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DYNAMICAL DEGREES OF BIRATIONAL TRANSFORMATIONS OF PROJECTIVE SURFACES

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1. Introduction

Given a birational transformation $f\colon X \dashrightarrow X$ of a projective surface, defined over a field $\mathbf{k}$, its dynamical degree $\lambda(f)$ is a positive real number that measures the complexity of the dynamics of f. If $\mathbf{k}$ is the field of complex numbers, the neperian logarithm $\log(\lambda(f))$ provides an upper bound for the topological entropy of $f\colon X(\mathbf{C}) \dashrightarrow X(\mathbf{C})$ and is equal to it under natural assumptions (see [3,23]). Our goal is to study the structure of the set of all dynamical degrees $\lambda(f)$, when f runs over the group of all birational transformations $\mathsf{Bir}(X)$ and X over the collection of all projective surfaces.

The dynamical degree $\lambda(f)$ is invariant under conjugacy. An important feature of our results may be summarized by the following slogan: *Precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f.* In particular, we shall obtain effective, quantitative bounds for the solutions of certain equations in $\mathsf{Bir}(X)$, like the conjugacy problem asking for a solution h of the equation $hfh^{-1} = g$.

Another motivation of the present paper is to develop the "dictionary" between groups of birational transformations of projective surfaces and mapping class groups of higher genus, closed, orientable surfaces. The dynamical degree $\lambda(f)$ plays a role which is similar to the dilatation factor $\lambda(\varphi)$ of pseudo-Anosov mapping classes (see Section 8 below).

As we shall see, our main results should be compared to two theorems proved by Thurston. The first one describes explicitly the set of topological entropies of post-critically finite, continuous, multimodal transformations of the unit interval as the set of logarithms of "weak Perron numbers." The second describes the structure of the set of volumes of hyperbolic manifolds of dimension 3; this set is a countable, non-discrete, and well ordered subset of the real line.

1.1. Dynamical degrees, Pisot numbers, and Salem numbers.

1.1.1. *Dynamical degrees.* Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$. In what follows, $\mathsf{NS}(X)$ denotes the Néron-Severi group of X. Given a ring $\mathbf{A}$, $\mathsf{NS}_{\mathbf{A}}(X)$ stands for $\mathsf{NS}(X) \otimes_{\mathbf{Z}} \mathbf{A}$; hence, $\mathsf{NS}_{\mathbf{Z}}(X)$ coincides with $\mathsf{NS}(X)$.

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Let f be a birational transformation of X defined over $\mathbf{k}$. It determines an endomorphism $f_*\colon \mathsf{NS}(X) \to \mathsf{NS}(X)$, and the **dynamical degree** $\lambda(f)$ of f is defined as the spectral radius of the sequence of endomorphisms $(f^n)_*$, as n goes to $+\infty$. More precisely, once a norm $\|\cdot\|$ has been chosen on the real vector space $\mathsf{End}(\mathsf{NS}_{\mathbf{R}}(X))$, one defines

$$\lambda(f) = \lim_{n\to\infty} \| (f^n)_* \|^{1/n} ;$$

this limit exists and does not depend on the choice of the norm. Moreover, for every ample divisor $D \subset X$

$$\lambda(f) = \lim_{n\to\infty} (D \cdot (f^n)_* D)^{1/n} ,$$

where $C \cdot D$ denotes the intersection number between divisors or divisor classes. By definition, f is **loxodromic** if $\lambda(f) > 1$.

The **dynamical spectrum** of X is defined as the set

$$\Lambda(X) = \{\lambda(f) \mid f \in \mathsf{Bir}(X)\}.$$

If one wants to specify the field $\mathbf{k}$, one may denote the dynamical spectrum by $\Lambda(X, \mathbf{k})$.

Example 1.1. The Néron-Severi group of $\mathbb{P}^2_{\mathbf{k}}$ coincides with the Picard group $\mathsf{Pic}(\mathbb{P}^2_{\mathbf{k}})$, has rank 1, and is generated by the class e_0 of a line:

$$\mathsf{NS}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{Pic}(\mathbb{P}^2_{\mathbf{k}}) = \mathbf{Z}e_0.$$

Fix a choice of homogeneous coordinates $[x : y : z]$ on the projective plane $\mathbb{P}^2_{\mathbf{k}}$. Let f be an element of $\mathsf{Cr}_2(\mathbf{k})$. One can then find three homogeneous polynomials P, Q, and R in the variables (x, y, z), of the same degree d, and without a common factor of positive degree, such that

$$f([x : y : z]) = [P(x, y, z) : Q(x, y, z) : R(x, y, z)].$$

This degree d does not depend on the choice of homogeneous coordinates; it is denoted by $\mathsf{deg}(f)$ and called the degree of f. On $\mathsf{Pic}(\mathbb{P}^2_{\mathbf{k}})$, f acts by multiplication by $\mathsf{deg}(f)$; thus, we have $\lambda(f) = \lim \mathsf{deg}(f^n)^{1/n}$. For instance, the **standard quadratic involution**

$$\sigma([x : y : z]) = [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}] = [yz : zx : xy]$$

satisfies $\mathsf{deg}(\sigma^n) = 1$ or 2, according to the parity of n; hence $\lambda(\sigma) = 1$.

1.1.2. *Pisot and Salem numbers (see* [8]*).* A **Pisot number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates lie in the open unit disk; the set of Pisot numbers includes all integers $d \geq 2$ as well as all reciprocal quadratic integers $\lambda > 1$. A **Salem number** is an algebraic integer $\lambda \in]1, \infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary; hence, the minimal polynomial of λ has at least two complex conjugate roots on the unit circle, its roots are permuted by the involution $x \mapsto 1/x$, and its degree is at least 4. We denote by Pis the set of Pisot numbers and by Sal the set of Salem numbers.

It is known that Pis is a closed subset of the real line. It is contained in the closure of Sal, and its infimum is equal to $\lambda_P \simeq 1.324717$, the unique root $\lambda_P > 1$ of the cubic equation $x^3 = x + 1$; this Pisot number is known as the **plastic number**, or **padovan number**. The smallest accumulation point of Pis is the golden mean $\lambda_G = (1 + \sqrt{5})/2$; all Pisot numbers between λ_P and λ_G have been listed.

Our present knowledge of Salem numbers is much weaker. Conjecturally, the infimum of Sal is larger than 1 and should be equal to the **Lehmer number**, i.e., to the Salem number $\lambda_L \simeq 1.176280$ obtained as the unique root > 1 of the irreducible polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$.

1.1.3. *Dynamical degrees and algebraic stability.* One says that $f \in \mathsf{Bir}(X)$ is **algebraically stable** when the endomorphism f_* of the Néron-Severi group $\mathsf{NS}(X)$ satisfies

$$(f^n)_* = (f_*)^n \tag{1.1}$$

for all positive integers n. If f is algebraically stable, then f^{-1} is also algebraically stable and $\lambda(f)$ is the spectral radius of the endomorphism f_* of $\mathsf{NS}(X)$; in particular, $\lambda(f)$ is an algebraic integer. Diller and Favre proved in [22] that every birational transformation of a projective surface X is conjugate by a birational morphism $\pi : Y \to X$ to an algebraically stable transformation $\pi^{-1} \circ f \circ \pi$. From this fact and the Hodge index theorem, they obtained the following result.

Theorem 1.2 (Diller and Favre)**.** *Let* $\mathbf{k}$ *be a field and let* f *be a birational transformation of a projective surface defined over* $\mathbf{k}$*. If* $\lambda(f)$ *is different from* 1*, then* $\lambda(f)$ *is a Salem or a Pisot number.*

In this article we initiate the study of the dynamical spectrum $\Lambda(X)$. By the Diller-Favre Theorem, $\Lambda(X)$ splits into two parts, its Pisot part $\Lambda^P(X)$ and its Salem part $\Lambda^S(X)$. The problem is to describe which numbers can appear in each of these sets, as well as the relationship between these two sets.

Example 1.3. When f is an algebraically stable transformation of $\mathbb{P}^2_{\mathbf{k}}$, one gets $\lambda(f) = \mathsf{deg}(f)$. For instance, the automorphism h of the affine plane defined by $h(X, Y) = (Y, X + Y^d)$ extends to a birational map of the projective plane such that $\mathsf{deg}(h^n) = d^n$ for all $n \geq 0$. In particular, $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ contains all integers $d \geq 1$, for all fields $\mathbf{k}$.

Example 1.4. Consider the group $\mathsf{GL}_2(\mathbf{Z})$ acting by (monomial) automorphisms of the multiplicative group $\mathbf{k}^* \times \mathbf{k}^*$: If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is an element of $\mathsf{GL}_2(\mathbf{Z})$ and (X, Y) denotes the coordinates on $\mathbf{k}^* \times \mathbf{k}^*$, the automorphism associated to A is defined by $f_A(X, Y) = (X^a Y^b, X^c Y^d)$. This provides an embedding of $\mathsf{GL}_2(\mathbf{Z})$ in the automorphism group $\mathsf{Aut}(\mathbf{k}^* \times \mathbf{k}^*)$, and thus in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}(\mathbf{k}))$.

For every A in $\mathsf{GL}_2(\mathbf{Z})$, the dynamical degree of f_A is equal to the spectral radius of the matrix A, i.e., to the modulus of its unique eigenvalue λ with $|\lambda| \geq 1$; this implies that f_A is not an algebraically stable transformation of $\mathbb{P}^2_{\mathbf{k}}$ as soon as $\lambda(f_A) > 1$, because $\lambda(f_A)$ is not an integer in that case.

As a by-product of this example, the dynamical spectrum of the plane contains all reciprocal quadratic integers, i.e., all roots $\lambda > 1$ of equations $x^2 + 1 = tx$ with t in $\mathbf{Z}$.

1.2. **Salem numbers and automorphisms.** The dynamical degree of an automorphism, if different from 1, is either a quadratic number or a Salem number (see [22]). Here we prove a converse statement.

Theorem A. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* f *be a birational transformation of a projective surface* X*, defined over* $\mathbf{k}$*. If* $\lambda(f)$ *is a Salem number, there exists a projective surface* Y *and a birational mapping* $\varphi\colon Y \dashrightarrow X$ *such that* $\varphi^{-1} \circ f \circ \varphi$ *is an automorphism of* Y*.*

Thus, one can decide whether a birational transformation is conjugate to an automorphism by looking at its dynamical degree, except when this degree is 1 or a quadratic integer. For the quadratic case, Examples 2.2 and 2.3 show that there are quadratic integers which are simultaneously realized as dynamical degrees of automorphisms and of birational transformations that cannot be conjugate to an automorphism. See Remark 2.4 for birational transformations with their dynamical degrees equal to 1.

Once Theorem A is proved, three corollaries can be deduced from results of McMullen and the second author (see [37] and [16]). The first corollary (see Section 2.6) is a **spectral gap property** for dynamical degrees: *There is no dynamical degree in the interval* $]1, \lambda_L[$. The second corollary does not seem to be related to values of dynamical degrees, but the simple proof given here makes use of the spectral gap. It asserts that *the centralizer, in the group* $\mathsf{Bir}(X)$*, of a loxodromic element* f *is finite by cyclic* (see Section 4.3). The third consequence is an effective and explicit bound for the optimal degree of a conjugacy (see Section 4.4).

Corollary 1.5. *Two loxodromic elements* f*,* $g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *of degree* $\leq d$ *are conjugate if and only if they are conjugate by an element* h *of degree* $\leq (2d)^{57}$*.*

1.3. **From projective surfaces to the projective plane.** Non-rational surfaces are easily handled with.

Theorem B. *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be a projective surface defined over* $\mathbf{k}$*. If* X *is not rational, then*

(1) $\Lambda(X) = \{1\}$ *if* X *is not birationally equivalent to an abelian surface, a K3 surface, or an Enriques surface;*
(2) $\Lambda(X) \setminus \{1\}$ *is made of quadratic integers and of Salem numbers of degree at most* 6 *(resp.* 22*, resp.* 10*) if* X *is birationally equivalent to an abelian surface (resp. a K3 surface, resp. an Enriques surface).*

The union of all dynamical spectra $\Lambda(X, \mathbf{k})$*, for all fields and all surfaces which are not geometrically rational, is a closed discrete subset of the real line.*

Remark 1.6. When the characteristic of the field $\mathbf{k}$ vanishes, the degree bounds of Assertion (2) become 4, 20, and 10 (in place of 6, 22, and 10).

This result, proved in Section 3, shows that the most interesting case is provided by rational surfaces. Thus, in the following statements, one can assume that X is birationally equivalent to the projective plane $\mathbb{P}^2_{\mathbf{k}}$; the dynamical spectrum is then equal to the set $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ of dynamical degrees of elements of the **Cremona group**

$$\mathsf{Cr}_2(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}}).$$

1.4. **Degrees and conjugacy classes.**

1.4.1. *Minimal degree in the conjugacy class.* Given an element f of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, define the **minimal degree** of f in its conjugacy class as the positive integer

$$\mathsf{mcdeg}(f) = \min \mathsf{deg}(g \circ f \circ g^{-1}),$$

where g describes $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ (thus, $\mathsf{mcdeg}(f)$ depends on the field and may decrease after a field extension). The function mcdeg is constant on conjugacy classes, and

$$\lambda(f) \leq \mathsf{mcdeg}(f) \leq \mathsf{deg}(f)$$

for all birational transformations of the plane. One of our main goals is to provide the following reverse inequality.

Theorem C. *Let* $\mathbf{k}$ *be an algebraically closed field and let* f *be a birational transformation of the plane* $\mathbb{P}^2_{\mathbf{k}}$.

(1) *If* $\lambda(f) \geq 10^6$, *then* $\mathsf{mcdeg}(f) \leq 4700\,\lambda(f)^5$.
(2) *If* $\lambda(f) > 1$, *then* $\mathsf{mcdeg}(f) \leq \cosh(18 + 345\log(\lambda(f))) \leq e^{18}\lambda(f)^{345}$.

On the other hand, there are sequences of elements $f_n \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $\mathsf{mcdeg}(f_n)$ goes to $+\infty$ with n while $\lambda_1(f_n) = 1$ for all n.

1.4.2. *Well ordered sets.* The set $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is a subset of $\mathbf{R}_+$ and, as such, is totally ordered. The following statement, which follows from Theorem C, asserts that $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is **well ordered**: Every non-empty subset of $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ has a minimum; equivalently, it satisfies the descending chain condition (if $(f_n)_{n\geq 0}$ is a sequence of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ and $\lambda(f_{n+1}) \leq \lambda(f_n)$ for each n, then $\lambda(f_n)$ eventually becomes constant).

Theorem D. *Let* $\mathbf{k}$ *be an algebraically closed field. The dynamical spectrum* $\Lambda(\mathbb{P}^2_{\mathbf{k}}) \subset \mathbf{R}$ *is well ordered, and it is closed if* $\mathbf{k}$ *is uncountable.*

In Theorem 7.4, we also show that $\Lambda^P(\mathbb{P}^2_{\mathbf{k}})$ *is contained in the closure of* $\Lambda^S(\mathbb{P}^2_{\mathbf{k}})$ *if* $\mathbf{k}$ *is algebraically closed and of characteristic* 0.

From Theorem B and Theorem D, one obtains the existence of gaps in the dynamical spectrum of projective surfaces: There are small intervals of real numbers that contain infinitely many Pisot and Salem numbers, but do not contain any dynamical degree.

Corollary 1.7. *Let* Λ *be the set of all dynamical degrees of birational transformations of projective surfaces, defined over any field. Then,*

(1) Λ *is a well ordered subset of* $\mathbf{R}_+$;
(2) *if* λ *is an element of* Λ, *there is a real number* $\epsilon > 0$ *such that* $]\lambda, \lambda + \epsilon]$ *does not intersect* Λ;
(3) *there is a non-empty interval* $]\lambda_G, \lambda_G + \epsilon]$, *on the right of the golden mean, that contains infinitely many Pisot and Salem numbers but does not contain any dynamical degree.*

In fact, gaps as in the third assertion of this corollary occur infinitely often, because there are infinitely many Pisot numbers that are limits of Pisot numbers from the right.

1.5. **Organization of the paper.** Section 2 provides a proof of Theorem A and its first corollary, the absence of a dynamical degree between 1 and $\lambda_L \simeq 1.17628$. Theorem B is proved in Section 3; this may be skipped on a first reading. Section 4 introduces the bubble space and an infinite dimensional hyperbolic space on which $\mathsf{Bir}(X)$ acts by isometries; as a first application, we obtain two corollaries of Theorem A. Section 5 contains preliminary results on the infinite Weyl group W_∞: This group is a Coxeter group on an infinite set of generators and plays a crucial technical role in the study of the Cremona group $\mathsf{Cr}_2(\mathbf{k})$. The proof of Theorem C is quite difficult even if, in spirit, it is a variation on the Noether-Castelnuovo proof of the fact that $\mathsf{PGL}_3(\mathbf{k})$ and the standard quadratic involution σ generate $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. This proof occupies Section 6, and Section 7 shows how Theorem D follows from Theorem C.

2. Salem numbers and automorphisms

This section is devoted to the proof of Theorem A. On our way, we introduce basic definitions that are used all along this article.

2.1. **Indeterminacy points, homaloidal nets and base points.** Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$. Let f be a birational transformation of X. We denote by $\mathrm{Ind}(f)$ the set of **indeterminacy points** of f; by convention, it is a proper subset of X and does not include infinitely near points.

The **base points** of f are defined as follows. Let D be a very ample divisor on X and $|D|$ be the complete linear system containing D. The image of $|D|$ by f is a linear system on X (which, in general, is not complete); when f is an element of the Cremona group and D is a line in $\mathbb{P}^2_{\mathbf{k}}$, this linear system $f_*|D|$ is the homaloidal net of f^{-1} (see [1]). The set of base points of f^{-1} (resp. the base ideal of f^{-1}) is defined as the set (resp. the ideal) of base points of this linear system: Base points may be infinitely near and come with a multiplicity. The notion of base points does not depend on the choice of a very ample divisor, but the multiplicities of the base points depend on this choice.

This distinction between base points and indeterminacy points is just used to emphasize the arguments for which it is important to know whether the point is a proper point of X.

2.2. **Algebraic stability and the intersection form.** One says that f is algebraically stable if the sequence $(f^n)_*$ of endomorphisms of $\mathsf{NS}(X)$ satisfies $(f^n)_* = (f_*)^n$ for all integers n (cf. Section 1.1.3). As explained in [22], f is not algebraically stable if and only if there is an indeterminacy point q of f^{-1} and a non-negative integer k such that f is well defined at q, $f(q)$, $\ldots$, $f^{k-1}(q)$, and $f^k(q)$ is an indeterminacy point of f. Blowing up q, $\ldots$, $f^k(q)$, the number of such "bad" indeterminacy points decreases and, in a finite number of steps, one constructs a birational morphism $\pi\colon X' \to X$ such that $\pi^{-1} \circ f \circ \pi$ is algebraically stable (see [22] for this proof).

Let us now assume that f is algebraically stable. The dynamical degree $\lambda(f)$ is then equal to the spectral radius of $f_* \in \mathsf{End}(\mathsf{NS}(X))$ and also to the spectral

radius of $f^* = (f^{-1})_*$ because these endomorphisms are adjoint for the intersection form:

$$f_*C \cdot D = C \cdot f^*D$$

for all pairs (C, D) of divisor classes.

Factorize f as $f = \epsilon \circ \pi^{-1}$ where $\pi\colon Z \to X$ and $\epsilon\colon Z \to X$ are birational morphisms. Write π as a composition $\pi_1 \circ \cdots \circ \pi_m$ of (the inverse of) point blowups, and denote by $F_j \subset Z$ the total transform of the indeterminacy point of π_j^{-1} under the map $\pi_j \circ \cdots \circ \pi_m$. Then, denote by E_j the direct image of F_j by ϵ, for $1 \leq j \leq m$. Each E_j, if not zero, is an effective divisor. According to [22], Theorem 3.3, one has

$$f_*f^*C = C + \sum_{j=1}^{m} (C \cdot E_j)E_j \tag{2.1}$$

for all curves (resp. divisor classes) C in X; this formula corresponds to the following fact: The preimage of C goes through the base points p_j of f with multiplicity $(C \cdot E_j)$; thus, the total transform of $f^{-1}C$ by f contains both C and $\sum_j (C \cdot E_j)E_j$. Taking the intersection, and using that f^* and f_* are adjoint endomorphisms of $\mathsf{NS}(X)$ for the intersection form, one gets

$$f^*C \cdot f^*C = C \cdot C + \sum_{j=1}^{m} (E_j \cdot C)^2. \tag{2.2}$$

In particular, f^* increases self-intersections. This property and Hodge index theorem, according to which the intersection form has signature $(1, \rho(X) - 1)$, are responsible for $\lambda(f)$ being a Pisot or Salem number (see [22], Theorem 5.1).

2.3. **Eigenvectors and automorphisms.** Since X has dimension 2, one easily shows that f^* and f_* preserve the pseudo-effective and nef cones of $\mathsf{NS}_{\mathbf{R}}(X)$. Assume that the dynamical degree $\lambda(f)$ is larger than 1. The Perron-Frobenius theorem assures the existence of an eigenvector $\Theta_X^+(f)$ for f^* in the nef cone of $\mathsf{NS}_{\mathbf{R}}(X)$ such that $f^*\Theta_X^+(f) = \lambda(f)\Theta_X^+(f)$; moreover, this vector is unique up to scalar factor (see [22]).

Theorem 2.1 (Diller-Favre)**.** *Let X be a projective surface, and f be a birational transformation of X, both defined over an algebraically closed field $\mathbf{k}$. Assume that the dynamical degree $\lambda(f)$ is larger than 1. Then*

(1) *$\Theta_X^+(f) \cdot \Theta_X^+(f) = 0$ if and only if $\Theta_X^+(f) \cdot E_j = 0$ for all E_j;*
(2) *if $\Theta_X^+(f) \cdot \Theta_X^+(f) = 0$, there exists a birational morphism $\eta\colon X \to Y$, such that $\eta \circ f \circ \eta^{-1}$ is an automorphism of Y.*

Sketch of the proof. Equation (2.2) and the eigenvector property

$$f^*\Theta_X^+(f) = \lambda(f)\Theta_X^+(f)$$

imply that

$$(\lambda(f)^2 - 1)\Theta_X^+(f) \cdot \Theta_X^+(f) = \sum_{j=1}^{m} (E_j \cdot \Theta_X^+(f))^2.$$

Hence, all divisors E_j are orthogonal to $\Theta_X^+(f)$ if, and only if, $\Theta_X^+(f)$ is an isotropic vector.

Let us now prove the second assertion. By the first assertion, every E_j is orthogonal to $\Theta_X^+(f)$; since the E_j are effective and $\Theta_X^+(f)$ is nef, all irreducible

components of the E_j are orthogonal to $\Theta_X^+(f)$; in other words, the $\mathbf{Q}$-vector subspace of $\mathsf{NS}_{\mathbf{Q}}(X)$ generated by the irreducible components of the divisors E_j is contained in the orthogonal complement $\Theta_X^+(f)^\perp$ of the isotropic vector $\Theta_X^+(f)$. On $\Theta_X^+(f)^\perp$, the intersection form is negative and its kernel is the line generated by $\Theta_X^+(f)$.

From Equation (2.1), one gets $f_*^k\Theta_X^+(f) = \lambda(f)^{-k}\Theta_X^+(f)$. Since $\lambda(f) > 1$ and f_* preserves the lattice $\mathsf{NS}_{\mathbf{Z}}(X)$, one deduces that $\Theta_X^+(f)$ is irrational: no scalar multiple of $\Theta_X^+(f)$ is contained in $\mathsf{NS}_{\mathbf{Z}}(X)$. Thus, the intersection form is negative definite on the $\mathbf{Q}$-vector space generated by all classes of irreducible components of the divisors E_j.

