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DECOUPLING, EXPONENTIAL SUMS AND THE RIEMANN ZETA FUNCTION

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0. Introduction

The main result of the paper is the essentially sharp bound on the mean-value expression for $r = 6$ (see [H] for details)

(0.1)

$$A_r\Big(\frac{1}{N^2}, \frac{1}{N}\Big) = \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1 \Bigg|\sum_{1\le n\le N} e(nx_1 + n^2x_2 + N^{\frac12}n^{3/2}x_3 + N^{\frac12}n^{\frac12}x_4)\Bigg|^{2r} dx_1dx_2dx_3dx_4.$$

It is proven indeed that $A_6 \ll N^{6+\varepsilon}$ (see Theorem 2 below). The bound $A_5(\delta, \delta N) \ll \delta N^{7+\varepsilon}, \frac{1}{N^2} \le \delta \le \frac{1}{N}$, established in [H-K], plays a key role in the refinement of the Bombieri-Iwaniec approach [B-I1] to bounding exponential sums as developed mainly by Huxley (see [H] for an expository presentation). As pointed out in [H], obtaining good bounds on A_6 leads to further improvements, and this objective was our main motivation.

In [B], we recovered the [H-K] A_5-result (in fact in a sharper form) as a consequence of certain general decoupling inequalities related to the harmonic analysis of curves in $\mathbb{R}^d$. Those inequalities were derived from the results in [B-D1] (see also [B-D2]). Theorem 2 will similarly be derived from a decoupling theorem, formulated as Theorem 1.

Let us next briefly recall the basic structure of the Bombieri-Iwaniec argument. Given an exponential sum $\sum_{m\sim M} e\big(TF(\frac{m}{M})\big)$ with $T > M$ and F a smooth function satisfying appropriate derivative conditions, the sum $\sum_{m\sim M}$ is replaced by shorter sums $\sum_{m\in I}$, I ranging over size-N intervals $\big($here N is a parameter to be chosen and is not the same as in (0.1)$\big)$. For each I, the phase may be replaced by a cubic polynomial and, by Poisson summation, the exponential sum $\sum_{m\in I} e\big(TF(\frac{m}{M})\big)$

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transformed (effectively) in a sum of the form

$$\sum_{h\leq H} e\big(x_1(I)h + x_2(I)h^2 + x_3(I)h^{3/2} + x_4(I)h^{1/2}\big), \tag{0.2}$$

where the vector $x(I) = \big(x_j(I)\big)_{1\leq j\leq 4} \in \mathbb{R}^4$ depends on the interval I.

At this point, one needs to analyze the distributions of

$$(h, h^2, h^{3/2}, h^{1/2}) \quad (1 \leq h \leq H) \tag{0.3}$$

and

(0.4) the vector function $x(I)$ of the interval I,

which Huxley refers to as the first and second spacing problems.

Before applying a large sieve estimate, one takes an r-fold convolution of (0.3) whose L^2-norm is expressed by mean values of the form (0.1) with N replaced by H. Roughly speaking, the L^2-norm of the distribution (0.4) is bounded by a certain parameter B, whose evaluation is highly non-trivial and so far sub-optimal. The only input of this paper is to provide an optimal result for the first spacing problem (see Corollary 3 below). Combined with available treatments of the second spacing problem, it leads to improved exponential sum estimates.

New exponential sum bounds are presented in Section 3. They are based on combining Corollary 3 with known estimates on the parameter B from the second spacing problem (see the Acknowledgment below).

The conclusion is stated as Theorem 4.

In Section 4, we establish our new estimate on $|\zeta(\frac{1}{2}+it)|$.

Theorem 5.

$$\Big|\zeta\Big(\frac{1}{2}+it\Big)\Big| \ll |t|^{\frac{13}{84}+\varepsilon}. \tag{0.5}$$

It is implied by the classical approximate functional equation (cf. [T]) together with Theorem 4 and some further (known) exponential sum bounds.

Recall that the original Bombieri-Iwaniec argument provided the estimate $\big|\zeta\big(\frac{1}{2}+it\big)\big| \ll |t|^{\frac{9}{56}+\varepsilon}$, $\frac{9}{56} = 0,16071$ (see [B-I1] and [B-I2]). The work of Huxley in [H1] (resp. [H2]) produced the exponents

$$\frac{89}{570} = 0.15614\ldots \quad \text{and} \quad \frac{32}{205} = 0.15609\ldots, \text{ resp.,}$$

while our A_6-bound leads to the exponent $\frac{13}{84} = 0.15476\ldots$, hence doubling the saving over $\frac{1}{6}$ obtained in [B-I1].

In Section 5 we highlight a new exponent pair that results from our work.

1. A decoupling inequality for curves

Let $\Phi = (\phi_1, \ldots, \phi_d) : [0,1] \to \Gamma \subset \mathbb{R}^d$ be a smooth parametrization of a non-degenerate curve in $\mathbb{R}^d$; more specifically we assume the Wronskian determinant

$$\det[\phi_j^{(s)}(t_s)_{1\leq j,s\leq d}] \neq 0 \text{ for all } t_1, \ldots, t_d \in [0,1]. \tag{1.1}$$

Let us assume moreover that d is even. For $\Omega \subset \mathbb{R}^d$ a bounded set of positive measure, denote by

$$\|f\|_{L^p_\#(\Omega)} = \Big(\fint_\Omega |f|^p dx\Big)^{\frac{1}{p}} = \Big(\frac{1}{|\Omega|}\int_\Omega |f|^p dx\Big)^{\frac{1}{p}}$$

the average L^p-norm, and let B_ρ be the ρ-cube in $\mathbb{R}^d$ centered at 0. We prove the following decoupling property in the spirit of results in [B] and [B-D2].

Theorem 1. *Let Γ be as above and $I_1, \dots, I_{\frac{d}{2}} \subset [0,1]$ subintervals that are $O(1)$-separated, and let N be large and $\{I_\tau\}$ a partition of $[0,1]$ in $N^{-\frac{1}{2}}$-intervals. Then for arbitrary coefficient functions $a_j = a_j(t)$*

$$\begin{aligned} &\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^{3d}_\#(B_N)} \\ (1.2) \qquad &\ll N^{\frac{1}{6}+\varepsilon} \prod_{j=1}^{d/2} \Big[\sum_{\tau; I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_\#(B_N)} \Big]^{\frac{1}{3d}} \end{aligned}$$

holds, with $\varepsilon > 0$ arbitrary.

Here $e(z)$ stands for $e^{2\pi i z}$ as usual. Strictly speaking, $L^6_\#(B_N)$ in the right hand side of (1.2) should be some weighted space $L^6_\#(w_N)$ with weight $1_{B_N}(x) \lesssim w_N(x) \leq (1 + \frac{|x|}{N})^{-10d}$, supp $\widehat{w}_N \subset B_{\frac{1}{N}}$ (cf. [B-D1] and [B-D2]). For simplicity, this technical point will be ignored here and in the sequel.

Let us say a few words about the role of Theorem 1 in the proof of Theorem 2 stated below. First, the inequality in Theorem 2 can be rewritten as

$$\left(\frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} |\sum_{n \leq N} e(x \cdot \Phi(\frac{n}{N}))|^{12} dx \right)^{1/12} \ll N^{\frac{1}{2}+\epsilon}$$

with

$$\tilde{\Omega} = [0, N] \times [0, N^2] \times [0, N^2] \times [0, N]$$

and

$$\Phi(t) = (t, t^2, t^{3/2}, t^{1/2}).$$

Theorem 1 implies via a standard discretization argument that the average over each ball B_{N^2} with radius N^2 satisfies

$$\left(\frac{1}{|B_{N^2}|} \int_{|B_{N^2}|} \left[\sqrt{\prod_{j=1}^{2} |\sum_{n \in I_j} e(x \cdot \Phi(\frac{n}{N}))|} \,\right]^{12} dx \right)^{1/12} \ll N^{\frac{1}{2}+\epsilon}.$$

This estimate is weaker than the one in Theorem 2 in two regards. First, in Theorem 2 one is interested in averages over the smaller region $\tilde{\Omega}$. This issue is dealt with in the first half of Section 2, by using two further standard two-dimensional (2D) decouplings and exploiting periodicity.

The second difference between Theorems 1 and 2 is that the former produces a bilinear estimate, while the latter requires a linear estimate. This issue is addressed in the second part of Section 2 and relies on a variant of the induction on scales from [B-G].

Remarks.

(1.3): Obviously (1.2) implies the same inequality for B_N replaced by any translate.

