JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, July 2016, Pages 883–891
http://dx.doi.org/10.1090/jams/840
Article electronically published on July 13, 2015

HYPERSURFACES THAT ARE NOT STABLY RATIONAL

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A fundamental problem of algebraic geometry is to determine which varieties are rational, that is, isomorphic to projective space after removing lower-dimensional subvarieties from both sides. In particular, we want to know which smooth hypersurfaces in projective space are rational. An easy case is that smooth complex hypersurfaces of degree at least $n+2$ in $\mathbf{P}^{n+1}$ are not covered by rational curves and hence are not rational.

By far the most general result on rationality of hypersurfaces is Kollár's theorem that for d at least $2\lceil (n+3)/3\rceil$, a very general complex hypersurface of degree d in $\mathbf{P}^{n+1}$ is not ruled and therefore not rational [13, Theorem 5.14]. Very little is known about rationality in lower degrees, except for cubic 3-folds and quintic 4-folds [4], [19, Chapter 3].

A rational variety is also stably rational, meaning that some product of the variety with projective space is rational. Many techniques for proving non-rationality give no information about stable rationality. Voisin made a breakthrough in 2013 by showing that a very general quartic double solid (a double cover of $\mathbf{P}^3$ ramified over a quartic surface) is not stably rational [23]. These Fano 3-folds were known to be non-rational over the complex numbers, but stable rationality was an open question. Voisin's method was to show that the Chow group of zero-cycles is not universally trivial (that is, the Chow group becomes non-trivial over some extension of the base field), by degenerating the variety to a nodal 3-fold which has a resolution of singularities X with nonzero torsion in $H^3(X,\mathbf{Z})$.

Colliot-Thélène and Pirutka simplified and generalized Voisin's degeneration method. They deduced that very general quartic 3-folds are not stably rational [6]. This was striking, in that non-rationality of smooth quartic 3-folds was the original triumph of Iskovskikh-Manin's work on birational rigidity, while stable rationality of these varieties was unknown [11]. Beauville applied the method to prove that very general sextic double solids, quartic double 4-folds, and quartic double 5-folds are not stably rational [1, 2].

In this paper, we show that a wide class of hypersurfaces in all dimensions are not stably rational. Namely, for all $d \geq 2\lceil (n+2)/3\rceil$ and $n \geq 3$, a very general complex hypersurface of degree d in $\mathbf{P}^{n+1}$ is not stably rational (Theorem 2.1). The theorem covers all the degrees in which Kollár proved non-rationality. In fact, we get a bit more, since Kollár assumed $d \geq 2\lceil (n+3)/3\rceil$. For example, very general quartic 4-folds are not stably rational, whereas it was not even known whether these varieties are rational.

Received by the editors February 12, 2015 and, in revised form, February 26, 2015 and May 27, 2015.

2010 *Mathematics Subject Classification.* Primary 14E08; Secondary 14J45, 14J70.

The method applies to some smooth hypersurfaces over $\overline{\mathbf{Q}}$ in each even degree. Section 3 gives some examples over $\mathbf{Q}$ which are not stably rational over $\mathbf{C}$.

The idea is that the most powerful results are obtained by degenerating a smooth complex variety to a singular variety in a positive characteristic, rather than to a singular complex variety. In fact, the best results arise by degenerating to characteristic 2. We find that very general hypersurfaces in the given degrees have a Chow group of zero-cycles not universally trivial, which is stronger than being not stably rational.

Kollár also proved non-rationality for several other classes of rationally connected varieties, such as many ramified covers of projective space or ramified covers of products of projective spaces [12, 14], [13, Section V.5]. The method of this paper should imply that those examples are also not stably rational.

Using Theorem 2.1, Corollary 4.1 shows for the first time that a family of rational projective varieties can specialize to a non-rational variety with Kawamata log terminal (klt) singularities. It remains unknown whether rationality specializes for smooth projective varieties, or for varieties with canonical singularities.

1. Notation

A property holds for *very general* complex points of a complex variety S if it holds for all points outside a countable union of lower-dimensional closed subvarieties of S. In particular, we can talk about properties of very general hypersurfaces of a fixed degree d in $\mathbf{P}^{n+1}_{\mathbf{C}}$, since the space S of all hypersurfaces of degree d is an algebraic variety (in fact, a projective space).

