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ON THE ERDŐS-SZEKERES CONVEX POLYGON PROBLEM

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1. Introduction

In their classic 1935 paper, Erdős and Szekeres [7] proved that, for every integer $n \geq 3$, there is a minimal integer $ES(n)$, such that any set of $ES(n)$ points in the plane in general position[1] contains n points in convex position; that is, they are the vertices of a convex n-gon.

Erdős and Szekeres gave two proofs of the existence of $ES(n)$. Their first proof used a quantitative version of Ramsey's theorem, which gave a very poor upper bound for $ES(n)$. The second proof was more geometric and showed that $ES(n) \leq \binom{2n-4}{n-2} + 1$ (see Theorem 2.2 in the next section). On the other hand, they showed that $ES(n) \geq 2^{n-2} + 1$ and conjectured this to be sharp [8].

Small improvements have been made on the upper bound $\binom{2n-4}{n-2} + 1 \approx \frac{4^n}{\sqrt{n}}$ by various researchers [3, 13, 17, 18, 22, 23], but no improvement in the order of magnitude has ever been made. The most recent upper bound, due to Norin and Yuditsky [18] and Mojarrad and Vlachos [17], says that

$$\limsup_{n\to\infty} \frac{ES(n)}{\binom{2n-4}{n-2}} \leq \frac{7}{16}.$$

In the present paper, we prove the following.

Theorem 1.1. *For all $n \geq n_0$, where n_0 is a large absolute constant, $ES(n) \leq 2^{n+6n^{2/3}\log n}$.*

The study of $ES(n)$ and its variants[2] has generated a lot of research over the past several decades. For a more thorough history on the subject, we refer the interested reader to [2, 15, 22]. All logarithms are to base 2.

2. Notation and tools

In this section, we recall several results that will be used in the proof of Theorem 1.1. We start with the following simple lemma.

Lemma 2.1 (see Theorem 1.2.3 in [14]). *Let X be a finite point set in the plane in general position such that every four members in X are in convex position. Then X is in convex position.*

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[1]No three of the points are on a line.

[2]Higher dimensions [11, 12, 21], for families of convex bodies in the plane [5, 9], etc.

FIGURE 1. A 4-cup and a 5-cap.

The next theorem is a well-known result from [7], which is often referred to as the Erdős-Szekeres cups-caps theorem. Let X be a k-element point set in the plane in general position. We say that X forms a *k-cup* (*k-cap*) if X is in convex position and its convex hull is bounded above (below) by a single edge. In other words, X is a cup (cap) if and only if for every point $p \in X$, there is a line L passing through it such that all of the other points in X lie on or above (below) L. See Figure 1.

Theorem 2.2 ([7])**.** *Let $f(k,\ell)$ be the smallest integer N such that any N-element planar point set in the plane in general position contains a k-cup or an ℓ-cap. Then*

$$f(k,\ell) = \binom{k+\ell-4}{k-2} + 1.$$

The next theorem is a combinatorial reformulation of Theorem 2.2 observed by Hubard *et al.* [10] (see also [9, 16]). A transitive 2-coloring of the triples of $\{1,2,\ldots,N\}$ is a 2-coloring, say with colors red and blue, such that, for $i_1 < i_2 < i_3 < i_4$, if triples (i_1,i_2,i_3) and (i_2,i_3,i_4) are red (blue), then (i_1,i_2,i_4) and (i_1,i_3,i_4) are also red (blue).

Theorem 2.3 ([7])**.** *Let $g(k,\ell)$ denote the minimum integer N such that, for every transitive 2-coloring on the triples of $\{1,2,\ldots,N\}$, there exists a red clique of size k or a blue clique of size ℓ. Then*

$$g(k,\ell) = f(k,\ell) = \binom{k+\ell-4}{k-2} + 1.$$

The next theorem is due to Pór and Valtr [20] and is often referred to as the positive-fraction Erdős-Szekeres theorem (see also [1, 19]). Given a k-cap (k-cup) $X = \{x_1,\ldots,x_k\}$, where the points appear in order from left to right, we define the *support of* X to be the collection of open regions $\mathcal{C} = \{T_1,\ldots,T_k\}$, where T_i is the region outside of conv(X) bounded by the segment $\overline{x_ix_{i+1}}$ and by the lines $x_{i-1}x_i$, $x_{i+1}x_{i+2}$ (where $x_{k+1} = x_1$, $x_{k+2} = x_2$, etc.). See Figure 2.

