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THE PARTIAL C^0-ESTIMATE ALONG THE CONTINUITY METHOD

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1. Introduction

A fundamental problem in Kähler geometry is the existence of Kähler-Einstein metrics on Fano manifolds. Yau [47] conjectured that the existence is related to the stability of the manifold in an algebro-geometric sense. A precise notion of stability, called K-stability, was defined by Tian [41], who also showed that Kähler-Einstein manifolds are K-stable. The other direction of the conjecture was recently obtained by Chen-Donaldson-Sun [11], showing that K-stable manifolds admit Kähler-Einstein metrics. The Kähler-Einstein metrics are constructed using a continuity method suggested by Donaldson [17], passing through singular Kähler-Einstein metrics with conical singularities along a divisor. One key ingredient of the proof is establishing the partial C^0-estimate conjectured by Tian [39] for such conical metrics. In this paper we extend the techniques of Chen-Donaldson-Sun [12, 13] to obtain the partial C^0-estimate along the more classical continuity method studied by Aubin [3], which uses smooth metrics.

First we recall the Bergman kernel. Let M be a Fano manifold, fix a metric $\omega \in c_1(M)$, and choose a metric h_k on K_M^{-1} whose curvature is $k\omega$. Define the L^2 inner product on $H^0(K_M^{-k})$ given by

$$\langle s, t\rangle = \int_M \langle s, t\rangle_{h^k} \frac{(k\omega)^n}{n!}. \tag{1}$$

The Bergman kernel $\rho_{\omega,k} : M \to \mathbf{R}$ can be defined as

$$\rho_{\omega,k}(x) = \sum_{i=0}^{N_k} |s_i(x)|^2_{h^k}, \tag{2}$$

where $\{s_0, \ldots, s_{N_k}\}$ is any orthonormal basis of $H^0(K_M^{-k})$.

Now fix another Kähler form $\alpha \in c_1(M)$ and suppose that $\omega_t \in c_1(M)$ solves the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha, \tag{3}$$

for $t \in [0, T)$, with $T \leqslant 1$. Our main result is that Tian's partial C^0-estimate holds for this family of metrics.

Theorem 1. *There is an integer k_0 and a constant $c > 0$ (depending on M, α), such that $\rho_{\omega_t,k_0}(x) > c$ for all $x \in M$ and $t \in [0, T)$.*

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This is a special case of a conjecture due to Tian [36], who conjectured more generally that the same estimate holds independently of the background Kähler form α. As has been explained by Tian in many places, e.g. [41, 43], this partial C^0-estimate along the continuity method is a crucial ingredient in relating the existence of Kähler-Einstein metrics to an algebro-geometric stability notion.

The continuity method through conical Kähler-Einstein metrics alluded to above can be thought of as an analog of (3) where α is a current of integration along a divisor. The advantage of this approach is that on "most" of the manifold the metrics are Einstein, and so powerful convergence results due to Anderson [1], Cheeger-Colding [7–9], and Cheeger-Colding-Tian [10] can be applied, at the price of having a singularity along the divisor.

It seems likely that Theorem 1 can be used to give an alternative proof of the existence of Kähler-Einstein metrics on K-stable manifolds, without the use of conical metrics, by adapting more arguments from [13]. We hope to address this in future work. If instead of K-stability we use the stability notion introduced by Paul [25], then using arguments due to Tian (see e.g. [37, 41, 44]), we obtain the following whose proof will be give in Section 5.

Corollary 2. *Suppose that M is a Fano manifold with no holomorphic vector fields. Then M admits a Kähler-Einstein metric if and only if M is stable in the sense defined by Paul* [25] *for all projective embeddings using powers of K_M^{-1}.*

Note that Paul's stability can also be tested using test-configurations, just like K-stability (see [25]); however, in general the "weight" associated to a test-configuration is different in the two theories. In particular, the weight in Paul's stability notion recovers precisely the asymptotics of the Mabuchi energy along a one-parameter family in a fixed projective space, in contrast with the generalized Futaki invariant of Tian [41] and Donaldson [16] in K-stability. Indeed, in general the latter is only related to Mabuchi energy asymptotics in a fixed projective space up to a correction term (see Paul-Tian [26] and Phong-Ross-Sturm [27]).

We will now sketch the general idea in obtaining a lower bound on the Bergman kernel for a fixed metric ω. The Bergman kernel can alternatively be written as

$$\rho_{\omega,k}(x) = \sup_{s\in H^0(K_M^{-k})} \frac{|s(x)|_h^2}{\|s\|_{L^2}^2}, \tag{4}$$

so to prove a lower bound on ρ we need to construct "peaked sections" for any point x, which are sections with small L^2-norm, but large value at x. The basic idea for this goes back to Tian [38], who used Hörmander's L^2-technique to construct peaked sections (see also Siu-Yau [34] for a precursor), and in the present context proved not only that $\rho_{\omega,k}$ is bounded below for large enough k, but also gave precise asymptotics as $k \to \infty$. These were later refined by Ruan [30], Zelditch [48], Lu [22], and many others.

The idea is that once k is large, the metric $k\omega$ is well approximated by the Euclidean metric on the unit ball B near any given point p (in practice a larger ball is used, with radius small compared to $\sqrt{k}$). The metric h_k in turn is well approximated by the metric $e^{-|z|^2}$ on the trivial bundle. The peaked section at p is then constructed by gluing the constant $\mathbf{1}$ section onto M using a cutoff function, and then perturbing it to a holomorphic section. The perturbation works because the exponential decay of the section $\mathbf{1}$ ensures that after the gluing our section is still approximately holomorphic in an L^2-sense. Thus the key property of a Kähler manifold (M, ω) is that there is some scale, depending on ω, at which a neighborhood of each point is well approximated by the Euclidean ball.

If we now have a family of Fano manifolds (M_i, ω_i), and we want to find a k_0 such that $\rho_{\omega_i, k_0} > c$ for a fixed positive c, then we need to find a scale at which neighborhoods of each point in each (M_i, ω_i) are well approximated by a model. However, in general, unless we have very good control of the metrics, the model cannot just be the Euclidean ball anymore. Under certain conditions, when the limit of the (M_i, ω_i) is an orbifold, the local model can be taken to be a quotient of the Euclidean ball, and the method can still be applied (see Tian [39, 40]). For more general sequences of metrics the fundamental tool is Gromov compactness [20], and the structure theorems of Cheeger-Colding [7–9] and Cheeger-Colding-Tian [10] on the Gromov-Hausdorff limits of manifolds with Ricci curvature bounded from below. Roughly speaking the consequence of this theory is that for suitable families of manifolds (M_i, ω_i), there is a fixed scale, independent of i, at which a neighborhood of each point is well approximated by a cone $C(Y) \times \mathbf{C}^{n-k}$, where $C(Y)$ is the metric cone over a $(2k-1)$-dimensional metric space Y. Under suitable conditions one can glue a section $\mathbf{1}$ with exponential decay over such cones onto the manifolds M_i, to obtain sections which are bounded away from zero at any given point.

This approach was first used by Donaldson-Sun [18] in the context of a family of Kähler-Einstein manifolds, proving the result analogous to Theorem 1 in this case (see Phong-Song-Sturm [28] for an extension of these ideas to Kähler-Ricci solitons). In the work of Chen-Donaldson-Sun [11] on the Yau-Tian-Donaldson conjecture, a key role is played by the extension of this result to Kähler-Einstein manifolds which have conical singularities along a divisor.

In Section 2 we review some of the results from the theory of Cheeger-Colding, and Chen-Donaldson-Sun that we will use. In Section 3 we will focus on obtaining estimates for solutions of (3) for $t < T$, with $T < 1$. This corresponds to Chen-Donaldson-Sun's paper [12] on conical metrics with the limiting cone angle being less than 2π. Finally in Section 4, we deal with the case $T = 1$ in parallel with [13] on the case when the cone angle tends to 2π.

2. Background

We first give a very quick review of the convergence theory for manifolds with Ricci curvature bounded from below. Recall that the Gromov-Hausdorff distance $d_{GH}(X, Y)$ between two compact metric spaces is the infimum of all $\delta > 0$, such that there is a metric on the disjoint union $X \sqcup Y$ extending the metrics on X, Y such that both X and Y are δ-dense. If a sequence of compact metric spaces X_i converge to X_∞ in the Gromov-Hausdorff sense, then we will always assume that we have chosen metrics d_i on $X_i \sqcup X_\infty$ extending the given metrics, such that both X_i and X_∞ are δ_i-dense, with $\delta_i \to 0$.

Suppose now that we have a sequence of n-dimensional Kähler manifolds (M_i, ω_i) satisfying

(1) $\mathrm{Ric}(\omega_i) \geqslant 0$,
(2) a non-collapsing condition $\mathrm{V}(B(p_i, 1)) > c > 0$, and
(3) a uniform diameter bound $\mathrm{diam}(M_i, \omega_i) < C$.

From Gromov's compactness theorem [20] it follows that up to passing to a subsequence, the (M_i, ω_i) converge to a compact metric space (Z, d_Z).