From the Grauert-Mumford contraction theorem (see [2], Theorem 2.1 p. 91), there is a birational morphism $\eta_0\colon X \to Y_0$ which contracts simultaneously all these components. Let f_0 be the birational transformation $\eta_0 \circ f \circ \eta_0^{-1}$. Since $\Theta_X^+(f)$ does not intersect the curves which are contracted by η_0, the class $(\eta_0)_*\Theta_X^+(f) \in \mathsf{NS}_{\mathbf{R}}(Y)$ is both isotropic and an eigenvector for $(f_0)_*$ with eigenvalue $\lambda(f)$. One can thus iterate this process until f_0^{-1} does not contract any curve; i.e., f_0 is an automorphism of Y_0. If Y_0 is singular, and Y is a minimal desingularization of Y_0, f_0 lifts to an automorphism f_Y of Y; one can then show that there is an intermediate birational morphism $\eta\colon X \to Y$ such that $\eta \circ f \circ \eta^{-1} = f_Y$. This concludes the proof. □

Example 2.2. Let E be the elliptic curve associated to the lattice of Gaussian (resp. Eisenstein) integers:

$$E = \mathbf{C}/\mathbf{Z}[\mathsf{i}] \quad (\text{resp. } \mathbf{C}/\mathbf{Z}[\mathsf{j}]),$$

where $\mathsf{i}^2 = -1$ (resp. $\mathsf{j}^3 = 1$, $\mathsf{j} \neq 1$). Let A be the abelian surface $E \times E$. The group $\mathsf{GL}_2(\mathbf{Z}[\mathsf{i}])$ (resp. $\mathsf{GL}_2(\mathbf{Z}[\mathsf{j}])$) acts by automorphisms on A and commutes to $\nu(x,y) = (\mathsf{i}x, \mathsf{i}y)$ (resp. $\nu(x,y) = (\mathsf{j}x, \mathsf{j}y)$). As a consequence $\mathsf{PGL}_2(\mathbf{Z}[\mathsf{i}])$ (resp. $\mathsf{PGL}_2(\mathbf{Z}[\mathsf{j}])$) acts by automorphisms on the (singular) rational surface $X_0 = A/\nu$ and on its minimal desingularization X. The surface X being rational, this construction provides an embedding of $\mathsf{PGL}_2(\mathbf{Z}[\mathsf{i}])$ (resp. $\mathsf{PGL}_2(\mathbf{Z}[\mathsf{j}])$) into the Cremona group. If M is an element of the linear group $\mathsf{GL}_2(\mathbf{Z}[\mathsf{i}])$ (resp. $\mathsf{GL}_2(\mathbf{Z}[\mathsf{j}])$), the associated birational transformation g_M has dynamical degree

$$\lambda(g_M) = \lambda(M)^2,$$

where $\lambda(M)$ is the spectral radius of the matrix M.

Example 2.3. Start with the matrix C defined by

$$C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Its spectral radius is the Golden mean λ_G. The square of λ_G can be realized as the dynamical degree of the monomial map f_{C^2} associated to the second power C^2 of C, as described in Example 1.4. It is also realized as the dynamical degree of the transformation g_C from Example 2.2. The birational transformation f_{C^2} is not conjugate to an automorphism of a rational surface Y, while g_C is.

Remark 2.4. The previous two examples show that *Theorem A does not extend to quadratic integers.*

If f is a birational transformation of a projective surface X with $\lambda(f) = 1$, then $\| (f^n)_* \|$ is bounded, or it grows linearly with n, or it grows quadratically: In the

first and third cases, f is conjugate to an automorphism of a projective surface Y by some birational transformation $\varphi\colon Y \dashrightarrow X$; in the second case, f is not conjugate to an automorphism (see Section 4.2.2 in [22]). Thus, again, the "degree growth" determines whether f is conjugate to an automorphism.

2.4. **Proof of Theorem A.** Let us now prove Theorem A. Assume $\lambda(f)$ is a Salem number. Let $\chi(t) \in \mathbf{Z}[t]$ be the minimal polynomial of $\lambda(f)$. By assumption, there exists a root α of χ with modulus 1; one can thus fix an automorphism σ of the field of complex numbers such that $\sigma(\lambda(f)) = \alpha$.

By the Diller-Favre Theorem, we may assume that f is algebraically stable. The eigenvector $\Theta_X^+(f)$ corresponds to the eigenvalue $\lambda(f)$; as such, it may be taken in $\mathsf{NS}_L(X)$, where L is the splitting field of χ. Our goal is to show that $\Theta_X^+(f)$ is orthogonal to all E_j, $1 \le j \le m$; the conclusion will follow from Theorem 2.1.

The automorphism σ of the field $\mathbf{C}$ acts on $\mathsf{NS}_{\mathbf{C}}(X)$, preserving $\mathsf{NS}(X)$ pointwise. Apply σ to both sides of $f^*\Theta_X^+(f) = \lambda(f)\Theta_X^+(f)$; since f^* is defined over $\mathbf{Z}$, one obtains

$$f^*\Psi = \alpha\Psi, \qquad \text{with} \qquad \Psi = \sigma(\Theta_X^+(f)).$$

Since the divisor classes of the E_j are in $\mathsf{NS}(X)$, all of them are σ-invariant. Thus, applying σ to Equation (2.1) we get

$$f_*f^*\Psi = \Psi + \sum_{j=1}^{m} (\Psi \cdot E_j)E_j.$$

Taking intersection with the complex conjugate $\bar{\Psi}$ of Ψ, and using $f^*\Psi = \alpha\Psi$, we get

$$(\alpha\bar{\alpha})\Psi \cdot \bar{\Psi} = f^*\Psi \cdot f^*\bar{\Psi} = \Psi \cdot \bar{\Psi} + \sum_{j=1}^{m} |E_j \cdot \Psi|^2.$$

Since α has modulus 1, all intersections $E_j \cdot \Psi$ vanish and, applying σ again, we deduce that $\Theta_X^+(f) \cdot E_j = 0$ for all $1 \le j \le m$. This concludes the proof.

2.5. **Salem numbers in $\Lambda(\mathbb{P}^2_{\mathbf{k}})$.** Let f be an element of $\mathsf{Cr}_2(\mathbf{k})$ such that $\lambda(f)$ is a Salem number. According to Theorem A, f is conjugate to an automorphism g of a smooth rational surface X; according to Kantor and Nagata [39, 40], X is a blowup of $\mathbb{P}^2_{\mathbf{k}}$ with Picard number $\rho(X) \ge 11$. Thus, the study of $\Lambda^S(\mathbb{P}^2_{\mathbf{k}})$ reduces to the following question: Which Salem numbers can be realized as spectral radii of linear transformations

$$g_* \in \mathsf{End}(\mathsf{NS}_{\mathbf{R}}(X)),$$

where X describes the set of blowups of $\mathbb{P}^2_{\mathbf{k}}$ and g runs over the group $\mathsf{Aut}(X)$? Recent results answer this question.

Write X as a blowup of the plane at n points $p_1, p_2, \ldots, p_n$; some of them may be infinitely near points, and we choose indices in such a way that $j \ge i$ if p_j is infinitely near p_i. Denote by $\pi : X \to \mathbb{P}^2_{\mathbf{k}}$ the birational morphism corresponding to this sequence of blowups. Let $e_i \in \mathsf{NS}(X)$ denote the Néron-Severi class of the total transform of p_i under π (for $1 \le i \le n$), and let $e_0 \in \mathsf{NS}(X)$ be the class of the total transform of a line in $\mathbb{P}^2_{\mathbf{k}}$. Then

$$\mathsf{NS}(X) = \mathsf{Pic}(X) = \mathbf{Z}e_0 \oplus \mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_n,$$

and the basis $(e_0, e_1, \dots, e_n)$ is orthogonal with respect to the intersection form on $\mathsf{Pic}(X)$. More precisely, we have

$$e_0 \cdot e_0 = 1,\ e_i \cdot e_i = -1 \text{ if } i \geq 1, \text{ and } e_i \cdot e_j = 0 \text{ if } i \neq j.$$

The canonical class of X is

$$k_X = -3e_0 + e_1 + e_2 + \cdots + e_n.$$

The automorphism group $\mathsf{Aut}(X)$ acts linearly on $\mathsf{Pic}(X)$, preserves k_X, and preserves the intersection form. As a consequence, $\mathsf{Aut}(X)$ preserves the orthogonal complement $k_X^\perp$ of k_X in $\mathsf{Pic}(X)$. The elements

$$\begin{aligned} v_0 &= e_0 - e_1 - e_2, \\ v_i &= e_i - e_{i+1}, \qquad 1 \leq i \leq n-1, \end{aligned}$$

form a basis of $k_X^\perp$, with respect to which the intersection form is given by the Dynkin diagram $T_{2,3,n-3}$ (see Figure 1 below).

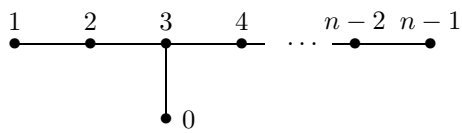

FIGURE 1. Coxeter-Dynkin diagram of type $T_{2,3,n-3}$

In other words,

$$\begin{aligned} v_k \cdot v_k &= -2, \quad \text{for all indices } k, \\ v_i \cdot v_j &= 0, \quad \text{if the vertices } i \text{ and } j \text{ are not linked by an edge,} \\ v_i \cdot v_j &= 1, \quad \text{if the vertices } i \text{ and } j \text{ are the end points of an edge.} \end{aligned}$$

The Weyl (or Coxeter) group W_X of X is the group of orthogonal transformations of $\mathsf{Pic}(X)$ generated by the involutions

$$s_i : u \mapsto u + (u \cdot v_i)v_i, \qquad 0 \leq i \leq n-1.$$

This group preserves the orthogonal decomposition $\mathsf{Pic}(X) = \mathbf{Z}k_X \oplus k_X^\perp$ and is isomorphic to the Coxeter group W_n of the Dynkin diagram $T_{2,3,n-3}$. It turns out that the definition of W_X does not depend on the choice of the realization of X as a blowup of the plane; as an abstract group, W_X depends only on the Picard number of X (see [25]).

Theorem 2.5 (Nagata, McMullen, Uehara). *Let* $\mathbf{k}$ *be an algebraically closed field, and* $n \geq 10$ *be an integer.*

(1) *Let* X *be a rational surface obtained from the projective plane* $\mathbb{P}^2_{\mathbf{k}}$ *by a sequence of blowups. The image of* $\mathsf{Aut}(X)$ *in* $\mathsf{GL}(\mathsf{NS}(X))$ *is contained in the Weyl group* W_X.
(2) *If* $\mathsf{char}(\mathbf{k}) = 0$ *and if* Φ *is an element of* W_n*, there exists a rational surface* Y *with Picard number* $n+1$ *and an element* g *of* $\mathsf{Aut}(Y)$ *such that the dynamical degree* $\lambda(g)$ *of* g *is equal to the spectral radius* $\lambda(\Phi)$ *of* Φ.
(3) *There are Salem numbers which are not contained in* $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ *(resp. in* $\Lambda(X)$ *for any projective surface* X*).*

When $\mathsf{char}(\mathbf{k}) = 0$, this theorem shows that the Salem part of $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is described in purely algebraic terms: It coincides with the set of spectral radii $\lambda(\Phi) > 1$, with Φ in some W_n, $n \geq 10$, and this set does not exhaust all Salem numbers.

Remark 2.6. Assertion (1) is due to Nagata (see [39, 40]). Assertion (2) is due to Uehara, based on previous works by McMullen and Bedford and Kim (see [42]). When the characteristic p of the field is positive, Harbourne proves a similar result, but for Φ in a normal subgroup $W_n(p)$ of W_n of finite index (the index goes to $+\infty$ with n; see Example 3.4 in [31]). Assertion (3) makes use of Theorem A to extend a former result of McMullen. More precisely, McMullen proves that there are Salem numbers between λ_L and λ_P which are not realized by eigenvalues of elements in the Coxeter groups W_n (see [36]), and we deduce from this that there are Salem numbers which are not realized by dynamical degrees of automorphisms of surfaces (see [37]); Theorem A implies that McMullen's result holds for dynamical degrees of birational transformations.

2.6. **Gaps in the dynamical spectrum.** As announced in the Introduction, we can now prove the following corollary to Theorem A.

Corollary 2.7. *Let* $\mathbf{k}$ *be an algebraically closed field.*

(1) *If* f *is a birational transformation of a projective surface* X *defined over* $\mathbf{k}$ *and* $\lambda(f)$ *is in the interval* $]1, \lambda_P[$*, then* f *is conjugate to an automorphism of a projective surface* Y *by a birational mapping* $\phi\colon X \dashrightarrow Y$*.*
(2) *There is no dynamical degree in the interval* $]1, \lambda_L[$*.*
(3) *If* $\mathsf{char}(\mathbf{k}) = 0$*, the minimum of the dynamical degree* $\lambda(f) > 1$ *among all birational transformations of projective surfaces defined over* $\mathbf{k}$ *(resp. of* $\mathbb{P}^2_{\mathbf{k}}$*) is equal to the Lehmer number* $\lambda_L \simeq 1.176280$*.*

Proof. Let f be a birational transformation of a projective surface X defined over an algebraically closed field $\mathbf{k}$. Assume that the dynamical degree $\lambda(f)$ is a Salem number. From Theorem A, f is conjugate to an automorphism of a smooth projective surface. Thus, Assertion (1) follows from the fact that $\lambda(f)$ is a Salem number if $1 < \lambda(f) < \lambda_P \simeq 1.324717$ (see Section 1.1.2).

From Theorem 1.2 in [37], we deduce that $\lambda(f) \geq \lambda_L$, where λ_L is the Lehmer number. Since all Pisot numbers are larger than λ_L, this proves Assertion (2).

If $\mathsf{char}(\mathbf{k}) = 0$, there is an automorphism g of a rational surface X such that $\lambda(g) = \lambda_L$ (see [4, 5] and [37]); McMullen recently announced that such an example also exists on a projective K3 surface (see [38]). In particular, the infimum of all dynamical degrees is a minimum and is equal to the Lehmer number. This proves Assertion (3). □

3. Surfaces which are not rational

In this section we prove Theorem B and provide an example of a K3 surface with automorphisms f whose dynamical degrees $\lambda(f)$ are Salem numbers of degree 22.

Let X be a projective surface defined over an algebraically closed field $\mathbf{k}$. In order to prove Theorem B, we consider the Kodaira dimension of X and refer to the classification of surfaces in Kodaira dimension 0 and $-\infty$ (see [2]).

3.1. **Ruled surfaces.** If the Kodaira dimension of X is $-\infty$ but X is not rational, then X is ruled in a unique, $\mathsf{Bir}(X)$-invariant way. This implies that all elements of $\mathsf{Bir}(X)$ have dynamical degree 1 (see Section 4.2.2 and Theorem 4.5 below).

3.2. **Minimal models and automorphisms.** If the Kodaira dimension of X is non-negative, X admits a unique minimal model X'. From now on, we replace X by X', so that we now have $\mathsf{Bir}(X) = \mathsf{Aut}(X)$. In particular, all elements of $\Lambda(X) \setminus \{1\}$ are Salem numbers, obtained from eigenvalues of linear transformations of $\mathsf{NS}(X)$ (preserving the intersection form).

3.2.1. *Positive Kodaira dimension.* If the Kodaira dimension is equal to 2, the automorphism group is finite, and $\Lambda(X)$ reduces to $\{1\}$. If the Kodaira dimension of X is equal to 1, the Kodaira-Iitaka fibration provides an $\mathsf{Aut}(X)$-equivariant fibration $X \to B$ from X to a curve B. The divisor class of the generic fiber of this fibration is an isotropic vector in $\mathsf{NS}(X)$. This vector is $\mathsf{Aut}(X)$-invariant and, consequently, all elements f in $\mathsf{Bir}(X)$ are elliptic or parabolic isometries of $\mathsf{NS}(X)$ (see Section 4.2.1). This implies that $\lambda(f) = 1$ for all f in $\mathsf{Bir}(X)$.

3.2.2. *Vanishing Kodaira dimension.* Let us now assume that (X is minimal and) the Kodaira dimension of X is equal to 0. According to the classification of surfaces, X is either

(i) an abelian surface;
(ii) a hyperelliptic surface, obtained as a quotient of an abelian surface by a fixed point free group of automorphisms;
(iii) a K3 surface;
(iv) or an Enriques surface.

Hyperelliptic surfaces do not have automorphisms with $\lambda(f) > 1$, as shown in [15]. In cases (i), (iii), and (iv), X has a Picard number bounded from above by 6, 22, and 10, respectively. This shows that $\lambda(f)$ is a Salem number of degree at most 22. Moreover, the Picard number is at most 20 if the characteristic of $\mathbf{k}$ vanishes, so that $\lambda(f)$ is an algebraic integer of degree at most 20 in this case.

Proposition 3.1. *In characteristic* 2*, there are examples of pairs* (X, f) *where* X *is a K3 surface,* $f\colon X \to X$ *is an automorphism, and* $\lambda(f)$ *is a Salem number of degree* 22*.*

To construct such an example, we make use of one of the main results of [24]: *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 2*. There exists a K3 surface* X*, defined over* $\mathbf{k}$*, such that*

(i) *the Picard number of* X *is equal to* 22*;*
(ii) *the automorphism group of* X *is infinite and does not preserve any proper subspace of* $\mathsf{NS}_{\mathbf{R}}(X)$*.*

Let $\mathsf{O}_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ be the Lie group of orthogonal endomorphisms of the Néron-Severi space with respect to the intersection form. This group is an algebraic group, and we denote by $\mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ its irreducible component that contains the identity. From the second property, we deduce that the image $\mathsf{Aut}(X)^\sharp$ of $\mathsf{Aut}(X)$ in $\mathsf{GL}_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ intersects $\mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ on a Zariski dense subgroup; indeed, if $G \subset \mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ is not Zariski dense, then G preserves a non-trivial, strict subspace of $\mathsf{NS}_{\mathbf{R}}(X)$ (see [7] for instance).

As $\mathsf{Aut}(X)^\sharp$ is Zariski dense, we can now prove that the characteristic polynomial of a "general" element of $\mathsf{Aut}(X)^\sharp$ is irreducible (over $\mathbf{Z}$), its degree is equal to 22, and its larger root is a Salem number. The proof relies on the following remark: If g^* is an element of $\mathsf{Aut}(X)^\sharp$, then g^* preserves the integral structure of $\mathsf{NS}(X)$, and preserves the intersection form, the signature of which is equal to $(1, 21)$; hence,

- if g^* has no eigenvalue of modulus > 1, the roots of χ_{g^*} are algebraic integers of modulus at most 1 and, by the Kronecker Lemma, are roots of 1; thus χ_{g^*} splits as a product of cyclotomic polynomials;
- if g^* has an eigenvalue of modulus > 1, it is unique and is either quadratic or a Salem number; hence, if $\chi_{g^*}(t)$ splits as a product of two non-constant polynomials $q(t)$ and $r(t)$ in $\mathbf{Z}[t]$, all the roots of r or q have modulus 1 and χ_{g^*} is divisible by a cyclotomic polynomial.

Thus, either there are elements g with the required properties or χ_{g^*} is divisible by a cyclotomic polynomial of degree at most 22 for every g in $\mathsf{Aut}(X)$. Since their degree is bounded by 22, there are only finitely many cyclotomic polynomials to consider. Let $V_{22} \subset \mathbf{R}[t]$ be the set of all monic polynomials of degree 22. Given a cyclotomic polynomial $r(t)$, the subset

$$V_{22}(r) = \{\chi(t) \mid r(t) \text{ divides } \chi(t)\}$$

is a proper algebraic subset of positive codimension; moreover, the image of $\mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ in V_{22} by the characteristic polynomial mapping is not contained in this set, because there are elements of $\mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$ without any eigenvalue being a root of unity (here we use that 22 is even). Since $\mathsf{Aut}(X)^\sharp$ is Zariski dense in $\mathsf{O}^0_{\mathbf{R}}(\mathsf{NS}_{\mathbf{R}}(X))$, we conclude that there are elements f^* of $\mathsf{Aut}(X)^\sharp$ such that $\chi_{f^*}(t)$ is not contained in any $V_{22}(r)$; the characteristic polynomial of such an element is irreducible (over $\mathbf{Z}$), its degree is equal to 22, and its larger root is a Salem number.

Remark 3.2. This argument has now been extended to other examples of K3 surfaces in positive characteristic by Esnault, Oguiso, and Yu (see [27]).

3.3. **Discrete spectrum.** To conclude the proof of Theorem B, we need to show that the union of all dynamical spectra $\Lambda(X)$ where X runs over the set of non-rational projective surfaces defined over $\mathbf{k}$, and $\mathbf{k}$ runs over the set of all fields, is a discrete subset of the real line. This follows from the upper bound 22 for the degrees of Salem numbers in $\Lambda(X)$ and the following lemma.

Lemma 3.3. *Let B be a positive number and Sal_B be the set of Salem numbers of degree at most B. Then Sal_B is a closed discrete subset of the real line.*

Proof. Let λ be such a Salem number, contained in the interval $[a^{-1}, a]$, with $a > 1$. Its minimum polynomial $\chi(t) \in \mathbf{Z}[t]$ has integer coefficients, and all of them are symmetric polynomials in λ, $\frac{1}{\lambda}$, and its conjugates of modulus 1. Since all these numbers have modulus at most a, all coefficients of χ are bounded by $C_B a^B$, where the constant C_B depends only on B. Since the coefficients of χ are integers, there is a finite list of possible coefficients, a finite list of possible minimum polynomials χ, and therefore a finite list of Salem numbers $\lambda \in [a^{-1}, a]$ of degree $\leq B$.

Thus, the intersection of Sal_B with any compact interval $[a^{-1}, a] \subset \mathbf{R}^*_+$ is finite, and Sal_B is discrete. $\square$

4. Blowups, bubbles, isometries

When X is a projective surface, the group $\mathsf{Bir}(X)$ acts faithfully by isometries on a hyperbolic space $\mathbb{H}_X$, the dimension of which is infinite when X is ruled or rational. This construction is described in [16] and [19]; in this section, we summarize the main facts and apply them to control centralizers and conjugacy classes in $\mathsf{Bir}(X)$. The reader may consult [14], [16], [19], and [28] for the results which are summarized in Sections 4.1 and 4.2.

4.1. **Bubbles and Picard-Manin space.** Let X be a projective surface, defined over an algebraically closed field $\mathbf{k}$. If $\pi\colon Y \to X$ is a birational morphism, one obtains an embedding of Néron-Severi groups $\pi^*\colon \mathsf{NS}(X) \to \mathsf{NS}(Y)$. Given two birational morphisms $\pi_1\colon Y_1 \to X$ and $\pi_2\colon Y_2 \to X$, one says that π_2 is above π_1 (or covers π_1) if $\pi_1^{-1} \circ \pi_2$ is regular. Starting with two birational morphisms $\pi_1\colon Y_1 \to X$ and $\pi_2\colon Y_2 \to X$, one can always find a third birational morphism $\pi_3\colon Y_3 \to X$ which covers π_1 and π_2. It follows easily that the inductive limit of all groups $\mathsf{NS}(Y_i)$, for all surfaces Y_i above X, is well defined. This limit is the **Picard-Manin space** Z_X of X; the intersection form determines a scalar product on Z_X, which we denote by $(v, w) \mapsto v \cdot w$.

The bubble space $\mathcal{B}(X)$ of X is defined as follows. Consider all surfaces Y above X, i.e., all birational morphisms $\pi\colon Y \to X$. Given p_1 on Y_1 and p_2 on Y_2, identify p_1 with p_2 if $\pi_1^{-1} \circ \pi_2$ is a local isomorphism in a neighborhood of p_2 and maps p_2 onto p_1. The bubble space $\mathcal{B}(X)$ is the union of all points of all surfaces above X modulo the equivalence relation generated by these identifications. If p is a point of the bubble space, represented by a point p on a surface $Y \to X$, one denotes by $E(p)$ the exceptional divisor of the blowup of p and by $e(p)$ its divisor class, viewed as a point in Z_X. These classes satisfy $e(p) \cdot e(p') = 0$ if $p \neq p'$ and $e(p) \cdot e(p) = -1$.

Remark 4.1 (see [43])**.** A point p of $\mathcal{B}(X)$ can also be seen as a divisorial valuation whose center in X is a closed point; the value of the valuation on a rational function is the order of vanishing (or pole) of this function along $E(p)$.

The Néron-Severi group $\mathsf{NS}(X)$ is naturally embedded as a subgroup of the Picard-Manin space. This finite dimensional lattice is orthogonal to $e(p)$ for all p in $\mathcal{B}(X)$, and the Picard-Manin space coincides with the direct sum

$$\mathsf{Z}_X = \mathsf{NS}(X) \oplus \bigoplus_p \mathbf{Z} e(p),$$

where p runs over the bubble space. Thus, each element v of the Picard-Manin space can be written as a finite sum

$$v = v_X + \sum_{p \in \mathcal{B}(X)} a(p) e(p).$$

The canonical form on Z_X is a linear form $\Omega\colon \mathsf{Z}_X \to \mathbf{Z}$, which is defined by

$$\Omega(v) = k_X \cdot v_X - \sum_p a(p),$$

where k_X is the canonical divisor of X.

There is a completion process for which the completion $\mathcal{Z}_X$ of $\mathsf{Z}_X \otimes_{\mathbf{Z}} \mathbf{R}$ is represented by square integrable sums:

$$\mathcal{Z}_X = \{w + \sum_p a(p) e(p) \mid w \in \mathsf{NS}_{\mathbf{R}}(X), \text{ and } \sum_p a(p)^2 < \infty\}.$$

The intersection form extends as a scalar product with signature $(1, \infty)$ on this space, but the canonical form Ω does not.

