

DISTANCE TO NORMAL ELEMENTS IN C^* -ALGEBRAS OF REAL RANK ZERO

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INTRODUCTION

The problem of estimating the distance from a bounded operator A to the set of normal operator in terms of $\|[A, A^*]\|$ dates back to Halmos [15]. The original question is as follows:

- (C) Is there a continuous function F on \mathbb{R}_+ with $F(0) = 0$ such that for each pair of Hermitian matrices X, Y with $\|X\| + \|Y\| \leq 1$ there exists a pair of commuting Hermitian matrices X', Y' satisfying the estimate $\|X - X'\| + \|Y - Y'\| \leq F(\|[X, Y]\|)$?

Introducing $A = X + iY$, one can reformulate (C) in terms of one operator as follows:

- (C') Is there a continuous function F on \mathbb{R}_+ with $F(0) = 0$ such that the distance from a matrix A with $\|A\| \leq 1$ to the set of normal matrices does not exceed $F(\|[A, A^*]\|)$?

Clearly, the answer is positive if we allow F to depend on the dimension. A survey of dimension-dependent results can be found in [11, Chapter I].

The problem of the existence of a dimension-independent function F is much more challenging. It was open until 1995, when Huaxin Lin found deep C^* -algebraic arguments showing that such a function F does exist [17]. Later, Friis and Rørdam gave a shorter proof of Lin's theorem [13]. Note that, without additional conditions on X and Y , Lin's theorem does not hold for non-Hermitian matrices or self-adjoint operators acting on a Hilbert space (see, for instance, [8, 9, 11]).

The proofs in [13, 17] are non-constructive and do not give any information about the function F . To the best of our knowledge, the only quantitative result in this direction is due to Hastings, who showed in [16] that the distance from A to the set of normal matrices is estimated by $C_\varepsilon \|[A, A^*]\|^{1/5-\varepsilon}$ for all finite matrices A with $\|A\| \leq 1$, where C_ε is a constant depending on $\varepsilon > 0$. For homogeneity reasons, the function F in (C) or (C') cannot decay faster than $t^{1/2}$ as $t \rightarrow 0$. If we drop the condition $\|A\| \leq 1$, it also cannot grow slower than $t^{1/2}$ as $t \rightarrow \infty$. In [11], Davidson and Szarek conjectured that for finite matrices one can indeed take $F(t) = Ct^{1/2}$.

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A closely related result is the famous Brown-Douglas-Fillmore (BDF) theorem [6]. Recall that an operator A on a Hilbert space is said to be essentially normal if its self-commutator $[A, A^*]$ is compact. The BDF theorem states that a bounded essentially normal operator A on a separable Hilbert space is a compact perturbation of a normal operator if and only if A has a trivial index function (that is, $\text{ind}(A - zI) = 0$ whenever the operator $A - zI$ is Fredholm). In [14], the authors gave a simple proof of this theorem, which essentially repeats their proof of Lin's theorem in [13].

In [4], Berg and Davidson obtained a quantitative version of the BDF theorem. They proved that for each bounded closed set $\Omega \subset \mathbb{C}$ there exists a continuous function F_Ω with $F_\Omega(0) = 0$ such that the following is true. If A is an essentially normal operator with a trivial index function, $\|[A, A^*]\|^{1/2} \leq \varepsilon$ and, in addition, $\|(A - \lambda I)^{-1}\| < (\text{dist}(\lambda, \Omega) - \varepsilon)^{-1}$ whenever $\text{dist}(\lambda, \Omega) > \varepsilon$, then there is a normal operator N_ε with a spectrum in Ω such that $A - N_\varepsilon$ is compact and $\|A - N_\varepsilon\| \leq F_\Omega(\varepsilon)$. Note that proofs in [4] could be simplified by applying Lin's theorem, which was not known at that time. Instead, the authors used an absorption result of Davidson [9].

The main result of this paper is Theorem 1.1, which gives explicit bounds for the distance to the set of normal elements in an abstract C^* -algebra. It refines and extends all the results mentioned above. In particular, Theorem 1.1 implies

- the estimate (1.3) showing that the conjecture from [11] is true,
- Berg and Davidson's theorem with $F_\Omega(\varepsilon) = C\varepsilon$, where C is a constant independent of Ω (see Remark 6.1), and
- a quantitative version of the BDF theorem, which holds for operators with non-compact self-commutators (see Corollary 1.4).

1. NOTATION AND RESULTS

1.1. Main theorem. Let \mathcal{A} be a unital C^* -algebra. Recall that \mathcal{A} is said to have *real rank zero* if any of its self-adjoint element can be approximated by self-adjoint elements with finite spectra. Further on

- $\text{GL}(\mathcal{A})$ denotes the group of invertible elements of \mathcal{A} ;
- $\text{GL}_0(\mathcal{A})$ is the connected component of $\text{GL}(\mathcal{A})$ containing the identity;
- $\text{N}(\mathcal{A})$ denotes the set of normal elements of \mathcal{A} ;
- $\text{N}_f(\mathcal{A})$ is the set of normal elements with finite spectra;
- $d_1(A) := \sup_{\lambda \in \mathbb{C}} \text{dist}(A - \lambda I, \text{GL}_0(\mathcal{A}))$.

Theorem 1.1. *For any unital C^* -algebra \mathcal{A} of real rank zero and all $A \in \mathcal{A}$*

$$(1.1) \quad \text{dist}(A, \text{N}_f(\mathcal{A})) \leq C \left(\|[A, A^*]\|^{1/2} + d_1(A) \right),$$

$$(1.2) \quad \text{dist}(A, \text{N}_f(\mathcal{A})) \geq \max \{ (5\|A\|)^{-1} \|[A, A^*]\|, d_1(A) \},$$

where C is a constant independent of \mathcal{A} and A .

Remark 1.2. All von Neumann algebras, including the algebra of bounded operators $\mathcal{B}(H)$ on a Hilbert space H , have real rank zero. If \mathcal{A} is a von Neumann algebra, then $\text{N}(\mathcal{A}) \subset \overline{\text{N}_f(\mathcal{A})}$ and $\text{GL}_0(\mathcal{A}) = \text{GL}(\mathcal{A})$, so that we can drop the subscripts f in (1.1), (1.2), and 0 in the definition of $d_1(A)$.

Remark 1.3. If the complement of the spectrum of A is connected and dense in \mathbb{C} , then $d_1(A) = 0$. Indeed, in this case $A - \lambda I$ can be approximated by invertible

elements of the form $A - \mu I$, and each invertible $A - \mu I$ belongs to $\text{GL}_0(\mathcal{A})$ because $-\mu^{-1}(A - \mu I) \rightarrow I$ as $\mu \rightarrow \infty$. In particular, $d_1(A) = 0$ for finite matrices and, more generally, compact operators A . If H is separable and $\mathcal{A} = \mathcal{B}(H)$, then $d_1(A) = 0$ if and only if A has a trivial index function (see, for instance, [12, Section 3.2] and references therein). In the general case, $d_1(A)$ can be estimated in terms of the so-called modulus of invertibility [5, Theorem 2].

In view of the above remark, Theorem 1.1 implies that for any two Hermitian matrices X, Y there exists a pair of commuting Hermitian matrices X', Y' such that

$$(1.3) \quad \|X - X'\| + \|Y - Y'\| \leq C \|[X, Y]\|^{1/2},$$

where the constant C does not depend on X, Y and the dimension. It also implies the following quantitative version of the BDF theorem.

Corollary 1.4. *Assume that H is separable and denote by $\mathcal{K}(H)$ the space of compact operators on H . If $A \in \mathcal{B}(H)$ and $d_1(A) = 0$, then there exists a normal operator $A' \in \mathcal{B}(H)$ such that*

$$(1.4) \quad \begin{aligned} \|A - A'\|_{\text{ess}} &\leq C \|[A, A^*]\|_{\text{ess}}^{1/2}, \\ \|A - A'\| &\leq C \left(\|[A, A^*]\|^{1/2} + \|A\|^{1/2} \|[A, A^*]\|_{\text{ess}}^{1/4} \right), \end{aligned}$$

where $\|S\|_{\text{ess}} := \inf_{K \in \mathcal{K}(H)} \|S - K\|$ is the essential norm and C is a constant independent of A .

Remark 1.5. The upper bound (1.1) is a difficult result, which is new even for finite matrices. The lower bound (1.2) is almost obvious and is proved in few lines (see Subsection 6.2). We have included it in Theorem 1.1 only for the sake of completeness. The example $A_\varepsilon = X + i\varepsilon Y$ with non-commuting Hermitian matrices X, Y shows that $(5\|A\|)^{-1} \|[A, A^*]\|$ in (1.2) cannot be replaced by $C \|[A, A^*]\|^{1/2}$.

On the other hand, if $A_\varepsilon = \begin{pmatrix} \varepsilon B & 0 \\ 0 & I \end{pmatrix}$ and $[B, B^*] \neq 0$, then the distance from A to the set of normal matrices is estimated from above and below by $C\varepsilon$ with some constants C depending on B . This shows that the distance from A to $\text{N}_f(\mathcal{A})$ may decay as $\|[A, A^*]\|^{1/2}$ when $[A, A^*]$ tends to zero.

1.2. Notation. By the Gelfand-Naimark theorem, every C^* -algebra \mathcal{A} can be isomorphically embedded into the algebra of bounded operators $\mathcal{B}(H)$ on a (not necessarily separable) Hilbert space H . In order to emphasize the operator-theoretic nature of our proofs, we shall always assume that $\mathcal{A} \subset \mathcal{B}(H)$ and refer to its elements as “operators.”

