JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 3, July 2015, Pages 617–667
S 0894-0347(2014)00806-7
Article electronically published on May 22, 2014

SOLUTION TO A NON-ARCHIMEDEAN MONGE-AMPÈRE EQUATION

SÉBASTIEN BOUCKSOM, CHARLES FAVRE, AND MATTIAS JONSSON

Contents

1. Introduction

The goal of this article is to construct continuous solutions to a non-Archimedean analog of certain complex Monge-Ampère equations on projective manifolds, which arose in complex geometry as more degenerate versions of the by-now classical equations considered by Aubin, Calabi, and Yau. More specifically, our main result can be understood as an analog of a fundamental result by S. Kołodziej [Koł98].

Let us briefly recall the complex statement that we have in mind. Let L be an ample line bundle on a smooth complex projective variety X of dimension n. Let μ be a positive measure on X, of mass equal to $c_1(L)^n$. It was shown in [Koł98] that under a mild regularity assumption on μ (which is, for instance, satisfied as soon as μ has L^p-density with respect to Lebesgue measure for some $p > 1$), there exists a continuous metric $\|\cdot\|$ on L, unique up to a multiplicative factor, whose curvature form $c_1(L, \|\cdot\|)$ is a closed positive $(1,1)$-current satisfying $c_1(L, \|\cdot\|)^n = \mu$ in the sense of pluripotential theory [BT82]. This result relied on the work of Aubin, Calabi, and Yau, which culminated in the celebrated article [Yau78], where the

Received by the editors December 30, 2011 and, in revised form, March 20, 2014.

2010 *Mathematics Subject Classification.* Primary 32P05, Secondary 32U05.

The first author was partially supported by the ANR projects MACK and POSITIVE..

The second author was supported by the ANR-grant BERKO, and by the ERC-starting grant project "Nonarcomp" no.307856.

The third author was partially supported by the CNRS and the NSF.

existence of a smooth solution was proved in the case when μ is a smooth positive volume form on X.

We next turn to the non-Archimedean analog, referring to Sec. 2 for more details. Let K be a complete discrete valuation field whose residue field k has characteristic zero, so that $K \simeq k((\varpi))$. Let X be a smooth projective variety over K, and write $n = \dim X$. Thanks to the non-Archimedean GAGA principle, it is reasonable to also denote by X the corresponding K-analytic space in the sense of Berkovich. Its underlying topological space is compact Hausdorff.

A *model* of X is a normal scheme $\mathcal{X}$ that is flat and projective over $S := \operatorname{Spec} k[\![\varpi]\!]$, and whose generic fiber can be identified with X. Consider an ample line bundle L on X. A *model metric* on L is a metric defined by an extension $\mathcal{L} \in \operatorname{Pic}(\mathcal{X})_{\mathbf{Q}}$ of L to some model $\mathcal{X}$. Such a metric is called *semipositive* if $\mathcal{L}$ is nef, i.e., has non-negative degree on all proper curves of the special fiber of $\mathcal{X}$. S.-W. Zhang introduced in [Zha95] the more flexible notion of *semipositive continuous* metric as the uniform limit of semipositive model metrics.[1] In this context, A. Chambert-Loir [CL06] defined the *Monge-Ampère measure* $c_1(L, \|\cdot\|)^n$ of a semipositive continuous metric $\|\cdot\|$ on L. It is a positive Radon measure on X, of mass $c_1(L)^n$.

V. Berkovich constructed in [Ber99] the *skeleton* associated to a polystable model of X. Since we are assuming K to have residue characteristic zero, it is easier to rely on resolution of singularities and instead consider *SNC models*, i.e., models whose special fiber has simple normal crossing support (but is not necessarily reduced, as opposed to a semistable model). To each SNC model $\mathcal{X}$ is associated a *dual complex* $\Delta_{\mathcal{X}}$ that encodes the combinatorics of the intersections of the components of the special fiber, and which embed in the Berkovich space X just as skeletons do. Any finite set of divisorial points is contained in the dual complex of some SNC model; in particular, $\bigcup_{\mathcal{X}} \Delta_{\mathcal{X}}$ is dense in X.

We can now state our main result.

Theorem A. *Let K be a complete discrete valuation field of residue characteristic zero and X a smooth projective K-variety satisfying the condition (†) below.*

Let $L \in \operatorname{Pic}(X)$ be an ample line bundle and μ a positive Radon measure on X of mass $c_1(L)^n$. Assume that μ is supported on the dual complex of some SNC model of X. Then there exists a continuous, semipositive metric $\|\cdot\|$ on L such that

$$c_1(L, \|\cdot\|)^{\dim X} = \mu. \tag{1.1}$$

This metric is unique up to a multiplicative constant.

Our statement relies on the following assumption.

Condition (†). *There exists a smooth projective curve C defined over k, a closed point $q \in C$, and a projective variety Y defined over the function field $k(C)$ such that X is isomorphic to the base change $Y_K = Y \otimes_k \operatorname{Spec} K$, where K is the fraction field of $\widehat{\mathcal{O}}_{C,q}$, the completion of the local ring of C at q.*

When X satisfies this condition, we will say for simplicity that it is defined over a function field.

[1] We refer to Table 1 in Sec. 2.6 below for a comparison of our terminology with existing terminology.

As we will explain below, this condition plays an essential role in our proof, even though the result is most likely true for all smooth projective K-varieties. Note that the line bundle L is not assumed to be defined over a function field.

The uniqueness part in Theorem A follows from a result of X. Yuan and S.-W. Zhang [YZ13a, Theorem 1.5] (see also [YZ13b]). Their result applies to an arbitrary complete non-Archimedean field K and says, in our language, that a continuous semipositive metric is uniquely determined, up to a constant, by its Monge-Ampère measure. Their proof is inspired by the one given by Błocki [Bło03] in the complex setting.

Our approach does not give any information on the regularity of the metric besides continuity. It would be interesting to further investigate this issue, for instance when μ is supported on finitely many divisorial points. We refer to Sec. 9 for a discussion of this problem in the case of toric varieties, based on the recent work [BPS13].

Versions of Theorem A are already known in a few cases. For curves (and in fact over any complete non-Archimedean, non-trivially valued field), it can easily be deduced from results of A. Thuillier [Thu05], who developed a theory of singular semipositive metrics on analytic curves that is completely analogous to the complex case. Solving (1.1) for curves boils down to a system of linear equations and relies on the negativity of the intersection form of the special fiber of a suitable model; see Sec. 9. Alternatively, one can exploit the structure of the Berkovich space as a metrized graph as in [BR10, FJ04].

In higher dimensions, Y. Liu [Liu11] treated the related case when X is a totally degenerate abelian variety over $\mathbf{C}_p$, and μ is a (smooth) measure supported on the dual complex of the canonical formal model of X, as constructed by Mumford. By exploiting the fact that this dual complex is a compact (real) torus, one can translate the equation $c_1(L, \|\cdot\|)^n = \mu$ into a (real) Monge-Ampère equation on this real torus, and apply Yau's result to its complexification to obtain the metric.

A statement very close to Theorem A also appears in an unpublished set of notes by M. Kontsevich and Y. Tschinkel [KT] dating from 2001, where the authors propose a detailed strategy of proof in the case μ is a Dirac mass at a divisorial point. Several ingredients in their approach also appear in this article (see Remark 8.7 below).

We are now going to present an outline of our proof of Theorem A, which consists in mimicking as far as possible the variational approach to complex Monge-Ampère equations of [BBGZ13] and the C^0-estimates of [Koł98]. To that end we will rephrase Theorem A in a more analytic language. Let us thus recall the notion of quasi-plurisubharmonic function that we developed in [BFJ09] and its main properties.

As a variant of [BGS95] we first define the space of *closed* $(1,1)$-*forms* on X as the direct limit

$$\mathcal{Z}^{1,1}(X) := \varinjlim_{\mathcal{X}} N^1(\mathcal{X}/S),$$

where $\mathcal{X}$ ranges over all models of X and the space of *numerical classes* $N^1(\mathcal{X}/S)$ is defined as $\mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$ modulo numerical equivalence on the special fiber. Each closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ defines a class $\{\theta\} \in N^1(X)$, which we refer to as its *de Rham class*. We say that $\theta \in \mathcal{Z}^{1,1}(X)$ is *semipositive* if it is determined by a nef numerical class on some model. Each model metric $\|\cdot\|$ on a line bundle L

over X defines a closed $(1,1)$-form $c_1(L, \|\cdot\|)$ that we call the *curvature form* of the metric. The de Rham class of $c_1(L, \|\cdot\|)$ is just $c_1(L) \in N^1(X)$, and the model metric $\|\cdot\|$ is semipositive (in the sense of Zhang) iff its curvature is. Each model metric on the trivial line bundle is of the form $e^{-\varphi}$ for some $\varphi \in C^0(X)$, which is then by definition a *model function.* Following complex notation, we write $dd^c\varphi$ for the curvature form of this metric, so that $c_1(L, \|\cdot\|e^{-\varphi}) = c_1(L, \|\cdot\|) + dd^c\varphi$.

Now let $\omega \in \mathcal{Z}^{1,1}(X)$ be a reference closed semipositive $(1,1)$-form on X, such that $\{\omega\} \in N^1(X)$ is furthermore ample. This situation arises for instance when ω is the curvature form of a semipositive model metric on an ample line bundle L. As was shown in [BFJ09], one may then define a class $\mathrm{PSH}(X,\omega)$ of ω-psh functions with the following properties.

- Each $\varphi \in \mathrm{PSH}(X,\omega)$ is an upper semicontinuous function $X \to [-\infty, +\infty[$ whose restriction to the faces of any dual complex is continuous and convex.
- The set $\mathrm{PSH}(X,\omega)$ is convex and stable under max.
- A model function φ is ω-psh iff $\omega + dd^c\varphi \in \mathcal{Z}^{1,1}(X)$ is semipositive.

The two main results of [BFJ09] further state that:

- $\mathrm{PSH}(X,\omega)/\mathbf{R}$ is compact with respect to the topology of uniform convergence on dual complexes; and
- every $\varphi \in \mathrm{PSH}(X,\omega)$ is the decreasing limit of a family of ω-psh model functions.

It follows from the latter property and Dini's lemma that every *continuous* ω-psh function is a *uniform* limit over X of ω-psh model functions. This shows, in particular, that our definition of continuous semipositive metrics is compatible with Zhang's. Chambert-Loir's definition of the Monge-Ampère measure of a continuous semipositive metric immediately extends to our setting and enables us to associate to any n-tuple of *continuous* ω-psh functions $\varphi_1, \dots, \varphi_n \in C^0(X) \cap \mathrm{PSH}(X,\omega)$ a (mixed) Monge-Ampère measure

$$(\omega + dd^c\varphi_1) \wedge \cdots \wedge (\omega + dd^c\varphi_n),$$

a positive Radon measure on X of mass $\{\omega\}^n$, which depends continuously on $(\varphi_1, \dots, \varphi_n)$ with respect to the topology of uniform convergence on X. As in the complex case, it is, however, not possible to define such mixed Monge-Ampère measures in a reasonable way for arbitrary ω-psh functions, as soon as $n \geq 2$.

The following result is a slight generalization of Theorem A phrased in the present language.

Theorem A'. *Let X be smooth projective K-variety of dimension n satisfying condition $(\dagger)$ as in Theorem A. Let $\omega \in \mathcal{Z}^{1,1}(X)$ be a closed semipositive $(1,1)$-form such that $\{\omega\} \in N^1(X)$ is ample and let μ be a positive Radon measure on X of mass $\{\omega\}^n$. If μ is supported on a dual complex then there exists a continuous ω-psh function φ such that*

$$(\omega + dd^c\varphi)^n = \mu. \tag{1.2}$$

The function φ is furthermore unique up to an additive constant.

This formulation is designed to emphasize the analogy with the complex case. However, it is important to keep in mind that the non-Archimedean Monge-Ampère operator is not a differential operator but rather defined in terms of intersection theory.

Let us now set up the variational approach we use to solve our non-Archimedean Monge-Ampère equation, following [BBGZ13]. A key feature of Monge-Ampère equations is that they may be written as Euler-Lagrange equations. This fact goes back at least to Alexandrov [Ale38] in the more classical case of real Monge-Ampère equations, while the relevant functional in the complex case has been well known in Kähler geometry since the works of Aubin, Calabi, and Yau. We introduce in our setting the *energy functional*

$$E(\varphi) := \frac{1}{n+1} \sum_{j=0}^{n} \int \varphi \, (\omega + dd^c\varphi)^j \wedge \omega^{n-j}, \tag{1.3}$$

defined for the moment for $\varphi \in C^0(X) \cap \mathrm{PSH}(X, \omega)$. An easy computation shows that

$$\left.\frac{d}{dt}\right|_{t=0_+} E((1-t)\varphi + t\psi) = \int (\psi - \varphi) \, (\omega + dd^c\varphi)^n \tag{1.4}$$

for any two $\varphi, \psi \in C^0(X) \cap \mathrm{PSH}(X, \omega)$, so that (1.2) is indeed the Euler-Lagrange equation of the functional

$$F_\mu(\varphi) := E(\varphi) - \int \varphi \, d\mu.$$

Observe that the compatibility condition $\mu(X) = \{\omega\}^n$ guarantees that F_μ is translation-invariant, i.e., $F_\mu(\varphi + c) = F_\mu(\varphi)$ for all $c \in \mathbf{R}$. As in the complex case, one shows that the functional E is concave on $C^0(X) \cap \mathrm{PSH}(X, \omega)$, so that any solution φ to (1.2) is necessarily a maximizer of F_μ. The variational method conversely amounts to proving the existence of a maximizer of F_μ and showing that it satisfies (1.2). But the lack of compactness of the space $C^0(X) \cap \mathrm{PSH}(X, \omega)$ where F_μ is defined so far makes it hard to construct a maximizer, while it is at any rate non-obvious that such a maximizer should satisfy the Euler-Lagrange equation, since it might belong to the boundary of $C^0(X) \cap \mathrm{PSH}(X, \omega)$. In order to circumvent these difficulties we are going to argue along the following three steps.

Step 1: Enlarge the space where the variational problem is being considered, in order to gain compactness and construct a maximizer φ_0 there.

Step 2: Show that the maximizer is in a natural way a generalized solution of the non-Archimedean Monge-Ampère equation (1.2).

Step 3: Show the regularity (i.e., continuity) of this generalized solution using capacity estimates.

The general strategy for Steps 1 and 2 follows [BBGZ13], whereas Step 3 follows [Koł98].

The condition that μ is supported on a dual complex makes Step 1 relatively easy in our case, granted the compactness property of $\mathrm{PSH}(X, \omega)/\mathbf{R}$ proved in [BFJ09]. Indeed, the support condition guarantees that the linear part $\varphi \mapsto \int \varphi \, d\mu$ of F_μ is finite valued and continuous on all of $\mathrm{PSH}(X, \omega)$. For this reason, several complications that occurred in [BBGZ13] to handle general measures disappear, since it is enough to extend E to a usc functional $E : \mathrm{PSH}(X, \omega) \to [-\infty, +\infty[$, which is done by setting

$$E(\varphi) := \inf \left\{ E(\psi) \mid \psi \geq \varphi, \ \psi \in C^0(X) \cap \mathrm{PSH}(X, \omega) \right\}.$$

Step 2 requires much more work and constitutes the main body of the article, in particular because virtually none of the more classical results in pluripotential theory on which [BBGZ13] was able to rely were available so far in our non-Archimedean context. Very recently, however, Chambert-Loir and Ducros have introduced a formalism for forms and currents on Berkovich spaces, see [CLD12, Gub13], that might shed some new light on this problem.

The only obvious information we have on the maximizer φ_0 of F_μ is that it lies in the set

$$\mathcal{E}^1(X, \omega) := \{\varphi \in \mathrm{PSH}(X, \omega),\ E(\varphi) > -\infty\}$$

of *ω-psh functions with finite energy.* In the complex case, $\mathcal{E}^1(X, \omega)$ was introduced in [Ceg98, GZ07] as a higher dimensional and non-linear generalization of the classical Dirichlet space from potential theory. The goal of Step 2 is to show that the Monge-Ampère operator can be naturally extended to $\mathcal{E}^1(X, \omega)$, and that φ_0 satisfies

$$(\omega + dd^c\varphi_0)^n = \mu$$

in this generalized sense.

In order to do so, we first extend the Monge-Ampère operator from continuous to bounded ω-psh functions, following the fundamental work of Bedford and Taylor [BT82, BT87]. As in the complex case, this mild generalization is in fact crucial in order to develop a reasonable capacity theory, and also because the natural bounded approximants $\max\{\varphi, -m\}$, $m \in \mathbf{N}$, of a given ω-psh function φ are not continuous in general. The bounded case is, however, substantially more involved than the continuous case, since uniform convergence has to be replaced with monotone convergence. The fact that any (bounded) ω-psh function can be written as a decreasing limit of a family of ω-psh model functions, proved in [BFJ09], plays a key role at this stage.

Of crucial importance is the following *locality property* of the Monge-Ampère operator: if φ, ψ are bounded ω-psh functions, then the restrictions of the measures $(\omega + dd^c \max\{\varphi, \psi\})^n$ and $(\omega + dd^c\varphi)^n$ to the Borel set $\{\varphi > \psi\}$ coincide. Note that, even when φ, ψ are model functions, this fact is not immediately clear from the definition in terms of intersection numbers.

Next, we further extend the Monge-Ampère operator from bounded ω-psh functions to functions with finite energy. The key observation, which goes back to [BT87], is the monotonicity of the sequence of measures

$$\mathbf{1}_{\{\varphi > -m\}}\,(\omega + dd^c \max\{\varphi, -m\})^n\ \ (m \in \mathbf{N})$$

a direct consequence of the locality property. This allows us to define $(\omega + dd^c\varphi)^n$ as the increasing limit of this sequence of measures, which is shown to be well behaved for $\varphi \in \mathcal{E}^1(X, \omega)$. More generally, mixed Monge-Ampère measures are shown to be well defined for functions in $\mathcal{E}^1(X, \omega)$, and (1.3), (1.4) are still valid in this generality.

As was already pointed out, these facts are however *a priori* not enough to show that the maximizer φ_0 of F_μ satisfies $(\omega + dd^c\varphi_0)^n = \mu$, because small perturbations of φ_0 cease to be ω-psh in general. In order to handle a similar difficulty in the setting of real Monge-Ampère equations, Alexandrov devised in [Ale38] an envelope argument, an analog of which was subsequently explored in the complex case in [BBGZ13]. Following the same lead, we introduce the *ω-psh envelope* $P(f)$

of a given continuous function f on X by setting for each $x \in X$:

$$P(f)(x) := \sup \{\varphi(x) \mid \varphi \in \mathrm{PSH}(X, \omega),\, \varphi \leq f\}\,.$$

It follows from [BFJ09] that $P(f)$ is the largest ω-psh function dominated by f on X. The key point is then the following *differentiability* property, whose complex analog was established in [BB10]:

$$\left.\frac{d}{dt}\right|_{t=0} E \circ P\,(f + tg) = \int_X g\,(\omega + dd^c P(f))^n \tag{1.5}$$

for any two $f, g \in C^0(X)$, which may more vividly be written as the chain rule-like formula $(E \circ P)' = E' \circ P$. Granted (1.5), a fairly direct argument based on the monotonicity of E implies $(\omega + dd^c\varphi_0)^n = \mu$ as desired.

The proof of (1.5) can be reduced by elementary arguments to the differentiability of $t \mapsto \int P(f + tg)\,(\omega + dd^c P(f))^n$, which, in turn, ultimately is a consequence of the following *orthogonality property*:

$$\int_X (f - P(f))\,(\omega + dd^c P(f))^n = 0. \tag{1.6}$$

Since $f \geq P(f)$, this relation means that $(\omega + dd^c P(f))^n$ is supported on the contact locus $\{f = P(f)\}$, a well-known fact in the complex case where the proof argues by balayage, using Bedford and Taylor's solution to the Dirichlet problem for the homogeneous complex Monge-Ampère equation on the ball. Such an approach seems far beyond reach in the non-Archimedean case. We proceed instead by translating (1.6) into an intersection theoretic statement on a model of X, where it boils down to the orthogonality of relative asymptotic Zariski decompositions for a line bundle that is ample on the generic fiber. It is precisely at this point that we use our condition (†) that X is defined over a function field. Indeed, this allows us to choose the model where we work to be algebraic, and therefore compactifiable into a projective variety over the residue field k. As explained in Appendix A, we can then reduce to the absolute case of big line bundles on projective varieties treated in [BDPP13].

Finally, Step 3 is handled by adapting in a fairly direct manner the capacity estimates of Kołodziej [Koł98, Koł03] to prove that φ_0 is actually continuous. The proof relies on the locality property in $\mathcal{E}^1(X, \omega)$. This shows the existence part of Theorem A'. Uniqueness is proved following [Bło03], as in [YZ13a].

Our result is not optimal, and we next discuss three important assumptions that we use in Theorems A and A'.

First, the condition that the measure μ be supported on a dual complex is probably unnecessarily strong. Relying on ideas of Cegrell [Ceg98], Guedj and Zeriahi [GZ07] defined in the case of compact Kähler manifolds a class $\mathcal{E}(X, \omega)$ of ω-psh functions where the Monge-Ampère operator is well defined and such that the measures $(\omega + dd^c\varphi)^n$, $\varphi \in \mathcal{E}(X, \omega)$ are exactly the positive measures μ on X giving zero mass to pluripolar[2] sets. The function φ is here again uniquely determined up to an additive constant by its Monge-Ampère measure, as was later shown by Dinew [Din09]. We expect the corresponding results to be true in our setting, too. The proof would probably require an even more systematic development of pluripotential theory in a non-Archimedean setting, something that is certainly of interest.

[2]A subset set $A \subset X$ is pluripolar if there exists an ω-psh function φ such that $A \subset \{\varphi = -\infty\}$.

Second, as explained above, the proof of the orthogonality property (1.6) relies in a crucial way on the assumption that X is defined over a function field. It would be interesting to drop this condition, which we expect to be an unnecessary restriction.

Finally, our variational approach uses the compactness of the space $\mathrm{PSH}(X, \omega)/\mathbf{R}$, which was obtained in [BFJ09]. The proof of this fact relied heavily on the existence of SNC models, which are so far only available in residue characteristic zero. It seems to be a challenging task to extend our methods and results to local fields and more general complete non-Archimedean fields. See [FJ04, BFJ08] for related work in the case of a trivially valued field.

Let us end this introduction by indicating the structure of the article.

In Sec. 2, we give the necessary background on Berkovich spaces, metrized line bundles, ω-psh functions and wedge-products of closed $(1,1)$-forms. We also recall some facts from measure theory.

The next three sections, Secs. 3–5, develop some of the basic Bedford-Taylor theory in our non-Archimedean setting. The definition of the Monge-Ampère operator on bounded functions and the continuity along decreasing families is carried out in Sec. 3. In Sec. 4, we introduce a Monge-Ampère capacity used to measures the size of subsets of X. We obtain the important result that any ω-psh function is quasicontinuous, i.e., continuous outside a set of arbitrarily small capacity. We also strengthen the regularization theorem of [BFJ09] and prove that any ω-psh function is a decreasing limit of a (countable) sequence of ω-psh model functions. Finally, in §5 we prove the locality property. The results in Secs. 3–5 and even some of the proofs parallel those in complex analysis (especially the ones on compact Kähler manifolds, see [GZ05]). However, the non-Archimedean results ultimately originate in basic properties of the intersection form on models whereas the basic results in the complex case concern differential operators.

The energy of an ω-psh function is introduced in Sec. 6. Following [Ceg98, GZ07], we extend the Monge-Ampère operator to the class $\mathcal{E}^1(X, \omega)$ of ω-psh functions with finite energy and prove that the locality property continues to hold.

In Sec. 7, we introduce ω-psh envelopes and reduce the differentiability property to the orthogonality property. This reduction, which follows [BB10] uses a version of the classical *comparison principle* which, in turn, is a consequence of the locality property mentioned above. The proof of the orthogonality property itself is relegated to Appendix A since it relies on methods that are quite different from those in the rest of the article.

Granted the differentiability result, we follow [BBGZ13] and prove Theorem A' in Sec. 8. We also explain how to get Theorem A from Theorem A'. Finally, in Sec. 9 we discuss the case of curves and toric varieties.

2. Background

For this section we refer to our companion paper [BFJ09] for details and further references.

