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CATEGORY FORCINGS, MM^{+++}, AND GENERIC ABSOLUTENESS FOR THE THEORY OF STRONG FORCING AXIOMS

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To Chiara and to our kids, Pietro and Adele.

The main objective of this paper is to show that there is a natural recursive extension T of $\mathsf{ZFC}+$ *large cardinals* which gives a complete theory of the Chang model $L([\mathrm{Ord}]^{\leq\aleph_1})$ with respect to the unique efficient method to produce consistency results for this structure, i.e., stationary set preserving forcings. In particular we will show that a closed formula ϕ relativized to this Chang model is first order derivable in T if and only if it is provable in T that $T+\phi$ is forceable by a stationary set preserving forcing. In our eyes the results of this paper give a solid a posteriori explanation of the success forcing axioms have met in providing at least one consistent solution to many ZFC-provably undecidable problems. The paper can be divided into six sections:

- An introduction (Section 1) shows how the above results stem out of Woodin's work on Ω-logic and of Woodin's absoluteness results for $L(\mathbb{R})$ and for the Chang model $L([\mathrm{Ord}]^{\aleph_0})$. We also try to explain in what terms we can assert that the main result of this paper is an optimal extension to the Chang model $L([\mathrm{Ord}]^{\leq\aleph_1})$ of Woodin's absoluteness results for the Chang model $L([\mathrm{Ord}]^{\aleph_0})$. We also try to argue that the results of this paper give an a posteriori explanation of the success that forcing axioms has met in solving a variety of problems showing up in set theory as well as in many other fields of pure mathematics.
- Section 2 presents some background material on stationary sets (Subsection 2.1), large cardinals (Subsection 2.2), posets and boolean completions (Subsection 2.3), stationary set preserving forcings (Subsection 2.4), forcing axioms (Subsection 2.5), and iterated forcing (Subsection 2.6) which will be needed in the remainder of the paper.
- Section 3 introduces the notion of category forcings. We look at subcategories of the category of complete boolean algebras (CBAs) with complete homomorphisms. Given a category (Γ,Θ) (where Γ is the class of objects and Θ the class of arrows) we associate to it the partial order $(\mathsf{U}^{\Gamma,\Theta},\leq_\Theta)$

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whose elements are the objects in Γ ordered by $\mathsf{B} \geq_\Theta \mathsf{C}$ iff there is an $i : \mathsf{B} \to \mathsf{C}$ in Θ. We also feel free to confuse a set sized partial order with its uniquely defined boolean completion. In this paper we self-contained as possible;focus on the analysis of the category $(\mathsf{SSP}, \mathsf{SSP})$ whose objects are the stationary set preserving (SSP) complete boolean algebras and whose arrows (still denoted by SSP) are the complete homomorphisms with a stationary set preserving quotient. The reasons are twofold:

- We aim at a generic absoluteness result for a strengthening of Martin's maximum. This naturally leads to an analysis of the category of forcings which are relevant for this forcing axiom, i.e., the SSP-forcings.
- We are able to predicate all the nice features we isolate for a category forcing just for the forcing $(\mathsf{U}^{\mathsf{SSP},\mathsf{SSP}}, \leq_{\mathsf{SSP}})$ (which we denote from now on just as $\mathsf{U}^{\mathsf{SSP}}$).

The following list sums up the main concepts and results we isolate on the combinatorial properties of these category forcings:

(1) We introduce the key concept (at least for our aims) of a totally rigid element of a category (Γ, Θ).
$\mathsf{B} \in \Gamma$ is Θ-totally rigid if it is fixed by any automorphism of some complete boolean algebra in Γ which absorbs B using an arrow in Θ.
We can formulate this property in purely categorical terms as follows:
    B object of Γ is Θ-totally rigid if for all $\mathsf{Q} \in \Gamma$ there is at most one arrow $i : \mathsf{B} \to \mathsf{Q}$ in Θ.
We show that in the presence of class many supercompact cardinals, the class of SSP-totally rigid partial orders is dense in $\mathsf{U}^{\mathsf{SSP}}$ (Theorem 3.8).

(2) We show that the cutoff $\mathsf{U}^{\mathsf{SSP}}_\delta = \mathsf{U}^{\mathsf{SSP}} \cap V_\delta$ of this category forcing at a rank initial segment δ which is an inaccessible limit of $< \delta$-supercompact cardinals is an SSP-totally rigid partial order which belongs to SSP and which absorbs all forcings in $\mathsf{SSP} \cap V_\delta$ (Theorem 3.5.1).

(3) We also show that $\mathsf{U}^{\mathsf{SSP}}_\delta$ forces MM^{++} in case δ is a supercompact limit of $< \delta$-supercompact cardinals (Theorem 3.5.2).

(4) We show that the quotient of the category forcing $(\mathsf{U}^{\mathsf{SSP}})^V$ with respect to a generic filter G for any of its elements $\mathsf{B} \in \mathsf{SSP}^V$ is the category forcing $(\mathsf{U}^{\mathsf{SSP}})^{V[G]}$ as computed in the generic extension $V[G]$ (see Theorem 3.9, where it is also given a precise definition of this statement).

- In Section 4 we sum up the relevant facts about towers of normal ideals we need in order to formulate the results of Section 5.
- Section 5 introduces and analyzes the forcing axiom MM^{+++}. First we observe that in the presence of class many Woodin cardinals the forcing axiom MM^{++} can be formulated as the assertion that the class of *presaturated towers* is dense in the category forcing $\mathsf{U}^{\mathsf{SSP}}$. What can be said about the intersection of the class of totally rigid posets and the class of presaturated towers? Can this intersection still be a dense class in $\mathsf{U}^{\mathsf{SSP}}$? Can $\mathsf{U}^{\mathsf{SSP}}_\delta$ belong to this intersection for some δ? The forcing axiom MM^{+++} arises as a positive answer to these questions and is a slight strengthening of the assertion that the class of presaturated tower forcings which are also totally rigid is dense in the category forcing $\mathsf{U}^{\mathsf{SSP}}$.

We first prove our main generic absoluteness result, i.e., that over any model of MM^{+++}+*large cardinals* any stationary set preserving forcing which preserves MM^{+++} does not change the theory of $L([\mathrm{Ord}]^{\leq\aleph_1})$ with parameters in $P(\omega_1)$ (Theorem 5.18). Then we turn to the proof of the consistency of MM^{+++} showing that any of the standard forcing methods to produce a model of MM^{++} collapsing an inaccessible δ to become ω_2 actually produces a model of MM^{+++} provided that δ is a super huge cardinal (Corollary 5.20 and Theorem 5.29). In particular Theorem 5.29 provides a rather extensive sample of notions which can be used to apply the generic absoluteness result.

- We end our paper in Section 6 with some comments regarding our results. In particular, note the following:
  - We outline that the usual forcing axioms can be seen as topological formulations of strengthenings of the axiom of choice and of Baire's category theorem. On the other hand, the category theoretic framework allows us to present MM^{+++} (and other types of forcing axioms) as a formulation in the language of categories of suitable strengthenings of many of the usual forcing axioms.
  - We outline the modularity of our results conjecturing that they ultimately can be predicated for many category forcings $(\mathsf{U}^{\Gamma}, \leq_{\Gamma})$ given by classes of forcings Γ satisfying certain natural requirements.
  - We give some heuristic argument suggesting that MM^{++} is an axiom really weaker than MM^{+++}.
  - We show that our results are optimal outlining that no axiom strictly weaker than MM^{++} can produce a generic absoluteness result for the theory of the Chang model $L([\mathrm{Ord}]^{\leq\aleph_1})$ with respect to SSP-forcings which preserve this axiom.
  - We also give a list of some possible lines of further investigations and link our results to other recent research in this area by Hamkins-Johnstone [13], Tsaprounis [25], and the author [28], [29] (the latter with Audrito).

Section 5 depends on the results of Section 3. Sections 2 and 4 present background material, and the reader acquainted with it can just skim through it. We tried to maintain this paper as self-contained as possible; nonetheless with the exception of the Introduction, the reader is expected to have a strong background in set theory and familiarity with forcing axioms, tower forcings, and large cardinals. Detailed references for the material presented in this paper are mentioned throughout the text; basic sources are Jech's set theory handbook [12], Larson's book on the stationary tower forcing [15], Foreman's handbook chapter on generic elementary embeddings [8], and our notes on semiproper iterations [30].

Contents

1. Introduction

Since its introduction by Cohen in 1963, forcing has been the key and most effective tool to obtain independence results in set theory. This method has found applications in set theory and in virtually all fields of pure mathematics: in the past 40 years natural problems of group theory, functional analysis, operator algebras, general topology, and many other subjects were shown to be undecidable by means of forcing. However, already in the early 1970s and with more evidence since the 1980s it became apparent that many consistency results could be derived by a short list of set theoretic principles which are known in the literature as forcing axioms. These axioms gave set theorists and mathematicians a very powerful tool to obtain independence results: for any given mathematical problem we are most likely able to compute its (possibly different) solutions in the constructible universe L and in models of strong forcing axioms. These axioms settle basic problems in cardinal arithmetic like the size of the continuum and the singular cardinal problem (see among others the works of Foreman, Magidor, and Shelah [9], Veličković [26], Todorčević [23], Moore [18], Caicedo and Veličković [5], and the author [27]), as well as combinatorially complicated ones like the basis problem for uncountable linear orders (see Moore's result [19] which extends previous work of Baumgartner [3], Shelah [21], Todorčević [22], and others). Interesting problems originating from other fields of mathematics and apparently unrelated to set theory have also been settled appealing to forcing axioms, as it is the case (to cite two of the most prominent examples) for Shelah's results [20] on Whitehead's problem in group theory and Farah's result [7] on the non-existence of outer automorphisms of the Calkin algebra in operator algebra. Forcing axioms assert that for a large class of compact topological spaces X Baire's category theorem can be strengthened to the statement that any family of $\aleph_1$-many dense open subsets of X has a non-empty intersection. In light of the success these axioms have met in solving problems, a convinced platonist may start to argue that these principles may actually give a "complete" theory of a suitable fragment of the universe. However, it is not clear how one could formulate such a completeness result. The aim of this Introduction is to explain in which sense we can show that these strong forcing axioms can give such a complete theory. Our argument will find its roots in the work of Woodin in Ω-logic. The basic observation is that the working tools of a set theorist are either first order calculus, by which he/she can justify his/her proofs over ZFC, or forcing, by which he/she can obtain his/her independence results over ZFC. However, it appears that there is still a gap between what we can achieve by ordinary proofs in some axiom system which extends ZFC and the independence results that we can obtain over this theory by means of forcing. More specifically to close the gap it appears that we are lacking two desirable features we would like to have for a complete first order theory T that axiomatizes set theory with respect to its semantics given by the class of boolean valued models of T:

- T is complete with respect to its intended semantics; i.e., for all statements ϕ only one among $T + \phi$ and $T + \neg\phi$ is forceable.
- Forceability over T should correspond to a notion of derivability with respect to some proof system, eventually derivability with respect to a standard first order calculus for T.

Both statements appear to be rather bold and have to be handled with care: Consider for example the statement $\omega = \omega_1$ in a theory T extending ZFC with the statements ω *is the first infinite cardinal* and ω_1 *is the first uncountable cardinal.* Then clearly T proves $|\omega| \neq |\omega_1|$, while if one forces with $Coll(\omega, \omega_1)$ one produces a model of set theory where this equality holds (however, the formula ω_1 *is the first uncountable cardinal* is now false in this model). On a first glance this suggests that as we expand the language for T, forcing starts to act randomly on the formulas of T switching the truth value of its formulas with parameters in ways which it does not seem simple to describe.[1] However, the above difficulties are raised essentially by our lack of attention to define the type of formulas for which we aim to have the completeness of T with respect to forceability. We shall show that when the formulas are prescribed to talk only about a suitable initial segment of the set theoretic universe and we consider only forcings that preserve the intended meaning of the parameters by which we enriched the language of T, this random behavior of forcing does not show up anymore.

We assume a platonistic stance toward set theory; thus we have one canonical model V of ZFC of which we try to uncover the truths. To do this we may allow ourselves to use all model theoretic techniques that produce new models of the truths of $Th(V)$ on which we are confident, which (if we are platonists) certainly include ZFC and all the axioms of large cardinals. We may start our quest for uncovering the truth in V by first settling the theory of $H^V_{\omega_1}$ (the hereditarily countable sets), then the theory of $H^V_{\omega_2}$ (the sets of hereditarily cardinality $\aleph_1$), and so on so forth covering step by step all infinite cardinals. To proceed we need some definitions.

[1]This is no longer the case for the closed formulas of T. Hamkins and Löwe represent the forceability of a closed formula ϕ as an interpretation of the modal statement $\diamond\phi$ and have shown that the closed sentences of ZFC and the family of all generic multiverses (i.e., the generic extensions of a given model V of ZFC) can be used to define a family of correct and complete frames for the propositional modal logic $S4.2$ [11].

Definition. *Given a theory $T \supseteq \mathsf{ZFC}$ and a family Γ of partial orders definable in T, we say that ϕ is Γ-consistent for T if T proves that there exists a partial order in Γ which forces ϕ.*

Given a model V of ZFC we say that V models that ϕ is Γ-consistent if ϕ is Γ-consistent for $Th(V)$.

Definition. *Given a partial order P and a cardinal λ, $\mathsf{FA}_\lambda(P)$ holds if for all $\{D_\alpha : \alpha < \lambda\}$ family of dense subsets of P, there is a $G \subset P$ filter which a has non-empty intersection with all D_α.*

Definition. *Let*

$$T \supseteq \mathsf{ZFC} + \{\lambda \textit{ is an infinite cardinal}\}.$$

Ω_λ is the definable (in T) class of partial orders P which satisfy $\mathsf{FA}_\lambda(P)$.

In particular Baire's category theorem amounts to say that $\Omega_{\aleph_0}$ is the class of all partial orders (denoted by Woodin as the class Ω). The following is a basic observation whose proof can be found in [28, Lemma 1.2].

Lemma (Cohen's absoluteness Lemma)**.** *Assume $T \supseteq \mathsf{ZFC} + \{p \subseteq \omega\}$ and $\phi(x,p)$ is a Σ_0-formula. Then the following are equivalent:*

- *$T \vdash \exists x\phi(x,p)$,*
- *$T \vdash \exists x\phi(x,p)$ is Ω-consistent.*

This shows that for Σ_1-formulas with real parameters the desired overlap between the ordinary notion of provability and the semantic notion of forceability is a provable fact in ZFC. Now it is natural to ask if we can expand the above in at least two directions:

(1) Increase the complexity of the formula,
(2) Increase the language allowing parameters also for other infinite cardinals.

The second direction requires almost no effort once one notices that in order to prove that $\exists x\phi(x,p)$ is provable if it is Ω-consistent, we used the fact that for all forcing P $\mathsf{FA}_{\aleph_0}(P)$ is provable in ZFC, and that the latter is just the most general formulation of Baire's category theorem. We can thus reformulate the above equivalence in a modular form as follows (see for a proof [28, Lemma 1.3]).

Lemma (Generalized Cohen's absoluteness Lemma)**.** *Assume*

$$T \supseteq \mathsf{ZFC} + \{p \subset \lambda\} + \{\lambda \textit{ is an infinite cardinal}\}$$

and $\phi(x,p)$ is a Σ_0-formula. Then the following are equivalent:

- *$T \vdash \exists x\phi(x,p)$,*
- *$T \vdash \exists x\phi(x,p)$ is Ω_λ-consistent.*

The extent by which we can increase the complexity of the formula requires once again some attention to the semantical interpretation of its parameters and its quantifiers. We have already observed that the formula $\omega = \omega_1$ is inconsistent but Ω-consistent in a language with parameters for ω and ω_1. One of Woodin's main achievements[2] in Ω-logic show that if we restrict the semantic interpretation of ϕ to range over the structure $L([\mathrm{Ord}]^{\aleph_0})$ and we assume large cardinal axioms, we can get a full correctness and completeness result [15, Corollary 3.1.7].

[2]We follow Larson's presentation as in [15].

Theorem (Woodin). *Assume*

$$T \supseteq \mathsf{ZFC} + \{p \subset \omega\} + \textit{there are class many Woodin cardinals which are limits of Woodin cardinals}$$

and $\phi(x)$ is any formula in one free variable. Then the following are equivalent:

- $T \vdash [L([\mathrm{Ord}]^{\aleph_0}) \models \phi(p)]$,
- $T \vdash [L([\mathrm{Ord}]^{\aleph_0}) \models \phi(p)]$ *is Ω-consistent.*

The natural question to address now is whether we can step up this result also for uncountable λ. If so in which form? Woodin [17, Theorem 3.2.1] has proved another remarkable absoluteness result for CH.

Theorem (Woodin). *Let T extend* $\mathsf{ZFC}+$ there are class many measurable Woodin cardinals.

A Σ^2_1-statement $\phi(p)$ with real parameter p is Ω-consistent for T if and only if

$$T + \mathsf{CH} \vdash \phi(p).$$

However, there are two distinct results that show that we cannot hope to obtain a complete (and unique) theory with respect to forceability which extends $\mathsf{ZFC}+\mathsf{CH}$:

- Asperó, Larson, and Moore [2] have shown that there are two distinct Π_2-statements ψ_0, ψ_1 over the theory of $H_{\aleph_2}$ such that $\psi_0 + \psi_1$ denies CH and $\psi_i + \mathsf{CH}$ is forceable by means of a proper forcing for $i = 0, 1$ over a model of $\mathsf{ZFC}+$ *large cardinal axioms*. This shows that any completion of $\mathsf{ZFC}+\mathsf{CH}+$ *large cardinal axioms* cannot simultaneously realize all Π_2-statements over the theory of $H_{\aleph_2}$, each of which is known to be consistent with CH (even consistent by means of a *proper* forcing).
- Woodin and Koellner [14] have shown that if there is an Ω-complete theory T for Σ^2_3-statements with real parameters which implies CH, then there is another Ω-complete theory T' for Σ^2_3-statements with real parameters which denies CH.

In particular the first result shows that we cannot hope to extend $\mathsf{ZFC}+\mathsf{CH}$ to a natural maximal completion which settles the Π_2-theory of the structure $H_{\aleph_2}$ at least with respect to the semantics given by forcing. Finally Woodin has proved a remarkable absoluteness result for a close relative of forcing axioms, Woodin's axiom[3] $(*)$. For this axiom Woodin can prove the consistency of a completeness and correctness result for $\mathsf{ZFC}+(*)$ with respect to a natural but non-constructive proof system and to Ω-consistency. This completeness result is very powerful, for it applies to the largest possible class of models produced by forcing, but it has two features which need to be clarified:

- It is not known if $(*)$ is Ω-consistent, i.e., if its consistency can be proved by forcing over an ordinary model of ZFC.
- The correctness and completeness results for $(*)$ are with respect to a natural but non-constructive proof system, and moreover the completeness theorem is known to hold only under certain assumptions on the set theoretic properties of V.

[3] See [17] or [32] for a detailed presentation of the models of this axiom.

Let us now focus on the first order theory with parameters in $P(\omega_1)$ of the structure H_{ω_2} or more generally of the Chang model $L([\mathrm{Ord}]^{\leq\aleph_1})$. A natural approach to the study of this Chang model is to expand the language of ZFC to include constants for all elements of H_{ω_2} and the basic relations between these elements.

Definition. *Let V be a model of ZFC and $\lambda \in V$ be a cardinal. The Σ_0-diagram of H^V_λ is given by the theory*

$$\{\phi(p) : p \in H^V_\lambda,\ \phi(p) \text{ a } \Sigma_0\text{-formula true in } V\}.$$

Following our approach, the natural theory of V we should look at is the theory

$$T = \mathsf{ZFC} + \text{ large cardinal axioms } + \Sigma_0\text{-diagram of } H_{\omega_2},$$

since we already know ZFC+*large cardinal axioms* settles the theory of $L([\mathrm{Ord}]^{\aleph_0})$ with parameters in H_{ω_1}. Now consider any model M of T we may obtain using model-theoretic techniques. In particular we may assume that M is a "monster model" which contains V such that for some completion $\bar{T}$ of T, M is a model of $\bar{T}$ which amalgamates "all" possible models of $\bar{T}$ and realizes all consistent types of $\bar{T}$ with parameters in $H^V_{\omega_2}$. If we have any hope that $\bar{T}$ is really the theory of V we are aiming for, we should at least require that $V \prec_{\Sigma_1} M$. Once we make this requirement we notice the following:

$$V \cap \mathsf{NS}^M_{\omega_1} = \mathsf{NS}^V_{\omega_1}.$$

If this were not the case, then for some S stationary and costationary in V, M models that S is not stationary, i.e., that there is a club of ω_1 disjoint from S. Since $V \prec_{\Sigma_1} M$ such a club can be found in V. This means that V already models that S is non-stationary. Now the formula $S \cap C = \emptyset$ *and C is a club* is Σ_0, and thus it is part of the Σ_0-diagram of $H^V_{\omega_2}$. However, this contradicts the assumption that S is stationary and costationary in V which is expressed by the fact that the above formula is not part of the Σ_0-elementary diagram of V. This shows that $V \not\prec_{\Sigma_1} M$. Thus any monster model M as above should be correct about the non-stationary ideal, so we better add this ideal as a predicate to the Σ_0-diagram of $H^V_{\omega_2}$, to rule out models of the completions of T which cannot even be Σ_1-superstructures of V. Remark that on the forcing side, this is immediately leading to the notion of the stationary set preserving forcing: if we want to use forcing to produce such approximations of monster models while preserving the fact of being a Σ_1-elementary superstructure of V with respect to T, we are forced to restrict our attention to stationary set preserving forcings. Let us denote by SSP the class of stationary set preserving posets and recall that Martin's maximum asserts that $\mathsf{FA}_{\aleph_1}(P)$ holds for all SSP-partial orders P. Subject to the limitations we have outlined, the best possible result we can hope for is to find a theory

$$T_1 \supset T$$

such that

(1) T_1 proves the strongest possible forcing axiom, i.e., some natural strengthening of Martin's maximum.

(2) For any $T_2 \supseteq T_1$ and any formula $\phi(S)$ relativized to the Chang model $L([\mathrm{Ord}]^{\leq\aleph_1})$ with parameter $S \subset \omega_1$ and a predicate for the non-stationary ideal NS_{ω_1}, $\phi(S)$ is provable in T_2 if and only if the theory $T_1 + \phi(S)$ is $\Omega_{\aleph_1}$-consistent for T_2.

We shall separately give arguments to justify these two requirements.

The first requirement above (1) is a natural maximality principle, since it can be argued that Martin's maximum MM is a natural strengthening of the axiom of choice and of the Baire category theorem:

- On the one hand, Todorčević [24] has noticed that the axiom of choice is equivalent (over ZF) to the global forcing axiom asserting that $\mathsf{FA}_\lambda(P)$ holds for each regular cardinal λ and for any $< \lambda$-directed closed poset[4] P.
- On the other hand, Shelah has shown that $\mathsf{FA}_{\aleph_1}(P)$ provably fails if P is not stationary set preserving below some of its conditions.
- It is also well known by means of Stone duality that for any compact Hausdorff topological space (X, τ), the intersection of a family of λ-many dense open sets is non-empty if and only if the partial order $P = (\tau \setminus \{\emptyset\}, \supseteq)$ is such that $\mathsf{FA}_\lambda(P)$ holds.

In particular, the first and the third items show that Martin's maximum can be seen as a natural topological formulation of a strengthening of the axiom of choice, since countably closed forcings are stationary set preserving. The second and the third items show that Martin's maximum is a maximal topological strengthening of Baire's category theorem. It has been shown by the work of Foreman, Magidor, and Shelah [9] that Martin's maximum is SSP-consistent with respect to $T = \mathsf{ZFC}+$*large cardinals*, and thus it is consistent. Denying it is not required by the known constraints we have to impose on T in order to get a complete extension of T.

The second requirement (2) above is the best possible form of completeness theorem we can currently formulate: there may be interesting model theoretic tools to produce models of T which are not encompassed by forcing, however, we haven't as yet developed powerful techniques to exploit them in the study of models of ZFC. Moreover the second requirement (2) shows that forcing becomes a powerful proof tool in the presence of strong forcing axioms, since it transforms a validity problem in a consistency problem.

MM^{++} is a well known "natural" strengthening of Martin's maximum (i.e., of the equality $\mathsf{SSP} = \Omega_{\aleph_1}$) shown to be consistent relative to the existence of a supercompact cardinal by the work of [9]. In the present paper we show that a further natural strengthening of MM^{++} which we call MM^{+++} (see Definition 5.9) enriched by suitable large cardinals axioms (see Definition 2.9 for the relevant notions) is already such a theory T_1, as we can prove the following theorem.

[4]Roughly the argument goes as follows: the axiom of choice is equivalent to the assertion that the axiom of dependent choice DC_λ holds for all infinite cardinals λ. It is almost immediate to check that DC_λ is equivalent to the assertion that $\mathsf{FA}_\lambda(P)$ holds for any $< \lambda$-directed closed poset P and that—assuming the amount of axiom of choice needed to implement Stone duality—DC_ω is an equivalent formulation of Baire's category theorem (see for more details http://www.personalweb.unito.it/matteo.viale/LUMINY2014viale.pdf).

Theorem 1. *Let* ZFC^* *stand for*[5]

$$\mathsf{ZFC}+ \text{ there are class many } \Sigma_2\text{-reflecting cardinals}$$

and T^* *be any theory extending*

$$\mathsf{ZFC}^* + \mathsf{MM}^{+++} + \omega_1 \text{ is the first uncountable cardinal } + S \subset \omega_1.$$

Then for any formula[6] $\phi(S)$ *the following are equivalent:*

(1) $T^* \vdash [L([\mathrm{Ord}]^{\leq \aleph_1}) \models \phi(S)]$,
(2) $T^* \vdash \mathsf{MM}^{+++}$ *and* $[L([\mathrm{Ord}]^{\leq \aleph_1}) \models \phi(S)]$ *are jointly* $\Omega_{\aleph_1}$*-consistent.*

We shall also see that the result is sharp in the sense that the work of Asperó [1] and Larson [16] shows that we cannot obtain the above completeness and correctness result relative to forcing axioms which are just slightly weaker than MM^{++}. It remains open whether our axiom MM^{+++} is really stronger than MM^{++} in the presence of large cardinals.