We shall use the following notation:

- $\mathcal{O}_r(\lambda) := \{z \in \mathbb{C} : |z - \lambda| < r\}$ is the open disc of radius r about λ , and $\mathcal{O}_r := \mathcal{O}_r(0)$.
- C and δ denote universal constants, which do not depend on the C^* -algebras and operators under consideration.
- $\sigma(A)$ is the spectrum of the operator A .
- $M_2(\mathcal{A})$ is the C^* -subalgebra of $\mathcal{B}(H \oplus H)$ formed by 2×2 -matrices whose entries belong to \mathcal{A} . If \mathcal{A} has real rank zero, then $M_2(\mathcal{A})$ also has real rank zero (see, for instance, [7]).
- $P := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathcal{A})$ is the projection onto the first component of $H \oplus H$.
- $\text{diag}_P T := PTP + (1 - P)T(1 - P)$ where $T \in M_2(\mathcal{A})$.

- $\text{GL}_0(\mathcal{A} \oplus \mathcal{A}) := \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \text{M}_2(\mathcal{A}) : A_1, A_2 \in \text{GL}_0(\mathcal{A}) \right\}$.
- $d_2(T, \lambda) := \text{dist}(T - \lambda I, \text{GL}_0(\mathcal{A} \oplus \mathcal{A}))$ and $d_2(T) := \sup_{\lambda \in \mathbb{C}} d_2(T, \lambda)$, where $T \in \text{M}_2(\mathcal{A})$. Note that

$$(1.5) \quad \|T - \text{diag}_P T\| = \|[P, T]\| \leq d_2(T, 0).$$

1.3. Plan of proof. The first step is a generalization of [9, Corollary 4.5]. In Section 3 we prove that, under a certain condition on $A \in \mathcal{A}$, there exists a normal operator $T \in \text{M}_2(\mathcal{A})$ such that $\|A - PTP\| \leq C \|[A, A^*]\|^{1/2}$ and $d_2(T) \leq d_1(A) + C \|[A, A^*]\|^{1/2}$.

The next step is the most difficult part of the proof. In Theorem 5.1 we show that for every normal operator $T \in \text{M}_2(\mathcal{A})$ with a sufficiently small $d_2(T)$ there exists a normal operator $T_0 \in \text{M}_2(\mathcal{A})$ with a finite spectrum such that $\|T - T_0\| \leq 3$ and $\|[P, T_0]\| \leq C d_2(T)$. If $d_2(T) = 0$, then Theorem 5.1 follows from [14, Theorem 3.2]. However, this does not help, since the operator T constructed in the first step is not block diagonal and, consequently, $d_2(T) > 0$.

Our proof of Theorem 5.1 uses the technique introduced in [13] and further developed in [12]. It is based on successive reductions of the operator to normal operators whose spectra do not contain certain subsets of the complex plane. One can think of this process as removing subsets from the spectrum.

In order to obtain the uniform estimates for $\|T - T_0\|$ and $\|[P, T_0]\|$ in Theorem 5.1, we use two auxiliary results proved in Section 4. Their proof follows the same lines as that of Lemmas 4.1 and 4.2 in [12], but with additional control of the commutator $[P, T_0]$.

Finally, in Section 6 we adjust the block PT_0P of the operator T_0 to obtain a normal operator lying within the prescribed distance from A . This yields Theorem 1.1. After that, we deduce Corollary 1.4 by combining (1.1) with [12, Theorem 3.8].

Throughout the paper, we shall be using various results on operator Lipschitz functions. Their statements and proofs are given in the next section.

2. OPERATOR LIPSCHITZ FUNCTIONS

Definition 2.1. Let $\mathfrak{F} \subset \mathbb{C}$ be a closed set. A continuous function $f: \mathfrak{F} \rightarrow \mathbb{C}$ belongs to the space $\text{OL}(\mathfrak{F})$ if

$$\|f\|_{\text{OL}(\mathfrak{F})} := \sup_{T_1, T_2} \frac{\|f(T_1) - f(T_2)\|}{\|T_1 - T_2\|} < \infty,$$

where the supremum is taken over all bounded normal operators T_i acting on an infinite dimensional Hilbert space H such that $\sigma(T_i) \subset \mathfrak{F}$ and $T_1 \neq T_2$.

Remark 2.2. In the above definition, one can assume that the space H is separable. Indeed, the C^* -algebra generated by two given operators T_1, T_2 and the identity operator is separable. Hence it is isomorphic to a subalgebra of $\mathcal{B}(H')$ for some fixed separable Hilbert space H' (see, for instance, [10, Theorem I.9.12]). It follows that for each pair of operators $T_1, T_2 \in \mathcal{B}(H)$ there exist operators $T'_1, T'_2 \in \mathcal{B}(H')$ such that $\|T_1 - T_2\| = \|T'_1 - T'_2\|$ and $\|f(T_1) - f(T_2)\| = \|f(T'_1) - f(T'_2)\|$.

The spaces $\text{OL}(\mathfrak{F})$ are complex quasi-Banach spaces, in which only constant functions have zero quasi-norms. The functions $f \in \text{OL}(\mathfrak{F})$ are said to be *operator*

Lipschitz. It is well known that an operator Lipschitz function f is commutator Lipschitz in the sense that

$$(2.1) \quad \|[X, f(N)]\| \leq \|f\|_{\text{OL}(\mathfrak{F})} \|[X, N]\|$$

for all $N \in \mathcal{N}(\mathcal{B}(H))$ and $X = X^* \in \mathcal{B}(H)$ (see, for instance, [3, Theorem 3.1]). The best known sufficient conditions for the inclusion of $f \in \text{OL}(\mathfrak{F})$ are given in [1, 2, 19] in terms of Besov spaces. For our purposes, it is sufficient to know that $C^2(\mathfrak{F}) \subset \text{OL}(\mathfrak{F})$ and $\|f\|_{\text{OL}(\mathfrak{F})} \leq C \|f\|_{C^2(\mathfrak{F})}$ for $\mathfrak{F} = \mathbb{R}$ and $\mathfrak{F} = \mathbb{C}$, where $\|f\|_{C^2(\mathfrak{F})} := \max_{0 \leq |\alpha| \leq 2} \sup_{\zeta \in \mathfrak{F}} |\partial_\zeta^\alpha f(\zeta)|$ and C is some constant.

We shall need the following simple lemmas.

Lemma 2.3. *If $\rho \in C(\mathbb{R})$, then, for all $T_1, T_2 \in \mathcal{B}(H)$,*

$$\begin{aligned} \|\rho(T_1^* T_1) - \rho(T_2^* T_2)\| &\leq \|\rho(t^2)\|_{\text{OL}(\mathbb{R})} \|T_1 - T_2\|, \\ \|T_1 \rho(T_1^* T_1) - T_2 \rho(T_2^* T_2)\| &\leq \|t \rho(t^2)\|_{\text{OL}(\mathbb{R})} \|T_1 - T_2\|. \end{aligned}$$

Proof. Consider the self-adjoint operators $X_j = \begin{pmatrix} 0 & T_j \\ T_j^* & 0 \end{pmatrix} \in \mathcal{B}(H \oplus H)$. Since

$$\rho(X_j^2) = \begin{pmatrix} \rho(T_j T_j^*) & 0 \\ 0 & \rho(T_j^* T_j) \end{pmatrix} \quad \text{and} \quad X_j \rho(X_j^2) = \begin{pmatrix} 0 & T_j \rho(T_j^* T_j) \\ T_j^* \rho(T_j T_j^*) & 0 \end{pmatrix},$$

we obtain

$$\begin{aligned} \|\rho(T_1^* T_1) - \rho(T_2^* T_2)\| &\leq \|\rho(X_1^2) - \rho(X_2^2)\| \\ &\leq \|\rho(t^2)\|_{\text{OL}(\mathbb{R})} \|X_1 - X_2\| = \|\rho(t^2)\|_{\text{OL}(\mathbb{R})} \|T_1 - T_2\| \end{aligned}$$

and, similarly,

$$\|T_1 \rho(T_1^* T_1) - T_2 \rho(T_2^* T_2)\| \leq \|X_1 \rho(X_1^2) - X_2 \rho(X_2^2)\| \leq \|t \rho(t^2)\|_{\text{OL}(\mathbb{R})} \|T_1 - T_2\|. \blacksquare$$

Lemma 2.4. *Suppose that $\chi \in C_0^\infty(\mathbb{C})$, $0 \leq \chi \leq 1$, and let $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{C}$ be a finite collection of points such that $\chi_i \chi_j \equiv 0$ for all $i \neq j$, where $\chi_j(z) := \chi(z - \lambda_j)$. Let also $M := \sum_{j=1}^k \chi_j(T) S_j \chi_j(T)$, where $T \in \mathcal{N}(\mathcal{B}(H))$ and $S_j \in \mathcal{B}(H)$. If $\|S_j\| \leq 1$ and $[S_j, \chi_j(T)] = 0$ for all $j = 1, \dots, k$, then*

$$\|[Q, M]\| \leq C_\chi \|[Q, T]\| \max_j \|S_j \chi_j(T)\| + \max_j \|[Q, S_j]\|$$

for all self-adjoint $Q \in \mathcal{B}(H)$, where C_χ is a constant depending only on χ .

Proof. We have $[Q, M] = R_1 + R_2 + R_3$, where $R_1 = \sum_{j=1}^k [Q, \chi_j(T)] S_j \chi_j(T)$, $R_2 = \sum_{j=1}^k \chi_j(T) [Q, S_j] \chi_j(T)$, and $R_3 = \sum_{j=1}^k \chi_j(T) S_j [Q, \chi_j(T)]$.