Theorem 2.4 (Proof of Theorem 4 in [20])**.** *Let $k \geq 3$, and let P be a finite point set in the plane in general position such that $|P| \geq 2^{32k}$. Then there is a k-element subset $X \subset P$ such that X is either a k-cup or a k-cap, and the regions $T_1,\ldots,T_{k-1}$ from the support of X satisfy $|T_i \cap P| \geq \frac{|P|}{2^{32k}}$. In particular, every $(k-1)$-tuple obtained by selecting one point from each $T_i \cap P$, $i = 1,\ldots,k-1$, is in convex position.*

Note that Theorem 2.4 does not say anything about the points inside region T_k. Let us also remark that in the proof of Theorem 2.4 in [20], the authors find a $2k$-element set $X \subset P$, such that k of the regions in the support of X each contain at least $\frac{|P|}{2^{32k}}$ points from P, and therefore these regions may not be consecutive. However, by appropriately selecting a k-element subset $X' \subset X$, we obtain Theorem 2.4.

3. Proof of Theorem 1.1

Let P be an N-element planar point set in the plane in general position, where $N = \lfloor 2^{n+6n^{2/3}\log n} \rfloor$ and $n \geq n_0$, where n_0 is a sufficiently large absolute constant. Set $k = \lceil n^{2/3} \rceil$. We apply Theorem 2.4 to P with parameter $k+3$ and obtain a subset $X = \{x_1, \ldots, x_{k+3}\} \subset P$ such that X is a cup or a cap, and the points in X appear in order from left to right. Moreover since $k = \lceil n^{2/3} \rceil$ is large, regions $T_1, \ldots, T_{k+2}$ in the support of X satisfy

$$|T_i \cap P| \geq \frac{N}{2^{40k}}.$$

Set $P_i = T_i \cap P$ for $i = 1, \ldots, k+2$. We will assume that X is a cap, since a symmetric argument would apply. We say that the two regions T_i and T_j are *adjacent* if i and j are consecutive indices.

Consider the subset $P_i \subset P$ and the region T_i, for some fixed $i \in \{2, \ldots, k+1\}$. Let B_i be the segment $\overline{x_{i-1}x_{i+2}}$. See Figure 2. The point set P_i naturally comes with a partial order $\prec$, where $p \prec q$ if $p \neq q$ and $q \in \text{conv}(B_i \cup p)$. Set $\alpha = 3n^{-1/3}\log n$. By Dilworth's theorem [4], P_i contains either a chain of size at least $|P_i|^{1-\alpha}$ or an antichain of size at least $|P_i|^{\alpha}$ with respect to $\prec$. The proof now falls into two cases.

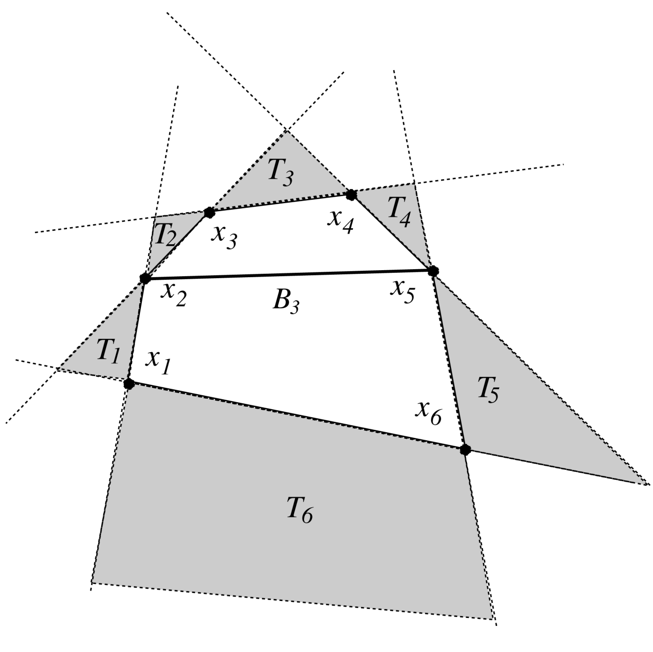

Figure 2. Regions $T_1, \ldots, T_6$ in the support of $X = \{x_1, \ldots, x_6\}$, and segment B_3.