Finally I think that the present results show that we have all reasons to expect that MM^{+++} (and most likely already MM^{++}) can decide Woodin's axiom $(*)$: any proof of the consistency of MM^{+++} with $(*)$ or with its negation obtained by an SSP-forcing would convert by the results of this paper into a proof of the provability of $(*)$ (or of its negation) from MM^{+++}.

2. Background material on large cardinals, generalized stationarity, forcing axioms

2.1. **Stationary sets and normal ideals.** We follow standard set theoretic notation as in [12]. In particular for an arbitrary set or class X and a cardinal λ

$$[X]^\lambda = \{Y \in P(X) : |Y| = \lambda\}.$$

$[X]^{\leq\lambda}$ and $[X]^{<\lambda}$ are defined accordingly. For $X \supseteq \lambda$

$$P_\lambda(X) = \{Y \in P(X) : |Y| < \lambda \text{ and } Y \cap \lambda \in \lambda\}.$$

We let

$$P_\lambda = \{Y \in V : |Y| < \lambda \text{ and } Y \cap \lambda \in \lambda\}.$$

Thus $P_\lambda(X) = P(X) \cap P_\lambda$.

For any $f : P_\omega(X) \to X$ we let $C_f \subset P(X)$ be the set of its closure points (i.e., the set of $y \subset X$ such that $f[P_\omega(y)] \subset y$).

Definition 2.1. S is stationary in X if $S \cap C_f$ is non-empty for all $f : P_\omega(X) \to X$. S is stationary if it is stationary in $\cup S$.

Definition 2.2. $I \subset P(P(X))$ is an ideal on X if it is closed under subsets and finite unions. The dual filter of an ideal I is denoted by $\breve{I}$. $I^+ = P(P(X)) \setminus I$.

I is normal if for all $S \in I^+$ and regressive $f : S \to X$, there is $T \in I^+$ on which f is constant.

I is $<\kappa$-complete if for all $J \subset I$ of size less than κ, $\cup J \in I$.

The completeness of I is the largest κ such that I is $<\kappa$-complete.

If $S \in I^+$, $I \restriction S$ is the ideal generated by $I \cup \{P(X) \setminus S\}$.

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[5] Σ_2-reflecting cardinals are defined in Definition 2.8.

[6] If we allow formulas of arbitrary complexity, we do not need to enrich the language with a predicate for the non-stationary ideal, since this ideal is a definable predicate over H_{ω_2} (though defined by a Σ_1-property) and thus can be incorporated as a part of the formula.

Definition 2.3. The non-stationary ideal NS_X on X is the ideal generated by the complements of sets of the form

$$C_f = \{Z \in P(X) : f[Z^{<\omega}] \subset Z\}$$

for some $f : X^{<\omega} \to X$. Its dual filter is the club filter on X.

An ideal I on X concentrates on $S \subset P(X)$, if $P(X) \setminus S \in I$.

Remark 2.4. $P_{\aleph_0}(X) = [X]^{<\aleph_0}$ for all $X \supseteq \omega$ and $P_{\aleph_1}(X)$ is a club subset of $[X]^{<\aleph_1}$ for all $X \supseteq \omega_1$. For other cardinals $\lambda > \aleph_1$, $[X]^{<\lambda} \setminus P_\lambda(X)$ can be stationary.

Lemma 2.5 (Pressing down Lemma). *Assume S is stationary and $f : S \to \cup S$ is such that $f(X) \in X$ for all $X \in S$. Then there is $T \subset S$ on which f is constant. In particular NS_X is a normal ideal for all X.*

We call functions f as in the Lemma regressive.

For a stationary set S and a set X, if $\cup S \subseteq X$, we let $S^X = \{M \in P(X) : M \cap \cup S \in S\}$; if $\cup S \supseteq X$, we let $S \restriction X = \{M \cap X : M \in S\}$. We define an order on stationary sets given by $T \leq S$ if letting $X = (\cup T) \cup (\cup S)$, there is $f : P_\omega(X) \to X$ such that $T^X \cap C_f \subseteq S^X$. We let $S \equiv T$ if $S \leq T$ and $T \leq S$.

In general if $\{S_i : i < \xi\}$ is a family of stationary sets, we let η be the least such that $\{S_i : i < \xi\} \in V_\eta$ and

$$\bigwedge\{S_i : i < \xi\} = \{M \prec V_\eta : \{S_i : i < \xi\} \in M \text{ and } \forall S_i \in M : M \cap \cup S_i \in S_i\},$$

$$\bigvee\{S_i : i < \xi\} = \{M \prec V_\eta : \{S_i : i < \xi\} \in M \text{ and } \exists S_i \in M : M \cap \cup S_i \in S_i\}.$$

It can be seen that these definitions are independent of the choice of the ordinal η.

We say that S and T are compatible if $S \wedge T$ is stationary. Moreover it can be checked that $\bigwedge$ and $\bigvee$ are exact lower and upper bounds for $\leq$.

2.2. **Large cardinals.** We shall repeatedly use supercompact cardinals which for us are defined as follows.

Definition 2.6. δ is γ-supercompact if for all $S \in V_\gamma$ there is an elementary

$$j : V_\eta \to V_\gamma$$

with $j(\text{crit}(j)) = \delta$ and $S \in j[V_\eta]$. δ is supercompact if it is γ-supercompact for all $\gamma \geq \delta$.

We shall also use this equivalent characterization of supercompactness.

Proposition 2.7. *The following are equivalent for a limit ordinal $\gamma > \delta$:*

- *δ is ξ-supercompact for all $\xi < \gamma$,*
- *For all $\xi < \gamma$ the set*

  $$\{M \prec V_\xi : M \cap \delta \in \delta \text{ and } (M, \in) \text{ is isomorphic to some } (V_\alpha, \in)\}$$

  is stationary.

We shall also repeatedly mention Woodin cardinals; however, we shall never actually need to employ them, so we dispense with their definition and we remark that if there is an elementary $j : V_{\alpha+1} \to V_{\delta+1}$, then both α and δ are Woodin cardinal and any normal measure on δ concentrates on Woodin cardinals below δ.

Definition 2.8. δ is a Σ_2-reflecting cardinal if it is inaccessible and for all formulas $\phi(p)$ with $p \in V_\delta$
$$\exists\gamma V_\gamma \models \phi(p)$$
if and only if there exists $\alpha < \delta$ such that
$$V_\alpha \models \phi(p).$$

Finally we will need the notion of super almost huge cardinals and of Laver function for almost huge embeddings which for us are defined as follows.

Definition 2.9. An elementary $j : V \to M$ with $\mathrm{crit}(j) = \delta$ is almost huge if $M^{<j(\delta)} \subset M \subset V$.

δ is super almost huge if for all $\lambda > \delta$ there is an almost huge $j : V \to M$ with $\mathrm{crit}(j) = \delta$ and $j(\delta) > \lambda$.

$f : \delta \to V_\delta$ is a Laver function for almost huge embeddings if for all X there is an almost huge $j : V \to M$ with $\mathrm{crit}(j) = \delta$ such that $j(f)(\delta) = X$.

Fact 2.10 ([6, Theorem 12, Fact 13])**.** *Assume $j : V \to M$ is elementary and such that $M^{j^2(\delta)} \subset M \subset V$. Then $V_{j(\delta)}$ models that δ carries a Laver function for almost huge embeddings.*

2.3. **Posets and their boolean completions.** We refer the reader to [30] for a detailed account on the results presented in this and the next subsections.

Given a partial order $(P, \leq_P)$ we let $\mathsf{RO}(P)$ denote its boolean completion given by the regular open set in the order topology on P (the topological space whose points are the elements of P and whose open sets are the downward closed subsets of P with respect to $\leq_P$).

Given partial orders $(P, \leq_P)$, $(Q, \leq_Q)$, $i : P \to Q$ is a regular embedding if it is order and incompatibility preserving and maps maximal antichain to maximal antichains.

Remark that if $i : P \to Q$ is a regular embedding, i gives rise to a complete injective homomorphism $\bar{i} : \mathsf{RO}(P) \to \mathsf{RO}(Q)$ mapping a regular open set $A \subseteq P$ (in the order topology on P) to the regular open set[7] $\mathring{\overline{i[A]}}$ (in the order topology on Q).

Let G be V-generic for P. The quotient forcing $(Q/i[G], \leq_{Q/_{i[G]}})$ is the partial order defined in $V[G]$ as follows:

- $q \in Q/i[G]$ if $q \in Q$ is compatible with all elements in $i[G]$,
- $q \leq_{Q/_{i[G]}} r$ if for all $s \in Q/i[G]$ such that $s \leq_Q q$ there is $t \leq_Q s, r$ such that $t \in Q/i[G]$.

We identify $Q/i[G]$ with its separative quotient given by equivalence classes of the form $[q]_{i[G]}$ induced by the order relation $\leq_{Q/_{i[G]}}$.

In almost all cases (with the notable exception of the discussion around Theorem 3.9) it will be more convenient for us to deal with these concepts in the language of boolean algebras rather than in the language of posets; thus we introduce also the following notation.

Let B and Q be complete boolean algebras. We say that a complete homomorphism (or complete embedding) $i : \mathsf{B} \to \mathsf{Q}$ is *regular* if it is injective. We remark

[7]For a given topological space (X, τ) and $B \subseteq X$ $\mathring{\overline{B}}$ denote the interior of the closure of B in the topology τ.

that if $i : \mathsf{B} \to \mathsf{Q}$ is a complete homomorphism and B is atomless, $i[\mathsf{B}]$ is a complete and atomless subalgebra of Q isomorphic via i to $\mathsf{B} \restriction \mathrm{coker}(i)$ where

$$\mathrm{coker}(i) = \neg_{\mathsf{B}} \bigvee_{\mathsf{B}} \{b : i(b) = 0_{\mathsf{Q}}\}.$$

We also remark that if $i : \mathsf{B} \to \mathsf{Q}$ and J is an ideal on B, letting $K = \downarrow i[J]$ (where $\downarrow X$ is the downward closure of X), we can define $i/J : \mathsf{B}/J \to \mathsf{Q}/K$ by $[b]_J \mapsto [i(b)]_K$.

The above notion of quotients introduced for posets and boolean algebras are correlated as follows: In the case that G is V-generic for $\mathsf{RO}(P)$, J is its dual ideal and $i : P \to Q$ is a regular embedding, we get that in $V[G]$, $\mathsf{RO}(Q/i[G])^{V[G]}$ is isomorphic to $\mathsf{RO}(Q)^V/i[J]$.

Also, assuming G is a filter on B and J is its dual ideal, we shall feel free to denote by i/G and B/G the boolean algebra B/J and the homomorphism i/J. We define V^{B} as the class of $\tau \in V$ such that $\tau : V^{\mathsf{B}} \to \mathsf{B}$. The canonical name in V^{B} for the V-generic filter is denoted by $\dot{G}_{\mathsf{B}}$, we will let $\mathrm{val}_G(\tau) = \{\mathrm{val}_G(\sigma) : \tau(\sigma) \in G\}$ denote the evaluation map induced by a V-generic filter G for B on B-names.

2.4. SSP-forcings and SSP-correct embeddings.

Proposition 2.11 ([30, Proposition 2.11]). *Assume $i : \mathsf{B} \to \mathsf{Q}$ is a complete homomorphism. Let $\hat{i} : V^{\mathsf{B}} \to V^{\mathsf{Q}}$ be defined by the requirement that*

$$\hat{i}(\tau)(\hat{i}(\sigma)) = i \circ \tau(\sigma)$$

for all $\sigma \in \mathrm{dom}(\tau) \in V^{\mathsf{B}}$. Then for all provably Δ_1-properties $\phi(x_0, \dots, x_n)$

$$i([\![\phi(\tau_0, \dots, \tau_n)]\!]_{\mathsf{B}}) = [\![\phi(\hat{i}(\tau_0), \dots, \hat{i}(\tau_n))]\!]_{\mathsf{Q}}.$$

We denote by SSP the class of stationary set preserving partial orders and by SP the class of semiproper partial orders (see [30, Definition 6.1, Definition 6.4] for a definition of semiproperness). We recall that semiproper posets are stationary set preserving and that P is stationary set preserving if

$$P \Vdash S \text{ is stationary}$$

for all S in the V stationary subset of ω_1.

Given $i : \mathsf{B} \to \mathsf{Q}$ we let $\mathsf{Q}/i[\dot{G}_{\mathsf{B}}]$ be a B-name for the quotient (living in $V[G]$ with G V-generic filter for B) of the boolean algebra Q by the ideal generated by the dual of $i[G]$ (which we shall denote by $\mathsf{Q}/i[G]$).

The following type of complete embeddings will be of central interest for us.

Definition 2.12. Let Γ be a definable class of partial orders. A complete homomorphism $i : \mathsf{B} \to \mathsf{Q}$ is Γ-correct if $\mathsf{Q} \in \Gamma$ and

$$[\![\mathsf{Q}/i[\dot{G}_{\mathsf{B}}] \in \Gamma]\!]_{\mathsf{B}} = 1_{\mathsf{B}}.$$

We shall repeatedly use the following facts.

Fact 2.13. *Assume $P \in \mathsf{SSP}$ and $\dot{Q} \in V^{\mathsf{RO}(P)}$ is a P-name for a partial order. Let $\mathsf{RO}(P) = \mathsf{B}$, $\mathsf{RO}(P * \dot{Q}) = \mathsf{Q}$, and $i : \mathsf{B} \to \mathsf{Q}$ be the associated canonical embedding between the respective boolean completions. Then $P \Vdash \dot{Q} \in \mathsf{SSP}$ (or equivalently $i : \mathsf{RO}(P) \to \mathsf{RO}(P * \dot{Q})$ is SSP-correct) iff for all $\dot{S} \in V^{\mathsf{RO}(P)}$*

$$i([\![\dot{S} \text{ is a stationary subset of } \omega_1]\!]_{\mathsf{B}}) = [\![\hat{i}(\dot{S}) \text{ is a stationary subset of } \omega_1]\!]_{\mathsf{Q}}.$$

Definition 2.14. Let H be V-generic for Q and $G \in V[H]$ be V-generic for B. We say that G is SSP-correct if[8] $\mathsf{NS}_{\omega_1}^{V[G]} = \mathsf{NS}_{\omega_1}^{V[H]} \cap V[G]$.

Fact 2.15. *Let $\mathsf{Q} \in \mathsf{SSP}$ and $i : \mathsf{B} \to \mathsf{Q}$ be a complete homomorphism. Then i is SSP-correct iff for all H V-generic filters for Q we have that $i^{-1}[H]$ is an SSP-correct V-generic filter for B.*

We feel free (except in some specific cases arising in the next section) to write $i : \mathsf{B} \to \mathsf{Q}$ is correct and $G \in V[H]$ is correct to abbreviate i, G are SSP-correct.

We shall also need the following proposition.

Proposition 2.16. *Let Γ be either the class SP or the class SSP. Let Q, Q_0, Q_1 be complete boolean algebras, and let G be a V-generic filter for Q. Let i_0, i_1, j form a commutative diagram of complete homomorphisms as in the following picture:*

$$\begin{tikzcd} \mathsf{Q} \arrow[r, "i_0"] \arrow[dr, "i_1"'] & \mathsf{Q}_0 \arrow[d, "j"] \\ & \mathsf{Q}_1 \end{tikzcd}$$

Then j, i_0, i_1 are Γ-correct homomorphisms in V iff in $V[G]$ $\mathsf{Q}_0/i_0[G], \mathsf{Q}_1/i_1[G]$ are both in Γ and $j/_G : \mathsf{Q}_0/i_0[G] \to \mathsf{Q}_1/i_1[G]$ is a Γ-correct homomorphism.

For the case $\Gamma = \mathsf{SP}$ see the proof of [30, Proposition 7.4]. The case $\Gamma = \mathsf{SSP}$ can be proved along the same lines.

We shall repeatedly apply the above proposition in the following context.

Proposition 2.17. *Assume G is V-generic for $\mathsf{B} \in \mathsf{SSP}$.*

(1) Assume $k_j : \mathsf{B} \to \mathsf{Q}_j$ and $l_j : \mathsf{Q}_j \to \mathsf{R}$ are correct homomorphisms in V such that $l_0 \circ k_0 = l_1 \circ k_1 = l$. Then in $V[G]$ both $l_j/_G : \mathsf{Q}_j/_{k_j[G]} \to \mathsf{R}/_{l[G]}$ are correct homomorphisms.

(2) Assume $k_j : \mathsf{B} \to \mathsf{Q}_j$ are correct homomorphisms in V, $i_j : \mathsf{Q}_j/_{k_j[G]} \to \mathsf{Q}$ are correct homomorphisms in $V[G]$ for $j = 0, 1$. Then there are in V the following:
- $\mathsf{R} \in \mathsf{SSP}$,
- *a correct homomorphism $l : \mathsf{B} \to \mathsf{R}$,*
- *correct homomorphisms $l_j : \mathsf{Q}_j \to \mathsf{R}$*

such that
- Q *is isomorphic to $\mathsf{R}/_{l[G]}$ in $V[G]$.*
- $l_j/_G = i_j$ *for $j = 0, 1$ (modulo the isomorphism identifying $\mathsf{R}/_{l[G]}$ and Q),*
- $l_j \circ k_j = l$ *for $j = 0, 1$,*
- $0_{\mathsf{R}} \notin l[G]$.

Proof. We sketch a proof of the second item. Let $\dot{\mathsf{Q}}, \dot{i}_j \in V^{\mathsf{B}}$ be such that for both j $\mathrm{val}_G(\dot{i}_j) = i_j$ and

$$b_j = [\![\dot{i}_j : \mathsf{Q}_j/_{k_j[\dot{G}_{\mathsf{B}}]} \to \dot{\mathsf{Q}} \text{ is a correct homomorphism}]\!]_{\mathsf{B}} \in G.$$

[8] NS_{ω_1} stands for the non-stationary ideal on ω_1.

Let $b = b_0 \wedge b_1$ and define $l_j : \mathsf{Q}_j \to \mathsf{R} = \mathsf{B} \upharpoonright b * \dot{\mathsf{Q}}$ by

$$l_j(q) = \langle b \wedge \bigwedge_{\mathsf{B}} \{a : k_j(a) \geq q\}, \dot{i}_j([q]_{k_j[\dot{G}_{\mathsf{B}}]})\rangle$$

and $l : \mathsf{B} \to \mathsf{B} \upharpoonright b * \dot{\mathsf{Q}}$ by $l(r) = \langle b \wedge r, 1_{\dot{\mathsf{Q}}}\rangle$. We leave to the reader to check that these definitions work. □

Definition 2.18. Assume λ, ν are regular cardinals.

B is $< \lambda$-CC if all antichains in B have sizes less than λ.

B is $(< \nu, < \lambda)$-presaturated if for all $\gamma < \lambda$, all families $\{A_\alpha : \alpha < \gamma\}$ of maximal antichains of B, and all $b \in \mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$, there is $q \leq b$ in B^+ such that

$$|\{a \in A_\alpha : a \wedge q > 0_{\mathsf{Q}}\}| < \nu$$

for all $\alpha < \gamma$.

Fact 2.19. *For all regular cardinals $\nu \geq \lambda$, B is $(< \nu, < \lambda)$-presaturated iff for all $\gamma < \lambda$ and all $\dot{f} : \gamma \to \nu$ in V^{B}, 1_{B} forces that $\operatorname{rng}(\dot{f})$ is bounded below ν.*

Definition 2.20. Let $A = \{\mathsf{B}_j : j \in J\}$ be a family of complete boolean algebras and $i_j : \mathsf{B} \to \mathsf{B}_j$ be a complete homomorphism for all $j \in J$. We let

- $\mathsf{B}_A = \bigvee_{j \in J} \mathsf{B}_j$ (the lottery sum of A) be defined as the boolean algebra given by the set of functions $f : \gamma \to \bigcup\{\mathsf{B}_i : i < \gamma\}$ such that $f(i) \in \mathsf{B}_i$ for all $i < \gamma$ with boolean operations given componentwise by
  - $f \wedge_{\mathsf{B}_A} g = (f(i) \wedge_{\mathsf{B}_i} g(i) : i < \gamma)$,
  - $\bigvee_{\mathsf{B}_A} \{f_\nu : \nu < \theta\} = (\bigvee_{\mathsf{B}_i} \{f_\nu(i) : \nu < \theta\} : i < \gamma)$,
  - $\neg_{\mathsf{B}_A} f = (\neg_{\mathsf{B}_i} f(i) : i < \gamma)$;
- $i_A : \mathsf{B} \to \mathsf{B}_A$ be defined by $i_A(b) = \langle i_j(b) : j \in J\rangle$.

We leave to the reader to check the following propositions.

Proposition 2.21. *Let $A = \{\mathsf{B}_j : j \in J\}$ be a family of complete boolean algebras and $i_j : \mathsf{B} \to \mathsf{B}_j$ be an* SSP *(or* SP*) correct homomorphism for all $j \in J$. Then i_A is also an* SSP *(*SP*) correct homomorphism.*

Proposition 2.22. *Assume A is a maximal antichain of B. Then B is isomorphic to $\bigvee_{a \in A} \mathsf{B} \upharpoonright a$.*

2.5. MM^{++}.

Definition 2.23. Let $\mathsf{B} \in V$ be a complete boolean algebra and $M \prec H_{|\mathsf{B}|^+}$.

$H \subset \mathsf{B} \cap M$ is M-generic for B if H is a filter on the boolean algebra $\mathsf{B} \cap M$ and $H \cap D \neq \emptyset$ for all $D \in M$ predense subsets of B.

Given an $M \prec H_{|\mathsf{B}|^+}$ such that $\omega_1 \subset M$, let $\pi_M : M \to N$ be the transitive collapse map.

$H \subset \mathsf{B}$ is a *correct* M-generic filter for B if it is M-generic for B, and letting $G = \pi_M[H]$ we have that $N[G]$ is stationarily correct, i.e.,

$$\mathsf{NS}_{\omega_1}^{N[G]} = \mathsf{NS}_{\omega_1}^V \cap N[G],$$

where $N[G] = \{\operatorname{val}_G(\tau) : \tau \in N^{\pi_M(\mathsf{B})}\}$.

$$T_{\mathsf{B}} = \{M \prec H_{|\mathsf{B}|^+} : M \in P_{\omega_2} \text{ and there is a correct } M\text{-generic filter for } \mathsf{B}\}.$$

We formulate MM^{++} as the assertion that $T_{\mathsf{RO}(P)}$ is stationary for all $P \in \mathsf{SSP}$. For an equivalence of this formulation with other more common formulations see [6, Lemma 3].

We also say that MM^{++} up to α holds if $T_{\mathsf{RO}(P)}$ is stationary for all $P \in \mathsf{SSP}$ with a dense subset of size less than α.

We shall need the following local version of the celebrated proof of the consistency of MM by Foreman, Magidor, and Shelah.

Theorem 2.24. *Assume δ is $\delta + \omega + 1$-supercompact and $\mathsf{B} \in \mathsf{SSP} \cap V_\delta$. Then there is Q of size δ and $i : \mathsf{B} \to \mathsf{Q}$ such that*

- $[\![\mathsf{Q}/_{k[\dot{G}_{\mathsf{B}}]}$ *is semiproper*$]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$,
- Q *forces* MM^{++} *up to* $\beth(\omega)$ *while collapsing δ to become ω_2.*

Proof. Sketch: adapt the proof of the original result as presented in [12, Theorem 37.9] or in [30, Theorem 8.5] to the different assumptions we are making on δ. □

2.6. **Iterated forcing.** We feel free to view iterations following the approach presented in [30] which expands on the work of Donder and Fuchs on revised countable support iterations [10]. We refer the reader to [30, Section 3] for the relevant definitions and to Sections 6 and 7 of the same paper for the relevant proofs. Here we recall the minimal amount of information we need to make sense of our use of these results in this paper.

Definition 2.25. Let $i : \mathsf{B} \to \mathsf{Q}$ be a regular embedding, the *retraction* associated to i is the map

$$\begin{array}{rccl} \pi_i : & \mathsf{Q} & \to & \mathsf{B} \\ & c & \mapsto & \bigwedge\{b \in \mathsf{B} : \ i(b) \geq c\}. \end{array}$$

Proposition 2.26 ([30, Proposition 2.6])**.** *Let $i : \mathsf{B} \to \mathsf{Q}$ be a regular embedding and $b \in \mathsf{B}$, $c, d \in \mathsf{Q}$ be arbitrary. Then,*

(1) $\pi_i \circ i(b) = b$; hence π_i is surjective;
(2) $i \circ \pi_i(c) \geq c$; hence π_i maps $\mathsf{Q}^+ = \mathsf{Q} \setminus \{0_{\mathsf{Q}}\}$ to $\mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$;
(3) π_i preserves joins, i.e., $\pi_i(\bigvee X) = \bigvee \pi_i[X]$ for all $X \subseteq \mathsf{Q}$;
(4) $i(b) = \bigvee\{e : \pi_i(e) \leq b\}$;
(5) $\pi_i(c \wedge i(b)) = \pi_i(c) \wedge b = \bigvee\{\pi_i(e) : e \leq c, \pi_i(e) \leq b\}$;
(6) π_i does not preserve and neither meets nor complements whenever i is not surjective, but $\pi_i(d \wedge c) \leq \pi_i(d) \wedge \pi_i(c)$ and $\pi_i(\neg c) \geq \neg\pi_i(c)$.

