0 \in S$  suppose that the derivative,  $D\psi_i(x_0)$ , of each component of  $\psi$  exists at  $x_0$ . If we write  $D\psi: \mathbb{R}^m \rightarrow \mathbb{R}^n$  for the linear map whose columns are the  $D\psi_i(x_0)$ , then for each  $v \in \mathbb{R}^m$ ,

$$\text{Lip}(D\psi(v) \cdot \varphi, x_0) = \text{Lip}(v \cdot \psi, x_0) \geq \eta \|v\| \text{Lip}(\psi, x_0)$$

and so

$$(6.7) \quad \|D\psi(v)\| \text{Lip}(\varphi, x_0) \geq \eta \|v\| \text{Lip}(\psi, x_0).$$

We also obtain

$$(6.8) \quad \lambda \|D\psi(v)\| \leq \|v\| \text{Lip}(\psi, x_0).$$

In particular,  $D\psi$  is injective and its inverse  $D\psi^{-1}: \text{Im } D\psi \rightarrow \mathbb{R}^n$  satisfies the inequalities corresponding to (6.7) and (6.8).

Now let  $w \in \mathbb{S}^{n-1} \cap \text{Im } D\psi$ ,  $0 < \theta < 1$  and suppose that  $\gamma \in \Gamma(X)$  satisfies  $\gamma(t_0) = x_0$  and that both  $(\varphi \circ \gamma)'(t_0)$  and  $(\psi \circ \gamma)'(t_0)$  exist. Then, for every  $v \in \mathbb{R}^m$ ,

$$v \cdot (\psi \circ \gamma)'(t_0) = D\psi(v) \cdot (\varphi \circ \gamma)'(t_0).$$

Therefore, if  $(\varphi \circ \gamma)'(t_0) \in C(w, \theta)$ , by equation (6.8),

$$\begin{aligned} D\psi(D\psi^{-1}(w)) \cdot (\varphi \circ \gamma)'(t_0) &= w \cdot (\varphi \circ \gamma)'(t_0) \\ &\geq (1 - \theta)\|(\varphi \circ \gamma)'(t_0)\| \\ &= (1 - \theta)\|D\psi^{-1} \cdot (\psi \circ \gamma)'(t_0)\| \\ &\geq (1 - \theta)\lambda\|(\psi \circ \gamma)'(t_0)\|/\text{Lip}(\psi, x_0). \end{aligned}$$

If we let  $\tilde{w} \in \mathbb{R}^m$  be a scalar multiple of  $D\psi^{-1}(w)$  with norm 1, then since

$$\|D\psi^{-1}\| \leq \text{Lip}(\varphi, x_0)/\eta \text{Lip}(\psi, x_0),$$

the previous inequality gives

$$\begin{aligned} \tilde{w} \cdot (\psi \circ \gamma)'(t_0) &\geq \frac{(1 - \theta)\lambda\|(\psi \circ \gamma)'(t_0)\|}{\|D\psi^{-1}(w)\| \text{Lip}(\psi, x_0)} \\ &\geq \frac{(1 - \theta)\lambda\eta\|(\psi \circ \gamma)'(t_0)\|}{\text{Lip}(\varphi, x_0)}. \end{aligned}$$

Therefore  $(\psi \circ \gamma)'(t_0) \in C(\tilde{w}, 1 - (1 - \theta)\lambda\eta/\text{Lip } \varphi)$ .

Similarly, if  $\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \text{Lip}(\varphi, x_0) \text{Lip}(\gamma, t_0)$ , then by equation (6.7) there exists a  $v \in \mathbb{S}^{n-1}$  such that

$$\begin{aligned} \|(\varphi \circ \gamma)'(t_0)\| &= |v \cdot (\varphi \circ \gamma)'(t_0)| \\ &= |D\psi^{-1}(v) \cdot (\psi \circ \gamma)'(t_0)| \\ &\leq \|(\psi \circ \gamma)'(t_0)\| \text{Lip}(\varphi, x_0)/\eta \text{Lip}(\psi, x_0). \end{aligned}$$

Therefore

$$(6.9) \quad \|(\psi \circ \gamma)'(t_0)\| \geq \delta\eta \text{Lip}(\psi, x_0) \text{Lip}(\gamma, t_0).$$

Let  $S = \cup_i S_i \cup N$  be a countable Borel decomposition where  $\mu(N) = 0$  and such that, for each  $i \in \mathbb{N}$ , there exists a  $w_i \in \mathbb{S}^{n-1}$  with

$$\left\| \frac{D\psi(x_0)(\tilde{w})}{\|D\psi(x_0)(\tilde{w})\|} - w_i \right\| < \epsilon.$$

Then if  $\gamma \in \Gamma(S_i)$  is in the  $\varphi$ -direction of  $C(w_i, \theta - \epsilon)$  with  $\varphi$ -speed  $\delta$ , for almost every  $t_0 \in \text{Dom } \gamma$ ,

$$(6.10) \quad (\varphi \circ \gamma)'(t_0) \in C(w_i, \theta - \epsilon) \subset C\left(\frac{D\psi(x_0)(\tilde{w})}{\|D\psi(x_0)(\tilde{w})\|}, \theta\right).$$

Therefore, by (6.9) and (6.10),  $\gamma$  is in the  $\psi$ -direction of  $C(\tilde{w}, 1 - (1 - \theta)\lambda\eta/\text{Lip } \varphi)$  with  $\psi$ -speed  $\delta\eta$ . □

As a consequence, we obtain the following theorem.

**Theorem 6.9.** *Any  $\tilde{A}$  subset of a Lipschitz differentiability space has measure zero.*

*Proof.* For  $\lambda > 0$  let  $(U, \varphi)$  be a  $\lambda$ -structured chart and  $U = \cup_i U_i$  be a countable Borel decomposition such that, for each  $i \in \mathbb{N}$ , there exist independent cones  $C_1, \dots, C_n$  and a  $\delta > 0$  such that  $\mu \llcorner U_i$  has Alberti representations in the  $\varphi$ -direction of each  $C_i$  with  $\varphi$ -speed  $\delta$ . Further, let  $0 < \epsilon < \theta < 1$  such that any cone in  $\mathbb{R}^n$  of width  $\theta - \epsilon$  completely contains one of the  $C_i$ .

We work with a fixed  $U_i$ . Suppose that for some Lipschitz  $\psi: X \rightarrow \mathbb{R}^m$  and  $\tilde{w} \in \mathbb{S}^{m-1}$ ,  $S \subset U_i$  satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Gamma(X)$  in the  $\psi$ -direction of  $C(\tilde{w}, 1 - (1 - \theta)\lambda\eta / \text{Lip } \varphi)$  with  $\psi$ -speed  $\delta\eta$ . Then by the previous lemma there exists a countable Borel decomposition  $S = \cup_i S_i \cup N$  and for each  $i \in \mathbb{N}$  cones  $C_i$  of width  $\theta - \epsilon$  such that  $\mu(N) = 0$  and  $\mathcal{H}^1(\gamma \cap S_i) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C_i$  with speed  $\delta$ . However, one of the Alberti representations of  $\mu \llcorner U_i$  is in the  $\varphi$ -direction of this cone with  $\varphi$ -speed  $\delta$  and so  $\mu(S_i) = 0$  and hence  $\mu(S) = 0$ .

Finally, let  $S' \in \tilde{A}$  and  $\eta > 0$  such that  $S' \in \tilde{A}(\delta', \theta', \eta)$  for any  $0 < \delta', \theta' < 1$ . In particular  $S \in \tilde{A}(\eta\delta, 1 - (1 - \theta)\lambda\eta / \text{Lip } \varphi, \eta)$ . Therefore, there exist a countable decomposition  $S' = \cup_j S_j$  and for each  $j \in \mathbb{N}$  a Lipschitz  $\psi_j: X \rightarrow \mathbb{R}^{n_j}$  such that each  $S_j$  has the form of  $S$  above. Therefore, each  $S_j$  has measure zero. In particular, any  $\tilde{A}$  subset of  $U_i$  and hence of  $(X, d, \mu)$  must be  $\mu$ -null.  $\square$

Finally, as a consequence of the existence of the above Alberti representations, we give a partial, positive answer to [Che99], Conjecture 4.63 regarding the image of a chart under the chart map.

**Corollary 6.10.** *For  $n = 1$  or  $2$  let  $(U, \varphi)$  be an  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$ . Then the pushforward  $\varphi_*(\mu \llcorner U)$  is absolutely continuous with respect to  $n$ -dimensional Lebesgue measure.*

*In particular, if  $\mu(U) > 0$ , then  $\mathcal{L}^n(\varphi(U)) > 0$ .*

*Proof.* Suppose that a measure  $\nu$  has an Alberti representation in the  $\varphi$ -direction of a cone  $C \subset \mathbb{R}^n$ . Then an easy application of Corollary 5.8 shows that  $\varphi_*\nu$  also has an Alberti representation in the direction of  $C$ . By Theorem 6.6 there exists a countable Borel decomposition  $U = \cup_m U_m$  such that each  $\mu \llcorner U_m$  has  $n$  independent Alberti representations, so that each  $\varphi_*(\mu \llcorner U_m)$  has  $n$  independent Alberti representations.

Any measure on  $\mathbb{R}$  with an Alberti representation is absolutely continuous with respect to Lebesgue measure. Further, results from [ACP] show that any measure on  $\mathbb{R}^2$  with two independent Alberti representations must also be absolutely continuous with respect to Lebesgue measure. Therefore, in either case,  $\varphi_*(\mu \llcorner U)$  is absolutely continuous with respect to Lebesgue measure.  $\square$

This improves the known cases proved by Keith ( $n = 1$ ; see [Kei04], page 282) and Gong ( $n = 2$ ; see [Gon11], Theorem 1.4) who prove Corollary 6.10 for Lipschitz differentiability spaces that satisfy the Poincaré inequality and possess a doubling measure (see Definition 10.1).

*Remark 6.11.* Using a recent announcement of Csörnyei and Jones (see [www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf](http://www.math.sunysb.edu/Videos/dfest/PDFs/38-Jones.pdf), pages 15–23) one may show that, for any  $n \in \mathbb{N}$ , any measure on  $\mathbb{R}^n$  with  $n$  independent Alberti representations is absolutely continuous with respect to Lebesgue measure. Therefore, the previous argument also proves the statements of Corollary 6.10 for any  $n \in \mathbb{N}$ .



One may also use this announcement to generalise the techniques of Keith and Gong to prove Corollary 6.10 for all  $n \in \mathbb{N}$  for Lipschitz differentiability spaces that satisfy the Poincaré inequality and possess a doubling measure.

## 7. A CHARACTERISATION VIA ALBERTI REPRESENTATIONS

We now show how the Alberti representations found in the previous section characterise the differentiability properties of a metric measure space.

We first introduce the notion of a *universal* set of Alberti representations for when a set of representations describes all of the Lipschitz functions on a metric measure space.

**Definition 7.1.** Let  $(U, \varphi)$  be a chart in a metric measure space  $(X, d, \mu)$  and  $\mathcal{A}_1, \dots, \mathcal{A}_n$  be a collection of  $\varphi$ -independent Alberti representations of  $\mu \llcorner U$ , each with strictly positive  $\varphi$ -speed. For  $\rho > 0$  we say that this collection of Alberti representations is  $\rho$ -*universal* (or just *universal* if such a  $\rho$  exists) if, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ , there exists a finite Borel decomposition  $X = X_1 \cup \dots \cup X_n$  such that the Alberti representation of  $\mu \llcorner X_i$  induced by  $\mathcal{A}_i$  has  $f$ -speed  $\rho$ .

*Remark 7.2.* A universal collection of Alberti representations is a much stronger concept than a *maximal* collection of representations (i.e. a collection for which there are no other independent representations). For example, any purely unrectifiable metric measure space has a maximal (empty) collection of Alberti representations. However, by Corollary 4.9, this collection is not universal as there exists a Lipschitz function with positive pointwise Lipschitz constant on a set of positive measure. (For a slightly less trivial example, one may consider the Cartesian product of  $\mathbb{R}$  and a purely unrectifiable space.)

