JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 30, Number 2, April 2017, Pages 561–599
http://dx.doi.org/10.1090/jams/865
Article electronically published on September 6, 2016

LENS RIGIDITY FOR MANIFOLDS WITH HYPERBOLIC TRAPPED SETS

COLIN GUILLARMOU

1. Introduction

In this work, we study a geometric inverse problem concerning the recovery of a Riemannian manifold (M, g) with boundary from information about its geodesic flow that can be read at the boundary. Different aspects of this problem have been extensively studied by [Mu, Mi, Cr1, Ot, Sh, PeUh, StUh1, BuIv, CrHe, SUV], among others. This study also has applications to applied inverse problems, in geophysics and tomography. Our results concern the case of negatively curved manifolds with strictly convex boundaries, and more generally manifolds with hyperbolic trapped sets and no conjugate points. In these settings we solve the deformation lens rigidity problem in all dimensions, and in dimension 2 we show that the lens data (and actually the scattering data) determine the Riemann surface up to conformal diffeomorphism. The important difference with most of the previous works on the subject is that we allow trapping and non-trivial topology; in this setting we obtain the first general results. This requires the introduction of new methods based on a systematic use of recent analytic techniques introduced in hyperbolic dynamical systems [BuLi, FaSj, DyZw, DyGu2].

1.1. **Negative curvature.** Let (M, g) be a smooth n-dimensional compact Riemannian manifold with strictly convex boundary ∂M (i.e., the second fundamental form is positive). In this work, we will always assume that either M is connected or each connected component of M has a non-empty boundary. The incoming $(-)$ and outgoing $(+)$ boundaries of the unit tangent bundle of M are defined and denoted by

$$\partial_\pm SM := \{(x, v) \in TM; x \in \partial M, |v|_{g_x} = 1, \mp g_x(v, \nu) > 0\},$$

where ν is the inward pointing unit normal vector field to ∂M. For all $(x, v) \in \partial_- SM$, the geodesic $\gamma_{(x,v)}$ with initial point x and tangent vector v either has infinite length or exits M at a boundary point $x' \in \partial M$ with tangent vector v' with $(x', v') \in \partial_+ SM$. We call $\ell_g(x, v) \in [0, \infty]$ the length of this geodesic. If $\Gamma_- \subset \partial_- SM$ denotes the set of $(x, v) \in \partial_- SM$ with $\ell_g(x, v) = \infty$, we call $S_g(x, v) := (x', v') \in \partial_+ SM$ the exit pair or scattering image of (x, v) when $(x, v) \notin \Gamma_-$. This defines the *length map* and the *scattering map*

$$\ell_g : \partial_- SM \to [0, \infty], \quad S_g : \partial_- SM \setminus \Gamma_- \to \partial_+ SM. \tag{1.1}$$

Received by the editors January 16, 2015 and, in revised form, July 12, 2016.
2010 *Mathematics Subject Classification.* Primary 35R30; Secondary 53C24, 53C65.
Key words and phrases. X-ray transform, lens rigidity, scattering rigidity, hyperbolic dynamics.
The research is partially supported by grants ANR-13-BS01-0007-01 and ANR-13-JS01-0006.

The lens data are the pair (ℓ_g, S_g). Notice that such data do not (a priori) contain information on closed geodesics of M, nor on geodesics not intersecting ∂M.

If (M,g) and (M',g') are two Riemannian manifolds with the same boundary N and $g|_{TN} = g'|_{TN}$, there is a natural identification between $\partial_- SM$ and $\partial_- SM'$. Indeed, $\partial_- SM$ can be identified with the boundary ball bundle $BN := \{(x,v) \in TN; |v|_g < 1\}$ via the orthogonal projection $\partial SM \to BN$ with respect to g (and similarly for (M',g')). The *lens rigidity problem* consists in showing that, if (M,g) and (M',g') are two Riemannian manifold metrics with strictly convex boundary and $\partial M = \partial M'$, then

$$\ell_g = \ell_{g'},\ S_g = S_{g'} \Longrightarrow \exists \phi \in \mathrm{Diff}(M';M),\ \phi^* g = g',\ \phi|_{\partial M'} = \mathrm{Id}. \tag{1.2}$$

When $(\ell_g, S_g) = (\ell_{g'}, S_{g'})$, we say that (M,g) and (M',g') are *lens equivalent*, while if $S_g = S_{g'}$ we say that they are *scattering equivalent*.

Our first result concerns deformation lens rigidity and holds in any dimension.

Theorem 1. *For $s \in (-1,1)$, let g_s be a smooth 1-parameter family of metrics with negative curvature on a smooth connected compact n-dimensional manifold M with strictly convex boundary. Assume that g_s is lens equivalent to g_0 for all s. Then there exists a family of diffeomorphisms ϕ_s satisfying $\phi_s|_{\partial M} = \mathrm{Id}$ and $\phi_s^* g_0 = g_s$.*

In dimension 2, we show that the scattering data determine the conformal structure.

Theorem 2. *Let (M,g) and (M',g') be two oriented negatively curved Riemannian surfaces with strictly convex boundary, and such that each connected component of M and M' has non-empty boundary. Assume also that $\partial M = \partial M'$ and $g|_{T\partial M} = g'|_{T\partial M'}$. If (M,g) and (M',g') are scattering equivalent, then there is a diffeomorphism $\phi : M \to M'$ such that $\phi^* g' = e^{2\omega} g$ for some $\omega \in C^\infty(M)$ and $\phi|_{\partial M} = \mathrm{Id}$, $\omega|_{\partial M} = 0$.*

In the special case of simple manifolds, these results correspond to the much studied boundary rigidity problem, which consists in determining a metric (up to a diffeomorphism which is the identity on ∂M) on an n-dimensional Riemannian manifold (M,g) with boundary ∂M from the distance function $d_g : M \times M \to \mathbb{R}$ restricted to $\partial M \times \partial M$. A *simple* manifold is a manifold with strictly convex boundary such that the exponential map $\exp_x : \exp_x^{-1}(M) \to M$ is a diffeomorphism at all points $x \in M$. Such manifolds have no conjugate points, no trapped geodesics (i.e., geodesics entirely contained in $M^\circ := M \setminus \partial M$), and there is a unique geodesic in M joining any given pair of boundary points $x, x' \in \partial M$. Simple manifolds are diffeomorphic to a ball. Boundary rigidity for simple metrics was conjectured by Michel [Mi] and has been proved in some cases:

(1) If (M,g) and (M,g') are conformal and lens equivalent simple manifolds, they are isometric; this is shown by Mukhometov-Romanov, Croke [Mu, MuRo, Cr2].
(2) If (M,g) and (M',g') are lens equivalent simple surfaces ($n = 2$), they are isometric. This was proved by Otal [Ot] in negative curvature and by Croke [Cr1] in non-positive curvature. For general simple metrics, Pestov-Uhlmann [PeUh] proved that the scattering data determine the conformal class and, combined with (1), this shows Michel's conjecture in dimension $n = 2$.

(3) If g and g' are simple metrics that are close enough to a given simple analytic metric g_0, and are lens equivalent, then they are isometric. This was proved by Stefanov-Uhlmann [StUh1]. All metrics C^2-close to a flat metric g_0 on a smooth domain of $\mathbb{R}^n$ are boundary rigid, as proved by Burago-Ivanov [BuIv].

(4) A 1-parameter smooth family of simple non-positive curved metrics with same lens data are all isometric; this was shown by Pestov-Sharafutdinov [PeSh].

Thus, Theorem 2 is similar to Pestov-Uhlmann result in (2), but for a class of non-simple surfaces, and Theorem 1 extends (4). We emphasize that in our case, there are typically infinitely many trapped (and closed) geodesics, and this provides the first general rigidity result in the presence of trapping. In fact, when there are trapped geodesics or when the flow has conjugate points, there exist lens equivalent metrics which are not isometric; see Croke [Cr2] and Croke-Kleiner [CrKl]. So far, only results of lens rigidity in very particular cases were proved in case of trapped geodesics:

(5) In dimension $n \geq 3$, Stefanov-Uhlmann [StUh2] proved lens rigidity near certain analytic metrics with trapped sets (a typical example is the solid torus $\mathbb{S}^1 \times \{z \in \mathbb{C}, |z| \geq 1\}$ with the flat metric).

(6) Croke-Herreros [CrHe] proved that a 2-dimensional negatively curved or flat cylinder with convex boundary is lens rigid. Croke [Cr3] showed that the flat product metric on $B_n \times S^1$ is scattering rigid if B_n is the unit ball in $\mathbb{R}^n$.

(7) In dimension $n \geq 3$, Stefanov-Uhlmann-Vasy [SUV] proved that the lens data near ∂M determine the metric near ∂M for metrics in a fixed conformal class. They also recover the metric outside the convex core of M under convex foliation assumptions.

(8) For the flat metric on $\mathbb{R}^n \setminus \mathcal{O}$ where $\mathcal{O}$ is a union of strictly convex domains, Noakes-Stoyanov [NoSt] show that the lens data for the billiard flow on $\mathbb{R}^n \setminus \mathcal{O}$ determine $\mathcal{O}$.

If $SM = \{(x, v) \in TM; |v|_{g_x} = 1\}$ is the unit tangent bundle and SM° its interior, the *trapped set* $K \subset SM^\circ$ of the geodesic flow is the set of points $(x, v) \in SM^\circ$ such that the geodesic passing through x and tangent to v does not intersect the boundary ∂SM; K is a closed flow-invariant subset of SM° which includes all closed geodesics. In results (5) and (6) above, the trapped set has an explicit simple structure; in (7), it can be anything but the result allows one to determine only the metric near ∂M, which is the region of M with no trapped geodesics. In comparison, in our case the trapped set is typically a closed set of fractal type. For instance, in constant negative curvature it has Hausdorff dimension given in terms of the convergence exponent of the Poincaré series for the fundamental group (see [Su]).

1.2. **More general results and the X-ray transform.** As we will show, the results obtained in negative curvature are valid in a more general setting. The only needed assumptions are that the metric has no conjugate points and the trapped set is a hyperbolic set (for the geodesic flow). Let us recall the definition of hyperbolicity of a set. For $t \in \mathbb{R}$, we denote by φ_t the geodesic flow at time t on SM, i.e., $\varphi_t(x, v) = (x(t), v(t))$ where $x(t)$ is the point at distance t on the geodesic generated by (x, v) and $v(t) = \dot{x}(t)$ the tangent vector. We say that the trapped set

K is a *hyperbolic set* if there exists $C > 0$ and $\nu > 0$ so that for all $y = (x, v) \in K$, there is a continuous flow-invariant splitting

$$T_y(SM) = \mathbb{R}X(y) \oplus E_u(y) \oplus E_s(y), \tag{1.3}$$

where $E_s(y)$ and $E_u(y)$ are vector subspaces satisfying

$$\begin{aligned} ||d\varphi_t(y)w|| &\leq Ce^{-\nu t}||w||, \quad \forall t > 0, \forall w \in E_s(y), \\ ||d\varphi_t(y)w|| &\leq Ce^{-\nu|t|}||w||, \quad \forall t < 0, \forall w \in E_u(y), \end{aligned} \tag{1.4}$$

with respect to any fixed metric on SM. This setting is quite natural and "interpolates" between the simple domain case (open, no trapped set) and the Anosov case (closed manifolds with hyperbolic geodesic flow). Negative curvature near the trapped set implies that K is a hyperbolic set (see [Kl2, Section 3.9 and Theorem 3.2.17]), but although this is the typical example, negative curvature is not necessary for that to happen.

The central tool to prove Theorems 1 and 2 is the X-ray transform on symmetric tensors on M. If $m \in \mathbb{N}_0$ is the order of the tensor, this operator associates to a symmetric tensor $f \in C^\infty(M; \otimes_S^m T^*M)$ a function $I_m f \in C^\infty(\partial_- SM \setminus \Gamma_-)$ describing all the possible integrals of f along geodesics of g with end points on ∂M,

$$I_m f(x, v) := \int_0^{\ell_g(x,v)} f(x(t))(\otimes^m v(t))dt,$$

where $\varphi_t(x, v) = (x(t), v(t))$ is the geodesic in SM with initial condition $(x, v) \in \partial_- SM \setminus \Gamma_-$.

Theorem 3. *Let (M, g) be a smooth compact connected Riemannian manifold with strictly convex boundary. Assume that g has a hyperbolic trapped set and no conjugate points. Then I_0 is injective and I_1 is injective on divergence-free 1-forms. If in addition g has non-positive curvature, then I_m is injective on divergence-free symmetric m-tensors for all $m \geq 2$.*

In Theorem 5, we actually obtain boundedness and injectivity of I_m on more general functional spaces. Similar results were proved for simple metrics in [MuRo, Mu, AnRo, PeSh, StUh1, PSU1, PSU2] and more recently in [UhVa] for metrics admitting foliations by convex hypersurfaces. The main new tool to show injectivity of I_m in our case is a Livsic theorem of a new type. Indeed, a Hölder Livsic theorem exists on the trapped set [HaKa, Theorem 19.2.4], but this is not very useful for our purpose. The result we need and prove in Proposition 5.5 is the following: if $f \in C^\infty(SM)$ integrates to 0 along all geodesics relating boundary points of M, then there exists $u \in C^\infty(SM)$ satisfying $Xu = f$ and $u|_{\partial SM} = 0$.

A straightforward consequence of Theorem 3 is the following deformation rigidity (from which Theorem 1 also follows).

Corollary 1.1. *Let M be a smooth connected compact manifold with boundary, equipped with a smooth 1-parameter family of lens equivalent metrics g_s for $s \in (-1, 1)$ and assume that ∂M is strictly convex for g_s for each s. Suppose that, for all s, g_s have hyperbolic trapped set.*

1. *If for all s, g_s is conformal to g_0 and has no conjugate points, then $g_s = g_0$.*
2. *If g_s has non-positive curvature, then there exists a family of diffeomorphisms ϕ_s that are equal to* Id *at ∂M and with $\phi_s^* g_0 = g_s$.*

Hyperbolicity of K is a stable condition by small perturbations of the metric, and there is structural stability of hyperbolic sets for flows (see [HaKa, Chapter 18.2] and [Ro]), which justifies the study of deformation rigidity in that class of metrics.

We also prove in Proposition 5.7 that $I_0^* I_0$ is an elliptic pseudo-differential operator of order -1, and we use this to deduce that I_0^* is surjective in Proposition 5.10. These results are the core to apply the method of Pestov-Uhlmann [PeUh] which relates in dimension 2 the scattering data to the set of boundary values of holomorphic functions on M. This set allows one to recover the conformal structure by using [Be].

Theorem 4. *Let (M, g) and (M', g') be two smooth oriented Riemannian surfaces with no conjugate points, and such that each connected component has non-empty strictly convex boundary. Assume that $\partial M = \partial M'$, $g|_{T\partial M} = g'|_{T\partial M'}$ and that the trapped sets of g and of g' are hyperbolic. If (M, g) and (M', g') are scattering equivalent, there is a diffeomorphism $\phi : M \to M'$ such that $\phi^* g' = e^{2\omega} g$ for some $\omega \in C^\infty(M)$ and $\phi|_{\partial M} = \mathrm{Id}$, $\omega|_{\partial M} = 0$.*

We emphasize that due to trapping, several important aspects of the proof of [PeUh] for simple metrics are much more difficult to implement in our setting. To obtain the desired result, we need to address delicate questions which are absent in the non-trapping case: we need to solve boundary value problems for the transport equations in low regularity spaces and understand the wave-front set of solutions, we need to describe boundary values of invariant distributions in SM with certain regularity only in terms of the scattering map S_g. The hyperbolicity assumption on K is essential. A novelty here is that we make use of the theory of anisotropic Sobolev spaces adapted to the dynamic, which appeared recently in the field of hyperbolic dynamical systems (typically on Anosov flows [BuLi, FaSj, DyZw]). More precisely, our analysis relies on microlocal tools developed recently in joint work with Dyatlov [DyGu2] for Axiom A type dynamical systems, in the same spirit as in the closed case [Gu] where we used the works [FaSj, DyZw]. A remarkable aspect of this setting with hyperbolic trapped set is that the X-ray transform still fits into a Fredholm type problem as it does for simple domains.

1.3. **Comments.**

(1) The assumption $g = g'$ on $T\partial M$ in Theorem 4 is not a serious one and could be removed by standard arguments since, by [LSU], the length function near $\partial_0 SM := \{(x, v) \in \partial SM; \langle \nu, v\rangle = 0\}$ determines the metric on $T\partial M$ (we would then have to change slightly the definition of S_g, as in [StUh2]).

(2) A part of this work deals with very general assumptions (no hyperbolicity assumption on K and no assumptions on conjugate point) to describe solutions of the boundary value problems for transport equations in SM.

(3) Contrary to the simple metric setting, the lens equivalence between two general metrics does not a priori induce a conjugation of their geodesic flows, which makes the problem more difficult.

(4) As pointed out to me by M. Salo, Theorem 3 is sharp in the sense that if there exists a flat cylinder $\mathcal{C} = ((-\epsilon, \epsilon)_\tau \times (\mathbb{R}/a\mathbb{Z})_\theta, d\tau^2 + d\theta^2)$ (with $a > 0$) embedded in a surface with strictly convex boundary, then it is easy to check that $\ker I_0$ is infinite dimensional and contains all functions

f compactly supported in $\mathcal{C}$, depending only on τ with $\int_{-\epsilon}^{\epsilon} f(\tau)d\tau = 0$. In this case the trapped set is not hyperbolic.

(5) A by-product of Theorem 3 (using [DKLS, Theorem 1.1]) is the existence of many new examples with non-trivial topology and a complicated trapped set where the Calderón problem can be solved in a conformal class.

(6) We are not yet able to prove that the lens data determine the conformal factor ω in Theorem 4. It likely does, but this seems to be a difficult problem.

2. Geometric setting and dynamical properties

2.1. **Geometry of SM and extensions.** We recall basic facts about the geometry of the unit tangent bundle and refer the reader to [Pa, Chapter 1] for details. Let

$$\pi_0 : SM \to M, \quad \pi_0(x, v) = x,$$

be the natural bundle projection on the base. The tangent space of SM has a natural splitting into vertical and horizontal smooth subbundles

$$T(SM) = \mathcal{V} \oplus \mathcal{H}, \tag{2.1}$$

where $\mathcal{V} = \ker d\pi_0$ and $\mathcal{H}$ is defined using the Levi-Civita connection (see [Pa, Chapter 1.3.1]). The connection induces in particular a map $\mathcal{K} : T(SM) \to TM$ which can be used to define the Sasaki metric on SM by

$$\langle \xi, \xi' \rangle_S := g(d\pi_0.\xi, d\pi_0.\xi') + g(\mathcal{K}\xi, \mathcal{K}\xi').$$

There is a natural contact 1-form α on SM called the *Liouville form*, satisfying $\alpha(X) = 1$ and $i_X d\alpha = 0$ if X is the geodesic vector field on SM. This induces an associated volume form and thus a measure $d\mu$ called the *Liouville measure* given by

$$d\mu := \frac{1}{(n-1)!}\alpha \wedge (d\alpha)^{n-1}, \tag{2.2}$$

which is also exactly the Sasaki volume form (here $\dim M = n$).

It is convenient to view (M, g) as a strictly convex region of a larger smooth manifold $(\hat{M}, \hat{g})$ with strictly convex boundary, and to extend the geodesic vector field X on SM into a vector field X_0 on $S\hat{M}$ which has complete flow, for instance by making X_0 vanish at $\partial S\hat{M}$. Let us describe this construction. Near the boundary ∂M, let (ρ, z) be normal coordinates to the boundary; i.e., ρ is the distance function to ∂M satisfying $|d\rho|_g = 1$ near ∂M and z are coordinates on ∂M. The metric then becomes $g = d\rho^2 + h_\rho$ in a collar neighborhood $[0, \delta]_\rho \times \partial M$ of ∂M for some smooth 1-parameter family h_ρ of metrics on ∂M, and the strict convexity condition means that the second fundamental form $-\partial_\rho h_\rho|_{\rho=0}$ is a positive definite symmetric tensor. We extend smoothly h_ρ from $\rho \in [0, \delta]$ to $\rho \in [-1, \delta]$ as a family of metrics on ∂M satisfying $-\partial_\rho h_\rho > 0$ for all $\rho \in [-1, 0]$. We can then view M as a strictly convex region inside a larger manifold M_e with strictly convex boundary as follows. First, let $E = \partial M \times [-1, 0]_\rho$ be the closed cylindrical manifold, and consider the connected sum $\hat{M} := M \sqcup E$ where we glue the boundary $\{\rho = 0\} \simeq \partial M$ of E to the boundary ∂M of M; then we put a smooth structure of manifold with boundary on $\hat{M}$ extending the smooth structure of M, and we extend the metric g smoothly from M to $\hat{M}$ by setting $\hat{g} = d\rho^2 + h_\rho$ in E. Each hypersurface $\{\rho = c\}$ with $c \in [-1, 0]$ is strictly convex. We now set the extension

$$M_e := M \cup \{y \in E;\ \rho(y) \in [-\epsilon, 0]\}$$

of M for $\epsilon > 0$ fixed small, so that (M_e, g) is a manifold with strictly convex boundary containing M and contained in $\hat{M}$. It is easily checked that the longest connected geodesic ray in $SM_e \setminus SM^\circ$ has length bounded by some $L < \infty$. When (M, g) has no conjugate point and hyperbolic trapped set, it is possible to choose ϵ small enough so that (M_e, g) has no conjugate point either (see Section 2.3), and we will do so each time we shall assume that (M, g) has no conjugate point. We denote by X the geodesic vector field on the unit tangent bundle $S\hat{M}$ of $\hat{M}$ with respect to the extended metric g. Let us define $\rho_0 \in C^\infty(\hat{M})$ so that near E, $\rho_0 = F(\rho)$ is a smooth nondecreasing function of ρ satisfying $F(\rho) = \rho + 1$ near $\rho = -1$, and so that $\{\rho_0 = 1\} = M_e$. Denote by $\pi_0 : S\hat{M} \to \hat{M}$ the projection on the base, then the rescaled vector field

$$X_0 := \pi_0^*(\rho_0)X$$

on $S\hat{M}$ has the same integral curves as X, it is complete, and $X_0 = X$ in the neighborhood SM_e of SM. The flow at time t of X_0 is denoted φ_t, and by strict convexity of M (resp. M_e) in $\hat{M}$, φ_t is also the flow of X in the sense that for all y in SM (resp. in SM_e) one has $\partial_t\varphi_t(y) = X(\varphi_t(y))$ for $t \in [0, t_0]$ as long as $\varphi_{t_0}(y) \in SM$ (resp. $\varphi_{t_0}(y) \in SM_e$).