For each $p \in Z$, and any sequence $r_k \to 0$, we can consider the sequence of pointed metric spaces $(Z, p, r_k^{-1} d_Z)$. The pointed version of Gromov's theorem implies that up to choosing a subsequence we can extract a limit $(Z_p, p_\infty, d_\infty)$, which by the results of Cheeger-Colding [7–9] is a metric cone $C(Y)$ over a $2n-1$-dimensional metric space Y. It is called a tangent cone to Z at p, and in general such tangent cones are not unique. The points in Z can be classified according to how "singular" their tangent cone is, giving rise to a stratification (we use the convention of [12] for the subscripts):

$$S_n \subset S_{n-1} \subset \cdots \subset S_1 = S \subset Z. \tag{5}$$

Here S_k consists of those points where there is no tangent cone of the form $\mathbf{C}^{n-k+1} \times C(Y)$. For instance at points in $S_1 \setminus S_2$ there must exist a tangent cone of the form $\mathbf{C}^{n-1} \times \mathbf{C}_\gamma$, where $\mathbf{C}_\gamma$ is the flat two-dimensional cone with cone angle $2\pi\gamma$. A key result due to Cheeger-Colding is that the Hausdorff codimension of S_k is at least $2k$. The fact that in the Kähler case only even dimensional cones occur, and complex Euclidean factors can be split off, is due to Cheeger-Colding-Tian [10].

The regular part $Z_{reg} \subset Z$ is defined to be $Z \setminus S$, and at each point of Z_{reg} every tangent cone is $\mathbf{C}^n$. This follows from results of Colding [14], which in effect say that in the presence of non-negative Ricci curvature, the Gromov-Hausdorff distance of a unit ball to the Euclidean unit ball is comparable to the difference in volumes of the two balls. An important improvement of this is obtained in the presence of an upper bound on $\mathrm{Ric}(\omega)$, using a result due to Anderson [1] (see Theorem 10.25 in Cheeger [6]).

Proposition 3. *There are constants $\delta, \theta > 0$ depending on $K > 0$ with the following property. Suppose that $B(p,1)$ is a unit ball in a Riemannian manifold with bounds $0 \leqslant \mathrm{Ric}(g) \leqslant Kg$ on its Ricci curvature.*

If $d_{GH}(B(p,1), B^{2n}) < \delta$, then for each $q \in B(p, \frac{1}{2})$, the ball $B(q,\theta)$ is the domain of a harmonic coordinate system in which the components of ω satisfy

$$\begin{aligned} &\frac{1}{2}\delta_{jk} < g_{jk} < 2\delta_{jk} \\ &\|g_{jk}\|_{L^{2,p}} < 2, \ \textit{for all } p. \end{aligned} \tag{6}$$

In the Kähler setting, the complex structure J can also be assumed to be close to the Euclidean complex structure J_0 in $C^{1,\alpha}$, using that it is covariant constant. For sufficiently small δ one can then construct holomorphic coordinates with respect to J which are close to the Euclidean coordinates in $C^{2,\alpha}$, using the families' version of Newlander-Nirenberg's interability result [23]. It follows that in the Kähler case we can assume that we have holomorphic coordinates on the ball $B(q,\theta)$ satisfying the properties (6) above, replacing θ by a smaller constant if necessary. The fact that we obtain holomorphic coordinates on a ball of a uniform size (once the complex structure J is sufficiently close to J_0) follows from the method of proof in [23] and is made more explicit in the sharper results of Hill-Taylor [21]; see for instance Proposition 4.1.

It follows from this result that in the setting above, where $(M_i, \omega_i) \to (Z, d_Z)$ in the Gromov-Hausdorff sense, in the presence of a 2-sided Ricci curvature bound we have the following:

- the regular set $Z_{reg} \subset Z$ is open, and
- the metrics ω_i converge in $C^{1,\alpha}$, uniformly on compact sets, to a Kähler metric ω_Z on Z_{reg} inducing the distance d_Z.

The same ideas also apply to scaled limit spaces, so in particular to tangent cones.

One of the key points in [12, 13] is to show that in their situation, even though there is not a two-sided Ricci curvature bound, the regular set is still open, and one obtains L^p convergence of the metrics on the regular part of limit spaces, for all p. This is also what our main goal is going to be.

We now review the construction in Donaldson-Sun [18] that we will need. First recall the following definition.

Definition 4. *Suppose that (Z, d_Z) is a Gromov-Hausdorff limit of Kähler manifolds as above, and $C(Y)$ is a tangent cone at $p \in Z$. We say that the tangent cone $C(Y)$ is good, if the following hold:*

1. *the regular set Y_{reg} is open in Y, and smooth,*
2. *the distance function on $C(Y_{reg})$ is induced by a Ricci flat Kähler metric,*
3. *for all $\eta > 0$ there is a Lipschitz function g on Y, equal to 1 on a neighborhood of the singular set $S_Y \subset Y$, supported on the η-neighborhood of S_Y and with $\|\nabla g\|_{L^2} \leqslant \eta$.*

With these definitions the following is the consequence of the main construction in [18] (see also [12, Proposition 7]).

Proposition 5. *Suppose that (M_i, ω_i) are as above (with non-negative Ricci curvature, non-collapsed, and with bounded diameter), and they converge to a limit space (Z, d_Z). Suppose that each tangent cone to Z is good and that the metrics (after rescaling) converge in L^p on compact sets of the regular part of the tangent cones (for some $p > 2n$) to the Ricci flat Kähler metric.*

Then for each $p \in Z$ there are $b(p), r(p) > 0$ and an integer $k(p)$ with the following property. There is a $k \leqslant k(p)$ such that for sufficiently large i there is a holomorphic section s of $K_{M_i}^{-k}$ with L^2-norm 1, and with $|s(x)| \geqslant b(p)$ for all $x \in M_i$ with $d_i(p, x) < r(p)$.

A compactness and contradiction argument in [18] then implies the lower bound for the Bergman kernel that we need in Theorem 1. With this said, our main goal is to prove the following.

Theorem 6. *Let (M, ω_t) be solutions of Equation 3 for $t \in [0, T)$, with $T \leqslant 1$. Suppose that a sequence (M, ω_{t_i}) converges to a limit space (Z, d_Z). Then each tangent cone of Z is good, and the metrics converge in L^p on compact subsets of the regular part of each tangent cone, for all p.*

As in [12, 13] the problem is somewhat different in the cases when $T < 1$ and $T = 1$, and we will deal with these separately in Section 3 and Section 4.

An important part of our approach is to treat α in (3) as a Kähler metric as well, and use that α is fixed. In a holomorphic chart we will get control of α through the following ϵ-regularity result (see Ruan [31]).

Proposition 7. *Suppose that α is a Kähler metric on an open subset $U \in \mathbf{C}^n$ such that $|\mathrm{Rm}(\alpha)| < K$. There are constants $\epsilon_0, C > 0$ depending on K, such that if $B_r^{Euc}(x) \subset U$ and*

$$r^{2-2n} \int_{B_r^{Euc}(x)} \alpha \wedge \omega_{Euc}^{n-1} < \epsilon_0, \tag{7}$$

then

$$\sup_{B^{Euc}_{r/2}(x)} \mathrm{tr}(\alpha) < Cr^{-2n} \int_{B^{Euc}_r(x)} \alpha \wedge \omega^{n-1}_{Euc} < C\frac{\epsilon_0}{r^2}, \tag{8}$$

where tr *denotes the trace with respect to* ω_{Euc}.

Proof. This follows from the ϵ-regularity for harmonic maps (Schoen-Uhlenbeck [32]), applied to the identity map $(U, \omega_{Euc}) \to (U, \alpha)$. The result also follows from the argument in Proposition 17. Indeed when B is a Euclidean ball, then the hypotheses are all satisfied, using also the monotonicity of $V(q, r)$ in that case. □

Following Chen-Donaldson-Sun [13] we introduce the invariant

$$I(\Omega) = \inf_{B(x,r)\subset\Omega} VR(x, r) \tag{9}$$

for any domain Ω, where $VR(x, r)$ is the ratio of the volumes of the ball $B(x, r)$ and the Euclidean ball rB^{2n}. Note that by Colding's volume convergence result, together with Bishop-Gromov volume comparison, if B is any unit ball in a manifold with non-negative Ricci curvature, then $1 - I(B)$ is controlled by the Gromov-Hausdorff distance $d_{GH}(B, B^{2n})$, which controls $d_{GH}(\rho B, \rho B^{2n})$ for any $\rho < 1$. We will need to obtain several variants of Proposition 7, where the crucial monotonicity property of the quantity in (7) is missing. Instead, the monotonicity of $I(\Omega)$ will be used.

3. The case $T < 1$

Suppose that α is a fixed Kähler form on M, and $0 < T_0 < T_1 < 1$. Suppose that for some $t \in (T_0, T_1)$ we have

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1 - t)\alpha. \tag{10}$$

To study the structure of the Gromov-Hausdorff limit of a sequence of such ω_{t_i} we need to study small balls with respect to these metrics, scaled to unit size. A scaling $\widetilde{\omega}_t = \Lambda\omega_t$ satisfies

$$\mathrm{Ric}(\widetilde{\omega}_t) = \Lambda^{-1}t\widetilde{\omega}_t + (1 - t)\alpha. \tag{11}$$

Because of this, in this section we work with a unit ball $B = B(p, 1)$ in a Kähler manifold with metric ω, such that

$$\mathrm{Ric}(\omega) = \lambda\omega + \alpha, \tag{12}$$

where $\lambda \in (0, 1]$ and α is a Kähler form on $B(p, 1)$ which we control in the following sense. There is a number $K > 0$, such that on each α-ball $B_\alpha \subset B(p, 1)$ of radius at most K^{-1} (in the α-metric) we have holomorphic coordinates, relative to which

$$\begin{gathered} \frac{1}{2}\delta_{j\bar{k}} < \alpha_{j\bar{k}} < 2\delta_{j\bar{k}}, \\ \|\alpha_{j\bar{k}}\|_{C^2} < K. \end{gathered} \tag{13}$$

In addition we can assume that $B(p, 1)$ is non-collapsed, i.e., $\mathrm{V}(B(p, 1)) > c > 0$. Note that this ball B is a scaled up version of a small ball with respect to a metric ω_t along the continuity method. We obtain the above estimates for any such ball with c, K depending on T_0, T_1, as long as $t \in (T_0, T_1)$.