(1.4): The case $d = 2$ is an immediate consequence of the L^6-decoupling inequality for planar curves Γ of non-vanishing curvature

$$\Big\| \int_0^1 a(t) e\big(x.\Phi(t)\big) dt \Big\|_{L^6(B_N)} \ll N^\varepsilon \Big(\sum_\tau \Big\| \int_{I_\tau} a(t) e\big(x.\Phi(t)\big) dt \Big\|^2_{L^6(B_N)} \Big)^{\frac{1}{2}}, \tag{1.5}$$

where $\Phi : [0,1] \to \Gamma \subset \mathbb{R}^2$ with $|\Phi''| \sim 1$ and $\{I_\tau\}$ as above, established in [B-D1]. In fact, (1.5) will be the main analytical input required for the proof of (1.2). We mention for future use also the following discrete version of (1.5):

$$\left(\frac{1}{|B_N|} \int_{B_N} |\sum_{n \leq N} a_n e(x \cdot \Phi(\frac{n}{N}))|^6 dx \right)^{1/6} \ll N^\epsilon \|a_n\|_{l^2},$$

for each complex coefficient a_n.

(1.6): In the language of [B-D1] and [B-D2], (1.2) may be reformulated as follows. Let $\Gamma_1, \ldots, \Gamma_{d/2} \subset \Gamma$ be $O(1)$-separated arcs and $f_1, \ldots, f_{\frac{d}{2}} \in L^1(\mathbb{R}^d)$ satisfy supp $\widehat{f_j} \subset \Gamma_j + B_{\frac{1}{N}}$. Denote $f_\tau = (\hat{f}|_{\Phi(I_\tau)+B_{\frac{1}{N}}})^\vee$ the Fourier restriction of f to the $\underbrace{\frac{1}{N} \times \cdots \times \frac{1}{N}}_{d-1} \times \frac{1}{\sqrt{N}}$ tube $\Phi(I_\tau) + B_{\frac{1}{N}}$.

Then

$$\Big\| \prod_{j=1}^{d/2} |f_j|^{2/d} \Big\|_{L^{3d}_\#(B_N)} \ll N^{\frac{1}{6}+\varepsilon} \prod_{j=1}^{d/2} \Big(\sum_\tau \|f_{j,\tau}\|^6_{L^6_\#(B_N)} \Big)^{\frac{1}{3d}}. \tag{1.7}$$

(1.8): It may be worthwhile to explain the relation between (1.7) and other known decoupling inequalities for curves in $\mathbb{R}^d$.

First, with Γ as above and $\Gamma_1, \ldots, \Gamma_d \subset \Gamma$ $O(1)$-separated, one has a d-linear inequality (the analogue of [B-C-T] for curves)

$$\Big\| \prod_{j=1}^{d} |f_j|^{\frac{1}{d}} \Big\|_{L^{2d}_\#(B_N)} \leq c_\Gamma \prod_{j=1}^{d} \Big(\sum_\tau \|f_{j,\tau}\|^2_{L^2_\#(B_N)} \Big)^{\frac{1}{2d}}. \tag{1.9}$$

This inequality turns out to be elementary. Using the fact that the map $I_1 \times \cdots \times I_d \to \mathbb{R}^d : (t_1, \ldots, t_d) \mapsto \Phi(t_1) + \cdots + \Phi(t_d)$ is a diffeomorphism for $I_1, \ldots, I_d$ $O(1)$-separated by assumption (1.1) and Parseval's theorem, one sees indeed that

$$\Big\| \prod_{j=1}^{d} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big| \Big\|_{L^2(B_N)} \leq c \prod_{j=1}^{d} \|a_j\|_{L^2(I_j)}. \tag{1.10}$$

On the other hand, one has the $(d-1)$-linear inequality (see [B-D2])

$$\Big\| \prod_{j=1}^{d-1} |f_j|^{\frac{1}{d-1}} \Big\|_{L^{2(d+1)}_\#(B_N)} \ll N^{\frac{1}{2(d+1)}} \prod_{j=1}^{d-1} \Big[\sum_\tau \Big\| f_{j,\tau} \Big\|^{\frac{2(d+1)}{d-1}}_{L_\#^{\frac{2(d+1)}{d-1}}(B_N)} \Big]^{\frac{1}{2(d+1)}}, \tag{1.11}$$

and one observes, for d even, that the pair $\big(2(d+1), \frac{2(d+1)}{d-1)}\big)$ in (1.11) is obtained by interpolation between the pairs $(2d, 2)$ from (1.9) and $(3d, 6)$ from (1.7). The issue of what is the analogue of Theorem 1 for odd d will not be considered here. In fact, our main interest is $d = 4$, which provides the required ingredient for the exponential sum application.

Before passing to the proof of Theorem 1, we make a few preliminary observations.

Note that in the setting of Theorem 1, (1.9) also implies the inequality

$$\begin{aligned}(1.12)\qquad &\Big\| \prod_{j=1}^{d/2} \Big| \sum_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^d_{\#}(B_N)} \\ &\leq c \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^2_{L^2_{\#}(B_N)} \Big]^{\frac{1}{d}}.\end{aligned}$$

To see this, take $f_j(x) = \frac{1}{\sqrt{N}} \sum_{\substack{0 \leq k \leq N \\ \frac{k}{N} \in I_j}} \varepsilon_k \, e\big(x.\Phi(\frac{k}{N})\big)$ for $j = \frac{d}{2}+1, \dots, d$ with $\varepsilon_k = \pm 1$ independent random variables and average over $\{\varepsilon_k\}$, noting that $\mathbb{E}_\varepsilon[|f_j|^2] \asymp 1$ and $\mathbb{E}_\varepsilon[|f_\tau|^2] \asymp N^{-\frac{1}{2}}$.

There is also the trivial bound

$$\begin{aligned}(1.13)\qquad &\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^\infty(B_N)} \\ &\leq N^{\frac{1}{2}} \prod_{j=1}^{d/2} \max_{I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^{2/d}_{L^\infty(B_N)}.\end{aligned}$$

Interpolation between (1.12) and (1.13) using appropriate wave packet decomposition as explained in [B-D1] (note that it is essential here that the I_τ are $N^{-\frac{1}{2}}$-intervals) gives

$$\begin{aligned}(1.14)\qquad &\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^{3d}_{\#}(B_N)} \\ &\leq C N^{\frac{1}{3}} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_{\#}(B_N)} \Big]^{\frac{1}{3d}}\end{aligned}$$

with $\{I_\tau\}$ a partition in $N^{-\frac{1}{2}}$-intervals.

More generally, if $\Delta = \Delta_K \subset \mathbb{R}^d$ is a K-cube, we have (by translation)

$$\begin{aligned}(1.15)\qquad &\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big|^{2/d} \Big\|_{L^{3d}_{\#}(\Delta)} \\ &\leq C K^{\frac{1}{3}} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_j} \Big\| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_{\#}(\Delta)} \Big]^{\frac{1}{3d}},\end{aligned}$$

where $\{I_\tau\}$ is now a partition in $K^{-\frac{1}{2}}$-intervals.

The main point of (1.15) is to provide a preliminary $L^6 - L^{3d}$ inequality; the prefactor $K^{1/3}$ is not important for what follows as it will be improved to $K^{\varepsilon+\frac{1}{6}}$ using a bootstrap argument.

Returning to (1.1), it follows from the mean value theorem that

$$|\det[\phi_i'(t_j)_{1\le i,j\le d}]| \sim \prod_{i\neq j} |t_i - t_j|. \tag{1.16}$$

By (1.16) and since $\phi''(t) = \lim_{s\to 0} \frac{1}{s}\big(\phi'(t+s) - \phi'(t)\big)$, it follows that for $t_1 < \cdots < t_{d/2} \in [0,1]$ $O(1)$-separated,

$$|\phi'(t_1)\wedge\phi''(t_1)\wedge\phi'(t_2)\wedge\phi''(t_2)\wedge\cdots\wedge\phi'(t_{\frac{d}{2}})\wedge\phi''(t_{\frac{d}{2}})| > c \tag{1.17}$$

holds.

Proof of Theorem 1. Introduce numbers $b(N) > 0$ for which the inequality, with arbitrary $\{a_j\}$,

$$\begin{aligned} &\Big\| \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big)dt\Big|^{2/d} \Big\|_{L^{3d}_{\#}(B_N)} \\ &\le b(N) N^{\frac{1}{6}} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau\subset I_j} \Big\| \int_{I_\tau} a_j(t) e\big(x.\Phi(t)\big)dt\Big\|^6_{L^6_{\#}(B_N)}\Big]^{\frac{1}{3d}} \end{aligned} \tag{1.18}$$

holds.