Case 1. Suppose there are $t = \lceil \frac{n^{1/3}}{2} \rceil$ parts P_i in the collection $\mathcal{F} = \{P_2, P_3, \ldots, P_{k+1}\}$, such that no two of them are in adjacent regions, and each such part contains a subset Q_i of size at least $|P_i|^\alpha$ such that Q_i is an antichain with respect to $\prec$. Let $Q_{j_1}, Q_{j_2}, \ldots, Q_{j_t}$ be the selected subsets.

For each Q_{j_r}, $r \in \{1, \ldots, t\}$, the line spanned by any two points in Q_{j_r} does not intersect the segment B_{j_r} and, therefore, does not intersect region T_{j_w} for $w \neq r$ (by the non-adjacency property). Since n is sufficiently large, we have $40k < n^{2/3} \log n$, and therefore

$$\begin{aligned}|Q_{j_r}| \geq |P_i|^\alpha \geq \left(\frac{N}{2^{40k}}\right)^\alpha \geq 2^{3n^{2/3}\log n + 15 n^{1/3} \log^2 n} \geq \binom{n + \lceil 2n^{2/3} \rceil - 4}{n-2} + 1 \\ = f(n, \lceil 2n^{2/3} \rceil).\end{aligned}$$

Theorem 2.2 implies that Q_{j_r} contains either an n-cup or a $\lceil 2n^{2/3} \rceil$-cap. If we are in the former case for any $r \in \{1, \ldots, t\}$, then we are done. Therefore we can assume Q_{j_r} contains a subset S_{j_r} that is a $\lceil 2n^{2/3} \rceil$-cap, for all $r \in \{1, \ldots, t\}$.

We claim that $S = S_{j_1} \cup \cdots \cup S_{j_t}$ is a cap, and therefore S is in convex position. Let $p \in S_{j_r}$. Since $|S_{j_r}| \geq 2$, there is a point $q \in S_{j_r}$ such that the line L supported by the segment $\overline{pq}$ has the property that all of the other points in S_{j_r} lie below L. Since L does not intersect B_{j_r}, all of the points in $S \setminus \{p, q\}$ must lie below L. Hence, S is a cap and

$$|S| = |S_{j_1} \cup \cdots \cup S_{j_t}| \geq \frac{n^{1/3}}{2}(2n^{2/3}) = n.$$

Case 2. Suppose we are not in Case 1. Then there are $\lceil n^{1/3} \rceil$ consecutive indices $j, j+1, j+2, \ldots,$ such that each such part P_{j+r} contains a subset Q_{j+r} such that Q_{j+r} is a chain of length at least $|P_{j+r}|^{1-\alpha}$ with respect to $\prec$. For simplicity, we can relabel these sets $Q_1, Q_2, Q_3, \ldots$.

Consider the subset Q_i inside the region T_i, and order the elements in $Q_i = \{p_1, p_2, p_3, \ldots\}$ with respect to $\prec$. We say that $Y \subset Q_i$ is a *right-cap* if $x_i \cup Y$ is in convex position, and we say that Y is a *left-cap* if $x_{i+1} \cup Y$ is in convex position. Notice that left-caps and right-caps correspond to the standard notion of cups and caps after applying an appropriate rotation to the plane so that the segment $\overline{x_i x_{i+1}}$ is vertical. Since Q_i is a chain with respect to $\prec$, every triple in Q_i is either a left-cap or a right-cap, but not both. Moreover, for $i_1 < i_2 < i_3 < i_4$, if $(p_{i_1}, p_{i_2}, p_{i_3})$ and $(p_{i_2}, p_{i_3}, p_{i_4})$ are right-caps (left-caps), then $(p_{i_1}, p_{i_2}, p_{i_4})$ and $(p_{i_1}, p_{i_3}, p_{i_4})$ are both right-caps (left-caps). By Theorem 2.3, if $|Q_i| \geq f(k, \ell)$, then Q_i contains either a k-left-cap or an ℓ-right-cap. We make the following observation.