2.1. **Berkovich space and models.** Let R be a complete discrete valuation ring with fraction field K and residue field k. We will assume that k has characteristic zero. We let $\varpi \in R$ be a uniformizing parameter and normalize the corresponding absolute value on K by $\log|\varpi|^{-1} = 1$. Note that $R \simeq k[\![\varpi]\!]$ and $K \simeq k(\!(\varpi)\!)$, see for instance [Ser68]. Write $S := \operatorname{Spec} R$.

Let X be a smooth projective K-variety, i.e., an integral (but not necessarily geometrically integral) smooth projective K-scheme. A *model* of X is a normal, flat and projective S-scheme $\mathcal{X}$, together with an isomorphism of its generic fiber onto X. We denote by $\mathcal{X}_0$ its special fiber, and by $\mathrm{Div}_0(\mathcal{X})$ the group of *vertical Cartier divisors*, i.e., those supported on $\mathcal{X}_0$. We write $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}$ accordingly.

Let $\mathcal{M}_X$ be the set of all isomorphism classes of models of X. Given $\mathcal{X}', \mathcal{X}$ in $\mathcal{M}_X$ we write $\mathcal{X}' \geq \mathcal{X}$ if there exists a morphism $\mathcal{X}' \to \mathcal{X}$ obtained by blowing up an ideal sheaf co-supported on the special fiber of $\mathcal{X}$. This turns $\mathcal{M}_X$ into a directed set.

Given a model $\mathcal{X}$, let $(E_i)_{i\in I}$ be the set of irreducible components of the special fiber. For each subset $J \subset I$ set $E_J := \bigcap_{j\in J} E_j$. A regular model $\mathcal{X}$ is an *SNC model* if the special fiber has simple normal crossing support and E_J is irreducible (or empty) for each $J \subset I$. By Hironaka's Theorem, every model is dominated by an SNC model.

As a topological space, the Berkovich space X^{an} attached to the given smooth projective K-variety X is compact and can be described as follows (cf. [Ber90, Theorem 3.4.1]). Choose a finite cover of X by affine open subsets of the form $U = \operatorname{Spec} A$ where A is a finitely generated K-algebra. The Berkovich space U^{an} is defined as the set of all multiplicative seminorms on A extending the given absolute value of K, endowed with the topology of pointwise convergence. The space X^{an} is obtained by gluing the open sets U^{an}.

There is a natural equivalence of categories between projective K-analytic spaces and projective K-schemes, see [Ber90, Sec. 3.4]. In the sequel we will therefore always identify a projective K-scheme with its associated Berkovich space and write $X^{\mathrm{an}} = X$.

Let $\mathcal{X}$ be a model of X. To each irreducible component E of the special fiber is associated a divisorial valuation ord_E of the function field of X. After rescaling and exponentiating, this gives rise to an element $x_E \in X$ called a *divisorial point*. The set X^{div} of divisorial points is dense in X.

When $\mathcal{X}$ is an SNC model, we can refine this construction. Write the special fiber as $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$. The *dual complex* $\Delta_{\mathcal{X}}$ of $\mathcal{X}$ is the simplicial complex whose vertices correspond to the irreducible components E_i and whose simplices correspond to nonempty intersections E_J. We can equip $\Delta_{\mathcal{X}}$ with an (integral) affine structure and embed it in the Berkovich space X as follows.

Consider a subset $J \subset I$ with $E_J \neq \emptyset$ and pick $s = (s_j)_{j\in J} \in \mathbf{R}^J$ with $s_j \geq 0$ and $\sum_{j\in J} b_j s_j = 1$. Let ξ_J be the generic point of E_J and pick a system $(z_j)_{j\in J}$ of regular parameters for $\mathcal{O}_{\mathcal{X},\xi_J}$ with z_j defining E_j. By Cohen's structure theorem, $\widehat{\mathcal{O}}_{\mathcal{X},\xi_J} \simeq \kappa(\xi_J)[\![z_j, j \in J]\!]$. Let $\mathrm{val}_{J,s}$ be the restriction to $\mathcal{O}_{\mathcal{X},\xi}$ of the monomial valuation on this power series ring, taking value s_j on z_j, i.e., $\mathrm{val}_{J,s}\left(\sum_{\alpha\in\mathbf{N}^J} c_\alpha z^\alpha\right) = \min\left\{\sum_{j\in J} s_j\alpha_j \mid c_\alpha \neq 0\right\}$. Then $e^{-\mathrm{val}_{J,s}} \in X$. This defines an embedding $\Delta_{\mathcal{X}} \hookrightarrow X$, and the parameters s equip $\Delta_{\mathcal{X}}$ with an affine structure. In order to keep notation light, we will identify $\Delta_{\mathcal{X}}$ with its image in X under this embedding. Note that this convention differs from the one adopted in [BFJ09].

There is also a *retraction* $p_{\mathcal{X}} : X \to \Delta_{\mathcal{X}} \subset X$, defined as follows. Any point $x \in X$ admits a *center* on $\mathcal{X}$. This is the unique point $\xi = c_{\mathcal{X}}(x) \in \mathcal{X}_0$ such that $|f(x)| \leq 1$ for $f \in \mathcal{O}_{\mathcal{X},\xi}$ and $|f(x)| < 1$ for $f \in \mathfrak{m}_{\mathcal{X},\xi}$. Let $J \subset I$ be the maximal subset such that $\xi \in E_J$. Then $p_{\mathcal{X}}(x) \in \Delta_{\mathcal{X}}$ corresponds to the monomial valuation with weight $-\log|z_j(x)|$, $j \in J$.

We have $p_{\mathcal{X}} = \mathrm{id}$ on $\Delta_{\mathcal{X}}$. If $\mathcal{Y}$ dominates $\mathcal{X}$, then $\Delta_{\mathcal{X}} \subset \Delta_{\mathcal{Y}}$ and $p_{\mathcal{X}} \circ p_{\mathcal{Y}} = p_{\mathcal{X}}$. The retractions induce a homeomorphism of X onto the inverse limit $\varprojlim \Delta_{\mathcal{X}}$.

2.2. **Model functions.** Let $\mathcal{X}$ be a model of X. A vertical fractional ideal sheaf $\mathfrak{a}$ is a finitely generated $\mathcal{O}_{\mathcal{X}}$-submodule of the function field of $\mathcal{X}$ such that $\mathfrak{a}|_X = \mathcal{O}_X$. Then $\mathfrak{a}$ defines a continuous function $\log|\mathfrak{a}| \in C^0(X)$ by setting

$$\log|\mathfrak{a}|(x) := \max\left\{\log|f(x)| \mid f \in \mathfrak{a}_{c_{\mathcal{X}}(x)}\right\}.$$

Note that each vertical Cartier divisor $D \in \mathrm{Div}_0(\mathcal{X})$ defines a vertical fractional ideal sheaf $\mathcal{O}_{\mathcal{X}}(D)$, hence a continuous function $f_D := \log|\mathcal{O}_{\mathcal{X}}(D)|$. Note that $f_{\mathcal{X}_0}$ is the constant function 1 since $\log|\varpi|^{-1} = 1$. The map $D \mapsto f_D$ extends by linearity to $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}} \to C^0(X)$.

Definition 2.1. *A function f on X is a* model function *if there exists a model $\mathcal{X}$ and a $\mathbf{Q}$-divisor $D \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ such that $f = f_D$. We then call $\mathcal{X}$ a* determination *of f. We let $\mathcal{D}(X) = \mathcal{D}(X)_{\mathbf{Q}}$ be the space of model functions on X.*

Proposition 2.2. [BFJ09, Corollary 2.3, Corollary 3.13] *The $\mathbf{Q}$-vector space $\mathcal{D}(X)$ of model functions is stable under max and (hence) dense in $C^0(X)$. If f is a model function and $\mathcal{X}$ is a determination of f, then f is affine on each face of $\Delta_{\mathcal{X}}$.*

Here, the density of $\mathcal{D}(X)$ follows from the 'boolean ring' version of the Stone-Weierstrass theorem.

2.3. **Forms and de Rham classes.** Let $\mathcal{X}$ be a model of X. The space $N^1(\mathcal{X}/S)$ of (relative, codimension 1) numerical equivalence classes on $\mathcal{X}$ is defined as the quotient of $\mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$ by the subspace spanned by numerically trivial line bundles, i.e. those $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{R}}$ such that $\mathcal{L} \cdot C = 0$ for all projective curves contained in a fiber of $\mathcal{X} \to S$. It is in fact enough to consider vertical curves, i.e., those contained in the special fiber $\mathcal{X}_0$. A class $\theta \in N^1(\mathcal{X}/S)$ is *nef* if $\theta \cdot C \geq 0$ for all such curves C.

Definition 2.3. *The space of* closed $(1,1)$-forms *on X is defined as the direct limit*

$$\mathcal{Z}^{1,1}(X) := \varinjlim_{\mathcal{X} \in \mathcal{M}_X} N^1(\mathcal{X}/S).$$

We say that a closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ is *determined* on a given model $\mathcal{X}$ if it is the image of an element $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$. By definition, two classes $\theta \in N^1(\mathcal{X}/S)$ and $\theta' \in N^1(\mathcal{X}'/S)$ define the same element in $\mathcal{Z}^{1,1}(X)$ iff they pull back to the same class on a model dominating both $\mathcal{X}$ and $\mathcal{X}'$.

Definition 2.4. *A closed $(1,1)$-form $\theta \in \mathcal{Z}^{1,1}(X)$ is* semipositive *if $\theta_{\mathcal{X}} \in N^1(\mathcal{X}/S)$ is nef for some (or, equivalently, any) determination $\mathcal{X}$ of θ.*

The natural map $N^1(\mathcal{X}/S) \to N^1(X)$ gives rise to a map $\mathcal{Z}^{1,1}(X) \to N^1(X)$ which in fact is surjective. We refer to $\{\theta\}$ as the *de Rham class* of the closed $(1,1)$-form θ. When θ is semipositive, the de Rham class $\{\theta\} \in N^1(X)$ is nef on X. In what follows, we will mainly work with forms having ample de Rham classes.

Any model function $f \in \mathcal{D}(X)$ induces a form $dd^c f \in \mathcal{Z}^{1,1}(X)$ as follows: for any determination $\mathcal{X}$ of f, $dd^c f$ is the class of the divisor $\sum_{i \in I} b_i f(x_i) E_i$, where $\mathcal{X}_0 = \sum_i b_i E_i$ and $x_i \in X$ is the divisorial point associated to E_i.

2.4. **θ-psh functions.** Fix a form $\theta \in \mathcal{Z}^{1,1}(X)$ with ample de Rham class $\{\theta\} \in N^1(X)$.

Definition 2.5. *A θ-psh function $\varphi : X \to [-\infty, +\infty)$ is an usc function such that for each SNC model $\mathcal{X}$ of X on which θ is determined we have:*

(i) *$\varphi \leq \varphi \circ p_{\mathcal{X}}$ on X; and*
(ii) *the restriction of φ to the dual complex $\Delta_{\mathcal{X}}$ is a uniform limit of restrictions of model functions ψ such that $\theta + dd^c\psi$ is a semipositive form.*

We write $\mathrm{PSH}(X, \theta)$ for the set of θ-psh functions on X.

It is a non-trivial fact that if φ is a θ-psh model function then the form $\theta + dd^c\varphi$ is in fact semipositive; see [BFJ09, Theorem 5.11]. In particular, a constant function is θ-psh iff θ is semipositive. In this case, $\max\{\varphi, c\}$ is θ-psh when φ is θ-psh and $c \in \mathbf{R}$.

Proposition 2.6. [BFJ09, Proposition 5.10]. *The space of model functions $\mathcal{D}(X)$ is spanned by θ-psh model functions.*

Proposition 2.7. [BFJ09, Proposition 7.4]. *The set $\mathrm{PSH}(X, \theta)$ is convex. If φ, ψ are θ-psh and $c \in \mathbf{R}$, then the functions $\max\{\varphi, \psi\}$ and $\varphi + c$ are also θ-psh.*

Proposition 2.8. [BFJ09, Proposition 7.5]. *Any $\varphi \in \mathrm{PSH}(X, \theta)$ is continuous on the dual complex of any SNC model $\mathcal{X}$, and convex on each of its faces.*

In fact, the continuity statement above can be made *uniform* in φ.

Theorem 2.9. [BFJ09, Corollary 7.8] *For any SNC model $\mathcal{X}$, the restrictions of all θ-psh functions to the dual complex $\Delta_{\mathcal{X}}$ form an equicontinuous family.*

We endow $\mathrm{PSH}(X, \theta)$ with the topology of uniform convergence on dual complexes. Notice that the divisorial points are dense on each dual complex Δ_X; see [BFJ09, Corollary 3.17] or [JM12, Remark 3.9]. As a consequence of equicontinuity, we thus have the following.

Theorem 2.10. [BFJ09, Theorem 7.10]. *For each model function ψ the map $\varphi \mapsto \sup_X(\varphi - \psi)$ is continuous and proper on $\mathrm{PSH}(X, \theta)$. In particular, the space $\mathrm{PSH}(X, \theta)/\mathbf{R}$ is compact. Further, the topology on $\mathrm{PSH}(X, \theta)$ is equivalent to the topology of pointwise convergence on X^{div}.*

Finally, we have the following regularization result. Its proof relies on multiplier ideals.

Theorem 2.11. [BFJ09, Theorem 8.7]. *For any θ-psh function φ, there exists a decreasing net $(\varphi_j)_j$ of θ-psh model functions that converges pointwise on X to φ.*

The complex analog of this result is due to Demailly [Dem92] (see also [GZ05, Appendix] and [BK07]). By Dini's lemma, we get the following as a consequence.

Corollary 2.12. [BFJ09, Corollary 8.8] *The set $\mathcal{D}(X) \cap \mathrm{PSH}(X, \theta)$ is dense in $C^0(X) \cap \mathrm{PSH}(X, \theta)$ with respect to uniform convergence on X.*

Proposition 4.7 below refines Theorem 2.11 and asserts that any θ-psh function is actually the decreasing limit of a *sequence* of θ-psh model functions (but the proof heavily uses Theorem 2.11).

2.5. **Envelopes.** Let θ be a form as in Sec. 2.4.

Proposition 2.13. [BFJ09, Theorem 7.11]. *If $(\varphi_\alpha)_{\alpha\in A}$ is a family of θ-psh functions that is uniformly bounded above, then the usc upper envelope $(\sup_\alpha \varphi_\alpha)^*$ is also θ-psh.*

Recall that the usc regularization u^* of a function $u : X \to [-\infty, +\infty[$ is the smallest usc function such that $u^* \geq u$.

Definition 2.14. *Let $f : X \to [-\infty, +\infty)$ be any function. We define its θ*-psh envelope *$P_\theta(f)$ as follows. If there does not exist any $\varphi \in \mathrm{PSH}(X, \theta)$ such that $\varphi \leq f$ on X then we set $P_\theta(f) \equiv -\infty$. Otherwise, we define $P_\theta(f)$ as the usc upper envelope of the set of all θ-psh functions φ such that $\varphi \leq f$ on X, i.e., we set*

$$P_\theta(f) := (\sup\{\varphi \mid \varphi \in \mathrm{PSH}(X, \omega),\, \varphi \leq f\})^*.$$

Thanks to Proposition 2.13 $P_\theta(f)$ is either $-\infty$ or belongs to $\mathrm{PSH}(X, \theta)$. If f is usc, then clearly $P_\theta(f) \leq f$ on X, and $P_\theta(f)$ is then the largest θ-psh function with this property.

Proposition 2.15. [BFJ09, Proposition 8.2]

(i) *P_θ is non-decreasing: $f \leq g \Rightarrow P_\theta(f) \leq P_\theta(g)$.*

(ii) *$P_\theta(f)$ is concave in both arguments:*

$$P_{t\theta+(1-t)\eta}(tf + (1-t)g) \geq tP_\theta(f) + (1-t)P_\eta(g)$$

for $0 \leq t \leq 1$.

(iii) *For each $c \in \mathbf{R}$ we have $P_\theta(f + c) = P_\theta(f) + c$.*

(iv) *P_θ is 1-Lipschitz continuous, i.e., $\sup_X |P_\theta(f) - P_\theta(g)| \leq \sup_X |f - g|$.*

(v) *Given a bounded function f, a determination $\mathcal{X}$ of θ, and a convergent sequence $\theta_m \to \theta$ in $N^1(\mathcal{X}/S)$, we have $P_{\theta_m}(f) \to P_\theta(f)$ uniformly on X.*

2.6. **Metrized line bundles and curvature forms.** We refer to [CL11] for a general account of metrized line bundles in a non-Archimedean context. Suffice it to say, a metric $\|\cdot\|$ on a line bundle L on X is a way to produce a local continuous function $\|s\|$ on (the Berkovich space) X from any local section s of L.

Let $\mathcal{X}$ be a model and $\mathcal{L}$ a line bundle on $\mathcal{X}$ such that $\mathcal{L}|_X = L$. To this data one can associate a unique metric $\|\cdot\|_{\mathcal{L}}$ on L with the following property: if s is a non-vanishing local section of $\mathcal{L}$ on an open set $\mathcal{U} \subset \mathcal{X}$, then $\|s\|_{\mathcal{L}} \equiv 1$ on $U := \mathcal{U} \cap X$. This makes sense since such a section s is uniquely defined up to multiplication by an element of $\Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{X}})$ and such elements have norm 1.

More generally, any $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ such that $\mathcal{L}|_X = L$ in $\mathrm{Pic}(X)_{\mathbf{Q}}$ induces a metric $\|\cdot\|_{\mathcal{L}}$ on L by setting $\|s\|_{\mathcal{L}} = \|s^{\otimes m}\|^{1/m}_{m\mathcal{L}}$ for any $m \in \mathbf{N}^*$ such that $m\mathcal{L}$ is an actual line bundle. Such a metric is called a *model metric* on L.

Given a model metric $\|\cdot\|$, any other continuous metric on L is of the form $\|\cdot\|e^{-\varphi}$, with $\varphi \in C^0(X)$. This is a model metric iff φ is a model function. By a *singular metric* on L we mean an expression of the form $\|\cdot\|e^{-\varphi}$ with $\varphi : X \to [-\infty, +\infty)$ an arbitrary function.

Fix a model metric $\|\cdot\|_{\mathcal{L}}$ on L associated to $\mathcal{L} \in \mathrm{Pic}_{\mathbf{Q}}(\mathcal{X})$. The numerical class associated to $\mathcal{L}$ in $N^1(\mathcal{X}/S)$ induces a form on X in the sense of Sec. 2.3. It does not depend on the choice of model $\mathcal{L}$ defining the metric. We call it the *curvature form* of the metric and denote it by $c_1(L, \|\cdot\|)$. By construction, its de Rham class is given by

$$\{c_1(L, \|\cdot\|)\} = c_1(L) \in N^1(X). \tag{2.1}$$

If $\varphi \in \mathcal{D}(X)$ is a model function, then

$$c_1(L, \|\cdot\| e^{-\varphi}) = c_1(L, \|\cdot\|) + dd^c\varphi,$$

where the form $dd^c\varphi \in \mathcal{Z}^{1,1}(X)$ is defined in Sec. 2.3.

Definition 2.16. *Fix a model metric $\|\cdot\|$ on L with curvature form θ. Then a singular metric $\|\cdot\|e^{-\varphi}$ is* semipositive *if the function φ is θ-psh.*

The results in Sec. 2.4 have obvious counterparts for singular metrics. In particular, we have the following.

Theorem 2.17. *Let $\|\cdot\|$ be a model metric on L, associated to a $\mathbf{Q}$-line bundle $\mathcal{L}$ on a model $\mathcal{X}$ of X. Then:*

(i) *the metric $\|\cdot\|$ is semipositive iff $\mathcal{L}$ is nef; and*
(ii) *a continuous metric $\|\cdot\|e^{-\varphi}$ is semipositive iff there exists a sequence of semipositive model metrics $\|\cdot\|_m = \|\cdot\|e^{-\varphi_m}$ such that $\varphi_m \to \varphi$ uniformly on X.*

This result implies that our definition of continuous semipositive metric coincides with that of Zhang and others. Unfortunately, the terminology is not uniform across the literature; see Table 1 below.

Model metric: [BFJ09, YZ13a, YZ13b]	*Continuous semipositive metric*: [BFJ09, CL06, CL11]
Algebraic metric: [BPS13, CL06, Liu11]	Approachable metric: [BPS13]
Smooth metric: [CL11]	Semipositive metric: [YZ13a, Liu11]
Root of an algebraic metric: [Gub08]	Semipositive admissible metric: [Gub08]

TABLE 1. Terminology for metrics on line bundles.

2.7. **Intersection numbers and Monge-Ampère measures.** The Monge-Ampère operator that we will use arises from intersection theory on models.

Let $\mathcal{X}$ be a model of X, and pick numerical classes $\theta_{1,\mathcal{X}}, \dots, \theta_{n,\mathcal{X}} \in N^1(\mathcal{X}/S)$. For any vertical divisor $D \in \mathrm{Div}_0(\mathcal{X})$ we define the intersection number

$$D \cdot \theta_{1,\mathcal{X}} \cdot \ldots \cdot \theta_{n,\mathcal{X}} := \sum_E \mathrm{ord}_E(D)\,(\theta_{1,\mathcal{X}}|_E \cdot \ldots \cdot \theta_{n,\mathcal{X}}|_E),$$

where E ranges over all irreducible components of the special fiber $\mathcal{X}_0$. We obtain a pairing that is linear in each entry and symmetric in the θ_i's.

Proposition-Definition 2.18. *To any n-tuple $(\theta_1, \dots, \theta_n)$ of closed $(1,1)$-forms we can associate a signed atomic measure $\theta_1 \wedge \cdots \wedge \theta_n$ supported on X^{div} such that*

$$\int_X f\,\theta_1 \wedge \cdots \wedge \theta_n = \sum_{i \in I} b_i f(x_i)\,(\theta_{1,\mathcal{X}}|_{E_i} \cdot \ldots \cdot \theta_{n,\mathcal{X}}|_{E_i}) \tag{2.2}$$

for any common determination $\mathcal{X}$ of the forms θ_i, and for any model function f. Here we have written the special fiber as $\mathcal{X}_0 = \sum_{i\in I} b_i E_i$ and $x_i = x_{E_i}$ is the divisorial point associated to E_i.

Further, $(\theta_1, \dots, \theta_n) \mapsto \theta_1 \wedge \cdots \wedge \theta_n$ is multilinear and symmetric.

Proof. Choose a common determination of the forms θ_i, and define $\int_X f\,(\theta_1 \wedge \cdots \wedge \theta_n)$ using (2.2). The fact that $\int_X f\,(\theta_1 \wedge \cdots \wedge \theta_n)$ does not depend on the choice of a determination $\mathcal{X}$ is a consequence of the projection formula

$$\pi_* D \cdot \theta_{1,\mathcal{X}} \cdot \ldots \cdot \theta_{n,\mathcal{X}} = D \cdot \pi^* \theta_{1,\mathcal{X}} \cdot \ldots \cdot \pi^* \theta_{n,\mathcal{X}}$$

if $\pi : \mathcal{X}' \to \mathcal{X}$, and D is any vertical divisor in $\mathcal{X}'$.

Then by construction $\theta_1 \wedge \cdots \wedge \theta_n$ can be identified with the atomic measure $\sum_i w_i \delta_{x_i}$ with $w_i = (\theta_{1,\mathcal{X}}|_{E_i} \cdot \ldots \cdot \theta_{n,\mathcal{X}}|_{E_i})$. This measure is supported on the divisorial points associated to the irreducible components of $\mathcal{X}_0$. The last statement is clear. $\square$

Proposition 2.19. *If the forms $\theta_1, \ldots, \theta_n$ are semipositive, then $\theta_1 \wedge \cdots \wedge \theta_n$ is a positive measure, of mass*

$$\int_X \theta_1 \wedge \cdots \wedge \theta_n = \{\theta_1\} \cdot \ldots \cdot \{\theta_n\}. \tag{2.3}$$

Proof. Pick a model $\mathcal{X}$ such that each θ_i is determined by a nef class $\theta_{i,\mathcal{X}} \in N^1(\mathcal{X}/S)$. The restriction of $\theta_{i,\mathcal{X}}$ to each component E of $\mathcal{X}_0$ is then also nef, and it follows that the intersection number $(\theta_{1,\mathcal{X}}|_E \cdot \ldots \cdot \theta_{n,\mathcal{X}}|_E)$ is non-negative, hence the first assertion. Since the constant function 1 corresponds to the vertical divisor $\mathcal{X}_0$ we have by definition

$$\int_X \theta_1 \wedge \cdots \wedge \theta_n = \mathcal{X}_0 \cdot \theta_1 \cdot \ldots \cdot \theta_n.$$

By [Ful98, Example 20.3.3] this is the same as the intersection number against the generic fiber of $\mathcal{X}$, and this is equal to $\{\theta_1\} \cdot \ldots \cdot \{\theta_n\}$ by definition. $\square$

As a special case, fix $\theta \in \mathcal{Z}^{1,1}(X)$. To any θ-psh model functions $\varphi_1, \ldots, \varphi_n$ we then associate a *mixed Monge-Ampère measure*

$$(\theta + dd^c \varphi_1) \wedge \cdots \wedge (\theta + dd^c \varphi_n).$$

This is an atomic positive measure on X of mass $\{\theta\}^n$.