Our aim is to establish a bootstrap inequality. By (1.14), $b(N) \le N^{1/6}$. With $K < N$ to specify, partition B_N in K-cubes $\Delta = \Delta_K$. We may bound for each Δ (since the inequalities for B_K and Δ_K are equivalent)

$$\begin{aligned} &⨍_\Delta \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big)dt\Big|^6 dx \le \\ &b(K)^{3d} K^{\frac{d}{2}} \prod_{j=1}^{d/2} \Big[\sum_{I_\sigma\subset I_j} \Big\| \int_{I_\sigma} a_j(t) e\big(x.\Phi(t)\big)dt\Big\|^6_{L^6_{\#}(\Delta)}\Big] \end{aligned} \tag{1.19}$$

with $\{I_\sigma\}$ a partition in $K^{-\frac{1}{2}}$-intervals. Summation over $\Delta \subset B_N$ implies then

(1.20)

$$⨍_{B_N} \prod_{j=1}^{d/2} \Big| \int_{I_j} a_j(t) e\big(x.\Phi(t)\big)dt\Big|^6 dx \le b(K)^{3d} K^{\frac{d}{2}}$$

$$\times \sum_{I_{\sigma_1}\subset I_1,\dots,I_{\sigma_{d/2}}\subset I_{d/2}} ⨍_{B_K^{d/2}} \Big\{ ⨍_{B_N} \prod_{j=1}^{d/2} \Big| \int_{I_{\sigma_j}} a_j(t) e\big((x+z_j).\Phi(t)\big)dt\Big|^6 dx\Big\} \prod_j dz_j.$$

Fix $I_{\sigma_j} = [t_j, t_j + K^{-\frac{1}{2}}] \subset I_j$ and write for $t = t_j + s \in I_{\sigma_j}$

$$(x+z_j).\Phi(t) = (x+z_j).\Phi(t_j) + (x+z_j).\Phi'(t_j)s + \frac{1}{2}(x+z_j).\Phi''(t_j)s^2 + o(1) \tag{1.21}$$

provided

$$N = o(K^{3/2}). \tag{1.22}$$

The inner integral in (1.20) may then be replaced by

$$\fint_{B_N} \prod_{j=1}^{d/2} \Big| \int_0^{K^{-\frac{1}{2}}} a_j(t_j+s) e\big((x+z_j).\Phi'(t_j)s + \frac{1}{2}(x+z_j).\Phi''(t_j)s^2\big) ds \Big|^6 dx, \tag{1.23}$$

the $o(1)$-term in (1.21) producing a harmless smooth Fourier multiplier that may be ignored.

Next, since $t_1 < t_2 < \cdots < t_{d/2}$ are $O(1)$-separated, (1.17) applies, and therefore the map $\mathbb{R}^d \to \mathbb{R}^d : x \mapsto \big(x.\Phi'(t_1), \frac{1}{2}x.\Phi''(t_1), \ldots, x.\Phi'(t_{d/2}), \frac{1}{2}x.\Phi''(t_{d/2})\big)$ is a linear homeomorphism. The image measure of the normalized measure on B_N may be bounded by the normalized measure on B_{CN}, up to a factor and

$$(1.23) \lesssim \prod_{j=1}^{d/2} \fint_{|u|,|v|<CN} \Big| \int_0^{K^{-\frac{1}{2}}} a_j(t_j+s)\, e(us+vs^2) ds \Big|^6 du dv. \tag{1.24}$$

This factorization is the main point in the argument.

We may now apply (after rescaling $s = k^{-\frac{1}{2}} s_1$) to each factor in (1.24) the 2D-decoupling inequality (1.5) with Γ the parabola (s_1, s_1^2) and perform a decoupling at scale $(\frac{K}{N})^{\frac{1}{2}}$. Thus, by another change of variables,

$$(1.24) \ll N^\varepsilon \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_{\sigma_j}} \Big\| \int_{I_\tau} a_j(t)\, e(ut) dt \Big\|^2_{L^6_\#[|u|<CN]} \Big]^3$$

with $\{I_\tau\}$ a partition in $N^{-\frac{1}{2}}$-intervals

$$\ll N^\varepsilon \Big(\frac{N}{K}\Big)^{\frac{d}{2}} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_{\sigma_j}} \Big\| \int_{I_\tau} a_j(t)\, e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_\#(B_N)} \Big]. \tag{1.25}$$

Substituting (1.25) in (1.20) leads to the estimate

$$b(K)^{3d} N^{\frac{d}{2}+\varepsilon} \prod_{j=1}^{d/2} \Big[\sum_{I_\tau \subset I_j} \Big\| \int_{I_\tau} a_j(t)\, e\big(x.\Phi(t)\big) dt \Big\|^6_{L^6_\#(B_N)} \Big]. \tag{1.26}$$

Recalling (1.22), one may conclude that

$$b(N) \ll b(N^{2/3}) N^\varepsilon$$

and Theorem 1 follows by iteration. □

2. A mean value theorem

From now on, we focus on $d = 4$ (in view of the application to exponential sums) and consider $\Phi : [0,1] \to \Gamma \subset \mathbb{R}^4$ satisfying (1.1). If $I_1, I_2 \subset \{1, \ldots, N\}$ are $\sim N$ separated, we get from Theorem 1

$$\begin{aligned} &\Big\| \prod_{j=1}^{2} \Big| \sum_{n \in I_j} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big).x\Big) \Big|^{\frac{1}{2}} \Big\|_{L^{12}_\#(B_N)} \\ &\ll N^{\frac{1}{6}+\varepsilon} \prod_{j=1}^{2} \Big(\sum_{J \subset I_j} \Big\| \sum_{n \in J} a_n e\Big(\Phi\Big(\frac{n}{N}\Big).x\Big) \Big\|^6_{L^6_\#(B_N)} \Big)^{\frac{1}{12}} \end{aligned} \tag{2.1}$$

with $\{J\}$ a partition of $\{1, \ldots, N\}$ in $N^{\frac{1}{2}}$-intervals.

Again in view of the application, specify

$$\phi_1(t) = t, \phi_2(t) = t^2 \tag{2.2}$$

and assume

$$|\phi_3'''| > c. \tag{2.3}$$

In order to perform a further decoupling in (2.1), we enlarge the domain B_N, considering first

$$\Omega = [0, N] \times [0, N^{3/2}] \times [0, N^{3/2}] \times [0, N],$$

which we partition in N-cubes Δ_N.

Let I_1, I_2 be as above. Application of (2.1) on Δ_N gives

$$\begin{aligned}
&\Big\| \prod_{j=1}^{2} \Big| \sum_{n\in I_j} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big).x\Big)\Big|^{\frac12} \Big\|_{L^{12}_{\#}(\Delta_N)} \\
&\ll N^{\frac16+\varepsilon} \Big[\prod_{j=1}^{2} \Big(\sum_{J\subset I_j} \Big\| \sum_{n\in J} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big).x\Big)\Big\|^6_{L^6_{\#}(\Delta_N)}\Big)\Big]^{\frac{1}{12}},
\end{aligned}$$

and summing over Δ_N

$$\begin{aligned}
&\Big\| \prod_{j=1}^{2} \Big| \sum_{n\in I_j} \cdots \Big|^{\frac12} \Big\|_{L^{12}_{\#}(\Omega)} \\
&\ll N^{\frac16+\varepsilon} \Bigg[\sum_{\substack{J_1\subset I_1\\ J_2\subset I_2}} \ \fint_{B_N\times B_N} dz dz' \fint_{\Omega} dx \Big| \sum_{n\in J_1} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big).(x+z)\Big)\Big|^6 \\
&\qquad \times \Big| \sum_{n\in J_2} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big)(x+z')\Big)\Big|^6 \Bigg]^{\frac{1}{12}}.
\end{aligned} \tag{2.4}$$

Let $J_1 = [h_1, h_1 + N^{\frac12}], J_2 = [h_2, h_2 + N^{\frac12}]$ with $h_1 - h_2 \asymp N$. Write for $n \in J_1$, $n = h_1 + m$, recalling (2.2)

$$\begin{aligned}
\Phi\Big(\frac{n}{N}\Big) \cdot (x+z) &= \Phi\Big(\frac{h_1}{N}\Big) \cdot (x+z) \\
&\quad + \frac{m}{N}\Big(x_1 + z_1 + 2\frac{h_1}{N}(x_2+z_2) + \phi_3'\Big(\frac{h_1}{N}\Big)(x_3+z_3) \\
&\quad + \phi_4'\Big(\frac{h_1}{N}\Big)(x_4+z_4)\Big) + \frac{m^2}{N^2}\Big(x_2 + \frac12 \phi_3''\Big(\frac{h_1}{N}\Big)x_3\Big) + O(1),
\end{aligned} \tag{2.5}$$

recalling that $|z|, |x_1|, |x_4| < N$ and $|x_2|, |x_3| < N^{3/2}$ while $|m| < N^{\frac12}$. Proceed similarly for $\Phi\Big(\frac{n}{N}\Big) \cdot (x + z'), n \in J_2$.