For a definition of semiproperness and of semiproper embedding see [30, Definition 6.1, Definition 6.4].

Definition 2.27.

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \leq \beta < \gamma\}$$

is an iteration system if each B_α is a complete boolean algebra and each $i_{\alpha,\beta}$ is a regular embedding such that

- whenever $\alpha \leq \beta \leq \gamma$, $i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma}$.
- $i_{\alpha,\alpha}$ is the identity mapping for all $\alpha < \gamma$.

- The full limit $T(\mathcal{F})$ of $\mathcal{F}$ is given by threads $f : \gamma \to V$ such that $\pi_{\alpha,\eta} \circ f(\eta) = f(\alpha)$ for all $\alpha \leq \eta < \gamma$, where $\pi_{\alpha,\eta}$ is the retraction associated to $i_{\alpha,\eta}$. For $f, g \in T(\mathcal{F})$, $f \leq_{T(\mathcal{F})} g$ iff for all $\alpha < \gamma$, $f(\alpha) \leq_{\mathsf{B}_\alpha} g(\alpha)$.
- The direct limit $C(\mathcal{F})$ of an iteration system

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \leq \beta < \gamma\}$$

is the partial order whose elements are the eventually constant threads $f \in T(\mathcal{F})$, i.e., threads f such that for some $\alpha < \gamma$ and all $\beta > \alpha$, $i_{\alpha\beta} \circ f(\alpha) = f(\beta)$. For $f \in C(\mathcal{F})$ the least such α is called the support of f.
- The revised countable support (RCS) limit is

$$\mathcal{RCS}(\mathcal{F}) = \{f \in T(\mathcal{F}) : \ f \in C(\mathcal{F}) \vee \exists \alpha \ f(\alpha) \Vdash_{\mathsf{B}_\alpha} \mathrm{cf}(\check{\lambda}) = \check{\omega}\};$$

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \leq \beta < \gamma\}$$

is a semiproper iteration system if for all $\alpha \leq \beta < \gamma$,

- $[\![\mathsf{B}_\beta / i_{\alpha\beta}[\dot{G}_\alpha] \text{ is semiproper}]\!]_{\mathsf{B}_\alpha} = 1_{\mathsf{B}_\alpha}$,
- $\mathsf{B}_{\alpha+1}$ forces that $|\mathsf{B}_\alpha| = \aleph_1$,
- B_α is the boolean completion of $\mathcal{RCS}(\mathcal{F} \restriction \alpha)$ if α is the limit.

We need the following two results of Shelah:

(1) (Shelah [30, Theorem 7.11]). Let

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \leq \beta < \gamma\}$$

be a semiproper iteration system. Then its RCS limit is a semiproper partial order.

(2) (Shelah [12, Theorem 39.10]) Assume α is a limit ordinal and T_{B} is stationary for all $\mathsf{B} \in \mathsf{SSP} \cap V_\alpha$. Then any $\mathsf{B} \in \mathsf{SSP} \cap V_\alpha$ is semiproper.

3. Category forcings

Assume Γ is a class of partial orders and Θ is a family of complete homomorphisms between the boolean completions of elements of Γ closed under composition and which contains all identity maps.

We let (Γ, Θ) denote the category whose objects are the complete boolean algebras in Γ and whose arrows are given by complete homomorphisms $i : \mathsf{B} \to \mathsf{Q}$ in Θ. We call embeddings in Θ, Θ-correct embeddings. Notice that these categories immediately give rise to natural class partial orders associated with them, partial orders whose elements are the complete boolean algebras in Γ and whose order relation is given by the arrows in Θ. We denote these class partial orders by $\mathsf{U}^{\Gamma,\Theta}$. Depending on the choice of Γ and Θ these partial orders can be trivial; for example, note the following remark.

Remark 3.1. Assume Γ is the class of all complete boolean algebras and Θ is the class of all complete embeddings, and then any two conditions in $\mathsf{U}^{\Gamma,\Theta}$ are compatible; i.e., $\mathsf{U}^{\Gamma,\Theta}$ is forcing equivalent to the trivial partial order. This is the case since for any pair of partial orders P, Q, and X of size larger than $2^{|P|+|Q|}$ there are regular embeddings of $\mathsf{RO}(P)$ and $\mathsf{RO}(Q)$ into the boolean completion of $Coll(\omega, X)$. These embeddings witness the compatibility of $\mathsf{RO}(P)$ with $\mathsf{RO}(Q)$.

Since we want to allow ourselves more freedom in the handling of our class forcings $\mathsf{U}^{\Gamma,\Theta}$, we allow elements of the category Γ to be arbitrary partial orders[9] in Γ and we identify the arrows in Θ between the objects P and Q in Γ to be the Θ-correct homomorphisms between the boolean completions of P and Q. In this paper we actually focus on the class forcing $\mathsf{U}^{\mathsf{SSP}}$ given by the class of stationary set preserving partial orders and the family of all SSP-correct homomorphisms between their boolean completions. The main reason is that it is just for this category that we can predicate all the properties of category forcings in which we are interested. We just write that i is a correct embedding whenever i is an SSP-correct embedding.

3.1. **Basic properties of $\mathsf{U}^{\mathsf{SSP}}$.** We start to outline some basic properties of this category forcing.

Incompatibility in $\mathsf{U}^{\mathsf{SSP}}$. First of all we show that $\mathsf{U}^{\mathsf{SSP}}$ is non-trivial.

Remark 3.2. $\mathsf{U}^{\mathsf{SSP}}$ seen as a class partial order is not a trivial partial order. For example, observe that if P is Namba forcing on $\aleph_2$ and Q is $Coll(\omega_1, \omega_2)$, then $\mathsf{RO}(P), \mathsf{RO}(Q)$ are incompatible conditions in $\mathsf{U}^{\mathsf{SSP}}$: If $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{RO}(P), \mathsf{RO}(Q)$, we would have that if H is V-generic for R, $\omega_1^{V[H]} = \omega_1$ (since $\mathsf{R} \in \mathsf{SSP}$) and there are $G, K \in V[H]$ V-generic filters for P and Q, respectively (since $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{RO}(P), \mathsf{RO}(Q)$). G allows us to define in $V[H]$ a sequence cofinal in ω_2^V of type ω while K allows us to define in $V[H]$ a sequence cofinal in ω_2^V of type $(\omega_1)^V$. These two facts entail that $V[H]$ models that $\mathrm{cof}(\omega_1^V) = \omega$ contradicting the assumption that $\omega_1^{V[H]} = \omega_1$.

Suprema in $\mathsf{U}^{\mathsf{SSP}}$. The lottery sum defines a natural $\bigvee$ operation of suprema on subsets of $\mathsf{U}^{\mathsf{SSP}}$.

Proposition 3.3. *Let $A = \{\mathsf{B}_i : i < \gamma\}$ be a family of stationary set preserving complete boolean algebras. Then B_A is the exact upper bound of A.*

Proof. Left to the reader. □

Why this ordering on SSP *partial orders?* Given a pair (Γ, Θ) as above, we can define two natural order relations $\leq_\Theta$ and $\leq_\Theta^*$ on Γ. The first one is given by complete homomorphisms $i : \mathsf{B} \to \mathsf{Q}$ in Θ (which is the one we described before), and the other is given by *regular* (i.e., complete and injective homomorphisms) embeddings $i : \mathsf{B} \to \mathsf{Q}$ in Θ. Both notions of orders are interesting, and as set theorists we are used to focusing on this second stricter notion of order since it is the one suitable to develop a theory of iterated forcing. However, in the present paper we focus mostly on complete (but possibly non-injective) homomorphisms because this notion of ordering will grant us that whenever B is put into a V-generic filter for $\mathsf{U}^{\mathsf{SSP}} \cap V_\delta = \mathsf{U}_\delta$, then this V-generic filter for U_δ will also add a V-generic filter for B. If we decided to order the family $\mathsf{SSP} \cap V_\delta$ using *regular* embeddings, we would get that a generic filter for this other category forcing defined according to this stricter notion of order will just give a directed system of SSP-partial orders with regular embeddings between them, without actually giving V-generic filters for the partial orders in this directed system. On the other hand, if we use the iteration theorems for the class of semiproper forcings we actually get the following proposition.

[9]Specifically our main aim is to show that for certain categories (Γ, Θ) $\Gamma \cap V_\delta \in \mathsf{U}^{\Gamma,\Theta}$. In general $(\Gamma \cap V_\delta, \leq_\Theta \cap V_\delta)$ is a non-separative partial order.

Proposition 3.4. *Let*

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \le \beta < \gamma\}$$

be an iteration system such that $i_{\alpha,\beta}$ is SP-correct for all $\alpha \le \beta < \gamma$ with $\mathsf{B}_0 \in \mathsf{SSP}$. Then $\mathcal{RCS}(\mathcal{F}) \in \mathsf{SSP}$ as well.

This gives quite easily that the class forcing $(\mathsf{SSP}, \le^*_{\mathsf{SP}})$ is closed under set sized descending sequences. This observation is useful to prove nice weak distributivity properties of the class forcing $(\mathsf{SSP}, \le_{\mathsf{SP}})$. On the other hand, there is a key combinatorial feature of the class forcing $(\mathsf{SSP}, \le_{\mathsf{SSP}})$ (freezeability; see Definition 3.10) which we are not able to predicate for the class forcing $(\mathsf{SSP}, \le_{\mathsf{SP}})$, unless we assume large cardinals. In the presence of supercompact cardinals the equality $\mathsf{SP} = \mathsf{SSP}$ can be forced by a semiproper forcing. In particular we aim to use large cardinals to expand the above equality to the extent to be able to identify the class forcings $(\mathsf{SSP}, \le_{\mathsf{SSP}})$ and $(\mathsf{SSP}, \le_{\mathsf{SP}})$ on a dense subset TR (the class of totally rigid elements of SSP which force MM^{++}; see Definition 3.6). In this way on $(\mathrm{TR}, \le_{\mathsf{SSP}}) = (\mathrm{TR}, \le_{\mathsf{SP}})$ we can have at the same time the nice closure properties of $\le^*_{\mathsf{SP}}$ with the nice combinatorial features of $\le_{\mathsf{SSP}}$.

Rank initial segments of $\mathsf{U}^{\mathsf{SSP}}$ are stationary set preserving posets. The first main result of this section is the following theorem.

Theorem 3.5. *Assume δ is an inaccessible limit of $< \delta$-supercompact cardinals. Then*

(1) $\mathsf{U}_\delta = \mathsf{U}^{\mathsf{SSP}} \cap V_\delta \in \mathsf{SSP}$ is totally rigid and collapses δ to become $\aleph_2$.
(2) If δ is supercompact, U_δ forces MM^{++}.

3.2. **Totally rigid partial orders for $\mathsf{U}^{\mathsf{SSP}}$.** Total rigidity is the key property which we would like to be able to predicate for a category forcing.

Definition 3.6. Assume P is a partial order in Γ. P is Θ-totally rigid if for no complete boolean algebra $\mathsf{C} \in \Gamma$ there are distinct complete homomorphisms $i_0 : \mathsf{RO}(P) \to \mathsf{C}$ and $i_1 : \mathsf{RO}(P) \to \mathsf{C}$ in Θ.

We shall see that the class of SSP-totally rigid partial orders is dense in $\mathsf{U}^{\mathsf{SSP}}$. This result will be the cornerstone on which we will elaborate to get the desired generic absoluteness theorem. As it will become transparent to the reader at the end of this paper, we should be able to prove the appropriate form of generic absoluteness for any "reasonable" class of forcings Γ for which we can predicate the existence of a dense class of totally rigid partial orders in U^Γ and for which we have an iteration theorem. We shall from now on just use "totally rigid" to abbreviate "SSP-totally rigid."

These properties give equivalent characterizations of totally rigid boolean algebras.

Lemma 3.7. *The following are equivalent:*

(1) For all $b_0, b_1 \in \mathsf{B}$ such that $b_0 \wedge_{\mathsf{B}} b_1 = 0_{\mathsf{B}}$ we have that $\mathsf{B} \restriction b_0$ is incompatible with $\mathsf{B} \restriction b_1$ in $\mathsf{U}^{\mathsf{SSP}}$.
(2) For all $\mathsf{Q} \le_{\mathsf{SSP}} \mathsf{B}$ and all H, V-generic filter for Q, there is just one $G \in V[H]$ correct V-generic filter for B.

(3) *For all* $\mathsf{Q} \leq \mathsf{B}$ *in* $\mathsf{U}^{\mathsf{SSP}}$ *there is only one complete homomorphism* $i : \mathsf{B} \to \mathsf{Q}$ *such that*

$$\mathsf{B} \Vdash \mathsf{Q}/i[\dot{G}_{\mathsf{B}}] \textit{ is stationary set preserving.}$$

Proof. We prove these equivalences as follows:

- We first prove 3 implies 2 by contraposition. Assume 2 fails for B as witnessed by some $\mathsf{Q} \leq_{\mathsf{SSP}} \mathsf{B}$, H V-generic filter for Q, and $G_1 \neq G_2 \in V[H]$ correct V-generic filters for B.

  Let $\dot{G}_1, \dot{G}_2 \in V^{\mathsf{Q}}$ and $q \in H$ be such that q is the boolean value of the statement

  $\dot{G}_1 \neq \dot{G}_2$ are V-generic filters for B and $V[\dot{H}]$ is a $V[\dot{G}_j]$-generic extension by an SSP forcing in $V[\dot{G}_j]$ for $j = 1, 2$.

  Notice that the above statement is expressible by a forcing formula in the parameters

  $$H_{\omega_2}^{V[\dot{G}_j]}, \omega_1, \dot{G}_1, \dot{G}_2, \dot{G}_{\mathsf{Q}}, \check{\mathsf{B}}, \check{H_{|\mathsf{B}|^+}}$$

  stating that

  for both $j = 1, 2$, $\dot{G}_j$ are distinct filters on $\check{\mathsf{B}}$ meeting all dense subsets of $\check{\mathsf{B}}$ in $\check{H_{|\mathsf{B}|^+}}$, and for all $\dot{S}$ in $H_{\omega_2}^{V[\dot{G}_j]}$, $\dot{S}$ is a stationary subset of $\check{\omega_1}$ in $H_{\omega_2}^{V[\dot{G}_j]}$ iff it is such in $V[\dot{G}_{\mathsf{Q}}]$.

  Let $r \leq_{\mathsf{Q}} q$ and $b \in \mathsf{B}$ be such that $r \Vdash_{\mathsf{Q}} b \in \dot{G}_1 \setminus \dot{G}_2$.

  Define $i_j : \mathsf{B} \to \mathsf{Q} \upharpoonright r$ by $a \mapsto [\![\check{a} \in \dot{G}_j]\!] \wedge r$.

  Then we get that $i_0(b) = r = i_1(\neg b)$, and thus $i_0 \neq i_1$. We can check that i_0, i_1 are correct embeddings as follows: First of all

  $$r \leq_{\mathsf{Q}} [\![i_j^{-1}[\dot{G}_{\mathsf{Q}}] = \dot{G}_j]\!]_{\mathsf{Q}}$$

  and

  $$r \leq_{\mathsf{Q}} [\![b \in \dot{G}_0 \setminus \dot{G}_1]\!]_{\mathsf{Q}}.$$

  Now observe that if $\dot{S} \in V^{\mathsf{B}}$ is such that

  $$[\![\dot{S} \text{ is a stationary subset of } \omega_1]\!]_{\mathsf{B}} = 1_{\mathsf{B}},$$

  we get that

  $$[\![\hat{i}_j(\dot{S}) \in V[\dot{G}_j]]\!]_{\mathsf{Q}} \geq r$$

  and

  $$[\![\hat{i}_j(\dot{S}) \text{ is a stationary subset of } \omega_1 \text{ in } V[\dot{G}_j]]\!]_{\mathsf{Q}} \geq r.$$

  Thus since r forces that $V[\dot{G}_{\mathsf{Q}}]$ is a generic extension of $V[\dot{G}_j]$ preserving stationary subsets of ω_1, we get that

  $$[\![\hat{i}_j(\dot{S}) \text{ is a stationary subset of } \omega_1 \ ]\!]_{\mathsf{Q}} \geq r.$$

  This shows that i_1, i_2 are distinct correct embeddings witnessing that 3 fails for B.

- Now we prove that 1 implies 3 again by contraposition. So assume 3 fails for B as witnessed by $i_0 \neq i_1 : \mathsf{B} \to \mathsf{Q}$. Let b be such that $i_0(b) \neq i_1(b)$.

  Without loss of generality (w.l.o.g.) we can suppose that $r = i_0(b) \wedge i_1(\neg b) > 0_{\mathsf{Q}}$. Then $j_0 : \mathsf{B} \upharpoonright b \to \mathsf{Q} \upharpoonright r$ and $j_1 : \mathsf{B} \upharpoonright \neg b \to \mathsf{Q} \upharpoonright r$ given by $j_k(a) = i_k(a) \wedge r$ for $k = 0, 1$ and a in the appropriate domain witness that $\mathsf{B} \upharpoonright \neg b$ and $\mathsf{B} \upharpoonright b$ are compatible in $\mathsf{U}^{\mathsf{SSP}}$; i.e., 1 fails.

- Finally we prove that 2 implies 1 again by contraposition.

  So assume 1 fails as witnessed by $i_j : \mathsf{B} \restriction b_j \to \mathsf{Q}$ for $j = 0, 1$ with b_0 incompatible with b_1 in B. Pick H V-generic for Q. Then $G_j = i_j^{-1}[H] \in V[H]$ are distinct and correct V-generic filters for B since $b_j \in G_j \setminus G_{1-j}$. □

The next subsections have as the objective the proof of the following theorems which are the cornerstones on which we shall develop our analysis of $\mathsf{U}^{\mathsf{SSP}}$ and are the other two main results of this section.

Theorem 3.8. *Assume there are class many supercompact cardinals. Let* TR *be the class of partial orders Q such that*

- *Q forces MM^{++} (and thus the equality $\mathsf{SP} = \mathsf{SSP}$),*
- *Q is totally rigid.*

Then TR *is dense in $\mathsf{U}^{\mathsf{SSP},\mathsf{SP}}$ and*[10] *in $\mathsf{U}^{\mathsf{SSP}}$.*

Theorem 3.9. *Assume there are class many supercompact cardinals and B is a stationary set preserving forcing. Then there are in V*

- *a stationary set preserving complete boolean algebra Q,*
- *a regular embedding*
$$i_0 : \mathsf{B} \to \mathsf{Q},$$
- *a regular embedding*
$$i_1 : \mathsf{B} \to \mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q},$$

such that whenever H is V-generic for B, then $V[H]$ models that the class forcing
$$(\mathsf{U}^{\mathsf{SSP}}_\delta \restriction \mathsf{Q})^V / i_1[H]$$
is identified with[11] *the class forcing*
$$(\mathsf{U}^{\mathsf{SSP}})^{V[H]} \restriction (\mathsf{Q}/i_0[H])$$
as computed in $V[H]$.

Moreover the above factorization property reflects down to $\mathsf{U}^{\mathsf{SSP}} \cap V_\delta = \mathsf{U}_\delta$ whenever the latter is stationary set preserving, and $\mathrm{TR} \cap \mathsf{U}_\delta$ is dense in U_δ and B in[12] *U_δ.*

This factorization property of U_δ is not shared by the other forcings which are used to prove the consistency of MM^{++}. It is due to this property of U_δ that we can prove the generic absoluteness results given in Theorem 1.

The following subsections show the proofs of Theorems 3.8, 3.9 and of Theorem 3.5.

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[10]We remark that there is no typo in the statement of the above theorem, i.e., for any B there is a *regular embedding* $i : \mathsf{B} \to \mathsf{Q}$ such that
$$[\![\mathsf{Q}/_{i[\dot{G}_\mathsf{B}]} \text{ is semiproper }]\!]_\mathsf{B} = 1_\mathsf{B}$$
and Q forces MM^{++}. Notice that on the class TR we have that $\mathsf{Q} \leq_{\mathsf{SSP}} \mathsf{B}$ iff $\mathsf{Q} \leq_{\mathsf{SP}} \mathsf{B}$ iff there is a unique SP-correct $i : \mathsf{B} \to \mathsf{Q}$.

[11]That is, we will show that there is an order and incompatibility preserving map with a dense image between these two class forcings.

[12]That is, with the same requirements for $\mathsf{B}, \mathsf{Q}, i_0, i_1$ as in the first conclusion of the theorem, $\mathsf{U}_\delta^{V[G]} \restriction (\mathsf{Q}/i_0[G])$ is forcing equivalent in $V[G]$ to $(\mathsf{U}_\delta^V \restriction \mathsf{Q})/i_1[G]$ whenever G is V-generic for $\mathsf{B} \in V_\delta$, $\mathsf{Q} \in \mathrm{TR} \cap V_\delta$, $i_0, i_1 \in V_\delta$.

3.2.1. *Freezeability.* We shall introduce the concept of freezeability and use it to prove Theorem 3.8.

Definition 3.10. Let (Γ, Θ) be a category of CBAs and complete homomorphisms. $k : \mathsf{B} \to \mathsf{Q}$ Θ-freezes B if for all $\mathsf{R} \leq_\Theta \mathsf{Q}$ and $i_j : \mathsf{Q} \to \mathsf{R}$ in Θ for $j = 0, 1$ we have that $i_0 \circ k = i_1 \circ k$. $\mathsf{Q} \leq_\Theta \mathsf{B}$ Θ-freezes B if there is a $k : \mathsf{B} \to \mathsf{Q}$ which Θ-freezes B.

B is Θ-freezeable if there is some $\mathsf{Q} \leq_\Theta \mathsf{B}$ which Θ-freezes B.

We just us "freezeable" when we mean "SSP-freezeable."

We show that all stationary set preserving posets are freezeable. We need an analogue of Lemma 3.7 to characterize freezeability.

Lemma 3.11. *Let $k : \mathsf{B} \to \mathsf{Q}$ be a correct homomorphism. The following are equivalent:*

(1) For all $b_0, b_1 \in \mathsf{B}$ such that $b_0 \wedge_{\mathsf{B}} b_1 = 0_{\mathsf{B}}$ we have that $\mathsf{Q} \restriction k(b_0)$ is incompatible with $\mathsf{Q} \restriction k(b_1)$ in $\mathsf{U}^{\mathsf{SSP}}$.

(2) For all $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$, all H V-generic filters for R, there is just one $G \in V[H]$ correct V-generic filter for B such that $G = k^{-1}[K]$ for all $K \in V[H]$ correct V-generic filters for Q.

(3) For all $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$ in $\mathsf{U}^{\mathsf{SSP}}$ and $i_0, i_1 : \mathsf{Q} \to \mathsf{R}$ witnessing that $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$ we have that $i_0 \circ k = i_1 \circ k$.

Proof. We prove these equivalences mimicking the proof of Lemma 3.7:

- We first prove 3 implies 2 by contraposition. Assume 2 fails for $k : \mathsf{B} \to \mathsf{Q}$ as witnessed by some $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$, H V-generic filter for R, and $K_0 \neq K_1 \in V[H]$ correct V-generic filters for Q with $b \in k^{-1}[K_0] \setminus k^{-1}[K_1]$. Let $\dot{K}_i \in V^{\mathsf{R}}$ be R-names for K_i and $r \in H$ force that
  $\dot{K}_i$ are correct V-generic filters for Q and $b \in k^{-1}[\dot{K}_0] \setminus k^{-1}[\dot{K}_1]$.
  Now define for $j = 0, 1$, $i_j : \mathsf{Q} \to \mathsf{R} \restriction r$ by
  $$c \mapsto [\![c \in \dot{K}_j]\!]_{\mathsf{R}} \wedge r.$$
  Now observe that $i_0 \circ k(b) = 1_{\mathsf{R}\restriction r}$ while $i_1 \circ k(b) = 0_{\mathsf{R}\restriction r}$. In particular i_0 and i_1 are correct embeddings of Q into R witnessing that 3 fails (we leave to the reader to check the correctness of i_0, i_1 along the same lines of what was done in Lemma 3.7).
- Now we prove that 1 implies 3 again by contraposition. So assume 3 fails for $k : \mathsf{B} \to \mathsf{Q}$ as witnessed by $i_0 \neq i_1 : \mathsf{Q} \to \mathsf{R}$. Let b be such that $i_0 \circ k(b) \neq i_1 \circ k(b)$.
  W.l.o.g. we can suppose that $r = i_0 \circ k(b) \wedge i_1 \circ k(\neg b) > 0_{\mathsf{Q}}$ Then $j_0 : \mathsf{B} \restriction b \to \mathsf{R} \restriction r$ and $j_1 : \mathsf{B} \restriction \neg b \to \mathsf{R} \restriction r$ given by $j_l(a) = i_l \circ k(a) \wedge r$ for $l = 0, 1$ and a in the appropriate domain witness that $\mathsf{B} \restriction \neg b$ and $\mathsf{B} \restriction b$ are compatible in $\mathsf{U}^{\mathsf{SSP}}$; i.e., 1 fails.
- Finally we prove that 2 implies 1 again by contraposition.
  So assume 1 fails as witnessed by $i_j : \mathsf{Q} \restriction k(b_j) \to \mathsf{R}$ for $j = 0, 1$ with b_0 incompatible with b_1 in B. Pick H V-generic for R. Then $G_j = i_j^{-1}[H] \in V[H]$ are distinct correct V-generic filters for Q and $b_j \in k^{-1}[G_j]$ for $j = 0, 1$. Since k is correct, we also have that $k^{-1}[G_j]$ are distinct correct V-generic filters for B in $V[H]$ witnessing that 2 fails (since $b_j \in k^{-1}[G_j] \setminus k^{-1}[G_{1-j}]$).

$\square$

Freezeable posets can be embedded in $\mathsf{U}^{\mathsf{SSP}}$ as follows.

