ENERGY SOLUTIONS OF KPZ ARE UNIQUE

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1. INTRODUCTION

The aim of this paper is to establish the well-posedness of the martingale problem for the stationary conservative stochastic Burgers equation (SBE) on $\mathbb{R}$,

$$\text{(1)} \qquad \mathrm{d}u_t = \nu\Delta u_t\mathrm{d}t + \lambda\partial_x u_t^2\mathrm{d}t + \sqrt{D}\partial_x\mathrm{d}W_t,$$

where $u\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous process in t taking values in the space of (Schwartz) distributions over $\mathbb{R}$, $\lambda \in \mathbb{R}$, $\nu, D > 0$, and W is a cylindrical Wiener process such that $\partial_t W$ is a space-time white noise. A direct consequence will be the well-posedness of the martingale problem for the quasi-stationary Kardar–Parisi–Zhang (KPZ) equation

$$\text{(2)} \qquad \mathrm{d}h_t = \Delta h_t\mathrm{d}t + \lambda((\partial_x h_t)^2 - \infty)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t,$$

where $h\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a continuous process, and for any t the law of $h_t(\cdot) - h_t(0)$ is a two-sided Brownian motion on $\mathbb{R}$. The SBE describes the evolution of the weak derivative $u_t(x) = \partial_x h_t(x)$ of the solution to the KPZ equation h. Our uniqueness proof also establishes that h is related to the solution of the linear multiplicative stochastic heat equation (SHE),

$$\text{(3)} \qquad \mathrm{d}Z_t = \Delta Z_t\mathrm{d}t + \sqrt{2}\lambda Z_t\mathrm{d}W_t$$

by the Cole–Hopf transformation

$$\text{(4)} \qquad h_t = \lambda^{-1}\log Z_t + \frac{\lambda^3}{12}t, \qquad t \geq 0.$$

The SHE allows a formulation via standard Itô calculus and martingale, weak, or mild solutions in suitable weighted spaces of continuous adapted processes. The SBE and the KPZ equation, on the other hand, cannot be studied in standard spaces due to the fact that the nonlinearity is ill-defined, essentially because the trajectories of the solutions do not possess enough spatial regularity. Indeed, solutions of the KPZ equation are of Hölder regularity less than $1/2$ in space, so a priori the pointwise square of their derivatives cannot be defined.

Despite this mathematical difficulty, the KPZ equation is expected to be a faithful description of the large scale properties of one-dimensional growth phenomena. This was the original motivation which led Kardar, Parisi, and Zhang [KPZ86] to

study the equation, and both experimental and theoretical physics arguments have, since then, confirmed their analysis. The rigorous study of the KPZ equation and its relation with the SHE started with the work of Bertini and Giacomin [BG97] on the scaling limit of the weakly asymmetric exclusion process (WASEP). Starting from this discrete Markov process on $\{0, 1\}^{\mathbb{Z}}$ and performing a suitable space-time rescaling and recentering, they were able to prove that its density fluctuation field converges to a random field u which is linked to the solution of the SHE by the Cole–Hopf transformation (4). Incidentally they had to add exactly the strange $1/12$ drift in order to establish their result. Their work clarifies that any physically relevant notion of solution to the (still conjectural) equations (1) and (2) needs to be transformed to the SHE by the Cole–Hopf transformation and also that the SBE should allow the law of the space white noise as invariant measure. A priori these insights are of little help in formulating the SBE/KPZ equation, since given a solution Z to the SHE it is not possible to apply Itô's formula to $\lambda^{-1} \log Z$, and in particular the inverse Cole–Hopf transformation is ill-defined. It should be noted that the main difficulty of equations (1) and (2) lies in the spatial irregularity and that no useful martingales in the space variable are known, a fact which prevents an analysis via Itô's stochastic integration theory. Moreover, the convergence result of [BG97] relies strongly on the particular structure of the WASEP and does not have many generalizations because most models behave quite badly under exponentiation (Cole–Hopf transformation); see [DT16, CT17, CST16, Lab17] for examples of models that do admit a useful Cole–Hopf transformation.