Let R be a discrete valuation ring with fraction field K and residue field k. Given a proper flat morphism $\mathcal{X} \to \operatorname{Spec}(R)$, we say that the general fiber $X = \mathcal{X} \times_R K$ *degenerates* to the special fiber $Y = \mathcal{X} \times_R k$. We also say that any base change of X to a larger field (perhaps algebraically closed) degenerates to Y.

Let X be a scheme of finite type over a field k. We say that the Chow group of zero-cycles of X is *universally trivial* if the flat pullback homomorphism $CH_0(X) \to CH_0(X_E)$ is surjective for every field E containing k. For X a smooth proper variety over k, universal triviality of CH_0 is equivalent to many other conditions: the degree map $CH_0(X_E) \to \mathbf{Z}$ is an isomorphism for every field E containing k, or X has a decomposition of the diagonal of a certain type (written out in the proof of Lemma 2.2), or X has trivial unramified cohomology with coefficients in any cycle module [17], or all Chow groups of X below the top dimension are universally supported on a divisor. A reference for these equivalences is [21, Theorem 2.1].

We use the following fact, due to Colliot-Thélène and Coray in characteristic zero and to Fulton in general [5, Proposition 6.3], [8, Example 16.1.12]. Fulton assumes that the base field is algebraically closed, but the proof works without that. These references only treat Theorem 1.1 for birational equivalence, but it follows for stable birational equivalence by the formula for the Chow groups of $X \times \mathbf{P}^n$ [8, Theorem 3.3].

Theorem 1.1. *The Chow group of zero-cycles is invariant under stable birational equivalence for smooth projective varieties over a field.*

2. Hypersurfaces

Theorem 2.1. *Let X be a very general hypersurface of degree d in $\mathbf{P}^{n+1}_{\mathbf{C}}$ with $n \geq 3$. If $d \geq 2\lceil (n+2)/3 \rceil$, then CH_0 of X is not universally trivial. It follows* *that X is not stably rational. For d even, the conclusions also hold for some smooth hypersurfaces over $\overline{\mathbf{Q}}$.*

Proof. We first prove this when the degree d is even, $d = 2a$. Then a smooth hypersurface of degree $2a$ can degenerate to a double cover of a hypersurface of degree a. We consider such a degeneration to an inseparable double cover in characteristic 2.

Explicitly, following Mori [18, Example 4.3], let R be a discrete valuation ring, and let S be the weighted projective space $\mathbf{P}(1^{n+2}a) = \mathbf{P}(x_0, \ldots, x_{n+1}, y)$ over R. Let $f, g \in R[x_0, \ldots, x_{n+1}]$ be homogeneous polynomials of degree $2a$ and a, respectively. Let t be a uniformizer for R. Let Z be the complete intersection subscheme of S defined by $y^2 = f$ and $g = ty$. Then the generic fiber of Z over R (where $t \neq 0$) is isomorphic to the hypersurface $g^2 - t^2 f = 0$ in $\mathbf{P}^{n+1}$ of degree $2a$, whereas the special fiber (where $t = 0$) is a double cover of the hypersurface $g = 0$ in $\mathbf{P}^{n+1}$ of degree a. We consider the case where the residue field k of R has characteristic 2; then the special fiber Y is an inseparable double cover of $\{g = 0\}$.

Assume that the residue field k is algebraically closed. Using only that $a \geq 2$, Kollár showed that for general polynomials f and g, the singularities of Y are etale-locally isomorphic to $0 = y^2 + x_1x_2 + x_3x_4 + \cdots + x_{n-1}x_n + f_3$ if n is even, or to $0 = y^2 + x_1^3 + x_2x_3 + x_4x_5 + \cdots + x_{n-1}x_n + f_3$ if n is odd [13, proof of Theorem V.5.11]. Here $f_3 \in (x_1, \ldots, x_n)^3$ and, for n odd, the coefficient of x_1^3 in f_3 is zero.

Then one computes that simply blowing up the singular points gives a resolution of singularities $Y' \to Y$. Moreover, each exceptional divisor of this resolution is isomorphic to a quadric Q^{n-1} over k, which is smooth if n is even and is singular at one point if n is odd. This quadric is a smooth projective rational variety over k for n even, and it is a projective cone for n odd. By Theorem 1.1 (for n even), the Chow group CH_0 of each exceptional divisor of $Y' \to Y$ is universally trivial. Therefore, for every extension field E of k, the pushforward homomorphism $CH_0 Y'_E \to CH_0 Y_E$ is an isomorphism.