Observe that z_1, z_1' have range $[0, N]$, so that periodicity considerations and a change of variables in z_1, z_1' permit one to replace the phase (2.5) by

$$\frac{m}{N} z_1 + \frac{m^2}{N^2}\Big(x_2 + \frac12 \phi_3''\Big(\frac{h_1}{N}\Big)x_3\Big)$$

and

$$\frac{m}{N}z_1' + \frac{m^2}{N^2}\Big(x_2 + \frac{1}{2}\phi_3''\Big(\frac{h_2}{N}\Big)x_3\Big).$$

Since $h_1 - h_2 \asymp N$ and (2.3), one more change of variables in x_2, x_3 gives the phases

$$\begin{cases} mu_1 + \frac{m^2}{N^{1/2}}w_1 \\ mu_2 + \frac{m^2}{N^{1/2}}w_2 \end{cases} \tag{2.6}$$

with u_1, u_2, w_1, w_2 ranging in $[0,1]$. Hence we obtain again a factorization of the integrand in (2.4), i.e.

$$\begin{aligned} \int_0^1\int_0^1\int_0^1\int_0^1 du_1du_2dw_1dw_2 \Big|\sum_{m_1<\sqrt{N}} a_{h_1+m_1}\, e\Big(m_1u_1 + \frac{m_1^2}{N^{1/2}}w_1\Big)\Big|^6 \\ \Big|\sum_{m_2<\sqrt{N}} a_{h_2+m_2}\, e\Big(m_2u_2 + \frac{m_2^2}{N^{1/2}}w_2\Big)\Big|^6, \end{aligned} \tag{2.7}$$

and the 2D-decoupling result applied to each factor enables to make a further decoupling at scale $N^{1/4}$. This clearly permits one to bound (2.4) by

$$N^{\frac{1}{6}+\varepsilon}N^{\frac{1}{12}+\varepsilon}\Big[\sum_{\substack{J_1'\subset I_1\\ J_2'\subset I_2}}\int_0^1\int_0^1\Big|\sum_{n\in J_1'} a_n\, e(nu_1)\Big|^6\Big|\sum_{n\in J_2'} a_n\, e(nu_2)\Big|^6 du_1du_2\Big]^{\frac{1}{12}} \tag{2.8}$$

with $\{J'\}$ a partition in $N^{\frac{1}{4}}$-intervals. The fact that we have dropped the term $\frac{n^2}{N^{1/2}}w_i$ that appears in (2.7) deserves a word of explanation. Note that if $n \in J_i'$, then $n = h_i + m$ with $m \le N^{1/4}$. Thus $\frac{n^2}{N^{1/2}}w = \big(\frac{h^2}{N^{1/2}} + \frac{2hm}{N^{1/2}}\big)w + O(1)$, which permits us to replace in (2.7) the argument by $m\big(u + \frac{2h}{N^{1/2}}w\big)$ and hence mu by change of variable.

If instead we consider a translate $\Omega + y$ of Ω, the expression (2.8) needs to be modified replacing a_n by $a_n\, e\big(\Phi(\frac{n}{N})\cdot y\big)$.

Finally, consider the domain (according to Huxley's A_6-problem)

$$\tilde\Omega = [0,N]\times[0,N^2]\times[0,N^2]\times[0,N],$$

which we partition in domains $\Omega_\alpha = \Omega + y_\alpha$ with Ω as above. Thus for each α (2.8) implies

$$\begin{aligned} &\Big\|\prod_{j=1}^{2}\Big|\sum_{n\in I_j}\cdots\Big|^{\frac{1}{2}}\Big\|_{L^{12}_{\#}(\Omega_\alpha)} \\ &\ll N^{\frac{1}{4}+\varepsilon}\Big[\sum_{\substack{J_1'\subset I_1\\ J_2'\subset I_2}}\int_0^1\int_0^1\Big|\sum_{n\in J_1'} a_n\, e\Big(\Phi\Big(\frac{n}{N}\Big).y_\alpha\Big)e(nu_1)\Big|^6\Big|\sum_{n\in J_2'}\cdots\Big|^6 du_1du_2\Big]^{\frac{1}{12}} \end{aligned}$$

and
(2.9)

$$\Big\|\prod_{j=1}^{2}\Big|\sum_{n\in I_j}\cdots\Big|^{\frac12}\Big\|_{L^{12}_{\#}(\tilde\Omega)}$$

$$\ll N^{\frac14+\varepsilon}\Big[\sum_{\substack{J_1'\subset I_1\\ J_2'\subset I_2}}\fint_{\tilde\Omega}\int_0^1\int_0^1\Big|\sum_{n\in J_1'}a_n\,e\Big(\Phi\Big(\frac{n}{N}\Big).y+nu_1\Big)\Big|^6\Big|\sum_{n\in J_2'}\cdots\Big|^6 dy\,du_1\,du_2\Big]^{\frac{1}{12}}.$$

Proceeding as before, let $J_1'=[h_1,h_1+N^{\frac14}], J_2'=[h_2,h_2+N^{\frac14}], h_1-h_2\asymp N$. Write for $n\in J_1', n=h_1+m$

$$\begin{aligned}&\Phi\Big(\frac{n}{N}\Big)\cdot y+nu_1\\ &=\Phi\Big(\frac{h_1}{N}\Big)\cdot y+h_1u_1\\ &\qquad+m\Big(u_1+\frac{y_1}{N}+2\Big(\frac{h_1}{N}\Big)y_2+\frac1N\phi_3'\Big(\frac{h_1}{N}\Big)y_3+\frac1N\phi_4'\Big(\frac{h_1}{N}\Big)y_4\Big)\\ &\qquad+\frac{m^2}{N^2}\Big(y_2+\frac12\phi_3''\Big(\frac{h_1}{N}\Big)y_3\Big)+O(1).\end{aligned}$$

Since $|y_2|,|y_3|<N^2,|y_4|<N$ and $|m|<N^{\frac14}$.

Again by periodicity, (2.3) and change of variables, we obtain the phases

$$mu_1+m^2w_1$$

and

$$mu_2+m^2w_2$$

with $u_1,u_2,w_1,w_2\in[0,1]$, and the L^6-norms are bounded by the ℓ^2-norms of the coefficients (see Remark 1.4). In conclusion, we proved that

$$\Big\|\prod_{j=1}^{2}\Big|\sum_{n\in I_j}a_n\,e\Big(\Phi\Big(\frac{n}{N}\Big)\cdot x\Big)\Big|^{\frac12}\Big\|_{L^{12}_{\#}(\tilde\Omega)}\ll N^{\frac12+\varepsilon}\|\bar a\|_\infty \tag{2.10}$$

with Φ satisfying (1.1), (2.2), (2.3), i.e.

$$\phi_1(t)=t,\phi_2(t)=t^2,|\phi_3''|>c\text{ and }\left|\begin{vmatrix}\phi_3'''(s)&\phi_4'''(s)\\ \phi_3''''(t)&\phi_4''''(t)\end{vmatrix}\right|>c\text{ for }s,t\in[0,1]. \tag{2.11}$$

The following statement is the mean value estimate for A_6 in [H].

Theorem 2.
(2.12)

$$\int_0^1\int_0^1\int_{-1}^1\int_{-1}^1\Big|\sum_{n\le N}e(nx_1+n^2x_2+N^{\frac12}n^{3/2}x_3+N^{\frac12}n^{\frac12}x_4)\Big|^{12}dx_1dx_2dx_3dx_4\ll N^{6+\varepsilon}.$$

Proof. Let $I\subset[1,N]$ be an interval of the form $[N_0,N_0+M]$, $100M<N_0\le N$, and assume $I_1,I_2\subset I$ subintervals of size $\sim M$ that are $\sim M$-separated. □

We first estimate

$$\int\Big\{\prod_{j=1}^{2}\Big|\sum_{n\in I_j}e(nx_1+n^2x_2+N^{1/2}n^{3/2}x_3+N^{1/2}n^{1/2}x_4)\Big|^6\Big\}dx. \tag{2.13}$$

Clearly (2.13) amounts to the number of solutions of the system

(2.14)

$$\begin{cases} m_1 + m_2 + m_3 - m_4 - m_5 - m_6 = m_7 + m_8 + m_9 - m_{10} - m_{11} - m_{12} \\ m_1^2 + m_2^2 + m_3^2 - m_4^2 - m_5^2 - m_6^2 = m_7^2 + m_8^2 + m_9^2 - m_{10}^2 - m_{11}^2 - m_{12}^2 \\ (N_0+m_1)^{3/2} + (N_0+m_2)^{3/2} + (N_0+m_3)^{3/2} \\ \quad - (N_0+m_4)^{3/2} - (N_0+m_5)^{3/2} - (N_0+m_6)^{3/2} \\ = (N_0+m_7))^{3/2} + (N_0+m_8)^{3/2} + (N_0+m_9)^{3/2} \\ \quad - (N_0+m_{10})^{3/2} - (N_0+m_{11})^{3/2} - (N_0+m_{12})^{3/2} + O(N^{-\frac12}) \\ (N_0+m_1)^{\frac12} + \cdots - (N_0+m_6)^{1/2} \\ = (N_0+m_7)^{\frac12} + \cdots - (N_0+m_{12})^{\frac12} + O(N^{-\frac12}), \end{cases}$$

with $m_1, \dots, m_6 \in I_1' = I_1 - N_0; m_7, \dots, m_{12} \in I_2' = I_2 - N_0$.

