

CONVEXITY OF THE K -ENERGY ON THE SPACE OF KÄHLER METRICS AND UNIQUENESS OF EXTREMAL METRICS

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1. INTRODUCTION

Let X be an n -dimensional compact complex manifold equipped with a Kähler form ω_0 . In the seminal work of Calabi [15, 16] the problem of finding a canonical Kähler metric in the corresponding cohomology class $[\omega_0] \in H^2(X, \mathbb{R})$ was proposed, in particular a metric with constant scalar curvature. As later shown by Mabuchi [41] such metrics are the critical points of a certain functional on the space of Kähler metrics in $[\omega_0]$ called the K -energy or the *Mabuchi functional*, which we will denote by \mathcal{M} , defined as follows. First recall that the space of all Kähler metrics in $[\omega]$ may be identified with the space $\mathcal{H}(X, \omega)$ of all Kähler potentials, modulo constants, i.e. the space of all functions u on X such that

$$\omega_u := \omega + dd^c u \quad (dd^c := \frac{i}{2\pi} \partial \bar{\partial}),$$

is positive, i.e. defines a Kähler form on X . The space $\mathcal{H}(X, \omega)$ admits a natural Riemannian metric g (of non-positive sectional curvature) that we will refer to as the *Mabuchi metric* [42], where the squared norm of a tangent vector $v \in C^\infty(X)$ at u is defined by

$$(1.1) \quad g_u(v, v) := \int_X v^2 \omega_u^n.$$

Now the Mabuchi functional \mathcal{M} on the infinite dimensional Riemannian manifold $\mathcal{H}(X, \omega)$ is uniquely defined, modulo an additive constant, by the property that its gradient is the normalized scalar curvature of the corresponding Kähler metric,

$$(1.2) \quad \nabla \mathcal{M}|_u := -(R_{\omega_u} - \bar{R}),$$

where \bar{R} denotes the average scalar curvature which, for cohomology reasons, is a topological invariant. The geometric role of the Mabuchi functional was elucidated by Donaldson [28] who showed that—from a dual point of view—it can be identified with the Kempf-Ness “norm-functional” for the natural action of the group of all Hamiltonian diffeomorphisms on the space of all complex structures on X compatible with the symplectic form ω_0 . This interpretation also provides a direct

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link between the Mabuchi functional and the notion of stability in geometric invariant theory (GIT), which in the case when the Kähler class in question is integral, i.e. equal to the first Chern class of an ample line bundle $L \rightarrow X$, has been made precise in the seminal Yau-Tian-Donaldson conjecture saying that $c_1(L)$ contains a Kähler metric with constant scalar curvature if and only if the polarized manifold (X, L) is K -stable [31, 56, 60].

1.1. Statement of the main results. As shown by Mabuchi [41, 42] the functional \mathcal{M} is convex along geodesics u_t in the Riemannian manifold $\mathcal{H}(X, \omega)$. Unfortunately, given u_0 and u_1 in \mathcal{H} there may be no geodesic u_t connecting them (see [26, 40] for recent counterexamples). Still by a result of Chen [18], with complements due to Blocki [14], there always exists a (unique) *weak* geodesic u_t connecting u_0 and u_1 defined as follows. First recall that, by an important observation of Semmes [48] and Donaldson [28], after a complexification of the variable t , the geodesic equation for u_t on $X \times [0, 1]$ may be written as the following complex Monge-Ampère equation on a domain $M := X \times D$ in $X \times \mathbb{C}$ for the function $U(x, t) := u_t(x)$:

$$(1.3) \quad (\pi^*\omega + dd^cU)^{n+1} = 0,$$

where $\pi : M \rightarrow X$ is the natural projection. As shown in [14, 18] for any smoothly bounded domain D in \mathbb{C} the corresponding boundary value problem on M admits a unique solution U such that $\pi^*\omega + dd^cU$ is a positive current with coefficients in L^∞ , satisfying Equation (1.3) almost everywhere. In particular, when D is an annulus in \mathbb{C} , this construction gives rise to the notion of a weak geodesic curve u_t in the extended space $\mathcal{H}_{1,1}$ of all functions u such that ω_u is a positive current with coefficients in L^∞ . Moreover, even if the original defining property (formula (1.2)) of the Mabuchi functional requires that ω_u be positive and C^2 -smooth (and in particular that u be C^4 -smooth), Chen went on to show [19] that the Mabuchi functional admits an explicit formula which is well-defined along a weak geodesic ray u_t as above. (This formula was also independently obtained by Tian; see [57].) Indeed,

$$(1.4) \quad \mathcal{M}(u) = \mathcal{E}(u) + \int_X \log\left(\frac{\omega_u^n}{\omega_0^n}\right)\omega_u^n,$$

where the first term $\mathcal{E}(u)$ is an explicit energy type expression involving the integral over X of a mixed Monge-Ampère expression of the form $u\omega_u^j \wedge \theta_j^{n-j}$ for $j \in [1, n]$, where θ_j are explicit smooth forms depending on ω_0 . The second term is the classical entropy of the measure ω_u^n relative to the reference volume form ω_0^n . As a consequence \mathcal{M} is naturally defined and finite on the space $\mathcal{H}_{1,1}$, where the weak geodesics live. It has been conjectured by Chen that $\mathcal{M}(\phi_t)$ is convex along any weak geodesic as above [19] (this was shown by Chen (for any class $[\omega]$), when $c_1(X) \leq 0$). Our main result confirms this conjecture.

Theorem 1.1. *For any Kähler class $[\omega]$ the Mabuchi functional \mathcal{M} is convex along the weak geodesic u_t connecting any two points u_0 and u_1 in the space \mathcal{H} of ω -Kähler potentials.*

We will also show (Theorem 3.3) that \mathcal{M} is “weakly subharmonic” (see Section 3 for precise definitions) along any curve u_τ satisfying the complex Monge-Ampère equation (1.3) on $X \times D$, as long as Chen’s regularity property holds; i.e. $\pi^*\omega + dd^cU$ is a positive current with coefficients in L^∞ . The subharmonicity of the Mabuchi functional under stronger regularity assumptions on the solution U to Equation

(1.3) (so called “almost smooth” solutions) has been shown by Chen-Tian [22]. The key point of the proof of Theorem 1.1 is a new local positivity property of the relative canonical line bundle $K_{M/D}$ along the one-dimensional current

$$S := (\pi^*\omega + dd^cU)^n$$

in the product $M = X \times D$. This can be seen as a generalization of a positivity property of Monge-Ampère foliations due to Bedford-Burns [2], further developed by Chen-Tian [22], since S can be realized as an average of the leaves of such a foliation, when it exists. But it should be stressed that one of the main points of our approach is that it does not require the existence of any sort of Monge-Ampère foliation. Our proof uses plurisubharmonic variation of local Bergman kernels [11, 44]; see Section 1.2 below for a sketch of the proof and Section 3.2 for comparison with previous results.

We will also give some applications of Theorem 1.1 to Kähler geometry, which have previously—in their full generality—only been shown using the partial regularity theory of Chen-Tian [22]. Very recently, however, it has been shown by Ross and Witt Nyström (see [46]) that the partial regularity results do not hold as stated in [22], so it seems that the earlier proofs are not complete.

We start with the following corollary which follows immediately from the previous theorem, using the “sub-slope property” of convex functions.

Corollary 1.2. *Any Kähler metric with constant scalar curvature minimizes the corresponding Mabuchi functional. More precisely, the following inequality holds:*

$$(1.5) \quad \mathcal{M}(u_1) - \mathcal{M}(u_0) \geq -d(u_1, u_0)\sqrt{\mathcal{C}(u_0)},$$

for any two Kähler potentials u_0 and u_1 on a Kähler manifold (X, ω) , where d is the distance function corresponding to the Mabuchi metric and \mathcal{C} denotes the Calabi energy, i.e. $\mathcal{C}(u) := \int (R_{\omega_u} - \bar{R})^2 \omega_u^n$.

The minimizing property above was first shown by Chen in the case when the first Chern class $c_1(X)$ is non-positive and by Donaldson [29, 30] in the case when the automorphism group of X is discrete and the Kähler class in question is integral, i.e. when it coincides with the first Chern class of an ample line bundle L over X . Shortly afterwards, Li [23] extended Donaldson’s method to also cover the case when the automorphism group is not discrete. (A version of Li’s result for extremal metrics was later given by Sano and Tipler [47].) The general case was treated by Chen-Tian in [22], using their partial regularity theory and approximation arguments, and the inequality (1.5) was then obtained by Chen, building on [22].

In the case of smooth geodesics it is well-known that the Mabuchi functional \mathcal{M} is *strictly* convex modulo automorphisms, or more precisely modulo the group $\text{Aut}_0(X)$ defined as the connected component of the identity in the group of all biholomorphisms of X (see [42] and [28]). If one could establish the corresponding strict convexity for *weak* geodesics—which seems very challenging—then it would immediately imply the uniqueness modulo $\text{Aut}_0(X)$ of the critical points of \mathcal{M} , i.e. of cohomologous Kähler metrics with constant scalar curvature. Here we will show that the conjectural general strict convexity result referred to above is not needed to establish the uniqueness result in question; it follows from a rather general argument combining the convexity in Theorem 1.1 with the well-known fact that

the strict convexity modulo $\text{Aut}_0(X)$ does hold at the linearized level (in other words, the Hessian of \mathcal{M} at a critical point of \mathcal{M} degenerates precisely along the action of holomorphic vector fields).

Theorem 1.3. *Given any two cohomologous Kähler metrics ω_0 and ω_1 on X with constant scalar curvature there exists an element g in the connected component $\text{Aut}_0(X)$ of the identity in the group of all biholomorphisms of X such that $\omega_0 = g^*\omega_1$.*

In the case when $[\omega] = c_1(X)$ this result is due to Bando-Mabuchi [1] while the case $[\omega] = c_1(L)$ with $\text{Aut}_0(X)$ trivial was shown by Donaldson [29], using approximation with so-called balanced metrics attached to high tensor powers of the line bundle L . After that, the case when the cohomology class $[\omega]$ is integral but the automorphism group is allowed to be non-discrete was treated by Mabuchi [43]; see also the very recent work of Seyyedali [49]. The general uniqueness result appears in [22].

Our approach to the uniqueness theorem consists in adding a small strictly convex perturbation to the Mabuchi functional. The perturbed functional is then strictly convex so it can then have at most one critical point. In case $\text{Aut}_0(X)$ is discrete, or equivalently there are no nontrivial holomorphic vector fields on X , it follows from the implicit function theorem that near any (smooth) critical point of the Mabuchi functional there is a critical point of such a perturbed functional, so the Mabuchi functional can also have at most one critical point. In the general case, when $\text{Aut}_0(X)$ is nontrivial, critical points of \mathcal{M} cannot in general be approximated by critical points of the perturbed functional. (Indeed, if this were possible, we would get absolute uniqueness instead of uniqueness modulo automorphisms.) However, we prove that such approximation is possible if we first move the critical point by a suitable automorphism, and this permits us to prove uniqueness modulo automorphisms in the general case. This is the principle of the proof, but in order to avoid technical complications (that arise when there are nontrivial holomorphic vector fields) we will instead work with “approximately critical points” so in the end we avoid the actual use of the implicit function theorem.

More specifically, we will consider the setting of Kähler metrics with constant α -twisted scalar curvature, defined with respect to a given “twisting form” α , i.e. a smooth closed non-negative $(1,1)$ -form on X (see Section 3.1.1), as well as Calabi’s extremal metrics (Section 4.1). As shown in [35] the twisted setting appears naturally in the case when X is realized as the base of a fibration whose fibers are equipped with constant scalar curvature metrics (then the role of the twisting form α is played by the corresponding Weil-Peterson metric on the base X describing the variation of the complex structures of the fibers); see also [52] for the relation to the Kähler-Ricci flow on varieties of positive Kodaira dimension and [53] for the relation to the algebro-geometric slope stability of Ross-Thomas. Let us finally point out that Theorem 1.1 can also be extended to Tian-Zhu’s modified K -energy functional [59], whose critical points are Kähler-Ricci solitons (details will appear elsewhere).