Analogously to the complex case we have the following *integration by parts* formula.

Proposition 2.20. *If $f, g \in \mathcal{D}(X)$ are model functions and $\theta_1, \ldots, \theta_{n-1}$ are closed $(1,1)$-forms then we have*

$$\int f\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = \int g\, dd^c f \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}.$$

Proof. Pick a common determination $\mathcal{X}$ of f, g and the θ_i's, and divisors D, D', D_i such that $f = \varphi_D$, $g = \varphi_{D'}$ and θ_i is the class in $N^1(\mathcal{X}/S)$ induced by D_i. Then by definition we have

$$\begin{aligned}
&\int f\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} \\
&= \sum_E \operatorname{ord}_E(D)(D'|_E \cdot D_1|_E \cdot \ldots \cdot D_{n-1}|_E) \\
&= \sum_{E,E'} \operatorname{ord}_E(D) \operatorname{ord}_{E'}(D')\, (D_1|_E)|_{E'\cap E} \cdot \ldots \cdot (D_{n-1}|_E)|_{E'\cap E} \\
&= \sum_{E,E'} \operatorname{ord}_E(D) \operatorname{ord}_{E'}(D')\, (D_1|_{E'})|_{E\cap E'} \cdot \ldots \cdot (D_{n-1}|_{E'})|_{E\cap E'} \\
&= \int g\, dd^c f \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}
\end{aligned}$$

where the third equality follows from [Ful98, Theorem 2.4]. □

The next result follows from the Hodge index theorem (see [YZ13a, Theorem 2.9]).

Proposition 2.21. *Suppose $\theta_1, \ldots, \theta_{n-1}$ are semipositive closed $(1,1)$-forms. Then the symmetric bilinear form*

$$(f,g) \mapsto \int_X f\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}$$

on $\mathcal{D}(X)$ is negative semidefinite. In particular, for any two model functions f, g, the following Cauchy-Schwarz inequality holds:

(2.4)

$$\begin{aligned}
&\left|\int_X f\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}\right| \\
&\le \left(-\int_X f\, dd^c f \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}\right)^{1/2} \left(-\int_X g\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1}\right)^{1/2}.
\end{aligned}$$

Proof of Proposition 2.21. Fix a model function f. We need to prove

$$I := \int_X f\, dd^c f \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} \le 0.$$

Choose a common determination $\mathcal{X}$ of φ and all the θ_i. By continuity, we may assume $\varphi = \varphi_D$ for some $D \in \operatorname{Div}_0(\mathcal{X})_{\mathbf{Q}}$, and each form θ_i is determined by a $\mathbf{Q}$-line bundle $\mathcal{L}_i$ on $\mathcal{X}$. Then $I = D^2 \cdot \mathcal{L}_1 \cdot \ldots \cdot \mathcal{L}_{n-1}$ and the result follows from [YZ13a, Theorem 2.9]. □

Remark 2.22. *In the complex case we have by Stokes' theorem*

$$\int f\, dd^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1} = -\int df \wedge d^c g \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1},$$

and negativity comes from the positivity of the $(1,1)$-form $df \wedge d^c f$. Recall also that $df \wedge d^c f \wedge \omega^{n-1} = |df|^2_\omega\, \omega^n$ when ω is a Kähler form, so that $\left(-\int f\, dd^c f \wedge \omega^{n-1}\right)^{1/2}$ is the L^2-norm of the gradient of f.

2.8. **Radon measures and convergence results.** We will make frequent use of basic integration and measure theory. Let X be a compact (Hausdorff) space. A *Radon measure* on X is a continuous linear functional $\mu : C^0(X) \to \mathbf{R}$. With this definition, it follows from the Riesz representation theorem that Radon measures are in 1-1 correspondence with regular Borel measures on X; see [Fol99, Sec. 7.1–2].

Since we will be dealing with (possibly uncountable) nets rather than sequences, one has to be careful using results from integration theory. For example, the monotone convergence theorem is of course not true for general nets. However, as the next results show, integration of semicontinuous functions against Radon measures is often well behaved.

Lemma 2.23. [Fol99, Proposition 7.12]. *If μ is a positive Radon measure on X and $(f_j)_j$ a decreasing net of usc functions on X, converging pointwise to a (usc) function f, then $\lim_j \int f_j\mu = \int f\mu$.*

In particular, one has the following.

Lemma 2.24. [Fol99, Corollary 7.13]. *If μ is a positive Radon measure on X and f is a usc function on X, then*

$$\int f\mu = \inf\left\{\int g\mu \,\middle|\, f \leq g,\, g \in C^0(X)\right\}$$

Corollary 2.25. *Let $(f_j)_j$ a decreasing net of usc functions on X converging pointwise to a (usc) function f, and $(\mu_j)_j$ a net of positive Radon measures on X converging weakly to a positive Radon measure μ. Then*

$$\limsup_j \int f_j\mu_j \leq \int f\mu.$$

Proof. Upon replacing μ_j with $(\int \mu_j)^{-1}\mu_j$ we may assume that the μ_j's are probability measures. Fix any $\varepsilon > 0$. By Lemma 2.24 there exists a continuous function $g \geq f$ on X such that $\int g\mu < \int f\mu + \varepsilon$. By Dini's lemma, we have $f_j < g + \varepsilon$ for all $j \gg 1$, hence

$$\limsup_j \int f_j\mu_j \leq \limsup_j \int g\mu_j + \varepsilon = \int g\mu + \varepsilon \leq \int f\mu + 2\varepsilon,$$

since $\int g\mu_j \to \int g\mu$ by the definition of weak convergence. The result follows. □

3. Monge-Ampère operator on bounded functions

From now on we fix a form $\theta \in \mathcal{Z}^{1,1}(X)$ whose de Rham class $\{\theta\} \in N^1(X)$ is *ample*. In the next three sections we will develop some of the Bedford-Taylor theory in our non-Archimedean setting.

Our first main objective is to extend the Monge-Ampère operator defined in Sec. 2.7 from θ-psh model functions to bounded θ-psh functions.

Theorem 3.1. *There exists a unique operator*

$$(\varphi_1, \ldots, \varphi_n) \mapsto (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$$

taking an n-tuple of bounded *θ-psh functions to a positive Radon measure on X of mass $\{\theta\}^n$ and such that:*

(i) *the definition is compatible with the one for θ-psh model functions given in Sec. 2.7; and*

(ii) *for any decreasing nets of bounded θ-psh functions $\psi^j \to \psi$, and $\varphi_i^j \to \varphi_i$ for $i = 1, \dots, n$ we have*

$$\int \psi^j\,(\theta + dd^c\varphi_1^j) \wedge \cdots \wedge (\theta + dd^c\varphi_n^j) \longrightarrow \int \psi\,(\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n).$$

Remark 3.2. *As in the usual Bedford-Taylor theory one can prove that the convergence in* (ii) *aso holds for* increasing *nets, but we will not need this.*

3.1. **Consequences.** Before proving Theorem 3.1 we derive some consequences.

Corollary 3.3. *If $(\varphi_i^j)_j$, $1 \le i \le n$ are decreasing nets of bounded θ-psh functions converging to bounded θ-psh functions φ_i, $1 \le i \le n$, then the Radon measures $\mu^j := (\theta + dd^c\varphi_1^j) \wedge \cdots \wedge (\theta + dd^c\varphi_n^j)$ converge to $\mu := (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$.*

Proof. By Theorem 3.1 we have $\lim_j \int \psi\,\mu^j = \int \psi\,\mu$ for every θ-psh model function ψ. By Proposition 2.6 and linearity we get the same convergence for any model function ψ. The result now follows since model functions are dense in $C^0(X)$, see Proposition 2.2. $\square$

Corollary 3.4. *For any bounded θ-psh functions φ_i and any constants c_i, $1 \le i \le n$ we have*

$$(3.1)\quad (\theta + dd^c(\varphi_1 + c_1)) \wedge \cdots \wedge (\theta + dd^c(\varphi_n + c_n)) = (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n).$$

Proof. By definition, (3.1) holds when the φ_i are model functions and the constants c_i are rational. In the general case, we use Theorem 2.11 to write the φ_i as decreasing limits of θ-psh model functions. Similarly, the constants c_i are decreasing limits of rational constants and so (3.1) follows from Corollary 3.3. $\square$

Corollary 3.5. *If $(\psi^j)_j$ and $(\varphi_i^j)_j$, $1 \le i \le n$, are nets of bounded θ-psh functions converging* uniformly *to bounded θ-psh functions ψ and φ_i, respectively, then the Radon measures $\mu^j := (\theta + dd^c\varphi_1^j) \wedge \cdots \wedge (\theta + dd^c\varphi_n^j)$ converge to $\mu := (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$ and we have $\lim_j \int \psi^j\,\mu^j = \int \psi\mu$.*

Proof. We may find constants $c_j > 0$ such that $c_j \to 0$ and the nets $(\tilde{\varphi}_i^j)_j$ are decreasing, where $\tilde{\varphi}_i^j = \varphi_i^j + c_j$ for $1 \le j \le n$. By (3.1) we get $\mu^j = (\theta + dd^c\tilde{\varphi}_1^j) \wedge \cdots \wedge (\theta + dd^c\tilde{\varphi}_n^j)$; hence μ^j converges to μ in view of Corollary 3.3. Since $\psi^j \to \psi$ uniformly, this implies that $\int \psi^j\,\mu^j \to \int \psi\,\mu$, as was to be shown. $\square$

Corollary 3.6. *The mapping*

$$(\varphi_1, \dots, \varphi_n) \mapsto (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$$

is symmetric in its arguments, and additive in the following sense:

$$\begin{aligned}&(\theta + t dd^c\varphi_1 + (1-t) dd^c\varphi_1') \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n)\\ &\quad = t\,(\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n) + (1-t)\,(\theta + dd^c\varphi_1') \wedge \cdots \wedge (\theta + dd^c\varphi_n)\end{aligned}$$

for $0 \le t \le 1$.

Proof. The properties hold when all the θ-psh functions involved are model functions. The general case follows from Theorem 3.1 and the fact that every θ-psh function is a decreasing limit of θ-psh model functions; see Theorem 2.11. $\square$

The additivity property in particular implies

$$(\theta + dd^c(t\varphi + (1-t)\psi))^n \geq t^n(\theta + dd^c\varphi)^n + (1-t)^n(\theta + dd^c\psi)^n \tag{3.2}$$

in the sense of measures, for all bounded θ-psh functions φ, ψ, and any $0 \leq t \leq 1$.

Given bounded θ-psh functions, one can now also define signed measures

$$dd^c\varphi_1 \wedge \cdots \wedge dd^c\varphi_p \wedge (\theta + dd^c\varphi_{p+1}) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$$

by writing $dd^c\varphi_i = (\theta + dd^c\varphi_i) - \theta$ and expanding the product formally using multilinearity. These products are also continuous along decreasing nets, and we thus obtain the following.

Corollary 3.7. *If $\varphi_1, \dots, \varphi_{n-1}$ are bounded θ-psh functions on X, then the bilinear form*

$$(\varphi, \psi) \mapsto \int (-\varphi)\, dd^c\psi \wedge (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_{n-1})$$

is well defined and positive semidefinite on the vector space spanned by the set of bounded θ-psh functions.

In particular, the Cauchy-Schwarz inequality (2.4) holds for all bounded θ-psh functions $\varphi_2, \dots, \varphi_n$ and for all functions ψ, φ that are linear combinations of bounded θ-psh functions.

3.2. **Proof of Theorem 3.1.** Note that the uniqueness part of Theorem 3.1 follows, as in the proof of Corollary 3.3, from Theorem 2.10 stating that every θ-psh function is the decreasing limit of a net of θ-psh model functions.

To prove existence, we adapt to our setting the Bedford-Taylor approach from [BT82].

Fix $0 \leq p \leq n$ and θ-psh *model functions* $\varphi'_{p+1}, \dots, \varphi'_n$. Consider the following statement.

Assertion A(p). To any p-tuple $\varphi_1, \dots, \varphi_p$ of bounded θ-psh functions is associated a positive Radon measure $\mathrm{M}(\varphi_1, \dots, \varphi_p)$ of mass $\{\theta\}^n$ such that:

(i) if $\varphi_1, \dots, \varphi_p$ are model functions then

$$\mathrm{M}(\varphi_1, \dots, \varphi_p) = (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_p) \wedge (\theta + dd^c\varphi'_{p+1}) \wedge \cdots \wedge (\theta + dd^c\varphi'_n); \text{ and} \tag{3.3}$$

(ii) the mapping

$$(\psi, \varphi_1, \dots, \varphi_p) \mapsto \int \psi\, \mathrm{M}(\varphi_1, \dots, \varphi_p)$$

is continuous along decreasing nets of bounded θ-psh functions.

We will prove $\mathrm{A}(p)$ by induction on p. Observe that for $p = n$, this proves Theorem 3.1.

The assertion $\mathrm{A}(0)$ is clear, since $\mathrm{M}(\varphi'_1, \dots, \varphi'_n)$ is a finite sum of Dirac masses at divisorial points of X. Assume that $\mathrm{A}(p-1)$ holds for any $(n-p+1)$-tuple of θ-psh model functions and let $\varphi'_{p+1}, \dots, \varphi'_n$ be θ-psh model functions.

Given bounded θ-psh functions $\varphi_1, \dots, \varphi_p$, we define $\mathrm{M}(\varphi_1, \dots, \varphi_p)$ by forcing the integration by parts formula

$$(\dagger) \quad \int \psi \, \mathrm{M}(\varphi_1, \dots, \varphi_{p-1}, \varphi_p) := \int \varphi_p \, (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_{p-1}) \wedge (\theta + dd^c\psi) \wedge (\theta + dd^c\varphi'_{p+1}) \wedge \cdots \wedge (\theta + dd^c\varphi'_n) + \int (\psi - \varphi_p)(\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_{p-1}) \wedge \theta \wedge (\theta + dd^c\varphi'_{p+1}) \wedge \cdots \wedge (\theta + dd^c\varphi'_n)$$

for every model function ψ.

Observe that the right-hand side is continuous along decreasing nets as a function of $(\varphi_1, \dots, \varphi_p)$ by the induction hypothesis $A(p-1)$. Since equality holds in $(\dagger)$ when all the φ_i are model functions and since $\mathrm{M}(\varphi_1, \dots, \varphi_p)$ is a positive measure of mass $\{\theta\}^n$, it follows by regularization (Theorem 2.11) that the right-hand side is also linear in ψ, and non-negative when $\psi \geq 0$.

Now the space of model functions is spanned by θ-psh model functions by Proposition 2.6; hence, $\mathrm{M}(\varphi_1, \dots, \varphi_p)$ is well defined as a positive measure of mass $\{\theta\}^n$ and is continuous along decreasing nets as a function of $(\varphi_1, \dots, \varphi_p)$.

It remains to show that

$$(\psi, \varphi_1, \dots, \varphi_p) \mapsto \int \psi \, \mathrm{M}(\varphi_1, \dots, \varphi_p)$$

is continuous along decreasing nets of bounded θ-psh functions. Thus, let $(\varphi_i^j)_j$, $i = 1, \dots, p$ and ψ^j be decreasing nets of θ-psh functions converging, respectively, to bounded θ-psh functions φ_i and ψ. Set

$$\mu^j := \mathrm{M}(\varphi_1^j, \dots, \varphi_p^j).$$

We already know that μ^j converges weakly to $\mu := \mathrm{M}(\varphi_1, \dots, \varphi_p)$. Since ψ^j is usc for each j, Corollary 2.25 yields

$$\limsup_j \int \psi^j \mu^j \leq \int \psi\mu.$$

For the reverse estimate, we rely on the following approximate monotonicity property.

Lemma 3.8. *Let ψ and $\chi_i \geq \varphi_i$, $i = 1, \dots, p$ be bounded θ-psh functions. Then we have*

$$\int \psi \, \mathrm{M}(\chi_1, \dots, \chi_p) \geq \int \psi \, \mathrm{M}(\varphi_1, \dots, \varphi_p) + \sum_{i=1}^p \int (\varphi_i - \chi_i) \, \mathrm{M}(\varphi_1, \dots, \varphi_{i-1}, 0, \chi_{i+1}, \dots, \chi_p).$$

The lemma implies that, for each j:

$$\int \psi^j \mu^j \geq \int \psi \mu^j \geq \int \psi\mu + \sum_{i=1}^p \int (\varphi_i - \varphi_i^j) \, \mathrm{M}(\varphi_1, \dots, \varphi_{i-1}, 0, \varphi_{i+1}^j, \dots, \varphi_p^j).$$

By the inductive hypothesis $A(p-1)$, the sum in the right-hand side tends to 0 as $j \to \infty$, so we infer as desired that $\liminf_j \int \psi^j \mu^j \geq \int \psi\mu$. This completes the proof of Theorem 3.1.

Proof of Lemma 3.8. Note first that ψ may be assumed to be a model function by the following.

Lemma 3.9. *Let ν be a positive Radon measure on X and let φ be a bounded θ-psh function. Then we have*

$$\int \varphi\nu = \inf_{\psi \geq \varphi} \int \psi\nu$$

where ψ ranges over all θ-psh model functions *such that $\psi \geq \varphi$.*

Since we already know that $(\varphi_1, \ldots, \varphi_p) \mapsto \mathrm{M}(\varphi_1, \ldots, \varphi_p)$ is continuous along decreasing nets, we may by regularization assume that all the φ_i and χ_i are also model functions. Integration by parts (†) then yields

$$\int \psi\, \mathrm{M}(\chi_1, \chi_2, \ldots, \chi_p) - \int \psi\, \mathrm{M}(\varphi_1, \chi_2, \ldots, \chi_p)$$
$$= \int (\chi_1 - \varphi_1)\, \mathrm{M}(\psi, \chi_2, \ldots, \chi_p) - \int (\chi_1 - \varphi_1)\, \mathrm{M}(0, \chi_2, \ldots, \chi_p),$$

hence

$$\int \psi\, \mathrm{M}(\chi_1, \ldots, \chi_p) \geq \int \psi\, \mathrm{M}(\varphi_1, \chi_2, \ldots, \chi_p) + \int (\varphi_1 - \chi_1)\, \mathrm{M}(0, \chi_2, \ldots, \chi_p).$$

We similarly have

$$\int \psi\, \mathrm{M}(\varphi_1, \chi_2, \chi_3, \ldots, \chi_p) \geq \int \psi\, \mathrm{M}(\varphi_1, \varphi_2, \chi_3, \ldots, \chi_p)$$
$$+ \int (\varphi_2 - \chi_2)\, \mathrm{M}(\varphi_1, 0, \chi_3, \ldots, \chi_p).$$

Iterating this argument and summing up then yields the desired result. ☐

Proof of Lemma 3.9. Let $\varepsilon > 0$. Since φ is usc, Lemma 2.24 shows that there exists a continuous function v on X such that $v \geq \varphi$ and $\int v\nu \leq \int \varphi\nu + \varepsilon$. The result now follows since [BFJ09, Corollary 8.6] yields a θ-psh model function ψ such that $\varphi \leq \psi \leq v + \varepsilon$. ☐

Definition 3.10. *A* pluripolar set *is a subset of $\{\psi = -\infty\}$ for some $\psi \in \mathrm{PSH}(X, \theta)$.*

Proposition 3.11. *Let $\varphi_1, \ldots, \varphi_n$ be bounded θ-psh functions. Then any $\psi \in \mathrm{PSH}(X, \theta)$ is integrable with respect to the measure $\mu := (\theta + dd^c\varphi_1) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$. In particular, μ does not put mass on pluripolar sets.*

Proof. Pick $\varphi_0 \in \mathrm{PSH}(X, \theta) \cap \mathcal{D}(X)$. Upon replacing θ, φ_i, and ψ with $\theta + dd^c\varphi_0$, $\varphi_i - \varphi_0$ and $\psi - \varphi_0$, respectively, we may assume that θ is semipositive. Adding constants to the φ_i and to ψ we may also assume that $\sup \varphi_i = 0$ for all i and $\sup \psi = 0$. Set $M := \max_i \sup |\varphi_i|$. First assume that ψ is bounded. We claim that $\int -\psi\mu$ is bounded by a constant depending only on M (but not on $\sup_X |\psi|$). Integrating by parts we obtain

$$0 \leq \int (-\psi)\mu = \int (-\psi)\theta \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$$
$$+ \int (-\varphi_1)(\theta + dd^c\psi) \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$$
$$+ \int \varphi_1\theta \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n).$$

Here the second to last integral is bounded by $M\{\theta\}^n$, while the last integral to the right is non-positive since $\theta \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n)$ is a positive measure. Hence,

$$0 \leq \int (-\psi)\mu \leq \int (-\psi)\theta \wedge (\theta + dd^c\varphi_2) \wedge \cdots \wedge (\theta + dd^c\varphi_n) + M\{\theta\}^n.$$

Iterating this argument yields

$$0 \leq \int (-\psi)\mu \leq \int (-\psi)\theta^n + nM\{\theta\}^n.$$

Now $\int(-\psi)\theta^n$ is bounded above by some $C > 0$ only depending on θ, by compactness of $\{\psi \in \mathrm{PSH}(X, \theta) \mid \sup_X \psi = 0\}$, see Theorem 2.10, and the fact that θ^n is an atomic measure supported at finitely many divisorial points. We conclude that

$$0 \leq \int (-\psi)\mu \leq C + nM\{\theta\}^n \tag{3.4}$$

for some constant $C > 0$ only depending on θ, as long as ψ is a bounded θ-psh function with $\sup_X \psi = 0$. If ψ is now a possibly unbounded θ-psh function normalized by $\sup_X \psi = 0$, ψ is the decreasing limit of the bounded θ-psh functions $\psi_m := \max\{\psi, -m\}$, so that (3.4) continues to hold, by monotone convergence. □

3.3. **The Chambert-Loir measure.** We follow the notation and terminology of Sec. 2.6. Consider an ample line bundle L on X and equip L with a model metric $\|\cdot\|$. Any continuous metric on L is then of the form $\|\cdot\|\,e^{-\varphi}$ where $\varphi \in C^0(X)$. Recall that this metric is semipositive iff the function φ is θ-psh, where $\theta := c_1(L, \|\cdot\|)$. In this case, set

$$c_1(L, \|\cdot\|e^{-\varphi})^n := (\theta + dd^c\varphi)^n,$$

where the right-hand side is the positive Radon measure in Theorem 3.1.

This is the same measure as the one defined by Chambert-Loir in [CL06]. Indeed, this is certainly true when φ is a model function, as seen by comparing (2.2) and [CL06, Définition 2.4]. In general, Corollary 2.12 yields a sequence $(\varphi_m)_{m=1}^\infty$ of θ-psh model functions converging uniformly to φ on X. The measure μ associated to $(L, \|\cdot\|e^{-\varphi})$ by Chambert-Loir is the limit of the measures $\mu_m := (\theta + dd^c\varphi_m)^n$; see [CL06, Proposition 2.7]. Thus, $\mu = (\theta + dd^c\varphi)^n$ by Corollary 3.5.

4. Capacity and quasicontinuity

Let $\omega \in \mathcal{Z}^{1,1}(X)$ be a closed $(1,1)$-form with ample de Rham class $\{\omega\} \in N^1(X)$. We assume that ω is *semipositive*, that is, $\mathbf{R} \subset \mathrm{PSH}(X, \omega)$. For simplicity, we also assume that ω is normalized in the sense that $\{\omega\}^n = 1$.

In this section, we introduce a capacity that will be used to measure the size of subsets of X. It is the analog of the Monge-Ampère capacity introduced in [BT82] and adapted to the case of compact Kähler manifolds in [GZ05].