We shall denote M° and M_e° for the interior of M and M_e.

2.2. **Incoming/outgoing tails and trapped set.** We define the incoming $(-)$, outgoing $(+)$, and tangent (0) boundaries of SM and SM_e,

$$\partial_\mp SM := \{(x, v) \in \partial SM; \pm d\rho(X) > 0\}, \quad \partial_\mp SM_e := \{(x, v) \in \partial SM_e; \pm d\rho(X) > 0\},$$
$$\partial_0 SM = \{(x, v) \in \partial SM; d\rho(X) = 0\}, \quad \partial_0 SM_e = \{(x, v) \in \partial SM; d\rho(X) = 0\}.$$

For each point $(x, v) \in SM$, define the time of escape of SM in positive $(+)$ and negative $(-)$ times,

$$\begin{aligned} \ell_+(x, v) &:= \sup\{t \geq 0; \varphi_t(x, v) \in SM\} \in [0, +\infty], \\ \ell_-(x, v) &:= \inf\{t \leq 0; \varphi_t(x, v) \in SM\} \in [-\infty, 0]. \end{aligned} \tag{2.3}$$

Definition 2.1. *The incoming $(-)$ and outgoing $(+)$ tails in SM are defined by*

$$\Gamma_\mp = \{(x, v) \in SM; \ell_\pm(x, v) = \pm\infty\} = \bigcap_{t \geq 0} \varphi_{\mp t}(SM),$$

and the trapped set for the flow on SM is the set

$$K := \Gamma_+ \cap \Gamma_- = \bigcap_{t \in \mathbb{R}} \varphi_t(SM). \tag{2.4}$$

We note that $\Gamma_\pm$ and K are closed sets and that K is globally invariant by the flow. By the strict convexity of ∂M, the set K is a compact subset of SM° since for all $(x, v) \in \partial SM$, $\varphi_t(x, v) \in S\hat{M} \setminus SM$ for either all $t > 0$ or all $t < 0$.

Moreover, it is easy to check ([DyGu2, Lemma 2.3]) that $\Gamma_\pm$ are characterized by

$$y \in \Gamma_\pm \iff d(\varphi_t(y), K) \to 0 \text{ as } t \to \mp\infty, \tag{2.5}$$

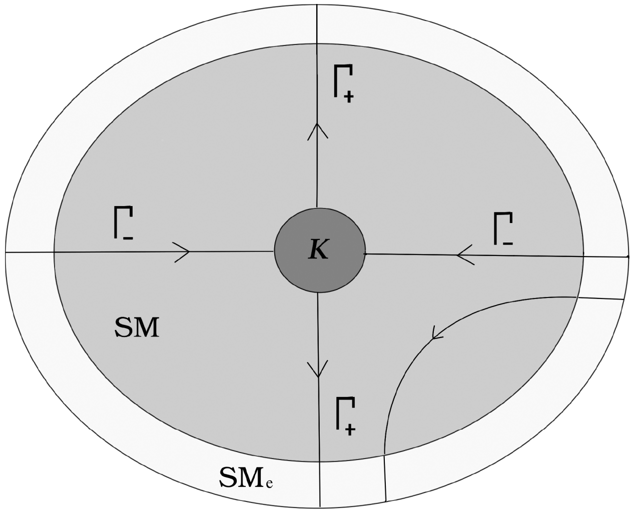

FIGURE 1. The manifold SM and SM_e.

where $d(\cdot,\cdot)$ is the distance induced by the Sasaki metric. We then extend $\Gamma_\pm$ to $S\hat{M}$ by using the characterization (2.5); the sets $\Gamma_\pm$ are closed flow-invariant subsets of the interior $S\hat{M}^\circ$ of $S\hat{M}$. By strict convexity of the hypersurfaces $\{\rho = c\}$ with $c \in (-1, 0]$, each point $y \in S\hat{M}$ with $\rho(y) \in (-1, 0]$ is such that $d(\varphi_t(y), \partial S\hat{M}) \to 0$ either as $t \to +\infty$ or $t \to -\infty$, and thus for all $c \in (0, 1)$

$$K = \bigcap_{t\in\mathbb{R}} \varphi_t(\{\rho_0 \geq c\}) = \bigcap_{t\in\mathbb{R}} \varphi_t(SM_e).$$

We also remark that the strict convexity of ∂M and ∂M_e implies

$$\Gamma_\mp \cap \partial SM = \Gamma_\mp \cap \partial_\mp SM, \quad \Gamma_\mp \cap \partial SM_e = \Gamma_\mp \cap \partial_\mp SM_e. \tag{2.6}$$

Using the flow invariance of Liouville measure in SM_e, it is direct to check that (see the proof of Theorem 1 in [DyGu1, Section 5.1])

$$\mathrm{Vol}(K) = 0 \iff \mathrm{Vol}(SM_e \cap (\Gamma_- \cup \Gamma_+)) = 0, \tag{2.7}$$

where the volume is taken with respect to the Liouville measure.

The hyperbolicity of the trapped set K is defined in the Introduction, and there is a flow-invariant continuous splitting of $T^*_K(SM)$ dual to (1.3), defined as follows: for all $y \in K$, $T^*_y(SM) = E^*_0(y) \oplus E^*_s(y) \oplus E^*_u(y)$ where

$$E^*_u(E_u \oplus \mathbb{R}X) = 0, \quad E^*_s(E_s \oplus \mathbb{R}X) = 0, \quad E^*_0(E_u \oplus E_s) = 0.$$

We note that $E^*_0 = \mathbb{R}\alpha$ where α is the Liouville 1-form.

2.3. **Stable and unstable manifolds.** Let us recall a few properties of flows with hyperbolic invariant sets; we refer to Hirsch-Palis-Pugh-Shub [HPPS, Sections 5 and 6], Bowen-Ruelle [BoRu], and Katok-Hasselblatt [HaKa, Chapters 17.4, 18.4] for details. For each point $y \in K$, there exist *global stable and unstable manifolds* $W_s(y)$ and $W_u(y)$ defined by

$$W_s(y) := \{y' \in S\hat{M}^\circ; d(\varphi_t(y), \varphi_t(y')) \to 0, t \to +\infty\},$$
$$W_u(y) := \{y' \in S\hat{M}^\circ; d(\varphi_t(y), \varphi_t(y')) \to 0, t \to -\infty\},$$

which are smooth injectively immersed connected manifolds. There are *local stable/unstable manifolds* $W_s^\epsilon(y) \subset W_s(y)$, $W_u^\epsilon(y) \subset W_u(y)$ which are properly embedded disks containing y, defined by

$$W_s^\epsilon(y) := \{y' \in W_s(y);\ \forall t \geq 0,\ d(\varphi_t(y), \varphi_t(y')) \leq \epsilon\},$$
$$W_u^\epsilon(y) := \{y' \in W_u(y);\ \forall t \geq 0,\ d(\varphi_{-t}(y), \varphi_{-t}(y')) \leq \epsilon\},$$

for some small $\epsilon > 0$,

$$\varphi_t(W_s^\epsilon(y)) \subset W_s^\epsilon(\varphi_t(y)) \text{ and } \varphi_{-t}(W_u^\epsilon(y)) \subset W_u^\epsilon(\varphi_{-t}(y)),$$
$$T_y W_s^\epsilon(y) = E_s(y), \text{ and } T_y W_u^\epsilon(y) = E_u(y).$$

The regularity of $W_u(y)$ and $W_s(y)$ with respect to y is Hölder. We also define

$$W_s(K) := \cup_{y\in K} W_s(y), \quad W_u(K) := \cup_{y \in K} W_u(y),$$
$$W_s^\epsilon(K) := \cup_{y\in K} W_s^\epsilon(y), \quad W_u^\epsilon(K) := \cup_{y \in K} W_u^\epsilon(y).$$

The incoming/outgoing tails are exactly the global stable/unstable manifolds of K.

Lemma 2.2. *If the trapped set K is hyperbolic, then the following equalities hold:*

$$\Gamma_- = W_s(K), \quad \Gamma_+ = W_u(K).$$

Proof. By (2.5), $W_s(K) \subset \Gamma_-$ and $W_u(K) \subset \Gamma_+$. Then $W_s^\epsilon(K) \cap W_u^\epsilon(K) \subset K$, and thus K has a local product structure in the sense of [HaKa, Definition p. 272]. Now from this local product structure, [HPPS, Lemma 3.2 and Theorem 5.2] show that for any $\epsilon > 0$ small, there is an open neighborhood V_K of K such that

$$\{y \in SM_e; \varphi_t(y) \in V_K, \forall t \geq 0\} \subset W_s^\epsilon(K), \tag{2.8}$$

which means that any trajectory which stays close enough to K is on the local stable manifold. The same holds for negative time and unstable manifold. A point $y \in \Gamma_-$ satisfies $d(\varphi_t(y), K) \to 0$ as $t \to +\infty$; thus for t large enough the orbit reaches V_K, and thus $\varphi_t(y) \in W_s^\epsilon(K)$ for $t \gg 1$ large. We conclude that $y \in W_s(K)$. Similarly $\Gamma_+ \subset W_u(K)$, and this achieves the proof. ☐

For each $y_0 \in K$, we extend the notion of stable subspace (resp. unstable subspace) to points on the $W_s^\epsilon(y_0)$ submanifold (resp. $W_u^\epsilon(y_0)$ submanifold), by

$$E_-(y) := T_y W_s^\epsilon(y_0) \text{ if } y \in W_s^\epsilon(y_0), \quad E_+(y) := T_y W_u^\epsilon(y_0) \text{ if } y \in W_u^\epsilon(y_0).$$

These subbundles can be extended to subbundles $E_\pm \subset T_{\Gamma_\pm} SM_e$ over $\Gamma_\pm$ in a flow-invariant way, and we can define the subbundles $E_\pm^* \subset T^*_{\Gamma_\pm} SM_e$ by

$$E_\pm^*(E_\pm \oplus \mathbb{R}X) = 0 \text{ over } \Gamma_\pm. \tag{2.9}$$

By [DyGu2, Lemma 2.10], these subbundles are continuous, are invariant by the flow, and satisfy the following properties (we use the Sasaki metric on SM):

(1) there exists $C > 0, \gamma > 0$ such that for all $y \in \Gamma_\pm$ and $\xi \in E_\pm^*(y)$, then

$$||d\varphi_t^{-1}(y)^T \xi|| \leq Ce^{-\gamma|t|}||\xi||, \qquad \mp t > 0; \tag{2.10}$$

(2) for $(y, \xi) \in T^*_{\Gamma_\pm} SM_e$ such that $\xi \notin E_\pm^*$ and $\xi(X) = 0$, then

$$||d\varphi_t^{-1}(y)^T\xi|| \to \infty \text{ and } \frac{d\varphi_t^{-1}(y)^T\xi}{||d\varphi_t^{-1}(y)^T\xi||} \to E_\mp^*|_K \text{ as } t \to \mp\infty; \tag{2.11}$$

(3) the bundles $E_\pm^*$ extend E_s^* and E_u^* in the sense that $E_-^*|_K = E_s^*$ and $E_+^*|_K = E_u^*$.

The dependence of $E^*_\pm(y)$ with respect to y is only Hölder continuous. The bundles $E^*_\pm$ can be thought of as conormal bundles to $\Gamma_\pm$ (this set is a union of smooth leaves parametrized by the set K). The differential of the flow $d\varphi_t$ is exponentially contracting on each fiber $E_-(y)$, and the proof of Klingenberg [Kl, Proposition p.6] shows

$$\varphi_t \text{ has no conjugate points } \Longrightarrow E_- \cap \mathcal{V} = \{0\}, \tag{2.12}$$

where we recall that $\mathcal{V} = \ker \pi_0$ is the vertical bundle. Similarly, $E_+ \cap \mathcal{V} = \{0\}$ in that case. These properties imply the following lemma.

Lemma 2.3. *If (M, g) has hyperbolic trapped set, strictly convex boundary, and no conjugate points, we can choose $\epsilon > 0$ small enough in Section 2.1 so that the extension (M_e, g) has no conjugate points.*

Proof. Indeed if it were not the case, there would be (by compactness) a sequence of points $(x_n, v_n) \in SM_e \setminus SM$ converging to $(x, v) \in \partial_- SM \cup \partial_0 SM$ and $(x'_n, v'_n) \in SM_e$ converging to $(x', v') \in SM$, and geodesics γ_n passing through (x_n, v_n) and (x'_n, v'_n), with x_n and x'_n being conjugate points for the flow of the extension of g. Note that $(x, v) = (x', v')$ is prevented by strict convexity of ∂M. By compactness, if the length of γ_n is bounded, we deduce that x, x' are conjugate points on M, which is not possible by assumption. There remains the case where the length of γ_n is not bounded, we can take a subsequence so that the length $t_n \to +\infty$. Then $(x, v) \in \Gamma_-$, and there is $w_n \in \mathcal{V} = \ker d\pi_0$ of unit norm for the Sasaki metric such that $d\varphi_{t_n}(x_n, v_n) \cdot w_n \in \mathcal{V}$. We can argue as in the proof of [DyGu2, Lemma 2.11]: by hyperbolicity of the flow on K, for n large enough, $d\varphi_{t_n}(x_n, v_n) \cdot w_n$ will be in an arbitrarily small conic neighborhood of E_+; thus it cannot be in the vertical bundle $\mathcal{V}$. This completes the argument. $\square$

Finally, let us denote by

$$\iota_\pm : \partial_\pm SM \to SM_e, \quad \iota : \partial SM \to SM_e, \tag{2.13}$$

the inclusion map, and define

$$E^*_{\partial,\pm} := (d\iota_\pm)^T E^*_\pm \subset T^*(\partial_\pm SM). \tag{2.14}$$

2.4. **Escape rate.** An important quantity in the study of open dynamical systems is the *escape rate*, which measures the amount of mass not escaping for long time. This quantity was studied for hyperbolic dynamical systems by Bowen-Ruelle, Young [BoRu, Yo]. First we define the *non-escaping mass function* $V(t)$ as follows:

$$\begin{gathered} V(t) := \mathrm{Vol}(\mathcal{T}_+(t)), \text{ with} \\ \mathcal{T}_\pm(t) := \{y \in SM;\ \varphi_{\pm s}(y) \in SM \text{ for } s \in [0, t]\}, \end{gathered} \tag{2.15}$$

and Vol being the volume with respect to the Liouville measure $d\mu$ defined in (2.2). The *escape rate* $Q \leq 0$ measures the exponential rate of decay of $V(t)$

$$Q := \limsup_{t \to +\infty} \frac{1}{t} \log V(t). \tag{2.16}$$

Notice that, since φ_t preserves the Liouville measure in SM, we have

$$\mathrm{Vol}(\mathcal{T}_+(t)) = \mathrm{Vol}(\mathcal{T}_-(t))$$

since the second set is the image of the first set by φ_t. Consequently, we also have $Q = \limsup_{t\to+\infty} \frac{1}{t}\log \mathrm{Vol}(\mathcal{T}_-(t))$. We define J_u the unstable Jacobian of the flow

$$J_u(y) := -\partial_t(\det d\varphi_t(y)|_{E_u(y)})|_{t=0},$$

where the determinant is defined using the Sasaki metric (to choose orthonormal bases in E_u). The *topological pressure* of a continuous function $f : K \to \mathbb{R}$ with respect to φ_t can be defined by the variational formula $P(f) := \sup_{\nu\in\mathrm{Inv}(K)}(h_\nu(\varphi_1) + \int_K f\,d\nu)$ where $\mathrm{Inv}(K)$ is the set of φ_t-invariant Borel probability measures and $h_\nu(\varphi_1)$ is the measure theoretic entropy of the flow at time 1 with respect to ν (e.g., $P(0)$ is just the topological entropy of the flow).

We gather two results of Young [Yo, Theorem 4] and Bowen-Ruelle [BoRu, Theorem 5] on the escape rate in our setting.

Proposition 2.4. *If M has strictly convex boundary, each connected component of M has non-empty boundary, and the trapped set K is hyperbolic, then the escape rate Q is negative and given by the topological pressure of the unstable Jacobian*

$$Q = P(J_u). \tag{2.17}$$

Proof. Formula (2.17) is proved by Young [Yo, Theorem 4] and follows directly from the volume lemma of Bowen-Ruelle [BoRu]. The pressure $P(J_u)$ of the unstable Jacobian J_u for φ_1 on K is equal to the pressure $P(J_u|_\Omega)$ of J_u for φ_1 on the non-wandering set $\Omega \subset K$ of φ_1; see [Wa, Corollary 9.10.1]. By the spectral decomposition of hyperbolic flows [HaKa, Theorem 18.3.1 and Exercise 18.3.7], the non-wandering set Ω decomposes into finitely many disjoint invariant topologically transitive sets $\Omega = \cup_{i=1}^N \Omega_i$ for φ_1. By [HaKa, Corollary 6.4.20], the periodic orbits of the flow are dense in Ω. By [HPPS, Proposition 7.2], each component Ω_i of Ω has local product structure, and thus, according to [HaKa, Theorem 18.4.1], it is locally maximal; each Ω_i is a basic set in the sense of Bowen-Ruelle [BoRu].

Then we can use the result of Bowen-Ruelle [BoRu, Theorem 5] which gives the following equivalence:

$$P(J_u|_{\Omega_i}) < 0 \iff \Omega_i \text{ is not an attractor for } \varphi_1 \iff \mathrm{Vol}(W_s(\Omega_i)) = 0, \tag{2.18}$$

where $W_s(\Omega_i) := \cup_{y\in\Omega_i} W_s(y)$ is the stable manifold of Ω_i. Suppose that one of the sets Ω_i is an attractor; then $W_s(\Omega_i)$ has positive Liouville measure, implying that $\mathrm{Vol}(\Gamma_-) > 0$, and thus $\mathrm{Vol}(K) > 0$ by (2.7). Since the Liouville measure is flow invariant on K, we have $\mathrm{Vol}(K) = \mathrm{Vol}(\Omega)$ by [Wa, Theorem 6.15], and thus there is Ω_j with positive Liouville measure. Now we can conclude with the argument of [BoRu, Corollary 5.7]: $\mathrm{Vol}(W_s(\Omega_j)) > 0$ and $\mathrm{Vol}(W_u(\Omega_j)) > 0$ so that Ω_j is an attractor for both φ_1 and φ_{-1} by (2.18), and this implies that $W_u(\Omega_j) = \Omega_j$ (as an attractor of φ_1) and $W_u(\Omega_j)$ is open (as an attractor of φ_{-1}); thus Ω_j is a whole connected component of SM. But this connected component has a strictly convex boundary which does not intersect K, and thus we obtain a contradiction. We conclude that $Q = P(J^u|_\Omega) < 0$. □

This of course implies that $\mathrm{Vol}(\Gamma_- \cup \Gamma_+) = 0$. Near $\partial_\pm SM$, we have $\{\varphi_{\mp t}(y) \in SM; t \in [0,\epsilon), y \in \partial_\pm SM \cap \Gamma_\pm\} \subset \Gamma_\pm$, and since for U a small open neighborhood of $\partial_\pm SM \cap \Gamma_\pm$ the map

$$(t, y) \in [0,\epsilon) \times U \mapsto \varphi_{\mp t}(y) \in SM$$

is a smooth diffeomorphism onto its image (the vector field X is transverse to $\partial_\pm SM$ near $\Gamma_\pm$ by (2.6)), we get

$$\mathrm{Vol}_{\partial SM}(\Gamma_\pm \cap \partial_\pm SM) = 0, \tag{2.19}$$

where the measure on ∂SM is the Riemannian measure induced by the Sasaki metric.