For a smooth n-fold X over a field, the *canonical bundle* is the line bundle $K_X = \Omega^n_X$. Let M be the pullback to Y' of the line bundle $K_{\{g=0\}} \otimes O(a)^{\otimes 2} \cong O(-n-2+3a)$ on the hypersurface $\{g = 0\} \subset \mathbf{P}^{n+1}$. Since $a \geq \lceil (n+2)/3 \rceil$, we have $H^0(Y', M) \neq 0$. Next, for $n \geq 3$, Kollár computed that there is a nonzero map from the line bundle M to $\Omega^{n-1}_{Y'}$ [13, proof of Theorem V.5.11]. In particular, it follows that the n-fold Y' has $H^0(Y', \Omega^{n-1}) \neq 0$.

Under the slightly stronger assumption that $a \geq \lceil (n+3)/3 \rceil$, the line bundle M is big, which Kollár used to show that Y is not separably uniruled. This is a strong conclusion. It follows, for example, that there is no generically smooth dominant rational map from a separably rationally connected variety to Y. In particular, Y is not stably rational. With that approach, however, it was not clear how to show that a lift of Y to characteristic zero is not stably rational.

Lemma 2.2. *Let X be a smooth projective variety over a field k. If $H^0(X, \Omega^i)$ is not zero for some $i > 0$, then CH_0 of X is not universally trivial. More precisely, if k has characteristic zero, then $CH_0 \otimes \mathbf{Q}$ is not universally trivial; and if k has characteristic $p > 0$, then CH_0/p is not universally trivial.*

Proof. This is part of the "generalized Mumford theorem" for k of characteristic zero. The proof by Bloch and Srinivas [22, Theorem 3.13], using the cycle class map in de Rham cohomology, is similar to what follows.

For k of characteristic $p > 0$, the hypothesis that $H^0(X, \Omega^i) \neq 0$ remains true after enlarging k, and so we can assume that k is perfect. In that case, we can use the cycle class map in de Rham cohomology constructed by Gros [9, section II.4]. Namely, for a smooth scheme X over k, let $\Omega^i_{\log}$ be the subsheaf of Ω^i on X in the etale topology generated by products $df_1/f_1 \wedge \cdots \wedge df_i/f_i$ for nonvanishing regular functions $f_1, \ldots, f_i$. This is a sheaf of $\mathbf{F}_p$-vector spaces, not of O_X-modules. Gros defined a cycle map from $CH^i(X)/p$ to $H^i(X, \Omega^i_{\log})$, which maps both to $H^i(X, \Omega^i)$ and to de Rham cohomology $H^{2i}_{\mathrm{dR}}(X/k)$.

As a result, for X smooth over k and Y smooth proper over k, with $y = \dim(Y)$, a correspondence α in $CH^y(X \times_k Y)/p$ determines a pullback map $\alpha^* \colon H^0(Y, \Omega^i) \to H^0(X, \Omega^i)$ for all i. Explicitly, α has a class in

$$\begin{aligned} H^y(X \times Y, \Omega^y_{X \times Y}) &= H^y(X \times Y, \oplus_j \Omega^j_X \otimes \Omega^{y-j}_Y) \\ &= \oplus_{i,j} H^i(X, \Omega^j) \otimes H^{y-i}(Y, \Omega^{y-j}). \end{aligned}$$

In particular, α determines an element α^* of $H^0(X, \Omega^i) \otimes H^y(Y, \Omega^{y-i})$, which is $\mathrm{Hom}(H^0(Y, \Omega^i), H^0(X, \Omega^i))$ by Serre duality, as we want. (The last step uses properness of Y over k.)