Denote by $\hat{\chi}$ the Fourier transform of $\chi(x + iy)$ as a function of two real variables x and y . Let $\lambda_j = x_j + iy_j$ and $T = X + iY$, where X, Y are self-adjoint. Then $[X, Y] = 0$ and

$$R_1 = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \hat{\chi}(s, t) [Q, e^{isX + itY}] \left\{ \sum_{j=1}^k e^{-isx_j - ity_j} S_j \chi_j(T) \right\} ds dt.$$

Since the operators $S_j \chi_j(T)$ act in mutually orthogonal subspaces, the norm of the sum in curly brackets does not exceed $\max_j \|S_j \chi_j(T - \lambda_j I)\|$. Also, $\|[Q, e^{isX + itY}]\| \leq (|s| + |t|) \|[Q, T]\|$ because

$$\|[Q, e^{isX}]\| = \|[e^{-isX} Q e^{isX} - Q]\| = \left\| \int_0^s e^{-itX} [Q, X] e^{itX} dt \right\| \leq |s| \| [Q, X] \|$$

and, similarly, $\| [Q, e^{itY}] \| \leq |t| \| [Q, Y] \|$. It follows that

$$(2.2) \quad \| R_1 \| \leq \frac{\| [Q, T] \|}{4\pi^2} \max_j \| S_j \chi_j(T) \| \int_{\mathbb{R}^2} (|s| + |t|) |\hat{\chi}(s, t)| ds dt.$$

Similar arguments show that $\| R_3 \|$ admits the same estimate. It remains to notice that $\| R_2 \| \leq \max_j \| [Q, S_j] \|$ because $\chi_i \chi_j \equiv 0$ for $i \neq j$. ■

Lemma 2.5. *Let $\rho \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\text{supp } \rho \subset [-1, 1]$ and $\sum_{n \in \mathbb{Z}} \rho_n^2(x) = 1$, where $\rho_n(x) = \rho(x - n)$. If $X, Y \in \mathcal{B}(H)$ are self-adjoint and $Y' = \sum_{n \in \mathbb{Z}} \rho_n(X) Y \rho_n(X)$, then*

- (d₁) $\| EY'E \| \leq \| EYE \|$ for any spectral projection E of the operator X ;
- (d₂) $\| [X, Y'] \| \leq \| [X, Y] \|$;
- (d₃) $\| Y - Y' \| \leq C_\rho \| [X, Y] \|$ where C_ρ is a constant depending only on ρ .

Proof. If $aI \leq EYE \leq bI$, then $a\|u\|^2 \leq (EY'E u, u) \leq b\|u\|^2$ for all $u \in H$. This implies (d₁). Since the commutator $[X, Y]$ is skew-adjoint, the inequality (d₂) follows from the identity $[X, Y'] = \sum_{n \in \mathbb{Z}} \rho_n(X) [X, Y] \rho_n(X)$ by similar arguments. Finally, since $Y - Y' = \sum_n [Y, \rho_n(X)] \rho_n(X)$, the estimate (d₃) is proved in the same way as (2.2). ■

3. AN EXTENSION THEOREM

The following theorem is a refined version of [9, Corollary 4.5]. The latter provides normal operators $N \in \mathcal{B}(H)$ and $T \in \mathcal{B}(H \oplus H)$ satisfying (e₁) with $C = 50\sqrt{2}$ and (e₂). If $\mathcal{A} = \mathcal{B}(H)$ and $\dim H < \infty$, then (e₃) follows from (e₁), so that [9, Corollary 4.5] is sufficient to prove (1.3).

Theorem 3.1. *Let \mathcal{A} be a unital C^* -algebra, and let $A \in \mathcal{A}$. If $\text{Re } A$ can be approximated by self-adjoint operators from \mathcal{A} with finite spectra, then there exist normal operators $N \in \mathcal{A}$ and $T \in M_2(\mathcal{A})$ such that*

- (e₁) $\| A \oplus N - T \| \leq C \| [A, A^*] \|^{1/2}$,
- (e₂) $\| N \| \leq \| A \|$ and $\| T \| \leq \| A \|$,
- (e₃) $d_2(T) \leq d_1(A) + C \| [A, A^*] \|^{1/2}$,

where C is a constant independent of \mathcal{A} and A .

Proof. Let $A = X + iY$, where X, Y are self-adjoint. First of all, let us make some reductions. Note that the statements (e₁)–(e₃) are invariant under multiplication of A by a scalar. Therefore, without loss of generality, we shall be assuming that $\| [A, A^*] \| = 1$. Since X is approximated by operators with finite spectra, we can also assume that X has a finite spectrum. Finally, (e₂) can be replaced with the weaker condition

$$(e'_2) \quad \| N \| \leq \| A \| + 1.$$

Indeed, since $\| [A, A^*] \| = 1$, from the estimates (e₁) and (e'_2) it follows that

$$(e''_2) \quad \| T \| \leq \| A \| + C + 1.$$

If $c = \frac{\| A \|}{\| A \| + C + 1}$, then, in view of (e₁), (e'_2), and (e''_2),

$$\begin{aligned} \| A \oplus cN - cT \| &\leq \| A \oplus N - T \| + (1 - c) (\| N \| + \| T \|) \\ &\leq C + 2(1 - c) (\| A \| + C + 1) = 3C + 2, \end{aligned}$$

$\|cN\| \leq \|A\|$, $\|cT\| \leq \|A\|$, and $d_2(cT) = cd_2(T) \leq d_2(T)$. Thus we see that (e₁)–(e₃) hold with the constant C replaced by $3C + 2$ for the normal operators cN and cT .

Let us fix a function ρ satisfying the conditions of Lemma 2.5, and denote by E_n the spectral projections of X corresponding to the intervals $[n; n + 1)$. Since $\sigma(X)$ is finite, the projections E_n belong to \mathcal{A} . Consider the operators

$$\Pi_n := \begin{pmatrix} \rho_n^2(X) & \psi_n(X) \\ \psi_n(X) & E_{n-1} + E_n - \rho_n^2(X) \end{pmatrix} \in M_2(\mathcal{A}),$$

where $\rho_n(x) = \rho(x - n)$ and $\psi_n(x) = (-1)^n \rho_n(x)(\rho_{n-1}(x) + \rho_{n+1}(x))$. One can easily see that Π_n are mutually orthogonal projections such that

$$(3.1) \quad \sum_{n \in \mathbb{Z}} \Pi_n = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \text{and} \quad \Pi_n = \begin{pmatrix} E_{n-1} + E_n & 0 \\ 0 & E_{n-1} + E_n \end{pmatrix} \Pi_n.$$

Note that $[\Pi_n, E_m \oplus E_m] = 0$ for all $n, m \in \mathbb{Z}$. It follows that $(E_n \oplus E_n)\Pi_m$ are mutually orthogonal projections such that

$$(3.2) \quad \sum_{n, m \in \mathbb{Z}} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \Pi_m = \sum_{|n-m| \leq 1} \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \Pi_m = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

Let Y' be the operator defined in Lemma 2.5, and let $Y'' = \sum_{m \in \mathbb{Z}} E_m Y' E_m$. We claim that the normal operators

$$N := \sum_{n \in \mathbb{Z}} E_n (nI + iY') E_n \quad \text{and} \quad T := \sum_{n \in \mathbb{Z}} \Pi_n \begin{pmatrix} nI + iY' & 0 \\ 0 & nI + iY'' \end{pmatrix} \Pi_n$$

satisfy the conditions (e₁), (e'₂), and (e₃).

First, let us prove (e₁). Since $E_n(X - nI)E_n \leq 1$, using (3.2), we obtain

$$(3.3) \quad \begin{aligned} \|\operatorname{Re}(A \oplus N) - \operatorname{Re} T\| &\leq 1 + \left\| \sum_n n(E_n \oplus E_n - \Pi_n) \right\| \\ &= 1 + \left\| \sum_{n, m} (n(E_n \oplus E_n)\Pi_m - n(E_m \oplus E_m)\Pi_n) \right\| \\ &= 1 + \left\| \sum_{|n-m| \leq 1} (n-m)(E_n \oplus E_n)\Pi_m \right\| \leq 2. \end{aligned}$$

For the imaginary part, in view of the estimate (d₃) in Lemma 2.5, we have

$$(3.4) \quad \|\operatorname{Im}(A \oplus N) - Y' \oplus Y''\| = \|Y \oplus Y'' - Y' \oplus Y''\| \leq C_\rho,$$

where C_ρ is a constant depending only on the choice of ρ . Note that $E_n Y' E_m = 0$ whenever $|n - m| \geq 2$. These identities and the second equality (3.1) imply that $\operatorname{Im} T = \sum_{|n-m| \leq 2} \Pi_n (Y' \oplus Y'') \Pi_m$ and, consequently,

$$(3.5) \quad \begin{aligned} \|Y' \oplus Y'' - \operatorname{Im} T\| &= \left\| \sum_{1 \leq |n-m| \leq 2} \Pi_n (Y' \oplus Y'') \Pi_m \right\| \\ &\leq 4 \max_{1 \leq |n-m| \leq 2} \|\Pi_n (Y' \oplus Y'') \Pi_m\| \leq 4 \max_n \|\Pi_n (Y' \oplus Y'')\|. \end{aligned}$$

Since $[E_n, Y''] = [E_{n-1}, Y''] = 0$, we have

$$(3.6) \quad [\Pi_n, Y' \oplus Y''] = \begin{pmatrix} [\rho_n^2(X), Y'] & \psi_n(X)Y'' - Y'\psi_n(X) \\ \psi_n(X)Y' - Y''\psi_n(X) & [Y'', \rho_n^2(X)] \end{pmatrix}.$$

The operator (3.6) is skew-adjoint and

$$(3.7) \quad \begin{aligned} \psi_n(X)Y'' - Y'\psi_n(X) \\ = (-1)^n ([\rho_n(X)\rho_{n+1}(X), Y']E_n + [\rho_n(X)\rho_{n-1}(X), Y']E_{n-1}). \end{aligned}$$

By Lemma 2.5, $\|[X, Y']\| \leq \|[X, Y]\| = \frac{1}{2} \| [A, A^*] \| = 1/2$ and, similarly,

$$\|[X, Y'']\| = \left\| \sum_{m \in \mathbb{Z}} E_m [X, Y'] E_m \right\| \leq \|[X, Y']\| \leq 1/2.$$

Since C^2 -norms of ρ_n^2 and $\rho_n \rho_{n+1}$ are estimated by a constant independent of n , the inequality (2.1) and (3.7) imply that the norms of all entries in the right hand side of (3.6) are estimated by constants depending only on ρ . Together with (3.3), (3.4), and (3.5), this yields (e₁).