The flow on SM_e shares the same properties as on SM, and the trapped set on SM and on SM_e are the same; the discussion above holds as well for SM_e, and in particular

$$Q = \limsup_{t\to+\infty} \frac{1}{t} \log \mathrm{Vol}(\{y \in SM_e;\ \varphi_{\pm s}(y) \in SM_e \text{ for } s \in [0,t]\}) < 0. \tag{2.20}$$

2.5. **Santalo formula.** There is a measure on ∂SM which comes naturally when considering geodesic flow in SM, we denote it $d\mu_\nu$, and it is given by

$$d\mu_\nu(x,v) := |g_x(v,\nu)\, \iota^* d\mu(x,v)|, \tag{2.21}$$

where ν is the inward unit normal vector field to ∂M in M, ι is defined in (2.13), and $d\mu$ is Liouville measure (2.2). This measure is also equal to $|\iota^*(i_X d\mu)|$. When $\mathrm{Vol}(\Gamma_- \cup \Gamma_+) = 0$, then (2.19) holds and we can apply Santalo formula [Sa] to integrate functions in SM; this gives us the following: for all $f \in L^1(SM)$

$$\int_{SM} f d\mu = \int_{\partial_- SM\setminus\Gamma_-} \int_0^{\ell_+(x,v)} f(\varphi_t(x,v))dt\, d\mu_\nu(x,v) \tag{2.22}$$

with ℓ_+ defined in (2.3). Extending f to $S\hat{M}$ by 0 in $S\hat{M} \setminus SM$, (2.22) can also be rewritten as

$$\int_{SM} f d\mu = \int_{\partial_- SM\setminus\Gamma_-} \int_{\mathbb{R}} f(\varphi_t(x,v))\, dt\, d\mu_\nu(x,v). \tag{2.23}$$

3. The scattering map and lens equivalence

In the setting of a compact Riemannian manifold (M,g) with strictly convex boundary ∂M, we define the *scattering map* by

$$S_g : \partial_- SM \setminus \Gamma_- \to \partial_+ SM \setminus \Gamma_+, \quad S_g(x,v) := \varphi_{\ell_+(x,v)}(x,v), \tag{3.1}$$

where $\ell_+(x,v)$ is the length of the geodesic $\pi_0(\cup_{t\in\mathbb{R}}\varphi_t(x,v))) \cap M$, as defined in (2.3).

Definition 3.1. *Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds with the same boundary and such that $g_1 = g_2$ on $T\partial M_1 = T\partial M_2$ and the boundary is strictly convex for both metrics. Let ν_i be the inward pointing unit normal vector field on ∂M_i and let $\Gamma_-^i \subset SM_i$ be the incoming tail of the flow for g_i. Let $\alpha : \partial SM_1 \to \partial SM_2$ be given by*

$$\alpha(x, v + t\nu_1) = (x, v + t\nu_2), \quad \forall (v,t) \in T_x\partial M_1 \times \mathbb{R},\ |v|^2_{g_1} + t^2 = 1. \tag{3.2}$$

Then (M_1,g_1) and (M_2,g_2) are said to be scattering equivalent *if*

$$\alpha(\Gamma_-^1 \cap \partial SM_1) = \Gamma_-^2 \cap \partial SM_2, \text{ and } \alpha \circ S_{g_1} = S_{g_2} \circ \alpha \text{ on } \partial SM_1 \setminus \Gamma_-^1.$$

Finally g_1 and g_2 are said to be lens equivalent *if they are scattering equivalent and for any $(x,v) \in \partial_- SM_1 \setminus \Gamma_-^1$, the length $\ell_+^1(x,v)$ of the geodesic generated by (x,v) in M_1 for g_1 is equal to the length $\ell_+^2(\alpha(x,v))$ of the geodesic generated by $\alpha(x,v)$ in M_2 for g_2.*

Let us show that for the case of surfaces, if K is hyperbolic and g has no conjugate points, then S_g determines the set $E^*_{\partial,\pm}$. This will be useful in Theorem 6.

Lemma 3.2. *Let (M,g) be a surface with strictly convex boundary. Assume that K is hyperbolic and that the metric has no conjugate points. Then the scattering map S_g determines $E^*_{\partial,\pm}$.*

Proof. All points in $\Gamma_+ \cap \partial SM$ are in some unstable leaf $W_u(p)$ for some $p \in K$. The unstable leaves are one-dimensional manifolds injectively immersed in SM_e, and they intersect ∂SM in a set of measure 0 in ∂SM. Above a point $y \in W_u(p) \cap \partial_- SM$, the fiber $E^*_{+,\partial}(y)$ is exactly one-dimensional since one has $T_y SM = \mathbb{R}X \oplus \mathcal{V} \oplus E_+(y)$ where $\mathcal{V} = \ker d\pi_0$ is the vertical bundle which is also tangent to ∂SM and $E^*_-(\mathcal{V}) \neq 0$ if there are no conjugate points (we refer the reader to the proof of Proposition 5.7 below for the discussion about that fact). Take a point $y \in W_u(p) \cap \partial_+ SM$ and a sequence $y_n \to y$ in $\partial_+ SM$ with $y_n \notin \Gamma_+$; then by compactness (by possibly passing to a subsequence) $z_n := S_g^{-1}(y_n)$ is converging to z in $\Gamma_- \cap \partial SM$ with $t_n := \ell_+(z_n) \to \infty$. We can write $S_g(z_n) = \varphi_{\ell_+(z_n)}(z_n)$. By Lemma 2.11 in [DyGu2] (in particular its proof), if $\xi_n \in T^*_{z_n} SM$ satisfies $\xi_n(X) = 0$ and $\mathrm{dist}(\xi_n/||\xi_n||, E^*_-) > \epsilon$ for some fixed $\epsilon > 0$, then $(d\varphi_{t_n}(z_n)^{-1})^T \xi_n / ||(d\varphi_{t_n}(z_n)^{-1})^T \xi_n||$ tends to $E_+(y)^* \cap S^*(SM)$. Then we compute for $w_n \in T_{y_n}(\partial SM)$

$$dS_g^{-1}(y_n).w_n = X(z_n) d\ell_-(y_n).w_n + d\varphi_{-t_n}(y_n).w_n,$$

and if $\xi_n \in T^*_{z_n}(\partial SM)$, we can define uniquely $\xi_n^\sharp \in T^*_{z_n} SM$ by $\xi_n^\sharp(X) = 0$ and $\xi_n^\sharp \circ d\iota = \xi_n$ (ι is defined in 2.13) so that $(dS_g(z_n)^{-1})^T \xi_n = (d\varphi_{t_n}(z_n)^{-1})^T \xi_n^\sharp$. We conclude that

$$(dS_g(z_n)^{-1})^T \xi_n / ||(dS_g(z_n)^{-1})^T \xi_n|| \to E^*_+(y), \quad n \to +\infty,$$

if ξ_n is such that $\mathrm{dist}(\xi_n^\sharp/||\xi_n^\sharp||, E^*_-) > \epsilon$. For instance, we can take ξ_n to be of norm 1 and in the annihilator of $\mathcal{V}$ in $T^*\partial SM$. Then the desired condition is satisfied, and this shows that we can recover $E^*_+(y)$ from S_g. The same argument with S_g^{-1} instead of S_g shows that S_g determines E^*_-. This ends the proof. ◻

We can define the *scattering operator* as the pullback by the inverse scattering map

$$\mathcal{S}_g : C_c^\infty(\partial_- SM \setminus \Gamma_-) \to C_c^\infty(\partial_+ SM \setminus \Gamma_+), \quad \mathcal{S}_g \omega_- = \omega_- \circ S_g^{-1}. \tag{3.3}$$

Lemma 3.3. *For any $\omega_\mp \in C_c^\infty(\partial_\mp SM \setminus \Gamma_\mp)$, there exists a unique function $w \in C_c^\infty(SM \setminus (\Gamma_- \cup \Gamma_+))$ satisfying*

$$Xw = 0, \quad w|_{\partial_\mp SM} = \omega_\mp, \tag{3.4}$$

and this solution satisfies $w|_{\partial_+ SM} = \mathcal{S}_g \omega_-$ (resp. $w|_{\partial_- SM} = \mathcal{S}_g^{-1} \omega_+$). The function w extends smoothly to SM_e in a way that $Xw = 0$, this defines two bounded operators

$$\mathcal{E}_\mp : C_c^\infty(\partial_\mp SM \setminus \Gamma_\mp) \to C^\infty(SM_e), \quad \mathcal{E}_\mp(\omega_\mp) := w, \tag{3.5}$$

which satisfy the identity $\mathcal{E}_+ \mathcal{S}_g = \mathcal{E}_-$.

Proof. The function $w = \mathcal{E}_\mp(\omega_\mp)$ is simply given by

$$w(x,v) = \omega_\mp(\varphi_{\ell_\mp(x,v)}(x,v)) \tag{3.6}$$

in SM, and is clearly unique in SM since it is constant on the flow lines. It is smooth in SM since $\ell_\pm$ is smooth when restricted to $\partial_\pm SM \setminus \Gamma_\pm$, by the strict convexity of ∂SM. Then $\mathcal{E}_\mp(\omega_\mp)$ can be extended in SM_e in a way that it is constant on the flow lines of X (i.e., $X\mathcal{E}_\mp(\omega_\mp) = 0$). The continuity and linearity of $\mathcal{E}_\pm$ is obvious, and the identity $\mathcal{E}_+\mathcal{S}_g = \mathcal{E}_-$ comes from uniqueness of w. Notice that $\mathrm{supp}(\mathcal{E}_\mp(\omega_\mp))$ is at positive distance from $\Gamma_- \cup \Gamma_+$ since $\omega_\mp$ has support not intersecting $\Gamma_\mp \cap \partial SM$. □

Denoting $\omega_\pm := \omega|_{\partial_\pm SM}$ if $\omega \in C_c^\infty(\partial SM \setminus (\Gamma_+ \cup \Gamma_-))$, we now define the space

$$C^\infty_{\mathcal{S}_g}(\partial SM) := \{\omega \in C_c^\infty(\partial SM \setminus (\Gamma_+ \cup \Gamma_-));\ \mathcal{S}_g\omega_- = \omega_+\}. \tag{3.7}$$

Using the strict convexity and fold theory, Pestov-Uhlmann [PeUh, Lemma 1.1.] prove[1]

$$\omega \in C^\infty_{\mathcal{S}_g}(\partial SM) \iff \exists w \in C_c^\infty(SM \setminus (\Gamma_- \cup \Gamma_+)),\ Xw = 0,\ w|_{\partial SM} = \omega. \tag{3.8}$$

Similarly to (3.7), we define the space

$$L^2_{\mathcal{S}_g}(\partial SM) := \{\omega \in L^2(\partial SM; d\mu_\nu);\ \mathcal{S}_g\omega_- = \omega_+\}. \tag{3.9}$$

We finally show the following lemma.

Lemma 3.4. *If $(\Gamma_+ \cup \Gamma_-) \cap \partial SM$ has measure 0 in ∂SM, the map $\mathcal{S}_g$ extends as a unitary map*

$$L^2(\partial_- SM, d\mu_\nu) \to L^2(\partial_+ SM, d\mu_\nu),$$

where $d\mu_\nu$ is the measure of (2.21).

Proof. Consider $\omega_-^1, \omega_-^2 \in C_c^\infty(\partial_- SM \setminus \Gamma_-)$ and ω_1, ω_2 their invariant extension as in (3.4). Then we have

$$\begin{aligned} 0 &= \int_{SM} Xw_1.\overline{w_2} + w_1.X\overline{w_2}\, d\mu = \int_{SM} X(w_1.\overline{w_2})d\mu \\ &= -\int_{\partial_- SM} \omega_-^1.\overline{\omega_-^2}\, |\langle X, N\rangle_S|\, d\mu_{\partial SM} + \int_{\partial_+ SM} \mathcal{S}_g\omega_-^1.\overline{\mathcal{S}_g\omega_-^2}\, |\langle X, N\rangle_S|\, d\mu_{\partial SM}, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_S$ is the Sasaki metric and N is the unit inward pointing normal vector field to ∂SM for S. But N is the horizontal lift of ν, and so $\langle X, N\rangle_S = \langle v, \nu\rangle_g$. This shows that $\mathcal{S}_g$ extends as an isometry by a density argument, and reversing the role of $\partial_- SM$ with $\partial_+ SM$ we see that $\mathcal{S}_g$ is invertible. □

[1] Their result is stated for a simple manifold, but the proof applies as well in our case since the analysis happens near $\partial_0 SM$ where the scattering map has the same properties as on a simple manifold by the strict convexity of ∂M.

4. Resolvent and boundary value problem

Most of this section is used to prove the results stated in the Introduction, except Section 4.3 which is only necessary for Theorems 2 and 4.

4.1. **Sobolev spaces and microlocal material.** For a closed manifold Y, the L^2-based Sobolev space of order $s \in \mathbb{R}$ is denoted $H^s(Y)$. If Z is a manifold with a smooth boundary, it can be extended smoothly across its boundary as a subset of a closed manifold Y of the same dimension; we denote by $H^s(Z)$ for $s \geq 0$ the L^2 functions on Z which admit an H^s extension to Y. The space $H^s_0(Z)$ is the closure of $C_c^\infty(Z^\circ)$ for the H^s norm on Y, and we denote by $H^{-s}(Y)$ the dual of $H^s_0(Y)$. We refer to [Ta, Chapters 3 to 5] for details and precise definitions. If Z is an open manifold or a manifold with boundary, we set $C^{-\infty}(Z)$ to be the set of distributions, defined as the dual of $C_c^\infty(Z^\circ)$. For $\alpha \geq 0$, the Banach space $C^\alpha(Z)$ is the space of α-Hölder functions. We will use the notion of a wave-front set of a distribution (see [Hö, Chap. 8]), the calculus of pseudo-differential operators (ΨDO in short); we refer the reader to the textbooks [GrSj, Zw]. We will also use the notion of support, singular support, and wave-front sets for a bounded operator $A : C_c^\infty(Z^\circ) \to C^{-\infty}(Z)$ which are by definition the support, singular support, and wave-front set of its Schwartz distributional kernel. In the particular case of a pseudo-differential operator A, we just say that A is supported in U if its Schwartz kernel has support in $U \times U$, and since the wave-front set of A is a subset of the conormal bundle $N^*\Delta(Z \times Z)$ of the diagonal $\Delta(Z \times Z)$ of $Z \times Z$, it is more convenient to reduce it to a subset of $T^*Z \setminus \{0\}$ by using the identification $(z, \xi) \in T^*Z \to (z, \xi, z, -\xi) \in N^*\Delta(Z \times Z)$. This amounts to say that $\mathrm{WF}(A)$ is the complement in $T^*Z \setminus \{0\}$ to the set of points $(y_0, \xi_0) \in T^*Z \setminus \{0\}$ such that there is a small neighborhood U_{y_0} of y_0 and a cutoff function $\chi \in C_c^\infty(U_{y_0})$ equal to 1 near y_0 such that $A_\chi := \chi A \chi$ can be written under the form (U_{y_0} is identified to an open set of $\mathbb{R}^n$ using a chart)

$$A_\chi f(y) = \int_{U_{y_0}} \int_{\mathbb{R}^n} e^{i(y-y').\xi} \sigma(y, \xi) f(y') d\xi dy'$$

for some smooth symbol σ satisfying the estimate in a conic neighborhood V_{ξ_0} of ξ_0,

$$\forall N > 0, \forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha,\beta,N} > 0, \forall \xi \in V_{\xi_0}, \quad |\partial_y^\alpha \partial_\xi^\beta \sigma(y, \xi)| \leq C_{\alpha,\beta,N} \langle \xi \rangle^{-N}.$$

4.2. **Resolvent.** We first define the resolvent of the flow in the physical spectral region. We will use the convention that when f is a function supported in SM_e, we freely view it as a function on $S\hat{M}$ by extending it by 0 outside SM_e.

Lemma 4.1. *For* $\mathrm{Re}(\lambda) > 0$*, the resolvents* $R_\pm(\lambda) : L^2(SM_e) \to L^2(SM_e)$ *defined by the following formula:*

$$R_+(\lambda) f(y) = \int_0^\infty e^{-\lambda t} f(\varphi_t(y)) dt, \quad R_-(\lambda) f(y) = -\int_{-\infty}^0 e^{\lambda t} f(\varphi_t(y)) dt, \tag{4.1}$$

are bounded. They satisfy in the distribution sense in SM_e°

$$\begin{aligned} &\forall f \in L^2(SM_e),\ (-X \pm \lambda) R_\pm(\lambda) f = f, \\ &\forall f \in H^1_0(SM_e),\ R_\pm(\lambda)(-X \pm \lambda) f = f, \end{aligned} \tag{4.2}$$

and we have the adjointness property

$$R_-(\overline{\lambda})^* = -R_+(\lambda) \text{ on } L^2(SM_e). \tag{4.3}$$

The expression (4.1) *gives an analytic continuation of* $R_\pm(\lambda)$ *to* $\lambda \in \mathbb{C}$ *as operators*

$$R_\pm(\lambda) : C_c^\infty(SM_e^\circ \setminus \Gamma_\pm) \to C^\infty(SM_e) \tag{4.4}$$

satisfying $(-X \pm \lambda)R_\pm(\lambda)f = f$ *in* SM_e *and, for* $\chi_\pm \in C_c^\infty(SM_e \setminus \Gamma_\mp)$*, one has an analytic continuation of* $R_\pm(\lambda)\chi_\mp$ *and* $\chi_\pm R_\pm(\lambda)$ *as operators*

$$R_\pm(\lambda)\chi_\mp : L^2(SM_e) \to L^2(SM_e), \quad \chi_\pm R_\pm(\lambda) : L^2(SM_e) \to L^2(SM_e). \tag{4.5}$$

Proof. The proof of (4.2) is straightforward. The boundedness on L^2 follows from the inequality (using Cauchy-Schwarz)

$$\int_{SM_e} \Big| \int_0^{\pm\infty} e^{-\lambda|t|} f(\varphi_t(x,v))dt \Big|^2 d\mu \leq C_\lambda \int_{SM_e} \int_0^{\pm\infty} e^{-\mathrm{Re}(\lambda)|t|} |f(\varphi_t(x,v))|^2 dt d\mu,$$

for some $C_\lambda > 0$ depending on $\mathrm{Re}(\lambda)$, and a change of variable $y = \varphi_t(x,v)$ with the fact that the flow φ_t preserves the measure $d\mu$ in SM_e gives the result. The adjoint property (4.3) is also a consequence of the invariance of $d\mu$ by the flow in SM_e. The identity $(-X \pm \lambda)R_\pm(\lambda)f = f$ holds for any $f \in C_c^\infty(SM_e^\circ)$; thus for $f \in L^2(SM_e)$ and any $\psi \in C_c^\infty(SM_e^\circ)$ ($\langle\cdot,\cdot\rangle$ is the distribution pairing)

$$\begin{aligned} \langle(-X \pm \lambda)R_\pm(\lambda)f, \psi\rangle &= \langle R_\pm(\lambda)f, (X \pm \lambda)\psi\rangle \\ &= \lim_{n\to\infty} \langle R_\pm(\lambda)f_n, (X \pm \lambda)\psi\rangle = \lim_{n\to\infty} \langle f_n, \psi\rangle \end{aligned}$$

if $f_n \to f$ in L^2 with $f_n \in C_c^\infty(SM_e)$. Thus $(-X \pm \lambda)R_\pm(\lambda)f = f$ in $C^{-\infty}(SM_e^\circ)$. The other identity in (4.2) is proved similarly. The analytic continuation of $R_\pm(\lambda)$ in (4.4) is direct to check by using that if $f \in C_c^\infty(SM \setminus \Gamma_\pm)$, then $\mathrm{supp}(f \circ \varphi_t) \cap SM = \emptyset$ for all $t > T$ for some $T > 0$. This is the same argument for (4.5). ☐

We next show that the resolvent at the parameter $\lambda = 0$ can be defined if the non-escaping mass function $V(t)$ in (2.15) is decaying enough as $t \to \infty$. Let us first define the maximal Lyapunov exponent of the flow near $\Gamma_- \cup \Gamma_+$,

$$\nu_{\max} = \max(\nu_+, \nu_-), \quad \text{if } \nu_\pm := \limsup_{t\to+\infty} \frac{1}{t} \log \sup_{(x,v)\in\mathcal{T}_\pm(t)} \|d\varphi_{\pm t}(x,v)\|, \tag{4.6}$$

where $\mathcal{T}_\pm$ is defined in (2.15).

Proposition 4.2. *Let* $\alpha \in (0,1)$*,* $Q < 0$*, and let* $\nu_{\max}$ *be the maximal Lyapunov exponent defined in* (4.6).

(1) *The family of operators* $R_\pm(\lambda)$ *of Lemma* 4.1 *extends as a continuous family in* $\mathrm{Re}(\lambda) \geq 0$ *of operators bounded on the spaces*

$$R_\pm(\lambda) : L^\infty(SM_e) \to L^p(SM_e), \quad \text{if} \int_1^\infty V(t)t^{p-1}dt < \infty \quad \text{with } p \in [1,\infty), \tag{4.7}$$

$$R_\pm(\lambda) : L^p(SM_e) \to L^1(SM_e), \quad \text{if} \int_1^\infty V(t)t^{\frac{1}{p-1}}dt < \infty \quad \text{with } p \in (1,\infty), \tag{4.8}$$

(4.9)

$$R_\pm(\lambda) : C_c^\alpha(SM_e^\circ) \to H^s(SM_e), \quad \text{if } V(t) = \mathcal{O}(e^{Qt}) \text{ with } s < \min\Big(\alpha, \frac{-Q}{2\nu_{\max}}\Big),$$

where $V(t)$ is the function of (2.15). *This operator satisfies* $(-X\pm\lambda)R_\pm(\lambda)f=f$ *in the distribution sense in* SM_e° *when* $f\in L^p$ *and* $\int_1^\infty V(t)t^{\frac{1}{p-1}}dt<\infty$ *for* $p\in(1,\infty)$.