Let X be a smooth projective variety over k with CH_0 universally trivial, or just with CH_0/p universally trivial. Since the diagonal Δ in $CH^n(X \times X)$ restricts (over the generic point of the first copy of X) to a zero-cycle of degree 1 on $X_{k(X)}$, our assumption implies that there is a zero-cycle x on X such that $\Delta = x$ in $CH_0(X_{k(X)})/p$. Equivalently, there is a decomposition of the diagonal,

$$\Delta = X \times x + B$$

in $CH^n(X \times X)/p$ for some cycle B supported on $S \times X$ with S a closed subset not equal to X. For any $i > 0$, the pullback Δ^* from the second copy of $H^0(X, \Omega^i)$ to the first is the identity, but the pullback by $X \times x$ or by B is zero. (For B, this uses that the restriction of B to $(X - S) \times X$ is zero, and hence its class in $H^n((X - S) \times X, \Omega^n)$ is zero. As a result, for any form θ in $H^0(X, \Omega^i)$, the restriction of $B^*\theta$ to $H^0(X - S, \Omega^i)$ is zero. But $H^0(X, \Omega^i) \to H^0(X - S, \Omega^i)$ is injective, and so $B^*\theta = 0$.) It follows that $H^0(X, \Omega^i) = 0$. $\square$

By Lemma 2.2, the resolution Y' discussed above has CH_0 not universally trivial. We also know that the resolution $Y' \to Y$ induces an isomorphism on CH_0 over all extension fields of k. It follows that any variety X over an algebraically closed field of characteristic zero that degenerates to Y has CH_0 not universally trivial, by the following result, due in this form to Colliot-Thélène and Pirutka [6, Théorème 1.12].

Theorem 2.3. *Let A be a discrete valuation ring with fraction field K and residue field k, with k algebraically closed. Let $\mathcal{X}$ be a flat proper scheme over A with geometrically integral fibers. Let X be the general fiber $\mathcal{X} \times_A K$ and Y the special fiber $\mathcal{X} \times_A k$. Assume that there is a proper birational morphism $Y' \to Y$ with Y' smooth over k such that $CH_0 Y' \to CH_0 Y$ is universally an isomorphism. Let $\overline{K}$ be an algebraic closure of K. Assume that there is a proper birational morphism $X' \to X$ with X' smooth over K such that CH_0 of $X'_{\overline{K}}$ is universally trivial. Then Y' has CH_0 universally trivial.*

As a result, a very general complex hypersurface of degree $d = 2a$ (where $a \geq \lceil (n+2)/3 \rceil$) has CH_0 not universally trivial. In particular, it is not stably rational, by Theorem 1.1.

It remains to consider hypersurfaces of degree $2a + 1$, for $a \geq \lceil (n+2)/3 \rceil$. The following approach seems clumsy, but it works.

If $2a + 1 > n + 1$, then every smooth hypersurface W of degree $2a + 1$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ has $H^0(W, K_W) \neq 0$, and so $CH_0 W$ is not universally trivial by Lemma 2.2. So we can assume that $2a + 1 \leq n + 1$.

Observe that a very general hypersurface of degree $2a + 1$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ degenerates to the union of a hypersurface X of degree $2a$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ and a hyperplane H. Since the special fiber of that degeneration is reducible, we need the following variant of Theorem 2.3, which I worked out with Colliot-Thélène.

Lemma 2.4. *Let A be a discrete valuation ring with fraction field K and algebraically closed residue field k. Let $\mathcal{X}$ be a flat proper scheme over A. Let X be the general fiber $\mathcal{X} \times_A K$ and Y the special fiber $\mathcal{X} \times_A k$. Suppose that X is geometrically integral and there is a proper birational morphism $X' \to X$ with X' smooth over K. Suppose that there is an algebraically closed field F containing K such that CH_0 of X'_F is universally trivial. Then, for every extension field l of k, every zero-cycle of degree zero in the smooth locus of Y_l is zero in $CH_0(Y_l)$.*

Proof. After replacing A by its completion, we can assume that A is complete. Then there is an algebraically closed field F containing the fraction field K of the new ring A such that CH_0 of X'_F is universally trivial. So the class of the diagonal in $CH_0(X'_{F(X')})$ is equal to the class of an F-point of X'. By a specialization argument, it follows that there is a finite extension L of K such that the class of the diagonal in $CH_0(X'_{L(X')})$ is equal to the class of an L-point of X'. Since A is complete, the integral closure of A in L is a complete discrete valuation ring [20, Proposition II.2.3]. Its residue field is k. We now replace A by this discrete valuation ring and $\mathcal{X}$ by its pullback to that ring, without changing the special fiber Y.