The **hyperbolic space** $\mathbb{H}_X$ of X is then defined by

$$\mathbb{H}_X = \{w \in \mathcal{Z}_X \mid w \cdot w = 1, \text{ and } w \cdot a > 0 \text{ for all ample classes } a \in \mathsf{NS}(X)\}.$$

This space $\mathbb{H}_X$ is an infinite dimensional analogue of the classical hyperbolic spaces $\mathbb{H}_n$: The distance dist on $\mathbb{H}_X$ is defined by

$$\cosh(\mathsf{dist}(v,w)) = v \cdot w$$

for all pairs of elements of $\mathbb{H}_X$; it is complete. If $\mathbb{H}_X$ is cut with a subspace of $\mathcal{Z}_X$ of dimension n, and the intersection is not empty, the result is a totally geodesic hyperbolic space of dimension $n-1$. In particular, geodesics are intersections of $\mathbb{H}_X$ with planes. The projection of $\mathbb{H}_X$ in the projective space $\mathbb{P}(\mathcal{Z}_X)$ is one to one, and the boundary of its image is the projection of the cone of isotropic vectors of $\mathcal{Z}_X$. Thus, we denote by $\partial\mathbb{H}_X$ the set of half-lines

$$\partial\mathbb{H}_X = \{\mathbf{R}_+ v \subset \mathcal{Z}_X \mid v \cdot v = 0, \text{ and } v \cdot a > 0 \text{ for all ample classes } a \subset \mathsf{NS}(X)\}.$$

Remark 4.2.
1. The hyperbolic space $\mathbb{H}_X$ is $\log(3)$-hyperbolic, in the sense of Gromov (see [20]).
2. Since $\mathbb{H}_X$ is Gromov hyperbolic, one can approximate configurations of points in $\mathbb{H}_X$ by configurations of points in metric trees (see [19] for instance).
3. The set $\partial\mathbb{H}_X$ coincides with the Gromov boundary of the hyperbolic space $\mathbb{H}_X$ (note that $\mathbb{H}_X$ is not locally compact).

4.2. **Isometries and dynamical degrees.** The important fact is that $\mathsf{Bir}(X)$ acts faithfully on $\mathcal{Z}_X$ by continuous linear endomorphisms, preserving the intersection form, the effective cone, and the nef cone; it also preserves the subset Z_X and canonical form $\Omega\colon \mathsf{Z}_X \to \mathbf{Z}$. In particular, it preserves the hyperbolic space $\mathbb{H}_X$.

Remark 4.3. Intuitively, elements of $\mathsf{Bir}(X)$ behave like automorphisms on $\mathcal{Z}_X$, because all points have been blown up to define $\mathcal{Z}_X$, so that all indeterminacy points have been resolved. When the Kodaira dimension of X is non-negative and X is minimal, then $\mathsf{Bir}(X)$ coincides with $\mathsf{Aut}(X)$. The space $\mathbb{H}_X$ can be replaced by the subset of $\mathsf{NS}(X,\mathbf{R})$ of elements v with $v \cdot v = 1$ and $v \cdot a > 0$ for all ample classes a; the action of $\mathsf{Bir}(X) = \mathsf{Aut}(X)$ is not always faithful but the kernel coincides with the connected component $\mathsf{Aut}^0(X)$ up to the finite index (see [18, 34]).

Let f be an element of $\mathsf{Bir}(X)$. Denote by $f_\bullet$ its action on $\mathcal{Z}_X$:

$$f_\bullet\colon \mathcal{Z}_X \to \mathcal{Z}_X$$

is a linear isometry with respect to the intersection form. We also denote by $f_\bullet$ the isometry of $\mathbb{H}_X$ that is induced by this endomorphism of $\mathcal{Z}_X$.

4.2.1. *Translation length and types of isometries.* The translation length of an isometry g of $\mathbb{H}_X$ is defined, as for all hyperbolic spaces, by

$$L(g) = \inf\{\mathsf{dist}(v, g(v)) \mid v \in \mathbb{H}_X\}.$$

If this infimum is a minimum, either it is equal to 0 and g has a fixed point in $\mathbb{H}_X$, in which case g is **elliptic**, or it is positive and g is **loxodromic** (also called hyperbolic). If g is loxodromic, the set of points $x \in \mathbb{H}_X$ such that $\mathsf{dist}(x, g(x))$ is equal to the translation length of g is a geodesic line $\mathsf{Ax}(g) \subset \mathbb{H}_X$; its boundary points are represented by isotropic vectors $a(g)$ and $b(g)$ in $\mathcal{Z}_X$ such that

$$g(a(g)) = e^{L(g)} a(g) \qquad \text{and} \qquad g(b(g)) = e^{-L(g)} b(g).$$

The axis of g is the intersection of $\mathbb{H}_X$ with the plane containing $a(g)$ and $b(g)$. Normalize the choice of $a(g)$ and $b(g)$ in such a way that $a(g) \cdot b(g) = 1$. Let x be a point of $\mathbb{H}_X$ or a point of the isotropic cone of $\mathcal{Z}_X$ that intersects $a(g)$ and $b(g)$

positively; then, the orbit $g^n(x)$ converges to the boundary point $\mathbf{R}_+a(g)$ when n goes to $+\infty$ and to $\mathbf{R}_+b(g)$ when n goes to $-\infty$. More precisely, in $\mathcal{Z}_X$ we have

$$\frac{1}{e^{nL(g)}}g^n(x) \to (x\cdot b(g))\,a(g), \qquad \text{and} \qquad \frac{1}{e^{nL(g)}}g^{-n}(x) \to (x\cdot a(g))\,b(g)$$

as n goes to $+\infty$.

When the infimum is not realized, $L(g)$ is equal to 0, and g is **parabolic**: It fixes a unique line in the isotropic cone of $\mathcal{Z}_X$; this line is fixed pointwise, and all orbits $g^n(x)$ in $\mathbb{H}_X$ accumulate to the corresponding boundary point when n goes to $\pm\infty$ (see [14] for examples of accumulation without convergence).

4.2.2. *Types of birational transformations (see* [16]*).* This classification of isometries into three types hold for all isometries of $\mathbb{H}_X$. For isometries $f_\bullet$ induced by birational transformations of X, there is a dictionary between this classification and the geometric properties of f. To state it, let us introduce the following definitions: A birational transformation f of a projective surface X is

(i) **virtually isotopic to the identity** if there is a positive iterate f^n of f and a birational mapping $\phi\colon Z \dashrightarrow X$ such that $\phi^{-1}\circ f^n\circ\phi$ is an element of $\mathsf{Aut}(Z)^0$;
(ii) a **Halphen twist** if f preserves a one parameter family of genus one curves on X but f is not virtually isotopic to the identity;
(iii) a **Jonquières twist** if f preserves a one parameter family of rational curves on X but f is not virtually isotopic to the identity.

When f is a Halphen (resp. a Jonquières twist), then, after conjugacy by a birational mapping $\phi\colon Z \dashrightarrow X$, f permutes the fibers of a genus 1 (resp. a rational) fibration $\pi\colon Z\to B$. Let z be the divisor class of the generic fiber of this fibration. Then z is an isotropic vector in $\mathcal{Z}_X$ that is fixed by $f_\bullet$; in particular, $f_\bullet$ cannot be loxodromic.

Remark 4.4. Let $f\colon X \dashrightarrow X$ be a Halphen twist, and let $\phi\colon Z\to X$ be a modification of X on which the f-invariant family of genus one curves form a fibration $\pi\colon Z\to B$. Let Z' be a relative minimal model of this genus one fibration (Z' is obtained from Z by blowing down exceptional divisors of the first kind that are contained in fibers of π, and iteration of this process). Then, f becomes an automorphism of Z'. On the other hand, Jonquières twists are not conjugate to automorphisms of projective surfaces (see [11]).

Theorem 4.5 (Gizatullin, Cantat, Diller-Favre; see [16, 22])**.** *Let* $\mathbf{k}$ *be an algebraically closed field. Let* X *be a projective surface defined over* $\mathbf{k}$*. Let* f *be birational transformation of* X*, let* $f_\bullet$ *be the isometry of* $\mathbb{H}_X$ *determined by* f*, and let* x *be a point of* $\mathbb{H}_X$*.*

(1) $f_\bullet$ *is elliptic if and only if* f *is virtually isotopic to the identity.*
(2) *If* $f_\bullet$ *is parabolic, either* $x\cdot f_\bullet^n(x)$ *grows linearly with* n*, and* f *is a Jonquières twist, or* $x\cdot f_\bullet^n(x)$ *grows quadratically with* n*, and* f *is a Halphen twist.*
(3) $f_\bullet$ *is loxodromic if and only if the dynamical degree* $\lambda(f)$ *is* >1*.*

In all cases, the translation length $L(f_\bullet)$ *is equal to the logarithm of* $\lambda(f)$*.*

Example 4.6. Let f be a birational transformation of the plane $\mathbb{P}^2_{\mathbf{k}}$. Let e_0 be the class of a line, viewed as a point in $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$. Then

$$f_\bullet(e_0) = \mathsf{deg}(f)e_0 - \sum a(p)e(p),$$

where $\mathsf{deg}(f)$ is the degree of f and $a(p)$ is the multiplicity of the homaloidal net $f_*|\mathcal{O}(1)|$ at the point p (p may be "infinitely near"). Since e_0 does not intersect any of the $e(p)$, one gets

$$\cosh(\mathsf{dist}(e_0, f_\bullet(e_0))) = e_0 \cdot f_\bullet(e_0) = \mathsf{deg}(f).$$

This establishes the link between $\mathsf{deg}(f^n)$ and $\mathsf{dist}(e_0, f_p^n(e_0))$ which leads to the equality $L(f) = \log(\lambda(f))$ (see Section 6.1 for details and complements).

Example 4.7. An element f of the Cremona group is virtually isotopic to the identity if and only if f has finite order or f is conjugate to an element of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}}) = \mathsf{PGL}_3(\mathbf{k})$ (see [11]).

4.3. **Centralizers.**

Corollary 4.8. *Let f be a birational transformation of a projective surface X. If f is loxodromic, the infinite cyclic group generated by f is a finite index subgroup of the centralizer of f in $\mathsf{Bir}(X)$.*

Proof. Let f be a loxodromic birational transformation of the surface X. Then f acts on the hyperbolic space $\mathbb{H}_X$ as a hyperbolic isometry, with an invariant axis $\mathsf{Ax}(f)$. The end points of $\mathsf{Ax}(f)$ correspond to two eigenvectors $b(f)$ and $a(f)$ in the isotropic cone of $\mathcal{Z}_X$, with

$$f_\bullet(b(f)) = \frac{1}{\lambda(f)} b(f), \qquad \text{and} \qquad f_\bullet(a(f)) = \lambda(f) a(f).$$

These vectors are unique up to scalar multiplication.

Let $\mathsf{Cent}(f)$ denote the centralizer of f in the group $\mathsf{Bir}(X)$. It preserves the eigenlines $\mathbf{R}b(f)$ and $\mathbf{R}a(f)$, acting on each of them by scalar multiplication. This provides a morphism $\theta\colon \mathsf{Cent}(f) \to \mathbf{R}_+^*$ such that

$$g_\bullet(a(f)) = \theta(g) a(f)$$

for all g in $\mathsf{Cent}(f)$. Moreover, $\theta(g)$ or its inverse coincides with the dynamical degree of g because, if g is loxodromic, then $g_\bullet$ has exactly two fixed points on the boundary of $\mathbb{H}_X$.

The image of θ is a subgroup of $\mathbf{R}_+^*$ which is contained in $\Lambda(X) \cup \{1\} \cup \Lambda(X)^{-1}$. From the spectral gap property, this image does not intersect the interval $]1, \lambda_L[$ and is consequently a discrete subgroup of $\mathbf{R}_+^*$. Since all infinite discrete subgroups of $\mathbf{R}_+^*$ are cyclic, the image $\theta(\mathsf{Cent}(f))$ is cyclic.

Let $\mathsf{Cent}(f)^0$ be the kernel of θ. All we need to prove is that $\mathsf{Cent}(f)^0$ is finite because, then, the exact sequence

$$1 \to \mathsf{Cent}(f)^0 \to \mathsf{Cent}(f) \xrightarrow{\theta} \mathbf{Z} \to 0$$

proves that $\mathsf{Cent}(f)$ is finite by cyclic.

The group $\mathsf{Cent}(f)^0$ preserves $\mathsf{Ax}(f)$ and fixes $a(f)$. It must therefore fix $\mathsf{Ax}(f)$ pointwise. Let q be a point of $\mathsf{Ax}(f)$ and let Δ be its distance to e_0 in $\mathbb{H}_X$ (where, as above, e_0 is the class of a multiple of an ample divisor D on X with $e_0^2 = 1$). Let h be an element of $\mathsf{Cent}(f)^0$. Then $\mathsf{dist}(e_0, h_\bullet(e_0)) \leq 2\mathsf{dist}(e_0, q) = 2\Delta$; hence

$$h_\bullet(e_0) \cdot e_0 \leq \cosh(2\Delta).$$

This shows that the degree of $h \in \mathsf{Cent}(f)^0 \subset \mathsf{Bir}(X)$ with respect to the polarization D is uniformly bounded by some explicit constant $M = \cosh(2\Delta)$.

Assume that the Kodaira dimension $\mathsf{Kod}(X)$ is non-negative. Changing X into its unique minimal model, we assume that X is minimal; this implies $\mathsf{Bir}(X) = \mathsf{Aut}(X)$ because $\mathsf{Kod}(X) \geq 0$. Thus, $\mathsf{Aut}(X)$ contains a loxodromic element (determined by f), and X is an abelian surface, a K3 surface, or an Enriques surface. The group $\mathsf{Cent}(f)^0$ is, now, a group of automorphisms of X with bounded degree with respect to a fixed polarization D on X. This implies that the intersection of $\mathsf{Aut}(X)^0$ with $\mathsf{Cent}(f)^0$ is a finite index subgroup of $\mathsf{Cent}(f)^0$. Moreover, the centralizer of f in $\mathsf{Aut}(X)^0$ is an algebraic group. Thus, either $\mathsf{Cent}(f)^0$ is finite or it contains a connected algebraic subgroup $G \subset \mathsf{Aut}(X)^0$ of positive dimension. In the latter case, X is an abelian variety and G acts by translations on X, because $\mathsf{Aut}(X)$ is discrete for K3 and Enriques surfaces. Let G_1 be a closed, one dimensional subgroup of G: Its orbits form a fibration of X by elliptic curves. Since f commutes to G_1, it preserves this fibration of X. This contradicts the fact that f is loxodromic and proves that $\mathsf{Cent}(f)^0$ is finite.

Assume that $\mathsf{Kod}(X)$ is negative. Since $\mathsf{Bir}(X)$ contains a loxodromic element f, the surface X is rational, and we can suppose that X is the projective plane and e_0 is the class of a line in $\mathbb{P}^2_{\mathbf{k}}$. The group $\mathsf{Cent}(f)^0$ is a subgroup of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of bounded degree. From Corollaries 2.8 and 2.18 of [12], we deduce that its Zariski closure in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is an algebraic subgroup of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. Denote by G the connected component of the identity in this group. If $\mathsf{Cent}(f)^0$ is infinite, the dimension of G is positive, and a result of Enriques shows that G is contained, after conjugation, in the group of automorphisms of a minimal, rational surface (see [9, 26]). As a consequence, G contains a Zariski closed abelian subgroup A whose orbits have dimension 1 in X. Those orbits are organized in a pencil of curves that is invariant under the action of f. This contradicts $\lambda(f) > 1$ and shows that $\mathsf{Cent}(f)^0$ is finite. □

4.4. **Conjugacy between loxodromic transformations.** Assertion (1) of Corollary 2.7 can be rephrased as follows: *For all loxodromic elements f in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ and all points x in $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$,*

$$\mathsf{dist}(x, f_\bullet(x)) \geq \log(\lambda_L),$$

where λ_L is the Lehmer number.

Lemma 4.9. *For all loxodromic elements f in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ and all points x in $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$,*

$$\mathsf{dist}(x, \mathsf{Ax}(f_\bullet)) \leq 28 \cdot \mathsf{dist}(x, f_\bullet(x)).$$

Remark 4.10. The constant 28 is the smallest integer m such that $m \log(\lambda_L) \geq 4\log(3)$, and the occurrence of $\log(3)$ comes from the fact that $(\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}, \mathsf{dist})$ is $\log(3)$-hyperbolic in the sense of Gromov (see Remark 4.2).

Proof. Let y be the projection of the point x on the axis of $f_\bullet$. Let n be the least positive integer which satisfies

$$\mathsf{dist}(y, f_\bullet^n(y)) \geq 8\log(3).$$

Consider the geodesic quadrilateral with vertices x, y, $f_\bullet^n(y)$, and $f_\bullet^n(x)$. By hyperbolicity, the geodesic segment $[x, f_\bullet^n(x)]$ is contained in the $(2\log(3))$-neighborhood of the other three, and its length is at least $8\log(3)$; hence, its middle point m is at most $(2\log(3))$ away from $[y, f_\bullet^n(y)]$. Let m' be the projection of m on the segment $[y, f_\bullet^n(y)]$. Then the distance from x to m' is equal to the sum of the distances

from x to y and from y to m', up to an error of $2\log(3)$. The same estimate in the triangle $(m', f_\bullet^n(y), f_\bullet^n(x))$ provides the inequality

$$\mathsf{dist}(x, f_\bullet^n(x)) \geq \mathsf{dist}(x,y) + \mathsf{dist}(y, f_\bullet^n(y)) + \mathsf{dist}(f_\bullet^n(y), f_\bullet^n(x)) - 8\log(3).$$

Since $f_\bullet$ is an isometry and y is the projection of x on its axis, the choice for n implies

$$n \cdot \mathsf{dist}(x, f_\bullet(x)) \geq 2\mathsf{dist}(x, \mathsf{Ax}(f_\bullet)).$$

On the other hand, Corollary 2.7 shows that n can be chosen to be the smallest integer above $8\log(3)/\log(\lambda_L) \simeq 54.13$; that is, $n = 55$. □

Theorem 4.11. *Let f and g be two loxodromic elements of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. If f is conjugate to g, one can find an element h of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $f = hgh^{-1}$ and*

$$\mathsf{deg}(h) \leq 2^{57}(\mathsf{deg}(f)\mathsf{deg}(g))^{29}.$$

The following argument provides also a proof of Corollary 1.5 (with $d \geq \mathsf{deg}(f)$, $\mathsf{deg}(g)$).

Proof. Let x be the projection of e_0 on the axis of f, and y be its projection on the axis of g. Let h_0 be an element of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ that conjugates f to g; it maps y onto a point $z_0 := (h_0)_\bullet(y)$ of $\mathsf{Ax}(f_\bullet)$. Let k be an integer such that $\mathsf{dist}(f_p^k(z_0), x) \leq \log(\lambda(f))$. Such a k exists because $f_\bullet$ acts by translation of length $\log(\lambda(f))$ on its axis. Changing h_0 into $h = f^k \circ h_0$, we obtain a new conjugacy from g to f that maps y onto a point $z = h(y)$ at a distance at most $\log(\lambda(f))$ from x. Now,

$$\mathsf{dist}(e_0, h_\bullet(e_0)) \leq \mathsf{dist}(e_0, x) + \mathsf{dist}(x, h_\bullet(y)) + \mathsf{dist}(h_\bullet(y), h_\bullet(e_0)).$$

Since $h_\bullet$ is an isometry, we get

$$\mathsf{dist}(e_0, h_\bullet(e_0)) \leq \mathsf{dist}(e_0, \mathsf{Ax}(f_\bullet)) + \log(\lambda(f)) + \mathsf{dist}(e_0, \mathsf{Ax}(g_\bullet)).$$

The previous lemma can then be applied to e_0 and gives

$$\mathsf{dist}(e_0, h_\bullet(e_0)) \leq \log(\lambda(f)) + 28 \cdot (\mathsf{dist}(e_0, f_\bullet(e_0)) + \mathsf{dist}(e_0, g_\bullet(e_0))).$$

The result follows from $\log(\lambda(f_\bullet)) \leq \mathsf{dist}(e_0, f_\bullet(e_0))$, $\cosh(\mathsf{dist}(e_0, f_\bullet(e_0))) = \mathsf{deg}(f)$, and easy estimates for the reciprocal function of $\cosh(\cdot)$. □

Let us add the following complement to Theorem 4.11, which shows that the hypothesis on f and g (being loxodromic) in the theorem can be checked in a finite number of steps.

Corollary 4.12. *An element $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is loxodromic if and only if $\mathsf{deg}(f^{400}) \geq 3^{19}\mathsf{deg}(f^{200})$.*

Proof. Let us write $g = f^{200}$, which is loxodromic if and only if f is.

One direction has been proved by Junyi Xie in [44]: If $\mathsf{deg}(g^2) \geq 3^{19}\mathsf{deg}(g)$, then $\mathsf{dist}(e_0, g_\bullet^2(e_0)) > \mathsf{dist}(e_0, g_\bullet(e_0)) + 18\log(3)$, and this implies that $g_\bullet$ is a loxodromic isometry because $\mathbb{H}_X$ is $\log(3)$-hyperbolic.

In the other direction, if $f_\bullet$ is loxodromic, the translation length of $f_\bullet$ is at least $\log(\lambda_L)$. Hence, the translation length of g is larger than $200\log(\lambda_L)$. Then, as in the proof of Lemma 4.9,

$$\mathsf{dist}(e_0, g_\bullet^2(e_0)) \geq \mathsf{dist}(e_0, g_\bullet(e_0)) + L(g) - 8\log(3),$$

and this implies that $\mathsf{deg}(g^2) \geq 3^{19}\mathsf{deg}(g)$ because $\lambda_L^{200} > 3^{8+19}$. □

Example 4.13. Let m be a positive integer and a be a non-zero element of the field $\mathbf{k}$. Let (x, y) be affine coordinates of the plane. Consider the transformation $f_a\colon (x, y) \mapsto (ax, y)$. This automorphism is conjugate to $g_{a,m}\colon (x, y) \mapsto (ax, a^m y)$ by the monomial transformation $h(x, y) = (x, x^m \cdot y)$. The degree of f_a and of $g_{a,m}$ is equal to 1, but the degree of h is $m + 1$. If $a^{\mathbf{Z}}$ is a Zariski dense subgroup of $\mathbf{k}^*$, one easily shows that there is no conjugacy h' of degree $< m + 1$. Thus, the degree of the conjugacy is not bounded by the degree of f_a and $g_{a,m}$ if one does not assume $\lambda(f) > 1$.

5. The Weyl group W_∞

We now define, and study, a group of linear transformations of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$, with integer coefficients, preserving the intersection form. This group of isometries W_∞ is a subgroup of $\mathsf{Isom}(\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}})$ that contains the image of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$.

5.1. **Definition of W_∞.** In what follows, $e_0 \in \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ is the class of a line. Let p_1, p_2, and p_3 be the three points of the plane defined by

$$p_1 = [1 : 0 : 0], \quad p_2 = [0 : 1 : 0], \quad p_3 = [0 : 0 : 1].$$

The "infinite Weyl group" W_∞ is the group of $\mathbf{Z}$-linear automorphisms of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ generated by

(1) the group $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ of permutations of the set $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ that acts on $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ by sending e_0 to itself and permuting the $e(p)$.
(2) the involution σ_0 that sends e_0 onto $2e_0 - e(p_1) - e(p_2) - e(p_3)$, sends $e(p_i)$ onto $e_0 - e(p_1) - e(p_2) - e(p_3) + e(p_i)$ for $i = 1, 2, 3$, and fixes $e(p)$ for all p in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1, p_2, p_3\}$.

Let p and q be two elements of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. The element $e(p) - e(q)$ of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ has self-intersection -2; as a consequence, the linear transformation

$$\tau_{p,q}\colon x \mapsto x + (x \cdot (e(p) - e(q)))(e(p) - e(q))$$

is the orthogonal reflection that maps $e(p) - e(q)$ to its opposite. The group generated by all these reflections is the subgroup of elements of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ with finite support. Similarly, σ_0 corresponds to the orthogonal reflection associated to $e_0 - e(p_1) - e(p_2) - e(p_3)$. This explains why W_∞ is considered as an infinite Weyl group (or Coxeter group).

By construction, W_∞ preserves the intersection form and the canonical form Ω, and extends as a group of isometries of $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$.

Lemma 5.1. *Let $\mathbf{k}$ be an algebraically closed field. If f is an element of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, the linear transformation $f_\bullet\colon \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}} \to \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ is an element of W_∞.*

Proof. If f has degree 1, it is an element of the group $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$, acts by permutation on $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, and fixes the class e_0. In other words, the map $f \mapsto f_\bullet$ provides an embedding of $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ into $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})) \subset \mathsf{W}_\infty$. If f is the standard quadratic transformation

$$[x : y : z] \dashrightarrow [yz : xz : xy],$$

it has three base points, namely $p_1 = [1 : 0 : 0]$, $p_2 = [0 : 1 : 0]$, and $p_3 = [0 : 0 : 1]$. Moreover, $f_\bullet$ acts as σ_0 on e_0 and the $e(p_i)$ for $i = 1, 2, 3$, and transforms each $e(q)$, $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{e(p_1), e(p_2), e(p_3)\}$, to some $e(q')$ with $q' \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{e(p_1), e(p_2), e(p_3)\}$. This implies that $f_\bullet$ is the composition of σ_0 with an element of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$,

so that $f_\bullet$ is in W_∞. The result follows from the Noether-Castelnuovo theorem, which asserts that $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is generated by $\mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}})$ and the standard quadratic transformation, when $\mathbf{k}$ is algebraically closed (see [32] or [1]). □

Remark 5.2.

1. The group W_∞ is strictly larger than $\mathsf{Cr}_2(\mathbf{k})$ because the elements of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ fix e_0 but most of them are not induced by projective linear transformations of the plane.