(2) *If* $\iota:\partial SM\to SM_e$ *is the inclusion map, then the operator* $\iota^*R_\pm(\lambda)$ *is a bounded operator on the spaces*

$$L^\infty(SM_e)\to L^p(\partial SM),\quad L^p(SM_e)\to L^1(\partial SM),\quad C_c^\alpha(SM_e^\circ)\to H^s(\partial SM) \tag{4.10}$$

under the respective conditions (4.7), (4.8), *and* (4.9) *on* V, p, *and* s; *the measure used on* ∂SM *is the measure* $d\mu_\nu$ *defined in* (2.21).

(3) *If the condition* (4.8) *is satisfied and* $f\in L^p(SM_e)$ *has* $\mathrm{supp}(f)\subset SM^\circ$, *then* $R_\pm(\lambda)f=0$ *in a neighborhood of* $\partial_\pm SM\cup\partial_0 SM$ *in* SM_e.

Proof. Let us denote $u_+(\lambda)=R_+(\lambda)f$ the L^2 function given by (4.1) for $\mathrm{Re}(\lambda)>0$, when $f\in L^\infty(SM_e)$. When $\int_1^\infty V(t)t^{p-1}dt<\infty$, the measure of $\Gamma_+\cup\Gamma_-$ is 0, and thus for $f\in L^\infty(SM_e)$ and $\lambda_0\in i\mathbb{R}$, the function $u_+(\lambda_0;x,v):=\int_0^\infty e^{-\lambda_0 t}f(\varphi_t(x,v))dt$ is finite outside a set of measure 0 since $\ell_+^e(x,v)$, defined as the length of the geodesic $\{\varphi_t(x,v);t\geq 0\}\cap SM_e$, is finite on $SM_e\setminus\Gamma_-$. If λ_n is any sequence with $\mathrm{Re}(\lambda_n)>0$ converging to λ_0, we have $u_+(\lambda_n)\to u_+(\lambda_0)$ almost everywhere in SM_e. Moreover $|u_+(\lambda_n)|\leq\int_0^\infty|f\circ\varphi_t|dt$ almost everywhere in SM_e for all $n>1$, and using Lebesgue theorem, we just need to prove that $||R_+(0)(|f|)||_{L^p}\leq C||f||_{L^\infty}$ to get that $||u_+(\lambda_0)||_{L^p}\leq C||f||_{L^\infty}$ and $u_+(\lambda_n)\to u_+(\lambda_0)$ in L^p. We have for almost every (x,v)

$$|u_+(0;x,v)|=\Big|\int_0^\infty f(\varphi_t(x,v))dt\Big|\leq||f||_{L^\infty}\,\ell_+^e(x,v). \tag{4.11}$$

Notice that, in view of our assumption on the metric in $SM_e\setminus SM$ we have $\ell_+(x,v)+L\geq\ell_+^e(x,v)\geq\ell_+(x,v)$ for some $L>0$ uniform in $(x,v)\in SM\setminus\Gamma_-$. Using the definition of $V(t)$ in (2.15), the volume of the set S_T of points $(x,v)\in SM_e$ such that $\ell_+^e(x,v)>T$ is smaller or equal to $2V(T-L)$ with L as above (independent of T). We apply the Cavalieri principle for the function $\ell_+^e(x,v)$ in $SM_e\setminus\Gamma_-$, and this gives

$$\int_{SM_e\setminus\Gamma_-}\ell_+^e(x,v)^pd\mu\leq C\Big(1+\int_1^\infty t^{p-1}V(t)dt\Big), \tag{4.12}$$

which shows (4.7) using (4.11). Notice that the same argument gives the same bound for the L^p norms of ℓ_-^e in $SM_e\setminus\Gamma_+$. The boundedness $L^p\to L^1$ of (4.8) is a direct consequence of (4.12) (with ℓ_- instead of ℓ_+) and the inequality

$$\int_{SM_e}\int_0^\infty|f(\varphi_t(x,v))|dtd\mu\leq\int_{SM_e}\int_0^\infty \mathbb{1}_{SM_e}(\varphi_{-t}(x,v))|f(x,v)|dtd\mu\leq||\ell_-^e||_{L^{p'}}||f||_{L^p}$$

for all $f\in C_c^\infty(SM_e^\circ)$ if $1/p'+1/p=1$. The fact that $\iota^*R_\pm(\lambda)f$ defines a measurable function in $L^1(\partial SM,d\mu_\nu)$ when $f\in L^1(SM_e)$ comes directly from Santalo formula (2.23) and Fubini theorem (note that ∂_0SM has zero measure in ∂SM). This shows the boundedness property of $\iota^*R_\pm(\lambda):L^p(SM_e)\to L^1(\partial SM,d\mu_\nu)$. Let us now prove the boundedness of the restriction $\iota^*R_\pm(0)f$ in L^p when $f\in L^\infty$. Since

$\ell_+^e(\varphi_t(x,v)) = (\ell_+^e(x,v)-t)_+$ for $t>0$, Santalo formula gives

$$\int_{\partial_- SM_e\setminus\Gamma_-}\int_0^{\ell_+^e(x,v)} \mathbb{1}_{[T,\infty)}(\ell_+^e(x,v)-t)dt|\langle v,\nu\rangle|d\mu_{\partial SM_e} = \mathrm{Vol}(S_T),$$

$$\int_{\partial_- SM\setminus\Gamma_-}\int_0^{\ell_+(x,v)} \mathbb{1}_{[T,\infty)}(\ell_+(x,v)-t)dtd\mu_\nu \leq \mathrm{Vol}(S_T)$$

for T large. From this, we get for large T

$$\int_{\partial_- SM\setminus\Gamma_-} \mathbb{1}_{[T,\infty)}(\ell_+(x,v))d\mu_\nu \leq 2V(T-L-1), \tag{4.13}$$

and using the Cavalieri principle, for any $\infty > p \geq 1$ there exists $C>0$ so that

$$\int_{\partial_- SM\setminus\Gamma_-} \ell_+(x,v)^p d\mu_\nu \leq C\Big(1+\int_1^\infty t^{p-1}V(t)dt\Big), \tag{4.14}$$

which shows, from (4.11) that $u_+|_{\partial SM\setminus\Gamma_-}\in L^p$ for any $1\leq p<\infty$ with a bound $\mathcal{O}(||f||_{L^\infty})$.

To prove that $(-X\pm\lambda)R_\pm(\lambda)f = f$ in $C^{-\infty}(SM_e^\circ)$ when $f\in L^p$ for $p\in(1,\infty)$ and the condition $\int_0^\infty V(t)t^{1/(p-1)}dt<\infty$ is satisfied, we take $\psi\in C_c^\infty(SM_e^\circ\setminus\Gamma_\mp)$ and write

$$\langle R_\pm(\lambda)f,(X\pm\lambda)\psi\rangle = \lim_{n\to\infty}\langle R_\pm(\lambda)f_n,(X\pm\lambda)\psi\rangle = \lim_{n\to\infty}\langle f_n,\psi\rangle = \langle f,\psi\rangle,$$

where $f_n\in C_c^\infty(SM_e^\circ\setminus\Gamma_\mp)$ converges in L^p to f; to obtain the second identity, we used (4.4) and the fact that $(-X\pm\lambda)R_\pm(\lambda)f_n = f_n$ in $SM_e^\circ\setminus\Gamma_\mp$.

Finally, we describe the case where the escape rate Q is negative (i.e., when $V(t)$ decays exponentially fast). We need to prove that u_+ is in $H^s(SM_e)$ for some $s>0$ if $f\in C_c^\alpha(SM_e^\circ)$. To prove that u_+ is $H^s(SM_e)$, it suffices to prove ([Hö, Chapter 7.9])

$$\int_{SM_e}\int_{SM_e}\frac{|u_+(y)-u_+(y')|^2}{d(y,y')^{N+2s}}dydy' < \infty$$

if $N=\dim(SM)$ and $d(y,y')$ denote the distance for the Sasaki metric on SM_e. Using that $f\in C^\alpha(SM_e)$, we have that for all $\alpha\geq\beta>0$ small, there exists $C>0$ such that for all $y,y'\in SM_e$, $\nu>\nu_{\max}$ and all $t\in\mathbb{R}$

$$|f(\varphi_t(y))-f(\varphi_t(y'))|\leq C||f||_{C^\beta}e^{\nu\beta|t|}d(y,y')^\beta;$$

thus for $\ell_+^e(y)<\infty$ and $\ell_+^e(y')<\infty$

$$|u_+(y)-u_+(y')|\leq C||f||_{C^\beta}\ell_+^e(y,y')e^{\nu\beta\ell_+^e(y,y')}d(y,y')^\beta,$$

where $\ell_+^e(y,y') := \max(\ell_+^e(y),\ell_+^e(y'))$. We then evaluate for $\beta-s>0$ and $\beta<\alpha$

$$\begin{aligned}\int\frac{|u_+(y)-u_+(y')|^2}{d(y,y')^{N+2s}}dydy' \leq & C_\beta||f||_{C^\beta}\int e^{2\nu\beta\ell_+(y,y')}d(y,y')^{2(\beta-s)-N}dydy'\\ \leq & 2C_\beta||f||_{C^\beta}\int_{\ell_+(y)>\ell_+(y')} e^{2\nu\beta\ell_+(y)}d(y,y')^{2(\beta-s)-N}dydy'\\ \leq & C_{s,\beta}||f||_{C^\beta}\int_{SM_e} e^{2\nu\beta\ell_+(y)}dy,\end{aligned}$$

and from the Cavalieri principle the last integral is finite if we choose $\beta>0$ small enough so that $0<s<\beta<-Q/2\nu$. Taking ν arbitrarily close to $\nu_{\max}$ gives that

$u_+ \in H^s(SM_e)$ if $s < -Q/2\nu_{\max}$. The same argument works for u_- and also for the boundary values $u_\pm|_{\partial SM}$.

The proof of part (3) is a direct consequence of the expression (4.1) for $R_\pm(\lambda)f$ since the positive (resp. negative) flow out of $\mathrm{supp}(f) \subset SM^\circ$ intersects ∂SM in a compact region of $\partial_+ SM$ (resp. $\partial_- SM$). $\square$

Remark. *Reasoning as in the proof of Proposition* 4.2*, it is straightforward by using Cauchy-Schwarz to check that if* $\mathrm{Vol}(K) = 0$*, then* $R_\pm(\lambda)$ *(restricted to functions on* SM*) extend continuously to* $\mathrm{Re}(\lambda) \geq 0$ *as a family of bounded operators*

$$\langle \ell_\pm \rangle^{-1/2-\epsilon} L^2(SM) \to \langle \ell_\pm \rangle^{1/2+\epsilon} L^2(SM)$$

for all $\epsilon > 0$ *where* $\ell_\pm$ *is the escape time function of* (2.3)*. This is comparable to the limiting absorption principle in scattering theory. The boundedness in Proposition* 4.2 *has the advantage that the resolvents map into Lebesgue spaces while functions in* $\langle \ell_\pm \rangle^{1/2+\epsilon} L^2(SM)$ *do not a priori have extensions as distributions in* SM*.*

The resolvents $R_\pm(0)$ have been defined under decay property of the non-escaping mass function. In the case where K is hyperbolic, we can actually say more about this operator.

Proposition 4.3 (Dyatlov-Guillarmou [DyGu2])**.** *Assume that the trapped set* K *is hyperbolic. There exists* $c > 0$ *such that for all* $s > 0$ *:*

(1) *The resolvents* $R_\mp(\lambda)$ *extend meromorphically to the region* $\mathrm{Re}(\lambda) > -cs$ *as a bounded operator*
$$R_\mp(\lambda) : H_0^s(SM_e) \to H^{-s}(SM_e)$$
with poles of finite multiplicity.

(2) *There is a neighborhood* $U_\mp$ *of* $E_\mp^*$ *such that for all pseudo-differential operators* $A_\mp$ *of order* 0 *with* $\mathrm{WF}(A_\mp) \subset U_\mp$ *and support in* SM_e°*,* $A_\mp R_\mp(\lambda)$ *maps continuously* $H_0^s(SM_e)$ *to* $H^s(SM_e)$*, when* λ *is not a pole.*

(3) *Assume that* λ_0 *is not a pole of* $R_\mp(\lambda)$*. Then the Schwartz kernel of* $R_\mp(\lambda_0)$ *is a distribution on* $SM_e^\circ \times SM_e^\circ$ *with wave-front set*
$$\mathrm{WF}(R_\mp(\lambda_0)) \subset N^*\Delta(SM_e^\circ \times SM_e^\circ) \cup \Omega_\pm \cup (E_\pm^* \times E_\mp^*), \tag{4.15}$$
where $N^*\Delta(SM_e^\circ \times SM_e^\circ)$ *is the conormal bundle to the diagonal* $\Delta(SM_e^\circ \times SM_e^\circ)$ *of* $SM_e^\circ \times SM_e^\circ$ *and*

$$\Omega_\pm := \{(\varphi_{\pm t}(y), (d\varphi_{\pm t}(y)^{-1})^T \xi, y, -\xi) \in T^*(SM_e^\circ \times SM_e^\circ);\ t \geq 0,\ \xi(X(y)) = 0\}.$$

Proof. Parts (1) and (2) are stated in [DyGu2, Proposition 6.1] (they actually follow from Lemmas 4.3 and 4.4 of that paper), while (3) is proved in [DyGu2, Lemma 4.5]. $\square$

We notice that a similar result in the closed Anosov setting was first proved in [DyZw] and was used for X-ray tomography in the work [Gu].

We can now combine Propositions 4.2 and 4.3 and obtain

Proposition 4.4. *Assume that the trapped set* K *is hyperbolic. Then we get for all* $p < \infty$*:*

(1) *The resolvent* $R_\pm(\lambda)$ *has no pole at* $\lambda = 0$*, and it defines for all* $s \in (0, 1/2)$ *a bounded operator* $R_\pm(0)$ *on the following spaces:*
$$R_\pm(0) : H_0^s(SM_e) \to H^{-s}(SM_e), \quad R_\pm(0) : L^\infty(SM_e) \to L^p(SM_e)$$

that satisfies $-XR_\pm(0)f = f$ *in the distribution sense. For* $f \in C^0(SM_e)$ *one has*

$$\forall y \in SM \setminus \Gamma_\mp, \quad (R_\pm(0)f)(y) = \int_0^{\pm\infty} f(\varphi_t(y))dt, \tag{4.16}$$

which is continuous in $SM \setminus \Gamma_\mp$ *and satisfies* $R_\pm(0)f|_{\partial_\pm SM} = 0$ *if* $\mathrm{supp}(f) \subset SM$.

2) *As a map* $H_0^s(SM_e) \to H^{-s}(SM_e)$ *for* $s \in (0, 1/2)$, *we have*

$$R_+(0) = -R_-(0)^*. \tag{4.17}$$

(3) *If* $f \in C_c^\infty(SM_e^\circ)$, *the function* $u_\pm := R_\pm(0)f$ *has wave-front set*

$$\mathrm{WF}(u_\pm) \subset E_\mp^*, \tag{4.18}$$

and the restriction $u_\pm|_{\partial SM} := \iota^* u_\pm$ *makes sense as a distribution satisfying*

$$u_\pm|_{\partial SM} \in L^p(\partial SM), \quad \mathrm{WF}(u_\pm|_{\partial SM}) \subset E_{\mp,\partial}^*. \tag{4.19}$$

(4) *If* $f \in C_c^\alpha(SM)$ *for* $\alpha > 0$, *then* $R_\pm(0)f \in H^s(SM_e)$ *and* $u_\pm|_{\partial SM} \in H^s(\partial_\pm SM)$ *for* $s < \min(\alpha, -Q/2\nu_{\max})$, *where* $\nu_{\max}$ *is the maximal Lyapunov exponent* (4.6) *and* $Q < 0$ *is the escape rate.*

Proof. Recall that for $\mathrm{Re}(\lambda) > 0$ we have for $f \in C_c^\infty(SM_e^\circ)$ and $\psi \in C_c^\infty(SM_e^\circ)$,

$$\langle R_+(\lambda)f, \psi\rangle = \int_0^\infty e^{-\lambda t}\langle f \circ \varphi_t, \psi\rangle\, dt.$$

By Proposition 4.2, then as $\lambda \to 0$ along any complex half-line contained in $\mathrm{Re}(\lambda) \geq 0$, we get $R_\pm(\lambda)f \to R_\pm(0)f$ in L^p (thus in the distribution sense). This implies that the extended resolvent $R_\pm(\lambda)$ of Proposition 4.3 cannot have poles at $\lambda = 0$ by the density of $C_c^\infty(SM_e^\circ)$ in any $H_0^s(SM_e)$. The same argument shows that $R_\pm(\lambda)$ is holomorphic in $\{\mathrm{Re}(\lambda) > Q\}$. The expression (4.16) comes from Proposition 4.2, which also implies the continuity of $R_\pm(0)f$ outside $\Gamma_\mp$ and its vanishing at $\partial_\pm SM$ when $\mathrm{supp}(f) \subset SM$.

Part (2) and (4.17) follow by continuity by taking $\lambda \to 0$ in (4.3) (and applying on $H_0^s(SM_e)$ functions instead of $L^2(SM_e)$).

For part (3), the wave-front set property of $u_\pm := R_\pm(0)f$ if $f \in C_c^\infty(SM_e^\circ)$ follows from the wave-front set description (4.15) of the Schwartz kernel of $R_\pm(0)$ and the composition rule of [Hö, Theorem 8.2.13]. The fact that $u_\pm$ restricts to ∂SM as a distribution which satisfies (4.19) comes from [Hö, Theorem 8.2.4] and the fact that $N^*(\partial SM) \cap E_\pm^* = 0$ if $N^*(\partial SM) \subset T^*(SM_e)$ is the conormal bundle to ∂SM (indeed a non-zero $\xi \in N^*(\partial SM)$ satisfies $\xi(X) \neq 0$ on $\partial SM \setminus \partial_0 SM$ but $E_\pm^*(X) = 0$). The $L^1(\partial SM)$ boundedness of the restriction follows from (4.10).

Part (4) follows from Proposition 4.2. □

Remark. *Since they will be useful for later purposes, we also want to make the following observations about the resolvents:*

(1) *If* $f \in C_c^\infty(SM_e^\circ)$ *has support in* SM, *then*

$$R_\pm(0)f \text{ vanishes to all order at } \partial_\pm SM. \tag{4.20}$$

(2) *The involution* $A : (x, v) \mapsto (x, -v)$ *on* SM_e *is a diffeomorphism and thus acts by pullback on distributions; it allows one to decompose distributions* u

on SM_e° into even and odd parts $u = u_{\rm ev} + u_{\rm od}$ where $u_{\rm od} := \frac{1}{2}(\mathrm{Id} - A^*)u$. If $f \in C_c^\infty(SM_e^\circ)$ is even, it is direct from the expression (4.16) that

$$(4.21)\quad (R_\pm(0)f)_{\rm ev} = \pm\tfrac{1}{2}(R_+(0) - R_-(0))f, \quad (R_\pm(0)f)_{\rm od} = \tfrac{1}{2}(R_+(0) + R_-(0))f,$$

and this extends by continuity to distributions. Similarly if f is odd, $(R_\pm(0)f)_{\rm ev} = \frac{1}{2}(R_+(0)+R_-(0))f$ and $(R_\pm(0)f)_{\rm od} = \pm\frac{1}{2}(R_+(0)-R_-(0))f$.

4.3. **Boundary value problem.** In this section, we extend the boundary value problem of Lemma 3.3 to the case of $L^2(\partial_\mp SM)$ boundary data. This analysis is only necessary to prove Theorems 2 and 4, but not Theorems 1 and 3 and Corollary 1.1.

We remark that each $w \in C^{-\infty}(SM_e^\circ)$ satisfying $Xw = 0$ has wave-front set $\mathrm{WF}(w) \subset \{\xi \in T^*(SM_e); \xi(X) = 0\}$ by ellipticity, and therefore the restriction $w|_{\partial_\pm SM}$ makes sense as a distribution by [Hö, Theorem 8.2.4] since $N^*(\partial_\pm SM) \cap \mathrm{WF}(w) = \emptyset$.