For any field extension l of k, there is a flat local A-algebra B such that $\mathfrak{m}_B = \mathfrak{m}_A B$ and B is a discrete valuation ring with residue field l, by "inflation of local rings" [3, Chapter IX, Appendice, Corollaire du Théorème 1]. Let C be the completion of B. Let M be the fraction field of C, which is an extension of the field L above. The residue field of C is l. Consider the C-scheme $\mathcal{X} \times_A C$. Its generic fiber is X_M, which has the resolution X'_M. Its special fiber is Y_l. Since the degree map $CH_0(X'_M) \to \mathbf{Z}$ is an isomorphism, every zero-cycle of degree zero in the smooth locus of Y_l is zero in $CH_0 Y_l$ [6, Proposition 1.9]. ☐

Since a very general hypersurface W of degree $2a+1$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ degenerates to the union of a very general hypersurface X of degree $2a$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ and a very general hyperplane H, Lemma 2.4 shows that CH_0 of W is not universally trivial if there is a zero-cycle of degree zero on $X_{k(X)} - (X \cap H)_{k(X)}$ which is not zero in $CH_0(X \cup H)_{k(X)}$. This holds if we can show that $CH_0(X \cap H)_{k(X)} \to CH_0 X_{k(X)}$ is not surjective.

If that homomorphism is surjective, then we have a decomposition of the diagonal

$$\Delta = A + B$$

in $CH_n(X \times X)$, where A is supported on $X \times (X \cap H)$ and B is supported on $S \times X$ for some closed subset S not equal to X. Let Y be the singular variety in characteristic 2 to which X degenerates (as in the earlier part of the proof). By the specialization homomorphism on Chow groups [8, Proposition 2.6, Example 20.3.5], the decomposition of the diagonal in $X \times X$ gives a similar decomposition of the diagonal in $Y \times Y$. That is, the class of the diagonal in $CH_0 Y_{k(Y)}$ is in the image of $CH_0(Y_H)_{k(Y)}$, where Y_H is the subvariety of Y to which $X \cap H$ degenerates. (Here Y_H is an inseparable double cover of a hypersurface, like Y itself. We can arrange for Y_H to be disjoint from the singular set of Y, but Y_H will still be singular, with finitely many singular points of the form described above for Y, one dimension lower.) Since the resolution Y' of Y we constructed has $CH_0 Y' \to CH_0 Y$ universally an isomorphism, it follows that the class of the diagonal in $CH_0 Y'_{k(Y')}$ is in the image of $CH_0(Y'_H)_{k(Y')}$, where the inverse image Y'_H of Y_H is isomorphic to Y_H. That determines a decomposition of the diagonal

$$\Delta = A + B$$

in $CH_n(Y' \times Y')$, where A is supported on $Y' \times Y'_H$ and B is supported on $S \times Y'$ for some closed subset S not equal to Y'.

Now consider the action of correspondences (by pullback from the second factor to the first) on $H^0(Y', \Omega^i)$, for any $i > 0$. Here Y'_H is singular, with only singularities of the form described above for Y, one dimension lower. Let Z' be the resolution of Y'_H obtained by blowing up the singular points. The diagonal Δ acts as the identity on $H^0(Y', \Omega^i)$, and B acts by 0. For any form β in $H^0(Y', \Omega^i)$ that pulls back to zero on Z', A acts by 0 on β, and so the decomposition above gives that $\beta = 0$. That is, we have shown that the restriction

$$H^0(Y', \Omega^i) \to H^0(Z', \Omega^i)$$

is injective for $i > 0$.

By viewing Y as a complete intersection in the smooth locus of a weighted projective space, we see that K_Y is isomorphic to $O(-n - 2 + 2a)$, and likewise K_{Y_H} is isomorphic to $O(-n-1+2a)$. Since we arranged that $2a+1 \leq n+1$, Y_H is Fano, and in particular $H^0(Y_H, K_{Y_H}) = 0$. It follows that the resolution Z' of Y_H has $H^0(Z', \Omega^{n-1}) = H^0(Z', K_{Z'}) = 0$. So the previous paragraph's injection gives that $H^0(Y', \Omega^{n-1}) = 0$. Since $a \geq \lceil (n+2)/3 \rceil$, this contradicts the calculation that $H^0(Y', \Omega^{n-1}) \neq 0$, as discussed before Lemma 2.2. We conclude that in fact CH_0 of a very general hypersurface of degree $2a + 1$ in $\mathbf{P}^{n+1}_{\mathbf{C}}$ is not universally trivial.