One of the key new results is the following, which essentially shows that at points in the regular set Z_{reg} we are in the setting of bounded Ricci curvature.

The analogous result for conical Kähler-Einstein metrics, [12, Proposition 3], is straightforward because $I(B)$ is bounded away from 1 by a definite amount for balls centered at a conical singularity. The proof of the result is reminiscent of arguments in Carron [5].

Proposition 8. *There is a $\delta > 0$, depending on K above, such that if $1-I(B) < \delta$, then*

$$|\mathrm{Ric}(\omega)| < 5, \qquad on\ \frac{1}{2}B. \tag{14}$$

Proof. We will bound α, which is equivalent to bounding $\mathrm{Ric}(\omega)$. Suppose that

$$\sup_B d_x^2 |\alpha(x)|_\omega = M, \tag{15}$$

where d_x denotes the distance to the boundary of the ball, and suppose that the supremum is achieved at $q \in B$. If $M > 1$, we can consider the ball

$$B\left(q, \frac{1}{2} d_q M^{-1/2}\right), \tag{16}$$

scaled to unit size $\widetilde{B}$ with metric $\widetilde{\omega} = 4Md_q^{-2}\omega$. By construction, we have

$$\begin{aligned} |\alpha|_{\widetilde{\omega}} &\leqslant 1 \text{ on } \widetilde{B}, \\ |\alpha(q)|_{\widetilde{\omega}} &= \frac{1}{4}. \end{aligned} \tag{17}$$

In particular, we have $\alpha \leqslant \widetilde{\omega}$ on $\widetilde{B}$, and using the equation satisfied by $\widetilde{\omega}$ we have $|\mathrm{Ric}(\widetilde{\omega})| < 2$ on $\widetilde{B}$. At the same time we have a unit vector v at q (with respect to $\widetilde{\omega}$), such that

$$\alpha(v, \bar{v}) \geqslant \frac{1}{16n}; \tag{18}$$

otherwise the norm of $\alpha(q)$ would be too small.

From the bound on $\mathrm{Ric}(\widetilde{\omega})$ on $\widetilde{B}$, and Anderson's result, we have holomorphic coordinates $\{z^1, \ldots, z^n\}$ on the ball $\theta\widetilde{B}$, with respect to which the components of $\widetilde{\omega}$ are controlled in $C^{1,\alpha}$. We can assume that $v = \partial_{z^1}$ at q. We can also assume without loss of generality that $K^{-1} < \theta$, so we have good holomorphic coordinates $\{w^1, \ldots, w^n\}$ for α on $K^{-1}\widetilde{B}$ (meaning that the components of α are controlled in C^2 as in (13)), since this is contained in a K^{-1}-ball with respect to α. The w^i are holomorphic functions of the z^i, and $|w_i| < 2K^{-1}$, so we obtain bounds on the derivatives $\partial w^i / \partial z^j$ in $\frac{1}{2}K^{-1}\widetilde{B}$. The upshot is that we can find a ball of a definite size $\rho\widetilde{B}$, on which

$$\alpha(\xi, \overline{\xi}) \geqslant \frac{1}{40n} \tag{19}$$

for any unit vector ξ with angle $\angle(\xi, \partial_{z^1}) < \pi/4$ (let us identify $T^{1,0}M$ with TM in the usual way). Making ρ smaller if necessary, we can also assume that on $\rho\widetilde{B}$, if $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = 0$ and $\angle(\dot{\gamma}(0), \partial_{z^1}) < \pi/8$, then $\angle(\dot{\gamma}(t), \partial_{z^1}) < \pi/4$ for $t < \rho$. Indeed, we have

$$\frac{d}{dt}\langle \dot{\gamma}(t), \partial_{z^1}\rangle = \langle \dot{\gamma}(t), \nabla_{\dot{\gamma}(t)} \partial_{z^1}\rangle, \tag{20}$$

and the covariant derivative of ∂_{z^1} is bounded since in the z^i coordinates the components of $\widetilde{\omega}$ are controlled in $C^{1,\alpha}$. It follows that we have a uniform bound on the derivative of the angle $\angle(\dot{\gamma}(t), \partial_{z^1})$ along the geodesic γ.

Since $\mathrm{Ric}(\widetilde{\omega}) > \alpha$, this gives a positive lower bound on the radial component of $\mathrm{Ric}(\widetilde{\omega})$ in these directions, and we can apply the Bishop-Gromov volume comparison to the corresponding spherical sector in $\rho\widetilde{B}$ (in the relevant Bochner formula, the lower bound on the Ricci curvature is only required in the radial directions). From this we obtain that the volume ratio

$$VR(\rho\widetilde{B}) < 1 - \epsilon \tag{21}$$

for some small (but definite) $\epsilon > 0$, which is a contradiction if we choose δ small enough in our assumptions. It follows that if δ is sufficiently small, then $M \leqslant 1$, which implies that $|\alpha|_\omega < 4$ in $\frac{1}{2}B$, and so $\alpha < 4\omega$ there. It follows that $\mathrm{Ric}(\omega) < 5\omega$ on $\frac{1}{2}B$. ☐

This proposition, together with Anderson's result, implies the following.

Proposition 9. *Suppose that we have a sequence of unit balls $B(p_i, 1)$, with ω_i, α_i satisfying the same conditions as ω, α above. Suppose that the $B(p_i, 1)$, with the metrics ω_i converge to the Euclidean ball B^{2n} in the Gromov-Hausdorff sense. Then the convergence is $C^{1,\alpha}$ on compact subsets.*

Proposition 10. *If $B_i(p_i, 1) \to Z$ in the Gromov-Hausdorff sense, then the regular set in Z is open, and the convergence on the regular set is locally $C^{1,\alpha}$.*

Next we examine the situation when $B_i(p_i, 1) \to Z$, with $p_i \to p$, and a tangent cone at $p \in Z$ is of the form $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$, where $\mathbf{C}_\gamma$ denotes the flat cone with cone angle $2\pi\gamma$. It follows from the previous proposition that a sequence of scaled balls converges to $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ in $C^{1,\alpha}$, uniformly on compact sets away from the singular set. The results of Chen-Donaldson-Sun [12] about good tangent cones can therefore be applied. In particular, the arguments in [12, Section 2.5] imply that for sufficiently large k, with i large enough (depending on k), we can regard $k\omega_i$ as a metric on the unit ball $B^{2n} \subset \mathbf{C}^n$. We use co-ordinates $(u, v_1, \ldots, v_{n-1})$ and let η_γ be the conical metric

$$\eta_\gamma = \sqrt{-1}\frac{du \wedge d\overline{u}}{|u|^{2-2\gamma}} + \sqrt{-1}\sum_{i=1}^{n-1} dv_i \wedge d\overline{v}_i. \tag{22}$$

The metric $\widetilde{\omega}_i = k\omega_i$ then satisfies, for some fixed constant C:

1. $\widetilde{\omega}_i = \sqrt{-1}\partial\overline{\partial}\phi_i$ with $0 \leqslant \phi_i \leqslant C$,
2. $\omega_{Euc} < C\widetilde{\omega}_i$,
3. Given δ and a compact set K away from $\{u = 0\}$, we can suppose (by taking i large once k is chosen sufficiently large) that $|\widetilde{\omega}_i - \eta_\gamma|_{C^{1,\alpha}} < \delta$ on K.

Proposition 11. *There are $0 < \gamma_1 < \gamma_2 < 1$ with the following property. If $B_i(p_i, 1) \to Z$, with $p_i \to p$, and a tangent cone at p is $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$, then $\gamma \in (\gamma_1, \gamma_2)$.*

Proof. The existence of γ_1 follows from the volume non-collapsing assumption. For the upper bound we use Proposition 8 and the fact that the unit balls in $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ converge to the Euclidean ball in the Gromov-Hausdorff sense as $\gamma \to 1$. In particular, if $B_i(p_i, 1) \to Z$ and a tangent cone at $p \in Z$ is $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ for γ

sufficiently close to 1, then for large i we can find small balls around p_i, which when scaled to unit size are sufficiently close to the Euclidean ball for Proposition 8 to imply that the Ricci curvature is bounded. In this case, however, there cannot be singularities of the form $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ in the Gromov-Hausdorff limit according to Cheeger-Colding-Tian [10]. This is a contradiction. □

Proposition 12. *There is a constant $c_0 > 0$ with the following property. If $B_i(p_i, 1) \to Z$, with $p_i \to p$, and a tangent cone at p is $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ with $\gamma \in (\gamma_1, \gamma_2)$ from the previous proposition, then*

$$\liminf_{i\to\infty} r^{2-2n} \int_{B_i(p_i,r)} \alpha_i \wedge \omega_i^{n-1} > c_0, \tag{23}$$

for all $r < 1$.