1.2. A sketch of the proof of Theorem 1.1. Let us sketch the proof of Theorem 1.1 in the special case when ω_{u_t} is continuous and strictly positive. The starting point is the following essentially well-known formula for the second order

variation of the Mabuchi functional:

$$(1.6) \quad d_t d_t^c \mathcal{M}(u_t) = \int_X T, \quad T := dd^c \Psi \wedge (\pi^* \omega + dd^c U)^n, \quad \Psi_t := \log(\omega_{u_t}^n).$$

Here Ψ denotes the local weight of the metric on the relative canonical line bundle $K_{M/D} \rightarrow M$ induced by the metrics ω_{u_t} on TX , and \int_X denotes the fiber-wise integral, i.e. the natural map pushing forward a form on $M := X \times D$ to a form on the base D . (This formula follows from (1.4), using that dd^c commutes with push-forwards.) The proof proceeds by showing that the integrand T in formula (1.6) is a non-negative top form on M and in particular its push-forward to D is also non-negative, as desired. First observe that we can locally write $\pi^* \omega + dd^c U = dd^c \Phi$ for a local plurisubharmonic function $\Phi(t, z) = \phi_t(z)$, defined on the unit-ball in \mathbb{C}^n . Accordingly, $\omega_{u_t}^n$ may be written as $(dd^c \phi_t)^n$ locally on X and by well-known convergence results for Bergman kernels going back to Hörmander, Bouche [13], and Tian [55], and the form T can thus be locally realized as the weak limit, as $k \rightarrow \infty$, of the forms T_k defined by

$$T_k := dd^c \log B_{k\phi_t} \wedge (dd^c \Phi)^n,$$

where $B_{k\phi} := K_{k\phi} e^{-k\phi}$ is the Bergman function (density of states function) for the Hilbert space of all holomorphic functions on the unit-ball equipped with the standard L^2 -norm weighted by the factor $e^{-k\phi}$. Finally, by the results on the plurisubharmonic variation of Bergman kernels in [11] the function $\log K_{k\phi_t}$ is plurisubharmonic on $X \times D$ and hence

$$(1.7) \quad dd^c \log B_{k\phi_t} = dd^c \log K_{k\phi_t} - k dd^c \Phi \geq 0 - k dd^c \Phi.$$

Since the latter form vanishes when wedged with $(dd^c \Phi)^n$ (by the geodesic equation), this shows that $T_k \geq 0$. Hence letting $k \rightarrow \infty$ shows that $T \geq 0$, which concludes the proof of Theorem 1.1 under the simplifying assumption that ω_{u_t} is continuous and strictly positive. The proof in the general case involves a truncation procedure (to compensate for the lack of strict positivity of the measures $\omega_{u_t}^n$) and a generalization of the Bergman kernel asymptotics used above to the case when the curvature form $dd^c \phi$ is merely in L_{loc}^∞ .

An intriguing aspect of our proof is that it relies on the individual positivity properties of the two currents $dd^c \log K_{k\phi_t}$ and $-k dd^c \Phi$ appearing in the decomposition (1.7) and these two currents diverge in the “semi-classical” limit $k \rightarrow \infty$ (contrary to their sum which converges to $dd^c \Psi$). Hence, our decomposition argument does not seem to have any direct analog for the current $dd^c \Psi$ itself.

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2. WEAK GEODESICS AND BERGMAN KERNEL ASYMPTOTICS

2.1. Preliminaries. We start by introducing the notation for (quasi-)psh functions (where psh stands for plurisubharmonic) and metrics on line bundles that we will use. Let (X, ω_0) be a compact complex manifold of dimension n equipped with a fixed Kähler form ω_0 , i.e. a smooth real positive closed $(1, 1)$ -form on X . Denote by $PSH(X, \omega_0)$ the space of all ω_0 -psh functions u on X ; i.e. $u \in L^1(X)$, u is upper semicontinuous (usc), and

$$\omega_u := \omega_0 + \frac{i}{2\pi} \partial \bar{\partial} u := \omega_0 + dd^c u \geq 0$$

holds in the sense of currents. We will write $\mathcal{H}(X, \omega_0)$ for the interior of $PSH(X, \omega_0) \cap C^\infty(X)$, i.e. the space of all Kähler potentials (with respect to ω_0). In the *integral case*, i.e. when $[\omega] = c_1(L)$ for a holomorphic line bundle $L \rightarrow X$, the space $PSH(X, \omega_0)$ may be identified with the space \mathcal{H}_L of (singular) Hermitian metrics on L with a positive curvature current. We will use additive notation for metrics on L ; i.e. we identify a Hermitian metric $\|\cdot\|$ on L with its “weight” ϕ . Given a covering (U_i, s_i) of X with local trivializing sections s_i of $L|_{U_i}$ the object ϕ is defined by the collection of open functions $\phi|_{U_i}$ defined by

$$\|s_i\|^2 = e^{-\phi|_{U_i}}.$$

The (normalized) curvature ω of the metric $\|\cdot\|$ is the globally well-defined $(1, 1)$ -current defined by the following local expression:

$$\omega = dd^c \phi.$$

The identification between \mathcal{H}_L and $PSH(X, \omega_0)$ referred to above is obtained by fixing ϕ_0 and identifying ϕ with the function $u := \phi - \phi_0$, so that $dd^c \phi = \omega_u$.

2.1.1. *Weak geodesics and the space $\mathcal{H}_{1,1}$.* As recalled in the Introduction of the paper equipping the space $\mathcal{H}(X, \omega_0)$ with the Mabuchi’s Riemannian metric a curve u_t in $\mathcal{H}(X, \omega_0)$ is a geodesic iff it satisfies a complex Monge-Ampère equation. More precisely, writing $t = \log |\tau|$ for $\tau \in \mathbb{C}$ so that u_t may be identified with an S^1 -invariant function U on $M := X \times D$, where D denotes the corresponding annulus in \mathbb{C} , the $\pi^*\omega$ -psh function U (with π denoting the natural projection from M to X) satisfies

$$(2.1) \quad (\pi^*\omega + dd^c U)^{n+1} = 0,$$

where U thus coincides at the boundary ∂M with the function determined by u_0 and u_1 . As shown in [14, 18] the previous boundary value problem always admits (for any bounded domain D in \mathbb{C}) a weak solution in the sense that $\pi^*\omega + dd^c U$ is a positive current with bounded coefficients, up to the boundary. We say that such functions have $\mathcal{C}_C^{1,1}$ -regularity. In particular, any given two points u_0 and u_1 in $PSH(X, \omega_0)$ are connected by a (unique) weak geodesic u_t as above, defining a curve in the space $\mathcal{H}_{1,1} \subset PSH(X, \omega_0)$ of all u such that $\omega + dd^c u$ is a positive current with components in L^∞_{loc} .

2.2. **Bergman kernel asymptotics.** Given a (possibly non-compact) complex manifold Y with a line bundle $L \rightarrow Y$ equipped with a (bounded) metric ϕ we denote by $K_{k\phi}$ the section of $(kL + K_Y) \otimes \overline{(kL + K_Y)} \rightarrow Y$ determined by the restriction to the diagonal of the Bergman kernel of the space $H^0(Y, kL + K_Y)$ of all global holomorphic sections of $kL + K_Y$ (viewed as holomorphic n -forms on Y with values in kL) equipped with the standard L^2 -norm determined by the metric ϕ (assumed to be finite),

$$(2.2) \quad K_{k\phi}(x) = \sup_{s \in H^0(Y, kL + K_Y)} \frac{s \wedge \bar{s}(x)}{\int_Y s \wedge \bar{s} e^{-k\phi}}.$$

In particular, contracting the corresponding metrics on kL gives a measure on Y that, after a scaling, we write as

$$(2.3) \quad \beta_k := \frac{n!}{k^n} K_{k\phi} e^{-k\phi}.$$

By well-known Bergman kernel asymptotics (due to Bouche [13] and Tian [55], independently) in the case when $Y = X$ the convergence $\beta_k \rightarrow (dd^c\phi)^n$ holds as $k \rightarrow \infty$, uniformly on X , if ϕ is C^2 -smooth and strictly positively curved, i.e. $dd^c\phi > 0$. However, in our setting ϕ will only have a Laplacian in L_{loc}^∞ (and not be strictly positively curved), i.e. ϕ will be in $\mathcal{H}_{1,1}$ and hence the convergence cannot be uniform in general. Moreover, unless the given class $[\omega]$ on X is integral, there will be no line bundle L over X , and then we will have to let Y be a small coordinate ball, identified with the unit-ball in \mathbb{C}^n , taking L as the trivial line bundle. In the next theorem we show that a sufficiently strong version of the convergence still holds in this setting.

Theorem 2.1. *Let $L \rightarrow Y$ be a line bundle over a (possibly non-compact) complex manifold Y and assume that L extends to a holomorphic line bundle over a compact complex manifold X equipped with a (singular) metric ϕ such that the curvature current $dd^c\phi$ is non-negative with components in L_{loc}^∞ (i.e. ϕ is in $\mathcal{H}_{1,1}$). Denote by β_k the Bergman measure on Y defined with respect to the restricted metric on Y . Then, given a smooth volume form dV on a compact subdomain E of Y there exists a constant C such that*

$$(2.4) \quad \beta_k \leq CdV$$

on E , where the constant C only depends on E and an upper bound on the sup-norm of $dd^c\phi$ on E . Moreover, $\beta_k(x) \rightarrow (dd^c\phi)^n$ in total variation norm on E .

Proof. Step one: upper bounds. We will start by showing the uniform upper bound (2.4) together with the following point-wise upper bound:

$$(2.5) \quad \limsup_{k \rightarrow \infty} \beta_k(x) \leq (dd^c\phi)^n(x)$$

at almost any point x of Y (recall that by assumption the right-hand side above has a density which is well-defined almost everywhere on X , so this statement indeed makes sense). The proof will be completely local. Given any point $x_0 \in X$ and local holomorphic coordinates z centered at x_0 we take a local trivializing section s of L such that ϕ is represented by a function $\phi(z)$ satisfying $\phi(0) = 0$. Any given holomorphic section of L may, locally, be written as $f(z)s$ for a local holomorphic function f . In particular, the function $\log|f|^2$ is subharmonic, and hence by the sub-mean inequality for subharmonic functions we have

$$\log|f|^2(0) \leq \int \log|f|^2 d\sigma_r,$$

where $d\sigma_r$ denotes the invariant probability measure on the sphere $|z| = r$. Writing $\log|f|^2 = \log(|f|^2 e^{-k\phi}) + k\phi$ in the right hand side above and applying Jensen's inequality gives

$$|f|^2(0) \exp\left(-\int k\phi d\sigma_r\right) \leq \int |f|^2 e^{-k\phi} d\sigma_r.$$

Accordingly, multiplying both sides with r^{2n-1} , integrating over $r \in [0, Rk^{-1/2}]$, and performing the change of variables $r \mapsto rk^{1/2}$ gives

$$(2.6) \quad |f|^2(0) \left(\int_{|z| \leq Rk^{-1/2}} |f|^2 e^{-k\phi} dV \right)^{-1} \leq C_{R,k} := \left(\int_0^R e^{-r^2 a_\phi(rk^{-1/2})} r^{2n-1} dr \right)^{-1},$$

where

$$a_\phi(r) = \frac{1}{r^2} \int_{|z|=r} \phi d\sigma_r.$$

We claim that

$$(2.7) \quad (i) |a_\phi(r)| \leq C, \quad (ii) \lim_{r \rightarrow 0} a_\phi(r) = a_\phi(0) = \frac{1}{n}(\Delta\phi)(0),$$

where C only depends on an upper bound on $\Delta\phi$ on $B(r) := \{|z| \leq r\}$ and where (ii) holds if 0 is a Lebesgue point for $\Delta\phi$. (Recall that 0 is a Lebesgue point for an L^1 -function h if

$$h(0) = \lim_{r \rightarrow 0} \frac{1}{V(B(r))} \int_{|z| \leq r} h dV,$$

where V denotes the volume of the ball $B(r)$.) Accepting this claim for the moment we can first set $R = 1$ and deduce from (i) that $\beta_k(x)$ is uniformly bounded on any compact subset E . Moreover, to get the precise point-wise bound (2.5) we assume that x is a Lebesgue point for the components of the current $(dd^c\phi)(x)$, i.e. that 0 is a Lebesgue point for the L^∞_{loc} -functions representing the distributional partial derivatives $\frac{\partial^2 \phi}{\partial z_i \partial \bar{z}_j}$. The complement of the set of all such points x is a null set for Lebesgue measure (as follows from Lebesgue's theorem).