Finally, in each even degree covered by this theorem, there are non-stably-rational smooth hypersurfaces over $\overline{\mathbf{Q}}$. Any hypersurface that reduces to a suitable double cover over $\overline{\mathbf{F}_2}$ is not stably rational, and most such hypersurfaces are smooth. We do not immediately get examples of odd degree over $\overline{\mathbf{Q}}$, since the argument involves two successive degenerations: first, degenerate a hypersurface of degree $2a+1$ to a hypersurface of degree $2a$ plus a hyperplane, and then degenerate the hypersurface of degree $2a$ to a double cover in characteristic 2. □

3. Examples over the rational numbers

In each even degree covered by Theorem 2.1, there are non-stably-rational smooth hypersurfaces over $\overline{\mathbf{Q}}$. We can use any hypersurface that reduces to a suitable double cover over $\overline{\mathbf{F}_2}$. In fact, if a double cover with only singularities as in the proof of Theorem 2.1 exists over $\mathbf{F}_2$, then there are smooth hypersurfaces over $\mathbf{Q}$ which are not stably rational over $\mathbf{C}$. We now use this method to give examples of smooth quartic 3-folds and smooth quartic 4-folds over $\mathbf{Q}$ that are not stably rational over $\mathbf{C}$.

Joel Rosenberg gave some elegant examples of inseparable double covers of projective space over $\mathbf{F}_2$ with only singularities as in the proof of Theorem 2.1 [16, Section 4.7]. Namely, for any even n and even d, the double cover of $\mathbf{P}^n$ over $\mathbf{F}_2$ ramified over the hypersurface $x_0^{d-1}x_1 + \cdots + x_{n-1}^{d-1}x_n + x_n^{d-1}x_0 = 0$ has the desired singularities. It would be interesting to find equally simple examples of inseparable double covers Y of smooth hypersurfaces over $\mathbf{F}_2$ such that Y has singularities as above; that is the problem we encounter here.

Example 3.1. Quartic 4-folds.

For quartic 4-folds, even non-rationality was previously unknown. The free program Macaulay2 shows that the 4-fold $\{y^2 = f, g = 0\}$ in $\mathbf{P}(1^6 2) = \mathbf{P}(x_0, \ldots, x_5, y)$ over $\mathbf{F}_2$ has only singularities as in the proof of Theorem 2.1 in the example

$$\begin{aligned} f &= x_0^3x_2 + x_0x_3^3 + x_1^3x_4 + x_1^2x_2x_4 + x_1x_4^3 + x_2^3x_5 + x_3x_5^3, \\ g &= x_0x_1 + x_2x_3 + x_4x_5. \end{aligned}$$

The proof of Theorem 2.1 shows that any hypersurface X in $\mathbf{P}^5_{\mathbf{Q}}$ of the form $g^2 - 4f = 0$, where g and f are lifts to $\mathbf{Z}$ of the polynomials above, has CH_0 of $X_{\mathbf{C}}$ not universally trivial. Therefore, $X_{\mathbf{C}}$ is not stably rational. For example, this applies to the following quartic 4-fold, which we compute is smooth:

$$\begin{aligned} 0 = {} & x_0^2x_1^2 - 4x_0^3x_2 + 2x_0x_1x_2x_3 + x_2^2x_3^2 - 4x_0x_3^3 - 4x_1^3x_4 \\ & - 4x_1^2x_2x_4 - 4x_1x_4^3 - 4x_2^3x_5 + 2x_0x_1x_4x_5 + 2x_2x_3x_4x_5 + x_4^2x_5^2 - 4x_3x_5^3. \end{aligned}$$

Example 3.2. Quartic 3-folds.

Here is an example of a smooth quartic 3-fold over $\mathbf{Q}$ that is not stably rational over $\mathbf{C}$. (Following a suggestion by Wittenberg, Colliot-Thélène and Pirutka gave examples over $\overline{\mathbf{Q}}$ [6, Appendice A].) Macaulay2 shows that the 3-fold $\{y^2 = f, g = 0\}$ in $\mathbf{P}(1^5 2) = \mathbf{P}(x_0, \ldots, x_4, y)$ over $\mathbf{F}_2$ has only singularities as in the proof of Theorem 2.1 in the example

$$\begin{aligned} f &= x_0^3x_2 + x_1^3x_2 + x_0x_1^2x_4 + x_0x_2^2x_4 + x_3^3x_4, \\ g &= x_0x_1 + x_2x_3 + x_4^2. \end{aligned}$$