Lemma 4.5. *Assume that $\int_1^\infty tV(t)dt < \infty$ if V is the function (2.15). The map $\mathcal{E}_\mp$ of (3.5) can be extended as a bounded operator $L^2(\partial_\mp SM, d\mu_\nu) \to L^1(SM_e)$, and $\mathcal{E}_\mp(\omega_\mp)$ satisfies $X\mathcal{E}_\mp(\omega_\mp) = 0$ in the distribution sense and $w|_{\partial_\mp SM} = \omega_\mp$ for $\omega_\mp \in L^2(\partial_\mp SM, d\mu_\nu)$.*

Proof. Using the expression (3.6), Santalo formula and Cauchy-Schwarz inequality, we see that there is $C > 0$ such that for all $\omega_\mp \in C_c^\infty(\partial_\mp SM)$

$$||\mathcal{E}_\mp(\omega_\mp)||_{L^1(SM_e)} \leq C(||\omega_\mp||_{L^2(\partial_- SM, d\mu_\nu)} + ||\ell_\pm^e||_{L^2(\partial_- SM, d\mu_\nu)}||\omega_\mp||_{L^2(\partial_- SM, d\mu_\nu)}),$$

where we used that there is $C' > 0$ such that $|\ell_\mp^e(x,v)| \leq C'$ on $\partial_\mp SM$. Using (4.14), we deduce the announced boundedness. The fact that $X\mathcal{E}_\mp = 0$ on L^2 follows from the same identity on $C_c^\infty(\partial_\mp SM)$. ☐

Next, we provide an alternative expression of the operators $\mathcal{E}_\mp$. Let U be defined by $U = \cup_{-\infty<t<\delta}\varphi_t(\partial_- SM) \cap SM_e^\circ$ for some small $\delta > 0$ so that $\overline{U} \cap \Gamma_+ = \emptyset$. Then U is diffeomorphic to an open subset V of $(-\infty, \delta) \times \partial_- SM$ by the map $\theta : (t,y) \mapsto \varphi_t(y)$. Let $\chi \in C^\infty(\mathbb{R} \times \partial_- SM)$ be constant in the $\partial_- SM$ variable with $\chi(t,y) = \chi(t) = 1$ near $t \in \mathbb{R}^-$, $\chi(t) = 0$ in $(\delta/2, +\infty)$. Let $\psi_- = \chi \circ \theta^{-1}$ and extend it by 0 in $SM_e \setminus U$; then we claim that

$$(4.22)\qquad \forall \omega_- \in L^2(\partial_- SM), \quad \mathcal{E}_-(\omega_-) = \psi_-\mathcal{E}_-(\omega_-) + R_-(0)(\mathcal{E}_-(\omega_-)X\psi_-).$$

The right-hand side defines a bounded operator from $L^2(\partial_- SM) \to L^1(SM_e)$: indeed $\mathcal{E}_-(\omega_-)X\psi_- \in L^2(SM_e)$ by using the explicit formula $\mathcal{E}_-(\omega_-)(x,v) = \omega_-(\varphi_{\ell_-(x,v)}(x,v))$ valid in U (since $\overline{U} \cap \Gamma_+ = \emptyset$), and $R_-(0)(\mathcal{E}_-(\omega_-)X\psi_-) \in L^1(SM_e)$ by (4.8) if $\int_1^\infty tV(t)dt < \infty$. It then suffices to check (4.22) for $\omega_- \in C_c^\infty(\partial_- SM)$ and to use a density argument. But if $\omega_- \in C_c^\infty(\partial_- SM)$, the right-hand side w of (4.22) satisfies $Xw = 0$ and $w|_{\partial_- SM} = \omega_-$; thus (4.22) holds true by using Lemma 3.3. Switching the role of $\partial_- SM$ with $\partial_+ SM$ and using the flow in backward time, we also have

$$(4.23)\qquad \forall \omega_+ \in L^2(\partial_+ SM), \quad \mathcal{E}_+(\omega_+) = \psi_+\mathcal{E}_+(\omega_+) + R_+(0)(\mathcal{E}_+(\omega_+)X\psi_+),$$

where ψ_+ is defined similarly to ψ_- but has support in $\cup_{-\delta<t<\infty}\varphi_t(\partial_+ SM) \cap SM_e^\circ$.

In the case of a hyperbolic trapped set, using the resolvents $R_\pm(0)$, we are able to construct invariant distributions in SM with a prescribed value on $\partial_- SM$, and we can describe (partly) its singularities.

Proposition 4.6. *Assume that K is hyperbolic, then:*

(1) *Let $\omega_- \in L^2(\partial_- SM)$ with compact support in $\partial_- SM$, satisfying*

$$\mathrm{WF}(\omega_-) \subset E^*_{\partial,-},\ \mathrm{WF}(\mathcal{S}_g \omega_-) \subset E^*_{\partial,+}. \tag{4.24}$$

Then the function $\mathcal{E}_-(\omega_-) \in L^1(SM_e)$ has a wave-front set which satisfies

$$\mathrm{WF}(\mathcal{E}_-(\omega_-)) \cap T^*(SM_e \setminus K) \subset E^*_- \cup E^*_+, \tag{4.25}$$

and the restriction $\mathcal{E}_-(\omega_-)|_{\partial_- SM}$ makes sense as a distribution in $L^2(\partial_- SM)$ and is equal to $\mathcal{E}_-(\omega_-)|_{\partial_- SM} = \omega_-$.

(2) *For $s > 0$, let $\omega_- \in H^s(\partial_- SM)$ with compact support in $\partial_- SM$, then $\mathcal{E}_-(\omega_-) \in H^s_{\rm loc}(SM_e^\circ)$. If $\pi_0 : SM_e \to M_e$ is the projection on the base and π_{0*} the pushforward defined in* (5.9), *then*

$$\pi_{0*}(\mathcal{E}_-(\omega_-)) \in H^{s+1/2}_{\rm loc}(M_e^\circ). \tag{4.26}$$

Proof. Assume that $\omega_- \in H^s(\partial_- SM)$ for some $s \geq 0$, and ω_- has compact support. Using the diffeomorphism $\theta : V \to U$ introduced before the Proposition, we see that $(\psi_- \mathcal{E}_-(\omega_-))(\theta(t,y)) = \omega_-(y)\chi(t)$, which is in $H^s(U)$, and therefore $\psi_-\mathcal{E}_-(\omega_-) \in H^s_{\rm loc}(SM_e^\circ)$. Moreover $(X\psi_-)\mathcal{E}_-(\omega_-)$ is supported in SM° and belongs to $H^s_0(SM_e)$.

We start by proving (1) by using (4.22), (4.23), and the propagation of singularities. First we claim that $\mathrm{WF}(\psi_-\mathcal{E}_-(\omega_-)) \subset E^*_-$. In the decomposition $V \subset (-\infty, \delta) \times \partial_- SM$ of U induced by the flow, the wave-front set of $\psi_-\mathcal{E}_-(\omega_-)$ is included in $\{0\} \times E^*_{\partial,-} \subset T^*V$. The map $d\theta(0,y)^T$ maps the annihilator of $\mathbb{R}X_y$ to $\{0\} \times T^*(\partial_- SM)$ and since $d\theta(0,y)\cdot(u,v) = uX(y) + d\iota(y)\cdot v$ where ι is the inclusion map, we have $(d\theta(0,y)^{-1})^T E^*_{\partial,-}(y) = E^*_-(y)$. And since the bundle E^*_- is invariant by the flow, $d\theta(t,y)^T E^*_- = \{0\} \times E^*_{\partial,-}$, thus we deduce that $\mathrm{WF}(\psi_-\mathcal{E}_-(\omega_-)) \subset E^*_-$ and that $\pi(\mathrm{WF}(\psi_-\mathcal{E}_-(\omega_-)))$ is at positive distance from Γ_+ if $\pi : T^*(SM_e) \to SM_e$ is the canonical projection. Similarly $\mathrm{WF}(\mathcal{E}_-(\omega_-)X\psi_-) \subset E^*_-$ and $\pi(\mathrm{WF}(\mathcal{E}_-(\omega_-)X\psi_-)) \subset SM^\circ$ is at positive distance from Γ_+. We recall the propagation of singularities for real principal type operators (see [DyZw, Proposition 2.5]): let $\Phi_t : T^*(S\hat{M}) \to T^*(S\hat{M})$ be the symplectic lift of φ_t; if $Xu = f$, then for each $T > 0$

$$\Phi_{\mp T}(y,\xi) \notin \mathrm{WF}(u),\ \bigcup_{t=0}^{T} \Phi_{\mp t}(y,\xi) \cap \mathrm{WF}(f) = \emptyset \Longrightarrow (y,\xi) \notin \mathrm{WF}(u). \tag{4.27}$$

Putting $u = R_-(0)(\mathcal{E}_-(\omega_-)X\psi_-)$, we have $u = 0$ near $\partial_- SM$ and thus all points $(y,\xi) \notin E^*_-$ with $y \notin \Gamma_+$ are not in $\mathrm{WF}(u)$ by (4.27). This implies by (4.22) that

$$\mathrm{WF}(\mathcal{E}_-(\omega_-)) \cap T^*(SM_e \setminus \Gamma_+) \subset E^*_-. \tag{4.28}$$

Since $\mathcal{E}_+\mathcal{S}_g = \mathcal{E}_-$ on $L^2(\partial_- SM, d\mu_\nu)$ by Lemmas 3.3 and 4.5, we have $\mathcal{E}_-(\omega_-) = \mathcal{E}_+(\omega_+)$ with $\omega_+ = \mathcal{S}_g\omega_-$. By assumption (4.24) and doing the same reasoning as we did but using the flow in the reverse direction and (4.23), we obtain directly that

$$\mathrm{WF}(\mathcal{E}_-(\omega_-)) \cap T^*(SM_e \setminus \Gamma_-) \subset E^*_+,$$

which combined with (4.28) proves (4.25).

Next we show (2). Let $\omega_- \in H^s(\partial_- SM)$ be compactly supported, with $s > 0$, and assume that $\omega_+ := \mathcal{S}_g\omega_- \in H^s(\partial_- SM)$. By (2) in Proposition 4.3 applied to $R_\pm(0)(\mathcal{E}_\pm(\omega_\pm)X\psi_\pm)$, we obtain that $A_\pm\mathcal{E}_\pm(\omega_\pm) \in H^s(SM_e)$ if $A_\pm$ is any 0th order ΨDO with $\mathrm{WF}(A_\pm)$ contained in a small enough conic neighborhood $V_\pm$ of $E_\pm^*$. We write $w = \mathcal{E}_-(\omega_-) = \mathcal{E}_+(\omega_+)$, and then using (4.22) and (4.23), we deduce that if B_1 is any 0th order ΨDO with $\mathrm{WF}(B_1)$ contained in a small open conic neighborhood V_1 of $E_+^* \cup E_-^*$, then $B_1 w \in H^s(SM_e)$. By ellipticity, we also have $B_0 w \in C^\infty(SM_e^\circ)$ if B_0 is any 0th order ΨDO compactly supported in SM_e° with $\mathrm{WF}(B_0)$ contained outside a small open conic neighborhood V_0 of the characteristic set $\{\xi \in T^*(SM_e); \xi(X) = 0\}$. Therefore, it remains to prove that $B_2 w \in H^s(SM_e)$ if B_2 is any 0th order ΨDO supported in SM_e° with wave-front set contained in the region $V_2 := W_0 \backslash W_1$ where W_0 is a conic open neighborhood of $\overline{V_0}$ in $T^*(SM_e)\backslash\{0\}$ and W_1 is a conic neighborhood of $E_-^* \cup E_+^*$ so that $\overline{W}_1 \subset V_1$. But this property will follow from propagation of singularities applied to w. Indeed, let $(y, \xi) \in V_2$, then the following alternative holds:

(1) if $y \notin K$, there is $T > 0$ such that either $\psi_+(\Phi_T(y,\xi)) = 1$ or $\psi_-(\Phi_{-T}(y,\xi)) = 1$;
(2) if $y \in K$, by (2.11) there is $T > 0$ such that either $\Phi_{-T}(y,\xi) \in V_-$ or $\Phi_T(y,\xi) \in V_+$.

In both cases we can apply [DyZw, Proposition 2.5]: since we know that $\psi_\mp w \in H^s$ and $B_1 w \in H^s$, then $B_2 w \in H^s(SM_e)$. This concludes the proof that $w \in H^s(SM_e)$.

To conclude, the 1/2 gain in Sobolev regularity in (4.26) follows from the averaging lemma of Gérard-Golse [GeGo, Theorem 2.1]: indeed, the geodesic flow vector field, viewed as a first order differential operator satisfies the transversality assumption of Theorem 2.1 in [GeGo]. Thus, after extending slightly w in an open neighborhood W of SM_e so that $Xw = 0$ in W and $w \in H^s(W)$, the averaging lemma implies that its average in the fibers $\pi_{0*} w$ restricts to M_e as an $H^{s+1/2}_{\mathrm{loc}}$ function. $\square$

Combining Proposition 4.6 with (3.8), we obtain (using notation (3.9)) the following existence result for invariant distributions on SM with prescribed boundary values. This will be fundamental for the resolution of the lens rigidity for surfaces.

Corollary 4.7. *Assume that the trapped set K is hyperbolic. For any $\omega \in L^2_{S_g}(\partial SM)$ satisfying $\mathrm{WF}(\omega) \subset E_{\partial,-}^* \cup E_{\partial,+}^*$, there exists $w \in L^1(SM_e)$ such that the restriction $w|_{\partial SM}$ makes sense as a distribution and*

$$Xw = 0 \text{ in } SM_e^\circ, \quad w|_{\partial SM} = \omega,$$
$$\mathrm{WF}(w) \cap T^*(SM_e \setminus K) \subset E_-^* \cup E_+^*.$$

If $\omega \in H^s(\partial SM)$ for $s > 0$, then $w \in H^s(SM_e)$ and $\pi_{0*} w \in H^{s+1/2}_{\mathrm{loc}}(M_e)$.

Proof. We decompose $\omega = \omega_1 + \omega_2$ where $\omega_1 \in C^\infty_{S_g}(\partial SM)$ with $\mathrm{supp}(\omega_1) \subset \partial SM \setminus (\Gamma_- \cup \Gamma_+)$ and ω_2 supported near $\partial SM \cap (\Gamma_- \cup \Gamma_+)$. We apply (3.8) to ω_1, and this produces $w_1 \in C^\infty(SM)$ which is flow invariant in SM and with boundary value ω_1. Then it is not difficult to extend w_1 smoothly in SM_e in a way that $Xw_1 = 0$. Next, we apply Proposition 4.6 to $\omega_2|_{\partial_- SM}$, and this produces $w_2 = \mathcal{E}_-(\omega_2|_{\partial_- SM})$ satisfying $Xw_2 = 0$ in SM_e and $w_2|_{\partial_- SM} = \omega_2|_{\partial_- SM}$. We set $w = w_1 + w_2$, and the wave-front set property of w and the regularity of $\pi_{0*} w$ follow from Proposition 4.6. $\square$

5. X-ray transform and the operator Π

Most of this section is used to prove the results stated in the Introduction, except Section 5.4 that is used only to prove Theorems 2 and 4. We start by defining *the X-ray transform* as the map

$$I : C_c^\infty(SM \setminus \Gamma_-) \to C_c^\infty(\partial_-SM \setminus \Gamma_-), \quad If(x,v) := \int_0^\infty f(\varphi_t(x,v))dt.$$

From the expression (4.16), we observe that

$$If = (R_+(0)f)|_{\partial_-SM\setminus\Gamma_-}. \tag{5.1}$$

Then I can be extended to more general spaces. For instance, Santalo formula implies directly that as long as $\mathrm{Vol}(K) = 0$ (and no other assumption on K),

$$I : L^1(SM) \to L^1(\partial_-SM; d\mu_\nu).$$

For our purposes, as we shall see later, there is an important condition on the non-escaping mass function which allows one to use TT^* type arguments and relate I^*I to the spectral measure at 0 of the flow. This condition is

$$\exists p \in (2,\infty], \quad \int_1^\infty t^{\frac{p}{p-2}} V(t)dt < \infty, \tag{5.2}$$

if V is the function defined in (2.15). It is always satisfied if K is hyperbolic. We have the following lemma.

Lemma 5.1. *Assume that* (5.2) *holds for some* $p > 2$*, then the X-ray transform* I *extends boundedly as an operator*

$$I : L^p(SM) \to L^2(\partial_-SM, d\mu_\nu).$$

Proof. Let $f \in L^p(SM)$, then using Hölder with $\frac{1}{p'} + \frac{1}{p} = 1$ and $\frac{r}{p'} = \frac{p-1}{p-2} > 1$,

$$\begin{aligned}
&\int_{\partial_-SM} \Big| \int_0^{\ell_+(y)} f(\varphi_t(y))dt \Big|^2 d\mu_\nu(y) \\
&\leq \int_{\partial_-SM} \Big(\int_0^{\ell_+(y)} |f(\varphi_t(y))|^p dt \Big)^{2/p} \ell_+(y)^{2/p'} d\mu_\nu(y) \\
&\leq \Big(\int_{\partial_-SM} \int_0^{\ell_+(y)} |f(\varphi_t(y))|^p dt d\mu_\nu(y) \Big)^{2/p} ||\ell_+||^{2/p'}_{L^{2r/p'}(\partial_-SM, d\mu_\nu)} \\
&\leq ||f||^2_{L^p(SM)} ||\ell_+||^{2/p'}_{L^{2r/p'}(\partial_-SM, d\mu_\nu)},
\end{aligned}$$

where we have used Santalo formula (2.22) to pass to the integral over SM. Since $\ell_+ \in L^q(\partial_-SM, d\mu_\nu)$ when $\int_1^\infty t^{q-1}V(t)dt$ by (4.14), we deduce the result. □

Assume that $\int_1^\infty t^{p/(p-2)} V(t)dt < \infty$ for some $p \in (2,\infty)$. Note that by Sobolev embedding $I : H_0^s(SM) \to L^2(\partial_-SM, d\mu_\nu)$ is bounded if $s = \frac{n}{2} - \frac{n}{p}$ for the $p \in (2,\infty)$ of Lemma 5.1. Since $H^{-s}(SM)$ is defined as the dual of $H_0^s(SM)$ and $L^{p'}$ is dual to L^p for $p \in (2,\infty)$ if $1/p + 1/p' = 1$, the adjoint of I, denoted I^*, is bounded as operators (for s as above)

$$I^* : L^2(\partial_-SM, d\mu_\nu) \to L^{p'}(SM), \quad I^* : L^2(\partial_-SM, d\mu_\nu) \to H^{-s}(SM). \tag{5.3}$$

In fact, a short computation gives the following lemma.

Lemma 5.2. *If* (5.2) *holds true, then* $I^* = \mathcal{E}_-$.

Proof. Let $\omega_- \in C_c^\infty(\partial_- SM \setminus \Gamma_-)$; then $\mathcal{E}_-(\omega_-) \in C^\infty(SM)$, and its support does not intersect $\Gamma_- \cup \Gamma_+$. By Green's formula, we have for $f \in C_c^\infty(SM^\circ)$

$$\int_{SM} f\mathcal{E}_-(\omega_-)d\mu = \int_{SM} -X(R_+(0)f)\cdot \mathcal{E}_-(\omega_-)d\mu = \int_{\partial_- SM} If \cdot \omega_- d\mu_\nu.$$

Using the density of $C_c^\infty(SM^\circ)$ in $L^p(SM)$ and of $C_c^\infty(\partial_- SM \setminus \Gamma_-)$ in $L^2(\partial_- SM, d\mu_\nu)$, we get the desired result. □

To describe the properties of I and I^*, it is convenient to define the operator

$$\Pi := I^* I : L^p(SM) \to L^{p'}(SM), \quad \text{when} \quad \int_1^\infty t^{\frac{p}{p-2}} V(t)dt < \infty, \tag{5.4}$$

for $p \in (2,\infty)$. We prove the following relation between Π and the resolvents.

Lemma 5.3. *Assuming* (5.2), *the operator* $\Pi = I^* I$ *of* (5.4) *is equal on* $L^p(SM)$ *to*

$$\Pi = R_+(0) - R_-(0).$$

Proof. Since $\langle R_+(0)f, f\rangle = -\langle f, R_-(0)f\rangle$ by (4.17), it suffices to prove the identity

$$\langle I^* If, f\rangle_{L^2(\partial_- SM, d\mu_\nu)} = 2\langle R_+(0)f, f\rangle$$

for all $f \in C_c^\infty(SM \setminus (\Gamma_- \cup \Gamma_+))$ real valued. We write $u = R_+(0)f$ and compute, using Green's formula,

$$\int_{SM} u.f d\mu = -\int_{SM} u.Xud\mu = -\tfrac{1}{2}\int_{SM} X(u^2)d\mu = \tfrac{1}{2}\int_{\partial_- SM} u^2 d\mu_\nu,$$

and this achieves the proof. □

With the assumption of Lemma 5.3, the operator Π can also be extended as a bounded operator Π^e on SM_e,

$$\Pi^e := R_+(0) - R_-(0) : L^p(SM_e) \to L^1(SM_e), \tag{5.5}$$

satisfying $\Pi^e f|_{SM} = \Pi f$ for all $f \in L^p(SM)$ extended by 0 on $SM_e \setminus SM$. As above, one directly sees that $\Pi^e = I^{e*} I^e$ if we call $I^e : L^p(SM_e) \to L^2(\partial_- SM_e; |\langle v,\nu\rangle| d\mu_{\partial SM_e})$ the X-ray transform on SM_e, defined just as on SM and satisfying the same properties. In particular this shows that $\Pi^e : L^p(SM_e) \to L^{p'}(SM_e)$ is bounded. We summarize the discussion by the following proposition.