Proof. Note that this would follow from the ϵ-regularity, if we knew that the expression in (23) is non-decreasing with r (a fact which is true in Euclidean space). Here we argue by contradiction. By Gromov compactness, scaling up the balls, and a diagonal argument, we will then have a sequence $B_i(p_i, 1) \to Z$, with $p_i \to p$, and with a tangent cone at p that is $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$, such that

$$\liminf_{i\to\infty} \int_{B_i(p_i,1)} \alpha_i \wedge \omega_i^{n-1} = 0. \tag{24}$$

For sufficiently large i, a sufficiently small ball $B_i(p_i, r_0)$, scaled to unit size, will be Gromov-Hausdorff close to the unit ball in the cone. By the discussion before Proposition 11 the scaled up metric $r_0^{-2}\omega_i$ can then be thought of as a metric on the Euclidean ball B^{2n} such that

$$\omega_{Euc} < C r_0^{-2} \omega_i. \tag{25}$$

If, with ϵ_0 from Proposition 7,

$$\int_{B^{2n}} \alpha_i \wedge \omega_{Euc}^{n-1} < \epsilon_0, \tag{26}$$

then $\alpha_i < C'\omega_{Euc} < C'Cr_0^{-2}\omega_i$ on $\frac{1}{2}B^{2n}$. If this happens for a subsequence of the i tending to infinity, then these ω_i have uniformly bounded Ricci curvature on these balls, and so by [10], no conical singularity can form. Therefore, for all sufficiently large i we have

$$\int_{B^{2n}} \alpha_i \wedge \omega_{Euc}^{n-1} \geqslant \epsilon_0, \tag{27}$$

and so

$$\int_{B(p_i,r_0)} \alpha_i \wedge \omega_i^{n-1} \geqslant \epsilon_1, \tag{28}$$

for some $\epsilon_1 > 0$ (depending on r_0 but not on i). This contradicts (24). □

Proposition 13. *There is a constant $A > 0$, such that if $B(p, 1)$ is sufficiently close to either the Euclidean unit ball or the unit ball in a cone $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$ with $\gamma \in (\gamma_1, \gamma_2)$, then*

$$\int_{B(p,\frac{1}{2})} \alpha \wedge \omega^{n-1} < A. \tag{29}$$

Proof. For the case of the Euclidean ball this follows from Proposition 8. For the cones, using Gromov compactness, it is enough to get a bound A for a fixed cone angle γ, and this will imply a uniform bound for $\gamma \in (\gamma_1, \gamma_2)$, and even for $\gamma \in (\gamma_1, 1]$.

We first show that

$$\int_{B(p,\frac{1}{2})} \mathrm{Ric}(\omega) \wedge \omega_{Euc}^{n-1} < A \tag{30}$$

for some constant A, once $B(p, 1)$ is sufficiently close to the unit ball in $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$.

After a further scaling by a fixed factor, we can assume that we are in the situation described before Proposition 11, and so ω can be thought of as a metric on the Euclidean ball. Because of the scaling, this will only bound the integral in (30) on a smaller ball, but we can cover $B(p, \frac{1}{2})$ with balls each of which is, up to scaling, very close to the unit ball in the cone, or the Euclidean unit ball. Adding up the contributions we will obtain (30).

Let us write

$$G = \frac{\omega^n}{\omega_{Euc}^n}, \tag{31}$$

so that

$$\begin{aligned} \mathrm{Ric}(\omega) &= -\sqrt{-1}\partial\overline{\partial} \log G, \\ \mathrm{Ric}(\omega) \wedge \omega_{Euc}^{n-1} &= -\frac{1}{n} \Delta \log G \, \omega_{Euc}^n, \end{aligned} \tag{32}$$

with Δ denoting the Euclidean Laplacian. We then have

$$\begin{aligned} \int_{B_r} \mathrm{Ric}(\omega) \wedge \omega_{Euc}^{n-1} &= -\frac{1}{n} \int_{\partial B_r} \nabla_n \log G \, dS \\ &= -\frac{1}{n} V(\partial B_r) ⨍_{\partial B_r} \nabla_n \log G \, dS \\ &= -\frac{1}{n} V(\partial B_r) \frac{d}{dr} ⨍_{\partial B_r} \log G \, dS, \end{aligned} \tag{33}$$

where everything is computed using the metric ω_{Euc}, and B_r denotes the Euclidean ball of radius r. In particular, the average $⨍_{\partial B_r} \log G \, dS$ is decreasing with r.

There is an $\epsilon > 0$ depending on the lower bound on the cone angle γ, such that the Euclidean ball B_ϵ is contained in the ball of radius $1/20$ in the conical metric. By volume convergence we have an upper bound on the volume of this ball with respect to ω. In sum we have

$$⨍_{B_\epsilon} G \, \omega_{Euc}^n < C_1, \tag{34}$$

so there is a point $q \in B_\epsilon$ satisfying $G(q) < C_1$. By translating our ball slightly, we can assume that $q = 0$. Define now the function

$$f(r) = ⨍_{\partial B_r} \log G \, dS. \tag{35}$$

We know that $f(r)$ is decreasing, and $f(0) = \log G(0) < C_1$.

Using that $\omega_{Euc} < C\omega$, we have a lower bound on G, and so, using volume convergence as well,

$$\int_{B_{3/4}\setminus B_{1/2}} \log G\, \omega^n > -C_2, \tag{36}$$

for some constant C_2. We have

$$\begin{aligned} -C_2 < \int_{B_{3/4}\setminus B_{1/2}} \log G\, \omega^n &= \int_{1/2}^{3/4} V(\partial B_r) \fint_{\partial B_r} \log G\, dS\, dr \\ &= \int_{1/2}^{3/4} V(\partial B_r) f(r)\, dr. \end{aligned} \tag{37}$$

In this range of r we have $c_0 < V(\partial B_r) < c_1$ for some fixed numbers c_0, c_1, and the fact that $f(0) < C$ and that f is decreasing implies that $f(\frac{1}{2}) < C$. It is then clear that we must have some $r \in \left(\frac{1}{2}, \frac{3}{4}\right)$ where we have a lower bound for $f'(r)$, and this, together with (33), implies (30).

Finally to obtain (30) with ω_{Euc} replaced by ω we can use the Chern-Levine-Nirenberg argument as in the proof of [12, Proposition 15], since we have a bound on the Kähler potential.

□

Suppose now that we have a sequence of balls B_i satisfying our hypotheses, and $B_i \to Z$ in the Gromov-Hausdorff sense. We can follow the arguments in Chen-Donaldson-Sun [12], Section 2.8, to show that all tangent cones of Z are good. As in [12] it is easier to explain the proof of a slightly different result. Namely, denote the singular set of Z by

$$S(Z) = S_2(Z) \cup \mathcal{D}, \tag{38}$$

where $S_2(Z)$ denotes the complex codimension 2 singularities, and $\mathcal{D}$ denotes the points that have a tangent cone of the form $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$.

Proposition 14. *For any compact set $K \subset Z$, the set $K \cap S(Z)$ has capacity zero; i.e., there is a cutoff function as in Definition* 4 *of good tangent cones.*

Proof. The proof is essentially identical to the one in [12], just defining, for $z \in Z$ and $0 < \rho < 1$,

$$V(i, z, \rho) = \rho^{2-2n} \int_{\widetilde{B}_i(z,\rho)} \alpha_i \wedge \omega_i^{n-1}, \tag{39}$$

where

$$\widetilde{B}_i(z, \rho) = \{x \in B_i \,:\, d_i(x, z) < \rho\}, \tag{40}$$

and as usual we fix a distance function d_i on $Z \sqcup B_i$ realizing the Gromov-Hausdorff convergence. By the Gromov-Hausdorff convergence, for each $\rho > 0$, the "ball" $\widetilde{B}_i(z, \rho)$ is comparable to a ball of radius ρ in B_i for sufficiently large i (i.e., it is contained between balls of radius $\rho/2$ and 2ρ).

Proposition 12 implies that for all $z \in \mathcal{D}$ we have

$$\liminf_{i\to\infty} V(i, z, \rho) > c_0' \tag{41}$$

for some c_0'. Proposition 13 implies that for each $z \in Z \setminus S_2(Z)$ there is a $\rho > 0$ such that $V(i, z, \rho)$ is bounded independently of i. These are analogous to [12, Propositions 18 and 19], and the rest of the proof is identical to the proof of [12, Proposition 20]. □

As explained in [12] a very similar proof shows that all tangent cones in a limit space Z of a sequence of balls $B_i(p_i, 1)$ are good, proving Theorem 6 in the case $T < 1$.

4. The case $T = 1$

When $T = 1$, then we need to study non-collapsed balls $B = B(p, 1)$ with metrics satisfying

$$\mathrm{Ric}(\omega) = \lambda\omega + (1-t)\alpha, \tag{42}$$

where $t < 1$ and $\lambda \in [0, 1]$. These will be small balls in (M, ω_t) along the continuity method, scaled up to unit size. We can still assume that α satisfies the bounds (13) in suitable holomorphic coordinates, on any ball of radius K^{-1} (measured using the metric α). The issue that arises when $T = 1$ is that $(1-t)\alpha$ no longer satisfies such bounds as $t \to 1$.