Letting $k \rightarrow \infty$ and applying the dominated convergence theorem for R fixed gives, by computing the Gaussian integral,

$$\lim_{R \rightarrow \infty} \lim_{k \rightarrow \infty} C_{R,k} = \left(\int_0^\infty e^{-r^2 a_\phi(0)} r^{2n-1} dr \right)^{-1} = \frac{(a_\phi(0))^n}{\pi^n}.$$

Now recall that $a_\phi(0) = \frac{1}{n}(\Delta\phi)(0)$, so what we need to do is to replace the Laplacian, i.e. the trace of $dd^c\phi$, by the determinant of the same form. For this we note that we can make an arbitrary linear change of variables in the coordinates z without changing the Bergman kernel estimate, if the determinant of the change of variables equals 1. First we change coordinates so that the Hessian of ϕ is diagonal at the origin. Then we apply a diagonal change of coordinates $z_j \rightarrow \mu_j z_j$ with determinant one. By the arithmetic-geometric mean inequality, the infimum

$$\inf_{\mu_j} (1/n) \sum \lambda_j \mu_j$$

over all positive μ_j with product 1 equals $(\prod \lambda_j)^{1/n}$, so taking the infimum over all such changes of coordinates we get that

$$\limsup_{k \rightarrow \infty} \beta_k \leq \det(\phi_{j,\bar{k}}).$$

This concludes the proof of Step one up to the proofs of (i) and (ii) in (2.7) to which we next turn.

First note that in order to establish (ii) it will be enough to show that the limit $a_\phi(0)$ exists and only depends on $(\Delta\phi)(0)$. Indeed, we can then replace $\phi(z)$ with $\phi_0(z) = |z|^2$ and note that, by symmetry, $a_{\phi_0}(0) = 1 = \frac{1}{n}(\Delta\phi_0)(0)$. Denote by $g(z)$ the standard spherical symmetric fundamental solution for the corresponding local Euclidean Laplacian $\Delta := \sum_i \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$ satisfying

$$(2.8) \quad g(1) = 0, \quad \frac{\partial}{\partial r} g(r) = c_n \frac{1}{r^{2n-1}}.$$

Using Green’s formula and integration by parts gives

$$(2.9) \quad c_n a_\phi(R) = R^{-2} \int_{|z| \leq R} (\Delta\phi) g dV = R^{-2} \int_0^R A_\phi(r) \frac{\partial}{\partial r} g(r) dr,$$

where

$$A_\phi(r) := \int_{|z| \leq r} \Delta\phi dV.$$

In particular, since $\Delta\phi \leq C$ on $B(r) =: B_r$ this proves (i) in (2.7). Moreover, if 0 is a Lebesgue point for $\Delta\phi$, then we get $A_\phi(r) = V(B_1)r^{2n}(\Delta\phi)(0) + o(r^{2n})$ and hence, using formula 2.9,

$$c_n a_\phi(R) = V(B_1)(\Delta\phi)(0)R^{-2} \int_0^R r(1 + o(1))dr \rightarrow \frac{1}{2}V(B_1)(\Delta\phi)(0),$$

as $R \rightarrow \infty$. This shows that the limit $a_\phi(0)$ exists and only depends on $(\Delta\phi)(0)$, which proves (ii) in (2.7).

Step two: convergence in total variation norm. First note that by the uniform and point-wise bounds on β_k established in the previous steps it will in order to prove the convergence in total variation norm be enough to show that, for any compact subdomain E of Y ,

$$(2.10) \quad \liminf_{k \rightarrow \infty} \int_E \beta_k \geq \int_E (dd^c \phi)^n.$$

Indeed, writing $\beta_k = f_k dV$ and $(dd^c \phi)^n = f dV$ we get

$$\|\beta_k - (dd^c \phi)^n\| = \int |f_k - f| dV = \int (f - f_k) dV + 2 \int (f - f_k)_-,$$

with $(f - f_k)_- = -\min(f - f_k, 0)$. The limsup of the first integral is less than or equal to zero by (2.10), and the limsup of the second integral is less than or equal to zero by Fatou’s lemma (cf. Lemma 2.2 in [4]).

Next we note that it will be enough to consider the case when Y is compact. Indeed, by assumption (L, ϕ) extends to a compact complex manifold X (with the same hypothesis on ϕ as on Y), and it follows immediately from the definition of Bergman measures that

$$\beta_k \geq \beta_{k,X},$$

where the right-hand side is the Bergman measure defined with respect to (X, L, ϕ) . Hence, once we have established that the bound (2.10) holds for $\beta_{k,X}$, it will automatically hold for β_k . Moreover, in the compact case of X it will be enough to establish the bound (2.10) for $E = X$. Indeed, as pointed out above it implies the convergence in total variation norm on X which in turn implies the lower bound (2.10) on E for $\beta_{k,X}$ and hence the same lower bound on E for β_k .

Finally, to prove the lower bound (2.10) for X compact we can exploit that $H^0(X, kL + K_X)$ is finite dimensional. Indeed, by the Hilbert-Samuel formula, $\dim H^0(X, kL + K_X) = k^n \int c_1(L)^n / n! + o(k^n)$. Moreover, by basic properties of Bergman kernels for finite dimensional Hilbert spaces $\int_X \beta_{k,X} = \frac{n!}{k^n} \dim H^0(X, kL + K_X)$ and hence

$$\lim_{k \rightarrow \infty} \int_X \beta_{k,X} = \int_X (dd^c \phi)^n,$$

which, as pointed out above, concludes the proof of the general convergence. □

For our purposes it will be enough to consider the case when Y is a Euclidean ball in \mathbb{C}^n .

Corollary 2.2. *Let ϕ be a plurisubharmonic function defined on the neighborhood of B_1 such that $\Delta\phi \in L^\infty_{\text{loc}}$ and denote by β_k the Bergman measure for the Hilbert space of all holomorphic functions f on B_1 equipped with the weighted L^2 -norm $\int_{B_1} |f|^2 e^{-k\phi} dV$, where dV denotes the Lebesgue measure. Then $\beta_k \leq C_E dV$ for any given compactly included subdomain E of B_1 and, after perhaps passing to a subsequence,*

$$\lim_{k \rightarrow \infty} \beta_k(x) = (dd^c\phi)^n(x)$$

for almost any x in B_1 .

Proof. Taking L to be the trivial holomorphic line bundle on $Y := B_1$ it will be enough to show the extension property required by the previous theorem. By assumption ϕ is in $\mathcal{H}_{1,1}(B_{1+\epsilon})$, and up to changing ϕ by a harmless additive constant we may assume that $\phi \geq \delta > 0$ on $B_{1+\delta}$. Hence for C sufficiently large $\psi_C := \max\{\phi, C \log |z|^2\}$ coincides with ϕ on a neighborhood of the closed unit-ball B_1 and with $C \log |z|^2$ outside $B_{1+\epsilon/2}$. Moreover, the same property holds when the max is replaced by a suitable regularized max ensuring that ψ_C is also in $\mathcal{H}_{1,1}(B_{1+\epsilon})$. Finally, for C a given positive integer we note that any function coinciding with $C \log |z|^2$ on the complement of a given ball B_R centered at 0 in \mathbb{C}^n extends, in the standard way, to define a metric on the m th tensor power $\mathcal{O}(m) \rightarrow \mathbb{P}^n$ of the hyperplane line bundle on complex projective space, which is smooth and of non-negative curvature on the complement of B_R . This gives the required extension and concludes the proof since L^1 -convergence implies almost everywhere convergence, after passing to a subsequence. (This reduction of a problem for local plurisubharmonic functions to a problem for global metrics on a line bundle was probably first used by Siu in [51].) □

3. CONVEXITY OF THE MABUCHI FUNCTIONAL ALONG WEAK GEODESICS

In this section we will prove our main result, stated as Theorem 1.1 in the Introduction, using the convergence results for local Bergman kernels proved in the previous section. We start by introducing some notation. If ω is a Kähler form on X , then it induces a metric ψ_ω on the anti-canonical line bundle $-K_X := \Lambda^n TX$ for which we will use the suggestive notation

$$\psi_\omega = -\log(\omega^n);$$

i.e. given local holomorphic coordinates ψ_ω is represented by $-\log(\omega^n/idz_1 \wedge d\bar{z}_1 \wedge \dots)$. More generally, given a measure μ , absolutely continuous with respect to the Lebesgue measure, we write ψ_μ for the corresponding metric on $-K_X$ which symbolically means that

$$\mu = e^{-\psi_\mu}.$$

By definition $\text{Ric } \omega$ is the curvature form of the metric ψ_ω , i.e. $\text{Ric } \omega = dd^c\psi_\omega$. The *Mabuchi functional* \mathcal{M} [41] is, with our normalization, the functional on $\mathcal{H} := \mathcal{H}(X, \omega)$ implicitly defined by

$$(3.1) \quad d\mathcal{M}|_u = -n\text{Ric}(\omega_u) \wedge \omega_u^{n-1} + \bar{R}\omega_u^n, \quad \bar{R} := \frac{nc_1(X) \cdot [\omega]^{n-1}}{[\omega]^n},$$

where $d\mathcal{F}|_u$ denotes the differential at ϕ of a given functional \mathcal{F} on the \mathcal{H} , i.e. the measure defined by the following property: for any $v \in C^\infty(X)$

$$\langle d\mathcal{F}|_u, v \rangle = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(u_t),$$

where u_t is any smooth curve in \mathcal{H} such that $\frac{d}{dt}\Big|_{t=0} u_t = v$ (assuming that the measure $d\mathcal{F}|_u$ exists). Given a curve u_t in \mathcal{H} we will identify it with a function U on $X \times D$, for D an annulus in \mathbb{C} (compare Section 2).

The starting point of the proof of Theorem 1.1 is the explicit formula for the Mabuchi functional in [19], which has an “energy part” and an “entropy part.” As there are many different notations (and normalizations) for the energy type functionals in question, we start by introducing our notation. Given a metric ϕ as above we will write

$$(3.2) \quad \mathcal{E}(u) := \int_X \sum_{j=0}^n u \omega_u^{n-j} \wedge \omega_0^j.$$

Similarly, given a closed $(1, 1)$ -form (or current) T we set

$$(3.3) \quad \mathcal{E}^T(u) := \int_X u \sum_{j=0}^{n-1} \omega_u^{n-j-1} \wedge \omega_0^j \wedge T.$$

A standard computation shows that the corresponding differentials are given by

$$(3.4) \quad d\mathcal{E}|_u = (n + 1)\omega_u^n, \quad d\mathcal{E}^T|_\phi = n\omega_u^{n-1} \wedge T.$$

Similarly, the second order variations are given by

$$(3.5) \quad d_\tau d_\tau^c \mathcal{E}(u_\tau) = \int_X (\pi^* \omega + dd^c U)^{n+1}, \quad d_\tau d_\tau^c \mathcal{E}^T(\phi_\tau) = \int_X (\pi^* \omega + dd^c U)^n \wedge \pi^* T,$$

where \int_X denotes the fiber-wise integral, i.e. the push-forward map induced by the natural projection π from $X \times D$ to X . Finally, we recall that the *entropy* of a measure μ relative to a reference measure μ_0 is defined as follows if μ is absolutely continuous with respect to μ_0 :

$$(3.6) \quad H_{\mu_0}(\mu) := \int_X \log \left(\frac{d\mu}{d\mu_0} \right) d\mu.$$

There is a well-known interpretation of the entropy functional as a Legendre transform that we will have to use at several occasions later on; see [39].

Proposition 3.1. *If μ_0 and μ are probability measures on X such that μ is absolutely continuous with respect to μ_0 , then*

$$H_{\mu_0}(\mu) = \sup_f \left(\int_X f d\mu - \log \int_X e^f d\mu_0 \right),$$

where the supremum is taken over all continuous functions on X .

Proof. First note that Jensen’s inequality gives

$$\exp \int_X (f - \log(d\mu/d\mu_0)) d\mu \leq \int_X e^f d\mu_0.$$

Taking logarithms and rearranging this gives the \geq direction of the inequality. The other direction follows by approximating $\log(d\mu/d\mu_0)$ by continuous functions f . □

For future use we record two immediate consequences of this: The entropy is a convex function of the measure μ for the natural affine structure on the space of probability measures. Second, as the supremum of a set of continuous functions, the entropy is lower semicontinuous with respect to the weak*-topology.

Now we can state the explicit formula in [19], written in our notation, for the Mabuchi functional \mathcal{M} implicitly defined (up to an additive constant) by formula (3.1).