Proposition 5.4. *Assume that* (5.2) *holds for* $p \in (2,\infty)$. *Then we obtain*

(1) *the operator* Π^e *is bounded and self-adjoint as a map*

$$\Pi^e : L^p(SM_e) \to L^{p'}(SM_e), \quad 1/p + 1/p' = 1;$$

it satisfies for each $f \in L^p(SM_e)$

$$X\Pi^e f = 0 \tag{5.6}$$

in the distribution sense and $\Pi^e f$ *is given, outside a set of measure* 0, *by the formula*

$$\Pi^e f(x,v) = \int_{-\infty}^\infty f(\varphi_t(x,v))dt. \tag{5.7}$$

(2) *If the trapped set K is hyperbolic, the operator* $\Pi^e : H^s_0(SM_e) \to H^{-s}(SM_e)$ *is bounded for all* $s \in (0, 1/2)$. *For each* $f \in C_c^\infty(SM_e^\circ)$, *the expression* (5.7) *holds in* $SM_e \setminus (\Gamma_+ \cup \Gamma_-)$, *and we have* $\mathrm{WF}(\Pi^e f) \in E_-^* \cup E_+^*$ *and* $\Pi^e f \in H^s(SM_e)$ *for all* $s < -Q/2\nu_{\max}$ *with* $\nu_{\max}$ *defined in* (4.6) *and* $Q < 0$ *the escape rate. The restriction* $\omega := (\Pi^e f)|_{\partial SM}$ *makes sense as a distribution and belongs to* $L^2_{S_g}(\partial SM) \cap H^s(\partial SM)$ *for all* $s < -Q/2\nu_{\max}$, *and* $\omega_\pm := \omega|_{\partial_\pm SM}$ *has wave-front set*

$$\mathrm{WF}(\omega_\pm) \subset E^*_{\partial,\pm}. \tag{5.8}$$

Proof. The boundedness and the self-adjoint property have already been proved. The property (5.6) is clear from the properties of $R_\pm(0)$ given in (1) of Proposition 4.4. The expression of $\Pi^e f$ follows from (4.16) (and the proof of Proposition 4.2 for the extension to L^p functions). If $f \in C_c^\infty(SM_e^\circ)$, the wave-front set property of $\Pi^e f$ follows from (4.18), and the wave-front set and regularity properties (5.8) of the restrictions $\omega_\pm$ are consequences of Proposition 4.4. The fact that $\omega \in L^2_{S_g}(\partial SM)$ comes from Lemma 5.1, the identity $\omega_+ = \omega_- \circ S_g^{-1}$ on $\partial_+ SM \setminus \Gamma_+$ (which follows from (5.7)) and Lemma 3.4. The H^s regularity of $\Pi^e f$ and $\omega_\pm$ follows from (4) in Proposition 4.4. □

Next, we describe the kernel of Π^e restricted to smooth functions supported in SM.

Proposition 5.5. *Assume that K is hyperbolic. Let $f \in C^\infty(SM)$ be extended by 0 in $SM_e \setminus SM$; if $\Pi^e f = 0$ in SM, there exists $u \in C^\infty(SM)$ vanishing at ∂SM such that $Xu = f$. If f vanishes to infinite order at ∂M, then u also does so.*

Proof. First, the extension of f by 0 can be viewed as an element in $H^s_0(SM_e)$ for $s < 1/2$ with $\mathrm{WF}(f) \subset N^*(\partial SM)$ where $N^*(\partial SM)$ is the conormal bundle of ∂SM in SM_e. By the composition law of the wave-front set in [Hö, Theorem 8.2.13] and (4.15), we deduce that

$$\begin{gathered}\mathrm{WF}(R_\mp(0)f) \subset N^*\partial SM \cup E_\pm^* \cup B_\mp,\\ B_\pm := \cup_{t\geq 0}\{(\varphi_{\pm t}(y), (d\varphi_{\pm t}(y)^{-1})^T\xi) \in T^*SM_e^\circ; y \in \partial_0 SM, \xi \in N^*(\partial SM)\}.\end{gathered}$$

Clearly, by strict convexity, $B_\pm$ projects down to $M_e \setminus M^\circ$. Now, the function $\ell_\pm$ is smooth in $SM \setminus (\partial_0 SM \cup \Gamma_- \cup \Gamma_+)$, and from the expression (4.16) and the smoothness of f, we then get that $R_\mp(0)f$ is smooth in $SM \setminus (\partial_0 SM \cup \Gamma_\pm)$ and $(R_\pm(0)f)|_{\partial_\pm SM} = 0$. To analyze the regularity at $\partial_0 SM$, we decompose $f = f_{\rm ev} + f_{\rm od}$, and we get by (4.21) that $(R_\pm(0)f_{\rm ev})_{\rm ev} = \pm\frac{1}{2}\Pi^e f = 0$ and similarly $(R_\pm(0)f_{\rm od})_{\rm od} = 0$. Now the argument of [SaUh, Lemma 2.3] shows that $(R_\pm(0)f_{\rm ev})_{\rm od}|_{SM}$ and $(R_\pm(0)f_{\rm od})_{\rm ev}|_{SM}$ are both smooth near $\partial_0 SM$, which implies that $R_\pm(0)f$ is smooth near $\partial_0 SM$ in SM. Since $R_+(0)f = R_-(0)f$ if $\Pi^e f = 0$, we deduce that $(R_\pm(0)f)|_{SM} \in C^\infty(SM \setminus K)$ and $(R_\pm(0)f)|_{\partial SM} = 0$. From the wave-front set description above and the fact that $E_+^* \cap E_-^* = \{0\}$ over K, we conclude that $(R_\mp(0)f)|_{SM} \in C^\infty(SM)$. It just suffices to set $u = R_+(0)f$ to conclude the proof. The fact that f vanishes to all order at ∂SM implies that $R_\pm(0)f$ vanishes to all order at $\partial_\pm SM$ by (4.20), and thus u vanishes to all order at ∂SM. □

5.1. **The operators I_0 and Π_0.** Here we deal with the analysis of X-ray transform acting on functions on M. The projection $\pi_0 : SM_e \to M_e$ on the base induces a pullback map

$$\pi_0^* : C_c^\infty(M_e^\circ) \to C_c^\infty(SM_e^\circ), \quad \pi_0^* f := f \circ \pi_0,$$

and a pushforward map π_{0*} defined by duality

$$\pi_{0*} : C^{-\infty}(SM_e^\circ) \to C^{-\infty}(M_e^\circ), \quad \langle \pi_{0*} u, f\rangle := \langle u, \pi_0^* f\rangle. \tag{5.9}$$

Pushforward corresponds to integration in the fibers of SM_e when acting on smooth functions. The pullback by π_0 also makes sense on M and gives a bounded operator $\pi_0^* : L^p(M) \to L^p(SM)$ for all $p \in (1,\infty)$. When (5.2) holds for some $p \in (2,\infty)$, we define the X-ray transform on functions as the bounded operator (see Lemma 5.1)

$$I_0 := I\,\pi_0^* : L^p(M) \to L^2(\partial_- SM, d\mu_\nu). \tag{5.10}$$

The adjoint $I_0^* : L^2(\partial_- SM, d\mu_\nu) \to L^{p'}(M)$ is bounded if $1/p' + 1/p = 1$ and it is given by $I_0^* = \pi_{0*} I^*$. The operator Π_0 is simply defined as the bounded self-adjoint operator for $p \in (2,\infty)$ and $1/p' + 1/p = 1$,

$$\Pi_0 := I_0^* I_0 = \pi_{0*} \Pi\, \pi_0^* : L^p(M) \to L^{p'}(M). \tag{5.11}$$

Similarly, we define the self-adjoint bounded operator

$$\Pi_0^e := \pi_{0*} \Pi^e \pi_0^* = (I^e \pi_0^*)^* I^e \pi_0^* : L^p(M_e) \to L^{p'}(M_e). \tag{5.12}$$

We first want to mention some boundedness result which holds in a general setting (no condition on conjugate points is required) and says that Π_0 is always regularizing if $V(t)$ decays sufficiently.

Lemma 5.6. *Assume that* (5.2) *holds for $p > 2$, then I_0^* and I_0 are bounded as maps*

$$I_0^* : L^2(\partial_- SM, d\mu_\nu) \to H_{\rm loc}^{-\frac{n-1}{2} + \frac{n}{p}}(M^\circ), \quad I_0 : H_{\rm comp}^{\frac{n-1}{2} - \frac{n}{p}}(M^\circ) \to L^2(\partial_- SM, d\mu_\nu),$$

and the same property holds for I_0^e with M_e replacing M.

Proof. It suffices to prove the boundedness for I_0^*. By Sobolev embedding, $I^* : L^2(\partial_- SM, d\mu_\nu) \to H_{\rm loc}^{-\frac{n}{2} + \frac{n}{p}}(M^\circ)$ is bounded, and using Lemma 5.2, we have $XI^* = 0$ as operators. Then applying [GeGo, Theorem 2.1] as in the proof of Proposition 4.4, we gain 1/2 derivative in the Sobolev scale by applying π_{0*}. This ends the proof. □

If $V(t) = \mathcal{O}(t^{-\infty})$, the Sobolev exponents are $H_{\rm comp}^{-1/2-\epsilon}(M^\circ)$ and $H_{\rm loc}^{1/2+\epsilon}(M^\circ)$ for all $\epsilon > 0$, and if $K = \emptyset$, we get $I_0^* I_0 : H_{\rm comp}^{-1/2}(M^\circ) \to H_{\rm loc}^{1/2}(M^\circ)$. Following the method of [Gu], we prove the following proposition.

Proposition 5.7. *Assume that the geodesic flow on SM has no conjugate points and that the trapped set K is hyperbolic. The operator $\Pi_0^e = \pi_{0*} \Pi^e \pi_0^*$ is an elliptic pseudo-differential operator of order -1 in M_e°, with principal symbol $\sigma(\Pi_0^e)(x,\xi) = C_n |\xi|_g^{-1}$ for some constant $C_n \neq 0$ depending only on n.*

Proof. First we choose the extension (M_e, g) so that the geodesic flow on M_e has non-conjugate points. Once we know the wave-front set of the Schwartz kernels of the resolvent $R_\pm(0)$, the proof is very similar to Theorems 3.1 and 3.4 in [Gu]. Therefore we do not write all the details but refer to that paper where this is done carefully for Anosov flows. It suffices to analyze $\chi\Pi_0^e\chi'$ where $\chi,\chi'\in C_c^\infty(M_e^\circ)$ are arbitrary functions. Its Schwartz kernel is given by $\chi(x)\chi'(x')((\pi_0\times\pi_0)_*\Pi^e)(x,x')$ where $\Pi^e=R_+(0)-R_-(0)$ is identified with its Schwartz kernel. We write for $\epsilon\geq 0$ small

$$R_+(0)=\int_0^\epsilon e^{tX}dt+e^{\epsilon X}R_+(0),$$

where e^{tX} is the pullback by the flow at time t. Using (4.15) and the computation of $\mathrm{WF}(e^{\epsilon X})$ which follows from [Hö, Theorem 8.2.4], the composition law of wave-front set [Hö, Theorem 8.2.14] can be used as in the proof of [Gu, Theorem 3.1]: we obtain

$$\begin{aligned}&\mathrm{WF}(\pi_0^*(\chi)e^{\epsilon X}R_+(0)\pi_0^*(\chi'))\\ &\quad\subset\Big(\{(\varphi_t(y),(d\varphi_t(y)^{-1})^T\eta,y,-\eta);\ t\leq-\epsilon,\ \eta(X(y))=0\}\\ &\qquad\cup\{(\varphi_{-\epsilon}(y),\eta,y,-d\varphi_{-\epsilon}(y)^T\eta);\ (y,\eta)\in T^*(SM)\setminus\{0\}\}\\ &\qquad\cup(E_-^*\times E_+^*)\Big)\cap\{(y,\eta,y',\eta');(\pi_0(y),\pi_0(y'))\in U\times U'\},\end{aligned}$$

where $U:=\mathrm{supp}(\chi)$ and $U'=\mathrm{supp}(\chi')$; here the wave-front set of an operator means the wave-front set of the Schwartz kernel of the operator. By applying the rule of pushforward of wave-front sets (given for example in [FrJo, Proposition 11.3.3.]), we get $\mathrm{WF}(\pi_{0*}e^{\epsilon X}R_0\pi_0^*)\subset S_1\cup S_2\cup S_3$ where

$$\begin{aligned}&S_1:=\{(\pi_0(y),\xi,\pi_0(y'),\xi')\in T_0^*(U\times U);\ (y,d\pi_0(y)^T\xi,y',d\pi_0(y')^T\xi')\in E_-^*\times E_+^*\},\\ &S_2:=\{(\pi_0(\varphi_t(y)),\xi,\pi_0(y),\xi')\in T_0^*(U\times U);\ \exists\, t\leq-\epsilon,\exists\,\eta,\eta(X(y))=0,\\ &\qquad\qquad d\pi_0(y)^T\xi'=-\eta,\ d\pi_0(\varphi_t(y))^T\xi=(d\varphi_t(y)^{-1})^T\eta\},\\ &S_3:=\{(\pi_0(\varphi_{-\epsilon}(y)),\xi,\pi_0(y),\xi')\in T_0^*(U\times U);\ (d(\pi_0\circ\varphi_{-\epsilon})(y))^T\xi=-d\pi_0(y)^T\xi'\}\end{aligned}$$

if we set $T_0^*(U\times U):=T^*(U\times U)\setminus\{0\}$. As before, $\mathcal{V}=\ker d\pi_0\subset T(SM_e)$ is the vertical bundle, $\mathcal{H}$ is the horizontal bundle, and $\mathcal{V}^*,\mathcal{H}^*\subset T^*(SM_e)$ are their dual spaces defined by $\mathcal{H}^*(\mathcal{V})=0$ and $\mathcal{V}^*(\mathcal{H})=0$. By (2.12), the absence of conjugate points for the flow in M_e implies that $T(SM_e)=\mathbb{R}X\oplus\mathcal{V}\oplus E_\pm$ at $\Gamma_\pm$ and thus $E_\pm^*\cap\mathcal{H}^*=\{0\}$. This implies that $S_1=\emptyset$. Similarly, it is direct to see that $S_2=\emptyset$ is equivalent to the absence of conjugate points for the flow (see the proof of [Gu, Theorem 3.1] for details). The last part is S_3. The proof is exactly the same as in [Gu, Theorem 3.1], and thus we do not repeat the details: the projection of S_3 on $M_e^\circ\times M_e^\circ$ is contained in $\Delta_\epsilon(M_e^\circ\times M_e^\circ):=\{(x,x')\in M_e^\circ\times M_e^\circ;d_g(x,x')=\epsilon\}$ where d_g is the Riemannian distance. The operator $L_\epsilon=\int_0^\epsilon\pi_{0*}e^{tX}\pi_0^*dt$ is explicit for small $\epsilon>0$ and is given by

$$L_\epsilon f(x):=\int_0^\epsilon\int_{S_xM_e}f(\varphi_t(x,v))dS_x(v)dt,$$

where $dS_x(v)$ is the volume measure on the sphere S_xM_e. The Schwartz kernel of L_ϵ has singular support included in $\Delta_\epsilon(M_e^\circ\times M_e^\circ)\cup\Delta_0(M_e^\circ\times M_e^\circ)$. Thus, $\epsilon>0$ being chosen arbitrary but small, the kernel of Π_0 has singular support on the diagonal $\Delta_0(M_e^\circ\times M_e^\circ)$. Now the kernel $\psi(x,x')L_\epsilon(x,x')$ is that of an elliptic pseudo-differential operator of order -1 if $\psi\in C_c^\infty(M_e^\circ\times M_e^\circ)$ is supported close enough to the diagonal $\{x=x'\}$ and equal to 1 in a neighborhood of the diagonal: the analysis is purely local and exactly the same as in [PeUh, Lemma 3.1], which also shows that the symbol of this ΨDO is $C_n|\xi|_g^{-1}$ for some $C_n>0$. It is direct to see (from $R_+(0)^* = -R_-(0)$) that $\Pi_0^e = 2\pi_{0*}R_+(0)\pi_0^*$, and we have then proved the claim. ☐

Since the Schwartz kernel of Π_0^e on M° is the restriction of the kernel of Π^e to $M^\circ \times M^\circ$, we deduce that in the case of hyperbolic trapped set and no conjugate points, Lemma 5.6 gives that $\Pi_0^e : H^{-1/2}_{\rm comp}(M^\circ) \to H^{1/2}_{\rm loc}(M^\circ)$ and the TT^* argument shows that for any compact domain $\mathcal{O} \subset M^\circ$ with non-empty interior and smooth boundary, we have

$$I_0 : H^{-1/2}(\mathcal{O}) \to L^2(\partial_- SM; d\mu_\nu), \quad I_0^* : L^2(\partial_- SM; d\mu_\nu) \to H^{1/2}(\mathcal{O}). \tag{5.13}$$

We can use Proposition 5.7 to prove the regularity property on elements in $\ker I_0$.

Corollary 5.8. *Assume that the trapped set K is hyperbolic, and the metric has no conjugate points. Let $f_0 \in L^p(M) + H^{-1/2}_{\rm comp}(M^\circ)$ for some $p > 2$ satisfying $I_0 f_0 = 0$. Then $f_0 \in C^\infty(M)$ and f_0 vanishes to all order at ∂M.*

Proof. First, $I_0 f_0 = 0$ in $L^2(\partial_- SM; d\mu_\nu)$ implies that $I_0^e f = 0$ if $I_0^e = I^e \pi_0^*$ is the X-ray transform on functions on M_e and f_0 is extended by 0 in $M_e \setminus M$. Thus $\Pi_0^e f_0 = 0$ in M_e°. This implies, by ellipticity of Π_0^e in M_e° that f_0 is smooth, and since it is equal to 0 in $M_e^\circ \setminus M$, we deduce that f_0 vanishes to all orders at ∂M. ☐

5.2. **X-ray transform on symmetric tensors.** For any $m \in \mathbb{N}$, symmetric cotensors of order m on M_e° can be viewed as functions on SM_e° via the map

$$\pi_m^* : C_c^\infty(M_e^\circ, \otimes_S^m T^* M_e^\circ) \to C_c^\infty(SM_e^\circ), \quad (\pi_m^* f)(x, v) := f(x)(\otimes^m v).$$

The dual operator is defined by

$$\pi_{m*} : C^{-\infty}(SM_e^\circ) \to C^{-\infty}(M_e^\circ, \otimes_S^m T^* M_e^\circ), \quad \langle \pi_{m*} u, f\rangle := \langle u, \pi_m^* f\rangle.$$

To define the distribution pairing, we have used the natural scalar product on the bundle $\otimes_S^m T^* M_e^\circ$ induced by the metric g. Next, we define the operator $D := \mathcal{S} \circ \nabla : C_c^\infty(M_e^\circ, \otimes_S^m T^* M_e) \to C_c^\infty(M_e^\circ, \otimes_S^{m+1} T^* M_e^\circ)$ by composing the Levi-Cività connection ∇ with the symmetrization of tensors $\mathcal{S} : \otimes^{m+1} T^* M_e^\circ \to \otimes_S^{m+1} T^* M_e^\circ$. The divergence of m-cotensors is the adjoint differential operator, which is given by $D^* f := -\mathcal{T}(\nabla f)$ where $\mathcal{T} : \otimes_S^m T^* M \to \otimes_S^{m-2} T^* M$ denotes the trace map defined by contracting with the Riemannian metric,

$$\mathcal{T}(q)(v_1, \dots, v_{m-2}) := \sum_{i=1}^n q(e_i, e_i, v_1, \dots, v_{m-2}), \tag{5.14}$$

if $(e_1, \dots, e_n)$ is a local orthonormal basis of TM_e. Each $u \in L^2(SM_e)$ function can be decomposed using the spectral decomposition of the vertical Laplacian Δ_v in the fibers of SM_e (which are spheres)

$$u = \sum_{k=0}^\infty u_k, \quad \Delta_v u_k = k(k + n - 2), \tag{5.15}$$

where u_k are L^2 sections of a vector bundle over M_e; see [GuKa2, PSU2].

When (5.2) holds for some $p \in (2, \infty)$, we define just as for $m = 0$ the X-ray transform on $\otimes_S^m T^* M$ as the bounded operator for all $p \in (2, \infty)$,

$$I_m := I\, \pi_m^* : L^p(M; \otimes_S^m T^* M) \to L^2(\partial_- SM, d\mu_\nu). \tag{5.16}$$

The adjoint $I_m^*: L^2(\partial_-SM, d\mu_\nu) \to L^{p'}(M; \otimes_S^m T^*M)$ is bounded if $1/p' + 1/p = 1$, and it is given by $I_m^* = \pi_{m*}I^*$. The operator Π_m is simply defined as the bounded self-adjoint operator for $p \in (2, \infty)$ and $1/p' + 1/p = 1$,

$$\Pi_m := I_m^* I_m = \pi_{0*}\Pi\, \pi_m^* : L^p(M; \otimes_S^m T^*M) \to L^{p'}(M; \otimes_S^m T^*M). \tag{5.17}$$

As for $m = 0$, we set $\Pi_m^e := \pi_{0*}\Pi^e\, \pi_m^*$, which can also be seen as $(I_m^e)^* I_m^e$ if $I_m^e = I^e \pi_m^*$ is the X-ray transform on m cotensors on M_e. Repeating the arguments of [Gu, Theorem 3.5] but adapted to our case we get directly the following proposition.

Proposition 5.9. *Assume that the geodesic flow on M has no conjugate points and that the trapped K is hyperbolic. For $m \geq 1$, the operator Π_m^e is a pseudo-differential operator of order -1 on the bundle $\otimes_S^m T^*M_e^\circ$, which is elliptic on $\ker D^*$ in the sense that for all $\psi_0 \in C_c^\infty(SM_e^\circ)$ there exist pseudo-differential operators Q, S, R on M_e° with respective order $1, -2, -\infty$ so that*

$$Q\psi_0 \Pi_m^e \psi_0 = \psi_0^2 + D\psi_0 S\psi_0 D^* + R. \tag{5.18}$$

The same result as (5.13) also holds for I_m and I_m^* since Π_m is a ΨDO of order -1: if $\mathcal{O} \subset M^\circ$ is any compact domain (with non-empty interior) with smooth boundary,

$$I_m : H^{-1/2}(\mathcal{O}, \otimes_S^m T^*M) \to L^2(\partial_-SM; d\mu_\nu). \tag{5.19}$$

5.3. **Injectivity of X-ray transform on symmetric tensors.** In this section, we use the Pestov identity and the smoothness property in Corollary 5.8 to prove injectivity of X-ray transform on functions and 1-forms in case of hyperbolic trapping. The proof is basically the same as in the simple domain setting, once we have proved the smoothness of elements in $\ker I_m \cap \ker D^*$.