We will see that a universal collection of Alberti representations is precisely the required concept so that the gradient (see Definition 2.11)  $\nabla f$  of a Lipschitz function  $f$  forms a derivative. We prove this in a very natural way: observe that the derivative of  $f - \nabla f \cdot \varphi$  along the Lipschitz curves defining the gradient is zero. Further, if the Alberti representations are sufficiently refined, then the derivative along almost every curve in the representation is also sufficiently small. Therefore, after a suitable limit of such gradients obtained from a sequence of Alberti representations, each refining the previous, we obtain a derivative.

We begin with a simple result that bounds the magnitude of the gradient using properties of Alberti representations. For this we must quantitatively describe an independent collection of Alberti representations.

**Definition 7.3.** For  $\xi > 0$  we say  $v_1, \dots, v_m \in \mathbb{R}^n$  are  $\xi$ -*separated* if, for any  $\lambda \in \mathbb{R}^m \setminus \{0\}$ ,

$$\left\| \sum_{i=1}^m \lambda_i v_i \right\| > \xi \max_{1 \leq i \leq m} \|\lambda_i v_i\|$$

and that closed cones  $C_1, \dots, C_m$  are  $\xi$ -separated if any choice of  $v_i \in C_i \setminus \{0\}$  is  $\xi$ -separated. Further, we say that Alberti representations  $\mathcal{A}_1, \dots, \mathcal{A}_m$  are  $\xi$ -*separated* if there exist  $\xi$  separated cones  $C_1, \dots, C_m$  such that each  $\mathcal{A}_i$  is in the  $\varphi$ -direction of  $C_i$ .

Observe that cones  $C_1, \dots, C_m$  are  $\xi$ -separated if and only if, for every  $1 \leq i \leq m$ , the distance of  $C_i \cap \mathbb{S}^{n-1}$  from those  $v$  in the symmetric convex hull of the  $C_j$  (for

$j \neq i$ ) with  $\|v\| \leq 1$  is strictly greater than  $\xi$ . In particular, any independent cones are  $\xi$ -separated for some  $\xi > 0$ .

Suppose that a metric measure space  $(X, d, \mu)$  has Alberti representations in the  $\varphi$ -direction of  $\xi$ -separated cones  $C(w_1, \theta), \dots, C(w_m, \theta)$ . Then there exists an  $\epsilon > 0$  such that  $C(w_1, \theta + \epsilon), \dots, C(w_m, \theta + \epsilon)$  are also  $\xi$ -separated and a finite cover of each  $C(w_i, \theta)$  by cones  $C_1^i, \dots, C_{N_i}^i$  of width  $\epsilon$  that are contained within  $C(w_i, \theta + \epsilon)$ . By applying Corollary 5.9 using the  $C_j^i$  we obtain *arbitrary* refinements of these Alberti representations that are also  $\xi$ -separated. Moreover, if for some Lipschitz  $\psi: X \rightarrow \mathbb{R}^n$  and  $\delta_1, \dots, \delta_n > 0$ , the original Alberti representations have  $\psi_i$ -speed strictly greater than  $\delta_i$  for each  $1 \leq i \leq N$ , then so do the refinements.

For this section we fix the following notation.

**Notation 7.4.** We fix a metric measure space  $(X, d, \mu)$  and for  $\rho, \delta, \lambda, \xi > 0$  let  $(U, \varphi)$  be an  $n$ -dimensional  $\lambda$ -structured chart in  $(X, d, \mu)$  such that  $\mu \llcorner U$  has a  $\rho$ -universal collection of  $n$ ,  $\xi$ -separated Alberti representations with  $\varphi$ -speed strictly greater than  $\delta$ .

Using the notion of separated Alberti representations, we may bound the magnitude of a gradient.

**Lemma 7.5.** *Let  $f: X \rightarrow \mathbb{R}$  be Lipschitz,  $x_0 \in U$  and  $\gamma_1, \dots, \gamma_n \in \Gamma(X)$  such that the  $(\varphi \circ \gamma_i)'(0)$  are  $\xi$ -separated and, for each  $1 \leq i \leq n$ ,  $\gamma_i^{-1}(x_0) = 0$  is a density point of  $\text{Dom } \gamma_i$  and*

$$(\varphi \circ \gamma_i)'(0) \geq \delta \text{Lip}(\varphi, x_0) \text{Lip}(\gamma_i, 0).$$

*Then the gradient  $\nabla f(x_0)$  of  $f$  at  $x_0$  with respect to  $\varphi$  and  $\gamma_1, \dots, \gamma_n$  satisfies*

$$\|\nabla f(x_0)\| \leq n \text{Lip}(f, x_0) / \xi \delta \lambda.$$

*Proof.* For any  $\xi$ -separated  $v_1, \dots, v_n$  and  $G \in \mathbb{R}^n$ , there exists a  $1 \leq i \leq n$  such that

$$\xi \|G\| / n \leq |G \cdot v_i|.$$

Therefore,

$$\begin{aligned} \text{Lip}(f, x_0) \text{Lip}(\gamma_i, 0) &\geq |(f \circ \gamma_i)'(0)| \\ &= |\nabla f(x_0) \cdot (\varphi \circ \gamma_i)'(0)| \\ &\geq \xi \|\nabla f(x_0)\| \|(\varphi \circ \gamma_i)'(0)\| / n \\ &\geq \xi \delta \|\nabla f(x_0)\| \text{Lip}(\varphi, x_0) \text{Lip}(\gamma_i, 0) / n. \end{aligned}$$

That is,

$$\|\nabla f(x_0)\| \leq \text{Lip } f n / \xi \delta \text{Lip}(\varphi, x_0)$$

as required. □

Next we refine a universal collection of Alberti representations whilst maintaining their speed with respect to a finite collection of Lipschitz functions.

**Lemma 7.6.** *Let  $\mathcal{F}$  be a finite collection of real valued Lipschitz functions defined on  $X$ ,  $f: X \rightarrow \mathbb{R}$  Lipschitz and  $\epsilon > 0$ . Then there exist a finite Borel decomposition  $X = X_1 \cup \dots \cup X_N$  and for each  $1 \leq i \leq N$ , Alberti representations  $\mathcal{A}'_1, \dots, \mathcal{A}'_n$  of  $\mu \llcorner X_i$  such that*

- (1) *The  $\mathcal{A}'_k$  are  $\xi$ -separated and have  $\varphi$ -speed strictly greater than  $\delta$ .*
- (2) *For every  $g \in \mathcal{F}$  there exists an  $\mathcal{A}'_k$  with  $g$ -speed  $\rho$ .*

- (3) If we write  $\Phi: X \rightarrow \mathbb{R}^{n+1}$  to be the function obtained by appending  $f$  to  $\varphi$ , then each  $\mathcal{A}'_k$  is in the  $\Phi$ -direction of a cone of width  $\epsilon$ .

*Proof.* The collection  $\mathcal{A}_1, \dots, \mathcal{A}_n$  is  $\rho$ -universal and so there exists a finite Borel decomposition  $X = Z_1 \cup \dots \cup Z_L$ , such that, for any  $g \in \mathcal{F}$  and  $1 \leq i \leq L$ , there exists a  $1 \leq k \leq n$  such that  $\mathcal{A}_k \llcorner Z_i$  has  $g$ -speed  $\rho$ .

We fix  $1 \leq i \leq L$  and for each  $1 \leq k \leq n$  let  $\mathcal{F}(k)$  be the set of  $g \in \mathcal{F}$  such that  $\mathcal{A}_k \llcorner Z_i$  has  $g$ -speed  $\rho$ . Then by Corollary 5.9 we may refine each  $\mathcal{A}_k \llcorner Z_i$  to obtain a finite decomposition  $Z_i = Y_k^1 \cup \dots \cup Y_k^{M_k}$  and Alberti representations  $\mathcal{A}'_k^1, \dots, \mathcal{A}'_k^{M_k}$  of  $\mu \llcorner Y_k^1, \dots, \mu \llcorner Y_k^{M_k}$  respectively, in the  $\Phi$ -direction of cones of width  $\epsilon$  and with  $\varphi$ -speed strictly greater than  $\delta$ , and such that each  $\mathcal{A}'_k^i$  has  $g$ -speed  $\rho$ , for each  $g \in \mathcal{F}(k)$ . Moreover, we may make these refinements such that the representations are  $\xi$ -separated, with respect to  $\varphi$ . Therefore, for these representations, both (1) and (3) are satisfied.

By taking the intersection of  $Y_k^m$  for  $1 \leq k \leq n$ , we obtain a finite Borel decomposition  $Z_i = X_1^i \cup \dots \cup X_{N_i}^i$  such that each  $X_j^i$  is a subset of some  $Y_k^m$ , for each  $1 \leq k \leq n$ . Therefore, for each  $1 \leq m \leq N_i$  and any  $g \in \mathcal{F}$ , there exists a  $1 \leq k \leq n$  such that  $\mathcal{A}_k \llcorner X_m^i$  has  $g$ -speed  $\rho$ . In particular, (2) is satisfied and so  $X = \cup_{i,m} X_m^i$  is a decomposition of the required form.  $\square$

Finally, we use this refinement to show that the gradient obtained from such Alberti representations is an  $\epsilon$ -derivative at almost every point.

**Lemma 7.7.** *Let  $f: X \rightarrow \mathbb{R}$  be Lipschitz. Then there exists an  $M \in \mathbb{R}$  such that, for every  $\epsilon > 0$  and almost every  $x_0 \in U$ , there exists a  $\nabla f(x_0) \in \mathbb{R}^n$  with*

$$\text{Lip}(f - \nabla f(x_0) \cdot \varphi, x_0) \leq M\sqrt{\epsilon}.$$

*Proof.* Fix  $\epsilon > 0$  and let  $\mathcal{Q}$  be a finite  $\epsilon$ -net of  $\overline{B}(0, \text{Lip } f n / \xi \delta \lambda) \subset \mathbb{R}^n$  and

$$\mathcal{F} = \{f - D \cdot \varphi : D \in \mathcal{Q}\}.$$

Then by the previous lemma there exists a countable decomposition  $U = \cup_i U_i$  such that each  $\mu \llcorner U_i$  has Alberti representations  $\mathcal{A}'_1, \dots, \mathcal{A}'_n$  satisfying the properties of the previous lemma for  $\epsilon, f$  and  $\mathcal{F}$ . We fix such a  $U_i$  and define  $\Phi: X \rightarrow \mathbb{R}^{n+1}$  to be the Lipschitz function obtained by appending  $f$  to  $\varphi$  and let  $C_1, \dots, C_n \subset \mathbb{R}^{n+1}$  have width  $\epsilon$  such that each  $\mathcal{A}_k$  is in the  $\Phi$ -direction of  $C_k$ .

By combining Proposition 2.9 and Lemma 7.5, for almost every  $x_0 \in U_i$  there exist  $\gamma_1, \dots, \gamma_n \in \Gamma(X)$  in the  $\Phi$ -direction of  $C_1, \dots, C_n$  respectively such that the gradient  $\nabla f(x_0)$  of  $f$  at  $x_0$  with respect to  $\varphi$  and  $\gamma_1, \dots, \gamma_n$  satisfies

$$\|\nabla f(x_0)\| \leq n \text{Lip}(f, x_0) / \xi \delta \lambda.$$

In particular, there exists a  $D \in \mathcal{Q}$  with  $\|D - \nabla f(x_0)\| < \epsilon$ . We write  $\mathbf{D}$  and  $\nabla \mathbf{f}(x_0) \in \mathbb{R}^{n+1}$  for the vectors obtained by appending  $-1$  to  $D$  and  $\nabla f(x_0)$  respectively.