Obviously, $\|E_n(X - nI)E_n\| \leq 1$. The estimate (d₁) in Lemma 2.5 implies that $\|E_n(nI + iY')E_n\| \leq \|E_n(nI + iY)E_n\|$. It follows that

$$\|N\| = \max_n \|E_n(nI + iY')E_n\| \leq \max_n \|E_n(nI + iY)E_n\| \leq \|A\| + 1.$$

Finally, $\sigma(N)$ is a bounded subset of $\mathbb{Z} + i\mathbb{R}$. By Remark 1.3, $d_1(N) = 0$. This equality and the estimate (e₁) imply (e₃). ■

4. TWO AUXILIARY THEOREMS

Recall that any invertible operator T has the polar decomposition $T = V|T|$, where $|T| = \sqrt{T^*T}$ and $V = T|T|^{-1}$ is a unitary operator from the same C^* -algebra as T . A normal operator T also admits the polar decomposition $T = V|T|$ with a unitary V . However, if T is not invertible, then, generally speaking, the unitary polar part V of T is not uniquely defined and may not belong to the same C^* -algebra. In both cases, the unitary operator V satisfies $V\rho(|T|) = \rho(|T^*|)V$ for all $\rho \in C(\mathbb{R}^1)$. If T is normal, this implies that V commutes with all continuous functions of T .

In the next two theorems V is a unitary polar part of T and Π_r is the spectral projection of T onto the disc \mathcal{O}_r .

Theorem 4.1. *There exist constants $\delta > 0$ and $C > 0$ such that for any normal $T \in M_2(\mathcal{A})$ with $d_2(T, 0) < \delta$ one can find a unitary operator $U \in M_2(\mathcal{A})$ satisfying the following conditions:*

- (u₁) $\text{diag}_P U \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$,
- (u₂) $[U, \Pi_r] = 0$ for $r \geq 1$,
- (u₃) $U(I - \Pi_1) = V(I - \Pi_1)$,
- (u₄) $\|[P, U]\| \leq Cd_2(T, 0)$.

Theorem 4.2. *There exist constants $\delta > 0$ and $C > 0$ such that the following is true. Let \mathcal{A} have real rank zero, and let $T \in M_2(\mathcal{A})$ be a normal operator such that $\sigma(T) \cap \mathcal{O}_3$ is a subset of the straight line $e^{i\theta}\mathbb{R}$, where $\theta \in [0, \pi)$. If $\|[P, T]\| < \delta$ and $\text{diag}_P(T \pm ie^{i\theta}I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$, then one can find a unitary operator $U \in M_2(\mathcal{A})$ satisfying (u₁) and the following conditions:*

- (u'₂) $[U, \Pi_r] = 0$ for $1 \leq r \leq 2$,
- (u'₃) $U(\Pi_2 - \Pi_1) = V(\Pi_2 - \Pi_1)$,
- (u'₄) $\|[P, U]\| \leq C\|[P, T]\|$,
- (u'₅) *the spectrum of $U|_{\text{Ran } \Pi_2}$ is contained in the intersection of $e^{i\theta}\mathbb{R}$ with the unit circle.*

The proofs of Theorems 4.1 and 4.2 use the following lemmas.

Lemma 4.3. *Suppose that \mathcal{A} is a C^* -algebra of real rank zero, and let $U \in \mathrm{GL}_0(\mathcal{A})$ be unitary. Then for any $\varepsilon > 0$ there exists a unitary operator $V_\varepsilon \in \mathrm{GL}_0(\mathcal{A})$ such that $-1 \notin \sigma(V_\varepsilon)$ and $\|U - V_\varepsilon\| \leq \varepsilon$.*

Lemma 4.3 is contained in [18]. See also [12, Lemma 1.8] for a more elementary proof.

Lemma 4.4. *Let $t \mapsto G_t$ be a continuous path in $\mathrm{GL}(\mathrm{M}_2(\mathcal{A}))$ such that $\|G_t^{-1}\| < \|[P, G_t]\|^{-1}$ for all $t \in [0, 1]$ and $\mathrm{diag}_P G_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. Then $\mathrm{diag}_P G_1 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.*

Proof. Since $\mathrm{diag}_P G_t = G_t (I - G_t^{-1}(G_t - \mathrm{diag}_P G_t))$ and $\|G_t - \mathrm{diag}_P G_t\| = \|[P, G_t]\|$, the operators $\mathrm{diag}_P G_t$ are also invertible. Hence, the path $t \mapsto \mathrm{diag}_P G_t$ connects G_0 and G_1 in $\mathrm{GL}(\mathcal{A}) \oplus \mathrm{GL}(\mathcal{A})$. As $G_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$, so does G_1 . ■

Lemma 4.5. *If there exists a unitary operator V such that $\|S - V\| \leq \varepsilon < 1$, then the operator S is invertible and $\|S - S|S|^{-1}\| < \frac{\varepsilon(1+\varepsilon)}{1-\varepsilon}$.*

Proof. If $S = V + R$ with $\|R\| \leq \varepsilon$, then $(1 - \varepsilon)^2 I \leq S^* S \leq (1 + \varepsilon)^2 I$. It follows that S is invertible. Since $(1 + \varepsilon)^{-1} I \leq |S|^{-1} \leq (1 - \varepsilon)^{-1} I$ and $\|S\| \leq (1 + \varepsilon)$, we obtain $\|S - S|S|^{-1}\| \leq (1 + \varepsilon) \|I - |S|^{-1}\| \leq (1 + \varepsilon) \varepsilon (1 - \varepsilon)^{-1}$. ■

Lemma 4.6. *Suppose that Γ is a simple closed curve given by an equation of the form*

$$(4.1) \quad \Gamma = \{z \in \mathbb{C} : |z - \lambda| = \varphi(\arg(z - \lambda))\},$$

where λ is an interior point of the domain bounded by Γ and $\varphi \in C^2(\mathbb{R})$ is a strictly positive 2π -periodic function. Let \mathcal{A} be a unital C^* -algebra of real rank zero, and let $T \in \mathrm{M}_2(\mathcal{A})$ be a normal operator such that $\sigma(T) \subset \Gamma$ and $\mathrm{diag}_P(T - \lambda I) \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$. Then for every $z_0 \in \Gamma$ there exists a normal operator T_0 such that $\sigma(T_0) \subset \Gamma \setminus \{z_0\}$, $T_0 - \lambda I \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$, and $\|T - T_0\| \leq C_\Gamma \|[P, T]\|$, where C_Γ is a constant depending only on Γ .

Proof. Without loss of generality, we can assume that $\lambda = 0$ and $z_0 \in \mathbb{R}_-$. Also, it is sufficient to prove the lemma assuming that $\|[P, T]\|$ is small enough, since we have $\|T_0 - T\| \leq C_\Gamma \|[P, T]\|$ for any normal operator T_0 with $\sigma(T_0) \subset \Gamma$ if $\|[P, T]\| \geq \varepsilon$ and $C_\Gamma = 2\varepsilon^{-1} \mathrm{diam} \Gamma$.

Let $\varphi_t(z) = z(t\varphi(\arg z) + 1 - t)^{-1}$. The functions φ_t belong to C^2 on an annulus Ω containing Γ , and their $C^2(\Omega)$ -norms are bounded by a constant depending only on Γ . Obviously, they can be extended to C^2 -functions on \mathbb{C} whose C^2 -norms admit a similar estimate.

The operator $\varphi_1(T)$ is unitary because φ_1 maps Γ onto the unit circle. By (2.1), we have $\|[\varphi_t(T), P]\| \leq C'_\Gamma \|[P, T]\|$ for all $t \in [0, 1]$ with a constant C'_Γ depending only on Γ . If $\|[P, T]\|$ is sufficiently small, Lemma 4.4 with $G_t = \varphi_t(T)$ implies that $S := \mathrm{diag}_P \varphi_1(T)$ belongs to $\mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Since $\|S - \varphi_1(T)\| = \|[\varphi_1(T), P]\| \leq C'_\Gamma \|[P, T]\|$, the operator S is close to the unitary operator $\varphi_1(T)$. If $U = S|S|^{-1}$, then $U \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and, by Lemma 4.5, $\|S - U\| \leq C''_\Gamma \|[P, T]\|$ where C''_Γ depends only on Γ .

Now, applying Lemma 4.3, we find a unitary $U_0 \in \mathrm{GL}_0(\mathcal{A} \oplus \mathcal{A})$ such that $\|U - U_0\| \leq \|[P, T]\|$ and $-1 \notin \sigma(U_0)$, and define $T_0 = \phi_1^{-1}(U_0)$ where $\phi_1^{-1}(z) = z(\varphi(\arg z))$ is the inverse function. Since ϕ_1^{-1} maps the unit circle onto Γ and

$\phi_1(-1) = z_0$, we have $\sigma(T_0) \subset \Gamma \setminus \{z_0\}$. Since φ_1^{-1} belongs to C^2 on a neighborhood of the unit circle, it can be extended to an operator Lipschitz function on \mathbb{C} . Therefore the inequality $\|T - T_0\| \leq C_\Gamma \| [P, T] \|$ follows from the estimate

$$\| \varphi_1(T) - U_0 \| \leq \| \varphi_1(T) - S \| + \| S - U \| + \| U - U_0 \| \leq (C'_\Gamma + C''_\Gamma + 1) \| [P, T] \|.$$

Finally, $T_0 \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ because the complement of its spectrum is a dense connected set (see Remark 1.3). ■

Remark 4.7. One can easily extend Lemma 4.6 to a much wider class of curves Γ , but (4.1) will be sufficient for our purposes.