Write $(N_0+m)^{3/2}, (N_0+m)^{1/2}$ in the form

(2.15)

$$(N_0+m)^{3/2} = N_0^{3/2} + \frac32 N_0^{\frac12} m + \frac38 N_0^{-\frac12} m^2 + M^3 N_0^{-3/2} \phi_3\Big(\frac{m}{M}\Big),$$

(2.16)

$$(N_0+m)^{1/2} = N_0^{1/2} + \frac12 N_0^{-\frac12} m - \frac18 N_0^{-\frac32} m^2 - M^3 N_0^{-5/2} \phi_3\Big(\frac{m}{M}\Big) + M^4 N_0^{-7/2} \phi_4\Big(\frac{m}{M}\Big),$$

where $\phi_3(t) \sim t^3\Big(1 + O\Big(\frac{M}{N_0}\Big)t + \cdots\Big), \phi_4(t) \sim t^4$.

Hence $\Phi(t) = \big(t, t^2, \phi_3(t), \phi_4(t)\big)$ satisfies (2.11).

From the first and second lines of (2.14) and from (2.15), (2.16), inequalities in the third and eighth lines of (2.14) may be replaced by

$$\phi_3\Big(\frac{m_1}{M}\Big) + \cdots + \phi_3\Big(\frac{m_{12}}{M}\Big) < O(N^{-\frac12} N_0^{3/2} M^{-3}), \tag{2.17}$$

$$\phi_4\Big(\frac{m_1}{M}\Big) + \cdots + \phi_4\Big(\frac{m_{12}}{M}\Big) < O(N^{-\frac12} N_0^{7/2} M^{-4}). \tag{2.18}$$

The number of solutions of the first and second lines of (2.14) and of (2.17), (2.18) may be evaluated by

(2.19)

$$\int_{[-1,1]^4} \Big\{ \prod_{j=1}^{2} \Big| \sum_{m\in I_j'} e(mx_1 + m^2 x_2 + \frac{N^{\frac12} M^3}{N_0^{3/2}} \phi_3\Big(\frac{m}{M}\Big) x_3 + \frac{N^{\frac12} M^4}{N_0^{7/2}} \phi_4\Big(\frac{m}{M}\Big) x_4\Big)\Big|^6 \Big\} dx.$$

According to (2.10), (2.19) and hence (2.13) are bounded by

$$M^{6+\varepsilon} \Big\{ 1 + \frac{N_0^{3/2}}{N^{\frac12} M} \Big\} \Big\{ 1 + \frac{N_0^{7/2}}{N^{1/2} M^3} \Big\} \ll N^{4+\varepsilon} M^2. \tag{2.20}$$

Returning to (2.12), let $b(N)N^6$ be a bound on the left hand side We use the same reduction procedure to multi-linear (here bilinear) inequalities as in [B] and [B-D2] (and originating from [B-G]). Denote K a large constant and partition $[0, N]$ in intervals $I_0, I_1, \dots, I_K$, where $|I_0| = \frac{100N}{K}$ and $|I_s| = \big(1 - \frac{100}{K}\big)\frac{N}{K} = M_0$ for $1 \le s \le K$.

Bound

$$\int \Big| \sum_{n\le N} \Big|^{12} \le 2^{12} \int \Big| \sum_{n\in I_0} \Big|^{12} + (2K)^{12} \sum_{1\le s\le K} \int \Big| \sum_{n\in I_s} \Big|^{12}. \tag{2.21}$$

The first term of (2.21) is bounded by $2^{12} 100^6 K^{-6} b\Big(\frac{100N}{K}\Big) N^6$.

For the remaining terms, write $I_s = [N_s, N_s + M_0]$, $N_s > 100M_0$, and make a further partition of I_s in consecutive intervals $I_{s,1}, \dots, I_{s,K}$ of size $M_1 = \frac{M_0}{K}$. The key point (going back to [B-G]) is an estimate of the form

(2.22)
$$\int \Big| \sum_{n\in I_s} \Big|^{12} \leq 4^{12} \sum_{s_1 \leq K} \int \Big| \sum_{n\in I_{s,s_1}} \Big|^{12} + K^{18} \sum_{\substack{s_1,s_2\leq K\\ |s_1-s_2|\geq 2}} \int \Big\{ \Big| \sum_{n\in I_{s,s_1}} \Big|^6 \Big| \sum_{n\in I_{s,s_2}} \Big|^6 \Big\}.$$

Recall that (2.22) follows from considering the (pointwise in x) decreasing rearrangement $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_K$ of the sequence $\big(\big|\sum_{n\in I_{s,s_1}}\big|\big)_{1\leq s_1\leq K}$ and distinguishing the cases $\eta_4 < \frac{1}{K_1}\eta_1$ and $\eta_4 \geq \frac{1}{K_1}\eta_1$.

Application of (2.20) gives for $|s_1 - s_2| \geq 2$

$$\int \Big\{ \Big| \sum_{n\in I_{s,s_1}} \Big|^6 \Big| \sum_{n\in I_{s,s_2}} \Big|^6 \Big\} \ll N^{6+\varepsilon},$$

and hence the second sum in (2.22) contributes at most for $C(K)N^{6+\varepsilon}$. Replace the second term in the right hand side of (2.21) by

$$(2K)^{12} 4^{12} \sum_{s\leq K, s_1\leq K} \int \Big| \sum_{n\in I_{s,s_1}} \Big|^{12}.$$

Repeating the procedure, partition each I_{s,s_1} in intervals I_{s,s_1,s_2} of size $M_2 = \frac{M_1}{K}$ and apply the decomposition (2.22) for each $\sum_{n\in I_{s,s_1}}$, etc.

In general, one gets bilinear contributions of the form

$$(2K)^{12} 4^{12\alpha} K^{18} \sum_{J,J'} \int \Big\{ \Big| \sum_{n\in J} \Big|^6 \Big| \sum_{n\in J'} \Big|^6 \Big\}, \tag{2.23}$$

where the sum extends over pairs J, J' of intervals of size $M_\alpha = \frac{N}{K^{\alpha+1}}, \alpha \geq 1$ that are at least M_α-separated and contained in an interval of the form $[N_0, N_0 + KM_\alpha], KM_\alpha < \frac{1}{100}N_0$. Again by (2.20)

$$\int \Big\{ \Big| \sum_{n\in J} \Big|^6 \Big| \sum_{n\in J'} \Big|^6 \Big\} \ll N^{4+\varepsilon} M_\alpha^2$$

implying that

$$(2.23) \ll C(K) 4^{12\alpha} \frac{N}{M_\alpha} N^{4+\varepsilon} M_\alpha^2 \ll N^{6+\varepsilon} \Big(\frac{4^{12}}{K}\Big)^\alpha.$$

Summing over α eventually leads to the bound

$$2^{12} 100^6 K^{-6} b\Big(\frac{100N}{K}\Big) N^6 + N^{6+\varepsilon} \tag{2.24}$$

as an estimate for the left hand side of (2.21). Therefore

$$b(N) \leq 2^{12} 100^6 K^{-6} b\Big(\frac{100N}{K}\Big) + C_\varepsilon N^\varepsilon,$$

implying $b(N) \ll N^\varepsilon$ and Theorem 2.

Using the notation from [H], Theorem 2 implies Corollary 3.

Corollary 3. *Let* $\frac{1}{N^2} \le \delta \le 1, \frac{1}{N} \le \Delta \le 1$. *Then*

$$A_6(N,\delta,\Delta) = \int_0^1\int_0^1\int_{-1}^1\int_{-1}^1 \times \Big|\sum_{n\le N} e\Big(nx_1 + n^2x_2 + \frac{1}{\delta}\Big(\frac{n}{N}\Big)^{3/2}x_3 + \frac{1}{\Delta}\Big(\frac{n}{N}\Big)^{1/2}x_4\Big)\Big|^{12} dx \ll \delta\Delta N^{9+\varepsilon}. \tag{2.25}$$

Considering the major arc contribution, (2.25) is clearly seen to be essentially the best possible.