2. Lemma 5.1 is implicitly contained in Noether's original "proof" of the Noether-Castelnuovo theorem.

5.2. **Degrees, multiplicity, base points.** Let h be an element of W_∞. We define the **degree** $\mathsf{deg}(h)$ by $\mathsf{deg}(h) = e_0 \cdot h(e_0)$; the degree is a positive integer because all elements of W_∞ preserve $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$. Writing

$$h(e_0) = \mathsf{deg}(h)e_0 - \sum_p a(p)e(p),$$

where p runs over $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, we say that p is a **base point** of h^{-1} if $a(p) \neq 0$; the integer $a(p)$ is the **multiplicity** of the base point. Since h is an isometry of $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$, it is elliptic, parabolic, or loxodromic; moreover, the **dynamical degree**

$$\lambda(h) = \lim_{k\to+\infty} (\mathsf{deg}(h^k)^{1/k})$$

is well defined, and its logarithm is the translation length $L(h)$.

It is not a priori clear that the multiplicities $a(p)$ are non-negative. For example, $\Lambda = 3e_0 + e(p_1) - \sum_{i=2}^7 e(p_i)$ satisfies $\Lambda^2 = 1$ and intersects the canonical form as e_0 does. To show that such an element cannot be sent onto e_0 by an element of W_∞, we need the following lemma.

Lemma 5.3. *Let $v \in \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ be one of the following vectors:*

$$e_0,\ \ e(q_1),\ \ e_0 - e(q_1),\ \ 3e_0 - \sum_{i=1}^{l} e(q_i)$$

for some distinct points $q_1, \dots, q_l \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. For any $h \in W_\infty$, the following holds:

(1) *There exists $n \geq 1$ and $s_1, \dots, s_n \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ satisfying*

$$\begin{aligned} h(v) &= s_n\sigma_0 s_{n-1}\sigma_0 \cdots s_2\sigma_0 s_1(v), \\ (s_i\sigma_0 \cdots \sigma_0 s_1)(v) \cdot e_0 &> (s_{i-1}\sigma_0 \cdots \sigma_0 s_1)(v) \cdot e_0 \end{aligned}$$

for all $i = 2, \dots, n$.

(2) *Either $h(v) = e(q)$ for some $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ or there exists $k \geq 0$, non-negative integers $d, a_1, \dots, a_k$, and a finite set of points $r_1, \dots, r_k \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, such that*

$$h(v) = de_0 - \sum_{i=1}^{k} a_i e(r_i).$$

Remark 5.4. Let us list the elements of degree 1 or 2 in W_∞; this will be useful for the proof of Lemma 5.3.

• Let h be an element of degree 1 in W_∞. This means that $h(e_0) = e_0 - \sum a(p)e(p)$ where $a(p) \in \mathbf{Z}$ vanishes for all but a finite number of points $p \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$.

Since the self-intersection is preserved under the action of h, the sum $\sum_p a(p)^2$ is equal to 0; this implies that $h(e_0) = e_0$. Let p be an element of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. From

$$h(e(p)) \cdot e_0 = h(e(p)) \cdot h(e_0) = 0$$

we deduce that $h(e(p)) = \sum b(q)e(q)$ with $b(q) \in \mathbf{Z}$, and then that $h(e(p)) = \pm e(q)$ for some point q in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, because the self-intersection of $h(e_p)$ is -1. Since the canonical linear form is preserved, one concludes that $h(e_p) = e_q$. In other words, h is an element of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$.

• Say that $h \in W_\infty$ is **quadratic** if its degree is equal to 2. Write $h(e_0) = 2e_0 - \sum a(p)e(p)$ with $a(p) \in \mathbf{Z}$. The invariance of the self-intersection provides

$$4 - \sum_p a(p)^2 = 1\,;$$

hence, there are exactly three base points, each with multiplicity 1. Composing h with an element of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$, one may assume that these three base points coincide with the base points p_1, p_2, and p_3 of σ_0. Then $\sigma_0 h$ has degree 1. We conclude that quadratic elements are composition $s\sigma_0 s'$ with s and s' in $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$.

Proof. The proof of this lemma parallels classical facts from Coxeter group theory.

We first prove that (1) implies (2). The proof proceeds by induction on the minimum number $n \geq 1$ for which h satisfies assertion (1). If $n = 1$, then $h(v) = s_1(v)$ for some element $s_1 \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$, and assertion (2) follows. Let us now assume that $n \geq 2$ and apply the induction hypothesis to $s_{n-1}\sigma_0 \cdots s_2\sigma_0 s_1$. Let $w = s_{n-1}\sigma_0 \cdots s_2\sigma_0 s_1(v)$. If w is equal to e_q for some $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, we apply $s_n\sigma_0$: $h(v) = s_n\sigma_0(w)$ is either $e(q')$ or $e_0 - e(q') - e(q'')$ for some $q', q'' \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, and the first case is in fact impossible by (1). Otherwise, we write $w = de_0 - \sum_{i=1}^k c_i e(r_i)$ for some points $r_1, \ldots, r_k \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ and non-negative integers d, c_i. Ordering the points and adding, if necessary, points with trivial coefficients $c_i = 0$, we assume that r_1, r_2, and r_3 are the three base points p_1, p_2, and p_3 of σ_0. By definition of σ_0, we find

$$\sigma_0(w) = (d + d')e_0 - \sum_{i=1}^{3}(c_i + d')e(r_i) - \sum_{i=4}^{m} c_i e(r_i),$$

where $d' = d - c_1 - c_2 - c_3$. Since

$$d + d' = s_n\sigma_0 \cdots s_2\sigma_0 s_1(v) \cdot e_0 > d,$$

we obtain $d' > 0$ and see that $c_i + d'$ is non-negative for $i = 1, 2, 3$. This proves that (1) $\Rightarrow$ (2) by induction on n.

We now prove assertion (1). By definition of W_∞, we can always write h as a composition $s_m\sigma_0 s_{m-1}\sigma_0 \cdots s_2\sigma_0 s_1$ for some $s_1, \ldots, s_m \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$. For $i = 1, \ldots, m$, write

$$w_i = s_i\sigma_0 \cdots \sigma_0 s_1(v) \quad \text{and} \quad d_i = w_i \cdot e_0.$$

Our aim is to replace the sequence $(s_i)_{i=1}^m$, keeping $s_m\sigma_0 s_{m-1}\sigma_0 \cdots s_2\sigma_0 s_1(v) = h(v)$, in order to assure that the sequence $(d_i)_{i=1}^m$ increases strictly.

Let s and s' be elements of $\mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$. One easily checks, for all possibilities of v, that

$$s'\sigma_0 s(v) \cdot e_0 > v \cdot e_0 \quad \text{or} \quad s'\sigma_0 s(v) = s''(v)$$

for some $s'' \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$. We can then change the sequence $(s_i)_{i=1}^m$ to assure that $m = 1$, in which case the result is obvious, or $d_2 > d_1$.

We set $S = \{i \in \{2, \dots, m-1\} \mid d_{i-1} < d_i \geq d_{i+1}\}$ and write $D = \max\{d_i | i \in S\}$ with the convention $D = 0$ if $S = \emptyset$. We then denote by l the number of elements $i \in \{2, \dots, m-1\}$ such that $d_i = D$ and prove assertion (1) by induction on the pairs (D, l), ordered lexicographically.

If $D = 0$ or $l = 0$, then $S = \emptyset$ and we have $d_1 < d_2 < \cdots < d_m$, which achieves the proof. We can thus assume that there is some $k \in S$ such that $d_k = D > 0$. Let us recall that

$$d_{k-1} < d_k \geq d_{k+1}.$$

Since $k < m$, the induction hypothesis and the proof of (1) $\Rightarrow$ (2) yield the existence of points $p_1, \dots, p_r$ such that

$$w_k = s_k \sigma_0 \cdots s_2 \sigma_0 s_1(v) = d_k e_0 - \sum_{i=1}^{r} a_i e(p_i)$$

for some non-negative integers $a_1, \dots, a_r$. We again assume that p_1, p_2, p_3 are the three base points of σ_0. Since $w_{k+1} = s_{k+1}\sigma_0(w_k)$ and $w_{k-1} = \sigma_0(s_k)^{-1}(w_k)$, we find

$$d_{k+1} = e_0 \cdot \sigma_0(w_k) = \sigma_0(e_0) \cdot w_k = 2d_k - \sum_{i=1}^{3} e(p_i) \cdot w_k = 2d_k - \sum_{i=1}^{3} a_i,$$

and

$$d_{k-1} = e_0 \cdot \sigma_0(s_k)^{-1}(w_k) = s_k \sigma_0(e_0) \cdot w_k = 2d_k - \sum_{i=1}^{3} e(q_i) \cdot w_k,$$

where $q_i = s_k(p_i)$ for $i = 1, 2, 3$. This implies

$$\sum_{i=1}^{3} e(q_i) \cdot w_k > d_k \quad \text{and} \quad \sum_{i=1}^{3} e(p_i) \cdot w_k \geq d_k.$$

Let $t + 2$ be the number of points of the set $\{p_1, p_2, p_3, q_1, q_2, q_3\}$. We can define t sets of 3 points $P_1, \dots, P_t \subset \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, such that $P_1 = \{q_1, q_2, q_3\}$, $P_t = \{p_1, p_2, p_3\}$, and

- $P_i \cap P_{i-1}$ contains 2 points and
- $\sum_{x \in P_i} w_k \cdot e(x) > d_k$ for $i = 1, \dots, t-1$.

For each $i = 1, \dots, t$, we fix an element $s'_i \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ that sends $\{p_1, p_2, p_3\}$ onto P_i; for $i = 1$, we choose s'_1 to be s_k and for $i = t$ we choose $s'_t = \mathrm{Id}$. We then write $w'_i = \sigma_0(s'_i)^{-1}(w_k)$ and $g_i = \sigma_0(s'_{i+1})^{-1} s'_i \sigma_0$. To illustrate this, we make a picture in the case where $t = 4$:

$$\begin{tikzcd}
 & & w_k \arrow[dll, "\sigma_0(s'_1)^{-1}"'] \arrow[dl, "\sigma_0(s'_2)^{-1}"] \arrow[dr, "\sigma_0(s'_3)^{-1}"'] \arrow[drr, "\sigma_0(s'_4)^{-1}"] & & \\
w_{k-1} = w'_1 \arrow[dr, "g_1"'] & & & & w'_4 \\
 & w'_2 \arrow[rr, "g_2"] & & w'_3 \arrow[ur, "g_3"'] &
\end{tikzcd}$$

For $i = 1, \dots, t-1$, the inequality $\sum_{x \in P_i} w_k \cdot e(x) > d_k$ yields $e_0 \cdot w'_i < d_k$. Moreover, the fact that $P_i \cap P_{i-1}$ contains two points implies that g_i is an element of W_∞ of degree 2; as such, it is equal to $a_i \sigma_0 b_i$ for some $a_i, b_i \in \mathsf{Sym}_{\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})}$ (see

Remark 5.4). Because s'_t preserves $P_t = \{p_1, p_2, p_3\}$, $s_{k+1}\sigma_0 s'_t \sigma_0$ is equal to some $a_t \in \mathsf{Sym}_{\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})}$. We can then write $s_{k+1}\sigma_0 s_k \sigma_0 \in W_\infty$ that sends w_{k-1} onto w_{k+1}, as

$$\begin{array}{rcl} s_{k+1}\sigma_0 s_k \sigma_0 & = & (s_{k+1}\sigma_0 s'_t \sigma_0)\sigma_0 (s'_t)^{-1} s'_1 \sigma_0 \\ & = & a_t(\sigma_0(s'_t)^{-1} s'_{t-1}\sigma_0)\cdots(\sigma_0(s'_3)^{-1}s'_2\sigma_0)(\sigma_0(s'_2)^{-1}s'_1\sigma_0) \\ & = & a_t g_t g_{t-1}\cdots g_1. \end{array}$$

For $i = 1, \dots, t-1$, $g_i \cdots g_1(w_{k-1}) = w'_i$, which has an intersection with e_0 smaller than d_k. The replacement above in the decomposition (writing each g_i as $a_i\sigma_0 b_i$ and rearranging the terms) either decreases D or decreases l, without changing D. □

5.3. **Noether inequality.** Let h be an element of W_∞ of degree d. By Lemma 5.3, there is a finite set of points $q_i \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, and positive integers a_i, $i = 1, \dots, k$, such that

$$h(e_0) = de_0 - \sum_{i=1}^{k} a_i e(q_i),$$

where the a_i are positive integers. Computing $h(e_0)^2 = e_0^2 = 1$ and applying the canonical form Ω to $h(e_0)$, we get the classical Noether equalities

$$\sum_{i=1}^{k}(a_i)^2 = d^2 - 1, \qquad \sum_{i=1}^{k} a_i = 3d - 3. \tag{5.1}$$

This already implies that there are at least three points q_i if $d \neq 1$.

Lemma 5.5 (Noether inequality)**.** *Let h be an element of W_∞ of degree $d \geq 2$, and let $a_1, \dots, a_k$ be the multiplicities of the base points of h.*

(1) *The following equality is satisfied:*

$$\begin{array}{rcl} (d-1)(a_1 + a_2 + a_3 - (d+1)) & = & (a_1 - a_3)(d - 1 - a_1) \\ & & +(a_2 - a_3)(d - 1 - a_2) \\ & & +\sum_{i=4}^{k} a_i(a_3 - a_i). \end{array}$$

(2) *For any i, j with $1 \leq i < j \leq k$, we have $a_i + a_j \leq d$.*

(3) *Ordering the a_i such that $a_1 \geq a_2 \geq a_3 \geq a_4 \cdots$ we have*

$$a_1 + a_2 + a_3 \geq d + 1.$$

Proof. To prove assertion (1), multiply the second Noether relation by a_3 and subtract it from the first; then rearrange the terms.

Assertion (2) is equivalent to $(e_0 - e(q_i) - e(q_j)) \cdot (h^{-1}(e_0)) \geq 0$. Take an element $s \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$ that sends q_i and q_j onto $p_1 = [1 : 0 : 0]$ and $p_2 = [0 : 1 : 0]$. This implies that $\sigma_0 s$ maps $e_0 - e(q_i) - e(q_j)$ onto $e(p_3)$, where $p_3 = [0 : 0 : 1]$. The inequality is now equivalent to $(\sigma_0 s h^{-1})(e_0) \cdot e(p_3) \geq 0$ and follows from Lemma 5.3.

Assertion (2) implies that $d - 1 - a_i \geq 0$ for all i, since the number of base points is bigger than 2. Then, Assertion (3) follows from the first one, because the right-hand side of the equality is non-negative. □

5.4. **Jonquières elements.** An element of W_∞ is called a **Jonquières element** with respect to $e_0 - e(p)$ (or to the point $p \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$) if $h(e_0 - e(p)) = e_0 - e(p)$. Jonquières twists f in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ are conjugate to Jonquières elements within $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ if $\mathbf{k}$ is algebraically closed.

Lemma 5.6. *Let $h \in W_\infty$, and let p_1, q_1 be points of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ such that $h(e_0 - e(p_1)) = e_0 - e(q_1)$. Let m be the degree of h. There exist two subsets of $2m-2$ points $\{p_2, \dots, p_{2m-1}\}$ and $\{q_2, \dots, q_{2m-1}\}$ in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, such that $q_1 \neq q_i$ and $p_1 \neq p_i$ for $i \geq 2$, and such that the following hold:*

$$\begin{array}{rcl} h(e_0) & = & me_0 - (m-1)e(q_1) - \sum_{i=2}^{2m-1} e(q_i); \\ h^{-1}(e_0) & = & me_0 - (m-1)e(p_1) - \sum_{i=2}^{2m-1} e(p_i); \\ h(e(p_1)) & = & (m-1)e_0 - (m-2)e(q_1) - \sum_{i=2}^{2m-1} e(q_i); \\ h^{-1}(e(q_1)) & = & (m-1)e_0 - (m-2)e(p_1) - \sum_{i=2}^{2m-1} e(p_i); \\ h(e(p_i)) & = & e_0 - e(q_1) - e(q_i) \ \ \textit{for } i = 2, \dots, 2m-1; \\ h^{-1}(e(q_i)) & = & e_0 - e(p_1) - e(p_i) \ \ \textit{for } i = 2, \dots, 2m-1. \end{array}$$

Proof. Write $h(e_0) = me_0 - \sum_{i=1}^k a_i e(q_i)$, where the a_i are positive integers. Since $m - a_1 = (e_0 - e(q_1)) \cdot h(e_0) = (e_0 - e(p_1)) \cdot e_0 = 1$, we obtain $a_1 = m-1$. From Noether equalities, one obtains $\sum_{i=2}^k a_i = \sum_{i=2}^k (a_i)^2 = 2m-2$. In particular, all a_i are equal to 1 and $k = 2m-2$:

$$h(e_0) = me_0 - (m-1)e(q_1) - \sum_{i=2}^{2m-2} e(q_i).$$

From $h(e(p_1)) = h(e_0) - (e_0 - e(q_1))$ we deduce $h(e(p_1)) = (m-1)e_0 - (m-2)e(q_1) - \sum_{i=2}^{2m-2} e(q_i)$.

Apply now Lemma 5.3 for the elements $e(q_i)$, $i = 1, \dots, 2m-2$. One finds a subset $\{p_1, \dots, p_l\}$ of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ and non-negative integers $b_i, c_{i1}, \dots, c_{il}$ such that

$$h^{-1}(e(q_i)) = b_i e_0 - \sum_{j=1}^{l} c_{ij} e(p_j).$$

Since $e_0 \cdot h^{-1}(e(q_i)) = h(e_0) \cdot e(q_i) = 1$ and $e(p_1) \cdot h^{-1}(e(q_i)) = h(e(p_1)) \cdot e(q_i) = 1$, we get $b_i = c_{i1} = 1$. From $(h^{-1}(e(q_i)))^2 = -1$ follows that $h^{-1}(e(q_i)) = e_0 - e(p_1) - e(p_i)$ for some point $p_i \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ distinct from p_1. Doing this for each i, this defines $2m-1$ points $p_2, \dots, p_{2m-1}$. Then

$$h(e(p_i)) = h(e_0 - e(p_1)) - h(e_0 - e(p_1) - e(p_i)) = e_0 - e(q_1) - e(q_i).$$

It remains to observe that $mh(e_0) - (m-1)h(e(p_1)) - \sum_{i=2}^{2m-1} h(e(p_i)) = e_0$, so $h^{-1}(e_0) = me_0 - (m-1)e(p_1) - \sum_{i=2}^{2m-1} e(p_i)$; the value of $h^{-1}(e(q_1))$ follows now directly from $h(e_0 - e(p_1)) = e_0 - e(q_1)$. □

Lemma 5.7. *Let $h_1, h_2 \in W_\infty$ be Jonquières elements with respect to the same point $p \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. Then*

$$\mathsf{deg}(h_1 h_2) < \mathsf{deg}(h_1) + \mathsf{deg}(h_2).$$

In particular, the sequence $\{\mathsf{deg}(h_1)^n\}_{n \in \mathbf{N}}$ grows at most linearly, and h_1 is not loxodromic.

Remark 5.8. Let $m \geq 2$ be an integer and h be an isometry of a finite dimensional hyperbolic space $\mathbb{H}_m$; here, $\mathbb{H}_m$ is one of the two connected components of the affine quadric $x_0^2 = x_1^2 + \cdots + x_m^2$ in $\mathbf{R}^{m+1}$, and h is the restriction of an element of $\mathsf{O}_{1,m}(\mathbf{R})$ preserving $\mathbb{H}_m$. Assume that h is parabolic; this means that the linear transformation h is not contained in a compact subgroup of $\mathsf{O}_{1,m}(\mathbf{R})$ but does not have any eigenvalue of modulus > 1. Then, $\| h^n \|$ grows quadratically

with n. In other words, given any base point e_0 in $\mathbb{H}_m$, the sequence of distances $\mathsf{dist}(e_0, h^n(e_0))$ grows like $\log(n^2)$ for some positive constant c.

Lemma 5.7 shows that the behavior of parabolic transformations may be different if $\mathbb{H}_n$ is replaced by its infinite dimensional sibling. For instance, consider the birational transformation $f\colon (x,y) \mapsto (xy, y)$. Then $f_\bullet$ determines a Jonquières element of W_∞ with $\mathsf{deg}(f_\bullet^n) = n$; equivalently, $\mathsf{dist}(e_0, f_p^n(e_0))$ grows like $\log(n)$.

Proof. Note that $\mathsf{deg}(h_1h_2) = h_1h_2(e_0)\cdot e_0 = h_2(e_0)\cdot (h_1)^{-1}(e_0)$. Let $q_2, \dots, q_k$ be the base points of h_1 or $(h_2)^{-1}$ which are distinct from p. Because $(e_0 - e(p))\cdot (h_i)^{\pm 1}(e_0) = (e_0 - e(p))\cdot e_0 = 1$, we get

$$\begin{aligned} (h_1)^{-1}(e_0) &= d_1e_0 - (d_1-1)e(p) - \textstyle\sum_{i=2}^k a_ie(q_i), \\ h_2(e_0) &= d_2e_0 - (d_2-1)e(p) - \textstyle\sum_{i=2}^k b_ie(q_i), \end{aligned}$$

for some non-negative integers d_1, d_2, a_i, and b_i. Moreover, $d_1 = \mathsf{deg}(h_1)$, $d_2 = \mathsf{deg}(h_2)$. Hence,

$$\mathsf{deg}(h_1h_2) = d_1d_2 - (d_1-1)(d_2-1) - \sum a_ib_i \le \mathsf{deg}(h_1) + \mathsf{deg}(h_2) - 1.$$

□

Proposition 5.9. *Let h be an element of W_∞ of degree $d \ge 3$. Let $q_1, \dots, q_k$ be the points which are base points of either h or h^{-1}. Let a_i and b_i be the multiplicities of the base points: $a_i = e(q_i)\cdot h^{-1}(e_0)$ and $b_i = e(q_i)\cdot h(e_0)$. Then one of the following properties holds:*

(1) *$d < 3(d-a_i)(d-b_i)$;*
(2) *$a_i = d-1 = b_i$ for at least one base point q_i; in that case h is a Jonquières element with respect to $e_0 - e(q_i)$.*

In particular, if h is not a Jonquières element of W_∞, then

$$d - (a_i+b_i)/2 > \sqrt{d/3}$$

for all $i = 1, \dots, k$.

Proof. To simplify the notation, write $e_i = e(q_i)$ for $i = 1, \dots, k$. Suppose that one of the a_i is equal to $d-1$, and order the points to assume $a_1 = d-1$. Since $a_1 = h(e_1)\cdot e_0$, we have $h(e_0 - e_1)\cdot e_0 = 1$; this implies that $h(e_0 - e_1) = e_0 - e_j$ for some j, and we deduce $b_j = d-1$. If $j = 1$, then h is a Jonquières element with respect to $e_0 - e_1$. Assume now that $j \ne 1$. Lemma 5.6 implies that the a_i for $i \ne 1$ and the b_i for $i \ne j$ are all equal to 0 or 1. In particular, for each i, we get $(d-a_i)(d-b_i) \ge d-1 > d/3$. This proves that either (1) or (2) is satisfied when some a_i is equal to $d-1$.

Assume now that $a_j < d-1$ for all indices j. In particular $h(e_0 - e_j)$ is distinct from $e_0 - e_j$. One only needs to show that $d < 3(d-a_i)(d-b_i)$ for all i. Reordering the points one may assume $i = 1$.

We have

$$h(e_0 - e_1)\cdot e_0 = (e_0 - e_1)\cdot h^{-1}(e_0) = d - a_1 \ge 2,$$

and we can write

$$h(e_0 - e_1) = (d-a_1)e_0 - \textstyle\sum_{i=1}^k r_ie_i$$

for some coefficients $r_i \ge 0$. Moreover

$$1 \le h(e_0 - e_1)\cdot (e_0 - e_1) = d - a_1 - r_1,$$

because $e_0 - e_1$ and $h(e_0 - e_1)$ are two isotropic elements of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ in the boundary of $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$ that are not orthogonal (hence their intersection is a positive integer). As h preserves the canonical linear form,

$$\sum_{i=1}^{k} r_i = 3(d - a_1) - 2.$$

From $h(e_0) \cdot h(e_0 - e_1) = e_0 \cdot (e_0 - e_1) = 1$ we get

$$d(d - a_1) - b_1 r_1 = \textstyle\sum_{i=2}^{k} b_i r_i + 1,$$

where $b_i = e_i \cdot h^{-1}(e_0) \geq 0$ is the multiplicity of p_i as a base point of h. Applying Lemma 5.5 to h we have $b_1 + b_i \leq d$ for $i = 2, \ldots, k$, so $b_i \leq d - b_1$. Because $d(d - a_1) - b_1 r_1 = d(d - a_1 - r_1) + (d - b_1) r_1 \geq d$, we get

$$d \leq 1 + (d - b_1) \sum_{i=2}^{k} r_i \leq 1 + (d - b_1) \cdot (3(d - a_1) - 2) < 3(d - b_1)(d - a_1).$$

This concludes the proof of the alternative. Then, Assertion (1) implies

$$\begin{array}{rcl} (d - (a_i + b_i)/2) & = & \frac{(d-a_i)+(d-b_i)}{2} \\ & \geq & \sqrt{(d - a_i)(d - b_i)} \\ & > & \sqrt{d/3}. \end{array}$$

This concludes the proof of the proposition. □

5.5. **Halphen elements.** Given nine distinct points q_i in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, the class

$$K = 3e_0 - \sum_{i=1}^{9} e(q_i)$$

is an isotropic vector of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$. An element h of W_∞ is a **Halphen element** with respect to such a class K if $h(K) = K$.