4.1. Proof of Theorem 4.1. Let $V \in \mathcal{B}(H \oplus H)$ be a unitary operator such that $T = V|T|$. Let us denote $\delta_1 := d_2(T, 0)$ and choose $T_0 \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ such that $\|T - T_0\| \leq 2\delta_1$. We have $T_0 = V_0|T_0|$ with $|T_0| \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and a unitary $V_0 \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Let $\rho_1 \in C^\infty(\mathbb{R}_+)$ be a nonincreasing function such that $\rho_1(t) = 1$ for $t \in [0, \frac{1}{2}]$ and $\rho_1(t) = 0$ for $t \geq 1$. Define $\rho_2 := \sqrt{1 - \rho_1^2}$ and consider the operator

$$(4.2) \quad S = \rho_1(T^*T)V_0\rho_1(T^*T) + V\rho_2^2(T^*T).$$

We have $V\rho_2^2(T^*T) = \chi(T)$, where $\chi(z) = z|z|^{-1}\rho_2^2(|z|^2)$ is a C^∞ -function. Thus $S \in \text{M}_2(\mathcal{A})$. Since $\rho_1(|z|^2) \in \text{OL}(\mathbb{C})$, $\chi(z) \in \text{OL}(\mathbb{C})$, and $[P, V_0] = 0$, from (2.1) and (1.5) it follows that

$$\| [P, S] \| \leq 2\| [P, \rho_1(T^*T)] \| + \| [P, \chi(T)] \| \leq C_1 \| [P, T] \| \leq C_1 \delta_1.$$

Here and in the rest of the proof C with a subscript denotes a constant depending only on the choice of ρ_1 .

By Lemma 2.3,

$$\begin{aligned} \| \rho_1(T^*T) - \rho_1(T_0^*T_0) \| &\leq C_2 \| T - T_0 \| \leq 2C_2 \delta_1, \\ \| \rho_1(TT^*) - \rho_1(T_0T_0^*) \| &\leq C_2 \| T - T_0 \| \leq 2C_2 \delta_1. \end{aligned}$$

Since $TT^* = T^*T$, these estimates and the identity $\rho_1(T_0T_0^*)V_0 = V_0\rho_1(T_0^*T_0)$ imply that

$$(4.3) \quad \| S - V_0\rho_1^2(T_0^*T_0) - V\rho_2^2(T^*T) \| \leq C_3 \delta_1.$$

The function $\rho_2(t)$ vanishes in a neighborhood of zero. Hence $\rho_2^2(t^2) = t\psi(t^2)$ with a smooth bounded function ψ and, by Lemma 2.3,

$$(4.4) \quad \begin{aligned} \| V_0\rho_2^2(T_0^*T_0) - V\rho_2^2(T^*T) \| &= \| V_0|T_0|\psi(T_0^*T_0) - V|T|\psi(T^*T) \| \\ &= \| T_0\psi(T_0^*T_0) - T\psi(T^*T) \| \leq C \| T - T_0 \| \leq 2C\delta_1. \end{aligned}$$

Combining (4.3) with (4.4) and using the identity $\rho_1^2 + \rho_2^2 \equiv 1$, we obtain $\| S - V_0 \| \leq C_4 \delta_1$.

Let us assume that δ in the statement of the lemma is so small that $C_4\delta_1 < 1$. If $U = S|S|^{-1}$, then, by Lemma 4.5,

$$(4.5) \quad \| U - V_0 \| \leq \| U - S \| + \| S - V_0 \| \leq C_5 \delta_1.$$

If δ is small, then the spectrum of S^*S lies in a small neighborhood of 1. Hence the operator $|S|^{-1}$ can be expressed as a smooth function of S^*S supported on a small interval containing 1. As $\| [P, S] \| \leq C_1 \delta_1$, we have $\| [P, |S|^{-1}] \| \leq C_6 \delta_1$. These two estimates imply (u₄).

Let $G_t = (1 - t)V_0 + tU$. If $C_5 \delta < \frac{1}{3}$, then, in view of (4.5), $\|[P, G_t]\| < \frac{2}{3}$ and $\|G_t^{-1}\| < \frac{3}{2}$ for all $t \in [0, 1]$. Applying Lemma 4.4, we obtain (u₁).

Finally, we have (u₂) and (u₃) because $[S, \Pi_r] = 0$ and $\rho_1(T^*T)\Pi_r = \rho_1(T^*T) = \Pi_r\rho_1(T^*T)$ for all $r \geq 1$. ■

4.2. Proof of Theorem 4.2. Multiplying T by a constant, we can assume that $\theta = 0$ and, consequently, $\sigma(T) \cap \mathcal{O}_3 \subset \mathbb{R}$. The proof consists of two parts.

4.2.1. Part 1. Suppose first that $\sigma(T) \subset \Gamma$, where Γ is a closed curve of the form (4.1) containing the line segment $[-3, 3]$.

Denote $\delta_1 := \|[P, T]\|$. By Lemma 4.6, there exists a normal operator $T_0 \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ such that $\sigma(T_0) \subset \Gamma \setminus \{0\}$ and $\|T - T_0\| \leq C_\Gamma \delta_1$. We have $T_0 = V_0|T_0|$ with $|T_0| \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and a unitary $V_0 \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

Let ρ_1, ρ_2 , and V be as in the proof of Theorem 4.1. Consider the operator

$$S = \rho_1(T^*T)(\text{Re } V_0)\rho_1(T^*T) + V\rho_2(T^*T)^2.$$

Since $\rho_1(T_0^*T_0)(\text{Re } V_0)\rho_1(T_0^*T_0) = V_0\rho_1^2(T_0^*T_0)$, the same arguments as in the proof of Theorem 4.1 show that

$$\|S - V_0\rho_1(T_0^*T_0)^2 - V\rho_2(T^*T)^2\| \leq C_{\Gamma,1} \delta_1$$

and $\|S - V_0\| \leq C_{\Gamma,2}\delta_1$, where $C_{\Gamma,j}$ are constants depending only on Γ .

If $U = S|S|^{-1}$, then, in the same way as in the proof of Theorem 4.1, we obtain (u₁)–(u₃) and (u'₄) with some constant C_Γ depending only on Γ . The condition (u'₅) is fulfilled because the operator $\Pi_2 S \Pi_2$ is self-adjoint, and so is the unitary operator $U|_{\text{ran } \Pi_2}$.

Remark 4.8. Note that

- (a) the above proof works under the weaker assumption $\sigma(T) \cap \mathcal{O}_2 \subset (-2, 2)$;
- (b) we only need to check that $\text{diag}_P(T_j - \lambda I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ for one point λ lying in the domain bounded by Γ ;
- (c) the constructed operator U satisfies stronger the conditions (u₂) and (u₃), which imply (u'₂), (u'₃).

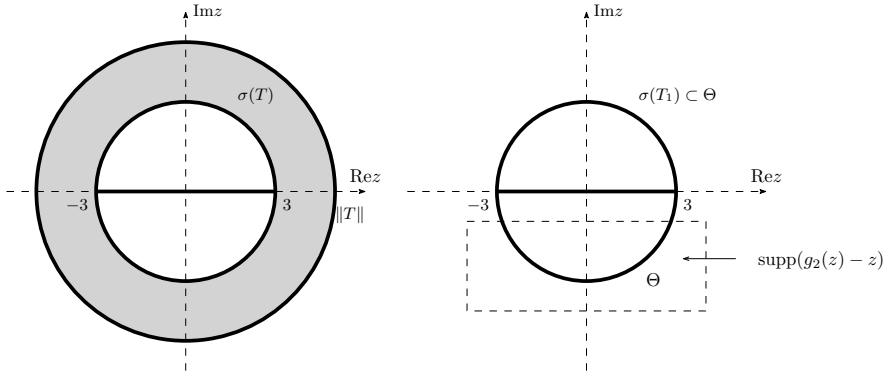
4.2.2. Part 2. Let now $\sigma(T)$ be an arbitrary closed bounded set such that $\sigma(T) \cap \mathcal{O}_3 \subset (-3, 3)$. First of all, let us note that the statement of Theorem 4.2 is local in the following sense.

Assume that we have found a unitary U_j satisfying the conditions of Theorem 4.2 for another normal operator T_j . If

- (i) the spectral projection of T_j corresponding to \mathcal{O}_2 coincides with Π_2 and $T_j\Pi_2 = T\Pi_2$,
- (ii) $\|[P, T_j]\| \leq C_j \|[P, T]\|$ where the constant C_j does not depend on \mathcal{A} and T ,
- (iii) $\text{diag}_P(T_j \pm iI) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$,

then (u₁) and (u'₂)–(u'₅) hold for T and $U = U_j$.

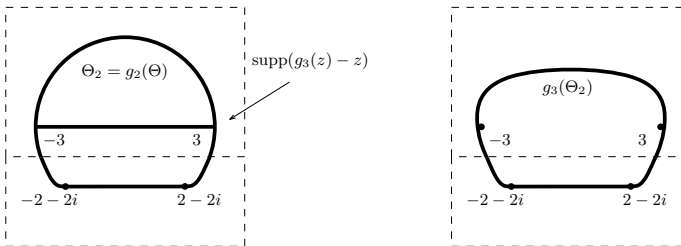
Hence, in view of Part 1, it is sufficient to find a normal operator T_j satisfying the conditions (i)–(iii) whose spectrum lies on a curve of the form (4.1) with $\lambda = i$. Furthermore, in view of Remark 4.8(b), if $\sigma(T_j) \subset \Gamma$, then we have to prove the inclusion (iii) only for $\text{diag}_P(T_j - iI)$. The construction proceeds in several steps.

FIGURE 1. The spectra of T and $T_1 = g_1(T)$.

First, let us choose a function $g_1 \in C^2(\mathbb{C})$ such that $g_1(z) = 3z/|z|$ for $|z| \geq 3$, $g_1(z) = z$ for $|z| \leq 2$, and $g_1(z)/z > 0$ for all $z \neq 0$. Put $T_1 = g_1(T)$. Then $\sigma(T_1) \subset \Theta = (-3; 3) \cup \partial\mathcal{O}_3$; see Figure 1. Clearly, T_1 satisfies (i). In view of (2.1), we also have (ii). Finally, if δ is small enough, then the paths $G_t = tg_1(T) + (1-t)T \pm iI$ satisfy the assumptions of Lemma 4.4, which implies the inclusions (iii).