In order to compactify notation, we will write

$$\mathrm{MA}(\varphi_1, \ldots, \varphi_n) := (\omega + dd^c\varphi_1) \wedge \cdots \wedge (\omega + dd^c\varphi_n)$$

as well as

$$\mathrm{MA}(\varphi) := \mathrm{MA}(\varphi, \ldots, \varphi) = (\omega + dd^c\varphi)^n,$$

for bounded ω-psh functions $\varphi_1, \ldots, \varphi_n, \varphi$.

Definition 4.1. *For any Borel set $E \subseteq X$, set*

$$\operatorname{Cap}(E) = \sup \left\{ \int_E \operatorname{MA}(u) \,\middle|\, u \in \operatorname{PSH}(X, \omega),\ -1 \le u \le 0 \right\}.$$

By Proposition 2.19 we have $0 \le \operatorname{Cap}(E) \le \{\omega\}^n = 1$. Note that if $E_1, E_2, \dots$ are Borel sets, then $\operatorname{Cap}(\bigcup E_j) \le \sum_j \operatorname{Cap}(E_j)$.

Remark 4.2. *The Monge-Ampère operator and the capacity of course depend on the choice of form ω, but we drop this dependence for notational simplicity.*

Lemma 4.3. *If $x \in X$ is a divisorial point, then* $\operatorname{Cap}\{x\} > 0$. *As a consequence, every non-empty open subset of X has strictly positive capacity.*

Proof. The second statement follows from the first since divisorial points are dense in X; see Sec. 2.1. To prove the first statement, pick an SNC model $\mathcal{X}$ of X such that $x = x_E$ is associated to an irreducible component E of the special fiber. By [BFJ09, Proposition 5.2] there exists a model function $u \in \mathcal{D}(X)$ determined on $\mathcal{X}$ such that $-1 \le u \le 0$ and $\omega + dd^c u$ is determined by an *ample* class $\theta \in N^1(\mathcal{X}/S)$. By the definition of capacity, we then have $\operatorname{Cap}\{x\} \ge \operatorname{MA}(u)\{x\} = b_E(\theta|_E)^n > 0$; see Sec. 2.7. □

Proposition 4.4. *If φ is a bounded ω-psh function then for each $\varepsilon > 0$ there exists an open subset $G \subseteq X$ with* $\operatorname{Cap}(G) < \varepsilon$ *and a decreasing sequence $(\varphi_m)_{m=1}^\infty$ of ω-psh model functions that converges uniformly to φ on G^c. In particular, φ is continuous on G^c.*

Definition 4.5. *A function $h : X \to \mathbf{R} \cup \{-\infty\}$ is said to be* quasicontinuous *iff it is continuous outside sets of arbitrarily small capacity.*

Proposition 4.6. *Every (not necessarily bounded) function $\varphi \in \operatorname{PSH}(X, \omega)$ is quasicontinuous.*

Using the Monge-Ampère capacity we will replace nets by *sequences* in the regularization result for ω-psh functions (Theorem 2.11). While not crucial, this result is psychologically satisfying and does simplify the proof of Corollary 7.3 below.

Proposition 4.7. *Any ω-psh function φ is the limit of a decreasing sequence $(\varphi_m)_{m=1}^\infty$ of ω-psh model functions.*

The rest of this section is devoted to the proof of these propositions. First we state and prove two estimates on special Monge-Ampère integrals.

Lemma 4.8. *The Monge-Ampère measure of any bounded ω-psh is linearly bounded by the capacity. More precisely, if u is an ω-psh function such that $-M \le u \le 0$, where $M \ge 1$, then*

$$\operatorname{MA}(u) \le M^n \operatorname{Cap}$$

on Borel sets.

Proof. Given a Borel set $E \subset X$ we have

$$\begin{aligned} \int_E \operatorname{MA}(u) &= \int_E (\omega + dd^c u)^n \le \int_E (M\omega + dd^c u)^n \\ &= M^n \int_E \left(\omega + dd^c \frac{u}{M}\right)^n \le M^n \operatorname{Cap}(E). \end{aligned}$$

Here the first inequality follows by writing $M\omega + dd^c u = (M-1)\omega + \omega + dd^c u$ and expanding the Monge-Ampère measure by multilinearity. □

Lemma 4.9. *Suppose φ, ψ and $u_1, \dots, u_n$ are bounded ω-psh functions such that $-M \le \varphi \le \psi \le 0$ and $-M \le u_i \le 0$, where $M \ge 1$. Then*

$$0 \le \int (\psi - \varphi)\, \mathrm{MA}(u_1, \dots, u_n) \le 4M \left(\int (\psi - \varphi)\, \mathrm{MA}\left(\frac{\varphi}{2}\right)\right)^{\frac{1}{2^n}}.$$

Proof. After regularizing we may assume that all functions involved are model functions. Write, symbolically, $T := (\omega + dd^c u_2) \wedge \cdots \wedge (\omega + dd^c u_n)$. Then

$$\int (\psi - \varphi)\, \mathrm{MA}(u_1, \dots, u_n) = \int (\psi - \varphi)\omega \wedge T + \int (\psi - \varphi)\, dd^c u_1 \wedge T.$$

Since $0 \le \int (\psi - \varphi)\omega \wedge T \le M$, the first term in the right-hand side satisfies

$$\int (\psi - \varphi)\omega \wedge T \le M^{\frac{1}{2}} \left(\int (\psi - \varphi)\omega \wedge T \right)^{\frac{1}{2}}.$$

By the Cauchy-Schwarz inequality (Corollary 3.7), the second term is bounded by

$$\left(\int (\psi - \varphi)\, dd^c(\varphi - \psi) \wedge T \right)^{\frac{1}{2}} \left(\int (-u_1)\, dd^c u_1 \wedge T \right)^{\frac{1}{2}}.$$

By the assumption that $-M \le u_1 \le 0$ and $\int \omega^n = 1$ we have

$$0 \le \int (-u_1)\, dd^c u_1 \wedge T = \int u_1 \omega \wedge T - \int u_1 (\omega + dd^c u_1) \wedge T \le M.$$

Similarly,

$$\begin{aligned} 0 \le \int (\psi - \varphi)\, dd^c(\varphi - \psi) \wedge T &= \int (\psi - \varphi)(\omega + dd^c \varphi) \wedge T \\ &\quad - \int (\psi - \varphi)(\omega + dd^c \psi) \wedge T \\ &\le \int (\psi - \varphi)(\omega + dd^c \varphi) \wedge T. \end{aligned}$$

Putting this together, and using the concavity of the square root, we get

$$\begin{aligned} &\int (\psi - \varphi)\, \mathrm{MA}(u_1, \dots, u_n) \\ &\le M^{\frac{1}{2}} \left(\left(\int (\psi - \varphi)\omega \wedge T \right)^{\frac{1}{2}} + \left(\int (\psi - \varphi)(\omega + dd^c \varphi) \wedge T \right)^{\frac{1}{2}} \right) \\ &\le 2M^{\frac{1}{2}} \left(\int (\psi - \varphi) \left(\omega + dd^c \frac{\varphi}{2} \right) \wedge T \right)^{\frac{1}{2}}. \end{aligned}$$

The lemma follows (with the constant $4M/(2M)^{\frac{1}{2^n}} < 4M$) by repeating this argument $n - 1$ times, successively replacing $u_2, \dots, u_n$ by $\varphi/2$. ☐

Proof of Proposition 4.6. If φ is bounded, then the result is an immediate consequence of Proposition 4.4. Now suppose that φ is unbounded. It suffices to show that the capacity of the sublevel sets $\{\varphi \le -t\}$ tend to 0 as $t \to \infty$, since the bounded ω-psh function $\max\{\varphi, -t\}$ is quasicontinuous.

We may assume $\sup\varphi = 0$. Pick $u \in \mathrm{PSH}(X,\omega)$ with $-1 \le u \le 0$. Using (3.4) with $\mu = \mathrm{MA}(u)$ and φ instead of ψ, we obtain the estimate

$$\int_{\{\varphi\le -t\}} \mathrm{MA}(u) \le \frac{1}{t}\int(-\varphi)\,\mathrm{MA}(u) \le \frac{C}{t}$$

for some constant $C > 0$ independent of φ. Taking the supremum over u we see that

$$\mathrm{Cap}\{\varphi \le -t\} \le \frac{C}{t} \tag{4.1}$$

for all $\varphi \in \mathrm{PSH}(X,\omega)$ such that $\sup\varphi = 0$ and all $t > 0$. In fact, this estimate also holds when $\sup\varphi \ge 0$, since we can then consider $\varphi - \sup\varphi$. $\square$

Proof of Proposition 4.4. Let $(\psi_j)_j$ be a decreasing *net* of ω-psh model functions converging to φ. After subtracting a positive constant from φ and all the ψ_j we may assume that $-M \le \psi_j \le 0$ for all j, where $M \ge 1$. For any ω-psh function u with $-1 \le u \le 0$ it follows from Lemma 4.9 that

$$0 \le \int(\psi_j - \varphi)\,\mathrm{MA}(u) \le 4M\left(\int(\psi_j-\varphi)\,\mathrm{MA}\left(\frac{\varphi}{2}\right)\right)^{\frac{1}{2^n}}$$

and the right-hand side tends to zero as $j \to \infty$ by Theorem 3.1. It therefore follows from the definition of the capacity and from Chebyshev's inequality that for each integer $m \ge 1$ there exists j_m such that the open set $G_m := \{\psi_{j_m} - \varphi > \frac{1}{m}\}$ has capacity $2^{-m}\varepsilon$. We can then set $G := \bigcup_m G_m$ and $\varphi_m := \psi_{j_m}$. $\square$

Proof of Proposition 4.7. As above, let $(\psi_j)_j$ be a decreasing *net* of ω-psh model functions converging to φ. After subtracting a constant we may assume that $\psi_j \le 0$ for all j. For each integer $m \ge 1$, the net $(\max\{\psi_j, -m\})_j$ decreases to the bounded ω-psh function $\max\{\varphi, -m\}$. By Theorem 3.1 we can therefore choose j_m such that

$$0 \le \int(\max\{\psi_{j_m}, -m\} - \max\{\varphi, -m\})\,\mathrm{MA}\left(\frac{\max\{\varphi,-m\}}{2}\right) \le (2m)^{-2^{n+1}}. \tag{4.2}$$

We may further assume $j_{m+1} \ge j_m$ for all m. Set $\varphi_m := \psi_{j_m}$. We claim that the decreasing sequence $(\varphi_m)_{m=1}^\infty$ converges to φ. By Theorem 2.10 it suffices to test this at any divisorial point $x \in X$. We have $0 \ge \varphi(x) > -\infty$ and $\varphi_m(x) \ge \varphi(x) \ge -m$ as long as $m \ge -\varphi(x) \ge 0$. By (4.2), Lemma 4.9 and the definition of capacity this yields

$$0 \le (\varphi_m(x) - \varphi(x))\,\mathrm{Cap}\{x\} \le \frac{1}{m}$$

as long as $m \ge -\varphi(x)$. Now $\mathrm{Cap}\{x\} > 0$ by Lemma 4.3, so we see that $\varphi_m(x)$ converges to $\varphi(x)$, which concludes the proof. $\square$

5. Locality and the comparison principle

Let ω be a form as in Sec. 4 with $\{\omega\}^n = 1$. In this section we prove the following analog of [BT87, Proposition 4.2]. We will refer to it as the *locality property* of the Monge-Ampère operator.

Theorem 5.1. *If φ and ψ are bounded ω-psh functions, then*

$$\mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}(\max\{\varphi,\psi\}) = \mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}(\varphi). \tag{5.1}$$

A first consequence of this result is the fact that our operator MA is indeed local in nature, something that is not immediate from the definition in Sec. 3.

Corollary 5.2. *Suppose φ, ψ are bounded ω-psh functions that agree on an open set $G \subseteq X$. Then* $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$ *on G.*

Proof. Given $\varepsilon > 0$ we apply Theorem 5.1 to $\varphi + \varepsilon$ and ψ. This gives $\mathrm{MA}(\max\{\varphi + \varepsilon, \psi\}) = \mathrm{MA}(\varphi)$ on $G \subseteq \{\varphi + \varepsilon > \psi\}$. Letting $\varepsilon \to 0$ and using Theorem 3.1 we get $\mathrm{MA}(\max\{\varphi, \psi\}) = \mathrm{MA}(\varphi)$ on G. Exchanging the roles of φ and ψ shows that $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$ on G. ☐

Another key consequence of Theorem 5.1 is the *comparison principle.*

Corollary 5.3. *If φ and ψ are bounded ω-psh functions, then*

$$\int_{\{\varphi<\psi\}} \mathrm{MA}(\psi) \leq \int_{\{\varphi<\psi\}} \mathrm{MA}(\varphi).$$

Proof. As in [GZ07, Theorem 1.5] the result easily follows from the locality property by integration. More precisely, for any $\varepsilon > 0$ we have

$$\begin{aligned} 1 = \int \mathrm{MA}(\max\{\varphi, \psi - \varepsilon\}) &\geq \int_{\{\varphi<\psi-\varepsilon\}} \mathrm{MA}(\max\{\varphi, \psi - \varepsilon\}) \\ &\quad + \int_{\{\varphi>\psi-\varepsilon\}} \mathrm{MA}(\max\{\varphi, \psi - \varepsilon\}) \\ &\overset{(5.1)}{=} \int_{\{\varphi<\psi-\varepsilon\}} \mathrm{MA}(\psi - \varepsilon) + \int_{\{\varphi>\psi-\varepsilon\}} \mathrm{MA}(\varphi) = \int_{\{\varphi<\psi-\varepsilon\}} \mathrm{MA}(\psi) + 1 \\ &\quad - \int_{\{\varphi\leq\psi-\varepsilon\}} \mathrm{MA}(\varphi), \end{aligned}$$

so we obtain the desired estimate by letting $\varepsilon \to 0$. ☐

The rest of this section is devoted to the proof of Theorem 5.1. We will use the following.

Lemma 5.4. *Let $(\varphi_j)_j$ be a uniformly bounded net of ω-psh functions, and assume that* $\mathrm{MA}(\varphi_j)$ *converges to* $\mathrm{MA}(\varphi)$ *in the weak sense of measures for some bounded ω-psh function φ. Then*

$$\int h \,\mathrm{MA}(\varphi_j) \to \int h \,\mathrm{MA}(\varphi) \quad \textit{as } j \to \infty$$

for every bounded, quasicontinuous function h.

Proof. We may assume $0 \leq h \leq 1$, $-M \leq \varphi \leq 0$ and $-M \leq \varphi_j \leq 0$ for all j, where $M \geq 1$. Given $\varepsilon > 0$, let G be an open set such that $\mathrm{Cap}(G) < \varepsilon$ and h is continuous on G^c; see Definition 4.5. Using the Tietze extension theorem, we extend $h|_{G^c}$ to a continuous function $\tilde{h}$ on all of X such that $0 \leq \tilde{h} \leq 1$. We then have

$$\int h\,\mathrm{MA}(\varphi_j) - \int h\,\mathrm{MA}(\varphi) = \int \tilde{h}\,\mathrm{MA}(\varphi_j) - \int \tilde{h}\,\mathrm{MA}(\varphi) + \int_G (h-\tilde{h})\,\mathrm{MA}(\varphi_j) - \int_G (h-\tilde{h})\,\mathrm{MA}(\varphi).$$

It follows from Lemma 4.8 that

$$\left|\int h\,\mathrm{MA}(\varphi_j) - \int h\,\mathrm{MA}(\varphi)\right| \le \left|\int \tilde{h}\,\mathrm{MA}(\varphi_j) - \int \tilde{h}\,\mathrm{MA}(\varphi)\right| + 2\sup|h-\tilde{h}|M^n\ \mathrm{Cap}(G).$$

Since $\tilde{h}$ is continuous, $\int \tilde{h}\,\mathrm{MA}(\varphi_j) \to \int \tilde{h}\,\mathrm{MA}(\varphi)$ as $j\to\infty$, so that

$$\limsup_j \left|\int h\,\mathrm{MA}(\varphi_j) - \int h\,\mathrm{MA}(\varphi)\right| \le 2M^n\varepsilon.$$

Letting ε tend to zero completes the proof. ☐

Proof of Theorem 5.1. We prove the result for successively more general functions φ, ψ.

Step 1. First assume φ, ψ are ω-psh model functions.

Pick an SNC model $\mathcal{X}$ on which φ, ψ, and $\max\{\varphi,\psi\}$ are determined by vertical divisors A, B, and C, respectively. These three functions are then affine on any face of the dual complex $\Delta_{\mathcal{X}}$. Further, $\mathrm{MA}(\varphi)$ and $\mathrm{MA}(\max\{\varphi,\psi\})$ are both atomic measures, supported on divisorial points corresponding to irreducible components of the special fiber; see Sec. 2.7. If E is such a component for which $\varphi(x_E) > \psi(x_E)$, then $\varphi(x_F) \ge \psi(x_F)$ and hence $\max\{\varphi(x_F),\psi(x_F)\} = \varphi(x_F)$ for all irreducible components F of the special fiber intersecting E, or else $\max\{\varphi,\psi\}$ would not be affine on the face $[x_E, x_F]$ in $\Delta_{\mathcal{X}}$. We have thus shown $\mathrm{ord}_F(A) = \mathrm{ord}_F(C)$ for all components F of $\mathcal{X}_0$ intersecting E. It follows that $A|_E = C|_E$ as numerical classes on E, and hence $\mathrm{MA}(\max\{\varphi,\psi\})\{x_E\} = \mathrm{MA}(\varphi)\{x_E\}$ by the definition of Monge-Ampère measures of model functions.

Step 2. Now suppose that φ is an ω-psh model function but that ψ is merely a bounded ω-psh function.

We may assume $-M \le \varphi,\psi < 0$, where $M \ge 1$. Note that the set $\Omega := \{\varphi > \psi\}$ is open since φ is continuous and ψ is usc. It suffices to prove that $\int h\,\mathrm{MA}(\max\{\varphi,\psi\}) = \int h\,\mathrm{MA}(\varphi)$ for all continuous functions h whose support is contained in Ω and such that $0 \le h \le 1$.

Fix a small number $\delta > 0$. By Proposition 4.4 there exists an open set $G \subseteq X$ and a decreasing sequence $(\psi_j)_{j=1}^\infty$ of ω-psh model functions on X such that $\mathrm{Cap}(G) < \delta$ and such that ψ_j converges uniformly to ψ on G^c. Pick $\varepsilon > 0$ small and rational and write $\Omega_j := \{\varphi+\varepsilon > \psi_j\}$. For $j \gg 0$, we have $\Omega \cap G^c \subseteq \Omega_j$. Since $\varphi+\varepsilon$ and ψ_j are both model functions, we have $\mathrm{MA}(\max\{\varphi+\varepsilon,\psi_j\}) = \mathrm{MA}(\varphi)$ on Ω_j by Step

1. For large enough j, we get

$$\begin{aligned}&\left|\int h\,\mathrm{MA}(\max\{\varphi+\varepsilon,\psi_j\})-\int h\,\mathrm{MA}(\varphi)\right|\\&=\left|\int_\Omega h\,\mathrm{MA}(\max\{\varphi+\varepsilon,\psi_j\})-\int_\Omega h\,\mathrm{MA}(\varphi)\right|\\&=\left|\int_{\Omega\cap G} h\,\mathrm{MA}(\max\{\varphi+\varepsilon,\psi_j\})-\int_{\Omega\cap G} h\,\mathrm{MA}(\varphi)\right|\\&\leq M^n\delta.\end{aligned}$$

Here the first equality holds since h is supported on Ω. The second equality follows from the inclusion $\Omega\cap G^c\subset\Omega_j$ together with the first step. The last inequality results from Lemma 4.8 in view of the inequalities $0\leq h\leq 1$ and $-M\leq\varphi+\varepsilon,\psi\leq 0$ and the estimate $\mathrm{Cap}(\Omega\cap G)\leq\mathrm{Cap}(G)\leq\delta$. Now $\max\{\varphi+\varepsilon,\psi_j\}$ decreases to $\max\{\varphi,\psi\}$ as $j\to\infty$ and $\varepsilon\to 0$, so Theorem 3.1 and the above inequality imply

$$\left|\int h\,\mathrm{MA}(\max\{\varphi,\psi\})-\int h\,\mathrm{MA}(\varphi)\right|\leq M^n\delta.$$

We obtain the desired equality letting $\delta\to 0$.

Step 3. Finally, we treat the general case when φ and ψ are bounded ω-psh functions.

Let $(\varphi_j)_j$ be a decreasing net of ω-psh model functions converging to φ. Write $\Omega_j:=\{\varphi_j>\psi\}$. This is an open set. Set $u:=\max\{0,\varphi-\psi\}$. Then

$$\{\varphi>\psi\}=\{u>0\}\subseteq\bigcap_j\Omega_j.$$

By what precedes, $\mathrm{MA}(\max\{\varphi_j,\psi\})=\mathrm{MA}(\varphi_j)$ on Ω_j. Moreover, $\max\{\varphi_j,\psi\}$ decreases to $\max\{\varphi,\psi\}$ and so the measure $\mathrm{MA}(\max\{\varphi_j,\psi\})$ converges weakly to $\mathrm{MA}(\max\{\varphi,\psi\})$. Let f be a continuous function on X. By Proposition 4.4 φ,ψ are quasicontinuous. It follows that u and fu are also quasicontinuous, and applying Lemma 5.4 twice we get that

$$\begin{aligned}\int fu\,\mathrm{MA}(\max\{\varphi,\psi\})&=\lim_{j\to\infty}\int fu\,\mathrm{MA}(\max\{\varphi_j,\psi\})\\&=\lim_{j\to\infty}\int fu\,\mathrm{MA}(\varphi_j)=\int fu\,\mathrm{MA}(\varphi).\end{aligned}$$

This holds for every $f\in C^0(X)$, so $\mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}(\max\{\varphi,\psi\})=\mathbf{1}_{\{\varphi>\psi\}}\,\mathrm{MA}(\varphi)$, as was to be shown. □

6. Energy

Let ω be a form as in Sec. 4 with $\{\omega\}^n=1$. As in the complex case, it turns out that the non-Archimedean Monge-Ampère operator admits a primitive, i.e., a functional whose directional derivatives at a given φ are given by integration against $\mathrm{MA}(\varphi)$. Adapting [GZ07, BEGZ10] to our case we introduce and study this functional, as well as the resulting class of ω-psh functions of finite energy. While such functions are unbounded in general, they behave from many points of view like bounded ω-psh functions.

6.1. **Energy of model functions.** For any model function φ we set

$$E(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j} \tag{6.1}$$

and call $E(\varphi)$ the *energy* of φ. Note that the energy depends on the choice of form ω, but we will not explicitly write out this dependence.

It follows formally from an integration by parts argument, see Proposition 2.20 and [Tia, Lemma 6.2] that if φ, ψ are any two model functions, then

$$E(\psi) - E(\varphi) = \frac{1}{n+1} \sum_{j=0}^{n} \int (\psi - \varphi)(\omega + dd^c\varphi)^j \wedge (\omega + dd^c\psi)^{n-j}. \tag{6.2}$$

Write $\varphi_t = (1-t)\varphi + t\psi$ for $0 \le t \le 1$. From (6.1) and (6.2) we see that $E(\varphi_t)$ is a polynomial in t of degree at most $n+1$ and that

$$E'(\varphi) \cdot (\psi - \varphi) := \left.\frac{d}{dt}\right|_{t=0+} E(\varphi_t) = \int (\psi - \varphi)\, \mathrm{MA}(\varphi); \tag{6.3}$$

$$E''(\varphi) \cdot (\psi - \varphi) := \left.\frac{d^2}{dt^2}\right|_{t=0+} E(\varphi_t) = n \int (\psi - \varphi) dd^c(\psi - \varphi)\, \mathrm{MA}(\varphi). \tag{6.4}$$

Proposition 6.1. *The restriction of E to the convex set $\mathrm{PSH}(X,\omega) \cap \mathcal{D}(X)$ is concave, non-decreasing, and satisfies $E(\varphi + c) = E(\varphi) + c$ for any constant $c \in \mathbf{R}$.*

Proof. Concavity follows from (6.4) and Proposition 2.21. Monotonicity is a consequence of (6.3), and the last equation follows from (6.2) since $(\omega + dd^c\varphi)^j \wedge \omega^{n-j}$ is a probability measure for each j thanks to Proposition 2.19 and the normalization $\{\omega\}^n = 1$. $\square$

6.2. **Energy of ω-psh functions.** For a general ω-psh function φ we set

$$E(\varphi) := \inf \{E(\psi) \mid \psi \in \mathrm{PSH}(X,\omega) \cap \mathcal{D}(X), \psi \ge \varphi\} \in [-\infty, +\infty[.$$

Proposition 6.2. *The extension $E : \mathrm{PSH}(X,\omega) \to [-\infty, +\infty[$ is non-decreasing, concave, and satisfies $E(\varphi + c) = E(\varphi) + c$ for any $c \in \mathbf{R}$. It is also upper semi-continuous, and continuous along decreasing nets.*

Proof. That E is non-decreasing, concave, and satisfies $E(\varphi + c) = E(\varphi) + c$ follows formally from Proposition 6.1 (using that $\mathrm{PSH}(X,\omega) \cap \mathcal{D}(X)$ is convex and invariant under addition of a constant).