Lemma 3.12. *Assume* Q *freezes* B. *Let* $k : \mathsf{B} \to \mathsf{Q}$ *be a correct and complete embedding of* B *into* Q *which witnesses it. Then the map* $i : \mathsf{B} \to \mathsf{U}^{\mathsf{SSP}} \upharpoonright \mathsf{Q}$ *which maps* $b \mapsto \mathsf{Q} \upharpoonright k(b)$ *is a complete embedding of partial orders.*

Proof. It is immediate to check that i preserve the order relation on B and $\mathsf{U}^{\mathsf{SSP}}$. Moreover Q freezes B if and only if i preserve the incompatibility relation. Thus we only have to check that $i[A]$ is a maximal antichain below Q in $\mathsf{U}^{\mathsf{SSP}}$ whenever A is a maximal antichain of B. If not, there is $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$ such that R is incompatible with $\mathsf{Q} \upharpoonright k(b)$ for all $b \in A$. This means that $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q}$ is incompatible with

$$\mathsf{Q} = \bigvee \{\mathsf{Q} \upharpoonright k(b) : b \in A\},$$

a contradiction. □

We refer the reader to Subsection 2.6 for the relevant definitions and results on iterations and to [30] for a detailed account.

Lemma 3.13. *Assume*

$$\{i_{\alpha\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha < \beta \leq \delta\}$$

is a complete iteration system such that for each α *there is* $\beta > \alpha$ *such that*

- B_β *freezes* B_α *as witnessed by the correct regular embedding* $i_{\alpha,\beta}$.
- B_δ *is the direct limit of the iteration system and is stationary set preserving.*

Then B_δ *is totally rigid.*

Proof. Assume the Lemma fails. Then there are f_0, f_1 incompatible threads in B_δ such that $\mathsf{B}_\delta \upharpoonright f_0$ is compatible with $\mathsf{B}_\delta \upharpoonright f_1$ in $\mathsf{U}^{\mathsf{SSP}}$. Now B_δ is a direct limit, so f_0, f_1 have support in some $\alpha < \delta$. Thus $f_0(\beta)$, $f_1(\beta)$ are incompatible in B_β for all $\alpha < \beta \leq \delta$. Now for eventually all $\beta > \alpha$, B_β freezes B_α as witnessed by $i_{\alpha,\beta}$. In particular, since $f_i = i_{\alpha,\delta} \circ f_i(\alpha)$ for $i = 0, 1$, we get that $\mathsf{B}_\delta \upharpoonright f_0$ cannot be compatible with $\mathsf{B}_\delta \upharpoonright f_1$ in $\mathsf{U}^{\mathsf{SSP}}$, contradicting our assumption. □

The above Lemma shows that in order to prove Theorem 3.8 we just need to exhibit an iteration system in which the direct limit is SSP and all its elements freeze their predecessors.

In the next subsection we shall define for any given SSP poset P a P-name $\dot{Q}$ for an SSP-poset which freezes P. In the subsequent one we will combine this result with the equality $\mathsf{SP} = \mathsf{SSP}$ (which holds in models of MM^{++}) to define the desired iteration systems whose direct limits are totally rigid and whose first factor can be any SSP poset.

3.2.2. *Freezing* SSP *posets.* We shall now define for any given stationary set preserving poset P a poset $\dot{R}_P \in V^P$ such that $\mathsf{Q}_P = P * \dot{R}_P$ freezes P.

Definition 3.14. For any regular cardinal $\kappa \geq \omega_2$ fix

$$\{S^i_\alpha : \alpha < \kappa, i < 2\}$$

a partition of E^ω_κ (the set of points in κ of countable cofinality) in pairwise disjoint stationary sets. Fix

$$\{A_\alpha : \alpha < \omega_1\}$$

partition of ω_1 in ω_1-many pairwise disjoint stationary sets such that $\min(A_\alpha) > \alpha$ and such that there is a club subset of ω_1 contained in

$$\bigcup\{A_\alpha : \alpha < \omega_1\}.$$

Given P a stationary set preserving poset, we fix in V a surjection f of the least regular $\kappa > |P|$ with P. Let $\dot{g}_P : \kappa \to 2$ be the P-name for a function which codes $\dot{G}_P$ using f; i.e., for all $\alpha < |P|$

$$p \Vdash_P \dot{g}_P(\alpha) = 1 \text{ iff } f(\alpha) \in \dot{G}_P.$$

Now let Q_P be the complete boolean algebra $\mathsf{RO}(P * \dot{R}_P)$ where $\dot{R}_P$ is defined as follows in V^P.

Let G be V-generic for P. Let $g = \mathrm{val}_G(\dot{g}_P)$. $R = \mathrm{val}_G(\dot{R}_P)$ in $V[G]$ is the poset given by pairs (c_p, f_p) such that for some countable ordinal α_p

- $f_p : \alpha_p + 1 \to \kappa$,
- $c_p \subseteq \alpha_p + 1$ is closed,
- for all $\xi \in c_p$

$$\xi \in A_\beta \text{ and } g \circ f_p(\beta) = i \text{ if and only if } \sup(f_p[\xi]) \in S^i_{f_p(\beta)}.$$

The order on R is given by $p \leq q$ if $f_p \supseteq f_q$ and c_p end extends c_q. Let

- $\dot{f}_{\mathsf{Q}_P} : \omega_1 \to \kappa$ be the $P * \dot{R}_P$-name for the function given by
$$\bigcup\{f_p : p \in \dot{G}_{P*\dot{R}_P}\},$$
- $\dot{C}_{\mathsf{Q}_P} \subset \omega_1$ be the $P * \dot{R}_P$-name for the club given by
$$\bigcup\{c_p : p \in \dot{G}_{P*\dot{R}_P}\},$$
- $\dot{g}_{\mathsf{Q}_P} \subset \omega_1$ be the $P * \dot{R}_P$-name for the function $\dot{g}_P$ coding a V-generic filter for P using f.

We are ready to show that all stationary set preserving posets are freezeable.

Theorem 3.15 (Freezing lemma)**.** *Assume P is stationary set preserving. Then P forces that $\dot{R}_P$ is stationary set preserving and $\mathsf{Q}_P = \mathsf{RO}(P * \dot{R}_P)$ freezes P as witnessed by the map $k : \mathsf{RO}(P) \to \mathsf{Q}_P$ which maps $p \in P$ to $\langle p, 1_{\dot{R}_P}\rangle$.*

Proof. It is rather standard to show that $\dot{R}_P$ is forced by P to be stationary set preserving. We briefly give the argument for $R = \mathrm{val}_G(\dot{R}_P)$ working in $V[G]$ where G is V-generic for P. First of all we observe that $\{S^i_\alpha : \alpha < \kappa, i < 2\}$ is still in $V[G]$ a partition of $(E^\omega_\kappa)^V$ in pairwise disjoint stationary sets, since P is $< \kappa$-CC and that $\{A_\alpha : \alpha < \omega_1\}$ is still a maximal antichain on $P(\omega_1)/\mathsf{NS}_{\omega_1}$ in $V[G]$ since $P \in \mathsf{SSP}$ and $\{A_\alpha : \alpha < \omega_1\}$ contains a club subset of ω_1.

Claim. *R is stationary set preserving.*

Proof of Claim. Let $\dot{E}$ be an R-name for a club subset of ω_1 and S be a stationary subset of ω_1. Then we can find α such that $S \cap A_\alpha$ is stationary. Pick $p \in R$ such that $\alpha \in \mathrm{dom}(f_p)$, Let $\beta = f_p(\alpha)$ and $i = g(\beta)$ where $g : \kappa \to 2$ is the function coding G by means of f. By standard arguments find $M \prec H_\theta^{V[G]}$ countable such that

- $p \in M$,
- $M \cap \omega_1 \in S \cap A_\alpha$,
- $\sup(M \cap \kappa) \in S^i_\beta$.

Working inside M build a decreasing chain of conditions $p_n \in R \cap M$ which seals all dense sets of R in M and such that $p_0 = p$. By density we get that

$$f_\omega = \bigcup_{n<\omega} f_{p_n} : \xi = M \cap \omega_1 \to M \cap \kappa$$

is surjective and that ξ is a limit point of

$$c_\omega = \bigcup_{n<\omega} c_{p_n}$$

which is a club subset of ξ. Set

$$q = (f_\omega \cup \{\langle \xi, 0\rangle\}, c_\omega \cup \{\xi\}).$$

Now observe that $q \in R$ since $\xi \in A_\alpha$ and $\sup(f_q[\xi]) \in S^{g(f_p(\alpha))}_{f_p(\alpha)}$, and c_q is a closed subset of $\xi + 1$. Now by density q forces that $\xi \in \dot{E} \cap S$, and we are done.

We now argue that Q_P freezes P. We do this by means of Lemma 3.11(2).

Assume that $\mathsf{R} \leq \mathsf{Q}_P$, let H be V-generic for R, and pick $G_0, G_1 \in V[H]$ distinct correct V-generic filters for Q_P. It is enough to show that

$$\bar{G}_0 = \bar{G}_1,$$

where

$$\bar{G}_j = \{p \in P : \exists \dot{q} \in V^P \text{ such that } \langle p, \dot{q}\rangle \in G_j\}.$$

Let $g_j : \kappa \to 2$ be the evaluation by G_j of the function $\dot{g}_P$ which is used to code $\bar{G}_j$ as a subset of κ by letting $g_j(\alpha) = 1$ iff $f^{-1}(\alpha) \in \bar{G}_j$. Let

$$h_j = \bigcup\{f_p : p \in G_j\},$$

$$C_j = \bigcup\{c_p : p \in G_j\},$$

In particular we get that C_0 and C_1 are club subsets of ω_1 in $V[H]$, and h_0, h_1 are bijections of ω_1 with κ.

Now observe that κ has size and cofinality ω_1 in $V[G_j]$, and thus (since $V[H]$ is a generic extension of $V[G_j]$ with the same ω_1) κ has size and cofinality ω_1 in $V[H]$. Observe also that in $V[H]$

> S is a stationary subset of κ iff $S \cap \{\sup h[\xi] : \xi < \omega_1\}$ is non-empty for any bijection $h : \omega_1 \to \kappa$.

Now the very definition of the h_j gives that for all $\alpha \in C_j$:

> $h_j(\alpha) = \eta$ and $g_j(\eta) = i$ if and only if $\sup h_j[\xi] \in S^i_\eta$ for all $\xi \in A_\alpha \cap C_j$.

Now the set

$$E = \{\xi \in C_0 \cap C_1 : h_0[\xi] = h_1[\xi]\}$$

is a club subset of ω_1, and the above observations show that

$$S^{g_j(\eta)}_\eta \supseteq \{\sup h_j[\xi] : \xi \in E \cap A_\alpha\} \neq \emptyset$$

for both j. In particular $g_0(\eta) = g_1(\eta)$ for all $\eta < \kappa$, else $S^0_\eta \cap S^1_\eta$ is non-empty for some η contradicting the very definition of the family of sets S^i_η. Thus $\bar{G}_0 = \bar{G}_1$.

The proof of Theorem 3.15 is completed. □

3.2.3. *Freezeability implies total rigidity.* We are now in the position to use the previous theorem and some elementary observations on iterations to conclude that the family of totally rigid partial orders is also dense in $\mathsf{U}^{\mathsf{SSP}}$ and $\mathsf{U}^{\mathsf{SP},\mathsf{SP}}$ provided there are enough large cardinals.

Theorem 3.16. *Assume δ is a limit of uncountable cofinality of cardinals α which are $\alpha + \omega + 1$-supercompact and is such that V_δ models* ZFC. *Then for every $\mathsf{B} \in \mathsf{SSP} \cap V_\delta$ there is $k_\mathsf{B} : \mathsf{B} \to \mathsf{R}_\mathsf{B}$ such that*

- $\mathsf{R}_\mathsf{B} \in \mathsf{SSP}$ *is a totally rigid complete boolean algebra of size δ,*
- $[\![\mathsf{R}_\mathsf{B}/_{k_\mathsf{B}[\dot{G}_\mathsf{B}]}$ *is semiproper*$]\!]_\mathsf{B} = 1_\mathsf{B}$.

We refer the reader to Subsection 2.6 for the relevant definitions and results on iterations and to [30] for a detailed account.

Proof. First notice that V_δ models that there are class many $\alpha+\omega+1$-supercompact cardinals α.

Now for any $\mathsf{B} \in \mathsf{SSP} \cap V_\delta$, let $i_\mathsf{B} : \mathsf{B} \to \mathsf{Q}_\mathsf{B}$ be the regular embedding defined in the previous subsection to freeze B and $\dot{R}_\mathsf{B}$ be the canonical name for the quotient forcing. Observe that $\dot{R}_\mathsf{B}$ can be chosen to be a B-name for a stationary set preserving poset in $V^\mathsf{B}_{\mathrm{rank}(\mathsf{B})+\omega}$. Moreover Q_B collapses B to have size ω_1.

Next observe the following: Assume B has inaccessible size, forces MM^{++} up to $\beth(\omega)$, and collapses its size to become ω_2. Then

$$\mathsf{B} \Vdash \dot{R}_{\mathsf{B}*\dot{Q}} \text{ is semiproper}$$

whenever $\dot{Q}$ is a B-name for a stationary set preserving poset of size less than $\beth(\omega)$ in V^B. Let $k_\mathsf{B} : \mathsf{B} \to \mathsf{C}_\mathsf{B} \in V_\delta$ be a regular embedding such that

- $[\![\mathsf{C}_\mathsf{B}/_{k_\mathsf{B}[\dot{G}_\mathsf{B}]}$ is semiproper$]\!]_\mathsf{B} = 1_\mathsf{B}$,
- C_B forces MM^{++} up to $\beth(\omega)$ while collapsing α to become ω_2 for some $\alpha \in (|\mathsf{B}|, \delta)$ which is $\alpha + \omega + 1$-supercompact.

Such a k_B can be found in V_δ applying Theorem 2.24 in V_δ.

Let

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha < \beta \le \delta\}$$

be defined as follows for all α limit and $n \in \omega$:

- $\mathsf{B}_0 = \mathsf{B}$,
- $i_{\alpha+2n,\alpha+2n+1} = k_{\mathsf{B}_{\alpha+2n}}$,
- $i_{\alpha+2n+1,\alpha+2n+2} = i_{\mathsf{B}_{\alpha+2n+1}}$.

The claim below follows in a rather straightforward manner from the above observation.

Claim. *For all $\alpha < \delta$,*

- $\mathsf{B}_{\alpha+2}$ *collapses* B_α *to have size* ω_1,
- $[\![\mathsf{B}_{\alpha+2}/_{i_{\alpha,\alpha+2}[\dot{G}_{\mathsf{B}_\alpha}]}$ *is semiproper* $]\!]_{\mathsf{B}_\alpha} = 1_{\mathsf{B}_\alpha}$,
- $\mathsf{B}_{\alpha+2}$ *has size less than* δ.

Then (by the standard arguments which are used to prove the consistency of MM^{++}; see [12, Theorem 37.9] or [30, Theorem 8.5]) this iteration will be such that

$$[\![C(\mathcal{F})/i_0[\dot{G}_{\mathsf{B}_0}] \text{ is semiproper}]\!]_{\mathsf{B}_0} = 1_{\mathsf{B}_0}.$$

Thus $C(\mathcal{F}) \in \mathsf{SSP}$ and $C(\mathcal{F}) \leq_{\mathsf{SSP}} \mathsf{B}_0$.

We can now prove the following claim.

Claim. *$C(\mathcal{F})$ is totally rigid.*

Proof of Claim. $i_{\alpha,\alpha+2}$ freezes B_α for any $\alpha < \delta$. Thus we get that $C(\mathcal{F})$ is a direct limit of freezed posets (since δ has uncountable cofinality). In particular Lemma 3.13 grants that $C(\mathcal{F})$ is totally rigid.

Theorem 3.16 is proved. □

We can now also prove Theorem 3.8.

Proof. Let δ be a supercompact cardinal. By Theorem 3.16 any $\mathsf{B} \in \mathsf{SSP} \cap V_\delta$ is absorbed by a totally rigid poset Q_B whose size is at most α and is such that the quotient forcing is semiproper, where $\alpha < \delta$ is the first $\alpha + \omega + 2$-supercompact cardinal larger than B. We fix $f : \delta \to V_\delta$ a Laver function, and we define an iteration system

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha < \beta \leq \delta\}$$

as follows for all α limit and $n \in \omega$:

- $\mathsf{B}_0 = \mathsf{B}$,
- $i_{\alpha,\alpha+1} : \mathsf{B}_\alpha \to \mathsf{Q}_{\mathsf{B}*f(\alpha)}$ is an SP-correct embedding, if $f(\alpha)$ is a B_α-name for a semiproper poset,
- $i_{\alpha,\alpha+1} : \mathsf{B}_\alpha \to \mathsf{RO}(\mathsf{B} * \mathit{Coll}(\omega_1, \mathsf{B}))$ is any SP-correct embedding otherwise.

By Theorem 3.16 this definition makes sense for all stages. Now we can mimic the consistency proof of MM^{++} as in [12, Theorem 37.9] or [30, Theorem 8.5] to argue that B_δ forces MM^{++}. By the same argument of the previous proof we can also grant that B_δ is totally rigid and that

$$[\![\mathsf{B}_\delta/_{i_{0,\delta}[\dot{G}_\mathsf{B}]} \text{ is semiproper }]\!]_\mathsf{B} = 1_\mathsf{B}.$$

Theorem 3.8 is completely proved. □

3.2.4. *Forcing properties of $\mathsf{U}_\delta^{\mathsf{SSP}}$.* In this subsection we assume δ is large, meaning that it is an inaccessible limit of $< \delta$-supercompact cardinals. With these assumptions at hand we have that the set of totally rigid partial orders Q in U_δ which force MM^{++} up to δ is dense in U_δ by Theorem 3.8. We shall limit ourselves to analyze U_δ restricted to this set which we denote by TR.

Fact 3.17. *The following holds for* TR.

(1) *For any* $\mathsf{B} \in \mathsf{U}_\delta$ *there is* $\mathsf{C} \leq_{\mathsf{SP}} \mathsf{B}$ *in* TR *by Theorem* 3.8.

(2) *For* $\mathsf{B}, \mathsf{Q} \in$ TR, $\mathsf{B} \leq_{\mathsf{SSP}} \mathsf{Q}$ *(*B *and* Q *are* $\leq_{\mathsf{SSP}}$*-incompatible) iff* $\mathsf{B} \leq_{\mathsf{SP}} \mathsf{Q}$ *(*B *and* Q *are* $\leq_{\mathsf{SP}}$*-incompatible).*

(3) *Let* $\mathcal{G} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta\} \subseteq$ TR *be an iteration system such that each* $i_{\alpha,\beta}$ *is* SSP*-correct. Then* $\mathcal{RCS}(\mathcal{G}) \in \mathsf{SSP}$ *is a lower bound for each* B_α *under* $\leq^*_{\mathsf{SSP}}$.

(4) *Assume* $A \subset$ TR *is an antichain of size less than* δ*. Then* $\bigvee A \in$ TR.

(5) *For any totally rigid* $\mathsf{C} \in \mathsf{U}_\delta$ *the map* $k_\mathsf{C} : \mathsf{C} \to \mathsf{U}_\delta \restriction \mathsf{C}$ *which maps* $c \in \mathsf{C}$ *to* $\mathsf{C} \restriction c$ *is a correct regular embedding.*

Proof. Left to the reader. $\square$

The next key observation is the following lemma.

Lemma 3.18. *Let $D \subset \mathrm{TR}$ be a dense open subset of $\mathrm{TR} \cap \mathsf{U}_\delta$. Then for every $\mathsf{B} \in \mathsf{U}_\delta$ there is $\mathsf{C} \in \mathrm{TR}$, an* injective SP-correct *complete homomorphism $i : \mathsf{B} \to \mathsf{C}$, and $A \subset \mathsf{C}$ maximal antichain of C such that $k_{\mathsf{C}}[A] \subset D$.*

Proof. Given $\mathsf{B} \in \mathrm{TR}$ find $\mathsf{C}_0 \leq \mathsf{B}$ in D. Let $i_0 : \mathsf{B} \to \mathsf{C}_0$ be a complete and SP-correct homomorphism of B into C_0 given by Theorem 3.8. Let $b_0 \in \mathsf{B}$ be the complement of $\bigvee_{\mathsf{B}} \ker(i_0)$ so that $i_0 \restriction b_0 : \mathsf{B} \restriction b_0 \to \mathsf{C}_0$ is an injective SP-correct homomorphism. Proceed in this way to define C_l and b_l such that

- $i_l \restriction b_l : \mathsf{B} \restriction b_l \to \mathsf{C}_l$ is an injective SP-correct homomorphism,
- $\mathsf{C}_l \in D \subset \mathrm{TR}$,
- $b_l \wedge_{\mathsf{B}} b_i = 0_{\mathsf{B}}$ for all $i < l$.

This procedure must terminate in $\eta < |\mathsf{B}|^+$ steps producing a maximal antichain $\{b_l : l < \eta\}$ of B and injective SP-correct homomorphisms $i_l : \mathsf{B} \restriction b_l \to \mathsf{C}_l$ such that $\mathsf{C}_l \in D \subset \mathrm{TR}$ refines B in the $\leq_{\mathsf{SP}}$ order. Then we get that

- $\mathsf{C} = \bigvee_{l<\eta} \mathsf{C}_l \in \mathrm{TR}$ by Fact 3.17.4.
- i is an injective SP-correct homomorphism where
$$\begin{aligned} i = \bigvee_{k<\eta} i_k : \mathsf{B} &\to \mathsf{C}_k \\ c &\mapsto \langle i_k(c \wedge_{\mathsf{B}} b_k) : k < \eta \rangle \end{aligned}$$
is such that
$$[\![\mathsf{B}/i[\dot{G}_{\mathsf{B}}] \in \mathsf{SP}]\!]_{\mathsf{B}} = 1_{\mathsf{B}},$$
since i is the lottery sum of the injective SP-correct homomorphisms i_l.
- $\mathsf{C} \restriction i(b_k) \in D$ for all $k < \eta$.

In particular we get that $A = i[\{b_k : k < \eta]$ is a maximal antichain of $\mathsf{C} \in \mathrm{TR}$ such that $\mathsf{C} \restriction c \in D$ for all $c \in A$ as was to be shown. $\square$

Lemma 3.19. U_δ *is $(<\delta, <\delta)$-presaturated.*

Proof. Let $\dot{f}$ be a U_δ-name for an increasing function from η into δ for some $\eta < \delta$. Given $\mathsf{B} \in \mathsf{U}_\delta$ let $A_i \subset \mathrm{TR}$ be the dense set of totally rigid partial orders in $\mathsf{U}_\delta \restriction \mathsf{B}$ which decide that $\dot{f}(i) = \alpha$ for some $\alpha < \delta$. Then using the previous lemma we can build inside V_δ an RCS iteration of complete boolean algebras
$$\{i_{\alpha,\beta} : \mathsf{C}_\alpha \to \mathsf{C}_\beta : \alpha \leq \beta \leq \eta\} \in V_\delta$$
such that for all $i < \eta$, $\mathsf{C}_{i+1} \in \mathrm{TR}$ and there is B_i maximal antichain of $\mathsf{C}_{i+1} \leq^*_{\mathsf{SP}} \mathsf{C}_i$ such that $k_{\mathsf{C}_{i+1}}[B_i] \subset A_i$. Then $\mathsf{C}_\eta \in \mathsf{SSP}$ forces that $\dot{f}$ has values bounded by
$$\sup\{\alpha : \exists c \in \mathsf{C}_i \text{ such that } \mathsf{C}_i \restriction c \text{ forces that } \dot{f}(i) = \alpha\} < \delta.$$
$\square$

Lemma 3.20. *Assume $\dot{f} \in V^{\mathsf{U}_\delta}$ is a name for a function in Ord^α for some $\alpha < \delta$. Then there is a dense set of $\mathsf{C} \in \mathrm{TR}$ with an $\dot{f}_{\mathsf{C}} \in V^{\mathsf{C}}$ such that*
$$[\![\hat{k}_{\mathsf{C}}(\dot{f}_{\mathsf{C}}) = \dot{f}]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{C}.$$

Proof. Given $\dot{f}$ as above, let for all $\xi < \alpha$

$$D_\xi = \{\mathsf{C} \in \mathrm{TR} : \exists\beta\, \mathsf{C} \Vdash_{\mathsf{U}_\delta} \dot{f}(\xi) = \beta\}.$$

Let $\mathsf{B} \in \mathrm{TR}$ be arbitrary. By the previous lemma we can find $\mathsf{C} \in \mathrm{TR}$ below B such that for all $\xi < \alpha$ there is a maximal antichain $A_\xi \subset \mathsf{C}$ such that $k_{\mathsf{C}}[A_\xi] \subset D_\xi$. Now let $\dot{f}_{\mathsf{C}}$ be the C-name

$$\{\langle(\xi,\eta),c\rangle : c \in A_\xi \text{ and } \mathsf{C} \restriction c \Vdash_{\mathsf{U}_\delta} \dot{f}(\xi) = \eta\}.$$

It is immediate to check that for all $\xi < \alpha$ and $c \in A_\xi$,

$$c \Vdash_{\mathsf{C}} \dot{f}_{\mathsf{C}}(\xi) = \eta \text{ iff } \mathsf{C} \restriction c \Vdash_{\mathsf{U}_\delta} \dot{f}(\xi) = \eta.$$

This gives that $[\![\hat{k}_{\mathsf{C}}(\dot{f}_{\mathsf{C}}) = \dot{f}]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{C}$. The lemma is proved. □

In particular we also get the following lemma.