Suppose that  $v, v' \in \mathbb{R}^{n+1}$  belong to a cone of width  $\epsilon$ . Then

$$\left\| \frac{v}{\|v\|} - \frac{v'}{\|v'\|} \right\| \leq \sqrt{\epsilon(2 - \epsilon)}$$

and so, for any  $a \in \mathbb{R}^{n+1}$ ,

$$|a \cdot v| \leq (|a \cdot v'| / \|v'\| + \sqrt{\epsilon(2 - \epsilon)}) \|v\|.$$

For almost every  $x_0 \in U$ ,  $\nabla f(x_0)$  is the gradient of  $f$  with respect to  $\varphi$  and  $\gamma_1, \dots, \gamma_n$ , and so  $\nabla f(x_0) \cdot (\Phi \circ \gamma_i)'(0) = 0$  for each  $1 \leq i \leq n$ . Therefore, if  $\tilde{\gamma} \in \Gamma(X)$  is in the  $\Phi$ -direction of some  $C_i$  with  $\tilde{\gamma}(t_0) = x_0$  such that  $(\Phi \circ \tilde{\gamma})'(t_0)$  exists, then

$$|\nabla f(x_0) \cdot (\Phi \circ \tilde{\gamma})'(t_0)| \leq \sqrt{\epsilon(2 - \epsilon)} \|\nabla f(x_0)\| \|(\Phi \circ \tilde{\gamma})'(t_0)\|.$$

In particular,

$$|\mathbf{D} \cdot (\Phi \circ \tilde{\gamma})'(t_0)| \leq \sqrt{\epsilon(2 - \epsilon)} \|\mathbf{D}\| \|(\Phi \circ \tilde{\gamma})'(t_0)\| + 2\epsilon (\Phi \circ \tilde{\gamma})'(t_0).$$

However, for each  $1 \leq i \leq n$ , almost every  $\tilde{\gamma} \in \mathcal{A}_i$  is of this form. Further, since  $D \cdot \varphi - f \in \mathcal{F}$ , one of the  $\mathcal{A}'_k$  has  $D \cdot \varphi - f$  speed  $\rho$ . Therefore,

$$\rho \operatorname{Lip}(D \cdot \varphi - f, x_0) \leq \sqrt{\epsilon(2 - \epsilon)} \|\mathbf{D}\| \operatorname{Lip} \Phi + 2\epsilon \operatorname{Lip} \Phi$$

for almost every  $x_0 \in U$ . Finally, since

$$\|\mathbf{D}\| \leq 1 + \|\nabla f(x_0)\| \leq 1 + n \operatorname{Lip} f / \xi \delta \lambda,$$

we have

$$\operatorname{Lip}(f - \nabla f(x_0) \cdot \varphi, x_0) \leq (\operatorname{Lip} \varphi + \operatorname{Lip} f)(2\epsilon + \sqrt{\epsilon(2 - \epsilon)}(1 + n \operatorname{Lip} f / \xi \delta \lambda))$$

almost everywhere in  $U$ , as required. □

Using this construction we may give our first characterisation of Lipschitz differentiability spaces. We now work without the fixed quantities given in Notation 7.4.

**Theorem 7.8.** *A metric measure space  $(X, d, \mu)$  is a Lipschitz differentiability space if and only if there exists a countable Borel decomposition  $X = \cup_i U_i$  such that each  $\mu \llcorner U_i$  has a finite universal collection of Alberti representations.*

*In this case, for each  $i \in \mathbb{N}$  let  $\varphi_i: X \rightarrow \mathbb{R}^{n_i}$  be Lipschitz such that the Alberti representations of  $\mu \llcorner U_i$  are  $\varphi_i$ -independent. Then each  $(U_i, \varphi_i)$  is a chart and the derivative of a Lipschitz function  $f: X \rightarrow \mathbb{R}$  is given by any gradient of  $f$  with respect to  $\varphi_i$  at almost every point.*

*Proof.* We first show that the condition is necessary for a metric measure space to be a Lipschitz differentiability space. For any  $i \in \mathbb{N}$  let  $\varphi_i: X \rightarrow \mathbb{R}^{n_i}$  be Lipschitz such that the Alberti representations of  $\mu \llcorner U_i$  described in the hypotheses are  $\varphi_i$ -independent. Then by applying Proposition 2.9, for almost every  $x_0 \in U_i$  there exist  $\gamma_1, \dots, \gamma_{n_i} \in \Gamma(X)$  such that the  $(\varphi \circ \gamma_i)'(0)$  form a basis of  $\mathbb{R}^{n_i}$  and, for each  $1 \leq i \leq n_i$ ,  $\gamma_i^{-1}(x_0) = 0$  is a density point of  $\operatorname{Dom} \gamma_i$ . Therefore, for almost every  $x_0 \in U_i$ , there exists a  $\lambda(x_0) > 0$  such that, for any  $v \in \mathbb{S}^{n_i-1}$ ,

$$\limsup_{X \ni x \rightarrow x_0} \frac{|(\varphi(x) - \varphi(x_0)) \cdot v|}{d(x, x_0)} \geq \lambda(x_0).$$

In particular, by Lemma 3.3, any possible derivative of a function at  $x_0$  is unique.

Now let  $\delta, \xi > 0$  such that the Alberti representations of  $\mu \llcorner U_i$  are  $\xi$ -separated with respect to  $\varphi_i$  and have  $\varphi_i$ -speed strictly greater than  $\delta$ . By taking a further decomposition (and allowing for a possible  $\mu$ -null subset of  $X$ ), we may suppose that  $(U_i, \varphi_i)$  is  $\lambda$ -structured, for some  $\lambda > 0$ . Then we are in the situation described by Notation 7.4 and so, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ , by Lemma 7.7 there exists

an  $M > 0$  such that, for any  $\epsilon > 0$  and almost every  $x_0 \in U$ , there exists a  $\nabla f(x_0) \in \mathbb{R}^{n_i}$  with  $\|\nabla f(x_0)\| \leq n_i \text{Lip } f / \xi \delta \lambda$  and

$$\text{Lip}(f - \nabla f(x_0) \cdot \varphi_i, x_0) \leq M\sqrt{\epsilon}.$$

Applying this with some sequence  $\epsilon_m \rightarrow 0$ , for almost every  $x_0 \in U_i$  we obtain a bounded sequence of such  $\nabla f(x_0)$  and so, after passing to a convergent subsequence, we obtain a  $Df(x_0) \in \mathbb{R}^{n_i}$  that is a derivative of  $f$  at  $x_0$  with respect to  $(U_i, \varphi_i)$ . Further, by Corollary 6.7, this derivative is given by any gradient of  $f$  with respect to  $\varphi_i$ .

Conversely, let  $(U, \varphi)$  be an  $n$ -dimensional,  $\lambda$ -structured chart in a Lipschitz differentiability space. Then by Theorem 6.6 and Corollary 5.9, there exist a countable Borel decomposition  $U = \cup_i U_i$  and for every  $i \in \mathbb{N}$  a  $\delta > 0$  such that, for each  $i \in \mathbb{N}$ ,  $\mu \llcorner U_i$  has  $n$   $\varphi$ -independent Alberti representations with  $\varphi$ -speed  $\delta$ .

Fix such a  $U_i$  and suppose such representations  $\mathcal{A}_1, \dots, \mathcal{A}_n$  of  $\mu \llcorner U_i$  are in the  $\varphi$ -direction of  $\xi$ -separated cones  $C_1, \dots, C_n$ . Then for any Lipschitz  $f: X \rightarrow \mathbb{R}$  and almost every  $x_0 \in U_i$ ,  $Df(x_0)$  exists. Moreover, there exists a  $1 \leq j \leq n$  such that, for any  $v \in C_j$ ,

$$\xi \|Df(x_0)\| \|v\| / n \leq |Df(x_0) \cdot v|.$$

For  $1 \leq j \leq n$  let  $V_j$  be the set of  $x_0$  that satisfies this for  $C_j$ . Therefore, if  $x_0 \in V_j$  and  $\gamma \in \Gamma(X)$  with  $\gamma(t_0) = x_0$ ,  $\|(\varphi \circ \gamma)'(t_0)\| \geq \delta \text{Lip}(\varphi, x_0) \text{Lip}(\gamma, t_0)$  and  $(\varphi \circ \gamma)'(t_0) \in C_j$ , then

$$\begin{aligned} (f \circ \gamma)'(t_0) &= |Df(x_0) \cdot (\varphi \circ \gamma)'(t_0)| \\ &\geq \|Df(x_0)\| \|(\varphi \circ \gamma)'(t_0)\| \\ &\geq \delta \text{Lip}(f, x_0) \text{Lip}(\varphi, x_0) \text{Lip}(\gamma, t_0) \\ &\geq \delta \lambda \text{Lip}(f, x_0) \text{Lip}(\gamma, t_0). \end{aligned}$$

That is,  $\mathcal{A}_j \llcorner V_j$  has  $f$ -speed  $\delta \lambda$ . Therefore, the  $\mathcal{A}_j$  are universal. □

### 8. CHARACTERISATIONS VIA NULL SETS

We now present other characterisations that prescribe a class of sets that determine if a metric measure space is a Lipschitz differentiability space. Such sets will be the singular sets found earlier when constructing Alberti representations of a measure. Therefore, in a metric measure space where such sets have measure zero, there exist many Alberti representations. Further, if these representations are sufficiently different, for almost every  $x_0$ , particular points of the Lipschitz curves obtained from the representations form a separated subset of balls centred at  $x_0$ . By combining this with a doubling condition on the metric space, we obtain a bound on the total number of such different Alberti representations. In turn, this leads to the existence of a derivative of a Lipschitz function at almost every point.

We begin by giving a method that constructs a chart structure in a metric measure space.

**Lemma 8.1.** *Let  $(X, d, \mu)$  be a metric measure space,  $Y \subset X$  Borel and  $N \in \mathbb{N}$ . Then the following are equivalent:*

- *For any Lipschitz  $\psi: X \rightarrow \mathbb{R}^{N+1}$ , for almost every  $x_0 \in Y$  there exists an  $a(x_0) \in \mathbb{S}^N$  such that*

$$\text{Lip}(a(x_0) \cdot \psi, x_0) = 0.$$

- There exist a countable Borel decomposition  $Y = \cup_i U_i$  and Lipschitz functions  $\varphi_i: X \rightarrow \mathbb{R}^{n_i}$  such that each  $(U_i, \varphi_i)$  is a chart of dimension at most  $N$ , with respect to which any Lipschitz  $f: X \rightarrow \mathbb{R}$  is differentiable at almost every point of  $U_i$ .

*Proof.* First suppose that the second statement is true,  $(U_i, \varphi_i)$  a chart of the decomposition and  $\psi: X \rightarrow \mathbb{R}^{N+1}$  Lipschitz. Then for almost every  $x_0 \in U_i$  and each  $1 \leq j \leq N + 1$ , the derivative of  $\psi_j$  exists at  $x_0$  and belongs to  $\mathbb{R}^{n_i}$ , for  $n_i < N + 1$ . Therefore, if we write  $D\psi$  for the matrix whose columns are the  $D\psi_i(x_0)$ , there exists an  $a(x_0) \in \mathbb{S}^N$  such that  $D\psi \cdot a(x_0) = 0$ . Then by the definition of a derivative,  $\text{Lip}(a(x_0) \cdot \psi, x_0) = 0$ .

Now suppose that the first statement holds and let  $U \subset Y$  be a Borel set of positive measure. First observe that either any Lipschitz function  $\psi: X \rightarrow \mathbb{R}$  satisfies  $\text{Lip}(\psi, x_0) = 0$  for almost every  $x_0 \in U$  in which case  $U$  is a chart of dimension zero (as described in Corollary 4.9) or there exist a Borel set  $U' \subset U$  of positive measure and a Lipschitz  $\varphi: X \rightarrow \mathbb{R}$  such that  $\text{Lip}(\varphi, x_0) > 0$  for every  $x_0 \in U'$ .

By our hypotheses on  $Y$ , there exists a maximal  $n \in \mathbb{N}$  such that there exist a Lipschitz  $\varphi: X \rightarrow \mathbb{R}^n$  and a  $U' \subset U$  of positive measure with  $\text{Lip}(v \cdot \varphi, x_0) > 0$  for every  $x_0 \in U'$  and every  $v \in \mathbb{S}^{n-1}$ . Further, we have  $n \leq N$ . Then for such an  $n, \varphi$  and  $U'$ , for any Lipschitz  $f: X \rightarrow \mathbb{R}$  and almost every  $x_0 \in U'$  there exists a  $Df(x_0) \in \mathbb{R}^n$  such that  $\text{Lip}(f - Df(x_0) \cdot \varphi, x_0) = 0$ . Moreover, by Lemma 3.3, the condition  $\text{Lip}(v \cdot \varphi, x_0) > 0$  for every  $v \in \mathbb{S}^{n-1}$  is equivalent to the uniqueness of such a  $Df(x_0)$ .

Therefore, within any  $U$  of positive measure, there exists a chart of positive measure with dimension less than  $N$ , with respect to which any real valued Lipschitz function is differentiable almost everywhere. Since  $\mu$  is finite we obtain a countable decomposition of  $U$  into charts with respect to which Lipschitz functions are differentiable almost everywhere. □

We now introduce additional properties of metric measure spaces that will allow us to satisfy the hypotheses of Lemma 8.1.