We follow the arguments in Chen-Donaldson-Sun [13], and we will point out the analogies with their results. One of the main difficulties when $T = 1$ is that we cannot control the integral of α on $B(p, 1)$ even when $B(p, 1)$ is Gromov-Hausdorff close to the Euclidean ball (note that $B(p, 1)$ is a possibly very small ball in the original metric along the continuity method scaled to unit size). When $T < 1$, we could achieve this in Proposition 8 using essentially that α controlled the Ricci curvature from above and below in that case.

A crucial tool in [13] is their Proposition 1, which does not make use of the conical singularity, and applies just as well in our situation. First we recall some definitions. For a subset A in a $2n$-dimensional length space P, and for $\eta < 1$, let $m(\eta, A)$ be the infimum of those M for which A can be covered by Mr^{2-2n} balls of radius r for all $\eta \leqslant r < 1$.

For $x \in B$ and $r, \delta > 0$ a holomorphic map $\Gamma : B(x, r) \to \mathbf{C}^n$ is called an (r, δ)-chart centered at x if

- $\Gamma(x) = 0$,
- Γ is a homeomorphism onto its image,
- For all $x', x'' \in B(x, r)$ we have $|d(x', x'') - d(\Gamma(x'), \Gamma(x''))| \leqslant \delta$,
- For some fixed $p > 2n$, we have $\|\Gamma_*(\omega) - \omega_{Euc}\|_{L^p} \leqslant \delta$.

With these definitions, [13, Proposition 1] is the following.

Proposition 15. *Given M, c there are $\rho(M), \eta(M, c), \delta(M, c) > 0$ with the following effect. Suppose that $1 - I(B) < \delta$ and $W \subset B$ is a subset with $m(\eta, W) < M$, such that for any $x \in B \setminus W$ there is a $(c\eta, \delta)$-chart centered at x. Then (if the constant K in the properties of α is large enough):*

1. *There is a holomorphic map $F : B(p, \rho) \to \mathbf{C}^n$ which is a homeomorphism to its image, $|\nabla F| < K$, and its image lies between $0.9\rho B^{2n}$ and $1.1\rho B^{2n}$.*
2. *There is a local Kähler potential ϕ for ω on $B(p, \rho)$ with $|\phi|\rho^{-2} < K$.*

The results which we have to modify in [13] are their Corollary 2, and Propositions 5, 6. For any $B(q, r) \subset B$, let us define the "volume density"

$$V(q, r) = r^{2-2n} \int_{B(q,r)} \alpha \wedge \omega^{n-1}. \tag{43}$$

The following is the analog of [13, Corollary 2].

Proposition 16. *Given M, suppose that the ball B satisfies the hypotheses of Proposition 15 for some $c > 0$. There are $A, \kappa > 0$, depending on M, such that if*

$$\int_B \alpha \wedge \omega^{n-1} < \kappa, \tag{44}$$

then $\alpha < A\kappa\omega$ on $\frac{1}{3}\rho B$, where $\rho = \rho(M)$ from Proposition 15.

Proof. From Proposition 15 we know that we can think of the metric ω as a metric on the Euclidean ball $0.9\rho B^{2n}$, where we have $\omega_{Euc} < K\omega$. We also think of α as being defined on this ball, and then (44) implies that

$$(0.9\rho)^{2-2n} \int_{0.9\rho B^{2n}} \alpha \wedge \omega_{Euc}^{n-1} < C_1\kappa, \tag{45}$$

for some C_1 (which depends on ρ, and thus on M). Our bounds on α imply that its curvature is bounded, so the ϵ-regularity, Proposition 7, implies that once κ is sufficiently small, we have

$$\alpha < C_2\kappa\,\omega_{Euc} < KC_2\kappa\,\omega \text{ on } 0.45\rho B^{2n}, \tag{46}$$

for some C_2 (depending on ρ). □

Our next goal is to obtain a weak form of monotonicity of the volume density (note that $V(q, r)$ is monotone in r if ω is the Euclidean metric), which is analogous to [13, Proposition 5]. For this we first need the following variant of the ϵ-regularity result, Proposition 7, which does not use monotonicity.

Proposition 17. *There are $\delta, \epsilon > 0$ with the following properties. Suppose that $1 - I(B) \leqslant \delta$, and*

$$\sup_{B(q,r)\subset B} V(q, r) < \epsilon. \tag{47}$$

Then $\alpha \leqslant 4\omega$ on $\frac{1}{2}B$.

Proof. The proof is similar to the proof of Proposition 8. Suppose that

$$\sup_B d_x^2 |\alpha|(x) = M, \tag{48}$$

where d_x is the distance to the boundary of B, and suppose that the supremum is achieved at $q \in B$. If $M > 1$, let $\widetilde{B}$ with metric $\widetilde{\omega}$ be the ball $B(q, 0.5 d_q M^{-1/2})$ scaled to unit size. On $\widetilde{B}$ we have $\alpha \leqslant \widetilde{\omega}$, and at the same time, at the origin we have

$$\alpha \wedge \widetilde{\omega}^{n-1}(0) \geqslant \frac{1}{4n}\widetilde{\omega}^n(0). \tag{49}$$

In particular, on $\widetilde{B}$ we have a two-sided Ricci bound, so Anderson's result gives good holomorphic coordinates on $\theta\widetilde{B}$ once δ is chosen small enough. By the same argument as in the proof of Proposition 8 we find a ball $\rho\widetilde{B}$ of a definite size, on which

$$\alpha \wedge \widetilde{\omega}^{n-1} \geqslant \frac{1}{10n}\widetilde{\omega}^n, \tag{50}$$

and so

$$\rho^{2-2n} \int_{\rho\widetilde{B}} \alpha \wedge \widetilde{\omega}^{n-1} \geqslant c_1, \tag{51}$$

where we also used the non-collapsing assumption (and c_1 depends on ρ, but ρ is a fixed number). If ϵ is chosen sufficiently small, then this contradicts our assumption (47). We then must have $M \leqslant 1$, and so $\alpha \leqslant 4\omega$ on $\frac{1}{2}B$. □

Proposition 18. *Given the $\epsilon > 0$ from Proposition 17, we have $\delta, \kappa > 0$ with the following properties. Suppose that $1 - I(B) \leqslant \delta$. If $B(q,r) \subset B(p,1/2)$ and $V(q,r) \geqslant \epsilon$, then $V(q,R) \geqslant \kappa$ whenever $R > r$ and $B(q,R) \subset B(p,1/2)$.*

Proof. Note first that $V(q,r) \to 0$ as $r \to 0$ for all $q \in B$. This means that if $V(q,r) > \epsilon$ and $q \in B(p,1/2)$, then r is bounded away from zero. Let us also fix $\rho \in (0,1)$ which we will choose later, and we will initially restrict attention to pairs r, R with $r < \frac{\rho}{6}R$. We can find a point q_0 and $r_0 \leqslant \frac{\rho}{6}R_0$ with $B(q_0,R_0) \subset \overline{B}(p,1/2)$ such that

$$V(q_0, r_0) = \epsilon, \tag{52}$$

and $V(q_0,R_0)$ is minimal in the sense that $V(q,R) \geqslant V(q_0,R_0)$ for all q, R for which

- $B(q,R) \subset B(p,1/2)$,
- and $V(q,r) = \epsilon$ for some $r < \frac{\rho}{6}R$.

Let $\widetilde{B}$ denote the ball $B(q_0,R_0)$ scaled to unit size, with metric $\widetilde{\omega}$, and denote

$$\kappa = V(q_0,R_0), \tag{53}$$

so

$$\int_{\widetilde{B}} \alpha \wedge \widetilde{\omega}^{n-1} = \kappa, \tag{54}$$

and we are trying to prove a lower bound for κ. We have

$$V(q,r) < \epsilon \ \text{ for all } r < \frac{\rho}{6}R, \text{ if } V(q,R) < \kappa \text{ and } B(q,R) \subset B(p,1/2). \tag{55}$$

For any $0 < \eta < 1/2$, let

$$Z_\eta = \{x \in \widetilde{B} \,:\, r^{2-2n} \int_{\widetilde{B}(x,r)} \alpha \wedge \widetilde{\omega}^{n-1} \geqslant 2^{2-2n}\kappa, \text{for all } r \in (\eta, 1/2)\}. \tag{56}$$

This set has two properties:

- Suppose that $q \in \widetilde{B} \setminus Z_\eta$ and $x \in B(q,\eta/2)$. There is a $\tau \in (\eta, 1/2)$ such that
$$\tau^{2-2n} \int_{\widetilde{B}(q,\tau)} \alpha \wedge \widetilde{\omega}^{n-1} < 2^{2-2n}\kappa, \tag{57}$$
and so using $\widetilde{B}(x,\tau/2) \subset \widetilde{B}(q,\tau)$ we get
$$\left(\frac{\tau}{2}\right)^{2-2n} \int_{\widetilde{B}(x,\tau/2)} \alpha \wedge \widetilde{\omega}^{n-1} < \kappa. \tag{58}$$
Using (55) we then have
$$r^{2-2n} \int_{\widetilde{B}(x,r)} \alpha \wedge \widetilde{\omega}^{n-1} < \epsilon, \tag{59}$$
for all $r < \frac{\rho\eta}{12}$. In particular, we can apply Proposition 17 to the ball $B(q, \rho\eta/12)$. It follows that we have $\alpha \leqslant 4 \cdot (\rho\eta/12)^{-2}\widetilde{\omega}$ on $\widetilde{B}(q, \rho\eta/24)$. Using Anderson's result we get a $(\theta\rho\eta, \delta')$-chart centered at q, for some $\theta, \delta' > 0$, where we can make δ' arbitrarily small by choosing δ small enough.