As a result, any hypersurface X in $\mathbf{P}^4_{\mathbf{Q}}$ of the form $g^2 - 4f = 0$, where g and f are lifts to $\mathbf{Z}$ of the polynomials above, has CH_0 of $X_{\mathbf{C}}$ not universally trivial. Therefore, $X_{\mathbf{C}}$ is not stably rational. For example, this applies to the following quartic 3-fold, which we compute is smooth:

$$\begin{aligned} 0 = {} & x_0^2x_1^2 - 4x_0^3x_2 - 4x_1^3x_2 - 8x_2^4 + 2x_0x_1x_2x_3 + x_2^2x_3^2 \\ & - 4x_0x_1^2x_4 - 4x_0x_2^2x_4 - 4x_3^3x_4 + 2x_0x_1x_4^2 + 2x_2x_3x_4^2 + x_4^4. \end{aligned}$$

4. Rationality in families

Given a family of projective varieties for which the geometric generic fiber is rational, are all fibers geometrically rational? The analogous question for geometric ruledness has a positive answer, by Matsusaka [13, Theorem IV.1.6]. The statement for geometric rationality is easily seen to be false if we make no restriction on the singularities; for example, a smooth cubic surface X can degenerate to the cone Y over a smooth cubic curve. Here X is rational over $\mathbf{C}$, but Y is not. In this example, Y is log canonical but not klt.

The following application of Theorem 2.1, suggested by Tommaso de Fernex, shows that rationality need not specialize even when the singularities allowed are mild, namely klt. It remains an open question whether rationality specializes when all fibers are smooth, or at least canonical. If the dimension is at most 3 and the characteristic is zero, rationality specializes when all fibers are klt, by de Fernex and Fusi [7, Theorem 1.3] and Hacon and McKernan [10, Corollary 1.5].

Corollary 4.1. *For any $m \geq 4$, there is a flat family of projective m-folds over the complex affine line such that all fibers over $A^1 - 0$ are smooth rational varieties, while the fiber over 0 is klt and not rational.*

Proof. For a variety X embedded in projective space $\mathbf{P}^N$ (with a proviso to follow), X degenerates to the projective cone over a hyperplane section $X \cap H$. To see this, consider the projective cone over X in $\mathbf{P}^{N+1}$, and intersect it with a pencil $\mathbf{P}^1$ of hyperplanes in $\mathbf{P}^{N+1}$. For most hyperplanes in the pencil, the intersection is isomorphic to X, while for a hyperplane through the node, the underlying set of the intersection is the projective cone over $X \cap H$. On the other hand, that intersection could be different from the projective cone over $X \cap H$ as a scheme because it is non-reduced at the cone point. That problem does not arise if $X \subset \mathbf{P}^N$ is projectively normal and $H^1(X, O(j)) = 0$ for all $j \geq -1$ [15, Proposition 3.10], as will be true in the following example.

By considering a Veronese embedding of a projective space $X = \mathbf{P}^m$, it follows that for any hypersurface W of degree d in $\mathbf{P}^m$, $\mathbf{P}^m$ degenerates to the projective cone Y over $(W, O(d))$. If the hypersurface W is not stably rational, then this is a degeneration of a smooth projective rational variety (in fact, projective space) to a variety Y which is not rational. Indeed, Y is birational to $\mathbf{P}^1 \times W$. (So just knowing that W is not rational would not be enough.)

It remains to describe the singularities of Y. Namely, the projective cone Y over $(W, O(d))$ is klt if and only if $d \leq m$, meaning that W is Fano [15, Lemma 3.1]. (Note that Y is *not* the projective cone over $(W, O(1))$ in $\mathbf{P}^{m+1}$, which has a milder singularity.) For every $m \geq 4$, Theorem 2.1 gives a smooth Fano hypersurface W in $\mathbf{P}^m$ which is not stably rational. (For $m = 4$, this is Colliot-Thélène and Pirutka's theorem [6].) Thus we have a degeneration of a smooth projective rational m-fold to a klt variety which is not rational. ☐

Acknowledgments

The author thanks Hamid Ahmadinezhad, Jean-Louis Colliot-Thélène, Tommaso de Fernex, János Kollár, Adrian Langer, and Claire Voisin for useful conversations. This work was supported by The Ambrose Monell Foundation and Friends, via the Institute for Advanced Study, and by NSF grant DMS-1303105.

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