Proposition 3.2. *Given a Kähler metric ω_0 on X with volume form $\mu_0 := \omega_0^n$ of total mass $[\omega]^n$ the following formula holds for the Mabuchi functional on the corresponding space \mathcal{H} of all Kähler potentials:*

$$(3.7) \quad \mathcal{M}(u) = \left(\frac{\bar{R}}{n+1} \mathcal{E}(u) - \mathcal{E}^{Ric\omega_0}(u) \right) + H_{\mu_0}(\omega_u^n), \quad \bar{R} := \frac{nc_1(X) \cdot [\omega_0]^{n-1}}{[\omega_0]^n}.$$

Proof. For completeness and as a way to check our normalizations we recall the proof. A direct calculation gives

$$\frac{d}{dt} H_{\mu_0}(\omega_{u_t}^n) = 0 + \int \log \frac{\omega_{u_t}^n}{\omega_0^n} \frac{d\omega_{u_t}^n}{dt} = -n \int_X \frac{du_t}{dt} Ric\omega_{u_t} \wedge \omega_{u_t}^{n-1} + n \int_X \frac{du_t}{dt} Ric\omega_0 \wedge \omega_{u_t}^{n-1}$$

(using, in the first equality, that ω_0^n has the same mass as $\omega_{u_t}^n$ and, in the second equality, one integration by parts). Hence, since $d\mathcal{E}_{|u}^T = nT \wedge \omega_u^{n-1}$ (formula (3.4)), we get $d(H_{\mu_0} - \mathcal{E}^{Ric\omega_0})|_u = -n Ric\omega_u \wedge \omega_u^{n-1}$, which coincides with the first term in the defining expression for $d\mathcal{M}|_u$ (formula (3.1)). Finally, since $d\mathcal{E}|_u = (n+1)\omega_u^n$ (formula (3.4)), this shows that the differential of the functional defined by the right hand side in formula (3.7) has the desired property. \square

Following Chen [19] we now extend the functional \mathcal{M} from \mathcal{H} to the space $\mathcal{H}_{1,1}$ of all u such that $\omega + dd^c u$ is a positive current with L^∞ -coefficients, using the formula in the previous proposition. Theorem 1.1 claims that this functional is convex along weak geodesics.

It is not a priori clear that the functional is continuous along weak geodesics. (We thank Sebastien Boucksom and Mihai Păun for pointing this out to us.) It does follow from pluripotential theory that the energy parts of the formula are continuous since the potential varies continuously from fiber to fiber. The entropy part, however, is only known to be lower semicontinuous. Therefore, we will first state the basic result concerning distributional derivatives and show the required continuity in our setting afterwards. In the theorem below we say that a function v of one complex variable is *weakly subharmonic* if $i\partial\bar{\partial}v \geq 0$ in the sense of currents. Similarly, we say that a function of one real variable is *weakly convex* if its second derivative in the sense of distributions is non-negative.

Theorem 3.3. *Let u_τ be a family of functions in $PSH(X, \omega)$ such that $\omega + dd^c U$ is a locally bounded current, $\pi^*\omega + dd^c U \geq 0$ and $(\pi^*\omega + dd^c U)^{n+1} = 0$ on $X \times D$. Then the Mabuchi functional $\mathcal{M}(u_\tau)$ is weakly subharmonic with respect to $\tau \in D$. In particular, $\mathcal{M}(u_t)$ is weakly convex along the weak geodesic u_t connecting any two given points in $\mathcal{H}(X, \omega)$.*

Proof. Let $\Psi = \Psi(\tau, x) = \psi_\tau(x)$ be a locally bounded singular metric on the relative canonical line bundle $K_{M/D}$, and denote by $f^\Psi(\tau)$ the following function

on D attached to Ψ :

$$f^\Psi(\tau) := \left(\frac{\bar{R}}{n+1} \mathcal{E}(u_\tau) - \mathcal{E} \text{Ric}_{\omega_0}(u_\tau) \right) + \int_X \log\left(\frac{e^{\psi_\tau}}{\omega_0^n}\right) \omega_{u_\tau}^n$$

(the definition is made so that $f^\Psi(\tau) = \mathcal{M}(u_\tau)$ if Ψ is the (unbounded) metric defined by $\omega_{u_\tau}^n$). Then we claim that

$$(3.8) \quad dd^c f^\Psi(\tau) = \int_X T, \quad T := dd^c(\Psi \wedge (\pi^* \omega_0 + dd^c U)^n),$$

where T is defined as an $(n+1, n+1)$ current (distribution), which a priori may not be of order zero. More precisely, for a local smooth test function v supported on a local coordinate neighborhood $V \subset M$ the current T is locally defined by

$$\langle T, v \rangle = \int \Psi_V (\pi^* \omega_0 + dd^c U)^n \wedge dd^c v,$$

where Ψ_V is a local function representing the metric Ψ on $K_{M/D}$ (given a local trivialization of $K_{M/D}$). To prove formula (3.8) take a sequence Ψ_j of uniformly bounded smooth metrics such that $\Psi_j \rightarrow \Psi$ almost everywhere on X for every τ (which may be constructed using local convolution and a partition of the unity). Then a direct calculation (using formula (3.5)) gives

$$(3.9) \quad dd^c f^{\Psi_j}(\tau) = \eta_j := \int_X T_j, \quad T_j := dd^c(\Psi_j \wedge (\pi^* \omega_0 + dd^c U)^n).$$

By the dominated convergence theorem $\eta_j \rightarrow \eta := \int_X T$ weakly on D (in the sense of distributions). Moreover, by the dominated convergence theorem $f^{\Psi_j}(\tau) \rightarrow f^\Psi(\tau)$ point-wise on D , in a dominated manner and hence, since the linear operator dd^c is continuous under such convergence, the desired formula (3.8) follows from formula (3.9).

We want to apply these considerations to $\Psi = \log(\omega_0 + dd^c_X U)^n$, but we cannot do so immediately since this metric is not locally bounded. For this reason we next introduce a truncation in the following way. For a fixed positive number A , we define

$$\Psi_A := \max\{\log(\omega_0 + dd^c_X u_\tau)^n, \chi - A\},$$

where χ denotes a suitable fixed continuous metric on $K_{M/D}$, to be constructed below. We claim that the current

$$T_A := dd^c \Psi_A \wedge (\pi^* \omega_0 + dd^c U)^n$$

satisfies $T_A \geq 0$, i.e. is defined by a positive measure, if χ is chosen to be continuous and such that

$$dd^c \chi \geq -k_0(\pi^* \omega_0 + dd^c U)$$

for some positive integer k_0 . As explained above this will imply that

$$f^{\Psi_A}(\tau) := \left(\frac{\bar{R}}{n+1} \mathcal{E}(u_\tau) - \mathcal{E} \text{Ric}_{\omega_0}(u_\tau) \right) + \int_X \log\left(\max\left\{\frac{\omega_{u_\tau}^n}{\omega_0^n}, \frac{\chi - A}{\omega_0^n}\right\}\right) \omega_{u_\tau}^n$$

is subharmonic for any $A > 0$. Letting $A \rightarrow \infty$ and invoking the dominated convergence theorem we get $f^{\Psi_A}(\tau) \rightarrow \mathcal{M}(u_\tau)$ which will conclude the proof of the theorem.

To construct χ we first let χ_0 be an arbitrary smooth metric on K_X . Then we set $\chi := \pi^* \chi_0 - k_0 U$ where k_0 is sufficiently large to ensure that $dd^c \chi_0 + k_0 \omega_0 \geq 0$. Then

$$dd^c \chi = \pi^* dd^c \chi_0 - k_0(\pi^* \omega_0 + dd^c U) + k_0 \pi^* \omega_0 \geq -k_0(\pi^* \omega_0 + dd^c U),$$

so χ fulfills our requirement.

Now, the claim that $T_A \geq 0$ is a local statement. Accordingly, we locally write

$$\pi^* \omega_0 + dd^c U = dd^c \Phi$$

for a local psh function Φ on M and write $\phi_\tau = \Phi(\cdot, \tau)$. Our proof proceeds by a local approximation argument involving the local Bergman measures $\beta_{k\phi_\tau}$ (that we identify with their density with respect to a volume form $c_n dz \wedge d\bar{z}$) for the Hilbert space of all holomorphic functions on the unit-ball in Euclidean \mathbb{C}^n equipped with the weight $k\phi_\tau$; see Section 2.2. More precisely, consider the following local current:

$$T_{A,k} := dd^c \Psi_{A,k} \wedge (dd^c \Phi)^n, \quad \Psi_{A,k} := \max\{\log \beta_k, \chi - A\}.$$

By Theorem 2.1 and the dominated convergence theorem

$$\lim_{k \rightarrow \infty} T_{k,A} = T_A$$

in the local weak topology of currents. Thus, to prove that $T_A \geq 0$ it will be enough to prove that the locally defined $(n + 1, n + 1)$ -current $T_{k,A}$ is a positive measure. To fix ideas we first observe that the following current is positive:

$$T_k := dd^c \Psi_k \wedge (dd^c \Phi)^n, \quad \Psi_k := \log(\beta_k)$$

(which formally corresponds to the case $A = \infty$). Indeed, by the results on the plurisubharmonic variation of Bergman kernels in [11] $dd^c \log K_{k\phi_t} \geq 0$ on $X \times D$ and hence

$$(3.10) \quad dd^c \log \beta_k \geq -k dd^c \Phi.$$

As a consequence,

$$T_k := dd^c \log \beta_k \wedge (dd^c \Phi)^n \geq -k(dd^c \Phi) \wedge (dd^c \Phi)^n = 0,$$

using the geodesic equation (2.1) in the last equality. Moving to the case when $A \neq \infty$ we note that, by construction, $\Psi_{A,k}$ is the max of two local functions whose curvature forms are bounded from below by $-k dd^c \Phi$ (for $k \geq k_0$), and hence $\Psi_{A,k}$ also satisfies

$$(3.11) \quad dd^c \Psi_{A,k} \geq -k dd^c \Phi.$$

Now arguing precisely as above (and using the inequality (3.11)) we see that $T_{k,A} \geq 0$. Moreover, by Corollary 2.2

$$e^{\Psi_{A,k}} := \max\left\{\frac{n!}{k^n} K_{k\phi_t} e^{-k\phi_t}, e^{-(\chi-A)}\right\} \rightarrow \max\{(dd^c \phi)^n, e^{-(\chi-A)}\},$$

as $k \rightarrow \infty$ point-wise almost everywhere on X and for every τ in a dominated fashion (after passing to a subsequence with respect to k). Hence, invoking the dominated convergence theorem gives the following local weak convergence:

$$\lim_{k \rightarrow \infty} T_{k,A} = T_A.$$

In particular, this shows that $T_A \geq 0$, and as explained above this concludes the proof of the theorem. □

Before going on to prove the continuity of the Mabuchi functional we point out that the previous proof simplifies somewhat in case the cohomology class of ω is integral. Then we can write

$$\omega_0 + dd^c u_\tau = dd^c \phi_\tau,$$

where ϕ_τ is for each τ the weight of a metric on a positive line bundle L . We can then consider the Bergman kernels for the spaces $H^0(X, K_X + kL)$, induced by the metrics $k\phi_\tau$ (instead of the local Bergman kernels that we used in the proof for the general case), and their Bergman measures

$$\beta_{k\tau} = K_{k\phi_\tau} e^{-k\phi_\tau} k^{-n}.$$

We define

$$\Psi_A = \max\{\log(dd^c \phi_\tau^n), \chi - A\}$$

and

$$\Psi_{A,k} = \max\{\log(\beta_{k\tau}), \chi - A\}$$

and use these metrics on the relative canonical bundle $K_{M/D}$ to define functions $f^{\Psi_A}(\tau)$ and $f^{\Psi_{A,k}}(\tau)$ as in the very beginning of the proof. We then get that point-wise $f^{\Psi_{A,k}}$ tends to $f^{\Psi_A}(\tau)$ as $k \rightarrow \infty$ and that $f^{\Psi_A}(\tau)$ tends to $\mathcal{M}(\tau)$ as $A \rightarrow \infty$. Moreover, $f^{\Psi_{A,k}}(\tau)$ is subharmonic by the same argument as before, and it follows that \mathcal{M} is at least weakly subharmonic. We will have use for this remark in the proof of the continuity.

Theorem 3.4. *\mathcal{M} is continuous along weak geodesics and therefore convex in the point-wise sense.*

Proof. Here we assume that the function U defines a weak geodesic, so we may assume that it depends only on $t := \text{Re } \tau$. We first consider the case when the class is integral. The functionals $f^{\Psi_{A,k}}(\tau)$ are then clearly continuous with respect to τ since by the continuity of the metric ϕ_τ , the Bergman kernels depend continuously on τ . Hence $f^{\Psi_{A,k}}$ are convex in the ordinary point-wise sense. These functions converge point-wise to f^{Ψ_A} as $k \rightarrow \infty$, so these functions are also convex. Finally, as $A \rightarrow \infty$, we get that the Mabuchi functional is also convex. As a convex function, \mathcal{M} is thus continuous on the open interval and upper semicontinuous on the closed interval. By the lower semicontinuity of the entropy, \mathcal{M} is always lower semicontinuous, so we conclude that \mathcal{M} is in fact continuous on the closed interval.