Lemma 5.10 (Growing of Halphen type maps)**.** *If h_1 and h_2 are Halphen elements of W_∞ with respect to the same isotropic class K, then*

$$\sqrt{\mathsf{deg}(h_1 h_2)} < \sqrt{\mathsf{deg}(h_1)} + \sqrt{\mathsf{deg}(h_2)}.$$

In particular, the sequence $\{\mathsf{deg}(h_1)^n\}_{n \in \mathbf{N}}$ grows at most quadratically, and h_1 is not loxodromic.

Proof. Note that $\mathsf{deg}(h_1 h_2) = h_1 h_2(e_0) \cdot e_0 = h_2(e_0) \cdot (h_1)^{-1}(e_0)$. For $i = 1, 2$, write $d_i = \mathsf{deg}(h_i)$ and define v_i by

$$(h_i)^{-1}(e_0) = \frac{d_i}{3} K + v_i.$$

This decomposition satisfies $e_0 \cdot v_i = 0$,

$$3 = K \cdot e_0 = K \cdot (h_i)^{\pm 1}(e_0) = K \cdot v_i$$

because $K \cdot K = 0$, and

$$1 = h_i(e_0)^2 = (v_i)^2 + 2\frac{d_i}{3}(K \cdot v_i) = (v_i)^2 + 2d_i.$$

Writing $v_1 = \sum a(p) e(p)$ and $v_2 = \sum b(p) e(p)$, one gets

$$(v_1 \cdot v_2)^2 = (\sum a(p) b(p))^2 \leq \sum a(p)^2 \cdot \sum b(p)^2 = (v_1)^2 \cdot (v_2)^2.$$

Hence,

$$\begin{aligned}\mathsf{deg}(h_1h_2) &= (\frac{d_1}{3}K+v_1)\cdot(\frac{d_1}{3}K+v_2)\\ &= d_1+d_2+v_1\cdot v_2\\ &\leq d_1+d_2+\sqrt{(2d_1-1)(2d_2-1)}\\ &< (\sqrt{d_1}+\sqrt{d_2})^2.\end{aligned}$$

□

Lemma 5.11. *Let h be an element of W_∞, and let $p_1,\dots,p_m$ be the base points of h and h^{-1}. Let a_k and b_k be the multiplicities of the base points: $a_k=h(e_0)\cdot e(p_k)$, $b_k=h^{-1}(e_0)\cdot e(p_k)$. If*

$$d/3\geq 3+(3+\sum_{j=10}^{m}b_j)\left(\max_{i=1}^{9}|3a_i-d|+\sum_{j=10}^{m}a_j\right),$$

then h is a Halphen element with respect to $K=3e_0-\sum_{i=1}^{9}e(p_i)$.

Proof. To simplify the notation, we write $e_k=e(p_k)$ for $k=1,\dots,m$. Thus,

$$\begin{aligned}h(e_0) &= de_0-\sum_{i=1}^{9}a_ie_i-\sum_{j=10}^{m}a_je(q_j),\\ h^{-1}(e_0) &= de_0-\sum_{i=1}^{9}b_ie_i-\sum_{j=10}^{m}b_je(q_j),\end{aligned}$$

where the multiplicities a_k and b_k are non-negative integers ($1\leq k\leq m$).

Denote by $K\in\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ the element $3e_0-\sum_{i=1}^{9}e_i$, and write

$$h(K)=ne_0-\sum_{i=1}^{m}c_ke_k\ ;$$

to obtain such a formula, we may have to allow new base points, and thus increase the number m. By Lemma 5.3, n is a positive integer, and the c_k are non-negative integers.

The canonical form Ω vanishes on K. The invariance of Ω gives

$$3n=\sum_{k=1}^{m}c_k$$

and $h(K)\neq e_0-e_k$ for all k (because $\Omega(e_0-e_k)=2\neq 0$). From Lemma 5.3 we get $K\cdot(e_0-e_k)=h(K)\cdot h(e_0-e_k)>0$ for all indices k:

$$c_k\leq n-1,\quad \forall 1\leq k\leq m.$$

Since h preserves the intersection form, the Hodge index theorem implies that $h(K)=K$ if and only if $h(K)\cdot K=0$, if and only if $h(K)$ is proportional to K. We now assume that h does not fix K; this implies that $h(K)\cdot K$ is positive, and hence that

$$1\leq 3n-\sum_{i=1}^{9}c_i\quad\text{and}\quad\sum_{i=1}^{9}c_i\leq 3n-1.$$

Since $h(K) \cdot e_0 = K \cdot h^{-1}(e_0)$, Noether equalities imply

$$n = 3d - \sum_{i=1}^{9} b_i = 3 + \sum_{j=10}^{m} b_j \geq 3.$$

We now compute $h(e_0) \cdot h(K)$ and get

$$3 = dn - \sum_{i=1}^{9} a_i c_i - \sum_{j=10}^{m} a_j c_j = \frac{d}{3}(3n - \sum_{i=1}^{9} c_i) - \sum_{i=1}^{9}(a_i - \frac{d}{3})c_i - \sum_{j=10}^{m} a_j c_j.$$

Then

$$\begin{aligned} d/3 &\leq 3 + \sum_{i=1}^{9}(a_i - \frac{d}{3})c_i + \sum_{j=10}^{m} a_j c_j \\ &\leq 3 + \left(\sum_{i=1}^{9} c_i\right) \max_{i=1}^{9} \left|a_i - \frac{d}{3}\right| + \max_{j\geq 10}(c_j) \sum_{j=10}^{m} a_j \\ &\leq 3 + (3n-1) \max_{i=1}^{9} |a_i - \frac{d}{3}| + (n-1) \sum_{j=10}^{m} a_j \\ &\leq \left(3 + \sum_{j=10}^{m} b_j\right)\left(3(\max_{i=1}^{9} |a_i - \frac{d}{3}|) + \sum_{j=10}^{m} a_j\right) + R, \end{aligned}$$

where

$$R = 3 - \max_{i=1}^{9} |a_i - \frac{d}{3}| - \sum_{j=10}^{m} a_j < 3.$$

This concludes the proof. □

5.6. **Base points of Jonquières transformations: From W_∞ to $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$.** Elements of Jonquières type in W_∞ are not all realized by birational transformations of the plane. The precise constraints that the base points must satisfy are listed in the following proposition. Both the statement and its proof are necessary to obtain Theorems C and D.

Proposition 5.12. *Let $p_1, \ldots, p_{2m-1} \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ be $2m-1$ distinct points. There exists a Jonquières element $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ whose base points are $p_1, \ldots, p_{2m-1}$, and such that*

$$f_\bullet{}^{-1}(e_0) = me_0 - (m-1)e(p_1) - \sum_{i=2}^{2m-1} e(p_i)$$

if and only if the points p_i can be ordered so as to satisfy the following properties:

1. *p_1 is a proper point of $\mathbb{P}^2_{\mathbf{k}}$;*
2. *for any $i \geq 2$, either p_i is a proper point of $\mathbb{P}^2_{\mathbf{k}}$ or p_i is in the first neighborhood of p_j for some $j < i$;*
3. *for all $i > j \geq 2$, there is no line of $\mathbb{P}^2_{\mathbf{k}}$ which passes through p_1, p_i, p_j;*
4. *for all triples $k > j > i \geq 2$, at least one of the two points p_j, p_k does not belong, as proper or infinitely near point, to the exceptional divisor associated to p_i;*

(5) *the number of points in* $\{p_2, \dots, p_{2m-1}\}$ *that belong, as proper or infinitely near points, to the exceptional divisor associated to* p_1 *is at most* $m-1$ *(these are points "proximate" to* p_1*);*
(6) *for any* $k \geq 1$*, each curve of* $\mathbb{P}^2_{\mathbf{k}}$ *of degree* k *with multiplicity* $k-1$ *at* p_1 *passes through at most* $k+m-1$ *of the points* $\{p_2, \dots, p_{2m-1}\}$*.*

For Property (1), the case $m = 2$ is special: There are three base points of multiplicity 1, and at least one of them is a proper point of the plane; we choose such a point and call it p_1.

Proof. This result is well known to specialists (see [1,30]), but it is hard to extract this precise statement from the literature.

We first verify that the six properties are necessary. The case $m = 1$ corresponds to linear projective transformations and is easily verified because there is no base point in this case. So, assume that f is a Jonquières transformation with base points p_i, degree $m \geq 2$, and

$$f_\bullet^{-1}(e_0) = me_0 - (m-1)e(p_1) - \sum_{i=2}^{2m-1} e(p_i).$$

Let C be a curve of degree k with multiplicity $k-1$ at p_1. Let I be the set of indices i with $p_i \in C$. The class $ke_0 - (k-1)e(p_1) - \sum_{i \in I} p_i$ is effective, and the class $f_\bullet^{-1}(e_0)$ is numerically effective; their intersection is equal to

$$mk - (m-1)(k-1) - |I| = m + k - 1 - |I|.$$

Since this number is non-negative, Property (6) is satisfied. Properties (3) and (4) are proved along the same lines. To prove (2), assume that p_i is not a proper point of the plane and is not in the first neighborhood of any other base point p_i. Then we find a point q which is not a base point such that $e(q) - e(p_i)$ is effective. Intersecting with $f_\bullet^{-1}(e_0)$ one gets $-1 \geq 0$, a contradiction. Properties (1) and (5) are proved in a similar way.

We now prove that Properties (1) to (6) are sufficient to construct such a Jonquières transformation. Denote by $\pi\colon X_2 \to \mathbb{P}^2_{\mathbf{k}}$ the blowup of p_1. The surface X_2 is the Hirzebruch surface $\mathbb{F}_1$: it admits a morphism $\eta_2\colon X_2 \to \mathbb{P}^1$, whose fibers correspond to the lines of $\mathbb{P}^2_{\mathbf{k}}$ through the point p_1.

By property (2), the point p_2 is a proper point of X_2. Consider the fiber

$$F_2 = (\eta_2)^{-1}(\eta_2(p_2));$$

since this curve is the strict transform of the line through p_1 and p_2, Property (3) implies that F_2 does not contain any of the p_i, $i \geq 3$, as proper or infinitely near point. Denote by $X_2 \dashrightarrow X_3$ the birational map which consists of the blowup of p_2, followed by the contraction of the strict transform of F_2. By construction X_3 is a Hirzebruch surface $\mathbb{F}_0$ or $\mathbb{F}_2$, and p_3 is now a proper point of X_3 by Property (2). The pencil of lines through p_1 correspond to the ruling $\eta_3\colon X_3 \to \mathbb{P}^1$, and the Assumptions (3) and (4) imply that the fiber F_3 of η_3 through p_3 does not contain any p_i with $i \geq 4$.

Iterating this process, one constructs a sequence of maps $X_3 \dashrightarrow X_4 \dashrightarrow \cdots \dashrightarrow X_{2m-1}$. For $j = 2, \dots, 2m-1$, the surface X_j is a Hirzebruch surface and comes with a morphism $\eta_j\colon X_j \to \mathbb{P}^1$, the fibers of which correspond to the lines of $\mathbb{P}^2_{\mathbf{k}}$

through p_1. Moreover, the point p_j is a proper point of X_j, and no other point of the fiber $F_j = (\eta_j)^{-1}(\eta_j(p_j))$ is one of the p_i; this latter condition is given by (3) and (4).

By construction, X_{2m-1} is isomorphic to $\mathbb{F}_r$, for some odd integer r. We claim that $r = 1$; this amounts to show that no section of η_{2m-1} has self-intersection ≤ -3. If this section corresponds to the curve of $X_2 = \mathbb{F}_1$ contracted by π onto p_1 (its exceptional curve), it implies that at least m of the points $p_2, \ldots, p_{2m-1}$ belong, as proper or infinitely near, to the section, contradicting hypothesis (5). We can thus assume that the hypothetic section of self-intersection ≤ -3 corresponds to a curve of $\mathbb{P}^2_{\mathbf{k}}$ of degree k passing through p_1 with multiplicity $k-1$. Such a curve has self-intersection $2k-1$ on $X_2 = \mathbb{F}_1$. It must pass through l of the points $p_2, \ldots, p_{2m-1}$, and it must have self-intersection $(2k-1) - l + (2m-2-l) = 2(k+m-l) - 3$ on $X_{2m-1} = \mathbb{F}_r$. Assumption (6) implies that $l < k+m$: This shows that the self-intersection is ≥ -1.

Therefore, X_{2m-1} is isomorphic to $\mathbb{F}_1$. Contracting the exceptional divisor, we obtain a birational morphism X_{2m-1} on a surface which is isomorphic to $\mathbb{P}^2_{\mathbf{k}}$; hence, we can identify this surface with the initial plane $\mathbb{P}^2_{\mathbf{k}}$ and assume that the section is contracted to the point p_1. The composition of the maps $\mathbb{P}^2_{\mathbf{k}} = X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_{2m-1} \to \mathbb{P}^2_{\mathbf{k}}$ is a birational map that preserves the pencil of lines through p_1, and whose base points are exactly $p_1, \ldots, p_{2m-1}$. Classical Noether equalities imply that the degree of the map is m, the multiplicity of p_1 is $m-1$, and the other multiplicities are 1. This achieves the proof. □

6. Dynamical degrees and conjugacy classes

Our goal is to prove Theorem C from the Introduction. Thus, given a birational transformation f of the projective plane with large dynamical degree $\lambda(f)$, we want to conjugate f by an element g of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ to obtain $\mathsf{deg}(gfg^{-1}) \leq C^{st}\lambda(f)^5$ (where the constant C^{st} does not depend on f and will be bounded by 4700).

Given $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ with $\lambda(f) > 1$, the main arguments may be summarized as follows. The degree of f is large, compared to $\lambda(f)$, if and only if the axis $\mathsf{Ax}(f_\bullet)$ is far away from the base point e_0 of the hyperbolic space $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$. Thus, we want to conjugate f by g so that the axis $g_\bullet(\mathsf{Ax}(f_p))$ of gfg^{-1} becomes closer to e_0. A similar problem occurs in the proof of the Noether-Castelnuovo theorem. When one wants to prove that quadratic transformations of the plane generate $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, one starts with an element f in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ and then looks for a quadratic map h such that $\mathsf{deg}(hf) < \mathsf{deg}(f)$; on $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$, the problem is to find a quadratic map such that $g_\bullet(f_\bullet(e_0))$ is closer to e_0 than $f_\bullet(e_0)$ is. We follow the same strategy as in Noether's proof. In other words, we first work with elements of W_∞ and produce elements $h \in W_\infty$ such that $h(\mathsf{Ax}(f_\bullet))$ is close to e_0; then, as Castelnuovo did to correct Noether's error, we have to modify h slightly in order to realize it as $g_\bullet$ for some g in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. Proposition 5.12 is used for this purpose.

Remark 6.1. The proof makes use of ideas from hyperbolic geometry (on the metric space $(\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}, \mathsf{dist})$). The distance is given by $\cosh(\mathsf{dist}(u,v)) = u \cdot v$, and it becomes rapidly annoying to transfer inequalities from distances to intersection numbers, and vice versa. This is the reason why computations are done with intersections, even if they correspond to simple inequalities between distances.

6.1. **Axis, degree, distance to e_0.**

6.1.1. *Isotropic eigenvectors of loxodromic elements.* Let h be a loxodromic element of W_∞, and let $\lambda(h)$ be its dynamical degree. We refer to Section 4.2.1 for the basic properties of loxodromic isometries of hyperbolic spaces.

Write

$$h(e_0) = de_0 - \sum_i a_i e(p_i),$$

where the $a_i \geq 0$ are the multiplicities of the base points $p_i \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. The positive integer d is the degree of h: $d = h(e_0)\cdot e_0$. As explained in Section 4.2.1, h preserves two isotropic lines $\mathbf{R}v_+$ and $\mathbf{R}v_-$, where v_+ and v_- are elements of $\mathcal{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ and

$$h(v_+) = \lambda(h)v_+, \qquad h(v_-) = v_-/\lambda(h).$$

With the normalization $v_+ \cdot e_0 = v_- \cdot e_0 = 1$, the vectors v_+ and v_- are uniquely defined. Moreover, one has

$$v_+ = \lim_{n\to\infty} \frac{h^n(e_0)}{\lambda^n}, \qquad v_- = \lim_{n\to\infty} \frac{h^{-n}(e_0)}{\lambda^n}.$$

Thus, we can write

$$v_+ = e_0 - \sum_i \alpha_i e(p_i), \qquad v_- = e_0 - \sum_i \beta_i e(p_i),$$

where the p_i form a countable subset of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$; the set $\{p_i\}$ is contained in the union of all base points of all iterates h^n for $n \in \mathbf{Z}$. The canonical form Ω is h-invariant, and hence $\Omega(v_+) = \Omega(v_-) = 0$; since v_+ and v_- are isotropic, we obtain

$$\sum_i \alpha_i^2 = \sum_i \beta_i^2 = 1 \quad \text{and} \quad \sum \alpha_i = \sum \beta_i = 3.$$

Since v_+ and v_- are limits of sequences $h^n(e_0)\lambda^{-|n|}$, Lemma 5.3 implies the following positivity statement.

Lemma 6.2. *Let $u \in \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ be one of the vectors*

$$e_0,\ e(p_1),\ e_0 - e(p_1),\ 3e_0 - \sum_{i=1}^{l} e(p_i)$$

for some distinct points $p_1, \dots, p_l \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. Then $u \cdot v \geq 0$ for all $v \in \mathsf{Ax}(h)$, where $\mathsf{Ax}(h)$ is the intersection of the plane generated by v_+ and v_- with $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$.

If $h = f_\bullet$ for some element $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ and $C \in \mathcal{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ is an effective divisor, then $C \cdot v \geq 0$ for every $v \in \mathsf{Ax}(h)$.

6.1.2. *Axis, and translation length.* The proof of the following lemma is straightforward (see [14, 19]).

Lemma 6.3. *Let h be a loxodromic element of W_∞ of degree d and dynamical degree λ. Denote by v_+ and v_- the eigenvectors of h in $\mathcal{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ for the eigenvalues λ and λ^{-1} such that $v_+ \cdot e_0 = v_- \cdot e_0 = 1$. Then*

(i) *$d = h(e_0)\cdot e_0 = e_0 \cdot h^{-1}(e_0)$, and this degree is equal to $\cosh(\mathsf{dist}(h(e_0), e_0))$ and to $\cosh(\mathsf{dist}(e_0, h^{-1}(e_0)))$;*

(ii) *$\log(\lambda(h))$ is the translation length of h, i.e., the minimum of $\mathsf{dist}(x, h(x))$ for x in $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$.*

(iii) *The set of points of* $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$ *that realize the translation length is the* **axis** *of* h*; it coincides with the geodesic line*

$$\mathsf{Ax}(h) = \left\{ \frac{1}{\sqrt{2v_+ \cdot v_-}} \left(tv_+ + \frac{v_-}{t} \right) \;\middle|\; t \in \mathbf{R}_{>0} \right\} \subset \mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}.$$

(iv) *The distance* δ *from* e_0 *to the axis* $\mathsf{Ax}(h)$ *satisfies*

$$\cosh(\delta) = \sqrt{\frac{2}{v_+ \cdot v_-}}.$$

It is realized by the projection of e_0 *on the axis, i.e., by the point*

$$E = \sqrt{\tfrac{2}{v_+ \cdot v_-}} \tfrac{v_+ + v_-}{2}.$$

6.1.3. *Approximation of the points of the axis.* Denote by $\|\cdot\|$ the Euclidean norm on $\mathcal{Z}_{\mathbb{P}^2_{\mathbf{k}}}$, defined by

$$\|u\| = a_0^2 + \sum_{p \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})} a_p^2$$

for $u = a_0 e_0 + \sum a_p e(p)$. If the degree of h is large, the Euclidean norm of $d^{-1}h(e_0) - v_+$ must be small.

Lemma 6.4 (Approximation of the axis)**.** *Let* $h \in W_\infty$ *be a loxodromic element of degree* d *and dynamical degree* λ*. Then*

$$\begin{array}{rclcrcl} \left\|\frac{1}{d}h^{-1}(e_0) - v_-\right\| & < & \sqrt{\frac{2}{\lambda d}} & ; & \left\|\frac{1}{d}h(e_0) - v_-\right\| & < & \sqrt{\frac{2\lambda}{d}}, \\ \left\|\frac{1}{d}h^{-1}(e_0) - v_+\right\| & < & \sqrt{\frac{2\lambda}{d}} & ; & \left\|\frac{1}{d}h(e_0) - v_+\right\| & < & \sqrt{\frac{2}{\lambda d}}. \end{array}$$

Moreover,

$$\left\| \left(\frac{h(e_0) + h^{-1}(e_0)}{2d} \right) - \left(\frac{v_+ + v_-}{2} \right) \right\| < \sqrt{\frac{2}{\lambda d}},$$

and

$$\frac{(\lambda - \frac{1}{\lambda})^2}{2d^2} < v_+ \cdot v_- < \frac{1}{d} \left(\frac{1}{\lambda} + \lambda + 2 \right).$$

In particular, Lemma 6.3 implies

$$\sqrt{\frac{2d}{\frac{1}{\lambda} + \lambda + 2}} < \cosh(\mathsf{dist}(e_0, \mathsf{Ax}(h))) < \frac{2d}{\lambda - \frac{1}{\lambda}}. \tag{6.1}$$

Remark 6.5. Note that the middle points $(v_+ + v_-)/2$ or $(h(e_0) + h^{-1}(e_0))/2$ are not contained in $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$. From a geometric point of view, it would be better to scale them (by the square root of their self-intersection), but the formulas would be difficult to read.

Proof. Let us derive the top four inequalities. From $b_i = e_i \cdot h^{-1}(e_0) = h(e_i) \cdot e_0$ we get

$$\lambda^{-1} = h(v_-) \cdot e_0 = v_- \cdot h^{-1}(e_0) = d - \sum_i b_i \beta_i.$$

With Noether equality $\sum(b_i)^2 = d^2 - 1$ and the relation $\sum(\beta_i)^2 = 1$ we deduce that

$$\begin{aligned}\sum_i \left(\frac{b_i}{d} - \beta_i\right)^2 &= \frac{\sum(b_i)^2}{d^2} - \frac{2\sum b_i\beta_i}{d} + \sum(\beta_i)^2 \\ &= \frac{2}{\lambda d} - \frac{1}{d^2} < \frac{2}{\lambda d}.\end{aligned}$$

This means that $\left\|\frac{h^{-1}(e_0)}{d} - v_-\right\| < \sqrt{\frac{2}{\lambda d}}$. If one replaces v_- by v_+, then λ^{-1} is changed into λ; replacing h with h^{-1} yields the three other inequalities.

The fifth inequality follows by the triangular inequality. We now estimate the intersection product $v_+ \cdot v_-$. On one hand,

$$\|v_+ - v_-\|^2 = 2v_+ \cdot v_- = 2 - 2\sum \alpha_i\beta_i$$

because v_+ and v_- are isotropic. On the other hand, the above inequalities yield

$$\|v_+ - v_-\|^2 < \left(\sqrt{\frac{2}{\lambda d}} + \sqrt{\frac{2\lambda}{d}}\right)^2 = \frac{2}{d}\left(\frac{1}{\lambda} + \lambda + 2\right).$$

Thus, altogether, one gets

$$v_+ \cdot v_- = 1 - \sum \alpha_i\beta_i < \frac{1}{d}\left(\frac{1}{\lambda} + \lambda + 2\right).$$

In the other direction, note that $\frac{1}{\lambda} = d - \sum_i a_i\alpha_i$ and $\lambda = d - \sum_i a_i\beta_i$, and deduce $\lambda - \frac{1}{\lambda} = \sum a_i(\alpha_i - \beta_i)$. The Cauchy-Schwarz inequality yields

$$(\lambda - \frac{1}{\lambda})^2 \leq \sum(a_i)^2 \cdot \sum(\alpha_i - \beta_i)^2 = (d^2 - 1) \cdot (2v_+ \cdot v_-),$$

so that

$$v_+ \cdot v_- \geq \frac{(\lambda - \frac{1}{\lambda})^2}{2(d^2 - 1)}.$$

This concludes the proof. □

6.2. **Decreasing the distance from e_0 to the axis.** We keep the same notation, h being a loxodromic element of degree d, dynamical degree $\lambda, \ldots$. In particular, E is the projection of e_0 to the axis $\mathsf{Ax}(h)$, $a_i = e(p_i) \cdot h(e_0)$ and $b_i = e(p_i) \cdot h^{-1}(e_0)$. The middle point of $h(e_0)$ and $h^{-1}(e_0)$ is

$$de_0 - \sum_i c_i e(p_i), \quad \text{with} \quad c_i = \frac{a_i + b_i}{2}.$$

6.2.1. *Strategy.* The following lemma provides a strategy to decrease the distance between $\mathsf{Ax}(h)$ and e_0 by conjugacy with a quadratic element g of W_∞. Similarly, if $\gamma_i = (\alpha_i + \beta_i)/2$, then

$$\frac{1}{2}(v_+ + v_-) = e_0 - \sum \gamma_i e(p_i).$$

Lemma 6.6. *Let p_1, p_2, p_3 be three distinct points of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. If*

$$\sum_{i=1}^{3} c_i \geq d + \frac{5}{2}\sqrt{\frac{d}{\lambda}},$$

then

(1) $(e(p_1)+e(p_2)+e(p_3)-e_0)\cdot E > \frac{\sqrt{2}(\frac{5}{2}-\sqrt{6})}{\sqrt{\lambda\left(\frac{1}{\lambda}+\lambda+2\right)}} = \frac{5-2\sqrt{6}}{\sqrt{2}(\lambda+1)}$,

(2) $\cosh(\mathsf{dist}(e_0,g(E))) < \cosh(\mathsf{dist}(e_0,E)) - \frac{5-2\sqrt{6}}{\sqrt{2}(\lambda+1)}$,
for any quadratic element $g\in W_\infty$ with base points at p_1,p_2,p_3.