Let g_2 be a smooth function mapping the arc of Θ between $(-3 - 3i)/\sqrt{2}$ and $(3 - 3i)/\sqrt{2}$ into the line segment $[-2 - 2i; 2 - 2i]$ such that $g_2(z) = z$ outside the lower rectangle in the right part of Figure 1. We have $g_2 \in \text{OL}(\mathbb{C})$ since it is a smooth compactly supported perturbation of z . Let $\Theta_2 = g_2(\Theta)$ and $T_2 = g_2(T_1)$, so that $\sigma(T_2) \subset \Theta_2$. For the same reasons as before, the operator T_2 satisfies (i)–(iii).

There is a function $g_3 \in \text{OL}(\mathbb{C})$ such that $g_3(z) = z$ outside the upper rectangle of Figure 2, $g(z) - z \in C^2(\mathbb{C})$ and $g_3(\Theta_2)$ is a simple curve given by an equation of the form (4.1). Note that $g_3: \mathbb{C} \rightarrow \mathbb{C}$ is not a diffeomorphism, as g_3 maps two arcs of Θ_2 into one. Let $T_3 = g_3(T_2)$. The same arguments as for T_1 show that T_3 satisfies (ii) and $\text{diag}_P T_3 - iI \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

FIGURE 2. The spectra of $T_2 = g_2(T_1)$ and $T_3 = g_3(T_2)$

Applying Part 1 to the operator $T_3 + 2iI$, we obtain a unitary operator U_3 satisfying (u_1) – (u_3) , (u'_4) and (u'_5) with Π_r being the spectral projections of $T_3 + 2iI$. Note that $g_2(z) \equiv z$ on $\mathcal{O}_2(-2i) \cap \sigma(T_2)$. Therefore Π_r coincide with the spectral projections $\Pi_r(-2i)$ of the operator T_2 corresponding to the discs $\mathcal{O}_r(-2i)$ for all $r \leq 2$.

Define

$$T_4 = (U_3 - 2iI)\rho(|T_2 + 2iI|) + T_2(I - \rho(|T_2 + 2iI|)),$$

where ρ is a nonincreasing C^∞ -function on \mathbb{R}_+ such that $\rho(t) = 1$ for $t \leq 1$ and $\rho(t) = 0$ for $t \geq 2$. Since U_3 commutes with $\Pi_r(-2i)$ for $r \leq 2$ and coincides with the polar part V_2 of $T_2 + 2iI$ on the range of $\Pi_2(-2i) - \Pi_1(-2i)$, the operator T_4 has the following block structure:

$$(4.6) \quad T_4 = (U_3 - 2iI)\Pi_1(-2i) \oplus \tilde{T}_2(\Pi_2(-2i) - \Pi_1(-2i)) \oplus T_2(I - \Pi_2(-2i)),$$

where $\tilde{T}_2 = T_2 + (V_2 - T_2 - 2iI)\rho(|T_2 + 2iI|)$. All operators on the right hand side of (4.6) are normal, and their spectra do not contain the point $-2i$. Thus T_4 is a normal operator whose spectrum is contained in Θ_2 with a part of the lower arc removed; see Figure 3. From the construction, it is clear that T_4 satisfies (i). The estimate (ii) for T_4 follows from the inclusion $\rho(|z|) \in C^2(\mathbb{C})$ and the estimate (u'_4) for the operator U_3 . If δ is small enough, then the path $G_t = tT_4 + (1-t)T_2 - iI$ satisfies the assumptions of Lemma 4.4, which implies that $\text{diag}_P T_4 - iI \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

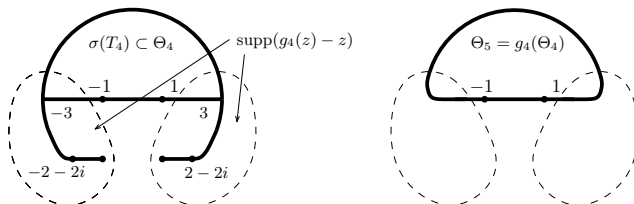


FIGURE 3. The spectra of T_4 and $T_5 = g_4(T_4)$.

Finally, there exists a smooth function $g_4 \in \text{OL}(\mathbb{C})$ such that $g_4(z) = z$ outside the oval-shaped areas in Figure 3, $g(z) - z \in C^2(\mathbb{C})$, and $g_4(\Theta_4)$ is given by an equation of the form (4.1). In simple words, g_4 maps the remaining parts of the lower arc into the end points of the central line segment and does not affect the rest of Θ_4 , including the segment $[-2; 2]$. The same arguments as before show that the operator $T_5 = g_4(T_4)$ satisfies (i), (ii), and $\text{diag}_P T_5 - iI \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ provided that δ is sufficiently small. Since $\sigma(T_5) \subset g_4(\Theta_4)$, this completes the proof. ■

Remark 4.9. If \mathcal{A} is a von Neumann algebra, then $\text{GL}_0(\mathcal{A} \oplus \mathcal{A}) = \text{GL}(\mathcal{A}) \oplus \text{GL}(\mathcal{A})$ (see Remark 1.2). In this case we do not need to check the condition (iii) and can simplify Part 2 by taking $T_5 = g(T)$, where $g \in C^2(\mathbb{C})$ is an arbitrary function such that $g(\sigma(T)) \subset \Theta_5$ and $g(z) = z$ near the line segment $[-2, 2]$.

Remark 4.10. Note that the constants C_j in the above proofs are determined only by the choice of the auxiliary functions ρ , ρ_1 , and g_j . It follows that C and δ in Theorems 4.1 and 4.2 are independent of \mathcal{A} and T .

5. APPROXIMATION BY OPERATORS WITH FINITE SPECTRA

The main result of this section is the following theorem.

Theorem 5.1. *There exist constants $\delta > 0$ and $C > 0$ such that the following is true. If \mathcal{A} has real rank zero and $T \in M_2(\mathcal{A})$ is a normal operator with $d_2(T) < \delta$,*

then one can find a normal operator $T_0 \in M_2(\mathcal{A})$ such that $\|T - T_0\| \leq 3$, $\|[P, T_0]\| \leq Cd_2(T)$, and $\sigma(T_0) \subset \mathbb{Z} + i\mathbb{Z}$.

Remark 5.2. The above theorem can be thought of as an approximation result, since it holds with the same constants for operators T with arbitrarily large norms. If $d_2(T)$ is sufficiently small, then, applying Theorem 5.1 to $\varepsilon^{-1}T$, we see that T is approximated by an almost block diagonal operator εT_0 with a finite spectrum $\sigma(\varepsilon T_0) \subset \varepsilon\mathbb{Z} + i\varepsilon\mathbb{Z}$.

As was mentioned in Section 1, Theorem 5.1 is proved by removing certain subsets from the spectrum $\sigma(T)$. Theorems 4.1 and 4.2 from Section 4 allow us to remove discs from $\sigma(T)$ or, in other words, to cut holes in the spectrum. Note that, for our purposes, it is important to be able to cut arbitrarily many holes in one go. If we removed discs one by one repeatedly applying Theorems 4.1 and 4.2 as in [12–14], then the error would accumulate and it would be difficult to effectively control the norm of the commutators with P . The following lemma deals with this issue, providing an estimate for $\|[P, T']\|$ independent of the number of holes.

Lemma 5.3. *There exist constants $\delta' > 0$ and $C' > 0$ such that the following is true for all C^* -algebras \mathcal{A} of real rank zero and all normal $T \in M_2(\mathcal{A})$. If $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ is a finite collection of points such that $\text{dist}(\lambda_i, \lambda_j) \geq 4$ for $i \neq j$ and each operator $T - \lambda_j I$ satisfies the conditions of Theorem 4.1 or Theorem 4.2 with $\delta < \delta'$, then one can find a normal $T' \in M_2(\mathcal{A})$ with the following properties:*

- (f₁) $\|T - T'\| \leq 3$.
- (f₂) $\sigma(T') \cap \mathcal{O}_1(\lambda_j) = \emptyset$ for all $j = 1, \dots, m$.
- (f₃) $\text{diag}_P(T' - \lambda_j I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ for $j = 1, \dots, m$.
- (f₄) $[T', \Pi_r^j] = 0$ and $T'|_{\text{Ran}(I - \sum_j \Pi_2^j)} = T|_{\text{Ran}(I - \sum_j \Pi_2^j)}$ for all $r \in [1, 2]$ and all $j = 1, \dots, m$, where Π_r^j denote the spectral projections of T corresponding to the discs $\mathcal{O}_r(\lambda_j)$.
- (f₅) $\|[P, T']\| \leq C'\delta$.
- (f₆) If $\sigma(T) \cap \mathcal{O}_3(\lambda_j)$ lies on a straight line of the form $e^{i\theta}\mathbb{R} + \lambda_j$, then $\sigma(T') \cap \mathcal{O}_2(\lambda_j) \subset e^{i\theta}\mathbb{R} + \lambda_j$.

Proof. Let U_j be the unitary operators obtained by applying Theorem 4.1 or Theorem 4.2 to $T - \lambda_j I$. Let us fix a nonincreasing function $\rho \in C^\infty(\mathbb{R}_+)$ such that $\rho(t) = 1$ for $0 \leq t \leq 1$ and $\rho(t) = 0$ for $t \geq 2$, and denote $\chi_j(z) = \rho(|z - \lambda_j|)$. We claim that (f₁)–(f₆) hold for the operator

$$(5.1) \quad T' = \sum_{j=1}^m (\lambda_j I + U_j) \chi_j^2(T) + T \left\{ I - \sum_{j=1}^m \chi_j^2(T) \right\}.$$

Since the function $\chi_j(z)$ vanishes for $|z - \lambda_j| \geq 2$ and is equal to 1 for $|z - \lambda_j| \leq 1$, the condition (u₂) implies that the operators in the first sum have block structure with respect to Π_1^j and Π_2^j . So do the operators in the second sum, and hence the operator T' . From (5.1) and (u₃) it follows that

- (1) the “small” blocks $T'|_{\text{Ran } \Pi_1^j}$ are the unitaries $U_j|_{\text{Ran } \Pi_1^j}$ shifted by λ_j ,
- (2) the “large” block $T'|_{\text{Ran}(I - \sum_j \Pi_2^j)}$ coincides with the corresponding block of T ,

- (3) the “intermediate” blocks $T'|_{\text{Ran}(\Pi_2^j - \Pi_1^j)}$ are functions of $T|_{\text{Ran}(\Pi_2^j - \Pi_1^j)}$ of the form $f_j(z) = \lambda_j + (z - \lambda_j)\{1 + (|z - \lambda_j|^{-1} - 1)\chi_j^2(z)\}$ with $\text{Ran } f_j \subset \overline{\mathcal{O}_2(\lambda_j)} \setminus \mathcal{O}_1(\lambda_j)$.