Theorem 5. *Let (M, g) be a compact Riemannian manifold with strictly convex boundary. Assume that the geodesic flow has no conjugate points, and that the trapped set K is hyperbolic.*

1. *Let $f_0 \in L^p(M) + H^{-1/2}_{\rm comp}(M^\circ)$ with $p > 2$ such that $I_0 f_0 = 0$, then $f_0 = 0$.*
2. *Let $f_1 \in C^\infty(M; T^*M) + H^{-1/2}_{\rm comp}(M^\circ; T^*M)$ such that $I_1 f_1 = 0$, then there exists $\psi \in C^\infty(M) + H^{1/2}_{\rm comp}(M^\circ)$ vanishing at ∂M such that $f_1 = d\psi$.*
3. *Assume that the sectional curvatures of g are non-positive; then if for $m > 1$, $f_m \in C^\infty(M; \otimes_S^m T^*M)$ satisfies $I_m f_m = 0$, then $f_m = Dp_{m-1}$ for some $p_{m-1} \in C^\infty(M; \otimes_S^{m-1} T^*M)$ which vanishes at ∂M.*

Proof. Let us first show (1) and (2). Using Hodge decomposition we write $f_1 = d\psi + f_1'$ with $f_1' \in C^\infty(M, T^*M) + H^{-1/2}_{\rm comp}(M^\circ, T^*M)$ satisfying $D^* f_1' = 0$ and $\psi \in C^\infty(M) + H^{1/2}_{\rm comp}(M^\circ)$ satisfying $\psi|_{\partial M} = 0$. This can be done by taking $\psi := \Delta_D^{-1}\delta f_1$ where Δ_D^{-1} is the inverse of the Dirichlet Laplacian on (M, g) and $\delta := d^* = D^*$ on 1-forms. Notice that f_1' is smooth near ∂M since f_1 is smooth near ∂M (using ellipticity of Δ_D). Since $I_1 d\psi = 0$ we get $\Pi_1 f_1' = 0$ and $\Pi_1^e f_1' = 0$. By applying (5.18) to f_1' with $\psi_0 = 1$ on M, we get that $f_1' \in C^\infty(M^\circ)$ and thus $f_1' \in C^\infty(M)$. Since also $\Pi_0 f_0 = 0$, Corollary 5.8 then implies that f_0 is smooth. By Proposition 5.5, we see that there exists $u_j \in C^\infty(SM)$ for $j = 0, 1$ such that $Xu_0 = \pi_0^* f_0$ and $Xu_1 = \pi_1^* f_1'$, with u_j vanishing on ∂SM. Now since the functions u_j are smooth and vanish at the boundary ∂SM, Pestov's identity

[PSU2, Proposition 2.2. and Remark 2.3] holds here in the same way as it does for simple manifolds with boundary or for closed manifolds,

$$||\nabla^v X u_j||^2_{L^2} = ||X\nabla^v u_j||^2_{L^2} - \langle R\nabla^v u_j, \nabla^v u_j\rangle + (n-1)||Xu_j||^2_{L^2}, \tag{5.20}$$

where ∇^v is the covariant derivative in the vertical direction of SM, mapping functions on SM to sections of the bundle $E \to SM$ with fibers

$$E_{(x,v)} := \{w \in T_xM; g_x(w,v) = 0\},$$

R is the curvature tensor acting on E by $R_{(x,v)}w := R(w,v)v \in E_{(x,v)}$, and X acts on sections of E by differentiating parallel transport along the geodesic (see Section 2 of [PSU2]). Then the proof of Lemma 11.2 of [PSU2] and Proposition 7.2 of [DKSU] is based on Santalo formula (2.22) and thus applies as well in our setting (i.e., the boundary is strictly convex, there are no conjugate points, and $\Gamma_+ \cup \Gamma_-$ has Liouville measure 0); then for all $Z \in C^\infty(SM, E)$

$$||XZ||_{L^2} - \langle RZ, Z\rangle \geq 0$$

with equality if and only if $Z = 0$. In particular, since $\nabla^v X u_0 = \nabla^v f_0 = 0$, we deduce from (5.20) that $f_0 = 0$, and since $||\nabla^v X u_1||^2_{L^2} = (n-1)||f_1||^2_{L^2}$, we deduce from (5.20) that $\nabla^v u_1 = 0$ and thus $u_1 = \pi_0^*\psi'$ for some smooth function ψ' on M which vanishes at ∂M; this implies that $Xu_1 = \pi_1^* d\psi'$. Notice that since $D^* f_1' = 0$, then $D^* f_1' = \Delta_g \psi' = 0$, and therefore $\psi' = 0$ since ψ' vanishes at ∂M. Thus $f_1' = 0$.

Finally, the case with $m > 1$ when the curvature of g is non-positive uses the proof of [PeSh] and [PSU2, Section 11]. If $I_m f = 0$, we also have $I_m^e f_m = 0$ and thus $\Pi^e \pi_m^* f_m = 0$. By Proposition 5.5, there exists $u = -R_+(0)\pi_m^* f_m = -R_-(0)\pi_m^* f_m$ smooth in SM such that $Xu = \pi_m^* f_m$ and $u|_{\partial SM} = 0$. Non-positive curvature implies that the flow is 1-controlled in the sense of [PSU2], and once we know that $Xu = \pi_m^* f_m$ with u smooth and vanishing at ∂M, the proof of Theorem 11.8 in [PSU2] (that proof is detailed in Sections 9 and 11) based on Pestov identity applies verbatim to our case. We do not repeat it here as it does not bring anything new. □

We get Corollary 1.1 and Theorem 1 as a direct corollary.

Proof of Corollary 1.1. We only prove (2) since the conformal case (1) is easier and a direct consequence of (1) in Theorem 5. If the metrics are lens equivalent, $\Gamma_\pm \cap \partial_\pm SM$ are the same for all metrics, and for a fixed $y := (x,v) \in \partial_- SM \setminus \Gamma_-$, the geodesic $\gamma_s(y;t)$ with $t \in [0, \ell_+(y)]$ depends smoothly on s (by general ODE arguments). By differentiating $\partial_s \ell_+(y)^2 = 0$, we obtain that $q_s := \partial_s g_s$ is a smooth symmetric 2-tensor satisfying $I_2^s q_s = 0$ if I_2^s is the X-ray for g_s on symmetric 2 cotensors. The argument is standard and detailed in [Sh, Section 1.1]. Applying Theorem 5 with $m = 2$ in non-positive curvature shows that $q_s = D_s p_s$ for some smooth 1-form p_s vanishing at ∂M. The tensor p_s can be written as $p_s = (\Delta_{D_s})^{-1} D_s^* q_s$ if $\Delta_{D_s} := D_s^* D_s$ with Dirichlet condition at ∂M (this is invertible; see [Sh]). Then we argue as in the proof of [GuKa1, Theorem 1]: by ellipticity of Δ_{D_s} and smoothness in s, p_s is smooth in s. Then one can construct a smooth family of diffeomorphisms ϕ_s which are the identity on ∂M so that $\partial_s \phi_s = p_s \circ \phi_s$ and $\phi_0 = \mathrm{Id}$ (here we view p_s as a vector field using the metric). This concludes the proof. □

Proof of Theorem 1. A negatively curved manifold with strictly convex boundary has hyperbolic trapped set K (see [Kl2, Section 3.9 and Theorem 3.2.17]) and no conjugate points (see [Kl]). Thus, Theorem 1 follows from Corollary 1.1 and Proposition 2.4. □

5.4. **Invariant distributions with prescribed pushforward.** We will show the existence of invariant distributions on SM with prescribed pushforward. This corresponds essentially to surjectivity of I_0^* and of I_1^* on $\ker D^*$. This section is only necessary to prove Theorems 2 and 4.

Proposition 5.10. *We make the same assumptions as in Theorem* 5.

(1) *For any $f_0 \in H^s(M)$ for $s > 1$, there exists $w \in (\cap_{u<0} H^u(SM_e)) \cap L^1(SM_e)$ such that $Xw = 0$ in SM_e° and $\pi_{0*}w = f_0$ in M. Moreover, if $f_0 \in C^\infty(M)$, then $w \in H^s(SM_e)$ for some $s > 0$ and has wavefront set satisfying* $\mathrm{WF}(w) \subset E_+^* \cup E_-^*$. *In addition, its boundary value $\omega = w|_{\partial SM}$ satisfies* (5.8) *and $\omega \in L^2_{S_g}(\partial SM) \cap H^s(\partial SM)$ for some $s > 0$.*

(2) *Let $f_1 \in C^\infty(M; T^*M)$ satisfy $D^* f_1 = 0$, then there exists $w \in L^{p'}(SM_e)$ such that $Xw = 0$ in SM_e° and $\pi_{1*}w = f_1$ in M, with* $\mathrm{WF}(w) \subset E_+^* \cup E_-^*$ *and $\omega := w|_{\partial SM}$ satisfies* (5.8) *and is in $L^2_{S_g}(\partial SM)$.*

Proof. Let Y be a closed manifold extending smoothly M_e across its boundary and extend the metric smoothly to Y (and still call the extension g). Let $\psi_0 \in C_c^\infty(Y)$ with support in M_e which is equal to 1 on a neighborhood of M and write $\psi := \pi_0^*(\psi_0)$ its lift to SY. Using Proposition 5.7, define the elliptic ΨDO of order -1 on Y

$$P_0 = \psi_0 \Pi_0^e \psi_0 + (1-\psi_0)(1+\Delta_g)^{-1/2}(1-\psi_0) : H^{-s}(Y) \to H^{-s+1}(Y)$$

bounded for all $s \geq 0$; here Δ_g is the Laplacian on (Y, g). Thus there exists $C > 0$ and $K : H^{-s}(Y) \to H^{-s+1}(Y)$ a bounded ΨDO (of order -1) such that for all $f \in H^{-s}(Y)$

$$||P_0 f||_{H^{1-s}(Y)} \geq C||f||_{H^{-s}(Y)} - ||Kf||_{H^{-s+1}(Y)},$$

and thus the range of P_0 is closed. Consequently, by Banach closed range theorem, $P_0^* : H^{s-1}(Y) \to H^s(Y)$ has closed range. Note that P_0^* has the same form as P_0, and to prove its surjectivity, it suffices to prove injectivity of P_0. If $P_0 f = 0$, then $f \in C^\infty(Y)$ by ellipticity of P_0, and $(1-\psi_0)f = 0$ since $(1+\Delta_g)^{-1/2}$ is injective, and $\langle \Pi_0^e(\psi_0 f), \psi_0 f\rangle_{L^2} = 0$. This implies that $I_0^e(\psi_0 f) = 0$ and by Theorem 5 applied with M_e instead of M, we get $\psi_0 f = 0$, and thus $f = 0$. We deduce that if $f_0 \in H^s(M)$, taking an extension $\tilde{f}_0 \in H^s(Y)$ supported in the region where $\psi_0 = 1$, there exists a unique $u \in H^{s-1}(Y)$ such that $P_0^* u = \tilde{f}_0$. Note that if f_0 is smooth, u is smooth by ellipticity of P_0^*. In particular, we get $\psi_0 \Pi_0^e(\psi_0 u) = \tilde{f}_0$; taking $w := \Pi^e(\pi_0^*(\psi_0 u))$, we get $Xw = 0$ in SM_e, $\pi_{0*}w = f_0$ in M; and by Proposition 5.4, we obtain the desired regularity for w and the properties of its restriction $w|_{\partial SM}$ and (5.8). This proves (1).

The proof of (2) is essentially the same as in [DaUh, Lemma 2.2] once we know Proposition 5.9 and the kernel of I_1. We just recall very briefly the argument and refer to [DaUh, Lemma 2.2] for details. First, by [KMPT, Corollary 3.3] (see also the last remark of that paper for the manifold case) there is a bounded extension operator $E : \ker D^*|_{L^2(M,T^*M)} \to \ker D^*|_{L^2(M_e^\circ, T^*M_e)}$ which restricts continuously to $E : \ker D^*|_{C^\infty(M,T^*M)} \to \ker D^*|_{C_c^\infty(M_e^\circ, T^*M_e)}$; then if $r_M : L^2(M_e, T^*M_e) \to L^2(M, T^*M)$ is the restriction to M, we get from Proposition 5.9 that $r_M \Pi_1^e \psi_0 Q^* E = \mathrm{Id} + r_M R^* E$ as a map on $\ker D^*|_{L^2(M,T^*M)}$ with R smoothing on M_e°. This implies that the range of $\mathrm{Id} + r_M R^* E$ is closed with finite codimension, and the same holds on $\ker D^*|_{C^\infty(M,T^*M)}$. Then $r_M \Pi_1^e \psi_0 Q^* E(\ker D^*|_{C^\infty(M,T^*M)})$ has closed range in $\ker D^*|_{C^\infty(M,T^*M)}$ with finite codimension, and thus

$r_M \Pi_1^e \psi_0 Q^*(C_0^\infty(M_e^\circ, T^*M_e))$ has closed range with finite codimension in ker $D^*|_{C^\infty(M,T^*M)}$. The kernel of the adjoint is trivial by using Theorem 5 just as in [DaUh, Lemma 2.2.]. This shows that there is $u \in C^\infty(M_e, T^*M_e)$ such that $r_M \Pi_1^e u = f_1$, and thus setting $w := \Pi^e \pi_1^* u$ we get the result. $\square$

6. Determination of the conformal structure for surfaces

In this section, we will study the lens rigidity for surfaces with strictly convex boundary, no conjugate points and hyperbolic trapped set. To recover the conformal structure from the scattering map, we shall use most of the results proved above together with the approach of Pestov-Uhlmann [PeUh] which reduces the scattering rigidity to the Calderón problem on surfaces.

For the oriented Riemannian surface M_e with boundary, the unit tangent bundle SM_e is a principal circle bundle, with an action

$$S^1 \times SM_e \to SM_e, \qquad e^{i\theta} \cdot (x, v) = (x, R_\theta v),$$

where R_θ is the rotation of angle $+\theta$. This induces a vector field V generating this action, defined by $Vf(x, v) = \partial_\theta (f(e^{i\theta}.(x, v))|_{\theta=0}$. We then define the vector field $X_\perp := [X, V]$, and the basis $(X, X_\perp, V)$ is an orthonormal basis of SM_e for the Sasaki metric. The space SM_e splits into $SM_e = \mathcal{V} \oplus \mathcal{H}$ where $\mathcal{V} = \mathbb{R}V = \ker d\pi_0$ is the vertical space, and $\mathcal{H} = \mathrm{span}(X, X_\perp)$ is the horizontal space. Following [GuKa1], there is an orthogonal decomposition (Fourier series in the fibers)

$$L^2(SM_e^\circ) = \bigoplus_{k\in\mathbb{Z}} \Omega_k, \quad \text{with } Vw_k = ikw_k \text{ if } w_k \in \Omega_k, \tag{6.1}$$

where Ω_k is the space of L^2 sections of a complex line bundle over M_e°. Similarly, one has a decomposition on ∂SM

$$L^2(\partial SM) = \bigoplus_{k\in\mathbb{Z}} \Omega'_k, \quad \text{with } V\omega_k = ik\omega_k \text{ if } \omega_k \in \Omega'_k, \tag{6.2}$$

using Fourier analysis in the fibers of the circle bundle.

6.1. Hilbert transform and Pestov-Uhlmann commutator relation. The Hilbert transform in the fibers is defined by using the decomposition (6.1),

$$H : L^2(SM_e^\circ) \to L^2(SM_e^\circ), \quad H(\sum_{k\in\mathbb{Z}} w_k) = -i \sum_{k\in\mathbb{Z}} \mathrm{sign}(k) w_k,$$

with $\mathrm{sign}(0) := 0$ by convention. It is skew-adjoint and $\overline{Hu} = H\overline{u}$; thus we can extend continuously H to $C^{-\infty}(SM_e^\circ) \to C^{-\infty}(SM_e^\circ)$ by the expression

$$\langle Hu, \psi \rangle := -\langle u, H\psi \rangle, \qquad \psi \in C_c^\infty(SM_e^\circ),$$

where the distribution pairing is $\langle u, \psi \rangle = \int_{SM_e} u\psi d\mu$ when $u \in L^2(SM_e^\circ)$. Similarly, we define the Hilbert transform in the fibers on ∂SM

$$H_\partial : C^\infty(\partial SM) \to C^\infty(\partial SM), \qquad H_\partial(\sum_{k\in\mathbb{Z}} \omega_k) = -i \sum_{k\in\mathbb{Z}} \mathrm{sign}(k) \omega_k$$

and its extension to distributions as for SM_e. For smooth $w \in C_c^\infty(SM_e^\circ)$ we have that

$$(Hw)|_{\partial SM} = H_\partial\, \omega, \quad \text{with } \omega := w|_{\partial SM}; \tag{6.3}$$

thus the identity extends by continuity to the space of distributions in SM_e° with a wave-front set disjoint from $N^*(\partial SM)$ since, by [Hö, Theorem 8.2.4], the restriction map $C^\infty(SM_e^\circ) \to C^\infty(\partial SM)$ obtained by pullback through the inclusion map ι of (2.13) extends continuously to the space of distributions on SM_e° with a wave-front set not intersecting $N^*(\partial SM)$. By [Gu, Lemma 3.5], we see that $\mathrm{WF}(Hu) \subset \mathrm{WF}(u)$ for all $u \in C^{-\infty}(SM_e)$, and the same holds for H_∂ and $u \in C^{-\infty}(\partial SM)$. The following commutator relation between Hilbert transform and flow follows easily from the Fourier decomposition and was proved by Pestov-Uhlmann [PeUh, Theorem 1.5]:

$$\text{if } w \in C^\infty(SM_e^\circ), \quad [H, X]w = X_\perp w_0 + (X_\perp w)_0, \tag{6.4}$$

where $w_0 = \frac{1}{2\pi}\pi_0^*(\pi_{0*}w)$ and $\pi_{0*}w(x) = \int_{S_xM_e} w(x,v)dS_x(v)$ for smooth w. Notice that $w \in C^\infty(SM_e^\circ) \mapsto w_0 \in C^\infty(SM_e^\circ)$ extends continuously to $C^{-\infty}(SM_e^\circ)$ since π_0 is a submersion (the pullback π_0^* extends to distributions), and then the relation (6.4) extends continuously to $C^{-\infty}(SM_e^\circ)$. We also have, for any $w \in C^{-\infty}(SM_e^\circ)$,

$$X_\perp w_0 = \frac{1}{2\pi}\pi_1^*(*d(\pi_{0*}w)), \tag{6.5}$$

where $* : T^*M_e \to T^*M_e$ is the Hodge-star operator on 1-forms. We use the odd/even decomposition of distributions with respect to the involution $A(x,v) = (x,-v)$ on SM_e, SM, and ∂SM, as explained at the end of Section 4.2. The operator X maps odd distributions to even distributions and conversely. The operator H maps odd (resp. even) distributions to odd (resp. even) distributions, we set $H_{\mathrm{ev}}w := H(w_{\mathrm{ev}})$ and $H_{\mathrm{od}}w := H(w_{\mathrm{od}})$. We write similarly $H_{\partial,\mathrm{ev}}$ and $H_{\partial,\mathrm{od}}$ for the Hilbert transform on ∂SM, and the relation (6.3) also holds with $H_{\partial,\mathrm{ev}}$ replacing H_∂ if w is even. Taking the odd part of (6.4), we have for any $w \in C^{-\infty}(SM_e^\circ)$

$$H_{\mathrm{od}}Xw - XH_{\mathrm{ev}}w = \frac{1}{2\pi}\pi_1^*(*d(\pi_{0*}w)) = X_\perp w_0. \tag{6.6}$$

6.2. **Determination of the conformal structure from scattering map.** For functions $\omega \in C^\infty(\partial SM)$, the function $\pi_{0*}\omega$ is smooth on ∂M, given by the expression $\pi_{0*}\omega(x,v) = \frac{1}{2\pi}\int_{S_xM_e} \omega(x,v)dS_x(v)$ and thus if $w \in C^\infty(SM_e^\circ)$ and $\omega = w|_{\partial SM}$, one has $\pi_{0*}\omega = (\pi_{0*}w)|_{\partial M}$. As above, the restriction map $C^\infty(SM_e^\circ) \to C^\infty(\partial SM)$, extends continuously to the space of distributions on SM_e° with wave-front set included in $E_+^* \cup E_-^*$ (since this does not intersect $N^*(\partial SM)$). Therefore, for $w \in C^{-\infty}(SM_e^\circ)$ with $\mathrm{WF}(w) \subset E_+^* \cup E_-^*$, we have

$$\pi_{0*}\omega = (\pi_{0*}w)|_{\partial M}, \qquad \text{with } \omega := w|_{\partial SM} \tag{6.7}$$

in the distribution sense (in fact, as in the proof of Proposition 5.10, it is easily checked that $\pi_{0*}w \in C^\infty(M_e^\circ)$).

For an oriented Riemannian surface (M,g) with boundary, the space of holomorphic functions can be described as follows: $f = f_1 + if_2$ is holomorphic if $*df_1 = df_2$ where $*$ is the Hodge star operator. We shall use the notation $\mathcal{P}(f) \in C^\infty(M)$ for the unique solution of $\Delta_g\mathcal{P}(f) = 0$ with $\mathcal{P}(f) = f$ on ∂M.