Observation 3.1. *Consider the (adjacent) sets Q_{i-1} and Q_i. If Q_{i-1} contains a k-left-cap Y_{i-1}, and Q_i contains an ℓ-right-cap Y_i, then $Y_{i-1} \cup Y_i$ forms $k + \ell$ points in convex position.*

Proof. By Lemma 2.1, it suffices to show every four points in $Y_{i-1} \cup Y_i$ are in convex position. If all four points lie in Y_i, then they are in convex position. Likewise if they all lie in Y_{i-1}, they are in convex position. Suppose we take two points $p_1, p_2 \in Y_{i-1}$ and two points $p_3, p_4 \in Y_i$. Since Q_{i-1} and Q_i are both chains with respect to $\prec$, the line spanned by p_1, p_2 does not intersect the region T_i, and the line spanned by p_3, p_4 does not intersect the region T_{i-1}. Hence p_1, p_2, p_3, p_4 are in convex position. Now suppose we have $p_1, p_2, p_3 \in Y_{i-1}$ and $p_4 \in Y_i$. Since the three lines L_1, L_2, L_3 spanned by p_1, p_2, p_3 all intersect the segment B_{i-1}, both x_i and p_4 lie in the same

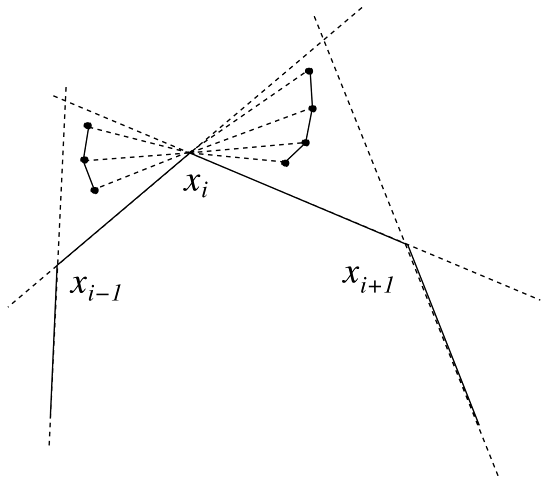

FIGURE 3. A 3-left-cap in Q_{i-1} and a 4-right-cap in Q_i, which forms 7 points in convex position.

region in the arrangement of $L_1 \cup L_2 \cup L_3$. Therefore p_1, p_2, p_3, p_4 are in convex position. The same argument follows in the case that $p_1 \in Y_{i-1}$ and $p_2, p_3, p_4 \in Y_i$. See Figure 3. □

We have for $i \in \{1, \ldots, \lceil n^{1/3} \rceil\}$,

$$|Q_i| \geq |P_i|^{(1-\alpha)} \geq \left(\frac{N}{2^{40k}} \right)^{1-\alpha} \geq 2^{n+2n^{2/3}\log n - 15n^{1/3}\log^2 n}. \tag{1}$$

Set $K = \lceil n^{2/3} \rceil$. Since n is sufficiently large, we have

$$|Q_1| \geq \binom{n+K-4}{K-2} + 1 = f(K, n),$$

which implies that Q_1 contains either an n-right-cap or a K-left-cap. In the former case we are done, so we can assume that Q_1 contains a K-left-cap. Likewise, $|Q_2| \geq \binom{n+K-4}{2K-2} + 1 = f(2K, n-K)$, which implies Q_2 contains either an $(n-K)$-right-cap or a $(2K)$-left-cap. In the former case we are done since Observation 3.1 implies that the K-left-cap in Q_1 and the $(n-K)$-right-cap in Q_2 form n points in convex position. Therefore we can assume Q_2 contains a $(2K)$-left-cap.

In general, if we know that Q_{i-1} contains an $(iK-K)$-left-cap, then we can conclude that Q_i contains an (iK)-left-cap. Indeed, for all $i \leq \lceil n^{1/3} \rceil$ we have

$$\binom{n+K-4}{iK-2} \leq 2^{n+\lceil n^{2/3} \rceil - 4}. \tag{2}$$

Since n is sufficiently large, (1) and (2) imply that

$$|Q_i| \geq 2^{n+2n^{2/3}\log n - 15n^{1/3}\log^2 n} \geq \binom{n+K-4}{iK-2} + 1 = f(iK, n-iK+K).$$

Therefore, Q_i contains either an $(n-iK+K)$-right-cap or an (iK)-left-cap. In the former case we are done by Observation 3.1 (recall that we assumed Q_{i-1} contains an $(iK-K)$-left-cap), and therefore we can assume Q_i contains an (iK)-left-cap.

Hence for $i = \lceil n^{1/3} \rceil$, we can conclude that $Q_{\lceil n^{1/3} \rceil}$ contains an n-left-cap. This completes the proof of Theorem 1.1. ☐

4. Concluding remarks

Following the initial publication of this work on arXiv, we have learned that Gábor Tardos has improved the lower order term in the exponent, showing that $ES(n) = 2^{n+O(\sqrt{n \log n})}$.

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