Upper semicontinuity is also a direct consequence of these formal properties of E and of Theorem 2.10. Indeed, pick $\varphi_0 \in \mathrm{PSH}(X,\omega)$ and $t \in \mathbf{R}$ such that $E(\varphi_0) < t$. We need to show that $E(\varphi) < t$ for φ in a neighborhood U of φ_0 in $\mathrm{PSH}(X,\omega)$. By definition, there exists $\psi_0 \in \mathrm{PSH}(X,\omega) \cap \mathcal{D}(X)$ such that $\psi_0 \ge \varphi_0$ and $E(\psi_0) < t - \varepsilon$ for some $\varepsilon > 0$. By Theorem 2.10, $U := \{\varphi \in \mathrm{PSH}(X,\omega) \mid \sup_X(\varphi - \psi_0) < \varepsilon\}$ is an open neighborhood of φ_0 in $\mathrm{PSH}(X,\omega)$. By (6.2) we have $E(\varphi) \le E(\psi_0) + \varepsilon < t$ for all $\varphi \in U$, which proves upper semicontinuity.

Finally, being usc and non-decreasing, E is automatically continuous along decreasing nets. $\square$

Proposition 6.3. *Formulas* (6.1)–(6.4) *are valid for bounded ω-psh functions φ and ψ. Further, $E((1-t)\varphi + t\psi)$ is a polynomial in t of degree at most $n+1$ for $0 \le t \le 1$.*

Proof. Let φ and ψ be bounded ω-psh functions and let $(\varphi_j)_j$, $(\psi_j)_j$ be decreasing nets of ω-psh model functions converging to φ and ψ, respectively. Write $\varphi_t := (1-t)\varphi+t\psi$ and $\varphi_{j,t} := (1-t)\varphi_j+t\psi_j$ for $0 \le t \le 1$. For each j, $h_j(t) := E(\varphi_{j,t})$ is a polynomial in t of degree at most $n+1$. Since E is continuous along decreasing nets, $h(t) := E(\varphi_t)$ is also a polynomial in t of degree at most $n+1$. This implies that $h_j'(0+)$ and $h_j''(0+)$ converge to $h'(0+)$ and $h''(0+)$, respectively. By (6.3) applied to φ_j and ψ_j we have $h_j'(0+) = \int(\psi_j - \varphi_j)\,\mathrm{MA}(\varphi_j)$, which tends to $\int(\psi-\varphi)\,\mathrm{MA}(\varphi)$ in view of Theorem 3.1. This proves (6.3). The proof of (6.4) is similar. ☐

6.3. **Non-pluripolar Monge-Ampère measures.** Let us introduce the class of ω-psh functions with finite energy

$$\mathcal{E}^1(X,\omega) := \{\varphi \in \mathrm{PSH}(X,\omega) \mid E(\varphi) > -\infty\}\,.$$

This is a convex set which contains all bounded ω-psh functions.

In this section and its sequel, we explain how to extend the Monge-Ampère operator to $\mathcal{E}^1(X,\omega)$ and prove that its basic properties continue to hold in this more general setting.

Consider an arbitrary ω-psh function φ. We will use the notation

$$\varphi^{\langle t\rangle} := \max\{\varphi, -t\}.$$

Note that if $s > t$, then $\{\varphi > -t\} = \{\varphi^{\langle s\rangle} > -t\}$ and $\max\{\varphi^{\langle s\rangle}, -t\} = \varphi^{\langle t\rangle}$; hence, Theorem 5.1 implies

$$\begin{aligned}\mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi^{\langle s\rangle}) &= \mathbf{1}_{\{\varphi^{\langle s\rangle}>-t\}}\,\mathrm{MA}(\varphi^{\langle s\rangle}) = \mathbf{1}_{\{\varphi^{\langle s\rangle}>-t\}}\,\mathrm{MA}(\varphi^{\langle t\rangle})\\ &= \mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi^{\langle t\rangle}).\end{aligned}$$

This equation allows us to introduce

Definition 6.4. [BT87,GZ07] *The* non-pluripolar Monge-Ampère measure $\mathrm{MA}(\varphi)$ *of any ω-psh function φ is the increasing limit of the measures* $\mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi^{\langle t\rangle})$ *as $t \to \infty$.*

Here the limit exists in a very strong sense: we have

$$\lim_{t\to\infty} \mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi^{\langle t\rangle})(F) = \mathrm{MA}(\varphi)(F) \tag{6.5}$$

for any Borel set F. Further, we have

$$\mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi) = \mathbf{1}_{\{\varphi>-t\}}\,\mathrm{MA}(\varphi^{\langle t\rangle}). \tag{6.6}$$

Remark 6.5. *The terminology non-pluripolar comes from the fact that* $\mathrm{MA}(\varphi)$ *does not put mass on pluripolar sets. This in turn follows from Proposition* 3.11 *applied to the bounded ω-psh function $\varphi^{\langle t\rangle}$ and from* (6.5)*.*

The measure $\mathrm{MA}(\varphi)$ is always defined and supported on the set $\{\varphi > -\infty\}$, but its total mass may be strictly less than one.

Definition 6.6. *A ω-psh function φ has* full Monge-Ampère mass *when* $\mathrm{MA}(\varphi)$ *is a probability measure.*

This is the case iff $\mathrm{MA}(\varphi^{\langle t\rangle})\{\varphi \le -t\} \to 0$ as $t \to \infty$, and implies that $\mathrm{MA}(\varphi^{\langle t\rangle})$ converges weakly to $\mathrm{MA}(\varphi)$.

Lemma 6.7. *If $\varphi \in \mathcal{E}^1(X,\omega)$, then* $\mathrm{MA}(\varphi^{\langle t\rangle})\{\varphi \le -t\} = o(t^{-1})$ *as $t \to \infty$; hence φ has full Monge-Ampère mass.*

Proof. We may assume $\varphi \leq 0$. Set $\mu_t := \mathrm{MA}(\varphi^{\langle t\rangle})$. Since (6.2) applies to bounded ω-psh functions by Proposition 6.3, we get

$$\begin{aligned} E(\varphi^{\langle t/2\rangle}) - E(\varphi^{\langle t\rangle}) &\geq \frac{1}{n+1}\int (\varphi^{\langle t/2\rangle} - \varphi^{\langle t\rangle})\mu_t \\ &= \frac{1}{n+1}\int_0^{t/2} \mu_t\big\{\varphi^{\langle t/2\rangle} - \varphi^{\langle t\rangle} \geq s\big\}\, ds \\ &\geq \frac{1}{n+1}\int_0^{t/2} \mu_t\big\{\varphi^{\langle t/2\rangle} - \varphi^{\langle t\rangle} \geq t/2\big\}\, ds \\ &= \frac{t}{2(n+1)}\mu_t\left\{\varphi \leq -t\right\}. \end{aligned}$$

Since $\lim_{t\to\infty} E(\varphi^{\langle t/2\rangle}) = \lim_{t\to\infty} E(\varphi^{\langle t\rangle}) = E(\varphi)$ by continuity of E along decreasing sequences, the proof is complete. $\square$

Lemma 6.8. *If $0 \geq \varphi \in \mathcal{E}^1(X,\omega)$ and $f \in \mathcal{D}(X)$, then*

$$\left|\int f\,\mathrm{MA}(\varphi^{\langle t\rangle}) - \int f\,\mathrm{MA}(\varphi)\right| \leq \frac{2(n+1)}{t}|E(\varphi)|\sup_X |f|$$

for any $t > 0$.

Proof. We may assume $\sup_X |f| = 1$. Pick $s \geq t$. The probability measures $\mu_t := \mathrm{MA}(\varphi^{\langle t\rangle})$ and μ_s agree on $\{\varphi > -t\}$. Hence,

$$\begin{aligned} \left|\int f\mu_t - \int f\mu_s\right| &\leq (\mu_t + \mu_s)\{\varphi \leq -t\} \leq \int \frac{-\varphi^{\langle t\rangle}}{t}\mu_t + \int \frac{-\varphi^{\langle s\rangle}}{t}\mu_s \\ &\leq \frac{n+1}{t}(|E(\varphi^{\langle t\rangle})| + |E(\varphi^{\langle s\rangle})|) \leq \frac{2(n+1)}{t}|E(\varphi)|. \end{aligned}$$

Here, the second inequality follows since $-\varphi^{\langle s\rangle} \geq -\varphi^{\langle t\rangle} \geq 0$ on X and $-\varphi^{\langle s\rangle} \geq -\varphi^{\langle t\rangle} \geq t$ on the set $\{\varphi \leq -t\}$. The result follows by letting $s \to \infty$. $\square$

Proposition 6.9. *If $\varphi \in \mathcal{E}^1(X,\omega)$ and $(\varphi_j)_j$ is a decreasing net of ω-psh functions converging to φ, then $\varphi_j \in \mathcal{E}^1(X,\omega)$ for all j and $\mathrm{MA}(\varphi_j) \to \mathrm{MA}(\varphi)$ as $j \to \infty$ in the weak sense of measures.*

Proof. Given $f \in \mathcal{D}(X)$, we have by definition that $\int f\,\mathrm{MA}(\varphi^{\langle t\rangle}) \to \int f\,\mathrm{MA}(\varphi)$ as $t \to \infty$ and $\int f\,\mathrm{MA}(\varphi_j^{\langle t\rangle}) \to \int f\,\mathrm{MA}(\varphi_j)$ as $t \to \infty$ for every j. Moreover, Lemma 6.8 shows that the latter convergence is uniform in j. Since for each t we have $\int f\,\mathrm{MA}(\varphi_j^{\langle t\rangle}) \to \int f\,\mathrm{MA}(\varphi^{\langle t\rangle})$ as $j \to \infty$ by Theorem 3.1, the result follows. $\square$

Lemma 6.10. *If $\varphi, \psi \in \mathcal{E}^1(X,\omega)$ and $\varphi, \psi \leq 0$, then we have the estimate*

$$-\infty < E\left(\frac{\varphi+\psi}{2}\right) \leq \frac{2^{-(n+1)}}{n+1}\int \psi\,\mathrm{MA}(\varphi).$$

Proof. Since E is concave, we have $E(\frac{\varphi+\psi}{2}) \geq \frac{1}{2}(E(\psi) + E(\psi)) > -\infty$. Now pick $s, t > 0$. Since (6.1) holds for bounded ω-psh functions, we see using (3.2) that

$$E\left(\frac{\varphi+\psi}{2}\right) \leq E\left(\frac{\varphi^{\langle t\rangle} + \psi^{\langle s\rangle}}{2}\right) \leq \frac{2^{-(n+1)}}{n+1}\int \psi^{\langle s\rangle}\,\mathrm{MA}(\varphi^{\langle t\rangle}).$$

As $s \to \infty$, $\psi^{\langle s\rangle}$ decreases to ψ pointwise on X, so the right-hand side converges to

$$\frac{2^{-(n+1)}}{n+1}\int \psi\, \mathrm{MA}(\varphi^{\langle t\rangle}) \le \frac{2^{-(n+1)}}{n+1}\int_{\{\varphi>-t\}} \psi\, \mathrm{MA}(\varphi^{\langle t\rangle}) = \frac{2^{-(n+1)}}{n+1}\int_{\{\varphi>-t\}} \psi\, \mathrm{MA}(\varphi)$$

by monotone convergence. We obtain the desired estimate by letting $t \to \infty$. □

6.4. **Locality and the comparison principle.**

Proposition 6.11. *For any $\varphi, \psi \in \mathcal{E}^1(X,\omega)$, we have*

$$\mathbf{1}_{\{\varphi>\psi\}}\, \mathrm{MA}(\max\{\varphi,\psi\}) = \mathbf{1}_{\{\varphi>\psi\}}\, \mathrm{MA}(\varphi), \tag{6.7}$$

and the comparison principle holds:

$$\int_{\{\varphi<\psi\}} \mathrm{MA}(\psi) \le \int_{\{\varphi<\psi\}} \mathrm{MA}(\varphi). \tag{6.8}$$

Proof. Set $u := \max\{\varphi,\psi\} \in \mathcal{E}^1(X,\omega)$. Then

$$\begin{aligned}\mathbf{1}_{\{\varphi^{\langle t\rangle}>\psi^{\langle t\rangle}\}}\, \mathrm{MA}(\varphi) &= \mathbf{1}_{\{\varphi^{\langle t\rangle}>\psi^{\langle t\rangle}\}}\, \mathrm{MA}(\varphi^{\langle t\rangle})\\ &= \mathbf{1}_{\{\varphi^{\langle t\rangle}>\psi^{\langle t\rangle}\}}\, \mathrm{MA}(u^{\langle t\rangle})\\ &= \mathbf{1}_{\{\varphi^{\langle t\rangle}>\psi^{\langle t\rangle}\}}\, \mathrm{MA}(u). \end{aligned}\tag{6.9}$$

Here, the first equality follows from (6.6) and the inclusion $\{\varphi^{\langle t\rangle} > \psi^{\langle t\rangle}\} \subset \{\varphi > -t\}$. The second equality is a consequence of the locality property (5.1) for bounded ω-psh functions, since $\max\{\varphi^{\langle t\rangle}, \psi^{\langle t\rangle}\} = u^{\langle t\rangle}$. The third equality again follows from (6.6) (with u instead of φ) in view of the inclusion $\{\varphi^{\langle t\rangle} > \psi^{\langle t\rangle}\} \subset \{u > -t\}$. Now

$$\mathbf{1}_{\{\varphi^{\langle t\rangle}>\psi^{\langle t\rangle}\}}\, \mathrm{MA}(u) = \mathbf{1}_{\{\varphi>-t\ge\psi\}}\, \mathrm{MA}(u) + \mathbf{1}_{\{\varphi>\psi>-t\}}\, \mathrm{MA}(u).$$

As $t \to \infty$, the first term on the right-hand side tends to 0 since $\mathrm{MA}(u)$ puts no mass on the pluripolar set $\{\psi = -\infty\}$ (see Remark 6.5), and the second term converges to $\mathbf{1}_{\{\varphi>\psi>-\infty\}}\, \mathrm{MA}(u) = \mathbf{1}_{\{\varphi>\psi\}}\, \mathrm{MA}(u)$. Thus, the right-hand side of (6.9) tends to $\mathbf{1}_{\{\varphi>\psi\}}\, \mathrm{MA}(u)$ as $t \to \infty$. Similarly, the left-hand side tends to $\mathbf{1}_{\{\varphi>\psi\}}\, \mathrm{MA}(\varphi)$, completing the proof of (6.7). Finally, the comparison principle (6.8) follows exactly as in the proof of Corollary 5.3. □

6.5. **Differentiability.**

Proposition 6.12. *For any $\varphi, \psi \in \mathcal{E}^1(X,\omega)$, the function $t \mapsto h_{\varphi,\psi}(t) := E((1-t)\varphi + t\psi)$ is a polynomial of degree at most $n+1$ for $0 \le t \le 1$. In particular, $h_{\varphi,\psi}$ is differentiable and we have*

$$E'(\varphi)\cdot(\psi-\varphi) := h'_{\varphi,\psi}(0+) = \int (\psi-\varphi)\, \mathrm{MA}(\varphi). \tag{6.10}$$

Proof. By (6.1), $h_{\varphi,\psi}$ is a polynomial of degree at most $n+1$ when φ and ψ are model functions. By continuity of the energy along decreasing nets, the same is true in general. In particular, $h_{\varphi,\psi}$ is differentiable on $[0,1]$.

Pick any decreasing net $(\psi_j)_j$ of ω-psh model functions converging to ψ. Note that $h_{\varphi^{\langle s\rangle},\psi_j} \to h_{\varphi,\psi}$ as polynomials when $s \to \infty$ and $j \to \infty$; hence $h'_{\varphi^{\langle s\rangle},\psi_j}(0+) \to h'_{\varphi,\psi}(0+)$. Since (6.10) holds true for bounded functions by Proposition 6.3, it suffices to show

$$\lim_{j\to\infty}\lim_{s\to\infty} \int (\psi_j - \varphi^{\langle s\rangle})\, \mathrm{MA}(\varphi^{\langle s\rangle}) = \int (\psi - \varphi)\, \mathrm{MA}(\varphi).$$

First, we have

$$\begin{aligned}\int \varphi^{\langle s\rangle}\,\mathrm{MA}(\varphi^{\langle s\rangle}) &= \int_{\{\varphi\le -s\}}(-s)\,\mathrm{MA}(\varphi^{\langle s\rangle}) + \int_{\{\varphi>-s\}}\varphi\,\mathrm{MA}(\varphi^{\langle s\rangle})\\ &= \int_{\{\varphi\le -s\}}(-s)\,\mathrm{MA}(\varphi^{\langle s\rangle}) + \int_{\{\varphi>-s\}}\varphi\,\mathrm{MA}(\varphi)\end{aligned}$$

by (6.7). By Lemma 6.7 the first term on the right-hand side tends to 0, and the second term converges to $\int \varphi\,\mathrm{MA}(\varphi)$ since $\mathrm{MA}(\varphi)$ puts no mass on $\{\varphi = -\infty\}$.

Second, for fixed j we have $\lim_{s\to\infty}\int \psi_j\,\mathrm{MA}(\varphi^{\langle s\rangle}) = \int \psi_j\,\mathrm{MA}(\varphi)$ since ψ_j is continuous.

Finally, Lemma 2.23 yields $\lim_{j\to\infty}\int \psi_j\,\mathrm{MA}(\varphi) = \int \psi\,\mathrm{MA}(\varphi)$, completing the proof. □

7. Envelopes and differentiability

Let ω be a form as in Secs. 4–6. As explained in the Introduction, the differentiability of the energy is not *a priori* sufficient to make the variational approach work, i.e., to infer that a maximizer of the relevant functional over $\mathcal{E}^1(X,\omega)$ is necessarily a critical point. In order to circumvent this difficulty, we establish as in [BB10] the differentiability of $E\circ P$, where P is the ω-psh envelope operator of Sec. 2.5. This idea was originally introduced by Alexandrov [Ale38] in the context of real Monge-Ampère equations.

Recall that the Monge-Ampère operator MA, the energy E and the envelope P all depend on the form ω, but we suppress this dependence in the notation.

Definition 7.1. *We say that ω has the* orthogonality property *if*

$$\int (f - P(f))\,\mathrm{MA}(P(f)) = 0$$

holds for every $f \in C^0(X)$.

Since $P(f) \le f$, this property means that $\mathrm{MA}(P(f))$ is concentrated on the contact locus $\{P(f) = f\}$. We refer to Appendix A for more information on the orthogonality property.

We can now state the main result of this section.

Theorem 7.2. *Assume that ω has the orthogonality property. Then the composition*

$$E\circ P : C^0(X) \to \mathbf{R}$$

is Gâteaux differentiable, with directional derivatives given by

$$\left.\frac{d}{dt}\right|_{t=0} E(P(f+tg)) = \int g\,\mathrm{MA}(P(f)).$$

Before giving a proof of this crucial result, we deduce a more general version that we will need when solving the Monge-Ampère equation.

If $\varphi \in \mathrm{PSH}(X,\omega)$ and $f \in C^0(X)$, observe that $P(\varphi+f)$ is ω-psh (i.e., is not identically $-\infty$) since $\varphi + f$ dominates the ω-psh function $\varphi + \inf_X f$. Furthermore, we have $P(\varphi+f) \le \varphi + f$ since the latter function is usc.

Corollary 7.3. *Assume that ω has the orthogonality property. Let $\varphi \in \mathcal{E}^1(X,\omega)$ and $g \in C^0(X)$. Then $P(\varphi+tg) \in \mathcal{E}^1(X,\omega)$ for all $t \in \mathbf{R}$, the function $t \mapsto E(P(\varphi+tg))$ is differentiable, and*

$$\frac{d}{dt}\bigg|_{t=0} E(P(\varphi+tg)) = \int g \,\mathrm{MA}(\varphi).$$

Proof. Note that $\varphi + tg \geq \varphi - |t|\sup_X |g|$ implies $P(\varphi+tg) \geq \varphi - |t|\sup_X |g|$; hence, $P(\varphi+tg) \in \mathcal{E}^1(X,\omega)$ for all t. We are going to show that

$$E(P(\varphi+tg)) = E(\varphi) + \int_0^t \left(\int g \,\mathrm{MA}(P(\varphi+sg))\right) ds \tag{7.1}$$

for all $t \in \mathbf{R}$, which will complete the proof.

If φ is continuous, then (7.1) is an immediate consequence of Theorem 7.2. In the general case, let $(\varphi_m)_{m=1}^\infty$ be a decreasing *sequence* of ω-psh model functions converging to φ; see Proposition 4.7.

For each $t \in \mathbf{R}$, $(P(\varphi_m + tg))_{m=1}^\infty$ is a decreasing sequence of ω-psh functions, and we claim that $\lim_m P(\varphi_m+tg) = P(\varphi+tg)$. Indeed, let $\tilde{\varphi}_t := \lim_m P(\varphi_m+tg)$. Since $\varphi_m + tg \geq \varphi + tg$, we have $P(\varphi_m+tg) \geq P(\varphi+tg)$; hence, $\tilde{\varphi}_t \geq P(\varphi+tg)$. For the reverse inequality, note that $\tilde{\varphi}_t \leq P(\varphi_m+tg) \leq \varphi_m + tg$ for all m, so that $\tilde{\varphi}_t \leq \varphi + tg$. Since $\tilde{\varphi}_t$ is ω-psh, we infer that $\tilde{\varphi}_t \leq P(\varphi+tg)$.

Now apply (7.1) to φ_m:

$$E(P(\varphi_m+tg)) = E(\varphi_m) + \int_0^t \left(\int f \,\mathrm{MA}(P(\varphi_m+sg))\right) ds. \tag{7.2}$$

As $m \to \infty$, $E(P(\varphi_m+tg))$ and $E(\varphi_m)$ decrease to $E(P(\varphi+tg))$ and $E(\varphi)$, respectively, and by Proposition 6.9 $\int g \,\mathrm{MA}(P(\varphi_m+sg))$ converges to $\int g\,\mathrm{MA}(P(\varphi+sg))$ for each s. Now (7.1) follows from (7.2) using dominated convergence, in view of the upper bound $|\int g\,\mathrm{MA}(P(\varphi_m+sg))| \leq \sup_X |g|$ for all m and all s. ☐

Proof of Theorem 7.2. We follow the exposition in [BB10, Sec. 4.3] very closely. Arguing as in Corollary 7.3 we may assume that $f, g \in \mathcal{D}(X)$. Set $\mu := \mathrm{MA}(P(f))$. We need to prove that

$$\frac{d}{dt}\bigg|_{t=0+} E(P(f+tg)) = \int g\,\mu. \tag{7.3}$$

As a first step, we linearize the problem and prove that

$$\frac{d}{dt}\bigg|_{t=0+} E(P(f+tg)) = \frac{d}{dt}\bigg|_{t=0+} \int P(f+tg)\,\mu. \tag{7.4}$$

Denote the left- and right-hand sides of (7.4) by a and b, respectively. Note that the one-sided derivatives exist since both E and P are concave.

The concavity of E also implies that the function

$$[0,1] \ni s \mapsto h(s) := E\left(sP(f+tg) + (1-s)P(f)\right)$$

is concave; hence,

$$E(P(f+tg)) = h(1) \leq h(0) + h'(0+) = E(P(f)) + \int (P(f+tg) - P(f))\mu$$

by Proposition 6.3. Letting $t \to 0$ yields $a \leq b$.

To prove the reverse inequality, fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$D := \int_X P(f+\delta g)\mu - \int_X P(f)\mu \geq \delta(b-\varepsilon).$$

Since μ is the differential of E, there exists $\gamma > 0$ such that

$$E\left((1-t)P(f) + tP(f+\delta g)\right) \geq E(P(f)) + t(D - \delta\varepsilon) \geq E(P(f)) + t\delta(b-2\varepsilon)$$

for $0 \leq t \leq \gamma$. The concavity of P yields $P(f+t\delta g) \geq (1-t)P(f) + tP(f+\delta g)$. Since E is non-decreasing we get

$$E(P(f+t\delta g)) \geq E\left((1-t)P(f) + tP(f+\delta g)\right) \geq E(P(f)) + t\delta(b-2\varepsilon)$$

for $0 \leq t \leq \gamma$. Letting $t \to 0$ and $\varepsilon \to 0$ we conclude $a \geq b$. This shows that (7.4) holds.