We will now sketch how this argument can be adapted to the general case. Then we define Ψ_A as in the proof of Theorem 3.3. It is enough to prove that the corresponding function f^{Ψ_A} is convex (in the point-wise sense) since then we can take the limit as $A \rightarrow -\infty$ and get that \mathcal{M} is convex, and we conclude as in the integral case that \mathcal{M} is continuous on the closed interval.

Let $\kappa_\epsilon(s)$ be a sequence of strictly convex functions with $\kappa'_\epsilon \geq 1$ on the real line tending to s as $\epsilon \rightarrow 0$; e.g. we may take $\kappa_\epsilon(s) = s + \epsilon e^s$. We define $f_\epsilon^{\Psi_A}$ just like f^{Ψ_A} , but replace the factor

$$\log\left(\frac{e^{\psi_A}}{\omega_0^n}\right)$$

in the entropy term by

$$\kappa_\epsilon\left(\log\left(\frac{e^{\psi_A}}{\omega_0^n}\right)\right).$$

It is enough to prove that these functions are convex for all $\epsilon > 0$, and we already know by the same argument as in the proof of Theorem 3.3 that they are weakly convex. We let ξ_j^2 be a partition of unity subordinate to a covering of coordinate patches over which L is trivial and consider the local entropy functions

$$H_j = \int_X \xi_j^2 \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A\tau}}}{\omega_0^n} \right) \right).$$

We define $H_j^{(k)}$ in a similar way, replacing Ψ_A by its k th approximation by local Bergman kernels. Taking the dd^c of $H_j^{(k)}$, using the plurisubharmonic variation of Bergman kernels, and using the strict convexity of κ_ϵ , a direct estimate shows that

$$dd^c H_j^{(k)} \geq -C_\epsilon,$$

so $H_j^{(k)} + C_\epsilon t^2$ is convex since our local Bergman kernels depend continuously on t . (Here we have no control of the size of C_ϵ , but this does not matter.) Letting $k \rightarrow \infty$ we find that $H_j + C_\epsilon t^2$ is also convex. We can then sum over j and conclude that

$$\int_X \kappa_\epsilon \left(\log \left(\frac{e^{\psi_{A\tau}}}{\omega_0^n} \right) \right) + C_\epsilon t^2$$

is convex and in particular continuous. Therefore, $f_\epsilon^{\Psi_A}$ is also continuous and thus convex in the point-wise sense, since we already know that it is weakly convex. (Notice that the size of C_ϵ plays no role here.) This completes the proof. \square

3.1. Proof of Corollary 1.2. Fix u_0 and u_1 in \mathcal{H} , and denote by u_t the corresponding weak geodesic. By the “sub-slope inequality” for the convex function $f(t) := \mathcal{M}(u_t)$, i.e. $f(1) - f(0) \geq f'(0)$, we have

$$\mathcal{M}(u_1) - \mathcal{M}(u_0) \geq f'(0) \geq \int_X (-R_{\omega_{u_0}} + \bar{R}) \frac{du_t}{dt} \Big|_{t=0} \omega_{u_0}^n,$$

where the lower bound for $f'(0)$ is obtained by direct differentiations as in the proof of Proposition 3.2 (see Lemma 3.5 below). In particular, if ω_{u_0} has constant scalar curvature, then it minimizes the Mabuchi functional. More generally, applying the Cauchy-Schwartz inequality to the right-hand side of the inequality above and using that $d(u_0, u_1)^2 = \int (\dot{u}_t|_{t=0})^2 \omega_{u_0}^n$ (see [18]) concludes the proof.

Lemma 3.5. *Given $u_0, u_1 \in \mathcal{H}$, let u_t be the corresponding weak geodesic curve. Then*

$$\lim_{t \rightarrow 0^+} \frac{\mathcal{M}(u_t) - \mathcal{M}(u_0)}{t} \geq \int_X (-R_{\omega_{u_0}} + \bar{R}) \frac{du_t}{dt} \Big|_{t=0} \omega_{u_0}^n.$$

Proof. This is shown by refining the argument in the proof of Proposition 3.2. We will first handle the entropy part, i.e. show that

$$\begin{aligned} & \lim_{t \rightarrow 0^+} (1/t)(H_{\mu_0}(\omega_{u_t}^n) - H_{\mu_0}(\omega_{u_0}^n)) \geq \\ & -n \int_X \frac{du_t}{dt} \Big|_{t=0} \text{Ric} \omega_{u_0} \wedge \omega_{u_0}^{n-1} + n \int_X \frac{du_t}{dt} \Big|_{t=0} \text{Ric} \omega_0 \wedge \omega_{u_0}^{n-1}. \end{aligned}$$

Here we use the fact that the entropy is convex with respect to the affine structure on the space of probability measures (cf. Proposition 3.1), so that

$$H_{\mu_0}(\nu_1) - H_{\mu_0}(\nu_0) \geq (d/ds)|_{s=0} H_{\mu_0}(\nu_s)$$

if $\nu_s = s\nu_1 + (1 - s)\nu_0$. Moreover, since $\log(\nu_s/\mu_0)\nu_s$ is convex in s , it follows from monotone convergence that

$$(d/ds)|_{s=0}H_{\mu_0}(\nu_s) = \int \log(\nu_0/\mu_0)(d\nu_1 - d\nu_0).$$

From this we get, choosing $\nu_1 = \omega_{u_t}^n$ and $\nu_0 = \omega_{u_0}^n$, that

$$\frac{1}{t} (H_{\mu_0}(\omega_{u_t}^n) - H_{\mu_0}(\omega_{u_0}^n)) \geq \int \log(\omega_{u_0}^n/\mu_0) \frac{1}{t} (\omega_{u_t}^n - \omega_{u_0}^n).$$

Expand $\omega_{u_t}^n - \omega_{u_0}^n = dd^c(u - u_t) \wedge (\omega_{u_0}^{n-1} + \dots + \omega_{u_t}^{n-1})$ and use integration by parts to let the dd^c -operator instead act on the smooth function $\log \frac{\omega_{u_0}^n}{\mu_0}$. Then letting $t \rightarrow 0$ we get the desired inequality for the entropy part of $\mathcal{M}(u_t)$. The calculation for the derivative of the “energy part” of \mathcal{M} follows immediately from the relations (3.4). □

3.1.1. *The twisted setting.* Later on we will also consider “twisted” versions of the Mabuchi functional. These are obtained simply as the sum of \mathcal{M} and another convex functional \mathcal{F} . We will consider two main cases. The first is to let μ be a strictly positive smooth volume form on X and put

$$\mathcal{F}(u) = \mathcal{F}_\mu(u) := \int_X u d\mu - c_\mu \mathcal{E}(u),$$

with c_μ chosen so that $\mathcal{F}_\mu(1) = 0$. Clearly \mathcal{F}_μ is convex along weak geodesics since its derivative is

$$(d/dt)\mathcal{F}_\mu(u_t) = \int_X u'_t d\mu - (d/dt)\mathcal{E}(u_t).$$

The first term here is increasing since u'_t is increasing, and the second term is constant since the energy is linear along weak geodesics. The next choice is to let α be a strictly positive $(1, 1)$ -form on X and let

$$\mathcal{F} = \mathcal{F}_\alpha := \mathcal{E}^\alpha - c_\alpha \mathcal{E},$$

the constant c_α again chosen so that \mathcal{F} vanishes on constants. By formula (3.5) \mathcal{F}_α is again convex along (sub)geodesics, since it is clearly continuous. (The strict convexity seems to be a more subtle issue that for simplicity we do not discuss here.) The critical points of $\mathcal{M}_\alpha := \mathcal{M} + \mathcal{F}_\alpha$ are said to have constant α -twisted scalar curvature; i.e. they satisfy an equation,

$$R_\omega - \text{tr}_\omega(\alpha) = \text{constant}_\alpha;$$

see [35, 53]. Just as before it follows that any metric with constant α -twisted scalar curvature minimizes \mathcal{M}_α . As a consequence, the α -twisted Mabuchi functional is bounded from below in any Kähler class containing a metric with constant α -twisted scalar curvature. As shown in [53] this leads to geometric obstructions for the existence of such metrics.

3.2. A positivity property for solutions to homogeneous Monge-Ampère equation and its relation to foliations. The proof of Theorem 3.3 yields the following positivity result of independent interest, for sufficiently regular solutions to the local homogeneous Monge-Ampère equation on a product domain (in the proof of Theorem 3.3 the role of the current S below is played by $(dd^c\Phi)^n$).

Theorem 3.6. *Let Φ be a plurisubharmonic function on $M := X \times D$ where X and D are domains in \mathbb{C}^n and \mathbb{C} , respectively, and assume that the positive current $dd^c\Phi$ has components in L_{loc}^∞ and satisfies $(dd^c\Phi)^{n+1} = 0$. Then the singular metric induced by the fiber-wise currents $\omega_\tau := dd^c\phi_\tau$ on the relative canonical line bundle $K_{M/D} \rightarrow M$ has non-negative curvature along any positive current S in M of bidimension $(1, 1)$ with the property that Φ is harmonic along S , i.e. $\langle dd^c\Phi, S \rangle = 0$. More precisely, for any positive number A*

$$i\partial\bar{\partial}\log_A \det\left(\frac{\partial^2\phi_\tau}{\partial z_i\partial\bar{z}_j}\right) \wedge S \geq 0,$$

in terms of the truncated logarithm defined by $\log_A t := \max\{\log t, -A\}$.

In particular, if Φ happens to admit a Monge-Ampère foliation, then the positivity result above holds along the leaves of the foliation. This observation is closely related to a previous local result of Bedford-Burns (see Proposition 4.1 in [2]) and Chen-Tian who considered the case when Φ corresponds to a global bona fide geodesic u_t in the space of Kähler potentials on a Kähler manifold (X, ω) (see Corollary 4.2.11 in [22]). Then, by a classical result of Bedford-Kalka (which only demands that Φ be C^3 -smooth), there is a foliation of $M := X \times D$ in one-dimensional complex curves \mathcal{L}_α (the leaves) such that the local potential Φ is harmonic along any leaf \mathcal{L}_α . Moreover, the leaves are transverse to the slice $X \times \{0\}$ (and hence the latter space can be used as the parameter space for the set of leaves). In this setting the results of Bedford-Burns and Chen-Tian referred to above may be formulated as the following special case of the previous theorem.

Proposition 3.7. *Consider the relative canonical line bundle $K_{M/A}$ with the smooth metric induced by the volume forms $(dd^c\phi_t)^n$. Then its restriction to any leaf \mathcal{L}_α has non-negative curvature.*

Interestingly, in the presence of a foliation as above the closed positive current $S := (dd^c\Phi)^n$ on M of dimension $(1, 1)$, appearing in the proof of Theorem 3.3, can be written as an average of the integration currents $[\mathcal{L}_\alpha]$ defined by the leaves of the foliation,

$$S = \int_{\alpha \in X} [\mathcal{L}_\alpha] \mu,$$

where $\mu := (dd^c\phi_0)^n$.

Another special case of Theorem 3.4, concerning the case when the current S is assumed to be a smooth complex curve (but not necessarily a leaf of a foliation) and Φ is C^2 -smooth has previously appeared in connection to the problem of constructing low regularity (i.e. not C^2) solutions to complex Monge-Ampère equations (see Lemma in [3] and Proposition 2.2 in [26]).

4. UNIQUENESS RESULTS

In this section we shall show how the convexity of the K -energy implies uniqueness of metrics, up to automorphisms, of metrics of constant scalar curvature and more generally extremal metrics. Recall that $\mathcal{H}(X, \omega)$ denotes the space of (smooth) potentials of Kähler metrics on X that are cohomologous to a fixed reference metric $\omega > 0$ (see the Introduction). The tangent space of \mathcal{H} is the space of smooth

functions on X , and we can identify the space of Kähler metrics cohomologous to ω with \mathcal{H} modulo constants. We will use the twisted Mabuchi functionals from Section 3.1.1 and start with some preparations.