**Definition 8.2.** We say that a metric measure space  $(X, d, \mu)$  is *pointwise doubling* if

$$\limsup_{r \rightarrow 0} \frac{\mu(B(x_0, r))}{\mu(B(x_0, r/2))} < \infty$$

for almost every  $x_0 \in X$ .

Further, for  $0 < \delta < 1$  and  $\eta \geq 1$ , we define  $M(\delta, \eta) = \eta^{2 - \log_2 \delta/5}$  and  $D_\eta$  to be the set of Borel subsets  $Y$  of  $X$  such that, for each  $x_0 \in Y$ ,  $0 < \delta < 1$  and  $r > 0$ , there exist  $x_1, \dots, x_M \in Y$  with

$$B(x_0, r) \cap Y \subset \bigcup_{i=1}^M B(x_i, \delta r),$$

for some  $M \in \mathbb{N}$  less than  $M(\delta, \eta)$ .

**Lemma 8.3.** For any pointwise doubling metric measure space  $(X, d, \mu)$ , there exists a Borel decomposition

$$X = N \cup \bigcup_{m=1}^{\infty} Y_m$$

where  $\mu(N) = 0$  and for each  $m \in \mathbb{N}$  there exists an  $\eta \geq 1$  such that  $Y_m \in D_\eta$ .

*Proof.* For  $\eta \geq 1$  and  $R > 0$  let

$$X_{R,\eta} := \{x_0 \in X : \eta\mu(B(x_0, r/2)) > \mu(B(x_0, r)) > 0, \forall 0 < r < R\}.$$

For any  $x_0 \in X$  and  $r > 0$ ,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, r/2))} = \lim_{r_m \nearrow r} \frac{\mu(B(x_0, r_m))}{\mu(B(x_0, r_m/2))}$$

and so, in the definition of  $X_{R,\eta}$ , it is equivalent to satisfying the condition for rational  $0 < r < R$ . Further, for a fixed  $r > 0$ ,  $x_0 \mapsto \mu(B(x_0, r))$  is lower semicontinuous. Therefore, each  $X_{R,\eta}$  is a Borel set.

Now fix  $\eta \geq 1$ ,  $R > 0$  and  $x_0 \in X_{R,\eta}$ . Then for any  $m \in \mathbb{N}$  and  $0 < r < R$ ,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, 2^{-m}r))} \leq \eta^m$$

and so, for any  $0 < \delta < 1$ ,

$$\frac{\mu(B(x_0, r))}{\mu(B(x_0, \delta r))} \leq \eta^M,$$

for  $M$  the least integer greater than  $-\log_2 \delta$ .

Now let  $0 < r < R/2$ . Then

$$B(x_0, r) \cap X_{R,\eta} \subset \bigcup \{B(x, \delta r/5) : x \in B(x_0, r) \cap X_{R,\eta}\}$$

and so there exists  $F \subset B(x_0, r) \cap X_{R,\eta}$  such that

$$B(x_0, r) \cap X_{R,\eta} \subset \bigcup \{B(x, \delta r) : x \in F\}$$

and such that the  $B(x_0, \delta r/5)$  are disjoint. Firstly, since  $\mu(B(x_0, 2r))$  is finite and for  $x \in F$  the  $B(x, \delta r/5)$  are disjoint subsets of  $B(x_0, 2r)$  with positive measure,  $F$  is countable. Further,

$$\eta^M \mu(B(x_0, r)) \geq \eta^M \sum_{x \in F} \mu(B(x, \delta r/5)) \geq \sum_{x \in F} \mu(B(x, 2r)) \geq \sum_{x \in F} \mu(B(x_0, r)),$$

for  $M$  the least integer greater than  $2 - \log_2 \delta/5$ . In particular, the cardinality of  $F$  is less than  $M(\delta, \eta)$ . Therefore, for any  $y \in X$ ,  $X_{R,\eta} \cap B(y, R/4)$  belongs to  $D_\eta$ .

Finally, for  $R_i \rightarrow 0$  and  $\eta_i \rightarrow \infty$ , the  $X_{R_i, \eta_i}$  cover almost all of  $X$  and so there exists a decomposition of  $X$  of the required form. □

Metric measure spaces in which all porous sets (see Definition 4.4) have measure zero are pointwise doubling (for example, see [MMPZ03], Theorem 3.6). Therefore, we obtain the following consequence of Corollary 4.5.

**Corollary 8.4** ([BS13], Corollary 2.6). *Any Lipschitz differentiability space is pointwise doubling.*

In addition, we also require concepts related to an  $\tilde{A}$  set (see Definition 5.11).

**Definition 8.5.** Let  $(X, d)$  be a metric space and  $\delta > 0$ . We define  $\tilde{B}_\delta$  to be the set of  $S \subset X$  for which there exist a countable Borel decomposition  $S = \cup_i S_i$  and Lipschitz functions  $f_i : X \rightarrow \mathbb{R}$  such that, for every  $i \in \mathbb{N}$ ,

$$S_i \subset \{x_0 \in X : \text{Lip}(f_i, x_0) > 0\}$$

and  $\mathcal{H}^1(\gamma \cap S_i) = 0$  for every  $\gamma \in \Gamma(X)$  with  $f_i$ -speed  $\delta$ . Further, we define  $\tilde{B}$  to be the set of  $S \subset X$  that belongs to  $\tilde{B}_\delta$  for every  $\delta > 0$ .

*Remark 8.6.* Note that for any  $0 < \delta \leq \delta'$  we have  $\tilde{B}_\delta \subset \tilde{B}_{\delta'}$ .

Observe that  $\tilde{B}_\delta \subset \tilde{A}(\delta, \theta, \lambda)$  for any  $0 < \delta, \theta, \lambda < 1$  and so  $\tilde{B} \subset \tilde{A}$ . In particular, any  $\tilde{B}$  subset of a Lipschitz differentiability space has measure zero. Also, by Corollary 5.8, metric measure spaces in which  $\tilde{B}_\delta$  sets are  $\mu$ -null have many Alberti representations with speed  $\delta$ . We now show how the curves obtained from such Alberti representations interact with the doubling properties of a metric measure space. This is done by creating a separated set from points belonging to curves obtained from such representations.

**Lemma 8.7.** *Let  $(X, d, \mu)$  be a metric measure space,  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz,  $Y \subset X$  Borel and  $\delta > 0$ . Suppose that  $\mathcal{A}_1, \dots, \mathcal{A}_n$  are Alberti representations of  $\mu \llcorner Y$  and  $w_1, \dots, w_n \in \mathbb{S}^{n-1}$  such that, for every  $1 \leq i \neq j \leq n$ , either*

$$\frac{|w_i \cdot (\varphi \circ \gamma_i)'(t_i)|}{\text{Lip}(\gamma_i, t_i)} > \frac{|w_i \cdot (\varphi \circ \gamma_j)'(t_j)|}{\text{Lip}(\gamma_j, t_j)} + \delta \text{Lip}(w_i \cdot \varphi, \gamma_i(t_i)) > 0,$$

*or the corresponding inequalities with  $i$  and  $j$  are exchanged. Then, for almost every  $x_0 \in X$  and any  $r > 0$ , there exist  $x_1, \dots, x_n \in B(x_0, r)$  such that*

$$d(x_i, x_j) \geq \delta r - \phi(r)$$

*for each  $1 \leq i \neq j \leq n$ , where  $\phi(r)/r \rightarrow 0$  as  $r \rightarrow 0$ .*

*In particular, if  $Y \subset W \in D_\eta$ , for some  $\eta \geq 1$ , then  $n \leq M(\delta/2, \eta)$ .*

*Proof.* Let  $\mathcal{Q}$  be a countable dense subset of  $\mathbb{S}^{n-1}$ . Then by standard techniques, there exists a Borel decomposition

$$Y = N \cup \bigcup_{m \in \mathbb{N}} Y_m$$

where  $\mu(N) = 0$  and for each  $m \in \mathbb{N}$  and  $q \in \mathcal{Q}$ ,  $\text{Lip}(q \cdot \varphi)$  is continuous on each  $Y_m$ . Therefore, for any  $\epsilon, \rho > 0$ ,  $m \in \mathbb{N}$  and  $q \in \mathcal{Q}$ , there exist a  $Z \subset Y_m$  with  $\mu(Y_m \setminus Z) < \rho$  and an  $R > 0$  such that

$$(8.1) \quad |q \cdot (\varphi(x) - \varphi(y))| \leq (\text{Lip}(q \cdot \varphi, x_0) + \epsilon)d(x, y)$$

for each  $0 < r < R$ ,  $x_0 \in Z$ ,  $y \in Z \cap B(x_0, r)$  and  $x \in B(y, r)$ . Therefore, by decomposing each  $Y_m$  further, we may suppose that for every  $\epsilon > 0$ ,  $q \in \mathcal{Q}$ ,  $m \in \mathbb{N}$  there exists an  $R > 0$  such that (8.1) holds for each  $0 < r < R$ ,  $x_0 \in Y_m$ ,  $y \in Z \cap B(x_0, r)$  and  $x \in B(y, r)$ .

We now work with a fixed  $Y_m$ . By Proposition 2.9, for almost every  $x_0 \in Y_m$  there exist  $\gamma_1, \dots, \gamma_n \in \Gamma(X)$  such that, for each  $1 \leq i \leq n$ ,  $\gamma_i^{-1}(x_0) = t_0$  is a density point of  $\gamma_i^{-1}(Y_m)$  and  $\text{Lip}(\gamma_i, t_0) = 1$ . Further, there exist  $w_1, \dots, w_n \in \mathcal{Q}$  such that, for each  $1 \leq i \neq j \leq n$ , either

$$(8.2) \quad |w_i \cdot (\varphi \circ \gamma_i)'(t_0)| > |w_i \cdot (\varphi \circ \gamma_j)'(t_0)| + \delta \text{Lip}(w_i \cdot \varphi, \gamma_i(t_0)) > 0,$$

or the corresponding inequalities with  $i$  and  $j$  exchanged.

Let  $\epsilon > 0$ . Then there exists an  $R_1 > 0$  such that, for any  $0 < r < R_1$ , if  $t$  belongs to each  $\text{Dom } \gamma_i$  with  $0 < |t - t_0| < r$ , then, after setting  $x_k = \gamma_k(t)$ ,

$$\begin{aligned} \text{Lip}(w_i \cdot \varphi, x_0)d(x_i, x_j) &\geq |w_i \cdot (\varphi(x_i) - \varphi(x_j))| - \epsilon r \\ &\geq |w_i \cdot (\varphi(x_i) - \varphi(x_0))| - |w_i \cdot (\varphi(x_j) - \varphi(x_0))| - \epsilon r \end{aligned}$$



for each  $1 \leq i, j \leq n$ . Further, there exists a  $0 < R_2 < R_1$  such that, for any  $0 < r < R_2$ ,  $1 \leq k \leq n$  and  $x_k = \gamma_k(t)$ ,

$$\|(\varphi \circ \gamma)'(t_0)(t_k - t_0) - \varphi(x_k) - \varphi(x_0)\| \leq \epsilon r.$$

In particular,

$$\text{Lip}(w_i \cdot \varphi, x_0)d(x_i, x_j) \geq |w_i \cdot (\varphi \circ \gamma_i)'(t_0)(t - t_0)| - |w_i \cdot (\varphi \circ \gamma_j)'(t_0)(t - t_0)| - 3\epsilon r.$$

Further, since  $t_0$  is a density point of each  $\text{Dom } \gamma_k$ , there exists a  $0 < R_3 < R_2$  such that, for any  $0 < r < R_3$  and  $1 \leq k \leq n$ , there exists a  $t$  in each  $\text{Dom } \gamma_k$  such that  $\gamma_k(t) \in B(x_0, r)$  and  $|t - t_0 - r| \leq \epsilon r$ . Therefore, if  $x_k = \gamma_k(t)$  for each  $1 \leq k \leq n$ ,

$$\text{Lip}(w_i \cdot \varphi, x_0)d(x_i, x_j) \geq r|w_i \cdot (\varphi \circ \gamma_i)'(t_0)| - r|w_i \cdot (\varphi \circ \gamma_j)'(t_0)| - 5\epsilon r.$$

By exchanging  $i$  and  $j$  if necessary, we may apply (8.2) as stated so that

$$\text{Lip}(w_i \cdot \varphi, x_0)d(x_i, x_j) \geq \delta r \text{Lip}(w_i \cdot \varphi, x_0) - 5\epsilon r.$$

Finally, also by our initial hypotheses,  $\text{Lip}(w_i \cdot \varphi, \gamma_i(t_0)) > 0$  and so we deduce that there exists  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  such that  $\phi(r)/r \rightarrow 0$  as  $r \rightarrow 0$  and

$$d(x_i, x_j) \geq \delta r - \phi(r).$$

The  $Y_m$  cover almost all of  $Y$  and so such  $x_1, \dots, x_n$  may be found for almost every  $x_0 \in Y$ , as required.