Theorem 6. *Let (M,g) and (M',g') be two oriented smooth compact Riemannian surfaces such that each connected component has non-empty strictly convex boundary. Assume that M and M' have the same boundary N, and that $g|_{TN} = g'|_{TN}$. For both surfaces, assume that the trapped sets are hyperbolic and the metrics have no conjugate points. If (M,g) and (M',g') are scattering equivalent, there exists a diffeomorphism $\phi : M \to M'$ with $\phi|_{\partial M} = \mathrm{Id}$ and such that $\phi^*g' = e^{2\eta}g$ for some $\eta \in C^\infty(M)$ satisfying $\eta|_{\partial M} = 0$.*

Proof. We shall follow the method of Pestov-Uhlmann [PeUh], and we will need to use most of the results from the previous sections. We work on (M,g) but all the results below apply as well on (M',g'). For $f\in C^\infty(N)$, the harmonic extension $\mathcal{P}(f)$ admits a harmonic conjugate $\mathcal{P}(f^*)$ if $*d\mathcal{P}(f)=d\mathcal{P}(f^*)$ or equivalently $\mathcal{P}(f+if^*)$ is holomorphic. We are going to prove the following statement: let $f^*\in C^\infty(N)$, then

$$2\pi(S_g^*-\mathrm{Id})(H_{\partial,\mathrm{ev}}\omega)=(S_g^*-\mathrm{Id})\pi_0^*f^* \tag{6.8}$$

holds for some $\omega\in L^2_{S_g}(\partial SM)\cap H^s(\partial SM)$ with $s>0$, satisfying $\mathrm{WF}(\omega_-)\subset E^*_{\partial,-}$ and $\mathrm{WF}(\omega_+)\subset E^*_{\partial,+}$, if and only if

$$I_0^*\omega_-=\mathcal{P}(f) \text{ with } \mathcal{P}(f-if^*) \text{ holomorphic}, \tag{6.9}$$

where $\pi_{0*}\mathcal{E}_-=I_0^*$ (see Lemma 5.2) and $\omega_\pm:=\omega|_{\partial_\pm SM}$.

Let us prove the first statement. Let $f\in C^\infty(N)$ so that $\mathcal{P}(f)$ admits a harmonic conjugate. Using Proposition 5.10, there exists $w\in H^s(SM_e)\cap C^\infty(SM_e\setminus(\Gamma_+\cup\Gamma_-))$ for some $s>0$, satisfying $Xw=0$ in SM_e° in the distribution sense with $\pi_{0*}w=\mathcal{P}(f)$ in M and

$$\omega:=w|_{\partial SM}\in L^2_{S_g}(\partial SM)\cap H^s(\partial SM),\quad \mathrm{WF}(\omega)\subset E^*_{\partial,+}\cup E^*_{\partial,-}, \tag{6.10}$$

$\omega_-:=\omega|_{\partial_-SM}$, where $E^*_{\partial,\pm}\subset T^*_{\Gamma_\pm}(\partial SM)$ are the bundles defined by (2.14) for the manifold M and π_{0*} is the pushforward defined by (5.9) on SM. From (6.6) and using that $H_{\mathrm{ev}}w$ is smooth in $SM\setminus(\Gamma_-\cup\Gamma_+)$, we get

$$XH_{\mathrm{ev}}w=-\frac{1}{2\pi}\pi_1^*(*d\mathcal{P}(f)) \tag{6.11}$$

as smooth functions on $SM\setminus(\Gamma_-\cup\Gamma_+)$. Now, for any $\psi\in C^\infty(SM\setminus(\Gamma_+\cup\Gamma_-))$,

$$IX\psi=(S_g^*-\mathrm{Id})(\psi|_{\partial SM\setminus(\Gamma_-\cup\Gamma_+)})$$

as a function on $\partial_-SM\setminus\Gamma_-$. Applying I to (6.11) and using that $\mathcal{P}(f-if^*)$ is holomorphic then gives (I_1 is the X-ray transform on 1-forms)

$$\begin{aligned}2\pi(S_g^*-\mathrm{Id})((H_{\mathrm{ev}}w)|_{\partial SM})&=-I_1(*d\mathcal{P}(f))=I_1(d\mathcal{P}(f^*))\\&=IX\pi_0^*(\mathcal{P}(f^*))=(S_g^*-\mathrm{Id})\pi_0^*f^*\end{aligned}$$

as smooth functions on $\partial_-SM\setminus\Gamma_-$ which are globally in $L^2(\partial_-SM,d\mu_\nu)$. Using (6.3) we thus obtain the identity (6.8).

Next, we prove the converse. Conversely, let $f^*\in C^\infty(N)$, let $q\in C^\infty(M)$ with $q|_{\partial M}=f^*$, and let $\chi\in C_c^\infty(SM^\circ)$ which is equal to 1 in $\{\rho>\epsilon\}$ with $\epsilon>0$ small (using ρ as in Section 2.1), thus on K. We write $w_1:=\chi\mathcal{E}_-\omega_-$ and $w_2:=(1-\chi)\mathcal{E}_-\omega_-$, and by (6.6), we get for $j=1,2$

$$HXw_j-XHw_j=\pi_1^*(*d\pi_{0*}w_j). \tag{6.12}$$

Since $\omega\in L^2_{S_g}(\partial SM)\cap H^s(\partial SM)$ for some $s>0$ and $\mathrm{WF}(\omega_\pm)\subset E^*_{\partial,\pm}$ by assumption, Proposition 4.6 tells us that $w_i\in H^s_{\mathrm{loc}}(SM_e^\circ)$ and $\mathrm{WF}(w_2)\subset E_+^*\cup E_-^*$; thus $\pi_{0*}w_2\in C^\infty(M_e^\circ)$ (using $(E_-^*\cup E_+^*)\cap\mathcal{H}^*=\{0\}$ if $\mathcal{H}^*\subset T^*(SM_e^\circ)$ is the annihilator of the vertical bundle $\mathcal{V}$), and $\pi_{0*}w_1\in H^{s+1/2}_{\mathrm{comp}}(M^\circ)$ with support containing K. We claim that we can apply I to (6.12) and view the result as a measurable function in $\partial_-SM\setminus\Gamma_-$: for $j=2$ we can apply I since all terms are smooth in $SM\setminus(\Gamma_-\cup\Gamma_+)$ and we get a smooth function on $\partial_-SM\setminus\Gamma_-$ that is in $L^2(\partial_-SM)$ (for example, using Lemma 5.1 and Sobolev embedding $H^s(SM)\subset L^p(SM)$ for some $p>2$), and for $j=1$ the only possible trouble is $I_1(*d\pi_{0*}w_1)$, but this makes sense since

$I_1 : H^{-1/2}_{\mathrm{comp}}(M^\circ, T^*M) \to L^2(\partial_- SM, d\mu_\nu)$ is bounded just as I_0 in (5.13) (see the remark after Proposition 5.9). Therefore, applying I to (6.12) and summing for $j = 1, 2$, we obtain almost everywhere on $\partial_- SM$

$$(S_g^* - \mathrm{Id})(H_{\partial,\mathrm{ev}}\omega) = IXH\mathcal{E}_-(\omega_-) = -\frac{1}{2\pi} I_1(*d\pi_{0*}w_1 + *d\pi_{0*}w_2);$$

this term is in $L^2(\partial_- SM, d\mu_\nu)$ and equal to $\frac{1}{2\pi}(S_g^* - \mathrm{Id})\pi_0^* f^* = \frac{1}{2\pi} I_1(dq)$ by our assumption. Since we know that this term is smooth on $\partial_- SM$ we obtain in $L^2(\partial_- SM, d\mu_\nu)$

$$I_1(*dI_0^*\omega_- + dq) = 0.$$

By Theorem 5 one has $*dI_0^*\omega_- + dq = d\psi$ for some $\psi \in C^\infty(M) + H^{1/2}_{\mathrm{comp}}(M^\circ)$ satisfying $\psi|_{\partial M} = 0$. Applying first d and then $d*$ to that equation and using ellipticity, we get $\psi - q \in C^\infty(M)$ and $I_0^*\omega_- \in C^\infty(M)$, and both functions are harmonic conjugate, which means that (6.9) holds with $f := (I_0^*\omega_-)|_{\partial M}$.

We can now finish the proof. All that we said above applies also on (M', g'), and we shall put a prime for objects related to g'. Let $\alpha : \partial SM' \to \partial SM$ be the map (3.2), so that $\alpha \circ S_{g'} = S_g \circ \alpha$ by assumption. Remark that for each $\omega \in C^\infty(\partial SM)$, $(\omega \circ \alpha)_k = \omega_k \circ \alpha$ in the Fourier decomposition (6.2), and thus

$$\alpha^*(H_{\partial,\mathrm{ev}}\omega) = H'_{\partial,\mathrm{ev}}(\alpha^*\omega). \tag{6.13}$$

This identity extends to $\omega \in L^2(\partial SM)$ by continuity. Let $f^* \in C^\infty(N)$, and assume that there exists $f \in C^\infty(N)$ so that $\mathcal{P}(f + if^*)$ is holomorphic in (M, g). Then we have proved that there is $\omega \in L^2_{S_g}(\partial SM)$ satisfying (6.8), $\pi_{0*}\omega = f$, and (6.10). Using $\alpha \circ S_{g'} = S_g \circ \alpha$ and $\pi_0 \circ \alpha = \pi_0$, together with (6.13), we get

$$(S_{g'}^* - \mathrm{Id})(H'_{\partial,\mathrm{ev}}\omega') = (S_{g'}^* - \mathrm{Id})\pi_0^* f^*, \tag{6.14}$$

with $\omega' := \alpha^*\omega$. We can use Lemma 3.2 which implies that $\mathrm{WF}(\omega') \subset E'^*_{\partial,+} \cup E'^*_{\partial,-}$, and since $\omega' \in L^2_{S_g}(SM')$, we get by (6.9) applied with (M', g') that $I_0'^*(\omega') - i\mathcal{P}'(f^*)$ is holomorphic in (M', g'). Since $I_0'^*(\omega')|_{\partial M} = \pi_{0*}\omega = f$, we have shown that all boundary values of a holomorphic function on (M, g) is also the boundary value of one on (M', g'). Exchanging the roles of (M, g) and (M', g'), we show that the space of boundary values of holomorphic functions on (M, g) and (M, g') are the same. The existence of the conformal diffeomorphism $\phi : M \to M'$ then follows from the work of Belishev [Be]. □

Acknowledgments

We thank S. Dyatlov for the work [DyGu2] which is fundamentally used here, and the referees for useful comments. Thanks also to V. Baladi, S. Gouëzel, M. Mazzucchelli, F. Monard, V. Millot, F. Naud, G. Paternain, M. Salo, S. Tapie, G. Uhlmann, and M. Zworski for useful discussions.

References

[AnRo] Yurii E. Anikonov and Vladimir G. Romanov, *On uniqueness of determination of a form of first degree by its integrals along geodesics*, J. Inverse Ill-Posed Probl. **5** (1997), no. 6, 487–490 (1998), DOI 10.1515/jiip.1997.5.6.487. MR1623603

[Be] Mikhail I. Belishev, *The Calderon problem for two-dimensional manifolds by the BC-method*, SIAM J. Math. Anal. **35** (2003), no. 1, 172–182 (electronic), DOI 10.1137/S0036141002413919. MR2001471

[BoRu] Rufus Bowen and David Ruelle, *The ergodic theory of Axiom A flows*, Invent. Math. **29** (1975), no. 3, 181–202. MR0380889

[BuIv] Dmitri Burago and Sergei Ivanov, *Boundary rigidity and filling volume minimality of metrics close to a flat one*, Ann. of Math. (2) **171** (2010), no. 2, 1183–1211, DOI 10.4007/annals.2010.171.1183. MR2630062

[BuLi] Oliver Butterley and Carlangelo Liverani, *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn. **1** (2007), no. 2, 301–322, DOI 10.3934/jmd.2007.1.301. MR2285731

[Cr1] Christopher B. Croke, *Rigidity for surfaces of nonpositive curvature*, Comment. Math. Helv. **65** (1990), no. 1, 150–169, DOI 10.1007/BF02566599. MR1036134

[Cr2] Christopher B. Croke, *Rigidity and the distance between boundary points*, J. Differential Geom. **33** (1991), no. 2, 445–464. MR1094465

[Cr3] Christopher Croke, *Scattering rigidity with trapped geodesics*, Ergodic Theory Dynam. Systems **34** (2014), no. 3, 826–836, DOI 10.1017/etds.2012.164. MR3199795

[CrHe] Christopher B. Croke and Pilar Herreros, *Lens rigidity with trapped geodesics in two dimensions*, Asian J. Math. **20** (2016), no. 1, 47–57, DOI 10.4310/AJM.2016.v20.n1.a3. MR3460758

[CrKl] Christopher B. Croke and Bruce Kleiner, *Conjugacy and rigidity for manifolds with a parallel vector field*, J. Differential Geom. **39** (1994), no. 3, 659–680. MR1274134

[DaUh] Nurlan Dairbekov and Gunther Uhlmann, *Reconstructing the metric and magnetic field from the scattering relation*, Inverse Probl. Imaging **4** (2010), no. 3, 397–409, DOI 10.3934/ipi.2010.4.397. MR2671103

[DKSU] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math. **178** (2009), no. 1, 119–171, DOI 10.1007/s00222-009-0196-4. MR2534094

[DKLS] David Dos Santos Ferreira, Yaroslav Kurylev, Matti Lassas, and Mikko Salo, *The Calderon problem in transversally anisotropic geometries*, to appear in J. Eur. Math. Soc.

[DyGu1] Semyon Dyatlov and Colin Guillarmou, *Microlocal limits of plane waves and Eisenstein functions* (English, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 2, 371–448. MR3215926

[DyGu2] Semyon Dyatlov, Colin Guillarmou, *Pollicott-Ruelle resonances for open systems.* Ann. Henri Poincaré, 1–18, DOI 10.1007/s00023-016-0491-8.

[DyZw] Semyon Dyatlov, Maciej Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. l'ENS **49** (2016), 543–577.

[FaSj] Frédéric Faure and Johannes Sjöstrand, *Upper bound on the density of Ruelle resonances for Anosov flows* (English, with English and French summaries), Comm. Math. Phys. **308** (2011), no. 2, 325–364, DOI 10.1007/s00220-011-1349-z. MR2851145

[FrJo] Friedrich G. Friedlander and Mark S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ. Press. (1999), pp. 188.

[GeGo] Patrick Gérard and François Golse, *Averaging regularity results for PDEs under transversality assumptions*, Comm. Pure Appl. Math. **45** (1992), no. 1, 1–26, DOI 10.1002/cpa.3160450102. MR1135922

[GrSj] Alain Grigis and Johannes Sjöstrand, *Microlocal Analysis for Differential Operators*, London Mathematical Society Lecture Note Series, vol. 196, Cambridge University Press, Cambridge, 1994. An introduction. MR1269107

[Gu] Colin Guillarmou, *Invariant distributions and X-ray transform for Anosov flows*, J. Differential Geom., to appear. arXiv:1408.4732.

[GuKa1] Victor Guillemin and David Kazhdan, *Some inverse spectral results for negatively curved 2-manifolds*, Topology **19** (1980), no. 3, 301–312, DOI 10.1016/0040-9383(80)90015-4. MR579579

[GuKa2] Victor Guillemin and David Kazhdan, *Some inverse spectral results for negatively curved n-manifolds*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 153–180. MR573432

[HaKa] Anatole Katok and Boris Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications, vol. 54, Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza. MR1326374

[HPPS] Morris Hirsch, Jacob Palis, Charles Pugh, and Michael Shub, *Neighborhoods of hyperbolic sets*, Invent. Math. **9** (1969/1970), 121–134. MR0262627

[Hö] Lars Hörmander, *The analysis of linear partial differential operators. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis. MR717035

[KMPT] Tosio Kato, Marius Mitrea, Gustavo Ponce, and Michael Taylor, *Extension and representation of divergence-free vector fields on bounded domains*, Math. Res. Lett. **7** (2000), no. 5-6, 643–650, DOI 10.4310/MRL.2000.v7.n5.a10. MR1809290

[Kl] Wilhelm Klingenberg, *Riemannian manifolds with geodesic flow of Anosov type*, Ann. of Math. (2) **99** (1974), 1–13. MR0377980

[Kl2] Wilhelm P. A. Klingenberg, *Riemannian Geometry*, 2nd ed., de Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., Berlin, 1995. MR1330918

[LSU] Matti Lassas, Vladimir Sharafutdinov, and Gunther Uhlmann, *Semiglobal boundary rigidity for Riemannian metrics*, Math. Ann. **325** (2003), no. 4, 767–793, DOI 10.1007/s00208-002-0407-4. MR1974568

[Mi] René Michel, *Sur la rigidité imposée par la longueur des géodésiques* (French), Invent. Math. **65** (1981/82), no. 1, 71–83, DOI 10.1007/BF01389295. MR636880

[MuRo] Ravil G. Mukhometov and Vladimir G. Romanov, *On the problem of finding an isotropic Riemannian metric in an n-dimensional space* (Russian), Dokl. Akad. Nauk SSSR **243** (1978), no. 1, 41–44. MR511273

[Mu] Ravil G. Mukhometov, *On a problem of reconstructing Riemannian metrics* (Russian), Sibirsk. Mat. Zh. **22** (1981), no. 3, 119–135, 237. MR621466

[NoSt] Lyle Noakes and Luchezar Stoyanov, *Rigidity of scattering lengths and travelling times for disjoint unions of strictly convex bodies*, Proc. Amer. Math. Soc. **143** (2015), no. 9, 3879–3893, DOI 10.1090/S0002-9939-2015-12531-2. MR3359579

[Pa] Gabriel P. Paternain, *Geodesic Flows*, Progress in Mathematics, vol. 180, Birkhäuser Boston, Inc., Boston, MA, 1999. MR1712465

[PSU1] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann, *Tensor tomography on surfaces*, Invent. Math. **193** (2013), no. 1, 229–247, DOI 10.1007/s00222-012-0432-1. MR3069117

[PSU2] Gabriel P. Paternain, Mikko Salo, and Gunther Uhlmann, *Invariant distributions, Beurling transforms and tensor tomography in higher dimensions*, Math. Ann. **363** (2015), no. 1-2, 305–362, DOI 10.1007/s00208-015-1169-0. MR3394381

[PeUh] Leonid Pestov and Gunther Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math. (2) **161** (2005), no. 2, 1093–1110, DOI 10.4007/annals.2005.161.1093. MR2153407

[PeSh] Leonid N. Pestov and Vladimir A. Sharafutdinov, *Integral geometry of tensor fields on a manifold of negative curvature* (Russian), Sibirsk. Mat. Zh. **29** (1988), no. 3, 114–130, 221, DOI 10.1007/BF00969652; English transl., Sib. Math. J. **29** (1988), no. 3, 427–441 (1989). MR953028

[Ro] Clark Robinson, *Structural stability on manifolds with boundary*, J. Differential Equations **37** (1980), no. 1, 1–11, DOI 10.1016/0022-0396(80)90083-2. MR583334

[SaUh] Mikko Salo and Gunther Uhlmann, *The attenuated ray transform on simple surfaces*, J. Differential Geom. **88** (2011), no. 1, 161–187. MR2819758

[Sh] Vladimir A. Sharafutdinov, *Integral Geometry of Tensor Fields*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 1994. MR1374572

[Sm] Stephen Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR0228014

[StUh1] Plamen Stefanov and Gunther Uhlmann, *Boundary rigidity and stability for generic simple metrics*, J. Amer. Math. Soc. **18** (2005), no. 4, 975–1003, DOI 10.1090/S0894-0347-05-00494-7. MR2163868

[StUh2] Plamen Stefanov and Gunther Uhlmann, *Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds*, J. Differential Geom. **82** (2009), no. 2, 383–409. MR2520797

[SUV] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy, *Boundary rigidity with partial data*, J. Amer. Math. Soc. **29** (2016), no. 2, 299–332, DOI 10.1090/jams/846. MR3454376

[Ot] Jean-Pierre Otal, *Sur les longueurs des géodésiques d'une métrique à courbure négative dans le disque* (French), Comment. Math. Helv. **65** (1990), no. 2, 334–347, DOI 10.1007/BF02566611. MR1057248

[Sa] Luis A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. With a foreword by Mark Kac; Encyclopedia of Mathematics and its Applications, Vol. 1. MR0433364

[Su] Dennis Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153** (1984), no. 3-4, 259–277, DOI 10.1007/BF02392379. MR766265

[Ta] Michael E. Taylor, *Partial Differential Equations. I*, Applied Mathematical Sciences, vol. 115, Springer-Verlag, New York, 1996. Basic theory. MR1395148

[Ta2] Michael E. Taylor, *Partial Differential Equations. III*, Applied Mathematical Sciences, vol. 117, Springer-Verlag, New York, 1997. Nonlinear equations; Corrected reprint of the 1996 original. MR1477408

[UhVa] Gunther Uhlmann and András Vasy, *The inverse problem for the local geodesic ray transform*, Invent. Math. **205** (2016), no. 1, 83–120, DOI 10.1007/s00222-015-0631-7. MR3514959

[Wa] Peter Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR648108

[Yo] Lai-Sang Young, *Large deviations in dynamical systems*, Trans. Amer. Math. Soc. **318** (1990), no. 2, 525–543, DOI 10.2307/2001318. MR975689

[Zw] Maciej Zworski, *Semiclassical Analysis*, Graduate Studies in Mathematics, vol. 138, American Mathematical Society, Providence, RI, 2012. MR2952218

DMA, U.M.R. 8553 CNRS, École Normale Superieure, 45 rue d'Ulm, 75230 Paris cedex 05, France

E-mail address: cguillar@dma.ens.fr