Let $\mu > 0$ be a smooth volume form on X that for simplicity we normalize so that

$$\int_X d\mu = \int_X \omega^n.$$

We have then defined the function

$$\mathcal{F}_\mu(u) = \int_X u d\mu - \mathcal{E}(u) := I_\mu(u) - \mathcal{E}(u)$$

in Section 3.1.1. The basic idea is to use the twisted Mabuchi functionals

$$\mathcal{M}_s := \mathcal{M} + s\mathcal{F}_\mu,$$

for $0 < s \ll 1$. The main difficulty in the proof is that although we know that \mathcal{M} is convex along generalized geodesics, we do not know when it is linear along the geodesics. (Conjecturally this holds only for geodesics that come from the flow of a holomorphic vector field.) Therefore, we perturb \mathcal{M} by adding $s\mathcal{F}_\mu$ which gives us a strictly convex functional. In case there are no nontrivial holomorphic vector fields on X , one can prove by the implicit function theorem that near each critical point of \mathcal{M} there is a critical point of \mathcal{M}_s . By strict convexity, there can be at most one critical point of \mathcal{M}_s , and it follows that there is at most one critical point of \mathcal{M} too. In case there are holomorphic vector fields it of course no longer holds that there are critical points of \mathcal{M}_s near each critical point of \mathcal{M} —if it did, we would get absolute uniqueness and not just uniqueness up to automorphisms. However, it turns out that each critical point of \mathcal{M} can be moved by an element in $\text{Aut}_0(X)$ to a new critical point, which *can* be approximated by critical points of \mathcal{M}_s , and this gives uniqueness up to automorphisms. The proof of this latter fact requires a rather sophisticated version of the implicit function theorem, so to simplify we shall instead work with “almost critical points,” which avoids the use of the implicit function theorem.

With our normalization, \mathcal{F}_μ vanishes on constants so it descends to a functional on the space of Kähler forms in $[\omega]$. We have already seen that \mathcal{F}_μ is convex; next we shall prove that it is strictly convex in a certain sense. Since \mathcal{E} is linear, this amounts to proving the strict convexity of I_μ .

Proposition 4.1. *I_μ is strictly convex along $C^{1,1}$ -subgeodesics in the sense that if u_t is a $C^{1,1}$ -subgeodesic and $f(t) := I_\mu(u_t)$ is affine, then $\omega_t = dd^c u_t + \omega$ is constant. More precisely, if $\omega_t = dd^c u_t + \omega \leq C\omega$ and $\mu \geq A\omega^n$, then*

$$f'(1) - f'(0) \geq \delta A / (C^{n+1}) d(\omega_0, \omega_1)^2,$$

where $\delta > 0$ only depends on μ, ω , and X , and $d(\omega_0, \omega_1)$ is the Mabuchi distance.

Proof. Assume first that u_t is a smooth subgeodesic and $\omega_t > 0$ for all t . Then

$$f''(t) = \int_X \ddot{u}_{tt} d\mu \geq \int_X |\bar{\partial}\dot{u}_t|_{\omega_t}^2 d\mu,$$

since u_t is a subgeodesic. Assume $\omega_t \leq C\omega$ for all t . Then

$$|\bar{\partial}\dot{u}_t|_{\omega_t}^2 \geq C^{-1} |\bar{\partial}\dot{u}_t|_{\omega}^2.$$

Since ω and μ are fixed and \dot{u}_t is a function, we have that

$$\int_X |\bar{\partial}\dot{u}_t|_\omega^2 d\mu \geq \delta \int_X |\dot{u}_t - a_t|^2 d\mu,$$

where a_t is the average of \dot{u}_t with respect to μ and δ only depends on μ, ω , and X . Hence

$$f''(t) \geq \delta/C \int_X |\dot{u}_t - a_t|^2 d\mu.$$

Clearly it follows that $\dot{u}_t = a_t$ if f is affine. If u_t is only of class $C^{1,1}$, we can write u_t as a decreasing limit of subgeodesics that converge uniformly in C^1 and are such that the constant C can be kept fixed. It then follows that $f(t)$ also converges in C^1 , and we get that

$$\int_0^1 dt \int_X |\dot{u}_t - a_t|^2 d\mu = 0,$$

so $\dot{u}_t = a_t$ again. Hence ω_t is independent of t .

For the second statement we notice that we also have proved that

$$f'(1) - f'(0) \geq (\delta/C) \int_0^1 dt \int_X |\dot{u}_t - a_t|^2 d\mu.$$

But

$$\int_X |\dot{u}_t - a_t|^2 d\mu \geq AC^{-n} \int_X |\dot{u}_t - a_t|^2 \omega_t^n \geq AC^{-n} \int_X |\dot{u}_t - b_t|^2 \omega_t^n,$$

where b_t is the average of \dot{u}_t with respect to ω_t^n . Since

$$\int_0^1 dt \int_X |\dot{u}_t - b_t|^2 \omega_t^n = d(\omega_1, \omega_0)^2,$$

we have also proved the second statement. □

We will also need a lemma on how \mathcal{F}_μ depends on μ .

Lemma 4.2. *Let μ and ν be two smooth volume forms with total mass equal to the mass of ω^n . Then*

$$|\mathcal{F}_\mu(u) - \mathcal{F}_\nu(u)| \leq C_{\mu,\nu}$$

for all u in \mathcal{H} .

Proof. By Yau’s solution of the Calabi conjecture, we can write

$$\mu = \omega_\mu^n, \quad \nu = \omega_\nu^n,$$

with ω_μ and ω_ν in $[\omega]$. (Of course, the proof does not really depend on the solution of the Calabi conjecture, since we could have used only volume forms that are given as powers of Kähler forms in the proof.) Then $\omega_\mu - \omega_\nu = dd^c v$ for some function v on X . Hence

$$\mathcal{F}_\mu(u) - \mathcal{F}_\nu(u) = \int_X u(\omega_\mu^n - \omega_\nu^n) = \int_X u(dd^c v \wedge \sum \omega_\mu^{n-k-1} \wedge \omega_\nu^k).$$

Integration by parts gives

$$\mathcal{F}_\mu(u) - \mathcal{F}_\nu(u) = \int_X v(dd^c u \wedge \sum \omega_\mu^{n-k-1} \wedge \omega_\nu^k) = \int_X v(\omega_u - \omega) \wedge \sum \omega_\mu^{n-k-1} \wedge \omega_\nu^k,$$

which is clearly bounded by a constant depending only on the sup-norm of v and the volume of $[\omega]$. □

Next we discuss briefly the Hessian of \mathcal{M} on the space of smooth Kähler potentials. Denote $F = d\mathcal{M}$, the differential of the Mabuchi functional. It is a 1-form on \mathcal{H} whose action on an element v of the tangent space of \mathcal{H} , i.e. a smooth function, is given by

$$F(u).v = - \int_X v(R_{\omega_u} - \hat{R}_{\omega_u})\omega_u^n.$$

The Mabuchi metric induces a connection D on the tangent bundle of \mathcal{H} , which in turn induces a (dual) connection on the space of 1-forms that we also denote by D . If v is a vector at a point u , we can then apply D_v to the 1-form F and get a new 1-form $D_v F$. By definition, if w is another vector at u , then

$$D_v F.w = H_{\mathcal{M}}(v, w)$$

is the Hessian of \mathcal{M} at u , which is a symmetric bilinear form. It was proved by Mabuchi [42] and Donaldson [28] that this equals

$$H_{\mathcal{M}}(v, w) = \text{Re} \int_X (\mathcal{D}_u^* \mathcal{D}_u v) w \omega_u^n,$$

where \mathcal{D}_u is a certain elliptic operator of second order. \mathcal{D}_u is defined by $\mathcal{D}_u w = \bar{\partial} V_w$ where V_w is the $(1, 0)$ vector field defined by

$$V \lrcorner \omega_u = i \bar{\partial} w$$

(the complex gradient of w). \mathcal{D}_u^* is the adjoint operator, so $\mathcal{D}_u^* \mathcal{D}_u$ is a self-adjoint elliptic operator of order 4. By definition,

$$\mathcal{D}_u w = 0$$

if and only if V_w is holomorphic, so $\mathcal{D}_u^* \mathcal{D}_u w = 0$ if and only if V_w is a holomorphic vector field. It is also clear that if w is real valued, $\text{Re} \mathcal{D}_u^* \mathcal{D}_u w = 0$ implies that V_w is holomorphic, since

$$\int_X \langle \bar{\partial} V_w, \bar{\partial} V_w \rangle \omega_u^n = \int_X (\text{Re} \mathcal{D}_u^* \mathcal{D}_u w) w \omega_u^n.$$

Moreover, since the highest order term in the fourth order elliptic operator $\mathcal{D}_u^* \mathcal{D}_u$ is real, it follows that $\text{Re} \mathcal{D}_u^* \mathcal{D}_u$ is also elliptic.

Proposition 4.3. *Let ν be a smooth volume form on X that defines a 1-form G_ν on the tangent space of $\mathcal{H}(X, \omega)$ at u by*

$$G_\nu.w = \int_X w d\nu,$$

for w smooth and real valued. Then there is a vector v at u such that

$$D_v F|_u = G_\nu$$

if and only if $G_\nu.w = 0$ for all real valued w such that the complex gradient of w is holomorphic.

Proof. We have

$$D_v F|_u.w = H_{\mathcal{M}}(v, w) = \text{Re} \int_X (\mathcal{D}_u^* \mathcal{D}_u v) w \omega_u^n.$$

Hence

$$D_v F|_u = G_\nu$$

means that

$$\operatorname{Re}(\mathcal{D}_u^* \mathcal{D}_u v) \omega_u^n = \nu.$$

Since $\operatorname{Re} \mathcal{D}_u^* \mathcal{D}_u$ is a self-adjoint elliptic operator, this equation is solvable if and only if ν annihilates the kernel of $\operatorname{Re} \mathcal{D}_u^* \mathcal{D}_u$, i.e. the space of (real valued) functions whose complex gradients are holomorphic. \square

We are now ready for the uniqueness, and we start with the case when there are no nontrivial holomorphic vector fields on X .

Theorem 4.4. *Assume ω_{u_0} and ω_{u_1} are metrics of constant scalar curvature on X and that X has no nontrivial holomorphic vector fields. Then $\omega_{u_0} = \omega_{u_1}$.*

Proof. By hypothesis u_0 and u_1 are both critical points of \mathcal{M} , so $F(u_0) = F(u_1) = 0$. Let μ be a strictly positive volume form normalized as in the beginning of this section. The differential of \mathcal{F}_μ at u_0 is $G(u_0) = G_\nu$, where $\nu = \mu - \omega_{u_0}^n$. Since by our normalization this measure annihilates constants, which are now the only functions with a holomorphic complex gradient, Proposition 4.3 implies that we can solve

$$D_{v_0} F|_{u_0} = -G(u_0).$$

Consider the functional

$$\mathcal{M}_s := \mathcal{M} + s\mathcal{F}_\mu,$$

and its differential

$$F_s(u) = F(u) + sG(u).$$

If w_s is a smooth curve of continuous functions, we have

$$(d/ds)|_0 F_s(u_0 + sv_0) \cdot w_s = D_{v_0} F|_{u_0} \cdot w_0 + F(u_0) \cdot D_{v_0} w_s + G(u_0) \cdot w_0 = 0,$$

since $F(u_0) = 0$ and $D_{v_0} F|_{u_0} = -G(u_0)$. Hence $F_s(u_0 + sv_0) \cdot w_s = O(s^2)$.

We can now do the same construction for the other critical point u_1 and obtain another function v_1 with similar properties. Then connect for $0 < s \ll 1$ the smooth points $u_0 + sv_0$ and $u_1 + sv_1$ by a $C^{1,1}$ -geodesic u_t^s . We need to relate F and F_s , the formal derivatives of \mathcal{M} and \mathcal{M}_s , to the actual one sided derivatives along the geodesics at the end points. It is not a priori clear that they coincide since the formal derivatives are the derivatives in \mathcal{H} , the space of smooth potentials, and the weak geodesic has less regularity. However, it follows from Lemma 3.5 that

$$(d/dt)|_{t=0}^+ \mathcal{M}(u_t^s) \geq F(u_0^s) \cdot (d/dt)|_{t=0} u_t^s,$$

and also that we have the converse inequality at the other end point. Since \mathcal{F}_s is differentiable one time on the closed geodesic, the same inequalities hold for \mathcal{M}_s as well.

Since $\mathcal{M}(u_t^s)$ is convex and $\mathcal{E}(u_t^s)$ is linear in t , we get that

$$0 \leq s((d/dt)|_1 - (d/dt)|_0) I_\mu(u_t^s) \leq ((d/dt)|_1 - (d/dt)|_0) \mathcal{M}_s(u_t^s) = F_s(u_1^s) \cdot (d/dt)|_{t=1} u_t^s - F_s(u_0^s) \cdot (d/dt)|_{t=0} u_t^s \leq O(s^2).$$

Dividing by s we get that

$$((d/dt)|_1 - (d/dt)|_0) I_\mu(u_t^s) \leq C' s,$$

so by Proposition 4.1, $d(\omega_{u_0^s}, \omega_{u_1^s})^2 \leq C'' s$. Here we have also used the fact that the constant C in Proposition 4.1 can be taken independent of s , i.e. that the L^∞ -bound on $\omega_{u_t^s}$ can be taken uniform in s ; see [7]. Hence $d(\omega_{u_0}, \omega_{u_1}) = 0$ which implies that $\omega_{u_0} = \omega_{u_1}$ by a result of Chen; see [18]. \square

Notice that there are two main points of the argument. Apart from the convexity along weak geodesics we also use that \mathcal{M} is strictly convex on \mathcal{H} (modulo constants) in the formal sense that its Hessian is strictly positive if there are no nontrivial holomorphic vector fields. This means that the derivative of $F = d\mathcal{M}$ is invertible which allows us to solve $D_{v_0}F|_{u_0} = -G(u_0)$. The same principle is illustrated in the next result which concerns uniqueness of metrics of constant α -twisted curvature (cf. Section 3.1.1).