Now suppose that for some  $\eta \geq 1$ ,  $Y \subset W \in D_\eta$ . Then for any  $0 < \epsilon < \delta$ , if  $R > 0$  such that  $|\phi(r)| < \epsilon r$  for each  $0 < r < R$ , the  $x_i$  found above all belong to  $B(x_0, r)$  but are separated by a distance of at least  $(\delta - \epsilon)r$ . Therefore no two  $x_k$  can belong to the same ball of radius  $(\delta - \epsilon)r/2$ . In particular  $n \leq N((\delta - \epsilon)/2, \eta)$  for all  $0 < \epsilon < \delta$  and so  $n \leq N(\delta/2, \eta)$ .  $\square$

We use this construction to satisfy the hypotheses of Lemma 8.1.

**Proposition 8.8.** *Let  $(X, d, \mu)$  be a metric measure space,  $\delta > 0$  and, for  $\eta \geq 1$ , let  $Y$  be a Borel subset of some  $W \in D_\eta$ . Suppose that, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ , there exists an Alberti representation of  $\mu \llcorner Y$  with  $f$ -speed  $\delta$ . Then, for any  $N > M(\delta/2, \eta)$  and Lipschitz  $\varphi: X \rightarrow \mathbb{R}^N$ , for almost every  $x_0 \in Y$  there exists an  $a(x_0) \in \mathbb{R}^N$  such that*

$$\text{Lip}(a(x_0) \cdot \varphi, x_0) = 0.$$

*Proof.* Suppose that  $\varphi: X \rightarrow \mathbb{R}^N$  is Lipschitz and, for some Borel  $Z \subset Y$  of positive measure,  $\text{Lip}(v \cdot \varphi, x_0) > 0$  for every  $x_0 \in Z$  and every  $v \in \mathbb{S}^{N-1}$ . By taking a countable decomposition of  $Z$  if necessary, we may suppose that there exists a  $\lambda > 0$  such that

$$\text{Lip}(v \cdot \varphi, x_0) \geq \lambda \text{Lip}(\varphi, x_0)$$

for every  $v \in \mathbb{S}^{N-1}$  and  $x_0 \in Z$ . We will prove the proposition by proving  $N \leq M(\delta, \eta)$ , using the assumption that there exists an Alberti representation of  $\mu \llcorner Z$  with  $v \cdot \varphi$ -speed  $\delta$ , for any  $v \in \mathbb{S}^{N-1}$ .

Let  $\epsilon > 0$  and  $\mathcal{A}_1$  be such a representation for an arbitrary choice of  $v_1 \in \mathbb{S}^{N-1}$ . Inductively, suppose that  $m \leq N$  and that there exists a countable Borel decomposition  $Z = \cup_i Z_i$  such that each  $\mu \llcorner Z_i$  has  $m - 1$  Alberti representations  $\mathcal{A}_1^i, \dots, \mathcal{A}_{m-1}^i$ . Then, by refining the representations if necessary, we may suppose that there exist  $w_1^i, \dots, w_{m-1}^i \in \mathbb{S}^{N-1}$  such that  $\mathcal{A}_k^i$  is in the  $\varphi$ -direction of  $C(w_k^i, \epsilon)$

for each  $1 \leq k < m$ . We choose  $w_m^i$  orthogonal to each  $w_j^i$  and let  $\mathcal{A}_m^i$  be an Alberti representation of  $\mu \llcorner Z_i$  with  $w_m^i \cdot \varphi$ -speed  $\delta$ . Then  $w_m^i$  satisfies

$$w_m^i \cdot (\varphi \circ \gamma_j)'(t) \leq \frac{\sqrt{\epsilon(2-\epsilon)}}{\lambda} \text{Lip}(\varphi, \gamma_j(t_j)) \text{Lip}(\gamma_j, t_j)$$

for almost every  $\gamma_j \in \mathcal{A}_j^i$  and almost every  $t_j \in \text{Dom } \gamma_j$ , for each  $1 \leq j < m$ . In particular,

$$\frac{w_m^i \cdot (\varphi \circ \gamma_m)'(t_m)}{\text{Lip}(\gamma_m, t_m)} > \frac{w_m^i \cdot (\varphi \circ \gamma_j)'(t_j)}{\text{Lip}(\gamma_j, t_j)} + \left( \delta - \frac{2\sqrt{\epsilon(2-\epsilon)}}{\lambda} \right) \text{Lip}(w_m^i \cdot \varphi, \gamma_m(t_m))$$

for every  $1 \leq j < m$ , almost every  $\gamma_m \in \mathcal{A}_m^i$  and  $\gamma_j \in \mathcal{A}_j^i$  and almost every  $t_m \in \gamma_m$  and  $t_j \in \gamma_j$ .

By repeating this process  $N - 1$  times, we obtain  $N$  Alberti representations that satisfy the hypotheses of Lemma 8.7 and so  $N \leq M(\delta/2 - \sqrt{\epsilon(2-\epsilon)}/\lambda, \eta)$ . This is true for every  $\epsilon > 0$  and so  $N \leq M(\delta/2, \eta)$ , as required.  $\square$

Using this result we may also bound the dimension of charts in Lipschitz differentiability spaces.

**Corollary 8.9.** *Let  $(U, \varphi)$  be a chart in a Lipschitz differentiability space and  $\eta \geq 1$ . Suppose that  $Y \subset U$  has positive  $\mu$  measure, is contained within some  $W \in D_\eta$  and that any  $\tilde{B}_\delta$  subset of  $Y$  is  $\mu$ -null. Then the dimension of  $(U, \varphi)$  is bounded above by  $M(\delta/2, \eta)$ .*

*Proof.* Suppose that  $(U, \varphi)$  is a chart of dimension  $n$ . Then by Lemma 3.3,  $\text{Lip}(v \cdot \varphi, x_0) > 0$  for all  $v \in \mathbb{S}^{n-1}$  and almost every  $x_0 \in U$ . Therefore, by Proposition 8.8,  $n \leq M(\delta/2, \eta)$ .  $\square$

Using the previous construction we may give our characterisation of Lipschitz differentiability spaces via null sets.

**Theorem 8.10.** *For  $(X, d, \mu)$  a metric measure space, the following are equivalent:*

- (1)  $(X, d, \mu)$  is a Lipschitz differentiability space.
- (2) Any  $\tilde{A}$  subset of  $X$  is  $\mu$ -null and  $X$  is pointwise doubling.
- (3) Any  $\tilde{B}$  subset of  $X$  is  $\mu$ -null and  $X$  is pointwise doubling.
- (4) There exist a countable Borel decomposition  $X = \cup_i X_i$  and sequences  $\eta_i \geq 1$  and  $\delta_i > 0$  such that
  - For any Lipschitz  $f: X \rightarrow \mathbb{R}$  and  $i \in \mathbb{N}$ , there exists an Alberti representation of  $\mu \llcorner X_i$  with  $f$ -speed  $\delta_i$ .
  - For every  $i \in \mathbb{N}$  there exists a  $W \in D_{\eta_i}$  such that  $\mu(X_i \setminus W) = 0$ .

*Proof.* We will prove the implications (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (1). Firstly, by Theorem 6.9, any  $\tilde{A}$  subset of a Lipschitz differentiability space is  $\mu$ -null. Further, by Corollary 8.4, any Lipschitz differentiability space is pointwise doubling, giving the first implication. The second implication is true since  $\tilde{B} \subset \tilde{A}$  for any metric measure space.

To prove (3)  $\Rightarrow$  (4) observe that, for any  $\delta > 0$ ,  $\tilde{B}_\delta$  is closed under countable unions. Therefore there exists a countable Borel decomposition  $X = \cup_i X_i \cup N$  where  $N \in \tilde{B}$  and such that, for each  $i \in \mathbb{N}$ , every  $\tilde{B}_{1/i}$  subset of  $X_i$  has measure zero. In particular, given a Lipschitz function  $f: X \rightarrow \mathbb{R}$ , we may apply Corollary 5.8 to obtain an Alberti representation of  $\mu \llcorner X_i$  with  $f$ -speed  $1/i$ . Further, by Lemma

8.3, for each  $i \in \mathbb{N}$  there exists a countable Borel decomposition  $X_i = \cup_j (X_i^j \cup N_i)$  where  $N_i$  is  $\mu$ -null and for each  $j \in \mathbb{N}$  there exists a  $W \in D_{1/j}$  such that  $X_i^j \subset W$ . Therefore, the decomposition  $X = \cup_{i,j} (X_i^j \cup N_i)$  is of the required form.

For the final implication, by applying Lemma 8.1 and Proposition 8.8, we see that  $(X, d, \mu)$  is a Lipschitz differentiability space. □

Finally, as mentioned above, metric measure spaces in which porous sets have measure zero are pointwise doubling. Moreover, by Corollary 4.5, porous sets in Lipschitz differentiability spaces have measure zero. Therefore, we obtain an additional characterisation of Lipschitz differentiability spaces entirely via null sets.

**Theorem 8.11.** *For  $(X, d, \mu)$  a metric measure space, the following are equivalent:*

- $(X, d, \mu)$  is a Lipschitz differentiability space.
- Any  $\tilde{A}$  and any porous set in  $X$  is  $\mu$ -null.
- Any  $\tilde{B}$  and any porous set in  $X$  is  $\mu$ -null.

### 9. ARBITRARY ALBERTI REPRESENTATIONS

We now extend the results from the previous sections to find Alberti representations in the direction of an arbitrary cone  $C$  in a chart  $(U, \varphi)$  of a Lipschitz differentiability space  $(X, d, \mu)$ . As before, we will produce this representation using Corollary 5.8 and so we are required to show that any Borel  $S \subset U$  that satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C$  has measure zero. This is achieved by giving a decomposition  $S = \cup_i S_i$  such that each  $S_i$  satisfies  $\mathcal{H}^1(\gamma \cap S_i) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of a cone  $C_i$  that defines the direction of an existing Alberti representation.

For this we will need to deduce properties of the derivative of a function obtained from Lemma 6.4. In particular, we require a bound on the directional derivative of the function in the direction  $w$ , for  $w$  as in the hypotheses of the lemma. This is possible for any  $x_0$  for which there exist  $x_m \rightarrow x_0$  such that  $\varphi(x_m) - \varphi(x_0)$  is parallel to  $w$ .

To begin we concatenate the curves obtained from our existing Alberti representations, in suitable ratios, to show the existence of such a sequence. Recall the notion of separated Alberti representations given in Definition 7.3.

**Lemma 9.1.** *Let  $(X, d, \mu)$  be a metric measure space,  $\varphi: X \rightarrow \mathbb{R}^n$  Lipschitz and  $\delta, \xi > 0$ . Suppose that  $X$  has  $n$   $\xi$ -separated Alberti representations with speed strictly greater than  $\delta$ . Then there exists an  $\eta > 0$  such that, for any measurable  $S \subset X$  and for almost every  $x_0 \in S$ , given any  $v \in \mathbb{S}^{n-1}$  there exist  $S \ni x_m \rightarrow x_0$  with*

$$\limsup_{m \rightarrow \infty} \left\| \frac{\varphi(x_m) - \varphi(x_0)}{d(x_m, x_0)} - v \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} \right\| = 0$$

and

$$\limsup_{m \rightarrow \infty} \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} \geq \eta \text{Lip}(\varphi, x_0).$$

*Proof.* Let  $S \subset X$  be measurable,  $w \in \mathbb{S}^{n-1}$  and  $0 < \epsilon < 1$ . By refining if necessary, we may suppose that the above Alberti representations are in the  $\varphi$ -direction of  $\xi$ -separated cones  $C_1 = C(w_1, \epsilon), \dots, C_n = C(w_n, \epsilon)$ . Further, for each  $1 \leq i \leq n$  let  $C'_i \supset C_i$  be open cones that are also  $\xi$ -separated.