In view of (7.4) it remains to show that

$$\int_X \left(P(f+tg) - P(f)\right)\mu = t\int_X g\mu + o(t) \tag{7.5}$$

as $t \to 0+$.

Since $P(f) \leq f$, the orthogonality property implies $P(f) = f$ for μ-a.e. point. We thus have $P(f+tg) \leq f + tg = P(f) + tg$ μ-a.e. We claim that $\mu(\Omega_t) = O(t)$, where

$$\Omega_t := \{P(f+tg) < P(f) + tg\}.$$

Observe that $|P(f+tg) - P(f)| \leq t \sup |g|$ so that the claim implies

$$\begin{aligned}\int_X (P(f+tg) - P(f) - tg)\mu &= \int_{\Omega_t} (P(f+tg) - P(f) - tg) \\ &\leq \mu(\Omega_t) \sup_X |P(f+tg) - P(f) - tg| \\ &= O(t^2)\end{aligned}$$

which proves (7.5).

The estimate of $\mu(\Omega_t)$ is based on the comparison principle. Since g is a model function, there exists $C \gg 1$, $\psi \in \mathcal{D}(X)$ such that ψ and $\psi + g$ are $C\omega$-psh by Proposition 2.6. Note that $\Omega_t = \{P(f+tg) + t\psi < P(f) + t(\psi+g)\}$, and both functions $P(f+tg) + t\psi$ and $P(f) + t(\psi+g)$ are $(1+Ct)\omega$-psh. The comparison principle then yields

$$\int_{\Omega_t} \left((1+Ct)\omega + dd^c\left(P(f) + t(\psi+g)\right)\right)^n \leq \int_{\Omega_t} \left((1+Ct)\omega + dd^c\left(P(f+tg) + t\psi\right)\right)^n.$$

By expanding as polynomials in t, we get

$$\left((1+Ct)\omega + dd^c\left(P(f) + t(\psi+g)\right)\right)^n = (\omega + dd^c P(f))^n + O(t)$$

and

$$\left((1+Ct)\omega + dd^c\left(P(f+tg) + t\psi\right)\right)^n = (\omega + dd^c P(f+tg))^n + O(t).$$

From these three estimates we conclude

$$\mu(\Omega_t) = \int_{\Omega_t} \mathrm{MA}(P(f)) \leq \int_{\Omega_t} \mathrm{MA}(P(f+tg)) + O(t).$$

But $\Omega_t \subseteq \{P(f+tg) < f + tg\}$, so the orthogonality property implies that the last integral vanishes. This concludes the proof. $\square$

Remark 7.4. *Conversely, the differentiability property of Theorem* 7.2 *implies the orthogonality property. Indeed, pick* $f \in C^0(X)$ *and set* $g := P(f) - f$*. We claim that* $\int g \, \mathrm{MA}(P(f)) = 0$*. It is enough to prove* $\int g \, \mathrm{MA}(P(f)) \geq 0$ *since* $g \leq 0$*. Now the differentiability property yields*

$$E(P(f+tg)) = E(P(f)) + t \int g \, \mathrm{MA}(P(f)) + o(t).$$

But we have

$$f + tg = (1-t)f + tP(f) \geq (1-t)P(f) + tP(f) = P(f),$$

hence $E\left(P(f+tg)\right) \geq E(P(f))$ *by monotonicity of* E*, and the result follows.*

8. The Monge-Ampère equation

In this section we prove the following.

Theorem 8.1. *Let* $\omega \in \mathcal{Z}^{(1,1)}(X)$ *be a semipositive closed* $(1,1)$*-form with ample de Rham class* $\{\omega\}$ *and normalized by* $\{\omega\}^n = 1$*. Assume that* ω *satisfies the orthogonality property (see Definition* 7.1*). Let* μ *be a probability measure on* X *supported on the dual complex of some SNC model of* X*. Then there exists a unique, continuous* ω*-psh function* $\varphi \in \mathrm{PSH}(X,\omega)$ *such that* $\mathrm{MA}(\varphi) = \mu$*, and* $\sup \varphi = 0$*.*

Let us explain how to deduce Theorems A and A' from the introduction. Let ω be any closed semipositive form with $\{\omega\}$ ample, and μ be a positive Radon measure of mass $\{\omega\}^n$. Set $\tilde{\omega} := \omega/(\{\omega\}^n)^{1/n}$, and $\tilde{\mu} = \mu/\{\omega\}^n$. Assume that X is defined over a function field (i.e., satisfies the condition (†) from the introduction). It follows from Appendix A that ω and $\tilde{\omega}$ satisfy the orthogonality property. Applying Theorem 8.1 to $\tilde{\omega}$ and $\tilde{\mu}$ yields a unique $\tilde{\varphi} \in \mathrm{PSH}(X, \tilde{\omega})$ such that $\sup \tilde{\varphi} = 0$ and $(\tilde{\omega} + dd^c \tilde{\varphi})^n = \tilde{\mu}$. Theorem A' follows since $(\omega + dd^c \varphi)^n = \mu$ with $\varphi = (\{\omega\}^n)^{1/n} \, \tilde{\varphi}$.

Now consider an ample line bundle $L \to X$ endowed with a semipositive model metric $\|\cdot\|$. The curvature form $\omega = c_1(L, \|\cdot\|)$ is semipositive and $\{\omega\}^n = c_1(L)^n$ in view (2.1). Given any positive Radon measure μ of mass $c_1(L)^n$ and supported on the dual complex of some SNC model of X, Theorem A' thus implies the existence of a unique continuous ω-psh function φ such that $\mathrm{MA}(\varphi) = \mu$, and $\sup \varphi = 0$. This statement implies Theorem A since $c_1(L, \|\cdot\| e^{-\varphi})^n = \mathrm{MA}(\varphi)$ by definition.

For the rest of this section, the closed $(1,1)$-form ω will be as in Secs. 4–7, that is, ω is semipositive, $\{\omega\}$ is ample and $\{\omega\}^n = 1$.

8.1. **Uniqueness.**

The uniqueness statement in Theorem 8.1 does not require the orthogonality property. Following [Bło03] as in [GZ07, YZ13a], we actually prove the following.

Proposition 8.2. *Let* ω *be any semipositive closed* $(1,1)$ *form. Suppose* $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$ *for any two functions* $\varphi, \psi \in \mathcal{E}^1(X,\omega)$*. Then* $\varphi - \psi$ *is constant.*

Proof. First we briefly indicate how to extend to ω-psh functions of finite energy the calculus that we developed in Sec. 3. Let $\varphi, \psi \in \mathcal{E}^1(X,\omega)$. Since E is concave, $(1-t)\varphi + t\psi \in \mathcal{E}^1(X,\omega)$ for any $t \in [0,1]$. Define $\omega_\varphi^i \wedge \omega_\psi^{n-i}$ for $0 \leq i \leq n$ to be the unique signed Radon measures such that

$$\sum_{i=0}^{n} \binom{n}{i} (1-t)^i t^{n-i} \omega_\varphi^i \wedge \omega_\psi^{n-i} = \mathrm{MA}((1-t)\varphi + t\psi) \tag{8.1}$$

for any $t = j/n$ with $0 \le j \le n$. In particular, $\omega_\varphi^n = (\omega + dd^c\varphi)^n$ and $\omega_\psi^n = (\omega + dd^c\psi)^n$. By Proposition 6.9, we get $\omega_{\varphi_j}^i \wedge \omega_{\psi_j}^{n-i} \to \omega_\varphi^i \wedge \omega_\psi^{n-i}$ for any decreasing nets of ω-psh functions $\varphi_j \to \varphi$ and $\psi_j \to \psi$. In particular, $\omega_\varphi^i \wedge \omega_\psi^{n-i}$ is a probability measure and in fact (8.1) holds for all $0 \le t \le 1$.

Replacing $\mathrm{MA}(\cdot) = (\omega + dd^c\cdot)^n$ by $(\omega + dd^c\cdot)^{i+j} \wedge \omega^{n-(i+j)}$ in (8.1), we can further define probability measures $\omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^{n-(i+j)}$ as soon as $i, j \ge 0$ and $i + j \le n$. Observe that by definition and Lemma 6.10, these measures integrate ω-psh functions of finite energy. It also follows that the Cauchy-Schwarz inequality

$$\int g dd^c h \wedge T \le \left(\int -g dd^c g \wedge T \right)^{1/2} \left(\int -h dd^c h \wedge T \right)^{1/2},$$

holds whenever g, h lie in the vector space generated by $\mathcal{E}^1(X, \omega)$ and T is a positive linear combination of measures of the type $\omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^{n-(i+j)}$ with $i + j \le n$.

We claim that

$$(8.2) \qquad \int (\psi - \varphi) dd^c(\varphi - \psi) \wedge \omega^{n-1} \le C \left(\int (\psi - \varphi)(\mathrm{MA}(\varphi) - \mathrm{MA}(\psi)) \right)^{2^{1-n}}$$

for some constant C depending on φ and ψ.

Grant (8.2) and suppose $\mathrm{MA}(\varphi) = \mathrm{MA}(\psi)$. We conclude the proof as in [YZ13a]. By the Cauchy-Schwarz inequality, for any model function h we get

$$(8.3) \qquad \int (\varphi - \psi) dd^c h \wedge \omega^{n-1} \le A^{1/2} \left(\int (\psi - \varphi) dd^c(\varphi - \psi) \wedge \omega^{n-1} \right)^{1/2} = 0,$$

with $0 \le A := \int -h dd^c h \wedge \omega^{n-1} < +\infty$.

It follows from [BFJ09, Proposition 5.2] that for any sufficiently high model $\mathcal{X}$ of X there exists a model function g determined on $\mathcal{X}$ such that $\tilde{\omega} := \omega + dd^c g$ is the class of an *ample* $\mathbf{R}$-line bundle $\mathcal{L}$ on $\mathcal{X}$. The functions $\tilde{\varphi} := \varphi - g$ and $\tilde{\psi} := \psi - g$ are both $\tilde{\omega}$-psh and satisfy $(\tilde{\omega} + dd^c\tilde{\varphi})^n = (\tilde{\omega} + dd^c\tilde{\psi})^n$. Write the special fiber as $\mathcal{X}_0 = \sum_E b_E E$. Let h be the model function determined in $\mathcal{X}$ such that $h(x_E) = \varphi(x_E) - \psi(x_E) = \tilde{\varphi}(x_E) - \tilde{\psi}(x_E)$ for any E. Then (8.3) gives

$$D^2 \cdot \mathcal{L}^{n-1} = 0,$$

where $D = \sum_E b_E h(x_E) E$. By the Hodge Index Theorem (see [YZ13a, Theorem 2.9]), this implies that D is proportional to $\mathcal{X}_0$, which translates into $\varphi - \psi$ being constant on the vertices of $\Delta_{\mathcal{X}}$. Varying $\mathcal{X}$, we see that $\varphi - \psi$ is constant on X^{div} and hence on X, since an ω-psh function is determined by its values on divisorial points, see [BFJ09, Corollary 7.7].

We now prove (8.2). For this we reproduce the argument of [Bło03]. By C we will denote possibly different constants depending on φ and ψ. (In fact, they will be bounded in terms of the energies $E(\varphi)$ and $E(\psi)$, but we do not need this.) Set $\rho = \varphi - \psi$. We will prove that if $i, j, k \ge 0$ and $i + j + k = n - 1$, then

$$(8.4) \qquad 0 \le \int -\rho dd^c \rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k \le C a^{2^{-k}}$$

where

$$a = \int (\psi - \varphi)(\mathrm{MA}(\varphi) - \mathrm{MA}(\psi)) = \int -\rho dd^c \rho \wedge T \ge 0$$

and

$$T = \sum_{l=0}^{n-1} \omega_\varphi^l \wedge \omega_\psi^{n-1-l}.$$

We prove (8.4) by induction on k. Note that $k = n-1$ yields the claim.

If $k = 0$, then

$$\int -\rho dd^c \rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \leq \int -\rho dd^c \rho \wedge T = a,$$

so (8.4) holds in this case (with $C = 1$). Now assume $0 < k \leq n-1$. We have

$$\omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k = \omega_\varphi^{i+k} \wedge \omega_\psi^j - dd^c\varphi \wedge \alpha,$$

where

$$\alpha = \omega_\varphi^i \wedge \omega_\psi^j \wedge \sum_{l=0}^{k-1} \omega_\varphi^l \wedge \omega^{k-1-l}.$$

Therefore,

$$\begin{aligned} 0 \leq \int -\rho dd^c \rho \wedge \omega_\varphi^i \wedge \omega_\psi^j \wedge \omega^k &\leq \int -\rho dd^c \rho \wedge (T - dd^c\varphi \wedge \alpha) \\ = a + \int \rho dd^c \varphi \wedge \alpha \wedge dd^c \rho &= a + \int \rho dd^c\varphi \wedge \alpha \wedge (\omega_\varphi - \omega_\psi). \end{aligned}$$

If η is equal to φ or ψ, the Cauchy-Schwarz inequality gives

$$\left| \int \rho dd^c \varphi \wedge \alpha \wedge \omega_\eta \right| \leq \left(\int -\rho dd^c \rho \wedge \alpha \wedge \omega_\eta \right)^{1/2} \left(\int -\varphi dd^c \varphi \wedge \alpha \wedge \omega_\eta \right)^{1/2}$$

By the inductive assumption, we have $\int -\rho dd^c \rho \wedge \alpha \wedge \omega_\eta \leq C a^{2^{-(k-1)}}$. Further, since $\varphi \in \mathcal{E}^1(X, \omega)$ we get $\int -\varphi dd^c \varphi \wedge \alpha \wedge \omega_\eta < +\infty$. The proof is complete. □

8.2. **Existence.** As in [BBGZ13], the strategy is to first use a variational argument (going back to Alexandrov [Ale38]) in order to produce a solution of finite energy.

Consider the functional $F_\mu : \mathrm{PSH}(X, \omega) \to [-\infty, +\infty[$ defined by

$$F_\mu(\varphi) := E(\varphi) - \int \varphi \mu. \tag{8.5}$$

We first claim that F_μ is usc on $\mathrm{PSH}(X, \omega)$. By Proposition 6.2, E is usc so that it suffices to prove $\varphi \mapsto \int \varphi\mu$ is continuous on $\mathrm{PSH}(X, \omega)$. But this is clear since the topology on $\mathrm{PSH}(X, \omega)$ is defined in terms of uniform convergence on dual complexes, and μ is supported on a dual complex by hypothesis.

By Theorem 2.10, the set $\mathrm{PSH}_0(X, \omega) := \{\varphi \in \mathrm{PSH}(X, \omega) \mid \sup \varphi = 0\}$ is compact so the usc functional F_μ attains its maximum there. On the other hand, Proposition 6.2 shows that $F_\mu(\varphi + c) = F_\mu(\varphi)$ for any constant c. Hence, we can find $\varphi \in \mathrm{PSH}_0(X, \omega)$ such that

$$F_\mu(\varphi) = \sup_{\mathrm{PSH}(X,\omega)} F_\mu.$$

Clearly, $E(\varphi) > -\infty$, so $\varphi \in \mathcal{E}^1(X, \omega)$. Let us show that $\mathrm{MA}(\varphi) = \mu$. Pick any model function f on X. For $t \in \mathbf{R}$, consider the function

$$h(t) := E(P(\varphi + tf)) - \int (\varphi + tf)\mu.$$

In view of Corollary 7.3, $h(t)$ is differentiable at $t = 0$ with derivative

$$h'(0) = \int f \ \mathrm{MA}(\varphi) - \int f \, \mu.$$

Now $P(\varphi + tf) \leq \varphi + tf$, so $h(t) \leq F_\mu(P(\varphi + tf)) \leq F_\mu(\varphi) = h(0)$ for all t. Thus, h has a local maximum at $t = 0$, so $h'(0) = 0$, that is, $\int f \mu = \int f \, \mathrm{MA}(\varphi)$. This implies $\mathrm{MA}(\varphi) = \mu$, as f was an arbitrary model function.

8.3. **Continuity.** Finally we show that φ is continuous. For this we use capacity estimates in the spirit of Kołodziej [Koł98, Koł03]; see also [EGZ09]. The following result (and its proof) is a translation of [EGZ09, Lemma 2.3].

Lemma 8.3. *Let $\varphi, \psi \in \mathcal{E}^1(X, \omega)$ with $\psi \leq 0$. Then*

$$\mathrm{Cap}\{\varphi < \psi\} \leq t^{-n} \int_{\{\varphi < (1-t)\psi + t\}} \mathrm{MA}(\varphi)$$

for $0 < t < 1$.

Proof. Fix $u \in \mathrm{PSH}(X, \omega)$ with $0 \leq u \leq 1$ and set $\psi_t := (1 - t)\psi + tu$. We have

$$\{\varphi < \psi\} \subseteq \{\varphi < \psi_t\} \subseteq \{\varphi < (1 - t)\psi + t\}$$

since $\psi \leq 0$. Now $\mathrm{MA}(\psi_t) \geq t^n \, \mathrm{MA}(u)$ by (3.2), so

$$\begin{aligned} t^n \int_{\{\varphi < \psi\}} \mathrm{MA}(u) &\leq \int_{\{\varphi < \psi\}} \mathrm{MA}(\psi_t) \leq \int_{\{\varphi < \psi_t\}} \mathrm{MA}(\psi_t) \\ &\leq \int_{\{\varphi < \psi_t\}} \mathrm{MA}(\varphi) \leq \int_{\{\varphi < (1-t)\psi + t\}} \mathrm{MA}(\varphi), \end{aligned}$$

where the third inequality follows from the comparison principle (6.8). Taking the supremum over all functions u as above completes the proof. □

As a consequence, we get the following version of the "domination principle," sufficient for our purpose.

Lemma 8.4. *Let $\varphi \in \mathrm{PSH}(X, \omega) \cap C^0(X)$ and $\psi \in \mathcal{E}^1(X, \omega)$. Assume that $\nu := \mathrm{MA}(\psi)$ is supported on the dual complex $\Delta_{\mathcal{X}}$ of some SNC model $\mathcal{X}$, and that $\varphi \leq \psi$ ν-a.e. Then $\varphi \leq \psi$ on X.*

Proof. Upon adding a constant we may assume that $0 \geq \varphi \geq -C$ for some constant $C > 0$. Let $\varepsilon > 0$. If we choose $0 < t \ll 1$ such that $t(C + 1) \leq \varepsilon/2$ then we have

$$\nu\{\psi + \varepsilon < (1 - t)\varphi + t\} \leq \nu\{\psi + \varepsilon/2 < \varphi\} = 0.$$

Since $\mathrm{MA}(\psi + \varepsilon) = \nu$, Lemma 8.3 implies that

$$\mathrm{Cap}\{\psi + \varepsilon < \varphi\} \leq t^{-n} \nu\{\psi + \varepsilon < (1 - t)\varphi + t\} = 0.$$

But $\{\psi + \varepsilon < \varphi\}$ is open by continuity of φ, hence empty by Lemma 4.3. We have thus proved that $\varphi \leq \psi + \varepsilon$ on X for all $\varepsilon > 0$, and the result follows. □

Now let $\varphi \in \mathcal{E}^1(X, \omega)$ be a solution to $\mathrm{MA}(\varphi) = \mu$, with μ supported on a dual complex $\Delta_{\mathcal{X}}$. We may normalize φ by $\sup_X \varphi = -1$. Let $(\varphi_j)_j$ be a decreasing net of ω-psh model functions converging to φ. We are going to show that $\varphi_j \to \varphi$ uniformly on X, which will in particular imply that φ is continuous.

By Theorem 2.10 we have $\sup_X \varphi_j \to \sup_X \varphi$, so we may assume $\varphi_j \leq 0$ for all j. Fix $\varepsilon > 0$. Since φ is continuous on $\Delta_{\mathcal{X}}$, the monotone convergence $\varphi_j \to \varphi$

is uniform on $\Delta_{\mathcal{X}}$ by Dini's lemma. We thus have $\varphi_j \le \varphi + \varepsilon$ μ-a.e. for $j \gg 1$. Lemma 8.4 now yields $\varphi_j \le \varphi + \varepsilon$ on X, which concludes the proof.

8.4. **The case of an atomic measure.** We now give a more explicit description of the solution to $\mathrm{MA}(\varphi) = \mu$, when μ is a finite sum of Dirac masses at divisorial points. Let ω be a semipositive closed $(1,1)$ form with ample de Rham class $\{\omega\}$. We do not assume that $\{\omega\}^n = 1$ but we do assume that ω satisfies the orthogonality property.

Lemma 8.5. *Let $S = \{x_1, \dots, x_N\} \subset X^{\mathrm{div}}$ be a finite set of divisorial points, and set for $t = (t_1, \dots, t_N) \in \mathbf{R}^N$*

$$\varphi_{S,t} := \sup \{\varphi \mid \varphi \in \mathrm{PSH}(X, \omega),\ \varphi(x_i) \le t_i \text{ for } i = 1, \dots, N\}. \tag{8.6}$$

Then $\varphi_{S,t}$ is a continuous ω-psh function and the support of $\mathrm{MA}(\varphi_{S,t})$ *is contained in S.*

Proof. Let $\mathcal{X}$ be an SNC model such that all the x_i appear as vertices of $\Delta_{\mathcal{X}}$. By Theorem 2.10 there exists a constant $C > 0$ such that $\sup_X \varphi \le \sup_S \varphi + C$ for all $\varphi \in \mathrm{PSH}(X, \omega)$. Since adding a constant c to the t_i only replaces $\varphi_{S,t}$ with $\varphi_{S,t} + c$, we may thus assume $t_i \le -1$ and $\varphi \le -1$ as soon as $\varphi \in \mathrm{PSH}(X, \omega)$ satisfies $\varphi(x_i) \le t_i$ for all i.

Let $f_{\mathcal{X}} \in \mathcal{D}(X)_{\mathbf{R}}$ be the unique model function that is determined on $\mathcal{X}$, takes value t_i at x_i for each i and vanishes at any other vertex of $\Delta_{\mathcal{X}}$. Note that $f_{\mathcal{X}}$ is affine on the faces of $\Delta_{\mathcal{X}}$ and that $f_{\mathcal{X}} = f_{\mathcal{X}} \circ p_{\mathcal{X}}$. In particular, $f_{\mathcal{X}}(x) < 0$ iff $c_{\mathcal{X}}(x) \in \overline{c_{\mathcal{X}}(x_i)}$ for some i, where $c_{\mathcal{X}}(x)$ (resp. $c_{\mathcal{X}}(x_i)$) denotes the center of x (resp. x_i) on $\mathcal{X}$, see [BFJ09, Sec. 2.3].

Since each $\varphi \in \mathrm{PSH}(X, \omega)$ is convex on the faces of $\Delta_{\mathcal{X}}$ and satisfies $\varphi \le \varphi \circ p_{\mathcal{X}}$, we have $\varphi(x_i) \le t_i$ for all i iff $\varphi \le f_{\mathcal{X}}$; hence $\varphi_{S,t} = P(f_{\mathcal{X}})$. This already shows that $\varphi_{S,t}$ is continuous and ω-psh, and the orthogonality property further shows that $\mathrm{MA}(\varphi_{S,t})$ is supported on $\{\varphi_{S,t} = f_{\mathcal{X}}\}$ for each SNC model $\mathcal{X}$ as above. We thus see that

$$\operatorname{Supp} \mathrm{MA}(\varphi_{S,t}) \subset \bigcap_{\mathcal{X}} \{f_{\mathcal{X}} < 0\} = \bigcap_{\mathcal{X}} \bigcup_{i=1}^{N} \{x \in X \mid c_{\mathcal{X}}(x) \in \overline{c_{\mathcal{X}}(x_i)}\}.$$

We claim that the latter intersection is in fact equal to S, which will conclude the proof of the lemma. Pick any $x \in X \setminus S$. It suffices to show that for each i there exists a model $\mathcal{X}_i$ such that $c_{\mathcal{X}_i}(x) \notin \overline{c_{\mathcal{X}_i}(x_i)}$, since if $\mathcal{X}$ is a model dominating all the $\mathcal{X}_i$ then $c_{\mathcal{X}}(x) \notin \overline{c_{\mathcal{X}}(x_i)}$ for all i, which is equivalent to $f_{\mathcal{X}}(x) = 0$.