- Using (54) we get that for any $r \in [\eta, 1)$, the set Z_η can be covered by Mr^{2-2n} balls of radius r for a universal constant M.

Using the number M in Proposition 15 we obtain a $\rho(M)$, which we fix as our choice of ρ. Feeding back M and $c = \theta\rho$ into Proposition 15 we get numbers $\eta(M,c), \delta(M,c)$. We can choose our δ so that it and δ' are less than $\delta(M,c)$. The second point above implies that we can use the set $W = Z_{\eta(M,c)}$ in Proposition 15, and so Proposition 16 applies to $\widetilde{B}$. In particular, if κ is sufficiently small, then $\alpha < A\kappa\widetilde{\omega}$ on $\frac{1}{3}\rho\widetilde{B}$, and so we have

$$r^{2-2n}\int_{\widetilde{B}(q_0,r)} \alpha \wedge \widetilde{\omega}^{n-1} < c_n A\kappa r^2 \tag{60}$$

for a dimensional constant c_n and all $r < \rho/3$. Translating back to the unscaled ball this means that

$$V(q_0, r) < c_n A\kappa \frac{R_0^2 r^2}{4}, \tag{61}$$

for all $r < (R_0\rho)/6$. If κ is too small, then this contradicts $V(q_0, r_0) = \epsilon$.

We have shown that we have a (universal) ρ with the following property. If $B(q,r) \subset B(p,1/2)$ and $V(q,r) \geqslant \epsilon$, then $V(q,R) \geqslant \kappa$ whenever $\frac{\rho}{6}R > r$ and $B(q,R) \subset B(p,1/2)$. Assume now that $V(q,R) < \kappa$, and $r < R$. If $r < \frac{\rho}{6}R$, then we know that $V(q,r) < \epsilon$; while if $r > \frac{\rho}{6}R$, then we have

$$V(q,r) < \left(\frac{R}{r}\right)^{2n-2} V(q,R) < \left(\frac{6}{\rho}\right)^{2n-2}\kappa. \tag{62}$$

Choosing κ sufficiently small we can therefore get $V(q,r) < \epsilon$ for all $r < R$. □

The following is analogous to [13, Proposition 6].

Proposition 19. *Let K be the number from Proposition* 15. *Given $A, \theta > 0$ there are $\sigma(K), \gamma(A,\theta), \delta_*(\theta) > 0$ with the following effect. Suppose that $\int_B \alpha\wedge\omega^{n-1} < A$ and that $1 - I(B) < \delta_*(\theta)$. Suppose that there is a holomorphic $F : B \to \mathbf{C}^n$ with $F(p) = 0$, which is a homeomorphism onto a domain between B^{2n} and $1.1B^{2n}$. In addition, assume $|\nabla F| < K$ and that ω has a Kähler potential with $|\phi| < K$. If $t > 1 - \gamma$, then there is a (σ,θ)-chart centered at p.*

Proof. Using the ϵ-regularity result Proposition 7, the proof is essentially the same as that in [13]. First, if A is sufficiently small, then the ϵ-regularity, as in the proof of Proposition 16, implies that α is actually bounded. We are in the situation of bounded Ricci curvature, so Anderson's result applies.

For large A we argue by contradiction, so we have a sequence of balls (B_i, ω_i) with additional Kähler forms α_i, satisfying (42), with $t_i \to 1$. By assumption we can think of the ω_i and α_i as metrics on B^{2n}, with ω_i having bounded Kähler potential, and $\omega_{Euc} < K\omega_i$. We have

$$\int_{B^{2n}} \alpha_i \wedge \omega_{Euc}^{n-1} < K^{n-1}\int_{B^{2n}} \alpha_i \wedge \omega^{n-1} < CA. \tag{63}$$

It follows that up to choosing a subsequence, the forms α_i converge weakly to a limiting current α_∞. By Siu's theorem [33], the set $V \subset B^{2n}$ where the Lelong numbers of α_∞ are at least $\epsilon_0/2$ (with ϵ_0 from Proposition 7) is an analytic subset. For any $p \in B^{2n} \setminus V$, there is a radius r_p, such that

$$r_p^{2-2n} \int_{B^{Euc}(p,r_p)} \alpha \wedge \omega_{Euc}^{n-1} \leqslant \epsilon_0/2, \tag{64}$$

and so for sufficiently large i, the same inequality holds for α_i with $\epsilon_0/2$ replaced by ϵ_0. Proposition 7 then implies that for $i > N_p$ we have

$$\sup_{B^{Euc}_{r_p/2}} \mathrm{tr}_{\omega_{Euc}} \alpha_i < C r_p^{-2}, \tag{65}$$

so for any compact subset of $B^{2n} \setminus V$ we can find a uniform bound on the Ricci curvature of the ω_i, and so by Anderson's result [1] (and using that the Euclidean r-ball contains the ball of radius $rK^{-1/2}$ in the metric ω_i) the metrics ω_i converge on $B^{2n} \setminus V$ locally in $C^{1,\alpha}$ to a limit ω_∞.

We now want to write $\alpha_i = \sqrt{-1}\partial\overline{\partial} f_i$ for all i, including $i = \infty$, so that $f_i \to f_\infty$ locally in L^1. We can do this by obtaining f_i for finite i, through the usual proof of the local $\partial\overline{\partial}$-lemma as in Griffiths-Harris [19] for instance. First we find 1-forms β_i such that $\alpha_i = d\beta_i$ using the proof of the usual Poincaré lemma, using a base point $p \in B^{2n} \setminus V$ in the argument. Since we control the α_i uniformly in a neighborhood of p, the β_i will also be controlled there. At the same time the integral bound (63) implies uniform L^1-bounds for the coefficients of β_i on B^{2n}. Now we use the proof of the $\overline{\partial}$-Poincaré lemma [19, p. 25] to obtain h_i with $\overline{\partial} h_i = \beta_i^{0,1}$. From the uniform control of β_i near p and the integral bound on B^{2n} it follows that we have uniform bounds on the h_i in a neighborhood of p. We can then take $f_i = 2\mathrm{Im}(h_i)$. The f_i are plurisubharmonic functions on B^{2n}, with uniform bounds on a neighborhood of p, so after taking a subsequence we can assume that $f_i \to f_\infty$ in L^1_{loc}, and consequently we have $\alpha_\infty = \sqrt{-1}\partial\overline{\partial} f_\infty$.

On any compact set in $B^{2n} \setminus V$ we have uniform bounds on Δf_i from (65) so on such compact sets we obtain $C^{1,\alpha}$ bounds on the f_i (independent of i). Using this, the rest of the proof in [13] can be followed closely.

Let us write $\omega_i = \sqrt{-1}\partial\overline{\partial}\phi_i$. Equation (42) can be written as

$$\det(\partial_j \overline{\partial}_k \phi_i) = e^{-\lambda_i \phi_i - (1-t_i) f_i} |U_i|^2, \tag{66}$$

where $\lambda_i \leqslant 1$ and U_i is a nowhere vanishing holomorphic function on $0.9B^{2n}$. As in [13] we can bound $|U_i|$ from above and below uniformly on a smaller ball $0.8B^{2n}$, and so on this ball we get

$$K^{-1}\omega_{Euc} \leqslant \omega_i \leqslant C' e^{-(1-t_i)f_i} \omega_{Euc}, \tag{67}$$

for some constant C'. Note that by Demailly-Kollár [15, Theorem 0.2], we have a constant $\kappa > 0$, such that $e^{-\kappa f_i} \to e^{-\kappa f_\infty}$ in L^1, over $0.8B^{2n}$. In particular, for any $q > 0$, once i is sufficiently large, we have $(1 - t_i)q < \kappa$, and so from (67) we have a uniform bound on the L^q-norm of ω_i. The $C^{1,\alpha}$ convergence of ω_i to ω_∞ away from V then implies that the ω_i converge to ω_∞ in L^p for any p. It follows that up to choosing a subsequence, the potentials ϕ_i for ω_i converge in $L^{2,p}$ to a potential ϕ_∞ for ω_∞. Up to choosing a further subsequence we can take the limit in (66) to see that ϕ_∞ is an $L^{2,p}$ solution of

$$\det(\partial_j \overline{\partial}_k \phi_\infty) = e^{-\lambda\phi_\infty} |U_\infty|^2 \tag{68}$$

on $0.7B^{2n}$ for some $\lambda \leqslant 1$ and nowhere-vanishing holomorphic function U_∞. Using Błocki [4, Theorem 2.5] this implies that ϕ_∞ is $C^{2,\alpha}$, and it follows that ω_∞ is a smooth Kähler-Einstein metric. In addition, passing to the limit in (67) and using that $t_i \to 1$, we get that the metric ω_∞ satisfies

$$K^{-1}\omega_{Euc} \leqslant \omega_\infty \leqslant C'\omega_{Euc}. \tag{69}$$

It remains to show that the ω_i converge to ω_∞ in the Gromov-Hausdorff sense on a smaller ball, since using Anderson's result, we will then obtain a (σ, θ)-chart centered at p for sufficiently large i, contradicting our assumption that no such chart exists.