Thus we see that T' is a normal operator satisfying (f₂) and (f₄). By (u'₅), we also have (f₆).

In view of (u'₂) and (u'₃), $[U_j, \chi_j(T)] = 0$ and, consequently,

$$(5.2) \quad T' - T = \sum_{j=1}^m \chi_j(T) U_j \chi_j(T) + \sum_{j=1}^m \chi_j(T) (\lambda_j I - T) \chi_j(T).$$

Since χ_j have mutually disjoint supports, the norm of the first sum is bounded by 1. The second sum is a function of T whose modulus does not exceed 2. These estimates imply (f₁).

In view of (1.5), (u₄), and (u'₄), we have $\|[P, T]\| \leq \delta$ and $\|[P, U_j]\| \leq C\delta$ for all j with some universal constant C . Therefore, the inequality (f₅) follows from (5.2) and Lemma 2.4 with $Q = P$ and $S_j = \lambda_j I + U_j - T$. Note that the constant C' in (f₅) depends only on C and the choice of ρ .

It remains to prove (f₃). Let us fix j , denote $\mathcal{T}_j = \Pi_2^j T' + (I - \Pi_2^j)T$, and consider the path $G'_t = \Pi_2^j T' + (I - \Pi_2^j)((1-t)T + tT')$ from \mathcal{T}_j to T' . From (1) to (3) it follows that the operators $G'_t - \lambda_j I$ are invertible with $\|(G'_t - \lambda_j I)^{-1}\| \leq 1$. The inequality $\|[P, U_j]\| \leq C\delta$ and (5.2) imply that $\|[P, \Pi_2^j(T - T')]\| \leq C_\rho \delta$, where C_ρ is a constant depending only on C and ρ . Hence $\|[P, G'_t]\|$ admits a similar estimate. If δ is smaller than a constant depending only on C and ρ , then $\|[P, G'_t]\|^{-1} > 1$. Thus, in view of Lemma 4.4, it is sufficient to show that $\text{diag}_P(\mathcal{T}_j - \lambda_j I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$.

If $T - \lambda_j I$ satisfies the conditions of Theorem 4.1, then $\text{diag}_P U_j$ belongs to $\text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ and, in view of (1)–(3) and (u₃), $U_j = (\mathcal{T}_j - \lambda_j I)|_{\mathcal{T}_j - \lambda_j I}^{-1}$. Choosing a homotopy $\varphi_t(|\mathcal{T}_j - \lambda_j I|)$ from $|\mathcal{T}_j - \lambda_j I|^{-1}$ to I with $\varphi_t \in C^2(\mathbb{R})$ and applying Lemma 4.4, we see that $\text{diag}_P(\mathcal{T}_j - \lambda_j I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$ provided that δ is smaller than a constant depending only on C and the choice of φ_t and ρ .

Assume now that $T - \lambda_j I$ satisfies the conditions of Theorem 4.2. Then the operators $e^{-i\theta} U_j|_{\text{Ran } \Pi_1^j}$ and $e^{-i\theta}(T - \lambda_j I)|_{\text{Ran } \Pi_2^j}$ are self-adjoint and $\text{diag}_P(T - \lambda_j I + ie^{i\theta} I) \in \text{GL}_0(\mathcal{A} \oplus \mathcal{A})$. Consider the path

$$(5.3) \quad G_t := t(\mathcal{T}_j - \lambda_j I) + (1-t)(T - \lambda_j I + ie^{i\theta} I), \quad t \in [0, 1].$$

Clearly, $\|[P, G_t]\| \leq C'_\rho \delta$ with a constant C'_ρ depending only on C and ρ .

In view of (2), $G_t|_{\text{Ran}(I - \Pi_2^j)} = (T - \lambda_j I + (1-t)ie^{i\theta} I)|_{\text{Ran}(I - \Pi_2^j)}$ and, consequently, the spectra of these restrictions are subsets of $\mathbb{C} \setminus \mathcal{O}_1$. Since the operator $e^{-i\theta}(T - \lambda_j I)|_{\text{Ran}(\Pi_2^j - \Pi_1^j)}$ is self-adjoint, (3) implies that the spectra of $G_t|_{\text{Ran}(\Pi_2^j - \Pi_1^j)}$ also lie outside \mathcal{O}_1 . Finally,

$$\begin{aligned} \text{Re} \left(e^{-i\theta} G_t|_{\text{Ran } \Pi_1^j} \right) &= (tU_j + (1-t)(T - \lambda_j I))|_{\text{Ran } \Pi_1^j}, \\ \text{Im} \left(e^{-i\theta} G_t|_{\text{Ran } \Pi_1^j} \right) &= (1-t)I \end{aligned}$$

because the operators $e^{-i\theta} U_j|_{\text{Ran } \Pi_1^j}$ and $e^{-i\theta}(T - \lambda_j I)|_{\text{Ran } \Pi_1^j}$ are self-adjoint. By (u'₅), $\sigma \left(e^{-i\theta} U_j|_{\text{Ran } \Pi_1^j} \right) \subset \{-1; 1\}$. The spectrum of $e^{-i\theta}(T - \lambda_j I)|_{\text{Ran } \Pi_1^j}$ is a

subset of $[-1, 1]$. These inclusions imply that $\sigma\left(\operatorname{Re}(e^{-i\theta} G_t|_{\operatorname{Ran}\Pi_1^j})\right)$ lie outside the interval $(-\frac{1}{3}, \frac{1}{3})$ for all $t > \frac{2}{3}$. It follows that $\sigma\left(G_t|_{\operatorname{Ran}\Pi_1^j}\right)$ are subsets of $\mathbb{C} \setminus \mathcal{O}_{1/3}$ for all $t \in [0, 1]$.

Thus we see that $\|G_t^{-1}\| \leq 3$. Therefore, the path (5.3) satisfies the conditions of Lemma 4.4 and, consequently, $\operatorname{diag}_P(\mathcal{T}_j - \lambda_j I) \in \operatorname{GL}_0(\mathcal{A} \oplus \mathcal{A})$ provided that δ is smaller than a constant depending only on C and ρ . ■

Proof of Theorem 5.1. Let

$$\mathcal{G} := (\mathbb{R} + i6\mathbb{Z}) \cup (6\mathbb{Z} + i\mathbb{R}) \quad \text{and} \quad \mathcal{L} := (6\mathbb{Z} + 3) + i(6\mathbb{Z} + 3).$$

The grid \mathcal{G} splits the complex plane into closed squares Ω_j of size 6×6 with vertices in $6\mathbb{Z} + i6\mathbb{Z}$. The lattice \mathcal{L} is formed by the center points z_j of Ω_j , and $\mathcal{G} = \cup_j \partial\Omega_j$.

Let us fix a nondecreasing function $\psi \in C^\infty(\mathbb{R})$ such that the derivative ψ' is periodic with period 6 and $\psi(t) = 6k$ for all $t \in [6k - \frac{5}{2}, 6k + \frac{5}{2}]$ and all $k \in \mathbb{Z}$. Define

$$g(z) := \psi(\operatorname{Re} z) + i\psi(\operatorname{Im} z) \quad \text{and} \quad g_t(z) = (1-t)z + tg(z).$$

Since $g(z) - z \in C^2(\mathbb{C})$, the functions g_t are operator Lipschitz. One can easily see that $g(\Omega_j \setminus \mathcal{O}_1(z_j)) \subset \partial\Omega_j$ and

$$(5.4) \quad g_t(\Omega_j \setminus \mathcal{O}_1(z_j)) \subset \Omega_j \setminus \mathcal{O}_1(z_j), \quad \forall t \in [0, 1].$$

Put $T_1 := 6T$ and consider a closed square Ω with vertices in $6\mathbb{Z} + i6\mathbb{Z}$, which contains $\sigma(T_1)$. Let us assume that $d_2(T_1) = 6d_2(T) < \delta'$, apply Lemma 5.3 with $\lambda_j = z_j \in \Omega \cap \mathcal{L}$ to the operator T_1 , and denote the obtained normal operator by T_1' . The operators T_1 and T_1' satisfy the conditions (f₁)–(f₅). In particular, $\|[P, T_1']\| \leq 6C'd_2(T)$, where C' is the constant from (f₅), and $\sigma(T_1') \subset \mathbb{C} \setminus (\cup_j \mathcal{O}_1(z_j))$, see Figure 4.

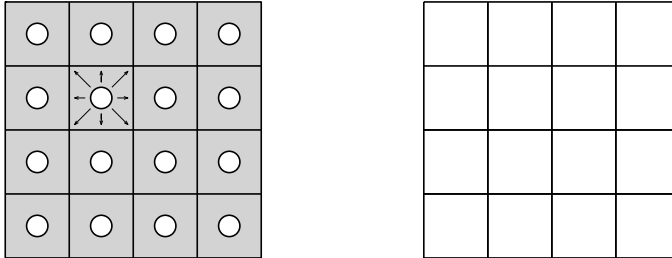


FIGURE 4. The spectra of T_1' and $T_2 = g(T_1')$.