For  $1 \leq i \leq n$  let  $\Gamma^i$  be the set of  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C_i$  with speed  $\delta$ . Further, for each  $0 < R < \epsilon$ , let  $G_R^i(S)$  be the set of  $x_0 \in S$  for which there exist a  $\gamma \in \Gamma^i$  and a  $t_0 \in \text{Dom } \gamma$  with  $\gamma(t_0) = x_0$  such that, for every  $0 < r < R/\|(\varphi \circ \gamma)'(t_0)\|$  and every  $t \in \text{Dom } \gamma$  with  $|t - t_0| \leq 5R/\|(\varphi \circ \gamma)'(t_0)\|$ , the following conditions hold:

- (1)  $\mathcal{L}^1(\gamma^{-1}(S) \cap B(t_0, 2r)) \geq 4r(1 - \epsilon)$ .
- (2)  $\|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - (\varphi \circ \gamma)'(t_0)(t - t_0)\| \leq \epsilon\|(\varphi \circ \gamma)'(t_0)\||t - t_0|$ .
- (3)  $\|(\varphi \circ \gamma)'(t_0)\||t - t_0| \leq 2\|\varphi(\gamma(t)) - \varphi(\gamma(t_0))\|$ .
- (4)  $\varphi(\gamma(t)) - \varphi(\gamma(t_0)) \in C'_i$ .
- (5)  $\|(\varphi \circ \gamma)'(t_0)\||t - t_0| \geq \delta \text{Lip}(\varphi, \gamma(t_0))d(\gamma(t), \gamma(t_0))/2$ .

Since  $\mu$  has an Alberti representation in the  $\varphi$ -direction of each  $C_i$  with speed  $\delta$ , by Proposition 2.9 each  $G_R^i(S)$  is measurable and monotonically increases to a set of full measure in  $S$  as  $R \searrow 0$ .

Now let  $x_0 \in G_R^i(S)$ , let  $\gamma \in \Gamma^i$  be as in the definition of  $G_R^i$  for  $x_0$  and write  $w = \sum_i \lambda_i w_i$ . Then for any  $0 < r < R/\max \lambda_i$ ,

$$r\lambda_i/\|(\varphi \circ \gamma)'(t_0)\| \leq R/\|(\varphi \circ \gamma)'(t_0)\|$$

and so, by (1), there exists a  $t \in \text{Dom } \gamma$  with

$$0 \leq t - t_0 - \frac{r\lambda_i}{\|(\varphi \circ \gamma)'(t_0)\|} \leq \frac{4r\epsilon}{\|(\varphi \circ \gamma)'(t_0)\|} \leq 4\epsilon(t - t_0).$$

In particular

$$(9.1) \quad |t - t_0| \leq 5R/\|(\varphi \circ \gamma)'(t_0)\|$$

and

$$(9.2) \quad \left| \|(\varphi \circ \gamma)'(t_0)\| - \frac{r\lambda_i}{|t - t_0|} \right| \leq 4\epsilon.$$

Note also that, since  $(\varphi \circ \gamma)'(t_0)$  belongs to  $C_i$ ,

$$(9.3) \quad \left\| \frac{(\varphi \circ \gamma)'(t_0)}{\|(\varphi \circ \gamma)'(t_0)\|} - w_i \right\| \leq \sqrt{2\epsilon}.$$

Therefore, by the triangle inequality, (9.2), (9.3), (2) and (3) give

$$\begin{aligned} \|\varphi(\gamma(t)) - \varphi(\gamma(t_0)) - r\lambda_i w_i\| &\leq 10\sqrt{\epsilon}\|(\varphi \circ \gamma)'(t_0)\||t - t_0| \\ &\leq 20\sqrt{\epsilon}\|\varphi(\gamma(t)) - \varphi(\gamma(t_0))\|. \end{aligned}$$

Moreover, by (5) and (9.1),

$$\delta \text{Lip}(\varphi, \gamma(t_0))d(\gamma(t), \gamma(t_0)) \leq 5R \leq 5\epsilon.$$

That is, for every  $x_0 \in G_R^i$ , there exists an  $x \in S$  with

$$(9.4) \quad \|\varphi(x) - \varphi(x_0) - r\lambda_i w_i\| \leq 20\sqrt{\epsilon}\|\varphi(x) - \varphi(x_0)\|$$

and

$$(9.5) \quad (x, x_0) \leq 5\epsilon/\delta \text{Lip}(\varphi, \gamma(t_0)).$$

Now define  $S_R^0 = S$  and for each  $0 \leq i < n$  define

$$S_R^{i+1} = G_R^{i+1}(S_R^i).$$

Then for any  $x_0 \in S_R^n$  and  $0 \leq i < n$  there exists  $x_i \in S_R^{n-i}$  such that the relations (9.4) and (9.5) hold between  $x_i$  and  $x_{i+1}$ . Therefore

$$\|\varphi(x_n) - \varphi(x_0) - rw\| \leq 20\sqrt{\epsilon} \sum_{0 \leq i < n} \|\varphi(x_i) - \varphi(x_{i+1})\|$$

and so, by the triangle inequality,

$$(9.6) \quad \|\varphi(x_n) - \varphi(x_0) - \|\varphi(x_n) - \varphi(x_0)\|w\| \leq 40\sqrt{\epsilon} \sum_{0 \leq i < n} \|\varphi(x_i) - \varphi(x_{i+1})\|.$$

However each  $\gamma$  above was chosen so that  $\varphi(x_i) - \varphi(x_{i-1}) \in C'_i$  for each  $0 \leq i < n$ . Since the  $C'_i$  form a collection of  $\xi$ -separated cones,

$$(9.7) \quad \begin{aligned} \|\varphi(x_0) - \varphi(x_n)\| &= \left\| \sum_{0 \leq i < n} \varphi(x_i) - \varphi(x_{i+1}) \right\| \\ &\geq \xi \max_{0 \leq i < n} \|\varphi(x_i) - \varphi(x_{i+1})\| \\ (9.8) \quad &\geq \xi\delta \operatorname{Lip}(\varphi, x_0) \max_{0 \leq i < n} d(x_i, x_{i+1})/4 > 0. \end{aligned}$$

Equations (9.6) and (9.7) give

$$\|\varphi(x_n) - \varphi(x_0) - \|\varphi(x_n) - \varphi(x_0)\|w\| \leq 40\sqrt{\epsilon}n \operatorname{Lip} \varphi d(x_n, x_0)/\xi$$

and (9.8) and the triangle inequality give

$$\|\varphi(x_n) - \varphi(x_0)\| \geq \xi\delta \operatorname{Lip}(\varphi, x_0)d(x_n, x_0)/4n.$$

Note also that we must have  $x_0 \neq x_n$  and

$$d(x_n, x_0) \leq \sum_{0 \leq i < n} d(x_i, x_{i+1}) \leq 5n\epsilon.$$

Finally,  $\cup_{m \in \mathbb{N}} S_{1/m}^n$  is a set of full measure in  $S$  and so for almost every  $x_0 \in S$  there exists an  $x_n \in S$  with the above properties, for a given  $0 < \epsilon < 1$ . Taking a countable intersection over  $\epsilon \in (0, 1) \cap \mathbb{Q}$  completes the proof for  $\eta = \xi\delta/n$ .  $\square$

**Corollary 9.2.** *Let  $(U, \varphi)$  be a  $\lambda$ -structured chart in a Lipschitz differentiability space and for some  $\xi, \delta > 0$ , let  $V \subset U$  be Borel such that  $\mu \llcorner V$  has  $n$   $\xi$ -separated Alberti representations with speed strictly greater than  $\delta$ . Suppose that, for some  $w \in \mathbb{S}^{n-1}$  and  $0 < \epsilon, \tau < 1$ ,  $S \subset V$  is closed and satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w, \epsilon)$  with  $\varphi$ -speed strictly greater than  $\tau$ . Then there exist an  $\eta > 0$  and, for any  $\zeta > 0$ , a  $(K(\epsilon) + 1 + \tau + \zeta)$   $\operatorname{Lip} \varphi$ -Lipschitz function  $f: U \rightarrow \mathbb{R}$  such that*

$$(1) \text{ For every } x_0 \in S \text{ and } x \in U \text{ with } (\varphi(x) - \varphi(x_0)) \cdot w \geq 0,$$

$$f(x) - f(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - \zeta.$$

$$(2) \text{ For almost every } x_0 \in S$$

$$Df(x_0) \cdot w \leq (\zeta + \tau)/\eta\lambda.$$

*Proof.* Let  $f: U \rightarrow \mathbb{R}$  be the Lipschitz function obtained from an application of Lemma 6.4, so that  $f$  automatically satisfies all of the required properties except for (2). Further, there exists a  $\rho > 0$  such that, for every  $x_0 \in S$  and  $y, z \in B(x_0, \rho)$ ,

$$|f(y) - f(z)| \leq K(\epsilon)\|\mathcal{P}(\varphi(y) - \varphi(z))\| + (\tau + \zeta)d(y, z)$$

for  $K(\epsilon) = (1 - \epsilon)/\sqrt{\epsilon(2 - \epsilon)}$  and  $\mathcal{P}: \mathbb{R} \rightarrow \mathbb{R}$  the orthogonal projection onto the plane orthogonal to  $w$  passing through the origin.

Now let  $\eta > 0$  such that the conclusion of Lemma 9.1 holds. Then for almost every  $x_0 \in V$ ,  $Df(x_0)$  exists and there exist  $U \ni x_m \rightarrow x_0$  with  $\|\varphi(x_m) - \varphi(x_0)\| \geq \eta d(x_m, x_0)$  such that

$$\frac{\|\varphi(x_m) - \varphi(x_0) - \|\varphi(x_m) - \varphi(x_0)\|w\|}{d(x_m, x_0)} \rightarrow 0.$$

In particular, by the triangle inequality,

$$\limsup_{m \rightarrow \infty} \frac{\|\mathcal{P}(\varphi(x_m) - \varphi(x_0))\|}{d(x_m, x_0)} \leq \|\mathcal{P}(w)\| \limsup_{m \rightarrow \infty} \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} = 0$$

and so

$$\limsup_{m \rightarrow \infty} \frac{|f(x_m) - f(x_0)|}{d(x_m, x_0)} \leq \tau + \zeta.$$

Therefore

$$\begin{aligned} \eta \operatorname{Lip}(\varphi, x_0) Df(x_0) \cdot w &\leq \limsup_{m \rightarrow \infty} \frac{\|\varphi(x_m) - \varphi(x_0)\|}{d(x_m, x_0)} Df(x_0) \cdot w \\ &\leq \limsup_{m \rightarrow \infty} \frac{Df(x_0) \cdot (\varphi(x_m) - \varphi(x_0))}{d(x_m, x_0)} \\ &= \limsup_{m \rightarrow \infty} \frac{|f(x_m) - f(x_0)|}{d(x_m, x_0)} \\ &\leq \tau + \zeta. \end{aligned}$$

Since  $(U, \varphi)$  is a  $\lambda$ -structured chart,  $\operatorname{Lip}(\varphi, x_0) \geq \lambda$  and so dividing by  $\eta\lambda$  completes the proof. □

Given a set  $S$  as in the hypotheses of Corollary 9.2, we will decompose it using the following lemma. In our application, the  $D_m$  in the hypotheses will be the derivatives of Lipschitz functions obtained from Corollary 9.2.

**Lemma 9.3.** *Let  $(X, d, \mu)$  be a metric measure space,  $U \subset X$  Borel and  $w \in \mathbb{S}^{n-1}$ . Suppose that, for some  $\beta \geq 0$  and  $E > 0$ , there exists a sequence of measurable functions  $D_m: X \rightarrow \mathbb{R}^n$  with essential supremum  $E$  such that*

$$\limsup_{m \rightarrow \infty} D_m(x) \cdot w + \beta \leq w \cdot w = 1$$

for almost every  $x \in U$ . Suppose further that, for some  $\xi > 0$  and  $\xi$ -separated  $w_1, \dots, w_k \in \mathbb{S}^{n-1}$ ,  $w$  belongs to the convex cone of the  $w_i$ . Then there exist a Borel decomposition

$$S = N \cup \bigcup_{j \in \mathbb{N}} S_j$$

with  $\mu(N) = 0$  and, for every  $j \in \mathbb{N}$ , a  $1 \leq i \leq n$  such that, for every  $0 < \theta < 1$ ,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{1 \leq k \leq m} D_k(x) \cdot v + \frac{\|v\|\beta\xi}{n} \leq w \cdot v + \sqrt{2\theta}(E + 1)\|v\|$$

uniformly for  $x \in S_j$  and  $v \in C(w_i, \theta)$ .