3. Applications to exponential sums

Let F be a smooth function on $[\frac{1}{2}, 1]$ satisfying, for some constant $c \in (0,1]$, the condition

$$\min\{|F''(x)|, |F'''(x)|, |F''''(x)|\} > c. \tag{3.1}$$

Given T sufficiently large, $M \ge 1$, put $f(u) = TF(u/M)$ with $\frac{M}{2} \le u \le M$ and

$$S = \sum_{m\sim M} e\big(f(m)\big). \tag{3.2}$$

In what follows, we assume $M \le \sqrt{T}$, in view of the application to $|\zeta(\frac{1}{2}+it)|$. We use notation and background from [H] and also rely on [H-W] and Sections 7 and 8 in [H1].

Once the parameter $N \in (1, M)$ is chosen, R is defined by the relation

$$R = \Big\lceil \Big(\frac{2M^3}{cNT}\Big)^{1/2}\Big\rceil, \tag{3.3}$$

so that, for each relevant size-N interval $I \subset [M/2, M]$, the corresponding "arc" $J(I) = \{f''(u)/2 : u \in I\}$ will be an interval of length exceeding $1/R^2$. We assume that N and R satisfy the conditions

$$R \le N \le R^2$$

(given that $\sqrt{T} \ge M$, these conditions imply that $2 \le R \le N \ll M^{3/2}T^{-1/2} \ll MT^{-1/4} \le M^{1/2}$ and $N \gg MT^{-1/3} \gg 1$). By following (in all but certain inessential respects) the steps of Section 4 in [H-W] that precede Equation (4.8) there, while slightly modifying the application of the lemma on "partial sums by Fourier transforms" (i.e. [H-W], Lemma 3.4), one obtains a result implying that, for some $Q, \ell, H, \alpha \in \mathbb{C}$ satisfying

$$Q, H \in \mathbb{N}, \ell \in \{0,1,2\}, \alpha \in \{e(-\eta) : -1/2 \le \eta \le 1/2\},$$
$$Q \ge R \gg \sqrt{Q} \quad\text{and}\quad H \ge NQ/R^2 \gg H,$$

one has an upper bound

$$|S| \ll \frac{M\log N}{N^{1/2}} + \frac{R\log^2 N}{Q^{1/2}} \sum_{I\in\mathcal{I}(Q,\ell)} \Big(\Big|\sum_{h\le H} \alpha^h e\big(\mathbf{x}(I)\cdot\big(h, h^2, h^{3/2}, h^{1/2}\big)\big)\Big| + \frac{Q}{R}\Big) \tag{3.4}$$

in which $\mathbf{x}$ is a certain mapping from the set $\mathcal{I}(Q,\ell)$ into $[-X_1, X_1]\times\cdots\times[-X_4, X_4] \subset \mathbb{R}^4$, where

$$X_1 = X_2 = \frac{1}{2} \quad\text{and}\quad X_3 = X_4 = \Big(\frac{R}{Q}\Big)^2 H^{1/2},$$

while $\mathcal{I}(Q,\ell)$ is the set of those $I \in \{[kN, N+kN] : k \in \mathbb{N}$ and $M/(2N) \le k \le (M-3N)/N\}$ that, via the procedures set out in [H-W], Section 4, Step 1, are associated with a reduced rational $a/q \in J(I)$ that happens to satisfy $Q \le q < 2Q$ and $Q \equiv \pm\ell (\operatorname{mod} 4)$ (in the terminology of [H-W], Section 4, Step 1, these I's are "minor arcs"). The details of the "second spacing problem" are not amongst the main points of interest in this paper, so we skip the definition of $\mathbf{x}(I)$ and mention only that this element of $\mathbb{R}^4$ is essentially identical to the vector $\mathbf{y} = \mathbf{y}^{(k)}$ defined in [H-W], Section 4, Step 4 (the index k there corresponds to our I).

By an appropriate application of [H-W], Lemma 2.1, one finds that

$$\sum_{\ell=0}^{2} |\mathcal{I}(Q,\ell)| \ll \frac{MR^2}{NQ^2} + \frac{R^2}{Q} \asymp \frac{MR^2}{NQ^2}.$$

Given this estimate that in (3.4) and the trivial upper bound for the modulus of the sum over h in (3.4), it follows by the sixth-power Hölder inequality that either

$$|S| \ll \frac{M \log^2 N}{N^{1/2}} \tag{3.5}$$

or else

$$R \le Q < R^{2/3} N^{1/3} \le N \tag{3.6}$$

and one has

$$|S|^6 \ll \Big(\frac{R\log^2 N}{Q^{1/2}}\Big)^6 \Big(\frac{MR^2}{NQ^2}\Big)^5 \sum_{I\in\mathcal{I}(Q,\ell)} \sum_{\mathbf{h}\in\mathbb{N}^6} e\big(\mathbf{x}(I)\cdot\mathbf{y}(\mathbf{h})\big)\omega(I)\Omega(\mathbf{h}), \tag{3.7}$$

where

$$\mathbf{y}(\mathbf{h}) = \sum_{j=1}^{6} (h_j, h_j^2, h_j^{3/2}, h_j^{1/2}) = (h_1 + \cdots + h_6, \ldots, h_1^{1/2} + \cdots + h_6^{1/2}) \in \mathbb{R}^4,$$

while ω is a certain complex-valued function that takes values that are (without exception) of modulus not exceeding unity, as does $\Omega(\mathbf{h})$ (which is equal to $\alpha^{y_1(\mathbf{h})}$ if $h_j \le H$ for $j = 1, \ldots, 6$, and is zero otherwise).

Suppose now that (3.6) and (3.7) hold. Then, similar to what is observed at the end of Step 4, in [H-W], Section 4, it follows by the Bombieri-Iwaniec "double large sieve" [B-I1], Lemma 2.4 (or see [H-W], Lemma 3.6) that one has

$$\Big| \sum_{I\in\mathcal{I}(Q,\ell)} \sum_{\mathbf{h}\in\mathbb{N}^6} e\big(\mathbf{x}(I)\cdot\mathbf{y}(\mathbf{h})\big)\omega(I)\Omega(\mathbf{h})\Big|^2 \ll AB_1 \prod_{j=1}^{4} (X_j Y_j + 1), \tag{3.8}$$

where

$$\begin{aligned} &Y_1 = 6H, \qquad Y_2 = 6H^2, \qquad Y_3 = 6H^{3/2}, \qquad Y_4 = 6H^{1/2}, \\ &B_1 = \Big|\big\{I, I') : I, I' \in \mathcal{I}(Q,\ell) \ \text{ and } \ |x_j(I) - x_j(I')| < \frac{1}{2Y_j} (j = 1, \ldots, 4)\big\}\Big| \end{aligned} \tag{3.9}$$

and

$$A = \Big|\big\{(\mathbf{h},\mathbf{h}') : \mathbf{h},\mathbf{h}' \in (\mathbb{N}\cap(0,H])^6 \ \text{ and } \ |y_j(\mathbf{h}) - y_j(\mathbf{h}')| < \frac{1}{2X_j} (j = 1, \ldots, 4)\big\}\Big|.$$

It is worth remarking here that the conditions on $(\mathbf{h},\mathbf{h}')$ in the above definition of the number A actually imply the equality of the ordered pairs $\big(y_1(\mathbf{h}), y_2(\mathbf{h})\big)$ and $\big(y_1(\mathbf{h}'), y_2(\mathbf{h}')\big)$ (both of which lie in $\mathbb{Z}^2$): hence the traditional definition of the "first spacing problem" as a question concerning the order of magnitude of the number of solutions in integers $h_1, h_1', \dots, h_r, h_r' \in (0, H]$ of a certain system of two equations and two inequalities (see for example [H], (11.1.1)–(11.1.5)). Given that $|y - y'| < \frac{1}{2X}$ implies $|y - y'| < \frac{1}{X}$ (whenever $X, y, y' \in \mathbb{R}$), it is a corollary of [H], Lemma 5.6.5 (for example) that we have here

$$\begin{aligned}
0 \le A &\le \Big(\frac{\pi^2}{2}\Big)^4 \Big(\frac{1}{X_1 \cdots X_4}\Big) \int_{-X_1}^{X_1} \int_{-X_2}^{X_2} \int_{-X_3}^{X_3} \int_{-X_4}^{X_4} \\
&\quad \times \Big| \sum_{\substack{\mathbf{h} \in \mathbb{N}^6 \\ h_j \le H\,(j=1,\dots,6)}} e(\mathbf{y}(\mathbf{h}) \cdot (z_1, \dots, z_4)) \Big|^2 dz_1 dz_2 dz_3 dz_4 \\
&= \frac{\pi^8 Q^4}{4HR^4} \int_0^1 \int_0^1 \int_{-\frac{R^2\sqrt{H}}{Q^2}}^{\frac{R^2\sqrt{H}}{Q^2}} \int_{-\frac{R^2\sqrt{H}}{Q^2}}^{\frac{R^2\sqrt{H}}{Q^2}} \\
&\quad \times \Big| \sum_{h \le H} e\big((h, h^2, h^{3/2}, h^{1/2}) \cdot (z_1, \dots, z_4)\big) \Big|^{12} dz_1 dz_2 dz_3 dz_4 \\
&= (\pi^8/4) A_6(H; \delta, H\delta),
\end{aligned}$$