Since model functions are dense in $C^0(X)$, we can, for each i, find $g_i \in \mathcal{D}(X)$ such that $g_i(x) = 0$ and $g_i(x_i) > 0$. Replacing g_i by $\max\{g_i, 0\}$ we may assume that $g_i \ge 0$. If $\mathcal{X}_i$ is a model in which g_i is determined, this implies that $c_{\mathcal{X}_i}(x) \notin \overline{c_{\mathcal{X}_i}(x_i)}$, as required. $\square$

As a consequence of this result, for any divisorial point $x \in X^{\mathrm{div}}$ the function

$$\varphi_x := \sup \{\varphi \in \mathrm{PSH}(X, \omega) \mid \varphi(x) \le 0\} \tag{8.7}$$

satisfied $\mathrm{MA}(\varphi_x) = \{\omega\}^n \, \delta_x$, since both sides of (8.7) are positive measures of mass $\{\omega\}^n$ supported at x. More generally, we have the following.

Proposition 8.6. *Let $S = \{x_1, \dots, x_N\} \subset X^{\mathrm{div}}$ be a finite set of divisorial points and let μ be a positive Radon measure of mass $\{\omega\}^n$ with support contained in*

$\{x_1, \dots, x_N\}$. *Then there exists* $t \in \mathbf{R}^N$ *such that the function* $\varphi_{S,t}$ *defined by* (8.6) *solves* $\mathrm{MA}(\varphi_{S,t}) = \mu$.

Proof. By Theorem A$'$, there exists a continuous ω-psh function φ satisfying $\mathrm{MA}(\varphi) = \mu$. Set $t_i = \varphi(x_i)$ for $i = 1, \dots, N$. We claim that $\varphi_{S,t} = \varphi$, which will conclude the proof. On the one hand, $\varphi \leq \varphi_{S,t}$ by (8.6), since φ is ω-psh and satisfies $\varphi(x_i) \leq t_i$. On the other hand, $\varphi_{S,t} = \varphi$ on the support of $\mathrm{MA}(\varphi)$; hence, $\varphi_{S,t} \leq \varphi$ by Lemma 8.4. $\square$

Remark 8.7. *Consider the setting of Theorem A, i.e.,* $\{\omega\}$ *is the class of an (ample) line bundle* L *on* X. *The strategy proposed in* [KT] *to solve Monge-Ampère equations mostly deals with the case of a Dirac mass* μ *at a divisorial point* $x \in X^{\mathrm{div}}$. *The authors introduce the envelope* (8.7)*, and assume by contradiction that* $\mathrm{MA}(\varphi_x)$ *is not supported at* x. *They define a limit functional* F *obtained by looking at the asymptotics of ball volumes in the space of sections of* mL *as* $m \to \infty$*, and indicate that* F *should satisfy* $F(\varphi_x + \varepsilon f) = F(\varphi_x) + \varepsilon \int f \,\mathrm{MA}(\varphi_x) + O(\varepsilon^2)$ *for each* $f \in C^0(X)$. *Comparing with* [BB10] *in the complex case,* F *is likely to coincide with* $E \circ P$*, so that a version of the differentiability property (Theorem* 7.2*) would also be a key ingredient in the approach proposed in* [KT].

Remark 8.8. *We do not know whether the function* φ_x *in* (8.7) *is necessarily a model function. This is the case on a toric variety, see Proposition* 9.1 *below, but we suspect the answer is no in general.*

Pick an SNC model $\mathcal{X}$*, an extension* $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})_{\mathbf{Q}}$ *of* L *and let* ω *be the curvature form of the model metric defined by* $\mathcal{L}$. *Let also* E *be a component of* $\mathcal{X}$ *corresponding to the divisorial point* $x = x_E$. *We have* $\varphi_x = P(-f_E)$ *up to a constant. On the other hand, it follows from* [BFJ09, Theorem 8.5] *that*

$$P(-f_E) = \lim_m \frac{1}{m} \log |\mathfrak{a}_m|,$$

where $\mathfrak{a}_m$ *denotes the base ideal of* $m\mathcal{L}'$ *with* $\mathcal{L}' := \mathcal{L} - E$. *As a consequence,* φ_x *is indeed a model function as soon as the graded* S*-algebra* $\bigoplus_{m \geq 0} H^0(\mathcal{X}, m\mathcal{L}')$ *is finitely generated. Building on Nakayama's counterexample to the existence of Zariski decompositions* [Nak04]*, it is reasonable to expect this algebra not to be finitely generated in general, and to subsequently prove that* φ_x *is not a model function.*

9. Curves and toric varieties

9.1. **Curves.** Potential theory on non-Archimedean analytic curves (over arbitrary complete valuation fields) was developed in detail by A. Thuillier in [Thu05]. Here we only indicate how to recover Theorem A' when $\dim X = 1$ following his approach.

Let X be a smooth projective curve over K. Thuillier defined the spaces $D^0(X)$ and $D^1(X)$ of distributions and currents on X as the duals of $\mathcal{Z}^{1,1}(X)$ and $\mathcal{D}(X)_{\mathbf{R}}$, respectively. An element of $D^0(X)$ can be viewed as an arbitrary function $X^{\mathrm{div}} \to \mathbf{R}$ [Thu05, Proposition 3.3.3]. The dd^c-operator extends to $dd^c : D^0(X) \to D^1(X)$, and its image is exactly the set of currents $\rho \in D^1(X)$ such that $\int_X \rho = 0$ [Thu05, Théorème 3.3.13]. By linearity, this fact easily reduces to the existence, for any two $x, y \in X^{\mathrm{div}}$, of a "Green function," i.e., a model function $g_{x,y}$ such that $dd^c g_{x,y} = \delta_x - \delta_y$. The existence of $g_{x,y}$ is in turn a consequence of the intersection form being negative definite on $\mathrm{Div}_0(\mathcal{X})_{\mathbf{R}}/\mathbf{R}\mathcal{X}_0$, for a model $\mathcal{X}$ such that x and y correspond to irreducible components of $\mathcal{X}_0$.

Now let ω be a closed $(1,1)$-form with ample de Rham class, so that $\{\omega\} > 0$. Let μ be an arbitrary positive Radon measure on X of mass $\{\omega\}$. The previous result shows the existence of a distribution φ_μ such that

$$\omega + dd^c\varphi_\mu = \mu. \tag{9.1}$$

By [Thu05, Lemme 3.4.1] the positivity of the current $\omega + dd^c\varphi_\mu$ shows that φ_μ uniquely extends to an ω-psh function, and we conclude that any positive Radon measure μ of mass $\{\omega\}$ satisfies (9.1) for some $\varphi_\mu \in \mathrm{PSH}(X,\omega)$, unique up to an additive constant.

Finally, suppose that μ is supported on a dual complex $\Delta_{\mathcal{X}}$. In order to see that φ_μ is continuous, we may assume that $\mathcal{X}$ is also a determination of ω. In this one-dimensional setting, it is easy to check that post-composition with the retraction $p_{\mathcal{X}} : X \to \Delta_{\mathcal{X}}$ preserves ω-psh functions, i.e., $\varphi\circ p_{\mathcal{X}}$ is ω-psh for every ω-psh function φ. Since μ is supported on $\Delta_{\mathcal{X}}$ we have $(p_{\mathcal{X}})_*\,\mu = \mu$; hence, $\omega + dd^c(\varphi_\mu \circ p_{\mathcal{X}}) = \mu = \omega + dd^c\varphi_\mu$. By uniqueness, $\varphi_\mu - \varphi_\mu \circ p_{\mathcal{X}}$ must be a constant, and this constant is zero since $p_{\mathcal{X}} = \mathrm{id}$ on $\Delta_{\mathcal{X}}$. Now $\varphi_\mu|_{\Delta_{\mathcal{X}}}$ is continuous, hence so is $\varphi_\mu = \varphi_\mu \circ p_{\mathcal{X}}$.

Let us now make the connection with the approach we followed in higher dimensions. In dimension 1, the energy is equal to $E(\varphi) = \int \varphi\omega + \frac{1}{2}\int \varphi dd^c\varphi$, so that an ω-psh function φ has finite energy iff φ is integrable with respect to the total variation measure of $dd^c\varphi$.

Now fix a positive Radon measure μ such that the solution φ_μ to (9.1) has finite energy. Then φ_μ is the unique ω-psh function realizing the infimum of the functional $E(\varphi) - \int \varphi\mu$; see [Thu05, Proposition 3.5.9].

Observe that if μ is supported on some dual complex, then φ_μ is continuous and in particular of finite energy; hence, Thuillier's result is stronger than ours in dimension 1.

We refer to [Thu05] for more on potential theory on non-Archimedean curves including the notion of harmonic functions, capacity, and the study of polar sets. See also [BR10] for the case of the projective line.

9.2. **Toric varieties.** We use [Ful93, KKMSD73, BPS13, Gub13] as references. Let $M \simeq \mathbf{Z}^n$ be a free abelian group, N its dual, and let $T = \operatorname{Spec} K[M]$ be the corresponding split K-torus. A (projective) toric K-variety X is described by a rational fan Σ in $N_{\mathbf{R}}$, and there is a natural embedding $j : N_{\mathbf{R}} \to X^{\mathrm{an}}$ given by monomial valuations that sends $n \in N_{\mathbf{R}}$ to the norm

$$K[M] \ni \sum a_m m \mapsto \max\{|a_m| \exp(-\langle m, n\rangle)\}.$$

In particular, $j(0) = x_G$, the Gauss point of the open T-orbit.

An ample T-line bundle L on X defines a rational polytope $\Delta \subset M_{\mathbf{R}}$ with normal fan Σ in such a way that points of $M \cap \Delta$ can be identified with T-eigensections of L.

According to [BPS13, Sec. 4] we have the following description of toric metrics on L. The polytope Δ is the Newton polytope of the piecewise $\mathbf{Q}$-linear convex function $g_\Delta = \sup_{m\in\Delta} m$ on the dual space $N_{\mathbf{R}} = M^*_{\mathbf{R}}$, and toric bounded (resp. model) metrics $\|\cdot\|$ on L correspond to bounded (resp. piecewise $\mathbf{Q}$-affine) functions f on $N_{\mathbf{R}}$ such that $f - g_\Delta$ is bounded. The metric $\|\cdot\|_f$ attached to a function f is semipositive iff f is convex.

The real Monge-Ampère measure of any convex function f on $N_{\mathbf{R}}$ is a well-defined positive Radon measure $\mathrm{MA}_{\mathbf{R}}(f)$ on $N_{\mathbf{R}}$ (see, e.g., [RT77]), while the growth condition $f = g_\Delta + O(1)$ further guarantees that

$$\int_{N_{\mathbf{R}}} \mathrm{MA}_{\mathbf{R}}(f) = \mathrm{Vol}(\Delta).$$

If f is a convex function on $N_{\mathbf{R}}$ with $f = g_\Delta + O(1)$, and if $\|\cdot\|_f$ is the corresponding continuous semipositive metric on L, then [BPS13, Theorem 4.7.4] relates their Monge-Ampère measures as follows:

$$c_1(L, \|\cdot\|_f)^n = n!\, j_* \,\mathrm{MA}_{\mathbf{R}}(f). \tag{9.2}$$

Since g_Δ is homogeneous, $\mathrm{MA}_{\mathbf{R}}(g_\Delta)$ is a Dirac mass at the origin of mass $\mathrm{Vol}(\Delta)$, so by (9.2) the corresponding metric $\|\cdot\|_{g_\Delta}$ on L satisfies

$$c_1(L, \|\cdot\|_{g_\Delta})^n = c_1(L)^n\, \delta_{x_G}.$$

Translating by any element in $N_{\mathbf{Q}}$ we get the following.

Proposition 9.1. *Let $x \in N_{\mathbf{Q}}$ and let $j(x) \in X^{\mathrm{div}}$ be the corresponding toric divisorial point*

$$c_1(L, \|\cdot\|_x)^n = c_1(L)^n \delta_{j(x)},$$

where $\|\cdot\|_x$ is the semipositive toric metric attached to the convex piecewise $\mathbf{Q}$-affine function $y \mapsto g_\Delta(y - x)$.

In particular, the metric in Proposition 9.1 is a *model* metric. More generally, we have the following.

Proposition 9.2. *Let (X, L) be a polarized toric K-variety. Pick $x_1, \dots, x_N \in N_{\mathbf{Q}}$ and set $\mu_w := \sum_i w_i \delta_{j(x_i)}$ for each $w \in \mathbf{R}_+^N$. Then for a dense set of $w \in \mathbf{R}_+^N \cap \{\sum_i w_i = c_1(L)^n\}$ the semipositive toric metric $\|\cdot\|$ for which*

$$c_1(L, \|\cdot\|_w)^n = \sum_i w_i \delta_{j(x_i)}$$

is a model metric.

Proof. For $t \in \mathbf{R}^N$ let f_t be the upper envelope of the family of piecewise $\mathbf{Q}$-affine convex functions f on $N_{\mathbf{R}}$ such that $f = g_\Delta + O(1)$ and $f(x_i) \le t_i$ for all i, and let $\|\cdot\|_t$ be the corresponding continuous toric semipositive metric. By Proposition 8.6, each measure μ_w with $w \in \mathbf{R}_+^N \cap \{\sum_i w_i = c_1(L)^n\}$ is of the form $c_1(L, \|\cdot\|_t)^n$ for some $t \in \mathbf{R}^N$. Now elementary Newton polytope considerations show that f_t is piecewise $\mathbf{Q}$-affine when all the t_i are rational, and the result follows by continuity of $t \mapsto c_1(L, \|\cdot\|_t)^n$. ☐

Remark 9.3. *The results of this section likely extend to the case of an arbitrary non-Archimedean complete non-trivially valued field. We refer to* [BPS13, Gub13] *for a discussion of toric varieties in this context.*

Appendix A. Orthogonality

The goal of this Appendix is to show that the crucial *orthogonality property* of Definition 7.1 is satisfied at least when X is defined over a function field (i.e., when the condition (†) from the introduction holds). Note that we shall impose no extra condition on the polarization.

Consider a closed form $\theta \in \mathcal{Z}^{1,1}(X)$ with ample de Rham class $\{\theta\} \in N^1(X)$, and a continuous function $f \in C^0(X)$. We say that the pair (θ, f) satisfies the *orthogonality property* if

$$\int_X (P_\theta(f) - f)\,(\theta + dd^c P_\theta(f))^n = 0. \tag{A.1}$$

In agreement with Definition 7.1 we also say that the form θ satisfies the orthogonality property if the pair (θ, f) does so for all $f \in C^0(X)$. It is convenient in what follows not to require that θ be semipositive, as opposed to the main body of the text.

A.1. **Preliminaries.** We first show some basic properties.

Lemma A.1. *Fix an ample class $\alpha \in N^1(X)$. Then the following assertions are equivalent:*

(1) *some form θ with $\{\theta\} = \alpha$ satisfies the orthogonality property;*
(2) *any form θ with $\{\theta\} = \alpha$ satisfies the orthogonality property;*
(3) *for any form θ with $\{\theta\} = \alpha$ and any model function f, the pair (θ, f) satisfies the orthogonality property; and*
(4) *for any form θ with $\{\theta\} = \alpha$, the pair $(\theta, 0)$ satisfies the orthogonality property.*

When (1)–(4) hold, we simply say that the class α satisfies the orthogonality property.

Proof. It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and that (2) $\Rightarrow$ (1). The implication (4) $\Rightarrow$ (3) follows from the equality $P_\theta(f) - f = P_{\theta + dd^c f}(0)$. The same argument shows that (1) $\Rightarrow$ (2) in view of the dd^c-lemma proved in [BFJ09, Theorem 4.3].

It remains to prove (3) $\Rightarrow$ (2). We may write any given $f \in C^0(X)$ as a uniform limit on X of a decreasing sequence $(f_m)_{m=1}^\infty$ of model functions. Then $(P_\theta(f_m))_m$ is a decreasing sequence of θ-psh functions on X, and $P_\theta(f_m) \to P_\theta(f)$ uniformly on X, thanks to the Lipschitz property of P_θ; see Proposition 2.15. Corollary 3.5 now yields

$$(\theta + dd^c P_\theta(f_m))^n \to (\theta + dd^c P_\theta(f))^n$$

in the weak topology of measures. Since $P_\theta(f_m) - f_m$ converges uniformly to $P_\theta(f) - f$, this implies that

$$\int (P_\theta(f) - f)\,(\theta + dd^c P_\theta(f))^n = \lim_m \int (P_\theta(f_m) - f_m)\,(\theta + dd^c P_\theta(f_m))^n = 0,$$

completing the proof. □

Lemma A.2. *The set of classes in $N^1(X)$ satisfying the orthogonality property is a closed subset of the ample cone.*

Proof. Pick any regular model $\mathcal{X}$. The linear map $N^1(\mathcal{X}/S) \to N^1(X)$ is surjective and hence open. It is thus enough to prove the following claim: let $\theta_\mathcal{X} \in N^1(\mathcal{X}/S)$ have ample image in $N^1(X)$, and assume that $\theta_\mathcal{X}$ is the limit of a sequence $\theta_{m,\mathcal{X}} \in N^1(\mathcal{X}/S)$. If the corresponding forms $\theta_m \in \mathcal{Z}^{1,1}(X)$ all satisfy the orthogonality property, then so does θ.

Let $f \in C^0(X)$. By Proposition 2.15 we have $P_{\theta_m}(f) \to P_\theta(f)$ uniformly on X. We claim that

$$(\theta_m + dd^c P_{\theta_m}(f))^n \to (\theta + dd^c P_\theta(f))^n$$

in the weak topology of Radon measures on X. Since $(P_{\theta_m}(f) - f) \to (P_\theta(f) - f)$ uniformly on X, this implies as before that

$$\int (P_\theta(f) - f)(\theta + dd^c P_\theta(f))^n = \lim_m \int (P_{\theta_m}(f) - f)(\theta_m + dd^c P_{\theta_m}(f))^n = 0.$$

To prove the claim, pick any model function $g \in \mathcal{D}(X)$, and fix $\varepsilon > 0$. By Corollary 2.12, we can find a θ-psh model function φ such that $\sup |\varphi - P_\theta(f)| \le \varepsilon$. We then have

$$\begin{aligned} I_m := & \left| \int g\,(\theta_m + dd^c P_{\theta_m}(f))^n - \int g\,(\theta + dd^c P_\theta(f))^n \right| \\ \le & \left| \int g\,(\theta_m + dd^c P_{\theta_m}(f))^n - \int g\,(\theta_m + dd^c \varphi)^n \right| \\ & + \left| \int g\,(\theta_m + dd^c \varphi)^n - \int g\,(\theta + dd^c \varphi)^n \right| \\ & + \left| \int g\,(\theta + dd^c \varphi)^n - \int g\,(\theta + dd^c P_\theta(f))^n \right|. \end{aligned}$$

Using integration by parts, the last term can be bounded as follows:

$$\begin{aligned} & \left| \int g\,(\theta + dd^c \varphi)^n - \int g\,(\theta + dd^c P_\theta(f))^n \right| \\ & = \left| \int (\varphi - P_\theta(f))\, dd^c g \wedge \sum_{i=0}^{n-1} (\theta + dd^c \varphi)^i \wedge (\theta + dd^c P_\theta(f))^{n-i-1} \right| \\ & \le C \sup |P_\theta(f) - \varphi| \le C\varepsilon \end{aligned}$$

for some constant C. Indeed, we can take $C = 2\{\omega\}\,\{\theta\}^{n-1}$, where ω is a fixed form such that ω and $(\omega + dd^c g)$ are semipositive.

The first term can be similarly bounded:

$$\left| \int g\,(\theta_m + dd^c P_{\theta_m}(f))^n - \int g\,(\theta_m + dd^c \varphi)^n \right| \le C \sup |P_{\theta_m}(f) - \varphi| \le 2C\varepsilon,$$

for m large enough. Finally, g and φ being model functions, the second term tends to zero as $m \to \infty$, and we get $\limsup_m I_m \le 3C\varepsilon$. We conclude by letting $\varepsilon \to 0$. $\square$

Finally, we investigate the behavior of envelopes under finite base change. Let K'/K be a finite Galois extension, and denote by $p : X' := X_{K'} \to X$ the natural projection. Let also $\theta \in \mathcal{Z}^{1,1}(X)$ be a given closed $(1,1)$-form on X with ample cohomology class, and set $\theta' := p^*\theta \in \mathcal{Z}^{1,1}(X')$.

Lemma A.3. *Let φ' be a θ'-psh function on X', and assume that φ' is* $\mathrm{Gal}(K'/K)$*-invariant. Then $\varphi' = p^*\varphi$ for a unique θ-psh function φ on X.*

Proof. Let φ'_j be a decreasing net of model θ'-psh functions converging to φ' pointwise on X'. Replacing φ'_j by the average of its Galois orbit, we may assume that each φ'_j is Galois invariant.

Let $\mathcal{X}'_j$ be a model of X' over $\mathcal{O}_{K'}$ such that φ'_j is determined by a vertical $\mathbf{Q}$-divisor $D'_j \in \mathrm{Div}_0(\mathcal{X}'_j)_{\mathbf{Q}}$. After replacing $\mathcal{X}'_j$ by a model dominating all Galois conjugates of $\mathcal{X}'_j$, we may assume that $\mathcal{X}'_j$ is Galois invariant, so that the invariance of φ'_j implies that of D'_j. By basic Galois descent, $(\mathcal{X}'_j, D'_j)$ is induced by a vertical divisor on a model of X. It follows that $\varphi'_j = p^*\varphi_j$ with $\varphi_j \in \mathcal{D}(X)$, which is θ-psh by the projection formula.

It is now clear that φ_j decreases pointwise to a θ-psh function φ, which must satisfy $p^*\varphi = \varphi'$. □

Lemma A.4. *If θ' has the orthogonality property, then so does θ.*

Proof. Pick $f \in C^0(X)$. We first claim that

$$P_{\theta'}(p^*f) = p^*P_\theta(f). \tag{A.2}$$

On the one hand, $p^*P_\theta(f)$ is θ'-psh and dominated by p^*f, and hence $p^*P_\theta(f) \leq P_{\theta'}(f)$ by the maximality property of envelopes. On the other hand, replacing each candidate in the supremum computing $P_{\theta'}(p^*f)$ by the average of its Galois orbit immediately shows that $P_{\theta'}(p^*f)$ is Galois invariant. By Lemma A.3 we thus have $P_{\theta'}(p^*f) = p^*\varphi$ with $\varphi \in \mathrm{PSH}(X, \theta)$. Since $P_{\theta'}(p^*f) \leq p^*f$, we must have $\varphi \leq f$, hence $\varphi \leq P_\theta(f)$, and (A.2) follows.

Next, for any continuous θ-psh function φ we have

$$p_*\left[(p^*\theta + dd^c p^*\varphi)^n\right] = \deg(p)(\theta + dd^c\varphi)^n.$$

This is a direct consequence of the projection formula when φ is a model function, and the general case follows by our (by now) usual regularization argument. Applying this to (A.2) yields

$$\deg(p)\int_X (P_\theta(f) - f)(\theta + dd^c P_\theta(f))^n = \int_{X'} (P_{\theta'}(p^*f) - p^*f)(\theta' + dd^c P_{\theta'}(p^*f))^n,$$

which shows that the orthogonality property holds for (θ', p^*f) iff it does so for (θ, f). □

A.2. **Geometric interpretation.** We next translate the orthogonality property into a geometric condition. Let $\mathcal{L}$ be a line bundle on a model $\mathcal{X}$ and assume that $L := \mathcal{L}|_X$ is ample. For each $m \in \mathbf{N}$, let $\mathfrak{b}_m$ be the base ideal of $m\mathcal{L}$, i.e., the image of the evaluation map

$$H^0(\mathcal{X}, m\mathcal{L}) \otimes \mathcal{O}_{\mathcal{X}}(-m\mathcal{L}) \to \mathcal{O}_{\mathcal{X}}.$$

Note that the ideal sheaf $\mathfrak{b}_m$ is vertical (i.e., cosupported on $\mathcal{X}_0$) for $m \gg 1$, thanks to the ampleness condition on the generic fiber. Let $\rho_m : \mathcal{X}_m \to \mathcal{X}$ be the normalized blow-up of $\mathcal{X}$ along $\mathfrak{b}_m$, so that the base scheme of $\rho_m^*(m\mathcal{L})$ is now a vertical Cartier divisor F_m on $\mathcal{X}_m$ satisfying

$$\mathfrak{b}_m \cdot \mathcal{O}_{\mathcal{X}_m} = \mathcal{O}_{\mathcal{X}_m}(-F_m).$$

Finally, set $M_m := \rho_m^*(m\mathcal{L}) - F_m$, so that

$$\rho_m^*(m\mathcal{L}) = M_m + F_m \tag{A.3}$$

is the decomposition into a 'mobile' (i.e., base point free) part and a fixed part.