Lemma 3.21. *Assume $\tau \in V^{\mathsf{U}_\delta}$ is a U_δ-name for an element of $(H_\delta)^{V^{\mathsf{U}_\delta}}$. Then there is a dense open set of $\mathsf{C} \in \mathrm{TR}$ and names $\sigma_{\mathsf{C}} \in V^{\mathsf{C}} \cap V_\delta$ such that*

$$[\![\tau = k_{\mathsf{C}}(\sigma_{\mathsf{C}})]\!]_{\mathsf{U}_\delta} \geq_{\mathsf{RO}(\mathsf{U}_\delta)} \mathsf{C}.$$

Proof. Left to the reader: observe that any such U_δ-name τ can be coded by a U_δ-name for a function from some $\alpha < \delta$ into δ. □

Lemma 3.22. *Assume G is V-generic for U_δ. Then for all $\mathsf{B} \in \mathsf{U}_\delta$, $\mathsf{B} \in G$ iff there is $H \in V[G]$ correct V-generic filter for B.*

Proof. We have to prove the following:

(1) Assume $\mathsf{B} \in G$. Then there is $H \in V[G]$ correct V-generic filter for B.
(2) Assume there is some $H \in V[G]$ correct V-generic filter for B. Then $\mathsf{B} \in G$.

We proceed as follows.

Proof of 1: Assume $\mathsf{B} \in G$. Then there is a totally rigid $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{B}$ freezing B in G as witnessed by $i : \mathsf{B} \to \mathsf{R}$, since the set of such R is dense in U_δ below B. In particular $H = \{b \in \mathsf{B} : \mathsf{R} \restriction i(b) \in G\}$ is a V-generic filter for B. We must show that it is also a correct V-generic filter, i.e., that for all B-names $\dot{S}$ for stationary subsets of ω_1, $\dot{S}_H$ is stationary in $V[G]$. So assume this is not the case. Then there is $\dot{S}$, B-name for a stationary subset of ω_1, and $\dot{C}$, U_δ-name for a club subset of ω_1, such that

$$V[G] \models \dot{C}_G \cap \dot{S}_H = \emptyset.$$

We can thus find $\mathsf{D} \leq_{\mathsf{SSP}} \mathsf{R}$ in G forcing the above statement, i.e., more precisely: Let for all $\mathsf{Q} \leq_{\mathsf{SSP}} \mathsf{R}$ $i_{\mathsf{Q}} : \mathsf{B} \to \mathsf{Q}$ be the unique correct homomorphism which factors through $i : \mathsf{B} \to \mathsf{R}$; recall also that $k_{\mathsf{Q}} : \mathsf{Q} \to \mathsf{U}_\delta \restriction \mathsf{Q}$ is the complete homomorphism defined by $d \mapsto \mathsf{Q} \restriction d$ for all $\mathsf{Q} \in \mathrm{TR}$. Then

$$[\![\dot{C} \cap \hat{k}_{\mathsf{R}} \circ \hat{i}(\dot{S}) = \emptyset]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{D} \in G.$$

We leave to the reader to check that

$$[\![\hat{k}_{\mathsf{R}} \circ \hat{i}(\dot{S}) = \hat{k}_{\mathsf{Q}} \circ \hat{i}_{\mathsf{Q}}(S)]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{Q}$$

holds for all $\mathsf{Q} \leq_{\mathsf{SSP}} \mathsf{R}$.

Applying Lemma 3.21 to the U_δ-name $\dot{C}$ for a subset of ω_1, we can find $\mathsf{C} \leq_{\mathsf{SSP}} \mathsf{D}$ in U_δ such that $\mathsf{C} \in \mathrm{TR}$ and for some $\dot{E}$ C-name for a subset of ω_1

$$[\![\hat{k}_{\mathsf{C}}(\dot{E}) = \dot{C}]\!]_{\mathsf{U}_\delta} \geq \mathsf{C}.$$

Since the formula $\phi(x, y, \omega_1)$ stating that x is a club subset of ω_1 disjoint from y is a Σ_0-statement in the parameter ω_1, we get that

$$k_{\mathsf{C}}([\![\dot{E} \cap \hat{i}_{\mathsf{C}}(\dot{S}) = \emptyset]\!]_{\mathsf{C}}) = \mathsf{C} \wedge [\![\dot{C} \cap \hat{k}_{\mathsf{C}} \circ \hat{i}_{\mathsf{C}}(\dot{S}) = \emptyset]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq_{\mathsf{SSP}}$$
$$\geq_{\mathsf{SSP}} \mathsf{C} \wedge [\![\dot{C} \cap \hat{k}_{\mathsf{R}} \circ \hat{i}(\dot{S}) = \emptyset]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq_{\mathsf{SSP}} \mathsf{C} \in G.$$

In particular we conclude that

$$[\![\dot{E} \cap \hat{i}_{\mathsf{C}}(\dot{S}) = \emptyset]\!]_{\mathsf{C}} = 1_{\mathsf{C}}.$$

This contradicts the very definition of $i_{\mathsf{C}} : \mathsf{B} \to \mathsf{C}$ being an SSP-correct homomorphism, concluding the proof of 1.

Proof of 2: Assume there is $H \in V[G]$ correct V-generic filter for B. Toward a contradiction assume that $\mathsf{B} \notin G$. Then there is some $\mathsf{R} \in G$ such that B and R are incompatible in U_δ since

$$\{\mathsf{R} \in \mathsf{U}_\delta : \mathsf{R} \text{ is orthogonal to } \mathsf{B} \text{ in } \mathsf{U}_\delta \text{ or } \mathsf{R} \leq_{\mathsf{SSP}} \mathsf{B}\}$$

is open dense in U_δ and G must meet it in some R satisfying the first clause of the above disjunction. Let $\dot{H}$ be a U_δ-name for H and $\mathsf{C} \in G \cap \mathrm{TR}$ be an element refining R and forcing in U_δ that $\dot{H}$ is a correct V-generic filter for B. In particular

- for all $\dot{S}$ B-name for a subset of ω_1 there is $\underline{\dot{S}}$ in V^{U_δ} such that

$$[\![\underline{\dot{S}} \text{ is the interpretation of the } \mathsf{B}\text{-name } \dot{S} \text{ by the } V\text{-generic filter } \dot{H}]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{C}$$

  and such that

$$[\![\dot{S} \text{ is stationary }]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$$

  if and only if for all $\dot{C}$ U_δ-name for a club subset of ω_1

$$[\![\underline{\dot{S}} \cap \dot{C} \neq \emptyset]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{C}.$$

- C forces in U_δ that $\dot{H}$ is an ultrafilter on B,
- C forces in U_δ that $\dot{H} \cap D \neq \emptyset$ for all $D \in V$ dense subset of B.

Now the family

$$\mathcal{A} = \{\underline{\dot{S}} : [\![\dot{S} \subset \omega_1]\!]_{\mathsf{B}} = 1_{\mathsf{B}}\} \cup \{\dot{H}\}$$

has size less than δ and is a family of names for sets of size less than δ. Thus, as in the proof of Lemma 3.19, we can apply iteratively Lemma 3.21 and Lemma 3.18 to find that the set of

$$\{\mathsf{D} \leq_{\mathsf{SSP}} \mathsf{C} : \forall \tau \in \mathcal{A}\, \exists \sigma(\tau) \in V^{\mathsf{D}}\ [\![\hat{k}_{\mathsf{D}}(\sigma(\tau)) = \tau]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{D}\}$$

is dense below C in U_δ. In particular we can find such a $\mathsf{D} \in G$. Notice that D is orthogonal to B in U_δ since C is.

Now let $\dot{K} \in V^{\mathsf{D}}$ be such that

$$[\![\hat{k}_{\mathsf{D}}(\dot{K}) = \dot{H}]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} \geq \mathsf{D},$$

and let

- $\phi_0(x, y, z)$ be the Σ_0-formula asserting x is a club subset of z and $x \cap y \neq \emptyset$,
- $\phi_1(x, y)$ be the Σ_0-formula asserting x is a ultrafilter on the boolean algebra y
- $\phi_2(x, y, z)$ be the Σ_0-formula asserting x meets y, and y is a dense subset of the boolean algebra z.

Notice that

$$k_{\mathsf{D}}([\![\phi_1(\dot{K},\mathsf{B})]\!]_{\dot{\mathsf{D}}} \geq \mathsf{D},$$

$$k_{\mathsf{D}}([\![\phi_2(\dot{K},D,\mathsf{B})]\!]_{\dot{\mathsf{D}}} \geq \mathsf{D},$$

$$k_{\mathsf{D}}([\![\phi_0(\dot{C},\sigma(\underline{\dot{S}}),\omega_1)]\!]_{\dot{\mathsf{D}}} \geq \mathsf{D}$$

for all D-names $\dot{C}$ for a club subset of ω_1 and all B-names $\dot{S}$ for a stationary subset of ω_1.

In particular we get that

$$[\![\dot{K} \text{ is a correct } V\text{-generic filter for } \mathsf{B}]\!]_{\mathsf{D}} = 1_{\mathsf{D}},$$

applying Fact 2.13. We reached a contradiction since the map

$$l : b \mapsto [\![b \in \dot{K}]\!]_{\mathsf{D}}$$

defines an SSP-correct homomorphism of B into D witnessing that $\mathsf{D} \leq_{\mathsf{SSP}} \mathsf{B}$, contrary to the already established fact that D is orthogonal to B in U_δ.

□

3.2.5. *Proof of Theorem* 3.5. In this section we prove the following theorem.

Theorem 3.23. *Assume δ is inaccessible and V_δ models that there are class many supercompact cardinals γ. Then $\mathsf{U}_\delta^{\mathsf{SSP}} \in \mathsf{SSP}$ and collapses δ to become $\aleph_2$. If δ is supercompact, then $\mathsf{U}_\delta^{\mathsf{SSP}}$ forces MM^{++}.*

Proof. We prove all items as follows:

U_δ **preserves the regularity of** δ**:** This follows immediately from the $(<\delta,<\delta)$-presaturation of U_δ.

U_δ **is stationary set preserving:** Fix $S \in V$ stationary subset of ω_1 and $\mathsf{B} \in \mathsf{U}_\delta$. Let $\dot{C}$ be a U_δ-name for a club subset of ω_1. We must find $\mathsf{C} \leq \mathsf{B}$ which forces in U_δ that $S \cap \dot{C}$ is non-empty.

Let $\dot{f}$ be a U_δ-name for the increasing enumeration of $\dot{C}$.

By Lemma 3.20 we can find $\mathsf{C} \leq \mathsf{B}$ in TR and $\dot{f}_{\mathsf{C}}$ such that C forces in U_δ that $\hat{k}_{\mathsf{C}}(\dot{f}_{\mathsf{C}}) = \dot{f}$.

Now observe that the statement

"$\dot{f}$ is a continuous strictly increasing map from ω_1 into ω_1"

is a Σ_0 statement in the parameters ω_1, $\dot{f}$ and is forced by C in U_δ. Since $\hat{k}_{\mathsf{C}}$ is Δ_1-elementary and $\hat{k}_{\mathsf{C}}(\dot{f}_{\mathsf{C}}) = \dot{f}$, we get that

$$[\![\dot{f}_{\mathsf{C}} \text{ is a continuous strictly increasing map from } \omega_1 \text{ into } \omega_1]\!]_{\mathsf{C}} = 1_{\mathsf{C}}.$$

This gives that

$$[\![\mathrm{rng}(\dot{f}_{\mathsf{C}}) \text{ is a club}]\!]_{\mathsf{C}} = 1_{\mathsf{C}}.$$

Since C is SSP, we get that

$$[\![S \cap \mathrm{rng}(\dot{f}_{\mathsf{C}}) \neq \emptyset]\!]_{\mathsf{C}} = 1_{\mathsf{C}}.$$

Now once again we observe that the above statement is Σ_0 in the parameters $S, \dot{f}_{\mathsf{C}}$ and thus, since $\hat{k}_{\mathsf{C}}$ is Δ_1-elementary, we can conclude that

$$[\![S \cap \mathrm{rng}(\dot{f}) \neq \emptyset]\!]_{\mathsf{RO}(\mathsf{U}_\delta \restriction \mathsf{C})} = 1_{\mathsf{RO}(\mathsf{U}_\delta \restriction \mathsf{C})},$$

which gives the thesis.

U_δ **forces** δ **to become** ω_2**:** This is immediate since there are densely many posets $\mathsf{B} \in \mathrm{TR} \cap \mathsf{U}_\delta$ which collapse their size to become at most ω_2 and embed completely into $\mathsf{U}_\delta \restriction \mathsf{B}$. Thus for unboundedly many $\xi < \delta$ we get that U_δ forces ξ to be an ordinal less than or equal to ω_2. All in all we get that U_δ forces δ to be less than or equal to ω_2.

U_δ **forces** MM^{++} **if** δ **is supercompact:** Let $\dot{\mathsf{R}} \in V^{\mathsf{U}_\delta}$ be a name for a stationary set preserving poset. Given B in U_δ find $j : V_\gamma \to V_\lambda$ in V such that $\mathrm{crit}(j) = \alpha$, $\mathsf{B} \in V_\alpha$, $j(\mathrm{crit}(j)) = \delta$, and $\dot{\mathsf{R}} \in j[V_\gamma]$.

Let $\dot{\mathsf{Q}} \in V^{\mathsf{U}_\alpha}$ be such that $j(\dot{\mathsf{Q}}) = \dot{\mathsf{R}}$.

By elementarity of j we get that $\mathsf{U}_\alpha \in \mathsf{SSP}$. Then $\mathsf{Q} = (\mathsf{U}_\alpha \restriction \mathsf{B}) * \dot{\mathsf{Q}} \leq_{\mathsf{SSP}} \mathsf{U}_\alpha, \mathsf{B}$ forces in U_δ that j lifts to

$$\bar{j} : V_\gamma[\dot{G}_{\mathsf{U}_\alpha}] \to V_\lambda[\dot{G}_{\mathsf{U}_\delta}].$$

Moreover let G be V-generic for U_δ with $\mathsf{Q} \in G$ and $G_0 = G \cap V_\alpha$. Then in $V_\lambda[G]$ there is a correct $V[G_0]$-generic filter K for $\mathsf{Q}/G_0 = \mathrm{val}_{G_0}(\dot{\mathsf{Q}})$.

Finally we get that in $V[G]$, $\bar{j}[K]$ is a correct $\bar{j}[V_\gamma[G_0]]$-generic filter for $\mathsf{R} = \mathrm{val}_G(\dot{\mathsf{R}}) = \bar{j} \circ \mathrm{val}_{G_0}(\dot{\mathsf{Q}})$ showing that T_{R} is stationary in $V[G]$. Since this holds for all V generic filter G to which $\mathsf{Q} \leq \mathsf{B}$ belongs, we have shown that for any $\dot{\mathsf{R}}$ U_δ-name for a stationary set preserving poset and below any condition B there is a dense set of posets Q in U_δ which forces that $T_{\dot{\mathsf{R}}}$ is stationary in $V[\dot{G}_{\mathsf{U}_\delta}]$.

The thesis follows.

The proof of Theorem 3.5 is completed. $\square$

We remark that all items except the last one required only that δ is an inaccessible limit of $< \delta$-supercompact cardinals.

3.3. **Proof of Theorem 3.9.**

Proof. For any $\mathsf{B} \in \mathsf{U}^{\mathsf{SSP}}$ we can find $\mathsf{Q} \leq_{\mathsf{SSP}} \mathsf{B}$ in $\mathsf{U}^{\mathsf{SSP}}$ which is totally rigid and freezes B as witnessed by some $r : \mathsf{B} \to \mathsf{Q}$ by the results of Section 3.2.1.

We just prove the theorem for such totally rigid Q, this simplifies the exposition since it allows us to dispense with several bits of heavy notation; the general case for arbitrary B is left to the reader.[13]

Since Q is totally rigid there is only one correct homomorphism

$$i_{\mathsf{R}} : \mathsf{Q} \to \mathsf{R}$$

for any $\mathsf{R} \leq \mathsf{Q}$. For each totally rigid $\mathsf{R} \in \mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q}$ let

$$k_{\mathsf{R}} : \mathsf{R} \to \mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{R}$$

be given by $r \mapsto \mathsf{R} \restriction r$. Then k_{R} is an order and incompatibility preserving embedding of R in the class forcing $\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{R}$ which maps maximal antichains to maximal antichains. Let us denote by k the map k_{Q}.

[13]The general case requires one to repeat the proof that follows replacing in the relevant places i_{R} with $i_{\mathsf{R}} \circ r$. The proof will go through also in this case using the fact that r is a freezing homomorphism for B.

Let G be V-generic for Q and J denote its dual prime ideal. We will show that $(\mathsf{U}^{\mathsf{SSP}})^{V[G]}$ can be identified in $V[G]$ to the quotient forcing given by $(\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q})^V/_{k[G]}$ as defined in Subsection 2.3. To this aim first observe that in $V[G]$

$$\downarrow k[J] = \{\mathsf{R} \in (\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{B})^V : \exists q \in J\, \mathsf{R} \leq^V_{\mathsf{SSP}} \mathsf{Q} \restriction q\}.$$

In $V[G]$ consider the map

$$i^* : (\mathsf{U}^{\mathsf{SSP}})^{V[G]} \to (\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q})^V/_{k[G]}$$

defined by $\mathsf{R}/_{i_{\mathsf{R}}[G]} \mapsto \mathsf{R}$. We must show that i^* is a total and surjective relation which preserves the order and incompatibility relation. If this is the case, i^* is an embedding with a dense image which witnesses that the two forcing notions are equivalent.

i^* **is a total and surjective relation:** Assume R is a non-trivial element in the quotient forcing $(\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q})^V/_{k[G]}$. Then R is positive with respect to the filter (generated by) $k[G]$, and thus $0_{\mathsf{R}} \notin i_{\mathsf{R}}[G]$: else $1_{\mathsf{R}} \in i_{\mathsf{R}}[J]$ which gives that $\mathsf{R} \leq_{\mathsf{SSP}} \mathsf{Q} \restriction q$ for some $q \in J$. In particular R would be in $\downarrow k[J]$, contradicting our assumptions. Since $0_{\mathsf{R}} \notin i_{\mathsf{R}}[G]$, we can conclude that $\mathsf{R}/_{i_{\mathsf{R}}[J]}$ is a non-trivial complete boolean algebra in $(\mathsf{U}^{\mathsf{SSP}})^{V[G]}$ and that the pair $(\mathsf{R}/_{i_{\mathsf{R}}[G]}, \mathsf{R}) \in i^*$.

i^* **is order and compatibility preserving:** Observe that if $j : \mathsf{R}_0/i_{\mathsf{R}_0}[G] \to \mathsf{R}_1/i_{\mathsf{R}_1}[G]$ is a correct complete homomorphism, we have (by Proposition 2.17(2) applied to $\mathsf{Q}, \mathsf{R}_0, \mathsf{R}_1$ in the place of $\mathsf{B}, \mathsf{Q}_0, \mathsf{R}$) that $j = l/_G$ for some complete homomorphism $l : \mathsf{R}_0 \to \mathsf{R}_1 \restriction r$ for some r such that $R_1 \restriction \neg r \in \downarrow k[J]$ with l in V and $0_{\mathsf{R}_1} \notin l[G]$. We can also check that j is correct in $V[G]$ iff l is correct in V. In particular l witnesses that $\mathsf{R}_0 \geq_{\mathsf{SSP}} \mathsf{R}_1 \restriction r \equiv \mathsf{R}_1$ and the fact that $0_{\mathsf{R}_1} \notin l[G]$ grants that R_1 is a non-trivial condition in $(\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q})^V/_{k[G]}$ refining R_0. This shows that i^* is order preserving and maps non-trivial conditions to non-trivial conditions. In particular we can also conclude that i^* maps compatible conditions to compatible conditions.

i^* **preserves the incompatibility relation:** We prove it by contraposition. Assume R_0 is compatible with R_1 in $(\mathsf{U}^{\mathsf{SSP}} \restriction \mathsf{Q})^V/_{k[G]}$. By definition of this quotient forcing there is some $\mathsf{C} \in \mathsf{SSP}$ in V such that $\mathsf{R}_l \geq_{\mathsf{SSP}} \mathsf{C}$ for $l = 0, 1$ and $1_{\mathsf{C}} \notin i_{\mathsf{C}}[J]$. We let $h_l : \mathsf{R}_l \to \mathsf{C}$ be the SSP-correct embeddings witnessing that $\mathsf{R}_l \geq_{\mathsf{SSP}} \mathsf{C}$ for $l = 0, 1$. Since Q is totally rigid, we get that $h_0 \circ i_{\mathsf{R}_0} = h_1 \circ i_{\mathsf{R}_1} = i_{\mathsf{C}}$. Now $\mathsf{C}/_{i_{\mathsf{C}}[G]}$ is a non-trivial SSP and complete boolean algebra in $V[G]$, since $1_{\mathsf{C}} \notin i_{\mathsf{C}}[J]$. Then, by Proposition 2.17(1), $\mathsf{C}/_{i_{\mathsf{C}}[G]} \leq^{V[G]}_{\mathsf{SSP}} \mathsf{R}_1/_{i_{\mathsf{R}_1}[G]}, \mathsf{R}_0/_{i_{\mathsf{R}_0}[G]}$ as witnessed by the correct homomorphisms $h_0/_G, h_1/_G$.

□

4. Background material on normal tower forcings

In this section we present definitions and results on normal tower forcings which are relevant for us. Since we depart in some cases from the standard terminology, we decided to be quite detailed in this presentation.[14] We assume the reader has some familiarity with tower forcings as presented in Foreman's handbook chapter

[14]We soon hope to have detailed notes available on the author's Web page concerning the material presented in this section.

on tower forcing [8] or in Larson's book on stationary tower forcing [15] which are reference texts for our treatment of this topic. Another source of our presentation is Burke's paper [4].

Recall that P_λ is the class of sets M such that $M \cap \lambda \in \lambda > |M|$ and $P_\lambda(X) = P_\lambda \cap P(X)$. The following fact is well known.

Fact 4.1. *Assume I is a normal ideal; then I is countably complete (i.e., $< \omega_1$-complete). A normal ideal I that concentrates on $P_\lambda(X)$ for some $X \supseteq \lambda$ has completeness λ.*

NS_X is a normal ideal and is the intersection of all normal ideals on X.

Definition 4.2. Let $X \subset Y$ and I, J be ideals on Y and X, respectively. I canonically projects to J if $S \in J$ if and only if $\{Z \in P(Y) : Z \cap X \in S\} \in I$.

Definition 4.3. $\{I_X : X \in V_\delta\}$ is a tower of ideals if I_Y canonically projects to I_X for all $X \subset Y$ in V_δ.

The following results are due to Burke [4, Theorems 3.1-3.3].

Lemma 4.4. *Assume I is a normal ideal on X. Then I is the canonical projection of $\mathsf{NS} \restriction S(I)$ where*

$$S(I) = \{M \prec H_\theta : M \cap X \notin A \text{ for all } A \in I \cap M\}$$

and θ is any large enough cardinal such that $I \in H_\theta$.

Assume δ is inaccessible and $\mathcal{I} = \{I_X : X \in V_\delta\}$ is a tower of normal ideals. Then each I_X is the projection of $\mathsf{NS} \restriction S(\mathcal{I})$ where

$$S(\mathcal{I}) = \{M \prec H_\theta : M \cap X \notin A \text{ for all } A \in I_X \cap M \text{ and } X \in M \cap V_\delta\}$$

and $\theta > |V_\delta|$ is a cardinal.

As usual if I is an ideal on X, there is a natural order relation on $P(P(X))$ given by $S \leq_I T$ if $S \setminus T \in I$.

There is also a natural order on towers of normal ideals. Assume $\mathcal{I} = \{I_X : X \in V_\delta\}$ is a tower of normal ideals. For S, T in V_δ, $S \leq_{\mathcal{I}} T$ if letting $X = \cup S \cup \cup T$, $S^X \leq_{I_X} T^X$. It is possible to check that $S \leq_{\mathcal{I}} T$ if and only if $S \wedge S(\mathcal{I}) \leq T \wedge S(\mathcal{I})$ as stationary sets.

Definition 4.5. Let δ be inaccessible and assume $\mathcal{I} = \{I_X : X \in V_\delta\}$ is a tower of normal ideals.

$\mathbb{T}_\delta^{\mathcal{I}}$ is the tower forcing whose conditions are the stationary sets $T \in V_\delta$ such that $T \in I^+_{\cup T}$.

We let $S \leq_{\mathcal{I}} T$ if $S^X \leq_{I_X} T^X$ for $X = \cup S \cup \cup T$.

For any A stationary such that $\cup A \supseteq V_\delta$, we denote by $\mathbb{T}_\delta^A$ the normal tower forcing $\mathbb{T}_\delta^{\mathcal{I}(A)}$, where

$$\mathcal{I}(A) = \{I_X^A : X \in V_\delta\}$$

is the tower of normal ideals given by the projection of $\mathsf{NS} \restriction A$ to X as X ranges in V_δ.

Notice that in view of Burke's representation theorem [4, Theorem 3.3] (see Lemma 4.4 above), any normal tower forcing induced by a tower of normal ideal

$$\mathcal{I} = \{I_X : X \in V_\delta\}$$

is of the form $\mathbb{T}_\delta^{S(\mathcal{I})}$.