For the Gromov-Hausdorff convergence, let us denote by d_i, d_∞ the distance functions induced by ω_i, ω_∞. Given $\epsilon > 0$ we will show that

$$d_i(q_1, q_2) \leqslant d_\infty(q_1, q_2) + \epsilon \tag{70}$$

for any $q_1, q_2 \in \frac{1}{4}B^{2n}$ and sufficiently large i. The converse inequality will be analogous.

For $\delta > 0$, let us denote by V_δ the Euclidean δ-neighborhood of V. Note that by (67) we have

$$\mathrm{Vol}(V_\delta, \omega_i) = \int_{V_\delta} \omega_i^n \leqslant (C')^n \int_{V_\delta} e^{-(1-t_i)nf_i}\, \omega_{Euc}^n, \tag{71}$$

and so for large enough i, by Hölder's inequality we obtain

$$\mathrm{Vol}(V_\delta, \omega_i) \leqslant \Psi(\delta), \tag{72}$$

where $\Psi(\delta)$ denotes a function which converges to zero with δ, and which we might change below. By non-collapsing, $\mathrm{Vol}(B_r(q_j, \omega_i)) > cr^{2n}$ for $j = 1, 2$, and so there is a function $\Psi(\delta)$, converging to zero with δ, such that if $q_j \in V_\delta$, then the ball $B_{\Psi(\delta)}(q_j, \omega_i)$ intersects the boundary of V_δ. In other words, there are points $q_j' \notin V_\delta$ such that $d_i(q_j, q_j') < \Psi(\delta)$ for $j = 1, 2$ and sufficiently large i. This means that we can replace q_j by $q_j' \in B \setminus V_\delta$ changing the distance $d_i(q_1, q_2)$ by only $\Psi(\delta)$. That $d_\infty(q_1, q_2)$ also only changes by $\Psi(\delta)$ follows from (67) and (69).

This means that we can assume that $q_1, q_2 \notin V_\delta$. We will use Cheeger-Colding's segment inequality, [6, Theorem 2.15], to show that for sufficiently large i we can find q_j' for $j = 1, 2$ such that $d_{Euc}(q_j, q_j') < \delta/2$, satisfying

$$d_i(q_1', q_2') \leqslant d_\infty(q_1', q_2') + \frac{\epsilon}{2}. \tag{73}$$

Note that the convergence of ω_i to ω_∞ is $C^{1,\alpha}$ outside $V_{\delta/2}$ and ω_∞ is uniformly equivalent to the Euclidean metric, so

$$d_i(q_j, q_j'),\ d_\infty(q_j, q_j') \leqslant \Psi(\delta). \tag{74}$$

It then follows from this that

$$d_i(q_1, q_2) \leqslant d_\infty(q_1, q_2) + \frac{\epsilon}{2} + \Psi(\delta), \tag{75}$$

which implies the result we want once δ is sufficiently small.

Let $g : 0.7B^{2n} \to \mathbf{R}$ be the function

$$g(x) = \sup_{\substack{v \in T_x B \\ |v|_{\omega_\infty} = 1}} \Big| |v|_{\omega_i} - |v|_{\omega_\infty} \Big|. \tag{76}$$

The L^p-convergence implies that by choosing i sufficiently large, we can make $\int_B g\,\omega_\infty^n$ arbitrarily small. Let $\gamma : [0,l] \to B^{2n}$ be a unit speed minimizing geodesic from q_1 to q_2 with respect to ω_∞. We then have

$$d_i(q_1,q_2) \leqslant d_\infty(q_1,q_2) + \int_0^l g(\gamma(\tau))\,d\tau. \tag{77}$$

The segment inequality allows us to perturb q_1, q_2 slightly such that the corresponding integral above is very small. Indeed, in the notation of [6, Theorem 2.1] we let $A_i = B_{\delta/2}(q_i)$ for $i = 1,2$ be Euclidean balls of radius $\delta/2$. Then

$$\int_{A_1\times A_2} \mathcal{F}_g(x_1,x_2) \leqslant C\Big(\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2)\Big) \int_{0.7B} g, \tag{78}$$

where the integrals and volumes are with respect to ω_∞ (which is uniformly equivalent to the Euclidean metric) and

$$\mathcal{F}_g(x_1,x_2) = \inf_\gamma \int_0^l g(\gamma(s))\,ds \tag{79}$$

is an infimum over all minimizing geodesics γ from x_1 to x_2 and l is the length of γ.

Given δ, we control the volumes of A_1 and A_2 (here these volumes only depend on δ since we are using ω_∞, but for the converse of (70) we need the volumes using the metric ω_i, which are nevertheless controlled by the L^p-convergence of ω_i to ω_∞). We can then choose i sufficiently large, so that the average of $\mathcal{F}_g$ satisfies

$$\frac{1}{\mathrm{Vol}(A_1\times A_2)} \int_{A_1\times A_2} \mathcal{F}_g(x_1,x_2) \leqslant \frac{\epsilon}{2}, \tag{80}$$

which in turn means that we can find $q_i' \in B_{\delta/2}(q_i)$ such that $\mathcal{F}_g(q_1',q_2') \leqslant \epsilon/2$. In particular,

$$d_i(q_1',q_2') \leqslant d_\infty(q_1',q_2') + \frac{\epsilon}{2}, \tag{81}$$

which is what we wanted to show. ☐

Finally we need the analog of [13, Proposition 8].

Proposition 20. *There is a $c > 0$ such that for any $\theta, A > 0$ we can find $\delta(\theta), \gamma(A,\theta)$ with the following property. If*

- $1 - I(B) \leqslant \delta$,
- $\int_B \alpha\wedge\omega^{n-1} \leqslant A$,
- $t > 1-\gamma$,

then there is a (c,θ)-chart centered at p.

Proof. The proof from [13] can be used almost verbatim. If for some $A > 0$ we have $V(p,1) \leqslant 2A$, then denote

$$Z_r = \{q \in B \,:\, V(q,r) \geqslant A\}. \tag{82}$$

If $\{rB_1, \ldots, rB_k\}$ is a maximal collection of disjoint r-balls with centers in Z_r, then $k \leqslant 2r^{2n-2}$. The balls $2rB_i$ cover Z_r, while we can cover each $2rB_i$ with a fixed number of r-balls. It follows that Z_r is covered by at most Mr^{2n-2} balls of radius r, where M is independent of r, A. Use this M in Proposition 15 to obtain $\rho = \rho(M)$.

From Proposition 19 we have a number $\sigma = \sigma(K)$ (with K from Proposition 15). Define $c = \rho\sigma$, and use M, c in Proposition 15 to obtain $\eta(M, c)$ and $\delta(M, c)$.

If A is sufficiently small, then for small enough $\delta(\theta)$ we can combine Propositions 17 and 18 with Anderson's result to get a (c, θ)-chart at p. Setting $\delta(\theta)$ even smaller, we let $\delta(\theta) < \delta(M, c)$ and $\delta(\theta) < \delta_*(\theta)$ (from Proposition 19). Let $\gamma(A, \theta)$ be the constant from Proposition 19. We say that A is good, if with these choices of constants the conclusion of Proposition 20 holds. If A is very small, then we have seen that A is good. We can also restrict ourselves to $\theta < \delta(M, c)$.

We suppose now that A is good and show that $2A$ is also good. Let $V(p, 1) \leqslant 2A$, and $W = \bigcap_{\eta \leqslant r < 1} Z_r$ with Z_r as in (82), so $m(\eta, W) < M$. If $x \in B \setminus W$, then $V(x, r) < A$ for some $r \in [\eta, 1)$. By assumption we can apply Proposition 20 to $B(x, r)$ scaled to unit size to obtain a $(c\eta, \delta(M, c))$-chart at x. This means that Proposition 15 applies, and its conclusion can be used in Proposition 19. This provides a (σ, θ)-chart at p, so $2A$ is also good. □

Given this Proposition, the argument in [13, Section 2.6] can be used verbatim to show the following. Let ω_t be metrics along the continuity method satisfying

$$\mathrm{Ric}(\omega_t) = t\mathrm{Ric}(\omega_t) + (1 - t)\alpha, \tag{83}$$

and let Z be the Gromov-Hausdorff limit of (M, ω_{t_i}) for some $t_i \to 1$. Then the regular set in Z is open, and the convergence of the metrics is locally in L^p. The same applies to any iterated tangent cone of Z. Indeed, suppose that p is a regular point in an iterated tangent cone of Z. Fix a small θ and let $\delta = \delta(\theta)$ from Proposition 20. A suitable ball B centered at p, scaled to unit size then satisfies $1 - I(B) < \delta/3$. By definition this ball is the Gromov-Hausdorff limit of a sequence of balls $B_i \subset Z$, scaled to unit size. In particular, $1 - I(B_i) < 2\delta/3$ for sufficiently large i. For a fixed such i, the ball B_i is the Gromov-Hausdorff limit of balls $B_{i,j} \subset (M, \omega_{t_j})$, with some radius r_i (the radius being independent of j). For sufficiently large j (depending on i), we will have $1 - I(B_{i,j}) < \delta$, and at the same time

$$r_i^{2-2n} \int_{B_{i,j}} \alpha \wedge \omega_{t_i}^{n-1} \leqslant r_i^{2-2n} \int_M \alpha \wedge \omega_{t_i}^{n-1} < C' r_i^{2-2n}, \tag{84}$$

for some C'. For fixed i we can therefore apply Proposition 20 to the balls $B_{i,j}$ scaled to unit size, for sufficiently large j to obtain a (c, θ)-chart at their centers. Since the original ball B in the iterated tangent cone is a limit of a sequence $B_{i,j(i)}$ where $j(i)$ can be taken arbitrarily large for any i, we obtain the required convergence on the ball cB.