Proof. If necessary, we enlarge the set of base points $\{p_i\}\subset\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ to include the three points p_1, p_2, and p_3 (allowing multiplicities equal to 0). From Lemma 6.4 we know that

$$\left\|\left(\frac{h^{-1}(e_0)+h(e_0)}{2d}\right)-\left(\frac{v_++v_-}{2}\right)\right\| < \sqrt{\frac{2}{\lambda d}},$$

which may be written as

$$\sum\left(\frac{c_i}{d}-\gamma_i\right)^2 < \frac{2}{\lambda d}.$$

Apply the Cauchy-Schwarz inequality for the scalar product between the vector $(1,1,1)\in\mathbf{R}^3$ and the vector $((c_i/d)-\gamma_i)_{i=1}^3$, to get

$$\sum_{i=1}^3\left|\frac{c_i}{d}-\gamma_i\right| < \sqrt{\frac{6}{\lambda d}}.$$

By assumption, we have $(\sum_{i=1}^3\frac{c_i}{d})-1\geq\frac{5}{2\sqrt{\lambda d}}$; hence

$$\gamma_1+\gamma_2+\gamma_3-1 > \frac{\frac{5}{2}-\sqrt{6}}{\sqrt{\lambda d}}.$$

Since $E=\sqrt{\frac{2}{v_+\cdot v_-}}\frac{v_++v_-}{2}$ and $v_+\cdot v_-<\frac{1}{d}\left(\frac{1}{\lambda}+\lambda+2\right)$ (Lemma 6.4), we obtain

$$(e(p_1)+e(p_2)+e(p_3)-e_0)\cdot E > \sqrt{\frac{2}{\frac{1}{d}\left(\frac{1}{\lambda}+\lambda+2\right)}}\left(\frac{\frac{5}{2}-\sqrt{6}}{\sqrt{\lambda d}}\right) = \frac{\sqrt{2}(\frac{5}{2}-\sqrt{6})}{\sqrt{\lambda\left(\frac{1}{\lambda}+\lambda+2\right)}}.$$

If g is a quadratic element of W_∞ with base points p_1,p_2,p_3, then $g^{-1}(e_0)=2e_0-e(p_1)-e(p_2)-e(p_3)$. Consequently,

$$\begin{aligned}\cosh(\mathsf{dist}(e_0,E))-\cosh(\mathsf{dist}(e_0,g(E))) &= e_0\cdot E-e_0\cdot g(E)\\ &= e_0\cdot E-g^{-1}(e_0)\cdot E\\ &= (e(p_1)+e(p_2)+e(p_3)-e_0)\cdot E,\end{aligned}$$

and the conclusion follows from the previous inequality. □

6.2.2. *Noether inequality for the axis of h.*

Lemma 6.7. *The coefficients $c_i=(a_i+b_i)/2$ of $(h(e_0)+h^{-1}(e_0))/2$ satisfy*

$$\begin{aligned}\textstyle\sum_{i=1}^k c_i &= 3d-3;\\ \textstyle\sum_{i=1}^k (c_i)^2 &> (d^2-1)-\frac{d}{2}\left(\lambda^{-1}+\lambda+2\right);\end{aligned}$$

and

$$\begin{aligned}(d-1)(c_1+c_2+c_3-(d+1)) &> (c_1-c_3)((d-1)-c_1)\\ &\quad+(c_2-c_3)((d-1)-c_2)\\ &\quad+\textstyle\sum_{i=4}^k c_i(c_3-c_i)\\ &\quad-\frac{d}{2}\left(\lambda^{-1}+\lambda+2\right).\end{aligned}$$

Proof. The first equality directly follows from Lemma 5.5, which asserts that $\sum_{i=1}^k a_i = \sum_{i=1}^k b_i = 3d-3$. By Lemma 6.4, we have

$$\sum_{i=1}^k \left(\frac{a_i}{d} - \frac{b_i}{d}\right)^2 < \left(\sqrt{\frac{2}{\lambda d}} + \sqrt{\frac{2\lambda}{d}}\right)^2 = \frac{2}{d}\left(\frac{1}{\lambda} + \lambda + 2\right).$$

Since $\sum (a_i)^2 = \sum (b_i)^2 = d^2-1$, we get $d^2 - 1 - \sum a_i b_i < d\left(\frac{1}{\lambda} + \lambda + 2\right)$; hence

$$\begin{aligned} \sum (a_i+b_i)^2 &= 2(d^2-1) + 2\sum a_i b_i \\ &> 4(d^2-1) - 2d\left(\tfrac{1}{\lambda} + \lambda + 2\right). \end{aligned}$$

Dividing by 4, we obtain the first inequality. Then, subtract $c_3 \sum c_i = c_3 \cdot 3(d-1)$ to obtain successively

$$\sum_{i=1}^k (c_i)^2 - c_3 \sum_{i=1}^k c_i > (d-1)((d+1) - 3c_3) - \frac{d}{2}\left(\lambda^{-1} + \lambda + 2\right)$$

and

$$(d-1)(3c_3 - (d+1)) + \sum_{i=1}^3 c_i(c_i - c_3) > \sum_{i=4}^k c_i(c_3 - c_i) - \frac{d}{2}\left(\lambda^{-1} + \lambda + 2\right).$$

The inequality follows by rearranging the terms as in the proof of Noether's inequality. □

6.2.3. *Decreasing the distance to the axis by conjugacy in W_∞.*

Proposition 6.8. *Let $h \in W_\infty$ be of degree d and dynamical degree $\lambda > 10^6$. Let $p_1, \dots, p_k$ be the base points of h and h^{-1}, and $c_i = (a_i + b_i)/2$ be the average of their multiplicities. Order the c_i in such a way that $c_1 \geq c_2 \geq c_3 \geq \cdots \geq c_k$. If $d > 24\lambda^3$, then*

$$c_1 + c_2 + c_3 \geq d + \frac{5}{2}\sqrt{\frac{d}{\lambda}}.$$

Remark 6.9. Together with Lemma 6.6, Assertion (2), this proposition provides a way to conjugate a loxodromic element h of W_∞ by a quadratic involution $g \in W_\infty$ so as to decrease the distance from e_0 to the axis.

Proof. We use the inequality of Lemma 6.7 and observe that $(c_2 - c_3)(d-1-c_2)$ can be removed, as it is non-negative. Indeed, by hypothesis we have $c_2 \geq c_3$, and Noether equalities (5.1) imply that $a_2 \leq d-1$ and $b_2 \leq d-1$. This yields

$$\sum_{i=1}^3 c_i - d > 1 + \frac{(c_1-c_3)(d-1-c_1)}{d-1} + \frac{\sum_{i=4}^k c_i(c_3-c_i)}{d-1} - \frac{d(\lambda^{-1}+\lambda+2)}{2(d-1)}. \tag{6.2}$$

As a consequence, the result follows from

$$(c_1 - c_3)(d-1-c_1) + \sum_{i=4}^k c_i(c_3 - c_i) > \frac{1}{2}d(\lambda^{-1} + \lambda + 2) + (d-1)\frac{5}{2}\sqrt{\frac{d}{\lambda}}.$$

The hypothesis on λ implies $\frac{5}{2}\sqrt{\frac{d}{\lambda}} < \sqrt{d}/400$; thus, it suffices to prove that

$$(c_1 - c_3)((d-1) - c_1) + \sum_{i=4}^k c_i(c_3 - c_i) > \frac{d\sqrt{d}}{200}. \tag{6.3}$$

Note that $2(d-1-c_1) = ((d-1)-a_1) + ((d-1)-b_1)$ is non-negative because $d-1 \geq a_1$ and $d-1 \geq b_1$ (this follows, for instance, from Lemma 5.5). Since $c_1 \geq c_3 \geq c_i$ for $i \geq 4$, every term of the left sum is non-negative.

We do a case by case study, and show that Inequality (6.3) holds in each case.

Step 1. Assume, first, that $c_3 \geq ((d+\sqrt{d}/400)-c_1)/2$. In this situation, the result directly follows:

$$c_1 + c_2 + c_3 \geq c_1 + 2c_3 \geq d + \sqrt{d}/400 \geq d + \frac{5}{2}\sqrt{\frac{d}{\lambda}}$$

because $\lambda > 10^6$.

Step 2. Hence, we assume $c_3 < ((d+\sqrt{d}/400)-c_1)/2$ in what follows. This yields

$$(c_1 - c_3)(d-1-c_1) > \frac{1}{2}(3c_1 - (d+\sqrt{d}/400))(d-1-c_1).$$

The right-hand side is a quadratic polynomial in the variable c_1; it vanishes at $d/3 + \sqrt{d}/1200$ and $d-1$ and is positive between these two roots. If $c_1 \geq d/3 + \sqrt{d}/100$, Proposition 5.9 implies that

$$c_1 \in [d/3 + \sqrt{d}/100, d - \sqrt{d/3}].$$

Both extremities of this interval are between the two roots of the above quadratic polynomial; thus, the infimum of this polynomial function on this interval is equal to its value at $d/3 + \sqrt{d}/100$ or at $d - \sqrt{d/3}$. One easily estimates these two values from below; the first one is

$$\frac{11}{800}\sqrt{d} \cdot (2/3d - \sqrt{d}/100 - 1) > \frac{d\sqrt{d}}{200}$$

because $d > 24 \cdot 10^{18}$, and the second one is

$$(2d - (\sqrt{3} + 1/400)\sqrt{d})/2 \cdot (\sqrt{d}/\sqrt{3} - 1) > \frac{d}{2} \cdot \frac{\sqrt{d}}{2} > \frac{d\sqrt{d}}{200}$$

for the same reason. This implies Inequality (6.3).

Step 3. We can then assume that

$$c_1 < d/3 + \sqrt{d}/100. \tag{6.4}$$

In particular, we have $d-1-c_1 \geq 2d/3 - \sqrt{d}/100 - 1 > (0.6)\cdot d$. If $c_1 - c_3 \geq \sqrt{d}/100$, we obtain Inequality (6.3); hence, we may add the assumption

$$c_3 > c_1 - \sqrt{d}/100. \tag{6.5}$$

Lemma 6.4 provides the inequality

$$\frac{1}{d^2}\sum_{i=1}^{k}(a_i - b_i)^2 < \left(\sqrt{\frac{2}{\lambda d}} + \sqrt{\frac{2\lambda}{d}}\right)^2 = \frac{2}{d}\left(\frac{1}{\lambda} + \lambda + 2\right).$$

In particular, $(a_r - b_r)^2 < 2d\left(\frac{1}{\lambda} + \lambda + 2\right) < 2.01 \cdot \lambda d$ for all indices r. Choosing r such that $a_r \geq a_j$ for all j, we know from Noether inequality that $a_r \geq d/3$ (see Lemma 5.5); this leads to $b_r > d/3 - 1.42\sqrt{\lambda d}$, $c_r = (a_r + b_r)/2 > d/3 - 0.71\sqrt{d\lambda}$, and $c_3 > c_1 - \sqrt{d}/100 \geq c_r - \sqrt{d}/100 > d/3 - (3/4)\cdot\sqrt{d\lambda}$. Hence,

$$c_3 > d/3 - (3/4)\cdot\sqrt{d\lambda} > 3d/10, \tag{6.6}$$

where the last inequality follows from $d > 24\lambda^3 > 10^{12}\lambda$.

For each $i \geq 4$, define $\epsilon_i = \min\{c_3 - c_i, c_i\}$, and note that $c_i \cdot (c_3 - c_i) \geq \epsilon_i \cdot c_3/2$. This gives

$$\sum_{i=4}^{k} c_i \cdot (c_3 - c_i) \geq (\sum_{i=4}^{k} \epsilon_i) \cdot c_3/2.$$

If $\sum_{i=4}^{k} \epsilon_i > \sqrt{d}/15$, Inequality (6.3) follows from $c_3 > 3d/10$ (Inequality (6.6)).

Step 4. We can now add the inequality

$$\sum_{i=4}^{k} \epsilon_i \leq \frac{1}{15}\sqrt{d} \tag{6.7}$$

to our assumptions. Our goal is to derive a contradiction from these assumptions.

Denote by l the largest index such that $c_l \geq c_3/2$. For $i = 4, \ldots, l$, the inequality $c_i \geq c_3/2$ corresponds to $c_i \leq c_3 - c_i$, and hence $\epsilon_i = c_3 - c_i$. This yields, together with Inequality (6.7), the following estimates for $\sum_{i=4}^{l} c_i$:

$$(l-3)c_3 - \sqrt{d}/15 < (l-3)c_3 - \sum_{i=4}^{l} \epsilon_i = \sum_{i=4}^{l} c_i \leq (l-3)c_3.$$

Moreover, $c_1 < c_3 + \sqrt{d}/100$ (Inequality 6.5), so $3c_3 \leq c_1 + c_2 + c_3 < 3c_3 + \sqrt{d}/50$. Adding the two estimates yields

$$lc_3 - \sqrt{d}/15 < \sum_{i=1}^{l} c_i < lc_3 + \sqrt{d}/50.$$

Because $\sum_{i=1}^{l} c_i \leq \sum_{i=1}^{k} c_i = 3d-3$ (Lemma 6.7) and $c_3 > 3d/10$ (Equation (6.6)), we have $l(3d/10) - \sqrt{d}/15 < 3d-3$, which gives $l < 10$, and hence $l \leq 9$.

From $\sum_{i=l+1}^{k} c_i = \sum_{i=l+1}^{k} \epsilon_i < \sqrt{d}/15$ (Inequality (6.7)), one gets $3d - 3 = \sum_{i=1}^{k} c_i < lc_3 + \sqrt{d}/50 + \sqrt{d}/15$. Together with $c_3 \leq c_1 < d/3 + \sqrt{d}/100$ (Inequality (6.4)), we get $3d-3 < l(d/3+\sqrt{d}/100)+\sqrt{d}/50+\sqrt{d}/15$, so $l \geq 9$. Since $l = 9$, Inequality (6.7) yields

$$\sum_{i=10}^{k} c_i = \sum_{i=10}^{k} \epsilon_i < \sqrt{d}/15.$$

In other words, there is a concentration of the multiplicities on the 9 points $p_1, \ldots, p_9$: h behaves like a Halphen element of W_∞.

Step 5. To derive a contradiction, we apply Lemma 5.11:

$$d/3 < (3 + \sum_{i=10}^{k} a_i)\left((3\max\{b_i\}_{i=1}^{9} - d) + \sum_{j=10}^{k} b_j\right) + 3, \tag{6.8}$$

because h is loxodromic.

Let us estimate $\sum_{i=10}^{k} c_i$ from below. For $i = 1, \ldots, 9$, write $\mu_i = d/3 - c_i$ and observe that Inequality (6.4) implies

$$-\sqrt{d}/100 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_9.$$

Lemma 6.7 yields $3d - 3 = \sum_{i=1}^{k} c_i = 3d - \sum_{i=1}^{9} \mu_i + \sum_{i=10}^{k} c_i$, so that

$$\sum_{i=1}^{9} \mu_i = 3 + \sum_{i=10}^{k} c_i < 3 + \sqrt{d}/15 < \sqrt{d}/14,$$

and

$$\mu_9 < \sqrt{d}/14 - \sum_{i=1}^{8} \mu_i < \sqrt{d}/14 + 8\sqrt{d}/100 < \sqrt{d}/6.$$

In particular, $\sum_{i=1}^{9}(\mu_i)^2 < 9(\sqrt{d}/6)^2 = d/4$.

We also have $\sum_{i=10}^{k}(c_i)^2 \leq (\sum_{i=10}^{k} c_i)^2 < d/225$. We compute

$$\begin{aligned}
\sum_{i=1}^{k}(c_i)^2 &= \sum_{i=1}^{9}(d/3-\mu_i)^2 + \sum_{i=10}^{k}(c_i)^2 \\
&= d^2 - \frac{2}{3}d\sum_{i=1}^{9}\mu_i + \sum_{i=1}^{9}(\mu_i)^2 + \sum_{i=10}^{k}(c_i)^2 \\
&< d^2 - \frac{2}{3}d(3+\sum_{i=10}^{k} c_i) + \frac{1}{4}d + \frac{1}{225}d \\
&< d^2 - \frac{2}{3}d(\sum_{i=10}^{k} c_i) - \frac{3}{2}d.
\end{aligned}$$

On the other hand, Lemma 6.7 yields

$$\sum_{i=1}^{k}(c_i)^2 > d^2 - 1 - \frac{d}{2}\left(\lambda^{-1}+\lambda+2\right) > d^2 - (0.501)\cdot(\lambda d),$$

and we obtain

$$d^2 - (0.501)\cdot(\lambda d) < d^2 - \frac{2}{3}d(\sum_{i=10}^{k} c_i) - \frac{3}{2}d;$$

hence

$$\sum_{i=10}^{k} c_i < (0.501)\frac{3}{2}\lambda - \frac{9}{4} < 0.7516\lambda.$$

In particular, either $\sum_{i=10}^{k} a_i < 0.7516\lambda$ or $\sum_{i=10}^{k} b_i < 0.7516\lambda$. We assume the first (otherwise we apply Lemma 5.11 to h^{-1} instead of h).

For each i, recall that $(a_i - b_i)^2 < 2d\left(\frac{1}{\lambda}+\lambda+2\right) < 2.01d\lambda$ and, consequently, $|a_i - b_i| < 1.42\sqrt{\lambda d}$; hence

$$b_i < c_i + 0.71\sqrt{\lambda d} < d/3 + \sqrt{d}/100 + 0.71\sqrt{\lambda d} < d/3 + 0.72\sqrt{\lambda d}.$$

This yields $(3\max\{b_i\}_{i=1}^{9} - d) < 2.16\sqrt{d\lambda}$. Equation (6.8) implies

$$\begin{aligned}
d &< 3(\textstyle\sum_{i=10}^{k} a_i)\left((3\max\{b_i\}_{i=1}^{9} - d) + \sum_{j=10}^{k} b_j\right) \\
&< 3\cdot(0.7516)\cdot\lambda(2.16\sqrt{d\lambda} + 2\cdot 0.7516\lambda) \\
&< 4.88\sqrt{d}\lambda^{3/2}.
\end{aligned}$$

In particular, $\sqrt{d} < 4.88\lambda^{3/2}$, contradicting the hypothesis $d > 24\lambda^3$. □

6.3. **From W_∞ to the Cremona group.** We can now prove Theorem C, Assertion (1), which we rephrase as follows.

Theorem 6.10. *Let $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ be a loxodromic element of dynamical degree $\lambda > 10^6$. There exists a birational map $g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $\mathsf{deg}(gfg^{-1}) < 4700\lambda^5$.*

To prove Theorem 6.10, we denote by $\mathsf{Ax}(f_\bullet)$ the axis of f and by E the projection of the point e_0 onto $\mathsf{Ax}(f_\bullet)$ (see Lemma 6.3). We fix a point $p_1 \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ such that $e(p_1)\cdot E \geq e(q)\cdot E$ for each $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$. We can choose p_1 so that it is a proper point of the plane.

6.3.1. *The involutions σ_Ω.* For each $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$, the vector

$$w_q := \frac{e_0 - e(p_1)}{2} - e(q) \in \mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}} \otimes \mathbf{Q}$$

has self-intersection -1; we denote by ν_q the orthogonal reflection of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}} \otimes \mathbf{Q}$ with respect to the hyperplane orthogonal to w_q. The $\mathbf{Q}$-linear automorphism ν_q is given by

$$u \mapsto u + 2(u\cdot w_q)w_q = u - 2\frac{(u\cdot w_q)}{(w_q\cdot w_q)}w_q.$$

The transformations ν_q, for q in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$, constitute a family of commuting involutions because w_q is orthogonal to $w_{q'}$ if $q \neq q'$.

For any finite set $\Omega \subset \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$ consisting of an even number $2m-2 \geq 0$ of points, we denote by σ_Ω the composition of all ν_q for q in Ω. By induction, the transformation σ_Ω is the automorphism of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ given by

$$\begin{array}{rcl}
\sigma_\Omega(e_0) & = & me_0 - (m-1)e(p_1) - \sum_{q\in\Omega} e(q) \\
 & = & e_0 + \sum_{q\in\Omega}(\frac{e_0 - e(p_1)}{2} - e(q)); \\
\sigma_\Omega(e(p_1)) & = & (m-1)e_0 - (m-2)e(p_1) - \sum_{q\in\Omega} e(q) \\
 & = & e(p_1) + \sum_{q\in\Omega}(\frac{e_0 - e(p_1)}{2} - e(q)); \\
\sigma_\Omega(e_q) & = & e_0 - e(p_1) - e(q) \text{ if } q \in \Omega; \\
\sigma_\Omega(e_q) & = & e_q \text{ if } q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash(\{p_1\}\cup\Omega).
\end{array}$$

The following statement is easily proved. It implies that the σ_Ω form a subgroup of W_∞.

Lemma 6.11. *For all pairs Ω, Ω' of subsets of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$ with an even number of elements, and for all pairs (q,q') of distinct points in $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$, we have*

(i) *σ_Ω of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$ is the product of all ν_q, $q \in \Omega$, and, as such, is an involution;*
(ii) *$\sigma_\Omega \cdot \sigma_{\Omega'} = \sigma_{\Omega\cup\Omega'\backslash\Omega\cap\Omega'}$;*
(iii) *$\sigma_{\{q,q'\}}$ is conjugate to the composition of $\sigma_0 \in W_\infty$ with a transposition $\eta \in \mathsf{Sym}(\mathcal{B}(\mathbb{P}^2_{\mathbf{k}}))$;*
(iv) *σ_Ω is an element of W_∞.*

6.3.2. *Minimal sets.* To simplify the notation, we define

$$\delta = \frac{\sqrt{2}(\frac{5}{2} - \sqrt{6})}{\sqrt{\lambda\left(\frac{1}{\lambda} + \lambda + 2\right)}} = \frac{5 - 2\sqrt{6}}{\sqrt{2}(\lambda+1)},$$

which is the value given by Lemma 6.6.

We denote by $\mathcal{S}$ the collection of all finite subsets $\Omega \subset \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})\backslash\{p_1\}$ consisting of an even number of points, and satisfying the following properties:

(1) Each point $q \in \Omega$ either is a proper point of $\mathbb{P}^2_{\mathbf{k}}$ or is in the first neighborhood of a point $q' \in \Omega$.
(2) There is no line of $\mathbb{P}^2_{\mathbf{k}}$ passing through p_1 and through two distinct points of Ω (taking here also infinitely near points).

(3) For any three distinct points $q_i, q_j, q_k \in \Omega$, the points q_j, q_k cannot simultaneously belong, as proper or infinitely near points, to the exceptional curve obtained by blowing up q_i (this means that q_j, q_k are not both proximate to q_i).

(4) Either $\Omega = \emptyset$ or

$$\cosh\left(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f_\bullet))\right) < \cosh\left(\mathsf{dist}(e_0, \mathsf{Ax}(f_\bullet))\right) - \delta$$

(note that $\sigma_\Omega(\mathsf{Ax}(f_\bullet)) = \mathsf{Ax}(\sigma_\Omega f_\bullet \sigma_\Omega^{-1})$)).

We put a partial order on $\mathcal{S}$, defined by $\Omega' < \Omega$ if and only if

$$\cosh\left(\mathsf{dist}(e_0, \sigma_{\Omega'} \mathsf{Ax}(f_\bullet))\right) < \cosh\left(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f_\bullet))\right) - \delta.$$

The set $\mathcal{S}$ is of course not empty, since it contains the set $\Omega = \emptyset$. The definition of the order implies that there is no infinite decreasing sequence $\Omega_1 > \Omega_2 > \cdots$ in $\mathcal{S}$, so $\mathcal{S}$ contains minimal elements. We now prove the following assertion:

($\star$) *Let Ω be a minimal element of $\mathcal{S}$, and let E be the projection of e_0 onto $\sigma_\Omega \mathsf{Ax}(f_\bullet)$. Either*

$$\mathsf{deg}(\sigma_\Omega f_\bullet \sigma_\Omega^{-1}) \leq 24\lambda^3$$

or there exists $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$ which satisfies $e(q) \cdot E > e(p_1) \cdot E$.