Observe also that modulo the identification of an element $S \in \mathbb{T}^{\mathcal{I}}_{\delta}$ with its equivalence class $[S]_{\mathcal{I}}$ induced by $\leq_{\mathcal{I}}$, and the adjunction of the class $[\emptyset]_{\mathcal{I}}$, $\mathbb{T}^{\mathcal{I}}_{\delta}$ can be viewed as a $<\delta$-complete boolean algebra $\mathsf{B}_{\mathcal{I}}$ with the boolean operations and constants given by

- $\neg_{\mathsf{B}_{\mathcal{I}}}[S]_{\mathcal{I}} = [P(\cup S) \setminus S]_{\mathsf{B}_{\mathcal{I}}}$,
- $\bigvee_{\mathsf{B}_{\mathcal{I}}}\{[S_i]_{\mathcal{I}} : i < \gamma\} = [\bigvee\{S_i : i < \gamma\}]_{\mathsf{B}_{\mathcal{I}}}$ for all $\gamma < \delta$,
- $0_{\mathsf{B}_{\mathcal{I}}} = [\emptyset]_{\mathcal{I}}$,
- $1_{\mathsf{B}_{\mathcal{I}}} = [P(X)]_{\mathcal{I}}$ for any non-empty $X \in V_{\delta}$.

There is a tight connection between normal towers and generic elementary embeddings. To formulate it we need to introduce the notion of V-normal towers of ultrafilters.

Definition 4.6. Assume $V \subset W$ are transitive models of ZFC and $X \in V$.

$G_X \in W$ is a V-normal ultrafilter on X if it is a filter contained in $P(P(X))$ such that for all regressive $f : P(X) \to X$ in V there is $x \in X$ such that $f^{-1}[\{x\}] \in G$.

$G \subset V_{\delta}$ in W is a V-normal tower of ultrafilters on δ if for all $X \subset Y$ in V_{δ}, $G_X = \{S \in G : S \subset P(X)\}$ is a V-normal ultrafilter and G_Y projects to G_X.

Proposition 4.7. *Let $V \subset W$ be transitive models of* ZFC.

- *Assume $j : V \to N$ is an elementary embedding which is a definable class in W and $\alpha = \sup\{\xi : j[V_{\xi}] \in N\}$. Let $S \in G$ if $S \in V_{\alpha}$ and $j[\cup S] \in j(S)$. Then G is a V-normal tower of ultrafilters on α.*
- *Assume G is a V-normal tower of ultrafilters on δ. Then G induces in a natural way a direct limit ultrapower embedding $j_G : V \to Ult(V, G)$, where $[f]_G \in Ult(V, G)$ if $f : P(X_f) \to V$ in V, $X_f \in V_{\delta}$ and for any binary relation R, $[f]_G\ R_G\ [h]_G$ if and only if for some $\alpha < \delta$ such that $X_f, X_h \in V_{\alpha}$ we have that*

$$\{M \prec V_{\alpha} : f(M \cap X_f)\ R\ h(M \cap X_h)\} \in G.$$

Recall that for a set M we let $\pi_M : M \to V$ denote the transitive collapse of the structure $(M, \in)$ onto a transitive set $\pi_M[M]$ and we let $j_M = \pi_M^{-1}$. We state the following general results about generic elementary embeddings induced by V-normal tower of ultrafilters.

Theorem 4.8. *Assume $V \subset W$ are transitive models of* ZFC*, δ is a limit ordinal, $\lambda < \delta$ is regular in V, $G \in W$ is a V-normal tower of ultrafilters on δ which concentrates on P_{λ}, and V is a definable class in W by the parameter p. Then*

(1) $j_G : V \to Ult(V, G)$ is a definable class in W in the parameters G, V_{δ}, p.

(2) $Ult(V, G) \models \phi([f_1]_G, \dots, [f_n]_G)$ if and only if for some $\alpha < \delta$ such that $f_i : P(X_i) \to V$ with $X_i \in V_{\alpha}$ for all $i \leq n$

$$\{M \prec V_{\alpha} : V \models \phi(f_1(M \cap X_1), \dots, f_n(M \cap X_n))\} \in G.$$

(3) For all $x \in V_{\delta}$, $x \in Ult(V, G)$ is represented by $[\rho_x]_G$ where $\rho_x : P(V_{\alpha}) \to V_{\lambda}$ is such that $x \in V_{\alpha}$ for some $\alpha < \delta$ and maps any $M \prec V_{\alpha}$ with $x \in M$ to $\pi_M(x)$.

(4) $(V_{\delta}, \in)$ is isomorphic to the substructure of $(Ult(V, G), \in_G)$ given by the equivalence classes of the functions $\rho_x : P(V_{\alpha}) \to V_{\alpha}$ described above.

(5) $\mathrm{crit}(j_G) = \lambda$ and δ is isomorphic to an initial segment of the linear order $(j_G(\lambda), \in_G)$.

(6) For all $x \in V$, $j_G(x) \in Ult(V,G)$ is represented by the equivalence class of the constant function $c_x : P(X) \to V$ which maps any $Y \subset X$ to x for an arbitrary choice of $X \in V_\delta$.

(7) For any $x \in V_\delta$, $j_G[x] \in Ult(V,G)$ is represented in $Ult(V,G)$ by the equivalence class of the identity function on $P(x)$.

(8) Modulo the identification of V_δ with the subset of $Ult(V,G)$ defined in 4, G is the subset of $Ult(V,G)$ defined by $S \in G$ if and only if $j_G[\cup(S)] \in_G j_G(S)$. Moreover $Ult(V,G) \models j_G(\lambda) \geq \delta$.

(9) For any $\alpha < \delta$, let $G_\alpha = G \cap V_\alpha$. Let $k_\alpha : Ult(V,G_\alpha) \to Ult(V,G)$ be defined by $[f]_{G_\alpha} \mapsto [f]_G$. Then k_α is elementary, $j_G = k_\alpha \circ j_{G_\alpha}$ and $\mathrm{crit}(k_\alpha) = j_{G_\alpha}(\mathrm{crit}(j_G)) \geq \alpha$.

A key observation is that V-generic filters for $\mathbb{T}^{\mathcal{I}}_\delta$ are also V-normal filters, but V-normal tower of ultrafilters G on δ which are contained in $\mathbb{T}^{\mathcal{I}}_\delta$ may not be fully V-generic for $\mathbb{T}^{\mathcal{I}}_\delta$. However, such V-normal tower of ultrafilters, while they may not be strong enough to decide the theory of $V^{\mathbb{T}^{\mathcal{I}}_\delta}$, are sufficiently informative to decide the behavior of the generic ultrapower $Ult(V, \dot{G}_{\mathbb{T}^{\mathcal{I}}_\delta})$ which can be defined inside $V^{\mathbb{T}^{\mathcal{I}}_\delta}$.

Lemma 4.9. *Assume V is a transitive model of* ZFC *and $\mathcal{I} \in V$ is a normal tower of height δ. The following holds:*

(1) For any V-normal tower of ultrafilters G on δ contained in $\mathbb{T}^{\mathcal{I}}_\delta$ the following are equivalent:
- *$Ult(V,G)$ models $\phi([f_1]_G, \ldots, [f_n]_G)$.*
- *There is $S \in G$ such that for all $M \in S$*

$$V \models \phi(f_1(M \cap X_{f_1}), \ldots, f_n(M \cap X_{f_n})).$$

(2) For any $S \in \mathbb{T}^{\mathcal{I}}_\delta$, $f_1 : X_{f_1} \to V, \ldots, f_n : X_{f_n} \to V$ in V, and any formula $\phi(x_1, \ldots, x_n)$ the following are equivalent:
- *For any α such that $X_{f_i} \in V_\alpha$ for all $i \leq n$*

$$\{M \prec V_\alpha : V \models \phi(f_1(M \cap X_{f_1}), \ldots, f_n(M \cap X_{f_n}))\} \geq_{\mathcal{I}} S.$$

- *For all V-normal tower of ultrafilters G on δ contained in $\mathbb{T}^{\mathcal{I}}_\delta$ to which S belongs we have that*

$$Ult(V,G) \models \phi([f_1]_G, \ldots, [f_n]_G).$$

In the situation in which W is a V-generic extension of V for some poset $P \in V$ and $G \in W$ is a V-normal tower of ultrafilters we can use G to define a tower of normal ideals in V as follows.

Lemma 4.10. *Assume V is a transitive model of* ZFC. *Let δ be an inaccessible cardinal in V and $\mathsf{Q} \in V$ be a complete boolean algebra. Assume H is V-generic for Q and $G \in V[H]$ is a V-normal tower of ultrafilters on δ. Let $\dot{G} \in V^{\mathsf{Q}}$ be such that* $\mathrm{val}_H(\dot{G}) = G$ *and*

$$[\![\dot{G} \text{ is a } V\text{-normal tower of ultrafilters on } \delta]\!]_{\mathsf{Q}} = 1_{\mathsf{Q}}.$$

Define

$$\mathcal{I}(\dot{G}) = \{I_X : X \in V_\delta\} \in V$$

by setting $A \in I_X$ *if and only if* $A \subset P(X)$ *and*

$$[\![A \in \dot{G}]\!]_{\mathsf{Q}} = 0_{\mathsf{Q}}.$$

Then $\mathcal{I}(\dot{G}) \in V$ *is a tower of normal ideals.*

In particular the above Lemma can be used to define $< \delta$-complete injective homomorphisms $i_{\dot{G}} : \mathbb{T}_\delta^{\mathcal{I}(\dot{G})} \to \mathsf{Q}$ whenever Q is a complete boolean algebra and $\dot{G} \in V^{\mathsf{Q}}$ is forced by Q to be a V-normal tower of ultrafilters on some δ which is inaccessible in V. These homomorphisms are defined by $S \mapsto [\![S \in \dot{G}]\!]_{\mathsf{Q}}$. It is easy to check that these maps are indeed $< \delta$-complete injective homomorphisms; however, without further assumptions on $\dot{G}$ these maps cannot in general be extended to complete homomorphisms of the respective boolean completions because we cannot control if $i_{\dot{G}}[\mathcal{A}]$ is a maximal antichain of Q whenever $\mathcal{A}$ is a maximal antichain of $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$. There are, however, two nice features of these mappings.

Proposition 4.11. *Assume* $\mathsf{Q} \in V$ *is a complete boolean algebra and* $\dot{G} \in V^{\mathsf{Q}}$ *is forced by* Q *to be a* V*-normal tower of ultrafilters on some* δ*. Then the following hold:*

- *The preimage of a* V*-generic filter for* Q *under* $i_{\dot{G}}$ *is a* V*-normal tower of ultrafilters contained in* $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$*;*
- $i_{\dot{G}} : \mathbb{T}_\delta^{\mathcal{I}(\dot{G})} \to \mathsf{Q}$ *extends to an isomorphism of* $\mathsf{RO}(\mathbb{T}_\delta^{\mathcal{I}(\dot{G})})$ *with* Q *if its image is dense in* Q.

Definition 4.12. Let $\lambda = \gamma^+$, δ be inaccessible and $\mathcal{I} = \{I_X : X \in V_\delta\}$ be a tower of normal ideals which concentrate on $P_\lambda(X)$. The tower $\mathbb{T}_\delta^{\mathcal{I}}$ is presaturated if δ is regular in $V[G]$ for all G V-generic for $\mathbb{T}_\delta^{\mathcal{I}}$.

The following theorem is well known.

Theorem 4.13. *Let* $\lambda = \gamma^+$*,* δ *be inaccessible and* $\mathcal{I} = \{I_X : X \in V_\delta\}$ *be a tower of normal ideals which concentrate on* $P_\lambda(X)$*.*

$\mathbb{T}_\delta^{\mathcal{I}}$ *is a presaturated normal tower if and only whenever* G *is* V*-generic for* $\mathbb{T}_\delta^{\mathcal{I}}$ $V[G]$ *models that* $Ult(V, G)$ *is closed under* $< \delta$*-sequences in* $V[G]$*.*

Theorem 4.14 (Woodin). [15, Theorem 2.7.7] *Let* $\mathbb{T}_\delta^{\omega_2}$ *be the tower given by stationary sets in* V_δ *which concentrate on* P_{ω_2}*. Assume* δ *is a Woodin cardinal. Then* $\mathbb{T}_\delta^{\omega_2}$ *is a presaturated tower.*

Lemma 4.15. *Assume* $\mathbb{T}_\delta^{\mathcal{I}}$ *is a presaturated tower such that* $\mathcal{I}$ *concentrates on* $P_{\omega_2}(H_{\delta^+})$*. Then* $\mathbb{T}_\delta^{\mathcal{I}}$ *is stationary set preserving.*

Proof. This is a consequence of the fact that if a tower is presaturated and concentrates on P_{ω_2}, the stationary subsets of ω_1 of the generic extension belong to the generic ultrapower. For more details see, for example, [28, Section 2.4]. $\square$

We also have the following duality property.

Lemma 4.16. *An* SSP *complete boolean algebra* B *which collapses* δ *to become the second uncountable cardinal is forcing equivalent to a presaturated normal tower of* *height δ if and only if there is a* B*-name* $\dot{G}$ *for a* V*-normal tower of ultrafilters on* δ *such that*

- $j_{\dot{G}} : V \to Ult(V, \dot{G})$ *is forced by* B *to be an elementary embedding such that* $j_{\dot{G}}(\omega_2) = \delta$ *and* $Ult(V, \dot{G})^{<\delta} \subset Ult(V, \dot{G})$*;*
- *the map* $S \mapsto [\![S \in \dot{G}]\!]_{\mathsf{B}}$ *which embeds* $\mathbb{T}_{\delta}^{\mathcal{I}(\dot{G})}$ *into* B *has a dense image.*

Finally we will need the following lemma to produce by forcing presaturated towers in generic extensions.

Lemma 4.17. *Assume* $j : V \to M$ *is an almost huge embedding with* $\delta = \mathrm{crit}(j)$*. Let* $P \subset V_\delta$ *be a forcing notion such that* $\mathsf{RO}(P)$ *is* $(<\delta, <\delta)$*-presaturated.*

Assume also that

- $j \restriction P$ *can be extended to a complete homomorphism between* $\mathsf{RO}(P)$ *and* $\mathsf{RO}(j(P)) \restriction q$ *for some* $q \in \mathsf{RO}(j(P))$*,*
- $\mathsf{RO}(j(P)) \restriction q$ *is* $(<j(\delta), <j(\delta))$*-presaturated.*

Let H *be* V*-generic for* $\mathsf{RO}(j(P))$ *with* $q \in H$ *and* $G = j^{-1}[H]$*. Then in* $V[H]$ *the map*

$$\bar{j} : V[G] \to M[H]$$

given by $\mathrm{val}_G(\tau) \mapsto \mathrm{val}_H(j(\tau))$ *is elementary,* $\mathrm{crit}\,(\bar{j}) = \delta$ *and* $M[H]^{<j(\delta)} \subset M[H]$*.*

Proof. Let $\mathsf{Q} = \mathsf{RO}(j(P)) \restriction q$ and $\mathsf{B} = \mathsf{RO}(P)$. The hypotheses grant that $j[G] \subset H$ and thus that

$$[\![\phi(\tau_1, \dots, \tau_n)]\!]_{\mathsf{B}} \in G \text{ iff } [\![\phi(j(\tau_1), \dots, j(\tau_n))]\!]_{\mathsf{Q}} \in H$$

for all formulas ϕ and $\tau_1, \dots, \tau_n \in V^{\mathsf{B}}$. This immediately gives that $\bar{j}$ is elementary and well defined.

Now we check that $M[H]^{<j(\delta)} \subset M[H]$ in $V[H]$.

Let $\tau : \gamma \to \mathrm{Ord}$ be in V^{Q} a name for a sequence of ordinals of length $\gamma < j(\delta)$. Then there is a family $\{A_i : i < \gamma)\} \in V$ of maximal antichains of $j(P) \restriction q \subseteq \mathsf{Q}$ such that for all $a \in A_i$ there is β such that

$$[\![\beta = \tau(i)]\!]_{\mathsf{Q}} \geq a.$$

By the $(<j(\delta, <j(\delta))$-presaturation of Q, there is $r \leq q$ in $H \cap j(P)$ such that

$$B_i = \{p \in A_i : p \wedge r > 0_{\mathsf{Q}}\} \subseteq j(P) \restriction q \subseteq M$$

has size less than $j(\delta)$ for all $i < \gamma$. This gives that $\{B_i : i < \gamma\} \in M$ and that if we set

$$\sigma = \{\langle (i, \beta), a\rangle : i < \gamma,\, a \in B_i,\, [\![\tau(i) = \beta]\!]_{\mathsf{Q}} \geq a\},$$

we have at the same time that $\sigma \in M$ and that

$$[\![\sigma = \tau]\!]_{\mathsf{Q}} \geq r.$$

In particular we get that $\mathrm{val}_H(\tau) = \mathrm{val}_H(\sigma) \in M[H]$ as was to be shown. □

5. MM^{+++}

In this section we introduce the forcing axiom MM^{+++} as a density property of the class forcing $\mathsf{U}^{\mathsf{SSP}}$, and we show how to derive the generic absoluteness of the Chang model $L(\mathrm{Ord}^{\omega_1})$ with respect to models of MM^{+++}+*large cardinals*. We first show that—in the presence of class many Woodin cardinals—MM^{++} can be characterized as the statement that the class of presaturated towers of normal filters is dense in $\mathsf{U}^{\mathsf{SSP}}$. Next we analyze the interactions between the category forcing

$\mathsf{U}^{\mathsf{SSP}}$ and the class forcing given by stationary sets S contained in $P_{\omega_2}(\cup S)$, an interaction which shows up only assuming MM^{++}. We then introduce MM^{+++} as (a slight strengthening of) the assertion that the class of totally rigid, presaturated towers of normal filters is dense in $\mathsf{U}^{\mathsf{SSP}}$. Next we prove that MM^{+++} is equivalent to the assertion that U_δ is a presaturated tower of normal filters for all δ which are Σ_2-reflecting cardinals. Then we show how to use the presaturation of U_δ to derive the absoluteness of the Chang model $L(\mathrm{Ord}^{\omega_1})$. Finally we prove the consistency of MM^{+++} relative to large cardinal axioms. The reader who is just interested in the results concerning MM^{+++} can skip the next subsection.

5.1. MM^{++} **as a density property of** $\mathsf{U}^{\mathsf{SSP}}$**.** We shall denote by $\mathbb{T}^{\omega_2}_\delta$ the stationary tower whose elements are stationary sets S in V_δ which concentrate on $P_{\omega_2}(\cup S)$. A key property of this partial order is that it is at the same stationary set preserving and a presaturated tower whenever δ is a Woodin cardinal. In [28, Theorem 2.16] we showed the following theorem.

Theorem 5.1. *Assume δ is a Woodin cardinal and let $P \in V_\delta$ be a partial order. Then the following are equivalent:*

(1) $T_{\mathsf{RO}(P)}$ is stationary,
(2) $\mathsf{RO}(P) \geq_{\mathsf{SSP}} \mathbb{T}^{\omega_2}_\delta \restriction S$ for some stationary set $S \in \mathbb{T}^{\omega_2}_\delta$.

In particular we have the following immediate corollary.

Corollary 5.2. *Assume there are class many Woodin cardinals. Then the following are equivalent:*

(1) T_{B} is stationary for all $\mathsf{B} \in \mathsf{SSP}$.
(2) The class of presaturated normal towers is dense in $\mathsf{U}^{\mathsf{SSP}}$.

Notice that in the presence of MM^{++} and class many Woodin cardinals we have a further characterization of total rigidity.

Proposition 5.3. *Assume MM^{++} and there are class many Woodin cardinals. Then the following are equivalent for a $\mathsf{B} \in \mathsf{SSP}$:*

(1) B is totally rigid.
(2) $G(M,\mathsf{B}) = \{b \in M : M \in T_{\mathsf{B}\restriction b}\}$ is the unique correct M-generic filter for B for a club of $M \in T_{\mathsf{B}}$.

Proof. We first show that $G(M,\mathsf{B})$ is a correct M-generic filter for B iff there is a unique such correct M-generic filter.

So assume there are two distinct correct M-generic filters for B H_0, H_1. Let $b \in H_0 \setminus H_1$. Then $M \in T_{\mathsf{B}\restriction b} \cap T_{\mathsf{B}\restriction \neg b}$ as witnessed by H_0, H_1; thus $b, \neg b \in G(M,\mathsf{B})$, and $G(M,\mathsf{B})$ is not a filter.

Conversely assume H is the unique correct M-generic filter for B. Then $b \in H$ gives that $M \in T_{\mathsf{B}\restriction b}$. Thus $H \subseteq G(M,\mathsf{B})$. Now if $c \in G(M,\mathsf{B}) \setminus H$, there is a correct M-generic filter H^* for M with $c \in H^* \setminus H$. This contradicts the uniqueness assumption on H. Thus $H = G(M,\mathsf{B})$ as was to be shown.

Now we prove the equivalence of total rigidity with 2.

Assume first that 2 fails. Let $S \subset T_{\mathsf{B}}$ be a stationary set such that for all $M \in S$ there are at least two distinct correct M-generic filters H_0^M, H_1^M. For each such M we can find $b_M \in M \cap (H_0^M \setminus H_1^M)$. By pressing down on S and refining S if necessary, we can assume that $b_M = b^*$ for all $M \in S$. Let $\delta > |\mathsf{B}|$ be a Woodin cardinal. For $j = 0, 1$ define $i_j : \mathsf{B} \to \mathbb{T}^{\omega_2}_\delta \restriction S$ by

$$b \mapsto \{M \in S : b \in H^M_j\}.$$

Then i_0, i_1 are complete homomorphisms such that

$$\mathsf{B} \Vdash \mathbb{T}^{\omega_2}_\delta \restriction S/_{i_j[\dot{G}_\mathsf{B}]} \text{ is stationary set preserving}$$

and $i_0(b^*) = S = i_1(\neg b^*)$. In particular we get that i_0 witnesses that $\mathsf{B} \restriction b^* \geq \mathbb{T}^{\omega_2}_\delta \restriction S$ and i_1 witnesses that $\mathsf{B} \restriction \neg b^* \geq \mathbb{T}^{\omega_2}_\delta \restriction S$. All in all we have that $\mathsf{B} \restriction b^*$ and $\mathsf{B} \restriction \neg b^*$ are compatible conditions in $\mathsf{U}^{\mathsf{SSP}}$; i.e., B is not totally rigid.

Now assume that B is not totally rigid. Let $i_0 : \mathsf{B} \restriction b \to \mathsf{C}$ and $i_1 : \mathsf{B} \restriction \neg_\mathsf{B} b \to \mathsf{C}$ be distinct complete homomorphisms of B into C such that for $j = 0, 1$

$$\mathsf{B} \Vdash \mathsf{C}/_{i_j[\dot{G}_\mathsf{B}]} \text{ is stationary set preserving.}$$

Then for all $M \in T_\mathsf{C}$ such that $i_0, i_1 \in M$ we can pick H_M a correct M-generic filter for C. Thus $G_j = i_j^{-1}[H_M]$ for $j = 0, 1$ are both correct and M-generic and such that $b \in G_0$ and $\neg_\mathsf{B} b \in G_1$. In particular we get that for a club of $M \in T_\mathsf{C} \restriction H_{|\mathsf{B}|^+} \subseteq T_\mathsf{B}$ there are at least two M-generic filters for B; i.e., 2 fails.

□

Duality between SSP*-forcings and stationary sets concentrating on* P_{ω_2}. We outline some basic properties which tie the arrows of $\mathsf{U}^{\mathsf{SSP}}$ with the order relation on stationary sets concentrating on P_{ω_2} assuming that MM^{++} holds.

Lemma 5.4. *Assume* MM^{++} *and there are class many Woodin cardinals. Then* $\mathsf{B}_0, \mathsf{B}_1$ *are compatible conditions in* $\mathsf{U}^{\mathsf{SSP}}$ *(i.e., there are arrows* $i_j : \mathsf{B}_j \to \mathsf{C}$ *for* $j = 0, 1$ *and* C *fixed in* $\mathsf{U}^{\mathsf{SSP}}$*) if and only if* $T_{\mathsf{B}_0} \wedge T_{\mathsf{B}_1}$ *is stationary.*

Proof. First assume that $\mathsf{C} \leq \mathsf{B}_0, \mathsf{B}_1$. We show that $T_{\mathsf{B}_0} \wedge T_{\mathsf{B}_1}$ is stationary. Let $i_0 : \mathsf{B}_0 \to \mathsf{C}$, $i_1 : \mathsf{B}_1 \to \mathsf{C}$ be atomless complete homomorphisms such that

$$\mathsf{B}_j \Vdash \mathsf{C}/_{i_j[\dot{G}_\mathsf{B}]} \text{ is stationary set preserving.}$$

For all $M \in T_\mathsf{C}$ such that $i_j \in M$ for $j = 0, 1$ pick G_M correct M-generic filter for C. Let $H^j_M = i_j^{-1}[G_M]$, and then H^j_M is a correct M-generic for B_j, since each i_j is such that

$$\mathsf{B}_j \Vdash \mathsf{C}/_{i_j[\dot{G}_\mathsf{B}]} \text{ is stationary set preserving.}$$

In particular $M \cap H_{|\mathsf{B}_j|^+} \in T_{\mathsf{B}_j}$ for $j = 0, 1$. Thus $T_\mathsf{C} \restriction H_{|\mathsf{B}_j|^+} \subset T_{\mathsf{B}_j}$ for $j = 0, 1$; i.e., $T_{\mathsf{B}_0} \wedge T_{\mathsf{B}_1}$ is stationary.

Conversely assume that $T_{\mathsf{B}_0} \wedge T_{\mathsf{B}_1}$ is stationary. For each $M \in S = T_{\mathsf{B}_0} \wedge T_{\mathsf{B}_1}$ pick H^j_M correct M-generic filter for B_j for $j = 0, 1$. Fix a Woodin cardinal $\delta > |S|$. Let $\mathbb{T}^{\omega_2}_\delta$ denote the stationary tower with critical point ω_2 and height δ. Let $i_j : \mathsf{B}_j \to \mathbb{T}^{\omega_2}_\delta \restriction S$ map

$$b \to \{M \in S : b \in H^j_M\}.$$

Then each i_j is a complete embedding such that

$$\mathsf{B}_j \Vdash \mathbb{T}^{\omega_2}_\delta \restriction S/_{i_j[\dot{G}_\mathsf{B}]} \text{ is stationary set preserving;}$$

i.e., $\mathsf{B}_j \geq \mathbb{T}^{\omega_2}_\delta \restriction S$ for $j = 0, 1$ showing that B_0 and B_1 are compatible conditions in $\mathsf{U}^{\mathsf{SSP}}$. □

We can also give a simple representation of totally rigid boolean algebras and a characterization of the complete embeddings between totally rigid complete boolean algebras.