Let $T_2 = g(T_1')$. Then $\sigma(T_2) \subset \mathcal{G}$ and, since g is operator Lipschitz, $\|[P, T_2]\| \leq C_\psi d_2(T)$ with a constant C_ψ depending only on C' and the choice of ψ . If $d_2(T) < C_\psi^{-1}$, then, in view of (5.4), the paths $g_t(T_1') - z_j I$ satisfy the conditions of Lemma 4.4. Hence $\operatorname{diag}_P(T_2 - z_j I) \in \operatorname{GL}_0(\mathcal{A} \oplus \mathcal{A})$ for all $z_j \in \Omega \cap \mathcal{L}$. Now, applying Lemma 4.4 to the paths $T_2 - (tz + (1-t)z_j)I$, we see that $\operatorname{diag}_P(T_2 - zI) \in \operatorname{GL}_0(\mathcal{A} \oplus \mathcal{A})$ whenever $\operatorname{dist}(z, \mathcal{G}) \geq 1$.

Let now λ_j be the points of the union $(\mathcal{L} + 3) \cup (\mathcal{L} + 3i)$ lying in Ω (in other words, λ_j are the middle points of the line segments forming the squares Ω_j). If $d_2(T)$ is smaller than a constant depending only on the choice of ψ , then, by the

above, the operators $T_2 - \lambda_j I$ satisfy the conditions of Theorem 4.2. Let us apply Lemma 5.3 to the operator T_2 and denote the obtained normal operator by T'_2 . The spectrum $\sigma(T'_2)$ is also a subset of \mathcal{G} , but now it does not contain central parts of the edges of Ω_j ; see Figure 5. By (f₅), we have $\|[P, T'_2]\| \leq C'_\psi d_2(T)$ with a constant C'_ψ depending only on C' and ψ .

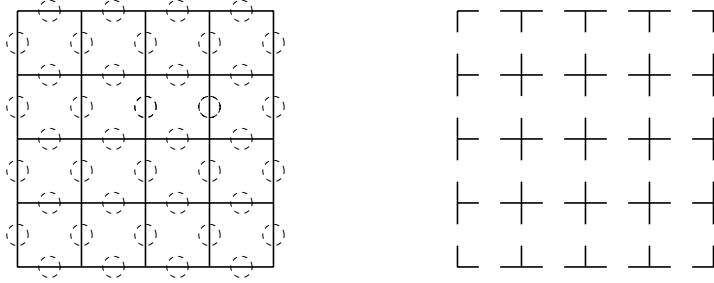


FIGURE 5. The spectra of T_2 and T'_2 .

The spectrum of T'_2 lies in the squares Ω'_k with edges of length 5 centered at the points $z'_k \in 6\mathbb{Z} + i6\mathbb{Z}$. On each of these squares the function g is identically equal to z'_k . Thus the spectrum of $g(T'_2)$ is a finite subset of $6\mathbb{Z} + i6\mathbb{Z}$. Let $T_0 = \frac{1}{6}g(T'_2)$. Then $\sigma(T_0) \subset \mathbb{Z} + i\mathbb{Z}$. Since g is operator Lipschitz, $\|[P, T_0]\| \leq C d_2(T)$ with a constant C depending only on C' in (f₅) and the choice ψ . Finally, the estimates (f₁) and $|g(z) - z| \leq 3\sqrt{2}$ imply that $\|T - T_0\| \leq 3$. ■

6. PROOFS OF THEOREM 1.1 AND COROLLARY 1.4

6.1. The upper bound (1.1). Denote $d'_1(A) := d_1(A) + \|[A, A^*]\|^{1/2}$. Applying Theorem 3.1 to the operator A , we find normal operators $N \in \mathcal{A}$ and $T \in M_2(\mathcal{A})$ such that

$$(6.1) \quad d_2(T) \leq C_1 d'_1(A) \quad \text{and} \quad \|T - A \oplus N\| \leq C_1 d'_1(A).$$

Here and further on, C with a subscript denotes a constant independent of \mathcal{A} and A .

Assume that $C_1 d'_1(A)$ is smaller than the constant δ in Theorem 5.1. Let T_0 be the normal operator with $\sigma(T_0) \subset \mathbb{Z} + i\mathbb{Z}$ given by that theorem, and let $PT_0P = X + iY$ where X, Y are self-adjoint. We have $\|T - T_0\| \leq 3$ and, in view of the first estimate (6.1), $\|T_0 - \text{diag}_P T_0\| = \|[P, T_0]\| \leq C_2 d'_1(A)$. These two inequalities, the second estimate (6.1) and the identity

$$2i[X, Y] = [PT_0^*P, PT_0P] = PT_0^*(I - P)T_0P - PT_0(I - P)T_0^*P$$

imply that

$$(6.2) \quad \|X + iY - A\| \leq 3 + C_1 d'_1(A) \quad \text{and} \quad \|[X, Y]\| \leq C_3 (d'_1(A))^2.$$

Assume also that $d'_1(A) \leq (3C_2)^{-1}$, so that $\|T_0 - \text{diag}_P T_0\| \leq \frac{1}{3}$. Since X is a block of $\text{Re}(\text{diag}_P T_0)$ and $\sigma(\text{Re} T_0) \subset \mathbb{Z}$, it follows that

$$(6.3) \quad \sigma(X) \subset \sigma(\text{Re}(\text{diag}_P T_0)) \subset \overline{\mathcal{O}}_{1/3}(\mathbb{Z}) \cap \mathbb{R},$$

where $\overline{\mathcal{O}}_{1/3}(\mathbb{Z})$ is the closed $\frac{1}{3}$ -neighborhood of \mathbb{Z} .

Let ρ be a function satisfying the conditions of Lemma 2.5 such that $\rho(t) = 1$ for $|t| \leq \frac{1}{3}$ and $\rho(t) = 0$ for $|t| \geq \frac{2}{3}$. If $\rho_n(t) := \rho(t - n)$ and $Y' = \sum_{n \in \mathbb{Z}} \rho_n(X) Y \rho_n(X)$, then, by (d₃),

$$(6.4) \quad \|Y - Y'\| \leq C_\rho \|[X, Y]\| \leq C_\rho C_3 (d'_1(A))^2.$$

In view of (6.3), $\rho_n(X)$ coincides with the spectral projections of X corresponding to the intervals $[n - \frac{1}{3}, n + \frac{1}{3}]$. If $X' := \sum_{n \in \mathbb{Z}} n \rho_n(X)$, then $[X', Y'] = 0$. It follows that $X' + iY'$ is a normal operator whose spectrum lies on vertical line segments passing through real integers.

Since \mathcal{A} has real rank zero, $X' + iY'$ belongs to the closure of $N_f(\mathcal{A})$. Now the estimates (6.2), (6.4), and $\|X - X'\| \leq \frac{1}{3}$ imply that

$$(6.5) \quad \text{dist}(A, N_f(\mathcal{A})) \leq 4 + C_1 d'_1(A) + C_\rho C_3 (d'_1(A))^2$$

whenever $d'_1(A) \leq \varepsilon$, where ε is a constant depending only on the constants δ and C in Theorems 3.1 and 5.1. For every $A \in \mathcal{A}$, the operator $\varepsilon (d'_1(A))^{-1} A$ satisfies the above condition. Substituting it into (6.5), we arrive at (1.1).

6.2. The lower bound (1.2). Denote $d := \text{dist}(A, N_f(\mathcal{A}))$, and let $N_\varepsilon \in N_f(\mathcal{A})$ be a normal operator with a finite spectrum such that $\|A - N_\varepsilon\| \leq d + \varepsilon$. If $R_\varepsilon = A - N_\varepsilon$, then $[A, A^*] = [R_\varepsilon, A^*] + [A, R_\varepsilon^*] - [R_\varepsilon, R_\varepsilon^*]$ and, consequently, $\|[A, A^*]\| \leq 4\|A\|(d + \varepsilon) + (d + \varepsilon)^2$. Since $d \leq \|A\|$, letting $\varepsilon \rightarrow 0$, we obtain $\|[A, A^*]\| \leq 5\|A\|d$. Now (1.2) follows from the inclusion $N_f(\mathcal{A}) \subset \overline{\text{GL}_0(\mathcal{A})}$ (see Remark 1.3).

6.3. Proof of Corollary 1.4. Let F be a continuous nondecreasing function on \mathbb{R}_+ such that $F(0) = 0$ and $0 \leq F \leq 1$. If

$$\text{dist}(A, N(\mathcal{A})) \leq F(\|[A, A^*]\|)$$

for any C^* -algebra \mathcal{A} of real rank zero and all $A \in \mathcal{A}$ with $d_1(A) = 0$ and $\|A\| \leq 1$, then, according to [12, Theorem 3.8], for every $\varepsilon > 0$ and each $A \in \mathcal{B}(H)$ satisfying the conditions $d_1(A) = 0$, $\|A\| \leq 1$, there exists a normal operator $A_\varepsilon \in \mathcal{B}(H)$ such that

$$(6.6) \quad \begin{aligned} \|A - A_\varepsilon\|_{\text{ess}} &\leq 2F(\|[A^*, A]\|_{\text{ess}}), \\ \|A - A_\varepsilon\| &\leq 5F(\|[A^*, A]\|) + 3F(2F(\|[A^*, A]\|_{\text{ess}})) + \varepsilon. \end{aligned}$$

Since $\text{dist}(A, N(\mathcal{A})) \leq \|A\|$, the estimate (1.1) implies that the function $F(t) = \min\{Ct^{1/2}, 1\}$ satisfies the above conditions. Applying (6.6) with this F and $\varepsilon = \|A\|^{-1} \|[A, A^*]\|^{1/2}$ to the operator $\|A\|^{-1} A$, we obtain (1.4).

Remark 6.1. If $\|(A - \lambda I)^{-1}\| < (\text{dist}(\lambda, \Omega) - \varepsilon)^{-1}$ whenever $\text{dist}(\lambda, \Omega) > \varepsilon$ and $\|[A^*, A]\|^{1/2} \leq \varepsilon$, then the spectrum of the normal operator A' given by Corollary 1.4 lies in a $C\varepsilon$ -neighborhood of Ω . Therefore it can be approximated by a normal operator with a spectrum in Ω .

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