Proposition 21. *In the limit space Z above, no iterated tangent cone can be of the form $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$.*

Proof. First, the discussion above means that one can use the arguments of [12, Section 2.5] to ensure that we are in the setting discussed before Proposition 11. In other words, if an iterated tangent cone of Z is $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$, then we can first find a ball in Z which scaled to unit size is very close to the unit ball in $\mathbf{C}_\gamma \times \mathbf{C}^{n-1}$, and this ball is a Gromov-Hausdorff limit of balls of some fixed radius r in (M, ω_{t_i}). After scaling up by a fixed factor the metrics $\widetilde{\omega}_i = k\omega_i$ on these small balls can then be thought of as metrics on the Euclidean ball B^{2n}, and they satisfy the properties

(1), (2), (3) before Proposition 11. Consider the set

$$V = \{(u, v_1, \ldots, v_{n-1}) \,:\, |u| \leqslant 1/4, |v_1|^2 + \cdots + |v_{n-1}|^2 \leqslant 1/4\}. \tag{85}$$

We want to bound

$$\int_V \mathrm{Ric}(\widetilde{\omega}_i) \wedge \omega_{Euc}^{n-1} \tag{86}$$

from below, and for this it is enough, for each $|\mathbf{a}|^2 \leqslant 1/4$, to bound

$$\int_{V_{\mathbf{a}}} \mathrm{Ric}(\widetilde{\omega}_i) \tag{87}$$

from below where $V_{\mathbf{a}}$ is the disk $V \cap \{(v_1, \ldots, v_{n-1}) = \mathbf{a}\}$. Using that

$$\mathrm{Ric}(\widetilde{\omega}_i) = -\sqrt{-1}\partial\overline{\partial} \log \frac{\widetilde{\omega}_i^n}{\omega_{Euc}^n}, \tag{88}$$

the integral (87) can be computed by an integral over $\partial V_{\mathbf{a}}$ of a term involving one derivative of $\widetilde{\omega}_i$. Since $\partial V_{\mathbf{a}}$ is a compact set disjoint from $\{u = 0\}$, we can assume that $\widetilde{\omega}_i$ is very close to the cone metric η_γ in $C^{1,\alpha}$, so the integral (87) can be assumed to be very close to the corresponding integral for η_γ. Unless $\gamma = 1$, this latter integral is non-zero. So for a sufficiently large scaling factor k, and large enough i, we have a lower bound

$$\int_V \mathrm{Ric}(\widetilde{\omega}_i) \wedge \omega_{Euc}^{n-1} > c_0, \tag{89}$$

which because of $\omega_{Euc} < C\widetilde{\omega}_i$ implies

$$\int_V \mathrm{Ric}(\widetilde{\omega}_i) \wedge \widetilde{\omega}_i^{n-1} > c_1, \tag{90}$$

for some other constant c_1 (depending on γ). We have

$$\mathrm{Ric}(\widetilde{\omega}_i) = k^{-1}t_i\widetilde{\omega}_i + (1 - t_i)\alpha_i \leqslant k^{-1}\widetilde{\omega}_i + (1 - t_i)\alpha_i, \tag{91}$$

where we write α_i to emphasize that the (fixed) Kähler metric α will depend on i when we are thinking of the $\widetilde{\omega}_i$ as metrics on the Euclidean ball B^{2n}. By volume convergence, we control the $\widetilde{\omega}_i$-volume of the set V, so from (90) we get

$$\int_V (1 - t_i)\alpha_i \wedge \widetilde{\omega}_i^{n-1} > \frac{c_1}{2} \tag{92}$$

once the scaling factor k is large enough (and also i is sufficiently large). But

$$\int_V \alpha_i \wedge \widetilde{\omega}_i^{n-1} \leqslant k^{n-1} \int_M \alpha \wedge \omega_{t_i}^{n-1}, \tag{93}$$

which contradicts (92) as $t_i \to 1$. □

From this result it is now clear that every tangent cone of Z is good, since the singular set in each tangent cone must have Hausdorff codimension at least 4, and so the argument of [18, Proposition 3.5] applies. This completes the proof of Theorem 6.

5. Proof of Corollary 2

We now give the proof of Corollary 2, relating Paul's version of stability with the existence of Kähler-Einstein metrics.

Proof. Paul [24] shows that under the assumption of his version of stability, for each $l > 0$, the Mabuchi energy is proper when restricted to the space of Bergman metrics, i.e., metrics obtained as pullbacks of the Fubini-Study metric under an embedding using K_M^{-l}. As explained in [44], the partial C^0-estimate allows us to compare the Mabuchi energy of the metrics ω_t with the Bergman metrics obtained using embeddings with L^2-orthonormal sections. The following argument is similar to that in Tian-Zhang [45]. Recall that if $\eta' = \eta + \sqrt{-1}\partial\overline{\partial}\phi$ are two metrics in $c_1(M)$, then the Mabuchi energy is defined by

$$\mathcal{M}(\eta, \eta') = \int_0^1 \int_M \phi(n - S(\eta_t))\, \eta_t^n \, dt, \tag{94}$$

where $\eta_t = \eta + t\sqrt{-1}\partial\overline{\partial}\phi$. Let us write $t_i \to T$ for a sequence converging to T, and let $\omega_i = \omega_{t_i}$. For the k_0 in Theorem 1 write

$$\eta_i = \omega_i + \frac{1}{k_0}\sqrt{-1}\partial\overline{\partial}\rho_{\omega_i, k_0} \tag{95}$$

for the corresponding Bergman metric, i.e., $\eta_i = \frac{1}{k_0}\Phi_i^*\omega_{FS}$, where $\Phi_i : M \to \mathbf{CP}^N$ are embeddings given by an orthonormal basis of $H^0(K_M^{-k_0})$ with respect to the metric ω_i. We can replace our sequence t_i by a subsequence to ensure that the varieties $\Phi_i(M)$ converge to a limit Z in projective space. Then Paul's result cited above implies that if M is stable, and Z is not in the $GL(N+1, \mathbf{C})$-orbit of the $\Phi_i(M)$, then $\mathcal{M}(\eta_1, \eta_i) \to \infty$.

The cocycle property of the Mabuchi energy implies that

$$\mathcal{M}(\eta_1, \eta_i) = \mathcal{M}(\eta_1, \omega_i) + \mathcal{M}(\omega_i, \eta_i), \tag{96}$$

and along the continuity method the Mabuchi energy is decreasing, so if $\mathcal{M}(\eta_1, \eta_i) \to \infty$, then necessarily $\mathcal{M}(\omega_i, \eta_i) \to \infty$. We will thus obtain a contradiction if we show that $\mathcal{M}(\omega_i, \eta_i)$ is bounded. To see this, as in [45] we can use the explicit formula for the Mabuchi energy from Tian [42] to obtain

$$\mathcal{M}(\omega_i, \eta_i) \leqslant \int_M \log \frac{\eta_i^n}{\omega_i^n}\eta_i^n + \int_M u_i(\eta_i^n - \omega_i^n), \tag{97}$$

where u_i is the Ricci potential of ω_i defined by

$$\sqrt{-1}\partial\overline{\partial}u_i = \omega_i - \mathrm{Ric}(\omega_i). \tag{98}$$

We can normalize u_i so that $\inf_M u_i = 0$. From the defining equation we have $\Delta u_i = n - S(\omega_i) \leqslant n$, which implies, through control of the Green's function, that

$$\int_M u_i\omega_i^n \leqslant C. \tag{99}$$

In addition, the partial C^0-estimate combined with a gradient estimate for holomorphic sections implies that $\eta_i < C\omega_i$ (see Donaldson-Sun [18, Lemma 4.2]). From this we easily obtain an upper bound on $\mathcal{M}(\omega_i, \eta_i)$. It follows therefore that the $\Phi_i(M)$ converge to a limit in the $GL(N+1, \mathbf{C})$-orbit of $\Phi_1(M)$, which in turn can be used to control the Kähler potentials of the η_i. The partial C^0-estimate then in turn controls the Kähler potentials of the ω_i. It follows that the continuity method can be continued past T (or we obtain a Kähler-Einstein metric when $T = 1$) using also the estimates of Yau [46] and Aubin [2].

Conversely an important result due to Tian [41] is that if M admits a Kähler-Einstein metric and has no holomorphic vector fields, then the Mabuchi energy is proper on the space of all Kähler metrics in $c_1(M)$, and this (see also Phong-Song-Sturm-Weinkove [29] for an improvement) implies the stability of M in the sense of Paul. See also the discussion in Section 6 of Tian [35] and [37]. □

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