Fact 5.5. *Assume* B *is totally rigid and* $T_{\mathsf{B}\restriction b}$ *is stationary for all* $b \in \mathsf{B}^+$. *Then* B *is isomorphic to the complete boolean subalgebra* $\{T_{\mathsf{B}\restriction b} : b \in \mathsf{B}\}$ *of the boolean algebra* $P(T_{\mathsf{B}})/\mathsf{NS}$.

Notice that in the above setting, $P(T_{\mathsf{B}})/\mathsf{NS}$ may not be a complete boolean algebra and may not be stationary set preserving, while $\{T_{\mathsf{B}\restriction b} : b \in \mathsf{B}\}$ is an SSP and complete boolean subalgebra.

Fact 5.6. *Assume* MM^{++} *holds. Assume* $\mathsf{B} \geq_{\mathsf{SSP}} \mathsf{Q}$ *are totally rigid and complete boolean algebras. Let* $i : \mathsf{B} \to \mathsf{Q}$ *be the unique* SSP*-correct homomorphism between* B *and* Q. *Then for all* $b \in \mathsf{B}$ *and* $q \in \mathsf{Q}$, $T_{\mathsf{B}\restriction b} \wedge T_{\mathsf{Q}\restriction q}$ *is stationary if and only if* $i(b) \wedge_{\mathsf{Q}} q > 0_{\mathsf{Q}}$.

Proof. Left to the reader. □

It is also immediate to check the following fact.

Fact 5.7. *For any family* $\mathcal{A} \subset \mathsf{U}^{\mathsf{SSP}}$

$$T_{\bigvee \mathcal{A}} = \bigvee \{T_{\mathsf{B}} : \mathsf{B} \in \mathcal{A}\}.$$

All in all assuming MM^{++} and class many Woodin cardinals we are in the following situation:

(1) MM^{++} can be defined as the statement that the class PT of presaturated towers of normal filters is dense in the category $\mathsf{U}^{\mathsf{SSP}}$.
(2) We also have in the presence of MM^{++} a functorial map

$$F : \mathsf{U}^{\mathsf{SSP}} \to \{S \in V : S \text{ is stationary and concentrate on } P_{\omega_2}\}$$

defined by $\mathsf{B} \mapsto T_{\mathsf{B}}$ which
- is order and incompatibility preserving,
- maps set sized suprema to set sized suprema in the respective class partial orders,
- gives a neat representation of the separative quotients of totally rigid partial orders and of the complete embeddings between them.

It is now tempting to conjecture that it is possible to reflect this to some V_δ and obtain that the map $F \restriction V_\delta$ defines a complete embedding of U_δ into $\mathbb{T}^{\omega_2}_\delta$. However, we just have that F preserves suprema of set sized subsets of $\mathsf{U}^{\mathsf{SSP}}$ which would reflect to the fact $F \restriction V_\delta$ defines a $< \delta$-complete embedding of U_δ into $\mathbb{T}^{\omega_2}_\delta$. However, we have no reason to expect that $F \restriction V_\delta$ extends to a complete homomorphism of the respective boolean completion because neither of the above posets is $< \delta$-CC.

On the other hand, we have shown in the previous sections that in the presence of class many supercompact cardinals we have that the class TR of totally rigid posets is dense in the category $\mathsf{U}^{\mathsf{SSP}}$. What about the intersection of the classes TR and PT? Can there be densely many presaturated towers which are also totally rigid in $\mathsf{U}^{\mathsf{SSP}}$?

A positive answer to this question leads to the definition of MM^{+++}.

5.2. **Strongly presaturated towers and MM^{+++}.**

Definition 5.8. Let $\mathcal{I} = \{I_X : X \in V_\delta\}$ be a tower of normal filters of height δ which concentrates on P_{ω_2}. S is a fixed point for $\mathcal{I}$ if $S \geq T_{\mathbb{T}^{\mathcal{I}}_\delta \restriction S}$.

A normal tower $\mathbb{T} = \mathbb{T}^{\mathcal{I}}_\delta$ is strongly presaturated[15] if it is presaturated, has a dense set of fixed points, and $T_{\mathbb{T}\restriction S}$ is stationary for all fixed points $S \in \mathbb{T}$ such that $S \notin I_{\cup S}$.

SPT denotes the class of strongly presaturated towers of normal filters.

We shall see below that for any B, T_{B} is a fixed point of $\mathbb{T}$ for any presaturated tower $\mathbb{T}$ to which T_{B} belongs. However, it is well possible (and we conjecture that) for all δ, $\mathbb{T}^{\omega_2}_\delta$ is never strongly presaturated, since it is likely that the family $\{T_{\mathsf{B}} : \mathsf{B} \in \mathsf{U}_\delta\}$ is not dense in $\mathbb{T}^{\omega_2}_\delta$. On the other hand, if U_δ is forcing equivalent to a presaturated normal tower $\mathbb{T}$, we shall see that $\{T_{\mathsf{B}} : \mathsf{B} \in \mathsf{U}_\delta\}$ is dense in $\mathbb{T}$. In particular this would give that $\mathsf{U}_\delta \in$ SPT. Moreover we shall see that (in almost all cases) a tower $\mathbb{T}$ of height δ is strongly presaturated iff there is a family $D \subset \mathsf{U}_\delta$ such that $\{T_{\mathsf{B}} : \mathsf{B} \in D\}$ is dense in $\mathbb{T}$. These comments are a motivation for the following definition.

Definition 5.9. MM^{+++} holds if SPT is a dense class in $\mathsf{U}^{\mathsf{SSP}}$.

Notice that MM^{+++} strengthens MM^{++} in view of Corollary 5.2, and we shall see in Lemma 5.15 below that MM^{++} entails that any $\mathbb{T} \in$ SPT is totally rigid. In particular assuming MM^{+++} the class of SPT posets is a subset of the intersection of the class of presaturated towers and the class of totally rigid posets; however, we haven't been able to establish whether it is a proper subset or a characterization of the intersection.

The plan of the remainder of this section is the following:

(1) We shall introduce some basic properties of the elements $\mathbb{T}$ of SPT.
(2) We will show that if $\mathsf{U}_\delta \in \mathsf{SSP}$ and SPT$\cap V_\delta$ is dense in U_δ, then actually U_δ is forcing equivalent to a strongly presaturated normal tower.
(3) We will use the previous item in combination with Theorem 3.9 to prove that the theory $\mathsf{ZFC}+\mathsf{MM}^{+++}+$*there are class many* Σ_2*-reflecting cardinals* δ *which are a limit of* $<\delta$*-supercompact cardinals* makes the theory of the Chang model $L(\mathrm{Ord}^{\leq\aleph_1})$ invariant under stationary set preserving forcings which preserve MM^{+++}.
(4) Finally we will show that essentially any "iteration" of length δ which produces a model of MM^{++} actually produces a model of MM^{+++}, provided δ is super almost huge.

Basic properties of SPT.

Lemma 5.10. *Assume* $\mathbb{T} = \mathbb{T}^{\mathcal{I}}_\delta$ *is a strongly presaturated tower of normal filters. Then for all* $M \in T_{\mathbb{T}}$

$$G(M, \mathbb{T}) = \{S \in M \cap V_\delta : M \cap \cup S \in S\}$$

is the unique correct M*-generic filter for* $\mathbb{T}$*.*

[15]There is a slight ambiguity between $\mathbb{T}$ meant as a partial order and $\mathbb{T}$ meant as a boolean algebra. We remark that any $S \in V_\delta$ such that $[S]_{\mathcal{I}} = 0_{\mathbb{T}}$ is a fixed point for $\mathbb{T}$ since $T_{\mathbb{T}\restriction S}$ and $T_{\mathbb{T}} \wedge S$ are always non-stationary in this case (see Fact 5.11 below). Thus this definition applies just to the $S \in V_\delta$ such that $S \notin I_{\cup S}$.

Proof. Observe that if $M \in T_{\mathbb{T}}$, G is a correct M-generic for $\mathbb{T}$, and $S \in G$ is a fixed point, we get that $M \in T_{\mathbb{T}\restriction S}$ and thus that $M \cap \cup S \in S$ since $S \geq T_{\mathbb{T}^{\mathcal{I}}_{\delta}\restriction S}$. In particular (since the set of fixed points is dense in $\mathbb{T}$) we get that $G(M, \mathbb{T}) \supseteq G$. Since G is an ultrafilter on the boolean algebra $\mathbb{T} \cap M$, this gives that $G = G(M, \mathbb{T})$ is a correct M-generic ultrafilter for $\mathbb{T}$. The thesis follows. $\square$

The following fact is almost self-evident.

Fact 5.11. *Assume $\mathcal{I} = \{I_X : X \in V_\delta\}$ is a tower of normal filters such that $\mathbb{T} = \mathbb{T}^{\mathcal{I}}_{\delta}$ is strongly presaturated. Let $S \in V_\delta$ be a fixed point for $\mathbb{T}$. Then $S \wedge T_{\mathbb{T}} = T_{\mathbb{T}\restriction S}$. Moreover the latter sets are stationary iff $S \in I^+_{\cup S}$.*

Proof. First assume $S \notin I^+_{\cup S}$. We show that $S \wedge T_{\mathbb{T}}$ and $T_{\mathbb{T}\restriction S}$ are both non-stationary. Since $S \in I_{\cup S}$, we get that $[P(\cup S) \setminus S]_{\mathcal{I}} = 1_{\mathbb{T}}$ and $[S]_{\mathcal{I}} = 0_{\mathbb{T}}$. Thus for all $M \in T_{\mathbb{T}}$ we get that $P(\cup S) \setminus S \in G(M, \mathbb{T})$, i.e., that $M \cap \cup S \notin S$. This gives at the same time that $T_{\mathbb{T}\restriction S}$ and $S \wedge T_{\mathbb{T}}$ are both non-stationary.

Next assume $S \in I^+_{\cup S}$, i.e., $[S]_{\mathcal{I}} > 0_{\mathbb{T}}$. Then $M \in T_{\mathbb{T}} \wedge S$ iff $S \in G(M, \mathbb{T})$ iff $M \in T_{\mathbb{T}\restriction S}$. This gives that $S \wedge T_{\mathbb{T}} = T_{\mathbb{T}\restriction S}$ also in this case. Finally by definition of strong presaturation, $T_{\mathbb{T}\restriction S}$ is always stationary if $S \in I^+_{\cup S}$. $\square$

Fact 5.12. *Assume $\mathbb{T} = \mathbb{T}^{\mathcal{I}}_{\delta}$ is a strongly presaturated tower of normal filters of height δ which makes δ the second uncountable cardinal. Then for all $M \in T_{\mathbb{T}}$ and all $h : P(X) \to \omega_2$ in $M \cap V_\delta$, we have that $h(M \cap X) < \operatorname{otp}(M \cap \delta)$.*

Proof. Pick $M \in T_{\mathbb{T}}$ (which is stationary by our assumption that $\mathbb{T}$ is strongly presaturated). Observe that $G(M, \mathbb{T})$ is the unique correct M-generic filter for $\mathbb{T}$. Let $H = \pi_M[G(M, \mathbb{T})]$ and $N = \pi_M[M]$. Then H is a correct N-generic filter for $\bar{\mathbb{T}} = \pi_M(\mathbb{T})$ and N models that $\bar{\mathbb{T}}$ is a presaturated tower of height $\bar{\delta} = \pi_M(\delta)$. In particular if $h : P(X) \to \omega_2$ is in M, we get that

$$Ult(N, H) \models [\pi_M(h)]_H < \pi_M(\delta) = \omega_2^{N[H]}.$$

Thus there is $\gamma \in M \cap \delta$ such that

$$[\pi_M(h)]_H = \pi_M(\gamma) = [\rho_{\pi_M(\gamma)}]_H.$$

This occurs if and only if

$$\begin{aligned} S = \{M' \prec V_{\gamma+1} : \operatorname{otp}(M' \cap \gamma) = \rho_\gamma(M') = h(M' \cap X)\} \in G(M, \mathbb{T}) \\ = \{S \in \mathbb{T} : M \cap \cup S \in S\}. \end{aligned}$$

The conclusion follows. $\square$

Fact 5.13. *For any $\mathsf{B} \in \mathsf{SSP}$, T_{B} is a fixed point of $\mathbb{T}$ for any presaturated normal tower $\mathbb{T} = \mathbb{T}^{\mathcal{I}}_{\delta}$ of height δ such that $\mathsf{B} \in V_\delta$.*

Proof. In case $T_{\mathsf{B}} \equiv_{\mathcal{I}} \emptyset$, $T_{\mathbb{T}\restriction T_{\mathsf{B}}}$ is non-stationary, and thus $T_{\mathsf{B}} \geq T_{\mathbb{T}\restriction T_{\mathsf{B}}}$ is a fixed point of $\mathbb{T}$.

Now observe that if G is a V-generic filter for $\mathbb{T}$, then

$$T_{\mathsf{B}} \in G \text{ iff } M = j_G[H_{|\mathsf{B}|^+}] \in j_G(T_{\mathsf{B}}) = T_{j_G(\mathsf{B})}.$$

This gives that H in $Ult(V, G)$ is a correct M-generic filter for $j_G(\mathsf{B})$ iff $\bar{H} = \pi_M(H)$ is a V-generic filter for B such that $V[\bar{H}]$ is correct about the stationarity of its subsets of ω_1. In particular we get that if $M \in T_{\mathbb{T}\restriction T_{\mathsf{B}}}$, then $N = \pi_M[M]$ has an N-generic filter for $\pi_M(\mathsf{B})$ which computes correctly the stationarity of its subsets of ω_1. Thus $M \cap \cup T_{\mathsf{B}} \in T_{\mathsf{B}}$. All in all we have shown that $T_{\mathsf{B}} \geq T_{\mathbb{T} \restriction T_{\mathsf{B}}}$, i.e., T_{B} is a fixed point of $\mathbb{T}$. □

Lemma 5.14. *Assume $\mathsf{U}_\delta \in \mathsf{SSP}$ forces that there is a U_δ-name $\dot{G}$ for a V-normal tower of ultrafilters on δ such that $j_{\dot{G}} : V \to Ult(V, \dot{G})$ is an elementary embedding with $j_{\dot{G}}(\omega_2) = \delta$. Then U_δ is strongly presaturated.*

Proof. Let for $X \in V_\delta$

$$I_X^{\dot{G}} = \{S \subset P(X) : [\![j_{\dot{G}}[X] \in j_{\dot{G}}(S)]\!]_{\mathsf{RO}(\mathsf{U}_\delta)} = 0_{\mathsf{RO}(\mathsf{U}_\delta)}\}$$

and

$$\mathcal{I}(\dot{G}) = \{I_X^{\dot{G}} : X \in V_\delta\}.$$

By Lemma 4.16 it is enough to show the following claim.

Claim. *For all $\mathsf{Q} \in \mathsf{U}_\delta$*

$$\mathsf{Q} = [\![T_{\mathsf{Q}} \in \dot{G}]\!]_{\mathsf{RO}(\mathsf{U}_\delta)}.$$

For suppose the claim holds, then we have that the map

$$i : \mathbb{T}_\delta^{\mathcal{I}(\dot{G})} \to \mathsf{RO}(\mathsf{U}_\delta)$$

which maps

$$S \mapsto [\![j_{\dot{G}}[\cup S] \in j_{\dot{G}}(S)]\!]_{\mathsf{RO}(\mathsf{U}_\delta)}$$

extends to an injective homomorphism of complete boolean algebras with a dense image, i.e., to an isomorphism. In particular this gives at the same time that $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$ is a presaturated tower forcing (since it preserve the regularity of δ) which is equivalent to U_δ and that

$$\{T_{\mathsf{B}} : \mathsf{B} \in \mathsf{U}_\delta\}$$

is a dense subset of $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$ since its image is U_δ which is, by definition, a dense subset of $\mathsf{RO}(\mathsf{U}_\delta)$. However,

$$\{T_{\mathsf{B}} : \mathsf{B} \in \mathsf{U}_\delta\}$$

is a family of fixed points of $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$ which gives that $\mathbb{T}_\delta^{\mathcal{I}(\dot{G})}$ is strongly presaturated. So we prove the following claim.

Proof of Claim. Let H be V-generic for U_δ and $G = \mathrm{val}_H(\dot{G})$. We show the following subclaim.

Subclaim. $\mathsf{B} \in H$ *if and only if* $T_{\mathsf{B}} \in G$.

Proof of Subclaim. First assume that $\mathsf{B} \in H$. Then we have that there is in $V_\delta[H]$ a V-generic filter H_0 for B such that $H_{|\mathsf{B}|^+}[H_0]$ computes correctly the stationarity of its subsets of ω_1 by Lemma 3.22.

Since $j_G(\omega_2) = \delta$ and $Ult(V, G)^{<\delta} \subset Ult(V, G)$, we have that

$$H_{\omega_2}^{V[H]} = V_\delta[H] = H_{\omega_2}^{Ult(V,G)}.$$

Since $\mathsf{B} \in V_\delta$, $H_0 \subset \mathsf{B} \in V_\delta[H] = H_{\omega_2}^{Ult(V,G)}$. Thus $H_0 \in Ult(V, G)$ and $Ult(V, G)$ models that $j_G[H_{|\mathsf{B}|^+}] \in T_{j(\mathsf{B})} = j(T_{\mathsf{B}})$ which occurs if and only if $T_{\mathsf{B}} \in G$.

Conversely assume that $T_{\mathsf{B}} \in G$. We get that $j_G[H_{|\mathsf{B}|^+}] \in T_{j(\mathsf{B})}$ and thus that in $Ult(V, G) \subset V[H]$ there is a V-generic filter H_0 for B such that $H_{|\mathsf{B}|^+}[H_0]$ computes correctly the stationarity of its subsets of ω_1; since

$$H_{\omega_2}^{V[H]} = V_\delta[H] = H_{\omega_2}^{Ult(V,G)},$$

we get that $H_0 \in V[H]$ is a correct V-generic filter for B. We conclude that $\mathsf{B} \in H$ by Lemma 3.22.

The claim is proved.

The Lemma is proved in all its parts. $\square$

Finally we cannot infer right away that a $\mathbb{T} \in \mathrm{SPT}$ is totally rigid since we are not able to exclude the case that there could be two distinct correct $i_0 : \mathbb{T} \to \mathsf{R}$ and $i_1 : \mathbb{T} \to \mathsf{R}$ for some $\mathsf{R} \in \mathsf{SSP}$ such that T_R is non-stationary. However, we can show that MM^{++} entails that any SPT tower is totally rigid by the following lemma.

Lemma 5.15. *Assume* MM^{++} *holds and there are class many Woodin cardinals. Then any* $\mathbb{T} \in \mathrm{SPT}$ *is totally rigid.*

Proof. Immediate by Lemmas 5.3 and 5.10. $\square$

5.3. MM^{+++} and the presaturation of U_δ.

Theorem 5.16. *Assume that any element of* SPT *is totally rigid, that* $\mathsf{U}_\delta \in \mathsf{SSP}$ *preserves the regularity of* δ*, and that* $\mathrm{SPT} \cap V_\delta$ *is dense in* U_δ*. Then* $\mathsf{U}_\delta \in \mathrm{SPT}$ *as well.*

Proof. Let for each $\mathsf{Q} \in \mathsf{U}_\delta$, $\mathsf{R}(\mathsf{Q},\gamma) \leq \mathsf{Q}$ be a strongly presaturated normal tower of height γ in U_δ.

Let H be V-generic for U_δ and G be the set of T_B for $\mathsf{B} \in H$. We aim to show that the upward closure of G generates a V-normal tower of ultrafilters $\bar{G}$ on δ and that $Ult(V,\bar{G})$ is $< \delta$-closed in $V[H]$. This suffices to prove the theorem by Lemma 5.14.

First we observe that for any $\mathsf{B} \in H$ the set of $\mathsf{R}(\mathsf{Q},\gamma) \in D = \mathrm{SPT} \cap V_\delta$ such that $\mathsf{B} \geq \mathsf{R}(\mathsf{Q},\gamma)$ is dense below B and thus H meets this dense set. In particular we get that

$$A = \{\gamma : \exists \mathsf{R}(\mathsf{Q},\gamma) \in D \cap H\}$$

is unbounded in δ.

Claim. *For all* $\gamma \in A$*,* $G \cap V_\gamma$ *generates the unique correct* V*-generic filter for* $\mathsf{R}(\mathsf{Q},\gamma)$ *in* $V[H]$*.*

Proof of Claim. Notice that for all $\mathsf{B} \in V_\gamma$, $T_\mathsf{B}, T_{\mathsf{R}(\mathsf{Q},\gamma)} \in G$ gives that $T_\mathsf{B} \wedge T_{\mathsf{R}(\mathsf{Q},\gamma)}$ is stationary. Now by Fact 5.11 we get that

$$T_{\mathsf{R}(\mathsf{Q},\gamma)\restriction T_\mathsf{B}} = T_\mathsf{B} \wedge T_{\mathsf{R}(\mathsf{Q},\gamma)}$$

and that these sets are stationary iff T_B is a positive element of the forcing $\mathsf{R}(\mathsf{Q},\gamma)$. This gives that $T_{\mathsf{R}(\mathsf{Q},\gamma)\restriction T_\mathsf{B}} \in G$ for all $\mathsf{B} \in H \cap V_\gamma$.

Now observe that by assumption $\mathsf{R}(\mathsf{Q},\gamma)$ is totally rigid, and thus the unique correct embedding i of $\mathsf{R}(\mathsf{Q},\gamma)$ into $\mathsf{U}_\delta \restriction \mathsf{R}(\mathsf{Q},\gamma)$ is given by $S \mapsto \mathsf{R}(\mathsf{Q},\gamma) \restriction S$. Thus for all $\mathsf{B} \in V_\gamma$, $\mathsf{B} \in H$ iff $T_\mathsf{B} \in G \cap V_\gamma$, and thus $i^{-1}[H] = G \cap V_\gamma$ generates a correct V-generic filter for $\mathsf{R}(\mathsf{Q},\gamma)$. Finally the strong presaturation of $\mathsf{R}(\mathsf{Q},\gamma)$ grants that $G \cap V_\gamma$ correct V-generic filter.

Since $G = \bigcup_{\gamma\in A} G \cap V_\gamma$ and $G \cap V_\gamma$ generates a V-normal tower of ultrafilters on γ for all $\gamma \in A$, we get that $G \in V[H]$ and its upward closure $\bar{G}$ is a V-normal tower of ultrafilters for δ.

Now we apply Fact 5.12 to each $\mathsf{R}(\mathsf{Q},\gamma) \in H$, and we get that for all $h : P(X) \to \omega_2$ in V, and $\gamma \in A$ such that $X \in V_\gamma$, we have that for all $M \in T_{\mathsf{R}(\mathsf{Q},\gamma)}$, $h(M\cap X) < \operatorname{otp}(M \cap \gamma)$.

This gives in particular that for all $h : P(X) \to \omega_2$ in V, and $X \in V_\delta$, we have that $[h]_{\bar{G}} < \delta$, since $T_{\mathsf{R}(\mathsf{Q},\gamma)} \in G$ witnesses that $[h]_{\bar{G}} < \gamma$ for all $h : P(X) \to \omega_2$ in V_γ.

All in all we get that $j_{\bar{G}} : V \to Ult(V, \bar{G})$ is such that $j_{\bar{G}}(\omega_2) = \delta$ and $Ult(V, \bar{G})^{<\delta} \subset Ult(V, \bar{G})$.

Since $G \in V[H]$ gives a V-normal tower of ultrafilters on δ with $j_G : V \to Ult(V, G)$ such that $j_G(\omega_2) = \delta = \omega_2^{V[H]}$, we can conclude that U_δ is a presaturated normal tower by Lemma 5.14. □

Corollary 5.17. *Assume there are class many supercompact cardinals δ which are limit of $< \delta$-supercompact cardinals. The following are equivalent:*

(1) SPT *is a dense subclass of* $\mathsf{U}^{\mathsf{SSP}}$ *consisting of totally rigid posets.*
(2) U_δ *is strongly presaturated for all Σ_2-reflecting cardinals δ which are limit of $< \delta$-supercompact cardinals.*

Proof. We prove just the non-trivial direction. Assume δ is a Σ_2-reflecting cardinal. The statement $\phi(i, \mathsf{Q}, \mathcal{I}, T_{\mathbb{T}}, \delta)$,

> $i : \mathsf{Q} \to \mathbb{T} = \mathbb{T}^{\mathcal{I}}_\delta$ *is an* SSP*-correct homomorphism and* $\mathbb{T}$ *is a strongly presaturated tower of height* δ*,*

is a statement formalizable in H_θ for any $\theta > 2^\delta$ using the parameters $i, \mathsf{Q}, \mathcal{I}, T_{\mathbb{T}^{\mathcal{I}}_\delta}$, since the statement $\mathbb{T}$ *is a strongly presaturated tower* can be phrased as

> δ is inaccessible and $\mathcal{I} = \{I_X : X \in V_\delta\}$ is a tower of normal filters of height δ and $T_{\mathbb{T}^{\mathcal{I}}_\delta}$ is stationary and $S \wedge T_{\mathbb{T}^{\mathcal{I}}_\delta} = T_{\mathbb{T}^{\mathcal{I}}_\delta \restriction S}$ for densely many $S \in \mathbb{T}^{\mathcal{I}}_\delta$.