To prove ($\star$), we take an element Ω of $\mathcal{S}$, we assume that $\mathsf{deg}(\sigma_\Omega f_\bullet \sigma_\Omega) > 24\lambda^3$ and that $e(p_1) \cdot E \geq e(q) \cdot E$ for all $q \in \mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$, and we show that Ω is not minimal in $\mathcal{S}$.

• Proposition 6.8 and Lemma 6.6 provide three distinct points q_1, q_2, and q_3 such that $(e(q_1) + e(q_2) + e(q_3) - e_0) \cdot E > \delta$. Because $e(p_1) \cdot E \geq e(q_i) \cdot E$ for all indices i, we can assume that $q_1 = p_1$. Since

$$(\sigma_{\{q_2,q_3\}})^{-1}(e_0) = 2e_0 - e(p_1) - e(q_2) - e(q_3),$$

we obtain $E \cdot e_0 - \sigma_{\{q_2,q_3\}}(E) \cdot e_0 = E \cdot e_0 - E \cdot (\sigma_{\{q_2,q_3\}})^{-1}(e_0) > \delta$; this implies that

$$e_0 \cdot \sigma_{\{q_2,q_3\}}(E) < \cosh(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f_\bullet))) - \delta.$$

As a consequence, if we define $\Omega' = \Omega \cup \{q_2, q_3\} \backslash (\Omega \cap \{q_2, q_3\})$, then $\sigma_{\Omega'} = \sigma_{\{q_2,q_3\}} \cdot \sigma_\Omega$, and the point $E' = (\sigma_\Omega)^{-1}(E) \in \mathsf{Ax}(f_\bullet)$ satisfies $\sigma_{\{q_2,q_3\}}(E) = \sigma_{\Omega'}(E')$; thus, the inequalities

$$\begin{aligned} \cosh(\mathsf{dist}(e_0, \sigma_{\Omega'} \mathsf{Ax}(f_\bullet))) &\leq e_0 \cdot \sigma_{\Omega'}(E') \\ &< \cosh(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f_\bullet))) - \delta \end{aligned}$$

imply that Ω is not a minimal element of S, or that Ω' does not satisfy one of the Assertions (1), (2), (3) in the definition of $\mathcal{S}$.

• We now replace Ω' by a new set Ω'' such that $\sigma_{\Omega''}(e_0) \cdot E'$ does not increase and Ω'' satisfies the defining properties of $\mathcal{S}$. From Section 6.3.1 we know that

$$e_0 \cdot \sigma_{\Omega'}(E') = \sigma_{\Omega'}(e_0) \cdot E' = (e_0 + \sum_{q \in \Omega'} ((e_0 - e(p_1))/2 - e(q))) \cdot E'.$$

If there are two distinct points $q, q' \in \Omega'$ such that

$$(e_0 - e(p_1) - e(q) - e(q')) \cdot E' \geq 0,$$

we can replace Ω' with $\Omega \backslash \{q, q'\}$, and this does not increase $\sigma_{\Omega'}(e_0) \cdot E'$. We can thus assume that $(e_0 - e(p_1) - e(q) - e(q')) \cdot E' < 0$ for all pairs of distinct points (q, q') of Ω'.

If $q \in \Omega$ is in the first neighborhood of a point q', the divisor $e(q)-e(q')$ is effective and intersects E' non-negatively, because E' is on the axis $\mathsf{Ax}(f_\bullet)$ of $f \in \mathsf{Cr}_2(\mathbf{k})$ (Lemma 6.2). If q' does not belong to Ω', we can thus replace Ω' with $\Omega' \cup \{q'\} \setminus \{q\}$; again, this does not increase $\sigma_{\Omega'}(e_0) \cdot E'$.

These replacements down, we get a new set Ω''. Let us show that Ω'' belongs to $\mathcal{S}$. Property (4) is obviously satisfied. The fact that $(e_0 - e(p_1) - e(q) - e(q')) \cdot E' < 0$ for all pairs of distinct points (q, q') in Ω'' implies that p_1, q, q' are not collinear (Assertion (2)). Similarly, the second family of modifications of Ω' shows that Ω'' satisfies Assertion (1). It remains to show that Assertion (3) holds for Ω''. Let q_i, q_j, and q_k be three distinct points of Ω'' such that q_j and q_k are proximate to q_i; then, the divisor $e(q_i) - e(q_j) - e(q_k)$ is effective and intersects thus E' non-negatively (Lemma 6.2). This yields

$$0 > (e_0 - e(p_1) - e(q_j) - e(q_k)) \cdot E' \geq (e_0 - e(p_1) - e(q_i)) \cdot E',$$

which is impossible. Indeed, this implies that $(e_0 - e(p_1) - e(q)) \cdot E' < 0$, where $q \neq p_1$ is a point which either is a proper point of $\mathbb{P}^2_{\mathbf{k}}$ or is in the first neighborhood of p_1 (choose either $q = q_i$ or q such that q_i is infinitely near to q), and this is impossible because $e_0 - e(p_1) - e(q)$ is effective, as it corresponds to a line of $\mathbb{P}^2_{\mathbf{k}}$. This concludes the proof of $(\star)$.

6.3.3. *Strategy.* To prove Theorem 6.10, we provide an algorithm which runs as follows. Start with f and choose a minimal configuration $\Omega \in \mathcal{S}$. If there is an element g in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $g_\bullet(e_0) = \sigma_\Omega(e_0)$, we prove that the distance from e_0 to the axis is decreased by a multiplicative factor that depends only on δ; we can thus replace f by gfg^{-1}. If this is not the case, it is proved that a conjugate of f satisfies $\mathsf{deg}(gfg^{-1}) < 4700\lambda^5$, and the algorithm stops.

6.3.4. *Algorithm: First case.* We now take an element $\Omega \subset \mathcal{S}$, which is minimal in $\mathcal{S}$. By Property $(\star)$, the set Ω is not empty (otherwise the Theorem is proved, with g equal to the identity), and hence we have

$$\cosh(\mathsf{dist}(e_0, \mathsf{Ax}(\sigma_\Omega f_\bullet \sigma_\Omega)) < \cosh(\mathsf{dist}(e_0, \mathsf{Ax}(f_\bullet))) - \delta.$$

We write now explicitly $\Omega = \{p_2, \dots, p_{2m-1}\}$, we denote by E the projection of e_0 on $\sigma_\Omega \mathsf{Ax}(f_\bullet)$, and we denote by $E' \in \mathsf{Ax}(f_\bullet)$ the element $(\sigma_\Omega)^{-1}(E)$; in general, this point differs from the projection of e_0 onto $\mathsf{Ax}(f_\bullet)$.

Suppose that there exists an element $g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ with $(g_\bullet)^{-1}(e_0) = \sigma_\Omega^{-1}(e_0) = \sigma_\Omega(e_0)$. The point $g_\bullet(E') \in \mathsf{Ax}((gfg^{-1})_\bullet)$ satisfies

$$e_0 \cdot g_\bullet(E') = (g_\bullet)^{-1}(e_0) \cdot E' = \sigma_\Omega(e_0) \cdot \sigma_\Omega(E) = e_0 \cdot E,$$

and this implies

$$\cosh(\mathsf{dist}(e_0, \mathsf{Ax}((gfg^{-1})_\bullet))) < \cosh(\mathsf{dist}(e_0, \mathsf{Ax}(f_\bullet))) - \delta. \tag{6.9}$$

We can thus replace f with gfg^{-1} and repeat the process (see below Section 6.3.6).

6.3.5. *Algorithm: Second case.* Suppose now that such a birational transformation g does not exist. Denote by p_i the elements of Ω (including the point p_1).

•An inequality. Recall that p_1 is a proper point of $\mathbb{P}^2_{\mathbf{k}}$. Since Ω is in the family $\mathcal{S}$, Properties (1) to (4) of Proposition 5.12 are fulfilled. Thus, if there is no birational transformation g such that

$$g^{-1}(e_0) = \sigma^{-1}(e_0) = me_0 - (m-1)e(p_1) - \sum_i e(p_i),$$

one of the two assumptions (5) and (6) of Proposition 5.12 is not satisfied. We now study these two possibilities.

Write

$$E = \alpha_0 e_0 - \sum \alpha_i e(p_i),$$

where $p_1, \dots, p_{2m-1}$ are as above and the remaining points p_k, $k \geq 2m$, are elements of $\mathcal{B}(\mathbb{P}^2_{\mathbf{k}})$; the α_i's are real non-negative numbers (apply Lemma 6.2).

If Assumption (5) is not fulfilled, the number of points in $\{p_2, \dots, p_{2m-1}\}$ which belong, as proper or infinitely near points, to the exceptional divisor associated to p_1 is equal to $m+l$ with $0 \leq l \leq m-2$; we write these points as $p_{i_1}, \dots, p_{i_{m+l}}$. The divisor $e(p_1) - \sum_{j=1}^{m+l} e(p_{i_j})$ is thus effective; hence, it intersects E' non-negatively.

Applying σ_Ω, we see that $E = \sigma_\Omega(E_f)$ intersects non-negatively

$$\begin{aligned} \sigma_\Omega(e(p_1) - \sum_{j=1}^{m+l} e(p_{i_j})) &= (m-1)e_0 - (m-2)e(p_1) \\ &\quad - \sum_{i=2}^{2m-1} e(p_i) - \sum_{j=1}^{m+l} (e_0 - e(p_1) - e(p_{i_j})). \end{aligned}$$

This gives $(-1-l)\alpha_0 + (l+2)\alpha_1 \geq 0$, i.e.,

$$\frac{\alpha_1}{\alpha_0} \geq \frac{l+1}{l+2}. \tag{6.10}$$

If Assumption (6) of Proposition 5.12 is not satisfied, we obtain the existence of a curve of degree $k \geq 1$ with multiplicity $k-1$ at p_1 which passes through $k+m+l$ of the points $\{p_2, \dots, p_{2m-1}\}$, for some $l \geq 0$; note that this curve is unique because it corresponds to the exceptional section of the Hirzebruch surface obtained by blowing up p_1 and performing elementary links at $p_2, \dots, p_{2m-2}$. As before, this implies that

$$ke_0 - (k-1)e(p_1) - \sum_{j=1}^{k+m+l} e(p_{i_j}) = k(e_0 - e(p_1)) + e(p_1) - \sum_{j=1}^{k+m+l} e(p_{i_j})$$

intersects non-negatively E'. Applying σ_Ω, we see that E intersects non-negatively

$$k(e_0 - e(p_1)) + (m-1)e_0 - (m-2)e(p_1) - \sum_{i=2}^{2m-1} e(p_i) - \sum_{j=1}^{k+m+l} (e_0 - e(p_1) - e(p_{i_j})),$$

and this leads again to $(-1-l)\alpha_0 + (l+2)\alpha_1 \geq 0$ and to Equation (6.10).

•Upper bound on the degree. Coming back to the proof of Proposition 5.12 we know that, in both cases, the problem is that the Hirzebruch surface obtained after blowing up p_1 and performing elementary links at $p_2, \dots, p_{2m-2}$ is equal to a new Hirzebruch surface $\mathbb{F}_{1+2l}$ which does not coincide with $\mathbb{F}_1$; so the map σ_Ω does not correspond to a geometric Jonquières map.

To recover a well defined birational transformation of the plane, we choose $2l$ general points of $\mathbb{P}^2_{\mathbf{k}}$ that we call $p_{2m}, \dots, p_{m+2l-1}$. Then, we obtain the existence of a Jonquières transformation $g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that

$$(g_\bullet)^{-1}(e_0) = (m+l)e_0 - (m+l-1)e(p_1) - \sum_{i=2}^{2m+2l-1} e(p_i).$$

We now estimate the degree of gfg^{-1} from above. The bound comes from the computation of the intersection of $g_\bullet(E') \in \mathsf{Ax}((gfg^{-1})_\bullet)$ with e_0, a number which is smaller than or equal to $\cosh(\mathsf{dist}(e_0, \mathsf{Ax}((gfg^{-1})_\bullet)))$. First,

$$g_\bullet^{-1}(e_0) = \sigma_\Omega(e_0) + l(e_0 - e(p_1)) - \sum_{i=2m}^{2m+2l-1} e(p_i),$$

and therefore

$$\sigma_\Omega(g_\bullet^{-1}(e_0)) = e_0 + l(e_0 - e(p_1)) - \sum_{i=2m}^{2m+2l-1} e(p_i).$$

In particular, we obtain

$$e_0 \cdot g_\bullet(E') = \sigma_\Omega(g_\bullet^{-1}(e_0)) \cdot E \leq \alpha_0 + l(\alpha_0 - \alpha_1).$$

Inequality (6.10) provides the estimate $\alpha_1 \geq \alpha_0 \cdot \frac{l+1}{l+2}$, and therefore

$$e_0 \cdot g_\bullet(E') \leq \alpha_0 \cdot (1 + l - l \cdot \frac{l+1}{l+2}) = 2\alpha_0 \cdot \frac{l+1}{l+2} < 2\alpha_0 = 2e_0 \cdot E.$$

Since E is the projection of e_0 on $\sigma_\Omega \mathsf{Ax}(f)$, one gets

$$\cosh(\mathsf{dist}(e_0, \mathsf{Ax}((gfg^{-1})_\bullet))) < 2\alpha_0 = 2\cosh(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f))).$$

Remark 6.12. We have $\alpha_i + \alpha_j \leq \alpha_0$ for all indices $i \neq j$. Indeed, one can choose p_i, p_j proper points and obtain that $e_0 - e(p_i) - e(p_j)$ is effective, and hence has a non-negative intersection with E. This also follows from Lemma 6.2, as $e_0 - e(p_i) - e(p_j)$ is in the same W_∞-orbit as $e(p_i)$.

From Equation (6.10), we get $\alpha_1 \geq \frac{\alpha_0}{2}$; hence the remark shows that $\alpha_1 \geq \alpha_i$ for all $i \geq 2$. By $(\star)$, this implies that $\mathsf{deg}(\sigma_\Omega f \sigma_\Omega) \leq 24\lambda^3$.

According to Inequality (6.1) we have

$$\alpha_0 = \cosh(\mathsf{dist}(e_0, \sigma_\Omega \mathsf{Ax}(f))) < \frac{2\mathsf{deg}(\sigma_\Omega f_\bullet \sigma_\Omega)}{\lambda - \frac{1}{\lambda}}$$

and

$$\sqrt{\frac{2\mathsf{deg}(gfg^{-1})}{\frac{1}{\lambda} + \lambda + 2}} < \cosh(\mathsf{dist}(e_0, \mathsf{Ax}((gfg^{-1})_\bullet))).$$

As a consequence,

$$\sqrt{\frac{2\mathsf{deg}(gfg^{-1})}{\frac{1}{\lambda} + \lambda + 2}} < \frac{4\mathsf{deg}(\sigma_\Omega h \sigma_\Omega)}{\lambda - \frac{1}{\lambda}}$$

and

$$\mathsf{deg}(gfg^{-1}) < \frac{8\mathsf{deg}(\sigma_\Omega h \sigma_\Omega)^2}{(\lambda - \frac{1}{\lambda})^2}(\frac{1}{\lambda} + \lambda + 2) < 4700\lambda^5.$$

Thus, we may stop the algorithm, since $\mathsf{deg}(gfg^{-1}) < 4700\lambda^5$.

6.3.6. *Conclusion.* Thus,

- either Section 6.3.4 applies, which means that we can find an element g in the Cremona group such that the hyperbolic cosine of the distance from e_0 to the axis of gfg^{-1} decreases by δ. We can then repeat the process for gfg^{-1} as long as $\mathsf{deg}(gfg^{-1}) > 24\lambda^3$;
- or the process stops, which means that Section 6.3.4 does not apply, and then Section 6.3.5 shows that there exists an element g in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ with $\mathsf{deg}(gfg^{-1}) < 4700\lambda^5$.

To sum up, the theorem 6.10 is proved in at most $\cosh(\mathsf{dist}(e_0, \mathsf{Ax}(f)))/\delta$ steps.

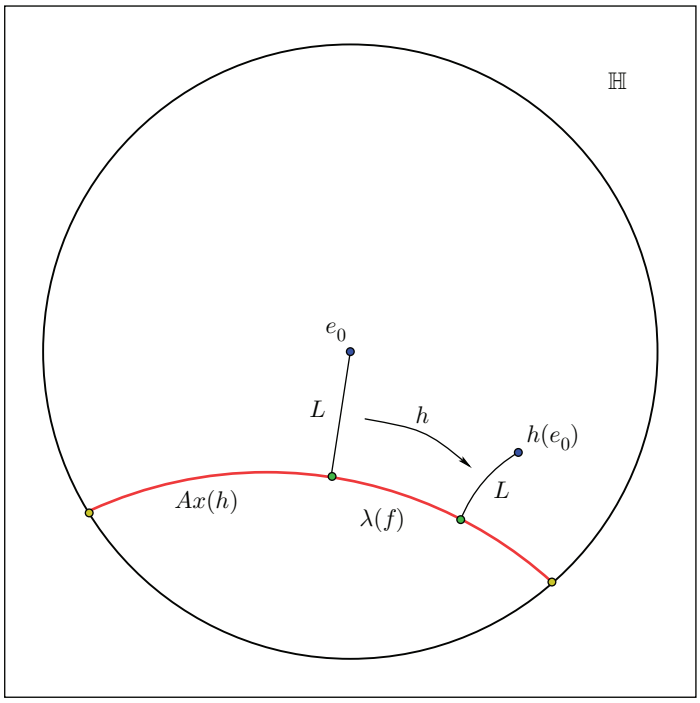

FIGURE 2. Axis and distances.– The blue points are the base point e_0 and its image by $h = gfg^{-1}$; the green points are the projections of these blue points onto the axis of h.

6.4. **Proof of Theorem C.** To conclude the proof of Theorem C, we need to prove the second assertion (using the first one, provided by Theorem 6.10). Let f be a loxodromic element of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$. By the spectral gap property (see Section 2.6), $\lambda(f) \geq \lambda_l \simeq 1.176280$; hence, $\lambda(f^{86}) > 10^6$ and the first assertion of Theorem C provides an element g in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that

$$\mathsf{deg}(gf^{86}g^{-1}) \leq 4700\,\lambda(f^{86})^5.$$

Let h be equal to gfg^{-1}; we have $\lambda(h) = \lambda(f)$. Both h and h^{86} have the same axis, and we denote by L the distance from e_0 to it (see Figure 2). By definition, the distance from e_0 to $(h^{86})_\bullet(e_0)$ is at most equal to $\log(2\mathsf{deg}(h^{86}))$ and, by hyperbolicity of $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$, it is bounded from below by $86\log(\lambda(f)) + 2L - 8\log(3)$. From the

last inequality, we get

$$\log(2\mathsf{deg}(h^{86})) \leq \log(9400) + 5 \cdot 86 \log(\lambda(f));$$

hence,

$$2L \leq 8\log(3) + \log(9400) + 4 \cdot 86 \log(\lambda(f)) \leq 18 + 344 \log(\lambda(f)).$$

With this upper bound in hand, we estimate

$$\begin{aligned} \mathsf{dist}(e_0, (h)_\bullet e_0) &\leq 2L + \log(\lambda(f)) \\ &\leq 18 + 345 \log(\lambda(f)), \end{aligned}$$

and we obtain the inequality $\mathsf{mcdeg}(f) \leq \cosh(18 + 345 \log(\lambda(f)))$. This concludes the proof of Theorem C.

7. Algebraic families of birational transformations and decreasing sequences of dynamical degrees

In this section, we prove the main corollaries of Theorem C. This includes Theorem D, which states that $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is well ordered and that it is closed if $\mathbf{k}$ is uncountable and algebraically closed.

7.1. The dynamical spectrum is well ordered.

7.1.1. *Well ordered subsets of* $\mathbf{R}$. Let Λ be a subset of the real line $\mathbf{R}$. By definition, Λ is well ordered if every subset Λ_0 of Λ has a minimum $\min(\Lambda_0) \in \Lambda_0$; equivalently, Λ is well ordered if it satisfies the descending chain condition: Every decreasing sequence (λ_n) of elements of Λ becomes eventually constant.

Example 7.1. Consider the set of volumes of all compact riemannian manifolds of dimension 3 with a metric of constant curvature -1. According to Jorgensen and Thurston, this set is infinite, is well ordered, and contains accumulation points.

7.1.2. *Dynamical degrees are well ordered.*

Theorem 7.2. *Let* $\mathbf{k}$ *be a field. The subset* $\Lambda \subset \mathbf{R}$ *of dynamical degrees of all birational transformations of projective surfaces defined over* $\mathbf{k}$ *is well ordered.*

Proof. We may assume that $\mathbf{k}$ is algebraically closed. Let λ_n, $n \geq 1$, be a sequence of dynamical degrees. Suppose that $\lambda_{n+1} < \lambda_n$ for all indices n. Our goal is to prove that the number of terms in this sequence is bounded.

Each element λ_n of this sequence is the dynamical degree of some birational transformation $f_n \colon X_n \dashrightarrow X_n$. The subsequence of dynamical degrees λ_n for which X_n is not geometrically rational is bounded, because the set of dynamical degrees of birational transformations of irrational surfaces is discrete. Thus, in what follows, we assume that X_n is equal to the projective plane $\mathbb{P}^2_{\mathbf{k}}$ for all $n \geq 1$.

If $\lambda_{n_0} = 1$ for some index n_0, the sequence contains n_0 terms, because all dynamical degrees are larger than or equal to 1. Thus, we assume that $\lambda_n > 1$ for all n. Let λ_∞ be the limit of the sequence (λ_n); by Corollary 2.7,

$$\lambda_\infty + 1 \geq \lambda_n \geq \lambda_\infty \geq \lambda_L \simeq 1.17628$$

if n is large enough ($n \geq n_0$). Theorem C provides conjugates g_n of f_n, such that

$$\mathsf{deg}(g_n) \leq \cosh(18 + 345 \log(\lambda_\infty + 1)).$$

Hence, the degree of g_n is bounded and, extracting a subsequence, we may assume that the degree of g_n does not depend on n: There exists a degree d such that g_n

is contained in the algebraic variety $\mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}})$ of elements of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of degree d, for $n \geq n_0$. Junyi Xie proved in [44] that the dynamical degree

$$\lambda\colon \mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}}) \to [1, +\infty[$$

is lower semi-continuous for the Zariski topology. In other words, the level subsets

$$L(\beta) = \{h \in \mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}}) | \quad \lambda(h) \leq \beta\}$$

are Zariski closed (for all $\beta \geq 1$). As the sequence of dynamical degrees (λ_n) is strictly decreasing, we deduce that the sequence of Zariski closed sets $L(\lambda_n) \subset \mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}})$ decreases strictly. Since the Zariski topology is Noetherian, the sequence (λ_n) contains only finitely many terms. □

7.2. **Small Pisot numbers and spectral gaps.** Theorem 7.2 implies that there are gaps in the dynamical spectrum on the right of every dynamical degree; the first gap occurs after the Lehmer number λ_L, and the first gap on the right of a Pisot number occurs on the right of the plastic number λ_P. If one restricts the study to dynamical degrees in the Pisot family, the following properties are corollaries of our previous results, known facts on Pisot numbers (see [8]), and a systematic study of quadratic birational transformations of the plane (see [21] and [10] for Assertion (b)):

(a) *The Golden mean λ_G is the smallest accumulation point in the set of Pisot numbers; it is an accumulation point from below and from above.*
(b) *All Pisot numbers below the Golden mean are realized as dynamical degrees of quadratic birational transformations of the plane.*
(c) *There is an $\epsilon > 0$ such that $]\lambda_G, \lambda_G + \epsilon]$ does not contain any dynamical degree; hence, the infimum of the set*

$$\{\lambda \in \mathsf{Pis} \mid \lambda \textit{ is not a dynamical degree}\}$$

is equal to λ_G.

7.3. **The dynamical spectrum is closed.** We now prove that *the dynamical spectrum $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ is closed if $\mathbf{k}$ is uncountable and algebraically closed.*

Let $(f_i)_{i\geq 0}$ be a sequence of birational transformations of the projective plane such that $\lambda(f_i)$ converges toward a real number λ_∞. Our goal is to construct an element h in $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ such that $\lambda(h) = \lambda_\infty$. Thus, we may (and do) assume that the numbers $\lambda_i := \lambda(f_i)$ form a strictly increasing sequence converging to $\lambda_\infty > 1$. Theorem C applies: Changing each f_i into a conjugate element of $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$, we assume that

$$\mathsf{deg}(f_i) \leq D,$$

where D depends on the supremum of the λ_i but does not depend on i. Replacing (f_i) by a subsequence, we assume that the f_i have the same degree d (with $2 \leq d \leq D$).

The set $\mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}})$ of all birational transformations of degree d is naturally endowed with the structure of an algebraic variety (see [12]); we denote by F the Zariski closure of the set $\{f_i\}_{i\geq 0}$ in $\mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}})$. The dimension of F is positive because the λ_i are pairwise distinct; extracting a new subsequence, we assume that F is irreducible.