*Proof.* Since each  $D_m$  is bounded, there exists a  $g \in L^2(U)$  such that, after passing to a subsequence,  $D_m \rightarrow g$  weakly. Then, by the Banach-Saks theorem, there exists a further subsequence such that the functions

$$\tilde{D}_m := \frac{1}{m} \sum_{1 \leq j \leq m} D_j$$

converge pointwise almost everywhere to  $g$ . Moreover, since

$$\limsup_{m \rightarrow \infty} D_m(x) \cdot w + \beta \leq w \cdot w,$$

$g(x) \cdot w + \beta \leq w \cdot w$  almost everywhere.

Let

$$S = N \cup \bigcup_{j \in \mathbb{N}} S_j$$

be a Borel decomposition of  $S$  where  $N$  is  $\mu$ -null and such that  $\tilde{D}_m \rightarrow g$  uniformly on each  $S_j$ . Further, since the  $w_i$  are  $\xi$ -separated and  $w$  lies in the convex cone of the  $w_i$ , there exist  $0 \leq \lambda_i \leq 1/\xi$  with  $w = \sum_i \lambda_i w_i$ . Therefore, for each  $x_0$  in some  $S_j$ , there exists a  $1 \leq i \leq n$  with

$$g \cdot w_i + \beta\xi/n \leq w \cdot w_i.$$

We may therefore decompose each  $S_j$  into the sets

$$S_j^i = \{x_0 \in S_j : g \cdot w_i + \beta\xi/n \leq w \cdot w_i\}$$

for  $1 \leq i \leq k$ . By taking a further decomposition, we may suppose that each  $S_j^i$  is compact.

Finally, for any  $0 < \theta < 1$ ,  $j \in \mathbb{N}$ ,  $1 \leq i \leq n$  and  $v \in C(w_i, \theta)$ ,

$$\begin{aligned} g(x) \cdot v + \|v\|\beta\xi/n &= \|v\|(g(x) \cdot w_i + \beta\xi/n) + g(x) \cdot (v - \|v\|w_i) \\ &\leq \|v\|(w \cdot w_i \sqrt{2\theta}E) \\ &\leq w \cdot v + \sqrt{2\theta}(E + 1)\|v\|. \end{aligned}$$

Since  $\tilde{D}_m \rightarrow g$  uniformly on each  $S_j^i$ , this decomposition is of the required form.  $\square$

Finally, we will show that a set with certain properties intersects a Lipschitz curve  $\gamma$  in a set of measure zero by applying the following lemma to the domain of  $\gamma$ .

**Lemma 9.4.** *Let  $S \subset \mathbb{R}$  be measurable and  $L > 0$ . Suppose that there exist a sequence of  $L$ -Lipschitz functions  $f_m : S \rightarrow \mathbb{R}$  and measurable functions  $\Phi, \Psi : S \rightarrow \mathbb{R}$  such that, for almost every  $t_0 \in S$ ,  $\Phi(t_0) < \Psi(t_0)$  and*

- (1) *There exist  $m_j \rightarrow \infty$  such that, for every  $t \in S$  with  $t \geq t_0$ ,*

$$f_{m_j}(t) - f_{m_j}(t_0) \geq \Psi(t_0)(t - t_0) - 1/m_j.$$

- (2) *There exists an  $M \in \mathbb{N}$  such that, for every  $m \geq M$ ,*

$$Df_m(t_0) \leq \Phi(t_0).$$

*Then  $S$  is Lebesgue null.*

*Proof.* Suppose that such an  $S$  has positive Lebesgue measure. Then there exist an  $M \in \mathbb{N}$ ,  $\alpha < \beta \in \mathbb{R}$  and  $S' \subset S$  of positive measure such that (1) and

$$Df_m(t_0) \leq \Phi(t_0) \leq \alpha < \beta \leq \Psi(t_0)$$

for all  $t_0 \in S'$  and  $m \geq M$ . Let  $t_0$  be a density point of  $S'$  and  $R > 0$  such that, for every  $t \in (t_0, t_0 + R)$ ,

$$\frac{\mathcal{L}^1(S' \cap (t_0, t))}{t} \geq 1 - (\beta - \alpha)/2L.$$

Then, for any  $m \geq M$  and  $t \in (t_0, t_0 + R) \cap S$ ,

$$\begin{aligned} f_m(t) - f_m(t_0) &= \int_{(t_0, t) \cap S'} Df_m + \int_{(t_0, t) \setminus S'} Df_m \\ &\leq \alpha(t - t_0) + (t - t_0)L(\beta - \alpha)/2L, \\ &\leq (\beta + \alpha)(t - t_0)/2. \end{aligned}$$

However, if  $m_j \geq M$  such that (1) holds for  $t_0$ , then for any  $t \in (t_0, t_0 + R) \cap S$ ,

$$(\beta + \alpha)(t - t_0)/2 \geq f_{m_j}(t) - f_{m_j}(t_0) \geq \beta(t - t_0) - 1/m_j.$$

In particular,  $|t - t_0| \leq 2/(\beta - \alpha)m_j$  for  $m_j \rightarrow \infty$  and so  $t = t_0$ , contradicting our assumption that  $t_0$  is a density point of  $S$ . □

We now apply these results to a chart in a Lipschitz differentiability space.

**Theorem 9.5.** *Let  $\lambda, \delta, \xi > 0, n \in \mathbb{N}$  and  $(U, \varphi)$  be a  $\lambda$ -structured  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$  with  $n$   $\xi$ -separated Alberti representations with speed strictly greater than  $\delta$ . Then there exists a  $\tau > 0$  (depending only upon  $\lambda, n, \xi$  and  $\delta$ ) such that, for any  $w \in \mathbb{S}^{n-1}$  and  $0 < \epsilon < 1$ ,  $\mu \llcorner U$  has an Alberti representation in the  $\varphi$ -direction of  $C(w, \epsilon)$  with speed  $\tau$ .*

*Proof.* Since we may combine Alberti representations using Lemma 2.4, by refining the Alberti representations we may suppose that they are in the  $\varphi$ -direction of  $\xi$ -separated cones  $C(w_1, \theta), \dots, C(w_n, \theta)$  with speed strictly greater than  $\delta$ , for  $\theta$  satisfying

$$\frac{\sqrt{2\theta}(2 + K(\epsilon) + \lambda)}{\lambda} \leq \frac{\xi}{4n}$$

and  $K(\epsilon) = (1 - \epsilon)/\sqrt{\epsilon(2 - \epsilon)}$ . Let  $\eta$  be obtained from Lemma 9.1 (which depends only upon  $\xi, \delta$  and  $n$ ) and set  $\tau = \min\{n\lambda, 1\}/2$ . We construct the Alberti representation using Corollary 5.8 and so are required to show that any compact  $S \subset U$  that satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$  direction of  $C(w, \epsilon)$  with speed strictly greater than  $\tau$  is in fact  $\mu$ -null.

We apply Corollary 9.2 (with  $\zeta = 1/m$  and  $m \geq 2$ ) to obtain a sequence of  $(2 + K(\epsilon))$ -Lipschitz functions  $f_m : U \rightarrow \mathbb{R}$  such that

- For every  $x_0 \in S'$  and  $x \in U$  with  $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$ ,

$$f_m(x) - f_m(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - 1/m.$$

- For almost every  $x_0 \in S'$ ,

$$\limsup_{m \rightarrow \infty} Df_m(x_0) \cdot w + \frac{1}{2} \leq w \cdot w.$$



Note that, since  $(U, \varphi)$  is a  $\lambda$ -structured chart, by Lemma 3.4,

$$\|Df_m(x_0)\| \leq (2 + K(\epsilon))/\lambda.$$

Therefore, by Lemma 9.3, there exists a countable Borel decomposition

$$S = N \cup \bigcup_{j \in \mathbb{N}} S_j$$

where  $\mu(N) = 0$  and for each  $j \in \mathbb{N}$  there exists a  $1 \leq i \leq n$  and an  $M \in \mathbb{N}$  such that, for every  $m \geq M$  and  $x_0 \in S_j$ ,

$$(9.9) \quad \frac{1}{m} \sum_{1 \leq k \leq m} Df_k(x_0) \cdot v + \frac{\xi}{2n} \|v\| \leq w \cdot v + \frac{\xi}{4n} \|v\|$$

for all  $v \in C(w_i, \theta)$ . Define, for each  $m \in \mathbb{N}$ ,

$$F_m = \frac{1}{m} \sum_{1 \leq k \leq m} f_k.$$

For each  $j \in \mathbb{N}$  and almost every  $x_0 \in S_j$  each  $Df_k(x_0)$  exists and so

$$DF_m = \frac{1}{m} \sum_{1 \leq k \leq m} Df_k.$$

Further, for every  $x_0 \in S$  and  $x \in U$  with  $(\varphi(x) - \varphi(x_0)) \cdot w \geq 0$ ,

$$F_m(x) - F_m(x_0) \geq (\varphi(x) - \varphi(x_0)) \cdot w - 1/m.$$

We fix  $j \in \mathbb{N}$  and let  $1 \leq i \leq n$  and  $M \in \mathbb{N}$  such that (9.9) holds for every  $m \geq M$  and  $x_0 \in S_j$ . It suffices to prove that  $\mu(S_j) = 0$ . Given the above Alberti representations of  $\mu \llcorner U_m$ , it suffices to prove that  $\mathcal{H}^1(\gamma \cap S_j) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w_i, \theta)$ .

To show this, fix  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C(w_i, \theta)$  and define, for each  $m \in \mathbb{N}$ , the  $(2 + K(\epsilon)) \text{Lip } \varphi \text{Lip } \gamma$ -Lipschitz function

$$g_m = F_m \circ \gamma: \text{Dom } \gamma \rightarrow \mathbb{R}.$$

Then, for any  $t_0 \in \text{Dom } \gamma$ , if  $(\varphi \circ \gamma)'(t_0)$  exists and  $\gamma(t_0) \in S_j$ ,

$$Dg_m(t_0) = DF_m(\gamma(t_0)) \cdot (\varphi \circ \gamma)'(t_0).$$

Therefore, if  $(\varphi \circ \gamma)'(t_0) \in C(w_i, \theta)$ ,

$$\begin{aligned} Dg_m(t_0) &\leq w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{4n} \|(\varphi \circ \gamma)'(t_0)\| \\ &\leq w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{4n} |w \cdot (\varphi \circ \gamma)'(t_0)| \\ &:= \Phi(t_0). \end{aligned}$$

Also, for any  $t, t_0 \in \text{Dom } \gamma$ ,

$$g_m(t) - g_m(t_0) \geq (\varphi(\gamma(t)) - \varphi(\gamma(t_0))) \cdot w - 1/m.$$

Suppose that  $\gamma^{-1}(S_j)$  has positive measure. Then there exist an  $R > 0$  and a  $T \subset \gamma^{-1}(S_j)$  of positive measure such that, for every  $t_0 \in T$  and  $t \in \text{Dom } \gamma$  with

$|t - t_0| \leq R$ ,  $(\varphi \circ \gamma)'(t_0) \in C(w_i, \theta)$  and

$$(\varphi(\gamma(t)) - \varphi(\gamma(t_0))) \cdot w \geq \left( w \cdot (\varphi \circ \gamma)'(t_0) - \frac{\xi}{8n} |w \cdot (\varphi \circ \gamma)'(t_0)| \right) (t - t_0) \\ := \Psi(t_0)(t - t_0).$$

We choose  $s \in T$  such that  $T' := T \cap B(s, R)$  has positive measure. Then for almost every  $t_0 \in T'$ ,  $\Psi(t_0) > \Phi(t_0)$  and

- For any  $m \geq M$  and every  $t \in T'$  with  $t \geq t_0$ ,

$$g_m(t) - g_m(t_0) \geq \Psi(t_0)(t - t_0) - 1/m.$$

- For every  $m \geq M$ ,

$$Dg_m(t_0) \leq \Phi(t_0).$$

Therefore, by Lemma 9.4,  $T'$  is Lebesgue null, a contradiction. □

We may use this theorem to improve our description of the local structure of a Lipschitz differentiability space.