Moreover the statement $i : \mathsf{Q} \to \mathbb{T}$ *is* SSP*-correct* is also definable in the parameters $i, \mathsf{Q}, \mathbb{T}$ in the structure H_θ using the relevant parameters. Now assume $\mathsf{Q} \in \mathsf{U}_\delta$. Then the statement

$$\exists\, H_\theta\, [H_\theta \models \exists \mathbb{T}, \delta, i, \mathcal{I}\, \phi(i, \mathsf{Q}, \mathcal{I}, T_{\mathbb{T}}, \delta)]$$

is a Σ_2-property in the parameter Q and holds in V; thus it holds in V_δ, since δ is Σ_2-reflecting. In particular this gives that $\mathrm{SPT} \cap V_\delta$ is dense in U_δ. Now by Theorem 3.5 we also have that $\mathsf{U}_\delta \in \mathsf{SSP}$ if δ is an inaccessible limit of $< \delta$-supercompact cardinals. We can now apply Theorem 5.16 to get that $\mathsf{U}_\delta \in \mathrm{SPT}$, completing the proof. □

5.4. MM^{+++} **and generic absoluteness of the theory of** $L([\mathrm{Ord}]^{\leq\aleph_1})$**.** Once we are able to infer that U_δ is a presaturated tower for any Σ_2-reflecting cardinal δ which is a limit of $< \delta$-supercompact cardinals, the generic aboluteness results is an easy consequence of Theorem 3.9.

Theorem 5.18. *Assume* MM^{+++} *holds and there are class many* Σ_2*-reflecting cardinals* δ *which are limits of* $<\delta$*-supercompact cardinals. Then*

$$\langle L([\mathrm{Ord}]^{\leq\aleph_1})^V, P(\omega_1)^V, \in\rangle \equiv \langle L([\mathrm{Ord}]^{\leq\aleph_1})^{V^P}, P(\omega_1)^V, \in\rangle$$

for all P *which are stationary set preserving and preserve* MM^{+++}.

We leave to the reader to convert a proof of this result in a proof of Theorem 1.

Proof. Let δ be a Σ_2-reflecting cardinal which is a limit of $<\delta$-supercompact cardinals. Let H be V-generic for U_δ. By Theorem 5.17 U_δ is forcing equivalent to a strongly presaturated normal tower. Thus in $V[H]$ we can define an elementary $j : V \to M$ such that $\mathrm{crit}(j) = \omega_2$ and $M^{<\delta} \subset M$. In particular we get that

$$L([\mathrm{Ord}]^{\leq\aleph_1})^M = L([\mathrm{Ord}]^{\leq\aleph_1})^{V[H]}.$$

Thus

$$\langle L([\mathrm{Ord}]^{\leq\aleph_1})^V, P(\omega_1)^V, \in\rangle \equiv \langle L([\mathrm{Ord}]^{\leq\aleph_1})^{V[H]}, P(\omega_1)^V, \in\rangle.$$

Now observe that if $P \in \mathsf{U}_\delta$ forces MM^{+++} and G is V-generic for P we have that δ is still a Σ_2-reflecting cardinal which is a limit of $<\delta$-supercompact cardinals in $V[G]$. We can first appeal to Lemma 3.9 to find:

- $\mathsf{Q} \in \mathsf{U}_\delta \restriction P$,
- $i_0 : \mathsf{RO}(P) \to \mathsf{Q}$ a regular embedding,
- $i_1 : \mathsf{RO}(P) \to \mathsf{U}_\delta \restriction \mathsf{Q}$ also a regular embedding,

such that whenever G is V-generic for P, then $V[G]$ models that the forcing

$$(\mathsf{U}_\delta^{\mathsf{SSP}} \restriction \mathsf{Q})^V / i_1[G]$$

is identified with the forcing

$$(\mathsf{U}_\delta^{\mathsf{SSP}})^{V[G]} \restriction (\mathsf{Q}/i_0[G])$$

as computed in $V[G]$.

Now let H be V-generic for $\mathsf{U}_\delta \restriction \mathsf{Q}$ and $G = i_1^{-1}[H]$ be V-generic for P. Then we have that $H/i_1[G]$ is $V[G]$-generic for

$$(\mathsf{U}_\delta)^{V[G]} \restriction (\mathsf{Q}/i_0[G])$$

as computed in $V[G]$ and that δ is a Σ_2-reflecting cardinal which is a limit of $<\delta$-supercompact cardinals in $V[G]$ as well. In particular by applying the above fact in V and in $V[G]$ which are both models of MM^{+++} where δ is a Σ_2-reflecting cardinal which is a limit of $<\delta$-supercompact cardinals we get that

$$\langle L([\mathrm{Ord}]^{\leq\aleph_1})^V, P(\omega_1)^V, \in\rangle \equiv \langle L([\mathrm{Ord}]^{\leq\aleph_1})^{V[H]}, P(\omega_1)^V, \in\rangle$$

and

$$\langle L([\mathrm{Ord}]^{\leq\aleph_1})^{V[G]}, P(\omega_1)^{V[G]}, \in\rangle \equiv \langle L([\mathrm{Ord}]^{\leq\aleph_1})^{V[H]}, P(\omega_1)^{V[G]}, \in\rangle.$$

The conclusion follows. □

5.5. **Consistency of MM^{+++}.** We now turn to the proof of the consistency of MM^{+++}. The guiding idea is to turn all the known proofs of the consistency of MM^{++} by means of iterations of length δ which collapse δ to ω_2 in methods to prove the consistency of MM^{+++} by just assuming stronger large cardinal properties of δ, i.e., super (almost) hugeness of δ. We shall give a fast and detailed proof and a more meditated and general (but sketchy) one.

5.5.1. *Fast proof of the consistency of* MM^{+++}.

Lemma 5.19. *Assume $j : V \to M \subset V$ is an almost huge embedding with* $\mathrm{crit}(j) = \delta$. *Assume that G is V-generic for U_δ.*

Then in $V[G]$ $\mathsf{U}_{j(\delta)}$ is a strongly presaturated tower of normal filters.

Proof. The almost hugeness of j grants that $j(\delta)$ is an inaccessible limit of $< j(\delta)$-supercompact cardinals, since M models that $j(\delta)$ is such, $V_{j(\delta)} \subset M$ and the closure of M grant that $j(\delta)^{<j(\delta)} \subset M$, so any sequence in V cofinal to $j(\delta)$ cannot have order type less than $j(\delta)$; otherwise it would be in M contradicting the inaccessibility of $j(\delta)$ in M.

Let G be V-generic for U_δ. Then in $V[G]$, $j(\delta)$ is still an inaccessible limit of $< j(\delta)$-supercompact cardinals. By Theorem 3.5, $\mathsf{U}_{j(\delta)}^{V[G]}$ is in $V[G]$ an SSP poset. By Theorem 3.9, $\mathsf{U}_{j(\delta)}^{V[G]}$ is isomorphic with $(\mathsf{U}_{j(\delta)} \upharpoonright \mathsf{U}_\delta)^V/G$ in $V[G]$. Moreover

$$\mathsf{U}_\delta \geq_{\mathsf{SSP}} \mathsf{U}_{j(\delta)} \upharpoonright \mathsf{U}_\delta$$

as witnessed by the map $\mathsf{B} \mapsto \mathsf{U}_\delta \upharpoonright \mathsf{B}$. Now let K be $V[G]$-generic for $\mathsf{U}_{j(\delta)}^{V[G]}$, $H_0 = \{\mathsf{B} : [\mathsf{B}]_G \in K\}$, and

$$H = \uparrow H_0 = \{\mathsf{Q} \in V_{j(\delta)} : \exists \mathsf{B} \in H_0, \mathsf{Q} \geq_{\mathsf{SSP}} \mathsf{B}\}.$$

Then H is V-generic for $(\mathsf{U}_{j(\delta)} \upharpoonright \mathsf{U}_\delta)^V$ and $j[G] = G \subset H$. Thus we can apply Lemma 4.17 and extend j to an elementary $\bar{j} : V[G] \to M[H]$ letting $\bar{j}(\mathrm{val}_G(\tau)) = \mathrm{val}_H(j(\tau))$. Lemma 4.17 grants that in $V[H]$ $\bar{j}$ is definable and $M[H]^{<\bar{j}(\delta)} \subset M[H]$.

Thus in $V[G]$:

- By Lemma 4.16 $\mathsf{U}_{j(\delta)}^{V[G]}$ is forcing equivalent to a presaturated normal tower, as witnessed by the tower of $V[G]$-normal ultrafilters
$$\{S \in V_{j(\delta)}[G] : \bar{j}[\cup S] \in \bar{j}(S)\}$$
living in $V[H]$.
- By Lemma 5.14, we get that $\mathsf{U}_{j(\delta)}^{V[G]}$ is in $V[G]$ a strongly presaturated tower.

□

Corollary 5.20. *Assume δ is super almost huge. Then U_δ forces* MM^{+++}.

Proof. Let G be V-generic for U_δ. The family

$$\{(\mathsf{U}_{j(\delta)} \upharpoonright \mathsf{B})^{V[G]} : \mathsf{B} \in \mathsf{U}_{j(\delta)}^{V[G]},\ j : V \to M \subset V \text{ is super almost huge with } \mathrm{crit}(j) = \delta\}$$

witnesses that the class of strongly presaturated towers is dense in $V[G]$. □

Remark 5.21. Notice that (in view of Theorem 5.16) in $V[G]$ the class of strongly presaturated towers includes $\mathsf{U}_\gamma^{V[G]}$ also for all γ which are not equal to $j(\delta)$ for some almost huge j but which are still Σ_2-reflecting cardinals which are limits of $< \gamma$-supercompact cardinals. It is the largeness of the family of γ for which we can predicate in models of MM^{+++} that U_γ is strongly presaturated that allows us to run the proof of Corollary 5.18: Assume $V, V[G]$ are both models of MM^{+++} with $V[G]$ a generic extension of V by an SSP partial order. Assume, on the other hand, that the class of γ such that U_γ is strongly presaturated in V and the class of η such that $\mathsf{U}_\eta^{V[G]}$ is strongly presaturated in $V[G]$ have bounded intersection. Then the proof of Corollary 5.18 would break down. The fact that these two classes have an unbounded overlap is essential and is the content of Theorem 5.16.

5.5.2. *A template for proving the consistency of* MM^{+++}. We want to show that the consistency proof of MM^{+++} does not depend on the particular choice we made among the poset that can be used to obtain a model of MM^{++} (in the previous proof we chose category forcing). So we shall devise a template to establish the consistency of MM^{+++} which can be applied to potentially all forcings which can prove the consistency of MM^{++}. To this aim, we need to introduce a slight strengthening of total rigidity. Since the arguments of this section are not essential to establish our main results but are just used to provide a deeper insight on the nature of the forcings that can produce a model of MM^{+++}, we feel free to omit all proofs.

Rigidly layered partial orders.

Definition 5.22. Given an atomless complete boolean algebra $\mathsf{B} \in \mathsf{SSP}$, let δ_{B} be the least size of a dense subset of $\mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$.

B is rigidly layered if there is a family $\{\mathsf{Q}_p : p \in D\}$ indexed by a dense subset of B such that each Q_p is a complete subalgebra of $\mathsf{B} \restriction p$ satisfying the following properties for all p, q in D:

(1) $1_{\mathsf{Q}_p} = p$, Q_p has size less than δ_{B} and is totally rigid.
(2) $\mathsf{B} \restriction p \leq_{\mathsf{SSP}} \mathsf{Q}_q$ iff $p \leq_{\mathsf{B}} q$ iff the map defined by $r \mapsto r \wedge_{\mathsf{B}} p$ is the unique correct homomorphism of Q_q into Q_p.
(3) Q_p is orthogonal to Q_q in $\mathsf{U}^{\mathsf{SSP}}$ iff $p \wedge_{\mathsf{B}} q = 0_{\mathsf{B}}$.

Definition 5.23. Let $\mathsf{B} \in \mathsf{SSP}$ be a complete boolean algebra. A family $\{\mathsf{B}_\alpha : \alpha < \delta_{\mathsf{B}}\}$ of complete subalgebras of B is a linear rigid layering of B if for all $\alpha < \delta_{\mathsf{B}}$:

(1) B_α has size less than δ and is totally rigid,
(2) $\mathsf{B}_\alpha \subset \mathsf{B}_\beta$ and the inclusion map of B_α in B_β is a regular embedding witnessing that $\mathsf{B}_\alpha \geq_{\mathsf{SSP}} \mathsf{B}_\beta$.
(3) $\bigcup_{\alpha<\delta} \mathsf{B}_\alpha$ is dense in B.
(4) Letting α_p be the least α such that $p \in \mathsf{B}_\alpha$, we have that the map $p \mapsto \alpha_p$ is order reversing.

Lemma 5.24. *Assume* $\{\mathsf{Q}_\alpha : \alpha < \delta_{\mathsf{B}}\}$ *is a linear rigid layering of* B *with union* D. *For each* $p \in D$ *let* $\mathsf{Q}_p = \mathsf{Q}_{\alpha_p} \restriction p$. *Then* $\{\mathsf{Q}_p : p \in D\}$ *is a rigid layering of* B.

The usual forcings which force MM^{++} are rigidly layered:

(1) Assume MM^{++} holds, $\mathsf{U}_\delta \in \mathsf{SSP}$, δ is inaccessible, and the set of totally rigid partial orders is dense in U_δ. We show that U_δ is rigidly layered.

    U_δ can be identified with the partial order $\mathit{Rep}(\mathsf{U}_\delta) = \{T_{\mathsf{B}} \wedge T_{\mathsf{U}_\delta} : \mathsf{B} \in \mathsf{U}_\delta\}$ with the ordering given by the usual ordering on stationary sets. Notice that this partial order is isomorphic to the separative quotient of U_δ. The family

$$\{\{T_{\mathsf{B}\restriction b} \wedge T_{\mathsf{U}_\delta} : b \in \mathsf{B}\} : \mathsf{B} \in \mathsf{U}_\delta \text{ is totally rigid}\}$$

defines a rigid layering of $\mathsf{RO}(\mathit{Rep}(\mathsf{U}_\delta))$.

(2) The standard RCS iteration[16]

$$\mathcal{F} = \{i_{\alpha,\beta} : \mathsf{B}_\alpha \to \mathsf{B}_\beta : \alpha \leq \delta\}$$

of length δ which uses a Laver function $f : \delta \to V_\delta$ to decide $\mathsf{B}_{\alpha+1}$ according to the value of $f(\alpha)$ shows that $C(\mathcal{F})$ forces that MM^{++} is rigidly layered. To see this observe that $\mathsf{B}_\alpha \subset V_\alpha$ is totally rigid for stationary many $\alpha < \delta$, and for all $\alpha < \beta$ $i_{\alpha,\beta}$ witnesses that $\mathsf{B}_\alpha \geq_{\mathsf{SSP}} \mathsf{B}_\beta$. Let Q_α be the boolean completion of

$$\{f \in C(\mathcal{F}) : \text{ the support of } f \text{ is at most } \alpha\}.$$

We get that the inclusion map is the unique correct regular embedding of Q_α into Q_β for all α such that Q_α is totally rigid. We leave to the reader to check that the family

$$\{\mathsf{Q}_\alpha : \alpha < \delta,\ \mathsf{Q}_\alpha \text{ is totally rigid}\}$$

is a linear rigid layering of $C(\mathcal{F})$.

(3) In a similar way we can define a linear rigid layering of the lottery preparation forcing to force MM^{++}.

Rigidly layered presaturated towers are strongly presaturated.

Lemma 5.25. *Assume that* B *is a rigidly layered complete boolean algebra which is forcing equivalent to a presaturated tower of normal filters of height* δ*. Then* B *is forcing equivalent to a strongly presaturated tower of normal filters.*

Linear rigid layerings are inherited by generic quotients.

Lemma 5.26. *Assume* B *is a linearly rigidly layered complete boolean algebra as witnessed by* $\{\mathsf{Q}_\alpha : \alpha < \delta_{\mathsf{B}}\}$*. Let* $D = \bigcup\{\mathsf{Q}_\alpha : \alpha < \delta_{\mathsf{B}}\}$*, and assume* $\mathsf{Q} \geq_{\mathsf{SSP}} \mathsf{Q}_\alpha \restriction p$ *for some* $p \in D$ *as witnessed by a correct homomorphism* $k : \mathsf{Q} \to \mathsf{Q}_\alpha \restriction p$*. We can extend* $k : \mathsf{Q} \to \mathsf{B} \restriction p$ *"composing" it with the inclusion of* $\mathsf{Q}_\alpha \restriction p$ *into* $\mathsf{B} \restriction p$*.*

Let G *be* V*-generic for* Q *and such that* $0_{\mathsf{B}} \notin k[G]$*. Then* $\mathsf{B} \restriction p/k[G]$ *is rigidly layered in* $V[G]$*.*

Remark 5.27. Also rigid layerings of B are inherited by generic quotients of B. The proof of this fact is slightly more elaborate, and we won't need it.

Other strongly presaturated towers. The hypotheses on $\mathcal{F}$ of the following propositions are satisfied by the standard forcings of size δ which produce a model of MM^{++} collapsing a super almost huge cardinal δ to become ω_2.

Lemma 5.28. *Assume that* $j : V \to M \subset V$ *is elementary with* $\mathrm{crit}(j) = \delta$ *and* $M^{<j(\delta)} \subset M$*. Let* $\mathcal{F} = \{i_{\alpha,\beta} : \alpha < \beta < \delta\}$ *be a semiproper iteration system contained in* V_δ *such that for stationarily many* α

$$\mathcal{RCS}(\mathrm{F} \restriction \alpha) = C(\mathcal{F} \restriction \alpha)$$

is totally rigid.

Then $C(\mathcal{F})$ *and* $C(j(\mathcal{F}))$ *are both in* SSP *and* $j \restriction C(\mathcal{F}) : C(\mathcal{F}) \to C(j(\mathcal{F}))$ *is a regular embedding.*

Assume finally that G *is* V*-generic for* $C(\mathcal{F})$*. Then in* $V[G]$ *we get* $C(j(\mathcal{F}))/j[G]$ *is a strongly presaturated tower of normal filters.*

[16] We refer the reader to Subsection 2.6 for the relevant definitions and results on iterations and to [30] for a detailed account.

Theorem 5.29. *Assume δ is super almost huge. Let*

$$\mathcal{F} = \{i_{\alpha,\beta} : \alpha < \beta < \delta\}$$

be a semiproper iteration system contained in V_δ such that

- *$C(\mathcal{F})$ is linearly rigidly layered.*
- *For all $\mathsf{Q} \leq_{\mathsf{SSP}} C(\mathcal{F})$ in SSP there is $j : V \to M$ such that*
  - *(1) $M^{<j(\delta)} \subseteq M$,*
  - *(2) $\mathrm{crit}(j) = \delta$,*
  - *(3) $\mathsf{Q} \geq_{\mathsf{SSP}} C(j(\mathcal{F})) \restriction q$ for some $q \in C(j(\mathcal{F}))$,*
  - *(4) $\mathsf{Q} \in V_{j(\delta)}$.*

Then, whenever G is V-generic for $C(\mathcal{F})$, we have that $V[G]$ models MM^{+++} as witnessed by the family of forcings $C(j(\mathcal{F})) \restriction q/j[G] \in \mathrm{SPT}^{V[G]}$ as $j : V \to M$ ranges among the witnesses of the super almost hugeness of δ and $q \in C(j(\mathcal{F}))$.

Remark 5.30. Theorem 5.29 establishes that the consistency of MM^{+++} can be obtained using Hamkins lottery preparation forcing or the standard iteration for producing a model of MM^{++} guided by a Laver function for a super almost huge cardinal. By [6, Theorem 12, Fact 13], this is a large cardinal assumption consistent relative to the existence of a 2-huge cardinal.

6. Some comments

We can sum up the results of this paper (as well as some other facts about forcing axioms) as density properties of the category $\mathsf{U}^{\mathsf{SSP}}$ as follows:

- $\mathsf{ZFC}+$ *there are class many supercompact cardinals* implies that the class of totally rigid partial orders which force MM^{++} is dense in $\mathsf{U}^{\mathsf{SSP}}$.
- $\mathsf{ZFC}+$ *δ is an inaccessible limit of $<\delta$supercompact cardinals* implies that
$$\mathsf{U}^{\mathsf{SSP}} \cap V_\delta = \mathsf{U}_\delta \in \mathsf{SSP}.$$
- MM^{++} is equivalent (over the theory $\mathsf{ZFC}+$ *there are class many Woodin cardinals*) to the statement that the class of presaturated towers is dense in the category $\mathsf{U}^{\mathsf{SSP}}$.
- MM^{+++} asserts that the class of strongly presaturated towers is dense in the category $\mathsf{U}^{\mathsf{SSP}}$.
- $\mathsf{ZFC}+$ *there is an almost superhuge cardinal* implies that the class of rigidly layered partial orders which force MM^{+++} is dense in $\mathsf{U}^{\mathsf{SSP}}$.
- $\mathsf{ZFC}+\mathsf{MM}^{+++}+$ *δ is a Σ_2-reflecting cardinal which is a limit of $<\delta$ supercompact cardinals* implies that U_δ is forcing equivalent to a presaturated tower.

In particular it appears that the categorial framework we introduced is particularly well suited to express strong forcing axioms as density properties of the category $\mathsf{U}^{\mathsf{SSP}}$. This approach is being pursued further in [29] where we merge this categorial approach with the research of Hamkins and Johnstone [13] and of Tsaprounis [25] on resurrection axioms and their unbounded versions.

Regarding the consistency strength of our results, it is likely that supercompactness is not sufficient to get the consistency of MM^{+++}. The problem is the following: Assume $P_\delta \subset V_\delta$ forces MM^{++} and collapses δ to become ω_2. Assume δ is supercompact but not almost superhuge, and then for any $j : V \to M$ such that $M^{<j(\delta)} \not\subseteq M$ we have no reason to expect that $j(P_\delta)$ is stationary set preserving in

V. We can just prove that it is stationary set preserving in M. On the other hand, if $P_\delta \subseteq V_\delta$ is stationary set preserving and $j : V \to M$ is an almost huge embedding (i.e., $M^{<j(\delta)} \subset M$), then $j(P_\delta)$ is stationary set preserving in V and we can argue that if G is V-generic for P_δ, then $j(P_\delta)/G \in V[G]$ is a strongly presaturated normal tower. This crucial difference suggests why MM^{+++} is likely to have a stronger consistency strength than MM^{++}. On the other hand, it can be checked that all the forcings considered in this paper to obtain the consistency of MM^{+++} collapsing an inaccessible δ to become ω_2 satisfy the δ-covering and δ-approximation property (see [31, Definition 4.5]). In particular by the results of [31], we can infer that such a δ must have at least a strongly compact cardinal. However, this conclusion can already be inferred for the models of MM^{++} obtained by such forcings, and in the present stage we are not able to extract any further indication regarding the consistency strength of MM^{+++} with respect to that of MM^{++}. It seems that we lack a combinatorial characterization of super almost hugeness in the same fashion as the one provided by the work of Jech, Magidor, and Weiss for supercompactness and strong compactness.

We also want to remark that the work of Larson [16] and Asperó [1] shows that our results are close to optimal and that we cannot hope to prove Theorem 1 for forcing axioms which are strictly weaker than MM^{++}:

- Larson showed that there is a Σ_3 formula $\phi(x)$ such that over any model V of ZFC with large cardinals and for any $a \in H_{\omega_2}^V$ there is a *semiproper* forcing extension $V[G]$ of V which models $\mathsf{MM}^{+\omega}$ and such that a is the unique set which satisfies $\phi(a)^{H_{\omega_2}^{V[G]}}$.
- Asperó showed that there is a Σ_5 formula $\psi(x)$ such that over any model V of ZFC with large cardinals and for any $a \in H_{\omega_2}^V$ there is a *semiproper* forcing extension $V[G]$ of V which models PFA^{++} and such that a is the unique set which satisfies $\phi(a)^{H_{\omega_2}^{V[G]}}$.

These results show that the theory of H_{ω_2} in models of $\mathsf{MM}^{+\omega}$ (respectively PFA^{++}) cannot be generically invariant with respect to SSP-forcings which preserve these axioms, since we would get otherwise that all elements of $H_{\aleph_2}$ could be defined as the unique objects satisfying ϕ (respectively ψ). It remains nonetheless open whether the results we got on the forcing $\mathsf{U}^{\mathsf{SSP}}$ can be declined for other category forcings given by suitable classes of forcings Γ. We conjecture that this is the case for many such Γ among which the proper posets. This requires us to investigate for which class of forcings we can predicate the freezeability property, since ultimately all the properties we were able to infer for $\mathsf{U}^{\mathsf{SSP}}$ were obtained appealing to the following:

- closure of $\leq^*_{\mathsf{SSP}}$ under set sized descending sequences (obtained by identifying SSP with SP),
- closure of SSP under two step iterations,
- closure of SSP under set sized lottery sums,
- a simple definition in terms of first order logic of the class SSP,
- the freezeability property.

Of the above list the unique property as yet not known to hold for many other interesting categories of forcing notions is the freezeability property. If we are able to infer such a property for other classes of forcings we are confident that the appropriate generic absoluteness result for the appropriate version of MM^{+++} declined for these categories is at reach.

Finally the theory of $L(\mathbb{R})$ in the context of large cardinals is generically invariant, and among other things this has led to the development of the rich theory of universally Baire subsets of $\mathbb{R}$, sets whose properties are generically invariant and which played an important role to understand the theory of $L(\mathbb{R})$ under determinacy axioms. The direction we want to investigate is that of isolating the correct notion of universally Baire subset of 2^{ω_1} and to understand the property of these sets in models of MM^{+++}, since for this theory we also have a notion of generic invariance. Most likely a theory of universally Baire subsets of 2^{ω_1} should complement the rich understanding we already have of the theory of $L(P(\omega_1))$ in the presence of strong forcing axioms.

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