Lemma A.5. *Let $\theta \in \mathcal{Z}^{1,1}(X)$ be the curvature form of the model metric on L induced by $\mathcal{L}$. Then*

$$\lim_{m\to\infty}\left(\frac{1}{m}M_m\right)^n \cdot \left(\frac{1}{m}F_m\right) = -\int P_\theta(0)(\theta + dd^c P_\theta(0))^n. \tag{A.4}$$

In particular, the limit in (A.4) *exists, and the pair* $(\theta, 0)$ *satisfies the orthogonality property iff* $\left(\frac{1}{m}M_m\right)^n \cdot \left(\frac{1}{m}F_m\right) \to 0$.

Proof. For $m \gg 1$ set $\varphi_m := \frac{1}{m} \log |\mathfrak{b}_m| = -\frac{1}{m}\varphi_{F_m}$. This is a θ-psh model function, and by [BFJ09, Theorem 8.5] we have $\varphi_m \to P_\theta(0)$ uniformly on X. Unravelling the definitions, we find

$$-\left(\frac{1}{m}M_m\right)^n \cdot \left(\frac{1}{m}F_m\right) = \int \varphi_m\,(\theta + dd^c\varphi_m)^n.$$

By Corollary 3.5 the right-hand side converges to $\int P_\theta(0)\,(\theta + dd^c P_\theta(0))^n$ as $m \to \infty$. Thus (A.4) holds, which completes the proof. $\square$

A.3. **Orthogonality for varieties defined over a function field.** Recall that X is assumed to be a smooth projective variety over the discretely valued field K, where the latter has valuation ring R and residue field k. Write $S = \operatorname{Spec} R$ as before.

We assume from now on that X is *defined over a function field*, in the sense that it satisfies the condition (†) from the introduction. In other words, we assume the existence of a smooth projective variety Y defined over a one-variable function field F/k having K as a completion, such that X is isomorphic to the base change Y_K.

Theorem A.6. *Assume that X is defined over a function field. Then all ample classes in $N^1(X)$ have the orthogonality property.*

Lemma A.7. *After perhaps replacing K by a finite Galois extension, we can arrange that $X = Y_K$ with Y defined over a function field F having K as a completion, and such that $N^1(Y/F) \to N^1(X/K)$ is furthermore surjective.*

Proof of Lemma A.7. Let $\bar{K}$ be an algebraic closure of K. By [Mat57], we have

$$N^1(X_{\bar{K}}) \simeq \left(\operatorname{Pic}(X_{\bar{K}})/\operatorname{Pic}^0(X_{\bar{K}})\right) \otimes_{\mathbf{Z}} \mathbf{R}.$$

By assumption, there exist a function field F having K as a completion and an F-variety Y such that $X = Y_K$. If we choose the algebraic closure $\bar{F}$ of F inside $\bar{K}$, then we similarly have

$$N^1(Y_{\bar{F}}) \simeq \left(\operatorname{Pic}(Y_{\bar{F}})/\operatorname{Pic}^0(Y_{\bar{F}})\right) \otimes_{\mathbf{Z}} \mathbf{R}.$$

We may now argue as in [MP12, Proposition 3.1]: since $\operatorname{Pic}(Y_{\bar{F}})$ is a group scheme locally of finite type over $\bar{F}$, its group of components is invariant under passing to the algebraically closed extension $\bar{K}$, hence $N^1(Y_{\bar{F}}) \simeq N^1(X_{\bar{K}})$.

It remains to choose a finite Galois extension K' over which all elements of a given basis of $N^1(X_{\bar{K}})$ are defined. $\square$

Proof of Theorem A.6. We argue in two steps.

Step 1. We first prove the orthogonality property in the following special case. Let C be a smooth projective curve over k whose function field F has K as its completion with respect to a closed point $0 \in C$ (so that S embeds into C). Let $\mathcal{Y}$ be a normal projective k-variety with a surjective morphism $\pi : \mathcal{Y} \to C$ such that $\mathcal{Y}_K \simeq X$, let $\mathcal{L}$ be a $\mathbf{Q}$-line bundle on $\mathcal{Y}$ which is ample on the generic fiber of π, and let $\theta \in \mathcal{Z}^{1,1}(X)$ be the curvature form of the model metric on L defined by restricting $\mathcal{L}$ to the model $\mathcal{X} := \mathcal{Y} \times_C S$ of X. We are going to prove that θ satisfies the orthogonality property.

By Lemma A.1, it suffices to prove that (θ, f) satisfies the orthogonality property for each $f \in \mathcal{D}(X)$. As in the proof of this lemma, we next reduce to the case $f = 0$. Indeed let $\mathcal{X}'$ be a determination of f, which may be taken to dominate $\mathcal{X}$. The model $\mathcal{X}'$ is then the blow-up of $\mathcal{X}$ along a vertical ideal sheaf $\mathfrak{a}$. Since $\mathfrak{a}$ is vertical on $\mathcal{X} = \mathcal{Y} \times_C S$, it comes from a vertical ideal sheaf on $\mathcal{Y}$, and the blow-up $\mathcal{Y}'$ of $\mathcal{Y}$ along this ideal satisfies $\mathcal{Y}' \times_C S = \mathcal{X}'$ since blow-ups commute with flat base change. Note also that $\mathcal{Y}'$ is normal; indeed, $\mathcal{Y}' \setminus \mathcal{Y}'_0 \simeq \mathcal{Y} \setminus \mathcal{Y}_0$ is normal, and $\mathcal{Y}'$ is normal along $\mathcal{Y}_0$ by regularity of the morphism $S \to C$. Replacing $\mathcal{Y}$ with $\mathcal{Y}'$, we may thus assume that f is already determined on $\mathcal{X}$, so that there exists a vertical $\mathbf{Q}$-divisor $E \in \mathrm{Div}_0(\mathcal{X})_{\mathbf{Q}}$ such that $f = f_E$. Since E is vertical on $\mathcal{X}$, it also comes from a vertical $\mathbf{Q}$-divisor on $\mathcal{Y}$, by which we may twist $\mathcal{L}$ to reduce to the case $f = 0$.

We are thus reduced to proving the orthogonality property for $(\theta, 0)$. For this, we rely on the geometric interpretation of Lemma A.5. Observe that $\mathcal{L}$ can be assumed to be an actual line bundle (as opposed to a $\mathbf{Q}$-line bundle), after multiplying by a large enough integer. For each $m \in \mathbf{N}$, let $\mathfrak{b}_m \subset \mathcal{O}_{\mathcal{Y}}$ be the relative base ideal of $m\mathcal{L}$ over C, i.e., the image of the evaluation map

$$\pi^* \pi_* \mathcal{O}_{\mathcal{Y}}(m\mathcal{L}) \otimes \mathcal{O}_{\mathcal{Y}}(-m\mathcal{L}) \to \mathcal{O}_{\mathcal{Y}}.$$

Note that $\mathfrak{b}_m$ is cosupported on finitely many fibers of π for $m \gg 1$, because $m\mathcal{L}$ is base point free on the generic fiber of π. Let $\rho_m : \mathcal{Y}_m \to \mathcal{Y}$ be the normalized blow-up of $\mathcal{Y}$ along $\mathfrak{b}_m$, let F_m be the corresponding effective Cartier divisor of $\mathcal{Y}_m$, and set $M_m := \rho_m^*(m\mathcal{L}) - F_m$, so that

$$\rho_m^*(m\mathcal{L}) = M_m + F_m$$

is the decomposition into a mobile part and a fixed part, relatively to C. By flat base change, the piece of F_m lying over $0 \in C$ induces the fixed part of the restriction of $m\mathcal{L}$ to $\mathcal{X} = \mathcal{Y} \times_C S$ in the sense of Lemma A.5. It is thus enough to show that

$$\lim_{m\to\infty} \left(\frac{1}{m} M_m\right)^n \cdot \left(\frac{1}{m} F_m\right) = 0, \tag{A.5}$$

where we stress that M_m and F_m denote divisors on $\mathcal{Y}$ whose restrictions to $\mathcal{X}$ are those of Lemma A.5. Indeed, there exists a finite subset $Z \subset C$ such that, for $m \gg 1$, F_m is supported on the fibers over the points in Z, and $(\frac{1}{m} M_m)^n \cdot (\frac{1}{m} F_m)$ can be written as a sum over Z of nonnegative terms. Therefore, if (A.5) holds, then the term corresponding to the point $0 \in C$ must tend to zero, which by Lemma A.5 shows that $(\theta, 0)$ has the orthogonality property.

We are going to prove (A.5) by reducing to the absolute case of a big line bundle on $\mathcal{Y}$. By Lemma A.8 below, there exists, after replacing $\mathcal{L}$ by a multiple, an ample line bundle $H \in \mathrm{Pic}(C)$ such that $D := \mathcal{L} + \pi^* H$ is big on the projective k-variety $\mathcal{Y}$ and such that

$$\pi_* \mathcal{O}_{\mathcal{Y}}(m\mathcal{L}) \otimes \mathcal{O}_C(mH)$$

is globally generated on C for all $m \gg 1$. The last property yields that the relative base ideal $\mathfrak{b}_m$ of $m\mathcal{L}$ coincides with the absolute base ideal of $mD = m\mathcal{L} + mH$, so that

$$\rho_m^*(mD) - F_m = M_m + (\pi \circ \rho_m)^*(mH)$$

is the base point free part of mD. Since F_m is π-vertical, we have

$$\left(\frac{1}{m} M_m\right)^n \cdot \left(\frac{1}{m} F_m\right) = \left(\frac{1}{m} M_m + (\pi \circ \rho_m)^* H\right)^n \cdot \left(\frac{1}{m} F_m\right),$$

which tends to 0 as $m \to \infty$ by [BDPP13, Theorem 4.1]. We have thus proved (A.5).

Step 2. We now consider the general case. Let $\alpha \in N^1(X)$ be an ample class, and let us prove that α satisfies the orthogonality property.

By Lemma A.2 we may assume $\alpha \in N^1(X)_{\mathbf{Q}}$. By Lemma A.4 and Lemma A.7, after replacing K with a finite Galois extension, we may further assume that $X = Y_K$ and that α is the image of a class in $N^1(Y)_{\mathbf{Q}}$, i.e., the image of a $\mathbf{Q}$-line bundle L on Y. It follows from [EGA, IV, Corollaire 2.7.2] that L is ample.

Picking a model $(\mathcal{Y}, \mathcal{L})$ of (Y, L) over the unique smooth projective curve C having F as its function field, we are now reduced to the situation of Step 1. $\square$

Lemma A.8. *Let $\mathcal{Y}$ be a projective k-variety with a surjective morphism $\pi : \mathcal{Y} \to C$ to a smooth projective curve over k. Assume that $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$ is ample on the generic fiber of π. Then, after replacing $\mathcal{L}$ by a suitable multiple, there exists an ample line bundle H on S such that the line bundle $\mathcal{L} + \pi^* H$ is big on $\mathcal{Y}$ and such that the sheaf*

$$\pi_* \mathcal{O}_{\mathcal{X}}(m\mathcal{L}) \otimes \mathcal{O}_C(mH)$$

is globally generated on C for all $m \gg 1$.

Proof. Set $\mathcal{F}_m := \pi_* \mathcal{O}_{\mathcal{Y}}(m\mathcal{L})$. Since $\mathcal{L}$ is ample over the generic point of C, the graded $\mathcal{O}_C$-algebra $\bigoplus_{m \in \mathbf{N}} \mathcal{F}_m$ is finitely generated at the generic point of C. After replacing $\mathcal{L}$ by a multiple, we may further assume that the generators have degree 1, so that $\mathcal{F}_m / \mathcal{F}_1^m$ has zero-dimensional support for all $m \geq 1$.

Since $\mathcal{L}$ is ample on the generic fiber, it is in particular π-big. We can therefore find a very ample line bundle H on S such that $\mathcal{L} + \pi^* H$ is big, and the same will be true after replacing H by any multiple.

By the Castelnuovo-Mumford criterion [Laz04, Theorem 1.8.5] it is now enough to show that

$$H^1(C, \mathcal{O}_C(mH) \otimes \mathcal{F}_m) = 0 \tag{A.6}$$

for all $m \gg 1$, after possibly replacing H by a multiple.

Since $\mathcal{F}_m / \mathcal{F}_1^m$ has zero-dimensional support, the map

$$H^1(C, \mathcal{O}_C(mH) \otimes \mathcal{F}_1^m) \to H^1(C, \mathcal{O}_C(mH) \otimes \mathcal{F}_m)$$

is surjective for all $m \geq 1$. We may thus replace $\mathcal{L}$ with $\mathrm{Proj}_C \left(\bigoplus_{m \in \mathbf{N}} \mathcal{F}_1^m \right)$, which reduces us to proving (A.6) when $\mathcal{L}$ is π-ample (i.e., ample on *all* fibers of π). After replacing H by a multiple, we then may assume that $\mathcal{L} + \pi^* H$ is ample on $\mathcal{Y}$.

By Serre vanishing, the fact that $\mathcal{L}$ is π-ample implies that $R^q \pi_* \mathcal{O}_{\mathcal{Y}}(m\mathcal{L}) = 0$ for $m \gg 1$ and $q > 0$. The degeneration of the Leray spectral sequence now yields

$$H^1(C, \mathcal{O}_C(mH) \otimes \mathcal{F}_m) \simeq H^1(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(m(\mathcal{L} + \pi^* H))),$$

which vanishes for all $m \gg 1$, since $\mathcal{L} + \pi^* H$ is ample on $\mathcal{Y}$. $\square$

Acknowledgment

This work has been strongly influenced by the work of M. Kontsevich and Y. Tschinkel. The 2001 colloquium talk of Kontsevich at the Institut de Mathématiques de Jussieu served as a guiding source for us. We are also grateful to him for showing to us the unpublished preprint [KT]. We further thank A. Thuillier for several interesting discussions, and J.-L. Colliot-Thélène for his help with Lemma A.7.

Our work was carried out at several institutions including the IHES, the École Polytechnique, and the University of Michigan. We gratefully acknowledge their support.

References

[Ale38] Alexandrov A. D., *On the theory of mixed volumes of convex bodies III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies. (Russian)*, Mat. Sbornik **3** (1938), 27–44. English translation available in selected works. Part I. Selected scientific papers. Gordon and Breach, Amsterdam, 1996.

[BR10] Matthew Baker and Robert Rumely, *Potential theory and dynamics on the Berkovich projective line*, Mathematical Surveys and Monographs, vol. 159, American Mathematical Society, Providence, RI, 2010. MR2599526 (2012d:37213)

[Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990. MR1070709 (91k:32038)

[Ber99] Vladimir G. Berkovich, *Smooth p-adic analytic spaces are locally contractible*, Invent. Math. **137** (1999), no. 1, 1–84, DOI 10.1007/s002220050323. MR1702143 (2000i:14028)

[BB10] Robert Berman and Sébastien Boucksom, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. **181** (2010), no. 2, 337–394, DOI 10.1007/s00222-010-0248-9. MR2657428 (2011h:32021)

[BBGZ13] Robert J. Berman, Sébastien Boucksom, Vincent Guedj, and Ahmed Zeriahi, *A variational approach to complex Monge-Ampère equations*, Publ. Math. Inst. Hautes Études Sci. **117** (2013), 179–245, DOI 10.1007/s10240-012-0046-6. MR3090260

[BT82] Eric Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. **149** (1982), no. 1-2, 1–40, DOI 10.1007/BF02392348. MR674165 (84d:32024)

[BT87] Eric Bedford and B. A. Taylor, *Fine topology, Šilov boundary, and* $(dd^c)^n$, J. Funct. Anal. **72** (1987), no. 2, 225–251, DOI 10.1016/0022-1236(87)90087-5. MR886812 (88g:32033)

[BGS95] S. Bloch, H. Gillet, and C. Soulé, *Non-Archimedean Arakelov theory*, J. Algebraic Geom. **4** (1995), no. 3, 427–485. MR1325788 (96g:14019)

[Bło03] Zbigniew Błocki, *Uniqueness and stability for the complex Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), no. 6, 1697–1701, DOI 10.1512/iumj.2003.52.2346. MR2021054 (2004m:32073)

[BK07] Zbigniew Błocki and Sławomir Kołodziej, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc. **135** (2007), no. 7, 2089–2093 (electronic), DOI 10.1090/S0002-9939-07-08858-2. MR2299485 (2008a:32029)

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248, DOI 10.1090/S1056-3911-2012-00574-8. MR3019449

[BEGZ10] Sébastien Boucksom, Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Monge-Ampère equations in big cohomology classes*, Acta Math. **205** (2010), no. 2, 199–262, DOI 10.1007/s11511-010-0054-7. MR2746347 (2011k:32049)

[BFJ08] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 449–494, DOI 10.2977/prims/1210167334. MR2426355 (2009g:32068)

[BFJ09] Sébastien Boucksom, Charles Favre, and Mattias Jonsson, *Differentiability of volumes of divisors and a problem of Teissier*, J. Algebraic Geom. **18** (2009), no. 2, 279–308, DOI 10.1090/S1056-3911-08-00490-6. MR2475816 (2009m:14005)

[BPS13] Burgos J. I., Philippon P., and Sombra M., *Arithmetic geometry of toric varieties. Metrics, measures, and heights*, available at `arXiv:1105.5584`. To appear in Astérisque, 212 pages.

[Ceg98] Urban Cegrell, *Pluricomplex energy*, Acta Math. **180** (1998), no. 2, 187–217, DOI 10.1007/BF02392899. MR1638768 (99h:32016)

[CL06] Antoine Chambert-Loir, *Mesures et équidistribution sur les espaces de Berkovich* (French), J. Reine Angew. Math. **595** (2006), 215–235, DOI 10.1515/CRELLE.2006.049. MR2244803 (2008b:14040)

[CL11] Antoine Chambert-Loir, *Heights and measures on analytic spaces. A survey of recent results, and some remarks*, Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume II, London Math. Soc. Lecture Note Ser., vol. 384, Cambridge University Press, Cambridge, 2011, pp. 1–50. MR2885340

[CLD12] Chambert-Loir A. and Ducros A., *Formes différentielles réelles et courants sur les espaces de Berkovich*, available at `arXiv:1204.6277`.

[Dem92] Jean-Pierre Demailly, *Regularization of closed positive currents and intersection theory*, J. Algebraic Geom. **1** (1992), no. 3, 361–409. MR1158622 (93e:32015)

[Din09] Sławomir Dinew, *Uniqueness in $\mathcal{E}(X,\omega)$*, J. Funct. Anal. **256** (2009), no. 7, 2113–2122, DOI 10.1016/j.jfa.2009.01.019. MR2498760 (2010e:32037)

[EGA] Grothendieck A., *Éléments de géométrie algébrique*, Publ. Math. IHES.

[EGZ09] Philippe Eyssidieux, Vincent Guedj, and Ahmed Zeriahi, *Singular Kähler-Einstein metrics*, J. Amer. Math. Soc. **22** (2009), no. 3, 607–639, DOI 10.1090/S0894-0347-09-00629-8. MR2505296 (2010k:32031)

[FJ04] Charles Favre and Mattias Jonsson, *The valuative tree*, Lecture Notes in Mathematics, vol. 1853, Springer-Verlag, Berlin, 2004. MR2097722 (2006a:13008)

[Fol99] Gerald B. Folland, *Real analysis*, 2nd ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999. Modern techniques and their applications; A Wiley-Interscience Publication. MR1681462 (2000c:00001)

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR1234037 (94g:14028)

[Ful98] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR1644323 (99d:14003)

[Gub08] Walter Gubler, *Equidistribution over function fields*, Manuscripta Math. **127** (2008), no. 4, 485–510, DOI 10.1007/s00229-008-0198-3. MR2457191 (2010a:14052)

[Gub13] Walter Gubler, *A guide to tropicalizations*, Algebraic and combinatorial aspects of tropical geometry, Contemp. Math., vol. 589, Amer. Math. Soc., Providence, RI, 2013, pp. 125–189, DOI 10.1090/conm/589/11745. MR3088913

[Gub13] Gubler W., *Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros)*. arXiv:1303.7364.

[GZ05] Vincent Guedj and Ahmed Zeriahi, *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. **15** (2005), no. 4, 607–639, DOI 10.1007/BF02922247. MR2203165 (2006j:32041)

[GZ07] Vincent Guedj and Ahmed Zeriahi, *The weighted Monge-Ampère energy of quasi-plurisubharmonic functions*, J. Funct. Anal. **250** (2007), no. 2, 442–482, DOI 10.1016/j.jfa.2007.04.018. MR2352488 (2008h:32056)

[JM12] Mattias Jonsson and Mircea Mustaţă, *Valuations and asymptotic invariants for sequences of ideals* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2145–2209 (2013), DOI 10.5802/aif.2746. MR3060755

[KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin-New York, 1973. MR0335518 (49 #299)

[Koł98] Sławomir Kołodziej, *The complex Monge-Ampère equation*, Acta Math. **180** (1998), no. 1, 69–117, DOI 10.1007/BF02392879. MR1618325 (99h:32017)

[Koł03] Sławomir Kołodziej, *The Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. J. **52** (2003), no. 3, 667–686, DOI 10.1512/iumj.2003.52.2220. MR1986892 (2004i:32062)

[KT] Kontsevich M. and Tschinkel Y., *Non-Archimedean Kähler geometry*. Unpublished.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in

Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series. MR2095471 (2005k:14001a)

[Liu11] Yifeng Liu, *A non-Archimedean analogue of the Calabi-Yau theorem for totally degenerate abelian varieties*, J. Differential Geom. **89** (2011), no. 1, 87–110. MR2863913

[Mat57] T. Matsusaka, *The criteria for algebraic equivalence and the torsion group*, Amer. J. Math. **79** (1957), 53–66. MR0082730 (18,602a)

[MP12] Davesh Maulik and Bjorn Poonen, *Néron-Severi groups under specialization*, Duke Math. J. **161** (2012), no. 11, 2167–2206, DOI 10.1215/00127094-1699490. MR2957700

[MN12] Mustaţă M. and Nicaise J., *Weight functions on non-archimedean analytic spaces and the Kontsevich-Soibelman skeleton*, available at `arXiv:1212.6328`.

[NX13] Nicaise J. and Xu C., *The essential skeleton of a degeneration of algebraic varieties*, available at `arXiv:1307.4041`.

[Nak04] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. MR2104208 (2005h:14015)

[RT77] Jeffrey Rauch and B. A. Taylor, *The Dirichlet problem for the multidimensional Monge-Ampère equation*, Rocky Mountain J. Math. **7** (1977), no. 2, 345–364. MR0454331 (56 #12582)

[Ser68] Jean-Pierre Serre, *Corps locaux* (French), Hermann, Paris, 1968. Deuxième édition; Publications de l'Université de Nancago, No. VIII. MR0354618 (50 #7096)

[Thu05] Thuillier A., *Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. Applications à la théorie d'Arakelov.* Ph.D. Thesis (2005), Université de Rennes I, available at `http://tel.archives-ouvertes.fr/docs/00/04/87/50/PDF/tel-00010990.pdf`.

[Tia] Gang Tian, *Canonical metrics in Kähler geometry*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2000. Notes taken by Meike Akveld. MR1787650 (2001j:32024)

[Yau78] Shing Tung Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411, DOI 10.1002/cpa.3160310304. MR480350 (81d:53045)

[YZ13a] Yuan X. and Zhang S.-W., *The arithmetic Hodge Theorem for adelic line bundles I*, available at `arXiv:1304.3538`.

[YZ13b] Yuan X. and Zhang S.-W., *The arithmetic Hodge Theorem for adelic line bundles II*, available at `arXiv:1304.3539`.

[Zha95] Shouwu Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), no. 1, 187–221, DOI 10.2307/2152886. MR1254133 (95c:14020)

CNRS–Université Pierre et Marie Curie, Institut de Mathématiques, F-75251 Paris Cedex 05, France

E-mail address: `boucksom@math.jussieu.fr`

CNRS–CMLS, École Polytechnique, F-91128 Palaiseau Cedex, France

E-mail address: `favre@math.polytechnique.fr`

Dept of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043

E-mail address: `mattiasj@umich.edu`