**Corollary 9.6.** *Let  $(U, \varphi)$  be an  $n$ -dimensional chart in a Lipschitz differentiability space  $(X, d, \mu)$ . Then for almost every  $x \in U$  there exists a  $\tau > 0$  such that, for any cone  $C \subset \mathbb{R}^n$ , there exists a  $\gamma^x \in \Gamma(X)$  with  $(\gamma^x)^{-1}(x) = 0$  a density point of  $(\gamma^x)^{-1}(U)$ ,  $(\varphi \circ \gamma^x)'(0) \in C$  and  $(\varphi \circ \gamma^x)'(0) \geq \tau \text{Lip}(\varphi, x) \text{Lip}(\gamma^x, 0)$ .*

We also obtain another characterisation of Lipschitz differentiability spaces corresponding to arbitrary Alberti representations.

**Definition 9.7.** For a metric measure space  $(X, d, \mu)$  we define  $\tilde{C}$  to be the set of Borel  $S \subset X$  for which there exist a  $0 < \eta < 1$  and, for every  $\tau > 0$ , a countable Borel decomposition  $S = \cup_k S_k$ , Lipschitz  $\psi_k: X \rightarrow \mathbb{R}^{n_k}$  and a cone  $C_k \subset \mathbb{R}^{n_k}$  (of any width) such that, for each  $k \in \mathbb{N}$ ,

- For every  $x_0 \in S_k$ ,

$$\text{Lip}(v \cdot \psi_k, x_0) > \eta \text{Lip}(\psi_k, x_0) \text{ for every } v \in \mathbb{S}^{n_k-1}.$$

- For every  $\gamma \in \Gamma(X)$  in the  $\psi$ -direction of  $C_k$  with  $\psi_k$ -speed  $\tau$ ,  $\mathcal{H}^1(\gamma \cap S_k) = 0$ .

**Theorem 9.8.** *For a metric measure space  $(X, d, \mu)$  the following are equivalent:*

- $(X, d, \mu)$  is a Lipschitz differentiability space.
- Every  $\tilde{C}$  subset of  $X$  is  $\mu$ -null and  $X$  is pointwise doubling.
- Every  $\tilde{C}$  subset of  $X$  and every porous subset of  $X$  is  $\mu$ -null.

*Proof.* These two conditions are sufficient since any  $\tilde{A}$  subset of  $X$  is a  $\tilde{C}$  set. Therefore, if any  $\tilde{C}$  subset of  $X$  is  $\mu$ -null, so is any  $\tilde{A}$  subset.

Conversely suppose that  $S \in \tilde{C}$  is contained within a chart of a Lipschitz differentiability space that satisfies the hypotheses of Theorem 9.5 and  $\tau > 0$  obtained from its conclusion. Then by Lemma 6.8 there exist a countable Borel decomposition  $S = \cup_k S_k$  and cones  $C_k$  such that  $\mathcal{H}^1(\gamma \cap S_k) = 0$  for any  $\gamma \in \Gamma(X)$  in the  $\varphi$ -direction of  $C_k$  with speed  $\tau$ . Therefore, by Theorem 9.5,  $S$  is  $\mu$ -null. □

## 10. RELATIONSHIP WITH OTHER WORKS

As an example of the use of our theory, we give an alternate proof of Cheeger's Differentiation Theorem. We begin by introducing the notion of a Poincaré inequality in a metric measure space.

**Definition 10.1.** Let  $(X, d)$  be a metric space and  $f: X \rightarrow \mathbb{R}$  Lipschitz. We say that a measurable function  $\rho: X \rightarrow \mathbb{R}$  is an *upper gradient of  $f$*  if, for any  $\gamma \in \Pi(X)$  that is parametrised by arc length,

$$|f(\gamma_e) - f(\gamma_s)| \leq \int_{\text{Dom } \gamma} \rho \circ \gamma \, d\mathcal{L}^1$$

where  $\gamma_e$  and  $\gamma_s$  are the endpoints of  $\gamma$ .

Further, for  $p, P \geq 1$  we say that a metric measure space  $(X, d, \mu)$  satisfies the  $p$ -Poincaré inequality with constant  $P$  if, for every closed ball  $B = \overline{B}(x_0, r) \subset X$ ,  $\mu(B) > 0$  and

$$\int_B |f - f_B| \leq Pr \left( \int_{PB} \rho^p \right)^{1/p}.$$

Here

$$f_B = \int_B f := \frac{1}{\mu(B)} \int_B f.$$

Finally, for  $C \geq 1$ , we say that a metric measure space  $(X, d, \mu)$  is  $C$ -doubling if

$$\mu(\overline{B}(x_0, r)) \leq C\mu(\overline{B}(x_0, r/2))$$

for every  $x_0 \in X$  and every  $r > 0$ .

**Theorem 10.2** (Cheeger [Che99]). *Any  $C$ -doubling metric measure space  $(X, d, \mu)$  that satisfies the  $p$ -Poincaré inequality with constant  $P$  is a Lipschitz differentiability space. Moreover, each chart is of a dimension bounded above by an integer depending only upon  $C$  and  $P$ , independent of the metric measure space.*

*Proof.* Since a  $C$ -doubling metric measure space is a  $D_\eta$  set, for some  $\eta \geq 1$  depending only upon  $C$ , by Theorem 8.11 and Corollary 8.9 it suffices to prove the existence of a  $\delta > 0$  depending only upon  $C$  and  $P$  such that any  $\tilde{B}_\delta$  subset of  $X$  is  $\mu$ -null.

Firstly, by Proposition 4.3.3 of [Kei04], for any  $C$ -doubling metric measure space, there exists a  $C'$  depending only upon  $C$  such that, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ ,

$$\text{Lip}(f, x_0) \leq C' \lim_{r \rightarrow 0} \frac{1}{r} \int_{\overline{B}(x_0, r)} |f - f_B|$$

for almost every  $x_0 \in X$ . Therefore, if  $\rho$  is an upper gradient of  $f$ , by using the Poincaré inequality and applying Lebesgue's differentiation theorem to  $\rho$ , we see that there exists a  $C'' > 0$  depending only upon  $C$  and  $P$  such that

$$(10.1) \quad \rho(x_0) \geq C'' \text{Lip}(f, x_0)$$

for almost every  $x_0 \in X$ . We set  $\delta = C''/2$ .

Now suppose that  $f: X \rightarrow \mathbb{R}$  is Lipschitz and  $S \subset \{x_0 : \text{Lip}(f, x_0) > 0\}$  satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Gamma(X)$  with  $f$ -speed  $\delta$ . Then for any  $\gamma \in \Gamma(X)$  and almost every  $t_0 \in \gamma^{-1}(S)$ ,

$$(f \circ \gamma)'(t_0) \leq \delta \text{Lip}(f, \gamma(t_0)) \text{Lip}(\gamma, t_0).$$

In particular this is true for any  $\gamma \in \Pi(X)$  that is parametrised by arc length and so

$$\rho(x) = \begin{cases} \delta \operatorname{Lip}(f, x) & x \in S \\ \operatorname{Lip}(f, x) & \text{otherwise} \end{cases}$$

is an upper gradient of  $f$ . However, for almost every  $x \in S$ , by equation (10.1),

$$\delta \operatorname{Lip}(f, x) = \rho(x) \geq 2\delta \operatorname{Lip}(f, x) > 0.$$

Therefore  $S$  must be  $\mu$ -null. In particular, any  $\tilde{B}_\delta$  subset of  $X$  is  $\mu$ -null, as required.  $\square$

As pointed out in the introduction, the existence of an Alberti representation of any doubling Lipschitz differentiability space  $(X, d, \mu)$  that satisfies the Poincaré inequality may be deduced from a theorem of Cheeger and Kleiner. To see this, suppose that  $S$  is compact, contained within a chart and satisfies  $\mathcal{H}^1(\gamma \cap S) = 0$  for every  $\gamma \in \Pi(X)$ . Then by applying [CK09], Theorem 4.2 (and adopting its terminology) with  $f$  a component of the chart map and the negligible set  $N = \{(\gamma, t) : \gamma(t) \in S\}$ , we see that the minimal upper gradient of  $f$  equals zero almost everywhere in  $S$ . However, the minimal upper gradient of  $f$  equals  $\operatorname{Lip}(f, \cdot) > 0$  almost everywhere and so  $S$  must be  $\mu$ -null. An application of Lemma 5.2 gives the required Alberti representation. In fact, almost every curve in this Alberti representation is defined on an interval.

In [Kei04], Keith introduced the Lip-lip condition (see below) on a metric measure space and showed that any doubling metric measure space with a Lip-lip condition is a Lipschitz differentiability space (via the Poincaré inequality). We now use our theory to give an alternate proof of this fact and to prove the converse statement. This gives a characterisation of Lipschitz differentiability spaces via the Lip-lip condition.

**Definition 10.3.** We say that a metric measure space  $(X, d, \mu)$  satisfies a *Lip-lip* condition if there exist a countable Borel decomposition  $X = \cup_i X_i$  and for each  $i \in \mathbb{N}$  a  $\delta_i > 0$  such that, for any Lipschitz  $f: X \rightarrow \mathbb{R}$ ,

$$\begin{aligned} \delta_i \operatorname{Lip}(f, x_0) &\leq \liminf_{r \rightarrow 0} \sup \left\{ \frac{|f(x) - f(x_0)|}{r} : 0 < d(x, x_0) < r \right\} \\ &:= \operatorname{lip}(f, x_0) \end{aligned}$$

for almost every  $x_0 \in X_i$ .

**Theorem 10.4.** *A metric measure space  $(X, d, \mu)$  satisfies a Lip-lip condition if and only if every  $\tilde{B}$  subset of  $X$  is  $\mu$ -null.*

*Proof.* First suppose that any  $\tilde{B}$  subset of  $X$  is  $\mu$ -null. Then by Theorem 8.11, there exist a countable Borel decomposition  $X = \cup_i X_i$  and for each  $i \in \mathbb{N}$  a  $\delta_i > 0$  such that, for any Lipschitz  $f: X \rightarrow \mathbb{R}$  and every  $i \in \mathbb{N}$ , there exists an Alberti representation of  $\mu \llcorner X_i$  with speed  $\delta_i$ . In particular, for every  $i \in \mathbb{N}$  and almost every  $x_0 \in X_i$ , there exist  $\gamma \in \Gamma(X)$  and  $t_0$  a density point of  $\operatorname{Dom} \gamma$  such that  $\gamma(t_0) = x_0$  and

$$\delta_i \operatorname{Lip}(f, x_0) \leq (f \circ \gamma)'(t_0).$$

However, since  $t_0$  is a density point of  $\text{Dom } \gamma$ ,

$$(f \circ \gamma)'(t_0) \leq \text{lip}(f, x_0)$$

and so  $X$  satisfies a Lip-lip condition.

Conversely, suppose that  $S \subset X$  belongs to  $\widetilde{B}$ , so that  $S$  belongs to  $\widetilde{A}$ . Then for any  $\epsilon > 0$  we may apply Lemma 6.4 to construct a sequence of functions that satisfy the hypotheses of Lemma 4.2 on some Borel  $S' \subset S$  with  $\mu(S') \geq \mu(S) - \epsilon$ . Therefore, by applying Lemma 4.2 with positive sequences  $R_i, r_i \rightarrow 0$  such that  $r_i/R_i \rightarrow 0$  as  $i \rightarrow \infty$ , for any  $\delta > 0$  we obtain a Lipschitz function  $f: X \rightarrow \mathbb{R}$  with

$$\delta \text{Lip}(f, x_0) > \text{lip}(f, x_0)$$

for almost every  $x_0 \in S'$ . In particular, if  $X$  satisfies a Lip-lip condition,  $S'$  and hence  $S$  are  $\mu$ -null.  $\square$

By combining this theorem with Theorem 8.11, we obtain the following corollary. Very shortly after the first preprint of this paper appeared, Gong gave a second, independent proof of this corollary in [Gon12].

**Corollary 10.5.** *A metric measure space  $(X, d, \mu)$  is a Lipschitz differentiability space if and only if it satisfies a Lip-lip condition and is pointwise doubling.*

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