Theorem 4.5. *Let α be a Kähler form on X . Then there is at most one metric ω_0 in a given Kähler class $[\omega]$ with constant α -twisted curvature.*

Proof. Recall that ω_0 is a critical point of the twisted Mabuchi functional $\mathcal{M}_\alpha = \mathcal{M} + \mathcal{E}^\alpha - c(\alpha)\mathcal{E}$ (cf. Section 3.1.1). One can check that the Hessian of \mathcal{E}^α (at a smooth point) is strictly positive, and we also have that the Hessian of \mathcal{E} is zero since \mathcal{E} is linear along geodesics. If we put $F_\alpha = d\mathcal{M}_\alpha$, it follows that we can solve $D_{v_0}F_\alpha|_{u_0} = -G(u_0)$ as in the proof of the previous theorem, and again we conclude by the convexity of \mathcal{M}_α . □

We finally turn to the case of metrics of constant scalar curvature when there are non-zero holomorphic vector fields on X . The argument is then essentially the same, with the additional difficulty that we cannot solve the equation

$$D_{v_0}F|_{u_0} = -G(u_0)$$

in general. Therefore, we shall make a preliminary modification of u_0 by applying an automorphism in $\text{Aut}_0(X)$, so that after the modification $G(u_0)$ annihilates all functions with a holomorphic complex gradient. In fact, we shall only consider a subgroup of $\text{Aut}_0(X)$.

Given ω_0 in $\mathcal{H}(X, \omega)$, following Mabuchi [42], we let

$$\mathfrak{g}_{\omega_0} := \{V \in H^0(X, T^{1,0}(X)); V \lrcorner \omega_0 = \bar{\partial}h, \text{ where } h \in C^\infty(X, \mathbb{R})\}.$$

(Here $V \lrcorner \omega$ is the contraction of the form ω with the vector field V .) The condition that $V \lrcorner \omega_0 = \bar{\partial}h$ is equivalent to $\text{Re}(V) \lrcorner \omega_0 = dh/2$ if h is real valued. Thus \mathfrak{g}_{ω_0} is the subspace of the Lie algebra of all holomorphic vector fields on X whose real parts are Hamiltonian with respect to ω_0 , hence a Lie algebra. Let $\Gamma_{\omega_0} = \exp(\mathfrak{g}_{\omega_0})$ be the corresponding connected Lie group.

For $g \in \Gamma_{\omega_0}$ we let $\omega_g = g_*(\omega_0) = (g^{-1})^*(\omega_0)$, and we let $S_{\omega_0} = \{\omega_g; g \in \Gamma_{\omega_0}\}$.

Lemma 4.6. *If $g \in \Gamma_{\omega_0}$, then $S_{\omega_0} = S_{\omega_g}$.*

Proof. It is readily verified that $\Gamma_{\omega_g} = g\Gamma_{\omega_0}g^{-1}$. Hence, if $\sigma \in \Gamma_{\omega_g}$, $\sigma = g\tau g^{-1}$ with τ in Γ_{ω_0} and

$$\sigma_*(\omega_g) = (g\tau)_*\omega_0$$

lies in S_{ω_0} . □

Any element, ω , in S_{ω_0} can be written $\omega = (\exp V)_*\omega_0$ for some V in \mathfrak{g}_{ω_0} . We can also write it as $(\exp i\text{Im} V)_*\omega_0$ since the real part of V is Hamiltonian. Thus S_{ω_0} is a union of rays $(\exp it\text{Im} V)_*\omega_0 = \omega_t$. By Proposition 3.5 in Mabuchi [42], these rays are geodesics, so that they can be written

$$\omega_t = i\partial\bar{\partial}u_t + \omega,$$

where u_t is a geodesic in $\mathcal{H}(X, \omega)$. In particular, $\mathcal{F}_\mu(\omega_t)$ is strictly convex.

Proposition 4.7. *Fix $\omega_0 = \omega_{u_0}$ in \mathcal{H} . Then \mathcal{F}_μ has a minimum and hence a critical point, ω_u , on S_{ω_0} . This implies that $G(u) = d\mathcal{F}_\mu|_u$ annihilates all real functions whose complex gradients with respect to ω_u are holomorphic.*

Proof. If $\mu = \omega_{u_0}^n$, then u_0 is a critical point of \mathcal{F}_μ . Since \mathcal{F}_μ is strictly convex along each ray in S_{ω_0} as just described, it follows that \mathcal{F}_μ is proper on each ray if $\mu = \omega_{u_0}^n$. Since S_{ω_0} is of finite dimension, it follows that \mathcal{F}_μ is proper on S_{ω_0} in this case. By Lemma 4.2 this implies that \mathcal{F}_μ is proper on S_{ω_0} for any choice of μ , and so must have a minimum, say ω_u .

Let h now be a real valued function such that the vector field V defined by

$$V \lrcorner \omega_u = \bar{\partial}h$$

is holomorphic. Then V lies in \mathfrak{g}_{ω_u} . Thus $(\exp it\text{Im } V)_*\omega_u$ lies in S_{ω_u} which equals S_{ω_0} by the lemma. Hence $f(t) := \mathcal{F}((\exp it\text{Im } V)_*\omega_u)$ has a minimum for $t = 0$. But, if we write $(\exp it\text{Im } V)_*\omega_u = i\partial\bar{\partial}u_t + \omega$, we have that

$$-\partial\bar{\partial}\dot{u} = -\partial\bar{\partial}\frac{du_t}{dt}\Big|_0 = \partial\bar{\partial}h,$$

so $-\dot{u} = h$ up to a constant. Hence $0 = f'(0) = -G(u) \cdot h$, which gives the last claim. □

By Proposition 4.3 this implies that we can find a v_0 that solves

$$D_{v_0}F|_{u_0} = -G(u_0).$$

We now apply this when u_0 is a critical point of \mathcal{M} . Notice that \mathcal{M} is invariant under the action of $\text{Aut}_0(X)$, since it is linear along the flow of holomorphic vector fields and is bounded from below; cf. Corollary 1.2. Hence the point we get after applying the automorphism is still a critical point of \mathcal{M} . If u_1 is another critical point, we can apply the same argument to u_1 . The proof of Theorem 4.4 then applies without change, and we see that after applying these automorphisms $\omega_{u_0} = \omega_{u_1}$. Therefore, we have proved the following.

Theorem 4.8. *Assume that ω_{u_0} and ω_{u_1} are metrics of constant scalar curvature. Then there is an automorphism g in $\text{Aut}_0(X)$ such that*

$$g^*(\omega_{u_1}) = \omega_{u_0}.$$

Notice that it follows from the proof that g can be taken in $\Gamma_{\omega_{u_0}}$.

4.1. Calabi’s extremal metrics. The *extremal* Kähler metrics (in a given Kähler class) introduced by Calabi [16], generalizing constant scalar curvature metrics, are defined as the critical points of the L^2 -norm of the scalar curvature, i.e. the functional $\omega \mapsto \int_X R_\omega^2 \omega^n$ on the space of Kähler metrics in a fixed Kähler class. As shown by Calabi, this equivalently means that the gradient of R_ω is a holomorphic vector field, or more precisely that the $(1, 0)$ field V with real part equal to the gradient of R_ω is holomorphic. We shall now generalize Theorem 4.8 to extremal Kähler metrics. This builds on the fundamental work in [32, 37, 50], which for completeness we develop from scratch in a form suitable in this context.

The holomorphic vector field V will in general depend on the extremal metric. The first step in the proof, following [37], is to prove that one can obtain a unique “extremal vector field” by fixing a compact subgroup K of $\text{Aut}_0(X)$ and requiring that the flow of $\text{Im } V$ lie in K . Once this is done, we, following [32, 50], modify the Mabuchi functional to obtain another functional \mathcal{M}_V , defined on \mathcal{H}_V , the space

of Kähler metrics invariant under $\text{Im } V$, by adding a term \mathcal{E}_V , depending on V . The extremal metrics corresponding to the now fixed field V are now critical points of \mathcal{M}_V on \mathcal{H}_V . The energy functional \mathcal{E}_V is linear, so \mathcal{M}_V is also convex along geodesics in \mathcal{H}_V . Given all this, the proof of the uniqueness of extremal metrics follows the same lines as before.

We start with a few preparations. Let ω be any Kähler form on X . Recall that if h is a complex valued function on X we define a vector field of type $(1, 0)$ by

$$V \lrcorner \omega = i\bar{\partial}h.$$

We have here changed the notation from the previous paragraph by multiplying with i in the right-hand side, and we denote $V = \nabla_\omega h$. Choosing local coordinates so that $\omega = i \sum dz_i \wedge d\bar{z}_i$ at a given point, we have

$$V = \sum \frac{\partial h}{\partial \bar{z}_i} \frac{\partial}{\partial z_i}$$

there. A direct computation shows that

$$2\text{Im } V \lrcorner \omega = d^c \text{Im } h + d\text{Re } h.$$

(Contrary to our earlier conventions we here write d^c for $i(\bar{\partial} - \partial)$.) Therefore, we see that the Lie derivative of ω along $\text{Im } V$, $L_V \omega = d\text{Im } V \lrcorner \omega = (1/2)dd^c \text{Im } h$ vanishes if and only if $\text{Im } h$ is a constant and in that case $\text{Re } h/2$ is a Hamiltonian of $\text{Im } V$. Then the real part of V is the real gradient of $h/2$, so with $h = R_\omega$ we see that ω is extremal if and only if the complex gradient of R_ω is holomorphic. We normalize by choosing h so that

$$\int_X h \omega^n = 0.$$

Then h is uniquely determined by V and ω , and we write $h = h_\omega^V$. Note that with this normalization, ω is invariant under the flow of $\text{Im } V$ if and only if h_ω^V is real valued. Denote by $H(X)$ the space of holomorphic vector fields that arise as complex gradients. Note that $h_\omega^{V+W} = h_\omega^V + h_\omega^W$ and we also have the following.

Lemma 4.9. *If $\omega_u = \omega_0 + i\partial\bar{\partial}u$ and $V \in H(X)$, then*

$$h_{\omega_u}^V = h_{\omega_0}^V + V(u).$$

In the proof of this we shall use a technical lemma that will reappear later several times.

Lemma 4.10. *If u and v are functions on X ,*

$$n \int_X v i\partial\bar{\partial}u \wedge \omega^{n-1} = - \int_X \nabla_\omega v(u) \omega^n.$$

Proof. This follows from integration by parts, and it is noted that $\nabla_\omega v(u) = \langle \partial u, \partial \bar{v} \rangle_\omega$. □

To prove Lemma 4.9 we first note that $i\bar{\partial}(h_{\omega_0}^V + V(u)) = V \lrcorner \omega_u$, so $h_{\omega_u}^V = h_{\omega_0}^V + V(u) + c(u)$, where $c(u)$ is constant on X . Moreover,

$$0 = (d/dt) \int_X h_{\omega_{tu}}^V \omega_{tu}^n = \int_X (V(u) + \dot{c}(tu)) \omega_{tu}^n + n \int_X h_{\omega_{tu}}^V i\partial\bar{\partial}u \wedge \omega_{tu}^{n-1}.$$

By the technical Lemma 4.10, $\dot{c}(tu) = 0$, so $c(u) = 0$ since $c(0) = 0$.

We next, following [37], define a bilinear form on $H(X)$ by

$$\langle V, W \rangle_\omega = \int_X h_\omega^V h_\omega^W \omega^n.$$

Proposition 4.11. $\langle \cdot, \cdot \rangle_\omega$ only depends on the cohomology class $[\omega]$.