with $A_6(L; \gamma, \Gamma)$ defined according to (2.28), and with

$$\delta = \frac{Q^2}{H^2 R^2}$$

so that $1/H \le H\delta \le Q/N \le 1$. Therefore it follows by Corollary 3 that we have

$$A \ll \delta^2 H^{10+\varepsilon}. \tag{3.10}$$

With regard to the second spacing problem one uses the treatment of the second spacing problem in [H-W], rather than the more advanced treatment in [H1]). One obtains

$$B_1 \ll \Delta_1 \Delta_2 \Big(\frac{M}{N}\Big)^2 \Big(\frac{Q}{R}\Big)^4 \ \text{if} \ \ N = MT^{-2/7},$$

where

$$\Delta_1 = \frac{1}{X_2 Y_2} = \frac{1}{3H^2} < \frac{R^4}{N^2 Q^2} \ \text{ and } \ \Delta_2 = \frac{1}{X_3 Y_3} = \frac{Q^2}{6R^2 H^2} < \frac{R^2}{N^2}.$$

Hence, with $N = MT^{-2/7}$

$$B_1 \ll \frac{M^2 R^2 Q^2}{N^6}. \tag{3.11}$$

It follows from (3.5)–(3.8), (3.10), and (3.11) that

$$|S|^6 \ll \max\Big\{\frac{M^{6+\varepsilon}}{N^3}, \Big(\frac{M^{5+\varepsilon} R^4 N}{Q^7}\Big)\Big(\frac{MRQ}{N^3}\Big)\Big\} \le \Big(\frac{M^{6+\varepsilon}}{N^3}\Big)\Big(\frac{N}{R}\Big) \ \text{ for } N = MT^{-2/7}. \tag{3.12}$$

Recalling (3.3) the above bound for $|S|^6$ implies

$$|S|^6 \ll \Big(\frac{M^{6+\varepsilon}}{N^3}\Big)\Big(\frac{N^3 T}{M^3}\Big)^{1/2} = \frac{M^{\varepsilon+9/2} T^{1/2}}{N^{3/2}} = M^{3+\varepsilon} T^{13/14}.$$

If $\sqrt{T} \ge M \ge cT^{3/7}$ (where c is the positive constant in (3.1) and (3.3)), then the conditions $N \in (1, M)$ and $R \le N \le R^2$ are satisfied, with $N = MT^{-2/7}$ and R as in (3.3). That is, we have

$$|S| \ll M^{\frac{1}{2}} T^{\varepsilon+13/84} \quad \text{for} \quad \sqrt{T} \geq M \geq cT^{3/7}. \tag{3.13}$$

The M-range for which the bound (3.13) holds may be extended by invoking the treatment in [H1] which we discuss next.

It was observed by Huxley, at the start of Section 7 in [H1], that for an arbitrary $V \geq 1$ the structure of the Bombieri-Iwaniec double large sieve implies that if the factor X_2Y_2+1 on the right-hand side of the bound (3.8) is increased to $X_2Y_2V+1 \leq (X_2Y_2+1)V$, then the adjacent term B_1 may be replaced by a term $B_V \leq B_1$, the definition of which differs from that of B_1 $\big($in (3.9)$\big)$ only insofar as it involves an upper bound on $|x_2(I) - x_2(I')|$ that is stronger by a factor V than is the case in (3.9). This observation plays a crucial part in Huxley's method of "resonance curves," through which the most recent progress [H1], [H], [H2] on the second spacing problem was achieved; we apply it here, in combination with (3.10) and the bounds $\prod_{j\leq 4}(X_jY_j+1) = (3H+1)(3H^2+1)(6\delta^{-1}+1)(6(H\delta)^{-1}+1) \ll H^2/\delta^2$, in order to deduce that

$$|S|^6 \ll \Big(\frac{M^5R^{16}\log^{12} N}{N^5Q^{13}}\Big)H^{6+\varepsilon}(VB_V)^{1/2} \ll \Big(\frac{M^5R^4N^{1+\varepsilon}}{Q^7}\Big)(VB_V)^{1/2} \quad (V \geq 1). \tag{3.14}$$

In [H1] Huxley invented an approach to the second spacing problem based on a theory involving certain resonance curves. In his first application of this, in [H1], Section 7, he obtained a result implying that, if $M \leq \sqrt{T}$ (as we suppose), and if one has either $V = N/Q \ll R^4/N^2$ or else $V = R^4/N^2$ (so that $V \geq 1$ in either case, given that $Q \leq N \leq R^2$), then

$$VB_V \ll \Big(\frac{VMR^2}{NQ^2} + \Delta_1\Delta_2\Delta_4^{2/3}\Big(\frac{M}{N}\Big)^2\Big)\Big(\frac{Q}{R}\Big)^4, \tag{3.15}$$

where Δ_1, Δ_2 are as above, while

$$\Delta_4 = \frac{1}{X_4Y_4} = \frac{Q^2}{6R^2H} < \frac{Q}{N}.$$

By (3.14) and (3.15), one obtains

$$\begin{aligned} |S|^6 &\ll \Big(\frac{M^{5+\varepsilon}R^4N}{Q^7}\Big)\Big(\min\Big\{\Big(\frac{N}{Q}\Big)^{1/2}, \frac{R^2}{N}\Big\}\Big(\frac{M}{N}\Big)^{1/2} + \Big(\frac{Q}{R}\Big)^{1/3}\Big(\frac{R}{N}\Big)^{7/3}\Big(\frac{M}{N}\Big)\Big) \\ &\leq \Big(\frac{M^{5+\varepsilon}N}{R^3}\Big)\Big(\min\Big\{\Big(\frac{N}{R}\Big)^{1/2}, \frac{R^2}{N}\Big\}\Big(\frac{M}{N}\Big)^{1/2} + \Big(\frac{R}{N}\Big)^{7/3}\Big(\frac{M}{N}\Big)\Big) \\ &= \min\Big\{\frac{M^{\varepsilon+11/2}N}{R^{7/2}}, \frac{M^{\varepsilon+11/2}}{RN^{1/2}}\Big\} + \Big(\frac{M^{6+\varepsilon}}{N^3}\Big)\Big(\frac{N}{R}\Big)^{2/3} \end{aligned} \tag{3.16}$$

$\big($with $M^{\varepsilon+11/2}/(RN^{1/2}) \asymp M^{4+\varepsilon}T^{1/2}$, by virtue of (3.3)$\big)$. When $M > T^{11/30}$ and R is given by (3.3), the upper bound (3.16) may be optimized by putting $N = \max\{MT^{-17/57}, M^{1/2}T^{-1/12}\}$: for each such M, T, N and R, the bound (3.16) is equivalent to

$$|S|^6 \ll \Big(\frac{M^{6+\varepsilon}}{N^3}\Big)\Big(\frac{N}{R}\Big)^{2/3} \asymp \frac{M^{5+\varepsilon}T^{1/3}}{N^2}. \tag{3.17}$$

One can check that if T is sufficiently large (in terms of c^{-1}), if $\sqrt{T} \geq M \geq T^{5/12}$, and if N is as assumed in (3.17), then N does satisfy our initial assumptions (that $1 < N < M$ and $R \leq N \leq R^2$, with R given by (3.3)). In fact our optimal choice of N in connection with the application of (3.15) is, unsurprisingly, identical to the choice of N found to be optimal in [H1], Section 8. The same is true in cases where $M < T^{5/12}$. The problem with such cases is that the choice of N assumed in (3.17) is too large to satisfy the condition $N \leq R^2$. The solution to this problem (utilized in [H1]) is to switch to a smaller value of N satisfying $N \leq R^2 \ll N$. This is achieved here by putting $N = \left(2M^3/(cT)\right)^{1/2}$, which satisfies all of our assumptions considering N and R whenever $T^{5/12} \geq M \geq 2(cT)^{1/3}$. It moreover follows from (3.16) and (3.3) that one obtains (3.17) for this alternate choice of N (satisfying $N \asymp R^2$).