For each positive integer n, consider the set

$$X_n = \{g \in \mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}}) \mid \mathsf{deg}(g^n) \leq \max_i(\mathsf{deg}(f_i^n))\}.$$

By [12], this set is Zariski closed[1]; hence F is contained in X_n for all $n \geq 1$. Similarly, the set

$$Y_n = \{g \in F \mid \mathsf{deg}(g^n) \leq \max_i(\mathsf{deg}(f_i^n)) - 1\}$$

is a Zariski closed subset of F and is a strict subset of F because its complement contains at least one f_j. Since $\mathbf{k}$ is not countable, there is at least one element h in $F \setminus \cup_{n\geq 1} Y_n$. This birational transformation satisfies

$$\mathsf{deg}(h^n) = \max_i(\mathsf{deg}(f_i^n))$$

for all $n \geq 1$. This implies that $\mathsf{deg}(f_i^n) \leq \mathsf{deg}(h^n)$ for all indices i and n; in particular, $\lambda(f_i) \leq \lambda(h)$ for all i, and

$$1 < \lambda_\infty \leq \lambda(h).$$

Thus, h is a *loxodromic* transformation of degree d.

Lemma 7.3. *Let $\mathbf{k}$ be a field and $d \geq 2$ be an integer. There exists a constant $\Delta(d)$ such that, for all loxodromic elements $g \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ of degree d,*

$$\mathsf{dist}(e_0, \mathsf{Ax}(g)) \leq \Delta(d)/2, \tag{7.1}$$

$$|\log(\mathsf{deg}(g^m)) - m\log(\lambda(g))| \leq \Delta(d) \tag{7.2}$$

for all $m \geq 1$.

Proof. From the spectral gap property $\lambda(g) \geq \lambda_L = 1.176280$. Thus, hyperbolic geometry implies that $\mathsf{dist}(e_0, g_\bullet(e_0))$ goes to infinity with $\mathsf{dist}(e_0, \mathsf{Ax}(g))$; more precisely, there is a uniform constant $\delta > 0$ such that

$$\mathsf{dist}(e_0, g_\bullet(e_0)) \geq 2\mathsf{dist}(e_0, \mathsf{Ax}(g)) + \log(\lambda(g)) - \delta.$$

Since $\mathsf{dist}(e_0, g_\bullet e_0)$ is bounded from above by $\log(2\mathsf{deg}(g))$, the first upper bound follows.

The triangular inequality implies

$$\mathsf{dist}(e_0, (g^m)_\bullet(e_0)) \leq 2\mathsf{dist}(e_0, \mathsf{Ax}(g_\bullet)) + m\log(\lambda(g))$$

and hyperbolicity implies

$$\mathsf{dist}(e_0, (g^m)_\bullet(e_0)) \geq m\log(\lambda(g)) + 2\mathsf{dist}(e_0, \mathsf{Ax}(g_\bullet)) - \delta.$$

The result follows. □

Apply this lemma to h and to the f_i:

$$m\log(\lambda(h)) - \Delta(d) \leq \log(\mathsf{deg}(h^m)) \leq m\log(\lambda(h)) + \Delta(d),$$
$$m\log(\lambda(f_i)) - \Delta(d) \leq \log(\mathsf{deg}(f_i^m)) \leq m\log(\lambda(f_i)) + \Delta(d).$$

Let ϵ be a positive real number, and m be a positive integer such that $\epsilon \geq 2\Delta(d)/m$. Then, there exists i such that $\mathsf{deg}(f_i^m) = \mathsf{deg}(h^m)$, and we get

$$m\log(\lambda(h)) - \Delta(d) \leq \log(\mathsf{deg}(h^m)) = \log(\mathsf{deg}(f_i^m)) \leq m\log(\lambda(f_i)) + \Delta(d).$$

[1]With the notation of [12], consider the set $\tilde{H}_d$ of triples (p, q, r) of homogeneous polynomials of degree d such that $f : [x : y : z] \mapsto [p : q : r]$ is a birational map; let H_d be the quotient of $\tilde{H}_d$ by the equivalence relation for which two triples are equivalent if they are multiples of each other by a non-zero constant. Then $(p, q, r) \mapsto f$ is a map from H_d to the set of birational transformations of degree $\leq d$. Denote by $H_{d,d}$ the subset of H_d made of triples of (p, q, r) that give rise to a birational map of degree d exactly; this set is a Zariski open subset of H_d, and the projection $\pi_d \colon H_{d,d} \to \mathsf{Bir}_d(\mathbb{P}^2_{\mathbf{k}})$ is an isomorphism. The map $H_d \to H_{d^n}$ that applies f to f^n is a morphism. Since $\mathsf{Bir}_{\leq \ell}(\mathbb{P}^2_{\mathbf{k}})$ is closed in $\mathsf{Bir}(\mathbb{P}^2_k)$ (see Corollary 2.8 of [12]), one deduces that X_n is closed.

Hence, $\log(\lambda(h)) \leq \log(\lambda(f_\infty)) + \epsilon$. Since this inequality holds for all $\epsilon > 0$, we obtain $\lambda(h) \leq \lambda_\infty$, and thus $\lambda(h) = \lambda_\infty$, as desired.

7.4. **Increasing approximation by Salem dynamical degrees.** The set Pis is contained in the closure of the set Sal. In this section, we show that the same property holds for dynamical degrees.

Theorem 7.4. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0*. Let* β *be an element of* $\Lambda^P(\mathbb{P}^2_{\mathbf{k}})$*. There exists a strictly increasing sequence* $(\alpha_n)_{n\geq 0}$ *of elements of* $\Lambda^S(\mathbb{P}^2_{\mathbf{k}})$ *that converges toward* β*.*

Corollary 7.5. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0*, and let* X *be a projective surface defined over* $\mathbf{k}$*. The dynamical spectrum* $\Lambda(X)$ *is contained in the closure of the set of dynamical degrees* $\lambda(f)$ *of automorphisms of surfaces which are birationally equivalent to* X*.*

Proof of the corollary. If X is rational, this is a consequence of Theorem 7.4 and Theorem A. If X is not rational, all dynamical degrees are realized by dynamical degrees of automorphisms of surfaces which are birationally equivalent to X (see Section 3). □

The proof of Theorem 7.4 is given in Section 7.4.1 when β is not a reciprocal quadratic integer. The case of reciprocal quadratic integers is dealt with in Section 7.4.2.

7.4.1. *Pisot numbers which are not reciprocal quadratic integers.* Let $\beta \in \Lambda^P(\mathbb{P}^2_{\mathbf{k}})$ be a Pisot number which is not a reciprocal quadratic integer; thus, β is the dynamical degree of a birational transformation of $\mathbb{P}^2_{\mathbf{k}}$, but is not the dynamical degree of an automorphism of a rational projective surface.

Choose $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ with $\lambda(f) = \beta$, denote by d the degree of f, denote by p_i and q_i the base points of f and f^{-1} $(1 \leq i \leq m)$, and write

$$\begin{aligned} f_\bullet(e_0) &= de_0 - \sum_{i=1}^{m} a_i e(q_i), \\ f_\bullet(e(p_i)) &= d_i e_0 - \sum_{i=1}^{m} c_{i,j} e(q_j). \end{aligned}$$

The multiplicities a_i are positive integers; the $c_{i,j}$ are non-negative integers.

Say that q_i has an **infinite length** if $f^l(q_i)$ is not a base point of f for all $l \geq 0$, and say that q_i has a **finite length** (equal to ℓ_i) if $f^l(q_i)$ is not a base point of f for $0 \leq l \leq \ell_i - 1$ but $f^{\ell_i}(q_i)$ is one of the base points p_j of f. If all the base points q_i, $1 \leq i \leq m$, have a finite length, one can blow up the points $f^l(q_i)$, $1 \leq l \leq \ell_i$, to get a new surface on which f is an automorphism. Since f is not conjugate to an automorphism, at least one of the base points q_i has an infinite length.

Order the base points q_i in such a way that $q_1, \ldots, q_n$ have infinite length and q_{n+j} has finite length ℓ_{n+j} for $j = 1, \ldots, m-n$. Then, number the p_j in such a way that $p_{n+j} = f^{\ell_j}(q_{n+j})$ for all $j \geq 1$. We shall now construct a sequence of birational transformations f_k such that

- each f_k is conjugate to an automorphism;
- $\lambda(f_k)$ converges to $\lambda(f) = \beta$ as k goes to $+\infty$.

The idea is to transform the points q_i into base points of finite length ℓ_i for $i \leq n$, but with length $\ell_i = k$ going to $+\infty$.

For this purpose, define

$$\begin{aligned} A &= \{e_0\} \cup \bigcup_{i=n+1}^{m} \left(\bigcup_{j=0}^{l_i-1} \{ f_\bullet^j(e(q_i)) \} \right), \\ B_j &= \bigcup_{i=1}^{n} \{ f_\bullet^j(e(q_i)) \} \text{ for any } j \geq 0, \\ C &= \bigcup_{i=1}^{n} \{e(p_i)\}. \end{aligned}$$

The elements of these three sets are linearly independent in $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$. In particular, A is a basis of the sub-module V_A spanned by A in $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$. Similarly, B_j (resp. C) is a basis of the sub-module V_{B_j} spanned by B_j (resp. C) for all $j \geq 0$. The map $f_\bullet$ restricts to an isomorphism between $V_C \oplus V_A$ and $V_A \oplus V_{B_0}$, and also to an isomorphism

$$V_C \oplus V_A \oplus V_{B_0} \oplus \cdots \oplus V_{B_k} \xrightarrow{f_*} V_A \oplus V_{B_0} \oplus \cdots \oplus V_{B_{k+1}}.$$

Writing $V_k = V_C \oplus V_A \oplus V_{B_0} \oplus \cdots \oplus V_{B_k}$, we define a linear transformation $F_k \in \mathsf{Aut}(V_k)$ by

$$F_k = \pi_k \circ f_\bullet,$$

where $\pi_k \colon V_{k+1} \to V_k$ is the isomorphism defined by $\pi_k(f_\bullet^{k+1}(e(q_i))) = e(p_i)$ for $i = 1, \ldots, n$ and $\pi_k(x) = x$ for $x \in V_A \oplus V_{B_0} \oplus \cdots \oplus V_{B_k} = V_k \cap V_{k+1}$.

Since $f_\bullet$ preserves the intersection form of $\mathsf{Z}_{\mathbb{P}^2_{\mathbf{k}}}$, F_k preserves the intersection form of V_k. This latter space is of Minkowski type: An orthonormal basis is given by e_0 (of self-intersection $+1$) and by the other elements of A, B_j, C (each of self-intersection -1). Since $f_\bullet$ satisfies the Noether equalities (5.1), so does F_k:

$$\sum_{i=1}^{m} a_i^2 = d^2 - 1; \quad \sum_{i=1}^{m} a_i = 3d - 3.$$

For a fixed integer $k \geq 1$, denote by $r+1$ the dimension of V_k. Then, denote by W_r the subgroup of W_∞ generated by

- the finite group of permutations of the set $e(q)$, where $e(q)$ runs over the elements of $A \setminus \{e_0\}$, of B_j ($j \leq k$), and of C;
- the involution σ_0 (with base points p_1, p_2, p_3 chosen among the base points p_j);
- the involutions $\tau_{p,q}$ for $e(p)$ and $e(q)$ in the sets $A \setminus \{e_0\}$, B_j ($j \leq k$), and C.

This group is isomorphic to the Coxeter group of the Dynkin diagram $T_{2,3,r-3}$ introduced in Section 2.5. Since F_k satisfies the Noether equalities, F_k is an element of the Coxeter group W_r (this is a version of Nagata's theorem mentioned in Section 2.5; see [25] and the proof of Lemma 5.3).

By Uehara's theorem (see [42]), there is an element $f_k \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ for which $\lambda(f_k)$ is equal to the spectral radius of the linear transformation F_k; moreover, f_k is conjugate to an automorphism of a projective rational surface X_k.

Lemma 7.6. *If β is not a reciprocal quadratic integer, the sequence $(\lambda(f_k))$ converges toward β as k goes to $+\infty$, and it contains a sub-sequence that increases strictly toward β.*

This lemma concludes the proof of Theorem 7.4 when β is not a reciprocal quadratic integer. Indeed, $\lambda(f_k)$ is not a quadratic integer if k is large, because the set of reciprocal quadratic integers is discrete; hence, $(\lambda(f_k))$ contains a strictly increasing sequence of Salem numbers that converges toward β.

Proof of Lemma 7.6. Let $\lambda_k = \lambda(f_k)$. This number is the largest real eigenvalue of $F_k \in \mathsf{Aut}(V_k \otimes \mathbf{R})$.

The map $f_\bullet$ preserves $V_C \oplus V_A \bigoplus_{i=1}^{\infty} V_{B_i}$, and its matrix can be written as a matrix by blocks as follows:

$$\begin{pmatrix} 0 & 0 & 0 & & \\ M & N & 0 & & \\ P & Q & 0 & & \\ 0 & 0 & I & \ddots & \\ & & & \ddots & \end{pmatrix}.$$

Let v be an eigenvector of $f_\bullet$ with eigenvalue $\lambda(f)$; such a vector exists (and is isotropic) in $\mathcal{Z}_{P^2_{\mathbf{k}}}$. Decompose v as $v = 0 + v_A + v_{B_1} + v_{B_2} + \cdots$ with respect to the direct sum $V_C \oplus V_A \bigoplus_{i=1}^{\infty} V_{B_i}$. We obtain the system of equations

$$\begin{pmatrix} 0 \\ \lambda(f)\cdot v_A \\ \lambda(f)\cdot v_{B_1} \\ \lambda(f)\cdot v_{B_2} \\ \cdots \end{pmatrix} = \lambda(f)\cdot v = f_*(v) = \begin{pmatrix} 0 \\ Nv_A \\ Qv_A \\ v_{B_1} \\ \cdots \end{pmatrix},$$

from which we deduce that v_A is an eigenvector of $N \in \mathsf{GL}(V_A)$, and $v_{B_i} = Qv_A/\lambda(f)^i$ for $i \geq 1$. Moreover, $v_A \neq 0$ because v intersects e_0 positively. Thus, β is a root of the characteristic polynomial $\det(tI - N)$.

The matrix of F_k, acting on $V_k = V_C \oplus V_A \oplus V_{B_1} \oplus \cdots \oplus V_{B_k}$, is

$$M_{F_k} \begin{pmatrix} 0 & 0 & & & & I \\ M & N & & & & \\ P & Q & 0 & & & \\ & & I & \ddots & & \\ & & & \ddots & 0 & \\ & & & & I & 0 \end{pmatrix}.$$

Its characteristic polynomial $\det(xI - M_{F_k})$ is equal to

$$\det\begin{pmatrix} xI & 0 & & & & -I \\ -M & xI-N & & & & 0 \\ -P & -Q & xI & x^2I & \cdots & x^kI \\ & & -I & 0 & & 0 \\ & & & \ddots & \ddots & \vdots \\ & & & & -I & 0 \end{pmatrix}$$

$$= \det\begin{pmatrix} xI & 0 & -I \\ -M & xI-N & 0 \\ -P & -Q & x^kI \end{pmatrix} = \det\begin{pmatrix} 0 & 0 & -I \\ -M & xI-N & 0 \\ -P+x^{k+1}I & -Q & x^kI \end{pmatrix}.$$

Let $P(s,t)$ be the polynomial function in two variables that is defined by

$$P(s,t) = \det\left(\begin{pmatrix} 0 & 0 & -I \\ -M & tI-N & 0 \\ -sP+tI & -sQ & I \end{pmatrix}\right).$$

The characteristic polynomial of F_k is equal to $x^kP(1/x^k, x)$. Hence,

$$P(\lambda(f_k)^{-k}, \lambda(f_k)) = 0.$$

Moreover, $P(0,t) = t^l \det(tI - N)$ for some integer l. Hence, the biggest real root of the polynomial $P(0,t)$ is β, and this root is simple.

We choose real numbers β^-, β^+ with $1 < \beta^- < \beta < \beta^+$ and define δ_k to be

$$\delta_k = \max_{t\in[\beta^-,\beta^+]} |P(0,t) - P(1/t^k, t)|.$$

By construction, $\lim_{k\to\infty} \delta_k = 0$. Hence, for large k the rational function $P(1/t^k, t)$ has a real root β_k near β, and $\lim_{k\to\infty} \beta_k = \beta$. Since F_k is an element of the Coxeter group W_r, it has at most one real root bigger than 1. Hence, $\beta_k = \lambda(f_k)$.

Thus, $\lambda(f_k)$ converges toward β. Since β is a Pisot number and is not a reciprocal quadratic integer, it is not equal to the dynamical degree of an automorphism. Thus, $\lambda(f_k) \neq \beta$ for all k, and one can extract a sub-sequence from $(\lambda(f_k))$ whose members are pairwise distinct. Theorem D implies that the sequence is strictly increasing. □

7.4.2. *Reciprocal quadratic integers.* It remains to prove Theorem 7.4 for reciprocal quadratic integers.

We fix integers $m, k \geq 2$ and choose a set $\Delta \subset \mathcal{B}(\mathbb{P}^2)$ of $2m - 1 + (m-2)k$ distinct points that we denote by

$$\Delta = \{q_i\}_{i=1}^{2m-1} \cup \{a_{i,j}\}_{i=1,\dots,m-2, j=1,\dots,k}.$$

We choose $2m-1$ from these points that we write $p_1, \dots, p_{2m-1}$. These are

$$p_i = a_{i,k} \text{ for } i = 1, \dots, m-2,\ p_i = q_i \text{ for } i = m-1, \dots, 2m-1.$$

Then we construct an element $h \in W_\infty$ defined by

$$\begin{array}{rcl} h(e_0) & = & me_0 - (m-1)e(q_1) - \sum_{i=2}^{2m-1} e(q_i); \\ h(e(p_1)) & = & (m-1)e_0 - (m-2)e(q_1) - \sum_{i=2}^{2m-1} e(q_i); \\ h(e(p_i)) & = & e_0 - e(q_1) - e(q_i) \ \text{ for } i = 2, \dots, 2m-1; \\ h(e(q_i)) & = & e(a_{i,1}) \ \text{ for } i = 1, \dots, m-1; \\ h(e(a_{i,j})) & = & e(a_{i,j+1}) \ \text{ for } i = 1, \dots, m-1, j = 1, \dots, k-1; \\ h(e(r)) & = & e(r) \ \text{ for } r \in \mathcal{B}(\mathbb{P}^2) \setminus \Delta. \end{array}$$

Note that h preserves the $\mathbf{Z}$-module W generated by e_0 and the $\{e(r)\}_{r\in\Delta}$. It corresponds then to an element of the Coxeter group associated to these points. By Uehara (see [42]), there is an element $f_{m,k} \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ for which $\lambda(f_{m,k})$ is equal to the spectral radius $\lambda_{m,k}$ of the linear transformation h; moreover, $f_{m,k}$ is conjugate to an automorphism of a projective rational surface. Hence, Theorem 7.4 follows from the following lemma in the case of reciprocal quadratic integers.

Lemma 7.7. *For integers $m \geq 2$, the sequence $(\lambda_{m,k})_{k\geq 1}$ converges toward the largest root $\lambda_{m,\infty}$ of $P_m(x) = x^2 - (m+1)x + 1$, and $\lambda_{m,k}$ is a Salem number if k is large enough.*

Proof. Denote by $W' \subset W$ the sub-$\mathbf{Z}$-module whose basis is

$$e(p_1),\ \sum_{i=2}^{m-2} e(p_i),\ e_0,\ \sum_{i=m-1}^{2m-1} e(p_i) = \sum_{i=m-1}^{2m-1} e(q_i),\ e(q_1),\ \sum_{i=2}^{m-2} e(q_i),$$
$$e(a_{1,1}),\ \sum_{i=2}^{m-2} e(a_{i,1}),\ \ldots,\ e(a_{1,k-1}),\ \sum_{i=2}^{m-2} e(a_{i,k-1}).$$

Then, W' is invariant by h, and the matrix of h relative to the above basis is

$$M_h = \left(\begin{array}{rrrr|ccc|cc}
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\hline
m-1 & m-3 & m & m+1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 0 & -1 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-(m-2) & -(m-3) & -(m-1) & -(m+1) & 0 & \cdots & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & & & 1 & 0 & 0
\end{array}\right).$$

A computation similar to the one done in the proof of Lemma 7.6 shows that its characteristic polynomial $\det(xI - M_h)$ is equal to

$$\det\begin{pmatrix}
x & 0 & 0 & 0 & -1 & 0 \\
0 & x & 0 & 0 & 0 & -1 \\
-(m-1) & -(m-3) & x-m & -(m+1) & 0 & 0 \\
1 & 0 & 1 & x+1 & 0 & 0 \\
m-2 & m-3 & m-1 & m+1 & x^k & 0 \\
1 & 1 & 1 & 0 & 0 & x^k
\end{pmatrix}$$

and is therefore equal to

$$x^{2k+2}(x^2 - x(m-1) + 1) + x^{k+1}((m-1)x^2 - 4x + (m-1)) + (x^2 - (m-1)x + 1).$$

Fixing m, we see that the sequence $(\lambda_{m,k})_k$ converges toward $\lambda_{m,\infty}$ and that $\lambda_{m,k} \neq \lambda_{m,\infty}$ for k large. Each $\lambda_{m,k}$ being the spectral radius of an element in a Coxeter group W_{r_k}, it is equal to 1, to a quadratic integer, or to a Salem number. The set of quadratic integers being discrete, and $\lambda_{m,k}$ being different from $\lambda_{m,\infty}$, the $\lambda_{m,k}$ are all Salem numbers for k large enough. $\square$

8. Appendix: Modular groups, dilatations, and volumes

8.1. The Cremona group $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ acts faithfully on the hyperbolic space $\mathbb{H}_{\mathbb{P}^2_{\mathbf{k}}}$; this space contains all classes

$$\frac{1}{\sqrt{C\cdot C}}[C],$$

where C is a curve with positive self-intersection on some rational surface. Similarly, the modular group (or mapping class group) $\mathrm{Mod}(g)$ of the closed, connected, and orientable surface Σ_g of genus $g \geq 2$ acts by isometries on several metric spaces, for instance on the Teichmüller space, endowed with its Teichmüller metric.[2]

The comparison of those two isometric actions provides a fruitful analogy between $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ and $\mathrm{Mod}(g)$ for $g \geq 2$ (see [16, 17]). In this analogy, loxodromic elements $f \in \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ correspond to pseudo-Anosov classes $\varphi \in \mathrm{Mod}(g)$. The dynamical degree $\lambda(f)$ may be compared to the dilatation factor $\lambda(\varphi)$ of φ; both $\lambda(f)$ and $\lambda(\varphi)$ are algebraic numbers: The degree of $\lambda(f)$ is bounded from above by the Picard number of a surface on which f is conjugate to an algebraically stable transformation, while the degree of $\lambda(\varphi)$ is at most $6g-6$.

Theorem A may be compared to the Franks and Rykken result, according to which a pseudo-Anosov homeomorphism $\Phi\colon \Sigma_g \to \Sigma_g$ with a quadratic dilatation factor and with orientable stable and unstable foliations is semi-conjugate, via a ramified cover, to a linear automorphism of a torus (see [29]). As for birational transformations, the infimum of $\lambda(\varphi)$ when φ describes the set of pseudo-Anosov classes that are a composition of Dehn-multitwists is the Lehmer number (see [33]).

8.2. Another measure of the complexity of a pseudo-Anosov isotopy class φ is obtained as follows. According to Thurston and Mostow, the three-dimensional manifold

$$M_\varphi = (\Sigma_g \times [0,1])\,/(x,0) = (\Phi(x),1)$$

(where Φ is a diffeomorphism of Σ_g in the isotopy class φ) admits a unique hyperbolic metric (a riemannian metric of constant curvature -1). The volume of M_φ with respect to this riemannian metric is a positive real number $\mathrm{vol}(\varphi)$; this volume is, up to a bounded multiplicative error, the translation length of φ on the Teichmüller space with respect to the Weil-Petersson metric (see [13]). Jorgensen and Thurston proved that the set of all volumes $\mathrm{vol}(M)$ of all compact hyperbolic three-manifolds is infinite countable, contains accumulation points, and is well ordered (see [6]). Thus, the set $\{\mathrm{vol}(\varphi)\}$ where φ describes the set of pseudo-Anosov classes of some higher genus surface is well ordered too; moreover, this set is not discrete (consider sequences $\mathrm{vol}(\phi\circ\tau^n)$ where τ is a Dehn twist). This parallels Theorem C. Moreover, as shown in Section 7.4, accumulation points in $\Lambda(\mathbb{P}^2_{\mathbf{k}})$ are obtained by replacing orbits of base points with an infinite length by orbits with a finite length. For volumes of hyperbolic manifolds, one obtains accumulation points by Dehn fillings of cusps. Thus, cusps correspond to base points of infinite length in this dictionary.

8.3. It may also be interesting to compare our results to the description obtained by Thurston of the possible topological entropies of multimodal continuous maps of the interval $[0,1]$ into itself which are post-critically finite (see [41]). Those

[2]It also acts on the complex of curves of the surface, a metric space which is Gromov hyperbolic (see [35]).

entropies are logarithms of Perron numbers, and all Perron numbers $\lambda > 1$ are realized. Thus, in this setting, there is no gap phenomenon similar to the gaps in the dynamical spectrum $\Lambda(\mathbb{P}^2_{\mathbf{k}})$.

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