Proof. We take a curve of metrics $\omega_t = \omega + i\partial\bar{\partial}u_t$ in $[\omega]$ and differentiate,

$$(d/dt) \int_X h_{\omega_t}^V h_{\omega_t}^W \omega_t^n = \int_X (V(\dot{u})h_{\omega_t}^W + W(\dot{u})h_{\omega_t}^V) \omega_t^n + n \int_X h_{\omega_t}^V h_{\omega_t}^W i\partial\bar{\partial}\dot{u} \wedge \omega_t^{n-1}.$$

By the technical lemma, this expression vanishes which proves the proposition. \square

Since the cohomology class is fixed in our discussion, we can thus consider the form as fixed and write $\langle \cdot, \cdot \rangle_\omega = \langle \cdot, \cdot \rangle$.

Let now K be a compact subgroup of $\text{Aut}_0(X)$ and denote by $H_K(X)$ the subspace of $H(X)$ consisting of holomorphic vector fields V such that the flow of $\text{Im } V$ lies in K .

Proposition 4.12. For any compact subgroup K of $\text{Aut}_0(X)$ the restriction of $\langle \cdot, \cdot \rangle$ to $H_K(X)$ is real valued and positive definite, in particular non-degenerate.

Proof. Taking averages of an arbitrary Kähler form in our class, we can represent our form by a K -invariant Kähler form ω . Then h_ω^V is real valued if V lies in H_K . Both claims of the proposition follow directly from this. \square

For a holomorphic vector field V in $H(X)$ we put

$$\mathcal{C}_V^\infty = \{u \in \mathcal{C}^\infty(X; \mathbb{R}); \text{Im}(Vu) = 0\}$$

and denote by $\text{Aut}_0(X, V)$ the subgroup of $\text{Aut}_0(X)$ of automorphisms commuting with the flow of V .

Proposition 4.13. Let V be a vector field in $H(X)$ and ω_0 a Kähler form invariant under the flow of $\text{Im } V$. Then a real valued function u lies in \mathcal{C}_V^∞ if and only if the vector fields $\text{Im } \nabla_{\omega_0} u$ and $\text{Im } V$ commute. If, moreover, $\nabla_{\omega_0} u$ is holomorphic, then W lies in the Lie algebra of $\text{Aut}_0(X, V)$.

Proof. For v real valued, let $W_v = 2\text{Im } \nabla_{\omega_0} v$ be the vector field determined by the Hamiltonian v and the symplectic form ω_0 , $W_v \lrcorner \omega_0 = dv$. Then

$$[W_u, \text{Im } V] = W_{\{u, h_{\omega_0}^V\}}$$

(where $\{ \cdot, \cdot \}$ is the Poisson bracket). Since for any u and v , $\{u, v\} = W_v u$, we see that $\{u, h_{\omega_0}^V\} = 2\text{Im } Vu$, so $\text{Im } Vu = 0$ if and only if W_u and $\text{Im } V$ commute. If we also assume that $\nabla_{\omega_0} u$ is holomorphic, then $\text{Im } \nabla_{\omega_0} u$ and $\text{Im } V$ commute if and only if $\nabla_{\omega_0} u$ and V commute, which means that $\nabla_{\omega_0} u$ lies in the Lie algebra of $\text{Aut}_0(X, V)$. \square

Finally, given a field V in $H(X)$ we define an associated energy functional \mathcal{E}_V by letting

$$d\mathcal{E}_V|_\omega \cdot \dot{u} := \int_X \dot{u} h_\omega^V \omega^n.$$

The next proposition shows that this indeed defines a function on the subspace \mathcal{H}_V of \mathcal{H} consisting of Kähler metrics invariant under $\text{Im } V$, and also computes its second derivative along a curve.

Proposition 4.14. *Let $\omega_u = \omega_0 + i\partial\bar{\partial}u$ where $u \in \mathcal{C}_V^\infty$ depends smoothly on two real parameters s and t . Assume that ω_0 is invariant under $\text{Im } V$. Then*

$$(d/ds) \int_X \dot{u}_t h_{\omega_u}^V \omega_u^n = \int_X (\ddot{u}_{st} - (\partial\dot{u}_t, \partial\dot{u}_s)_{\omega_u}) h_{\omega_u}^V \omega_u^n,$$

where $(\cdot, \cdot)_{\omega_u} = \text{Re} \langle \cdot, \cdot \rangle_{\omega_u}$ is the real scalar product defined by ω_u .

Proof. A direct computation using the technical lemma shows that

$$(d/ds) \int_X \dot{u}_t h_{\omega_u}^V \omega_u^n = \int_X (\ddot{u}_{st} - \langle \partial\dot{u}_t, \partial\dot{u}_s \rangle_{\omega_u}) h_{\omega_u}^V \omega_u^n.$$

If u lies in \mathcal{C}_V^∞ , then $h_{\omega_u}^V = h_{\omega_u}^V + V(u)$ is real valued for all s and t . Hence the proposition follows by taking real parts. \square

Since this expression is symmetric in s and t , it follows that

$$\int_0^1 dt \int_X \dot{u}_t h_{\omega_u}^V \omega_u^n$$

is independent of the choice of path between $u_0 = 0$ and u_1 , so \mathcal{E}_V is a well defined function. Note also that $d\mathcal{E}_V \cdot \dot{u}_t$ vanishes if \dot{u}_t is a constant, so $\mathcal{E}_V(u)$ descends to a function on the space of Kähler forms in $[\omega]$. In addition, we see from Proposition 4.13 that

$$(d/dt)^2 \mathcal{E}_V(u) = \int_X (\ddot{u}_{tt} - |\bar{\partial}\dot{u}_t|_{\omega_u}^2) h_{\omega_u}^V \omega_u^n.$$

This formula extends to curves in $\mathcal{C}_\mathbb{C}^{1,1}$, if we define $h_{\omega_u}^V = h_{\omega_0}^V + V(u)$, simply by approximation. Thus we see that \mathcal{E}_V is linear along a $\mathcal{C}_\mathbb{C}^{1,1}$ -geodesic in \mathcal{H}_V . For any pair of metrics in \mathcal{H}_V the weak geodesic between them will remain in \mathcal{H}_V , so \mathcal{E}_V is linear along the connecting geodesic. From this we also conclude that the Hessian of \mathcal{E}_V at ω_0 restricted to \mathcal{C}_V^∞ vanishes.

We are now ready for the proof of the uniqueness of extremal metrics. Following [37] we shall first see that the holomorphic vector field that arises as the complex gradient of the scalar curvature R_ω of an extremal metric is uniquely determined by the given Kähler class, modulo $\text{Aut}_0(X)$.

Proposition 4.15. *Let K be a maximal compact subgroup of $\text{Aut}_0(X)$, and let ω_0 be an extremal metric in $[\omega]$. Let V_0 be the associated vector field $V_0 = \nabla_{\omega_0} R_{\omega_0}$. Then*

1. *There is an element g in $\text{Aut}_0(X)$ such that after replacing ω_0 by $g^*\omega_0$ the flow of $\text{Im } V_0$ lies in K ,
and*
2. *If ω_1 is another extremal metric in the same cohomology class, with associated vector field V_1 such that the flow of $\text{Im } V_1$ also lies in K , then $V_0 = V_1$.*

Hence, given K , we may speak of “the” extremal vector field.

Proof. Since V_0 is the complex gradient of a real valued function R_{ω_0} , the flow of $\text{Im } V_0$ is an isometry as we have seen. Hence the flow of $\text{Im } V_0$ lies in some maximal compact group K_0 . By a fundamental theorem of Iwasawa [38], the two groups K and K_0 are conjugate under some automorphism g . This proves 1.

Let V_0 now be the holomorphic vector field associated to ω_0 . Then V_0 lies in $H_K(X)$. Let W be an arbitrary field in H_K . Then

$$-\langle V_0, W \rangle = \int_X (R_{\omega_0} - \hat{R}_{\omega_0}) h_{\omega_0}^W \omega_0^n.$$

By definition, this is nothing but the negative of the Futaki invariant of W [36], which is well-known not to depend on the choice of Kähler metric. In particular, it also equals $-\langle V_1, W \rangle$, so since the bilinear form is non-degenerate on H_K , it follows that $V_0 = V_1$. □

The main theorem of this section generalizes Theorem 4.7.

Theorem 4.16. *Given any two extremal Kähler metrics ω_0 and ω_1 in a given cohomology class, there exists an element $g \in \text{Aut}_0(X)$ such that $\omega_0 = g^*\omega_1$.*

Following [32, 50] we modify the Mabuchi functional to obtain another functional which has our extremal metrics as critical points. By Proposition 4.13 we may assume that the vector fields associated to ω_0 and ω_1 are the same field V . Then both ω_0 and ω_1 are invariant under $\text{Im } V$ and hence invariant under the closure of the one parameter subgroup of $\text{Aut}_0(X)$ generated by $\text{Im } V$, which we call T . Let $\mathcal{M}_V := \mathcal{M} + \mathcal{E}_V$, where \mathcal{E}_V is the previously introduced energy functional associated to the extremal field V . \mathcal{M}_V is defined on the subspace \mathcal{H}_V of \mathcal{H} of Kähler potentials invariant under $\text{Im } V$. Then both ω_0 and ω_1 are critical points of \mathcal{M}_V on \mathcal{H}_V . We now let μ be a smooth T -invariant volume form, normalized as before and consider, following the proof of Theorem 4.7, the functionals

$$\mathcal{M}_V + s\mathcal{F}_\mu,$$

where s is a small positive number, and let $F_V(u, s) := d(\mathcal{M}_V + s\mathcal{F}_\mu)|_u$. We shall prove that if $\omega_0 = \omega + i\partial\bar{\partial}u_0$, then there exists a smooth function v_0 such that $F_V(u_0 + sv_0, s) = O(s^2)$, and as before this amounts to solving the equation

$$D_{v_0}d\mathcal{M}_V|_{u_0} = -d\mathcal{F}_\mu|_{\omega_0} = -(\mu - \omega_0^n).$$

Moreover, we look for v_0 such that $\text{Im } V(v_0) = 0$. We proceed as in the proof of Theorem 4.7, but this time we first replace ω_0 by $g^*\omega_0$ where $g \in \text{Aut}_0(X, V)$ is chosen to give the minimum of \mathcal{F}_μ on the orbit $\text{Aut}_0(X, V)\omega_0$; i.e. we use the subgroup $\text{Aut}_0(X, V)$ instead of the full group $\text{Aut}_0(X)$. Notice that \mathcal{M}_V is invariant under the action of $\text{Aut}_0(X, V)$ by the same reason as before: It is linear along the flow of vector fields that commute with V and is bounded from below on \mathcal{H}_V (this can be proved in the same way that we proved Corollary 1.2). Therefore, $g^*\omega_0 =: \omega'_0$ is still critical for \mathcal{M}_V .

Then $d\mathcal{F}_\mu|_{\omega'_0}$ annihilates all real valued functions whose complex gradients lie in the Lie algebra of $\text{Aut}_0(X, V)$; cf. the proof of Proposition 4.6. By Proposition 4.11 it follows that $d\mathcal{F}_\mu$ annihilates all functions in \mathcal{C}_V^∞ with holomorphic complex gradients. But, if h is a general real valued function with a holomorphic complex gradient, and $\text{Av}_T(h)$ denotes the average of h over T , then (since $\mu - \omega_0^n$ is T -invariant)

$$\int_X h(\mu - \omega_0^n) = \int_X \text{Av}_T(h)(\mu - \omega_0^n) = 0,$$

since $\text{Av}_T(h)$ is annihilated by $\text{Im } V$. Hence $\mu - \omega_0^n$ annihilates all real functions with a holomorphic complex gradient, which by Proposition 4.3 means that we can

solve

$$-D_{v_0} d\mathcal{M}|_{u_0} = \mu - \omega_0^n.$$

Replacing v_0 by its average over T we can also find a solution that is T -invariant, i.e. annihilated by $\text{Im } V$. Finally, we recall that by our formula for the second derivative of \mathcal{E}_V , the Hessian of \mathcal{E}_V restricted to \mathcal{C}_V^∞ vanishes, so we have also solved

$$-D_{v_0} d\mathcal{M}_V|_{u_0} = \mu - \omega_0^n.$$

The proof is then completed in the same way as before: After applying an element of $\text{Aut}_0(X, V)$ to ω_1 we may solve in the same way

$$-D_{v_1} d\mathcal{M}_V|_{u_1} = \mu - \omega_1^n.$$

We then let $u_0^s = u_0 + sv_0$ and $u_1^s = u_1 + sv_1$ and connect with a geodesic u_t^s . By uniqueness the geodesics lie in \mathcal{H}_V so $\mathcal{E}_V(u_t^s)$ is linear in t . It then follows again from Proposition 4.1 that the square of the distance between $\omega_{u_0^s}$ and $\omega_{u_1^s}$ is of order s , and hence $\omega_0 = \omega_1$.

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