By the above observations, and the further observation that $MT^{-17/57} \gg M^{1/2}T^{-1/12}$ if and only if $M \gg T^{49/114}$, we arrive at the upper bounds,

$$|S|^6 \ll \begin{cases} M^{3+\varepsilon}T^{53/57} & \text{if } \sqrt{T} \geq M > T^{49/114}; \\ M^{4+\varepsilon}T^{1/2} & \text{if } T^{49/114} \geq M \geq T^{5/12}; \\ M^{2+\varepsilon}T^{4/3} & \text{if } T^{5/12} > M \geq 2(cT)^{1/3}. \end{cases} \tag{3.18}$$

Note that this bound is $\min\{M^{3+\varepsilon}T^{53/57}, M^{4+\varepsilon}T^{1/2}\}$ when $\sqrt{T} \geq M \geq T^{5/12}$.

Using (3.18) if $M < T^{3/7}$ (noting the inequalities $\frac{49}{114} > \frac{3}{7} > \frac{5}{12}$), one verifies that the bound on $|S|$ in (3.13) holds if $\sqrt{T} \geq M \geq T^{17/42}$. Thus we establish Theorem 4.

Theorem 4. *With the above notation, one has that*

$$|S| \ll M^{1/2}T^{\varepsilon+13/84} \quad \textit{if} \quad \frac{1}{2} \geq \alpha = \frac{\log M}{\log T} \geq \frac{17}{42}. \tag{3.19}$$

4. Bounding the zeta-function on the critical line

For the application to $|\zeta(1/2+it)|$ one wants to show that (3.19) also holds when $17/42 > \alpha \geq 0$. The cases with $0 \leq \alpha \leq 13/42$ are trivial (there one can just use $|S| \leq M$), so all that remains to be done is establishing that (3.19) holds when α lies in the interval $(13/42, 17/42)$. To achieve this one can employ the bound

$$|S| \ll T^{\frac{1}{128}(4+103\alpha)+\varepsilon} \qquad (12/31 < \alpha \leq 1), \tag{4.1}$$

which is [H1], Theorem 3, in combination with the exponent pair estimate

$$|S| \ll \left(\frac{T}{M}\right)^{1/9} M^{13/18} = M^{11/18}T^{1/9} \quad (0 \leq \alpha \leq 1), \tag{4.2}$$

which corresponds to the exponent pair $(\frac{1}{9}, \frac{13}{18}) = ABA^2B(0,1)$ mentioned in [T], Section 5 20. It should be noted that (4.2) (and also (4.1)) assume additional hypotheses concerning the function F, beyond condition (3.1). This, however, is not an obstacle to the application to $|\zeta(1/2+it)|$, since that only requires consideration of cases in which $F(x) = \log x$ (a function that does satisfy all the unmentioned conditions attached to (4.1) and (4.2)). Assume henceforth that F is "a suitable function" such that (4.1) and (4.2) are applicable. A calculation shows that (3.19) is implied by (4.1) for all α in the interval $(12/31, 332/819]$, and is implied by (4.2) for all α in the interval $[0, 11/28]$: noting that $11/28 =$

$0.39285\ldots > 0.38709\ldots = 12/31$, we find that the union of these two intervals is $[0, 332/819] = [0, 0.40537\ldots] \supset (0.30952\ldots, 0.40476\ldots) = (13/42, 17/42)$.

By the preceding one has the bound (3.19) whenever $0 \le \alpha \le 1/2$ (at least this is so in the case $F(x) = \log x$). It follows from the "approximate functional equation" for $\zeta(s)$ in the critical strip (see [T] (4.12.4)) that

$$\left|\zeta\Big(\frac{1}{2}+it\Big)\right| \le 2\Big|\sum_{n\le\sqrt{t/2\pi}} n^{-\frac{1}{2}+it}\Big| + O(1)\ (t\to\infty). \tag{4.3}$$

From partial summation and dyadic dissection, Theorem 5 now follows in the usual way.

Remark. Bombieri and Iwaniec achieved the exponent $9/56 = (1-1/28)/6$ using an essentially optimal bound on $A_4(\delta, H\delta)$, so that the exponent $13/84 = (1-1/14)/6$ $\big($achieved with an essentially optimal bound for $A_6(\delta, H\delta)\big)$ represents exactly a doubling of Bombieri and Iwaniec's improvement over the classical "1/6" (with their essentially optimal bound on $A_5(\delta, H\delta)$ Huxley and Kolesnik, in 1991, got the exponent $17/108 = (1-1/18)/6$, and would in fact have got $11/70 = (1-2/35)/6$, except for the fact that cases with α near to $23/54$ were a problem at that time).

5. Further comments

Recalling (3.2), the preceding shows that one has the estimate

$$|S| \ll M^{\frac{1}{2}} T^{\varepsilon+\frac{13}{84}} \quad \text{if} \quad \frac{1}{2} \ge \alpha = \frac{\log M}{\log T} > 0 \tag{5.1}$$

provided f is in the class of functions to which the exponent pair theory applies (see for instance [G-K], Chapter 3 for details).

Theorem 6. $\big(\varepsilon + \frac{13}{84}, \varepsilon + \frac{55}{84}\big)$ *is an exponent pair.*

One needs to obtain the bound $|S| \ll (T/M)^{\frac{13}{84}+\varepsilon} M^{\frac{55}{84}+\varepsilon} = M^{\frac{1}{2}} T^{\frac{13}{84}+\varepsilon}$ subject to conditions that are weaker, in two respects, than the conditions under which direct application of (5.1) gives this bound on $|S|$. More specifically

(a) the relevant M may exceed the square root of the relevant T (although one will at least not have $M > T$);

(b) the summation may not be over $[M/2, M] \cap \mathbb{Z}$ $\big($it may just be over some set $[a, b] \cap \mathbb{Z}$, where $[a, b]$ is some proper subset of $(M/2, M)\big)$. Moreover the function f that one is "given" may only be defined on the subinterval $[a, b]$ (this is, for example, what occurs in the theory of exponent pairs developed in [G-K]).

As a first step to getting around the problem (a), one can note that in the absence of problem (b) the desired result in any cases with $T < M^2 \ll T$ can be seen to follow from (5.1). For in such cases one may replace c, F, T by $c_1 = TM^{-2}c, F_1 = TM^{-2}F$ and $T_1 = M^2$ without invalidating (3.1) or causing any change in the value of the sum S.

Second, just to secure any extreme cases, one can deal with the cases in which $M \ge T^{9/10}$ (say) simply by an appeal to the exponent pair $(1/2, 1/2)$ (this is analogous to using the trivial bound $|S| \le M$ when α is in a neighborhood of 0).

Cases with $T^{1/2} < M < T^{9/10}$ become manageable after they are converted, through Poisson summation and partial summation, into cases involving (in place of

M and T) an $M' \asymp T/M$ and a $T' \asymp T$, so that one has $(T')^{1/10} \ll M^1 \ll (T')^{1/2}$. The details of this conversion are essentially the same as what goes on in the verification of the "B-process" of exponent pair theory (see, for example, [G-K], Section 3.5); its efficacy, in disposing of problem (a), is related to the fact that, when $0 \le k \le \frac{1}{2}$ and $l = k + \frac{1}{2}$, one has $B(k,l) := (l - \frac{1}{2}, k + \frac{1}{2}) = (k,l)$.

Problem (b) is also remarked upon in Sargos's paper [S] (see the remark on page 310). First, one constructs a suitable extension of the function $f(x)$, so that the resulting function $f_1(x)$ has domain $[M/2, M]$, is identical to $f(x)$ on the subinterval $[a,b]$, and satisfies (on $[M/2, M]$) the requisite set of conditions on its derivatives (these conditions being such as to make the theory of exponent pairs applicable). If $s > 0$ and yx^{-s} is the relevant "monomial" approximation to $f'(x)$ on $[a,b]$ (such as must be present when the exponent pair theory is applicable), then it is enough to consider an extension f_1 of f that, for $b < x \le M$, satisfies $f_1(x) = y\int_b^x u^{-s}du + a_0 + a_1 x + \cdots + a_P x^P$, where $a_0, a_1, \ldots, a_P$ are certain constants (determined by the requirement that $f_1^{(P)}(x)$ be continuous at $x = b$): given that the derivatives $f'(x), \ldots, f^{(P)}(x)$ satisfy the requisite conditions on the interval $[a,b]$ (for which see [G-K], Condition (3.3.3)) one may deduce that the constants $a_1, \ldots, a_P$ are small enough not to prevent those same conditions being satisfied by $f_1'(x), \ldots, f_1^{(P)}(x)$ on the longer interval $[a, M]$. By a similar construction one obtains an extension of f (and so also of f) that has domain $[M/2, M]$ and is of the class to which the exponent pair theory applies.

Once the extension of f to $\left[\frac{M}{2}, M\right]$ is obtained, one can employ [S], Lemma 2.1 to solve problem (b) at the cost of losing a harmless factor $O(\log M)$ in the final estimate.

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