After the work of Bertini and Giacomin, there have been various attempts to study the SBE via Gaussian analysis tools taking into account the necessary invariance of the space white noise. A possible definition based on the Wick renormalized product associated to the driving space-time white noise has been ruled out because it lacks the properties expected from the physical solution [Cha00]. Assing [Ass02] has been the first, to our knowledge, to attempt a martingale problem formulation of the SBE. He defines a formal infinite-dimensional generator for the process essentially as a quadratic form with dense domain, but he has not been able to prove its closability. The singular drift, which is ill-defined pointwise, make sense as a distribution on the Gaussian Hilbert space associated to the space white noise, however this distributional nature prevents the identification of a suitable domain for the formal generator.

The martingale problem approach has been subsequently developed by Gonçalves and Jara [GJ10, GJ14].[1] Their key insight is that while the drift in (1) is difficult to handle in a Markovian picture (that is, as a function on the state space of the process), it makes perfect sense in a pathwise picture. They proved in particular that a large class of particle systems (which generalize the WASEP studied by Bertini and Giacomin) has fluctuations that subsequentially converge to random fields u, which are solutions of a generalized martingale problem for (1) where the singular nonlinear drift $\partial_x(u(t, x)^2)$ is a well-defined *space-time* distributional random field. Avoiding a description of a Markovian generator for the process, they manage to introduce an auxiliary process which plays the same role in the formulation of the martingale problem. Subsequent work of Jara and Gubinelli [GJ13]

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[1]The paper [GJ14] is the revised published version of [GJ10].

gave a different definition of the martingale problem via a forward-backward description. The solution of the martingale problem is a Dirichlet process, that is the sum of a martingale and a zero quadratic variation process. This property and the forward-backward decomposition of the drift are reminiscent of Lyons–Zheng processes and in general of the theory of Markov processes described by Dirichlet forms; however, a complete understanding of the matter is at the moment not well developed, and the martingale problem formulation avoids the subtleties of the Markovian setting. Gonçalves and Jara called the solutions of this generalized martingale problem *energy solutions* for the SBE/KPZ equation.

Following [GJ14], it has been shown for a variety of models that their fluctuations subsequentially converge to energy solutions of the KPZ equation or the SBE, for example for zero range processes and kinetically constrained exclusion processes in [GJS15], various exclusion processes in [GJS17, FGS16, BGS16, GJ16], interacting Brownian motions in [DGP17], and Hairer–Quastel type SPDEs in [GP16]. This is coherent with the conjecture that the SBE/KPZ equation describes the universal behavior of a wide class of conservative dynamics or interface growth models in the particular limit where the asymmetry is "small" (depending on the spatial scale), the so-called *weak KPZ universality conjecture*; see [Cor12, Qua14, QS15, Spo16]. In order to fully establish the conjecture for the models above, the missing step was a proof of uniqueness of energy solutions. This question remained open for some time during which it was not clear if the notion is strong enough to guarantee uniqueness or if it is too weak to expect well-posedness. Here we present a proof of uniqueness for the refined energy solutions of [GJ13], on the full line and on the torus, thereby finally establishing the well-posedness of the martingale problem and its expected relation with the SHE via the Cole–Hopf transform. In fact we even prove that the equation formulated in [GJ13] leads to *strongly unique* solutions, which directly gives uniqueness in law. The reason we emphasize the (weaker) uniqueness in law is that energy solutions frequently arise as scaling limits for particle systems.

Our proof follows the strategy developed by Funaki and Quastel in [FQ15]. Namely, we map a mollified energy solution to the SHE via the Cole–Hopf transform, and we use a version of the Boltzmann–Gibbs principle to control the various error terms arising from the transformation and to derive the relation (4) in the limit as we take the mollification away. A direct corollary of our results is the proof of the weak KPZ universality conjecture for all the models in the literature which have been shown to converge to energy solutions.

Shortly after the introduction of energy solutions, the fundamental work [Hai13] of Hairer on the KPZ equation appeared, where he established a pathwise notion of solution using Lyons's theory of rough paths to provide a definition of the nonlinear term as a continuous bilinear functional on a suitable Banach space of functions. Existence and uniqueness were then readily established by fixed point methods. This breakthrough developed into a general theory of singular SPDEs, Hairer's theory of regularity structures [Hai14], which provides the right analytic setting to control the singular terms appearing in stochastic PDEs, such as the SBE/KPZ equations and their generalizations, but also in other important SPDEs, such as the stochastic Allen–Cahn equation in dimensions $d = 2, 3$ and the (generalized) parabolic Anderson model in $d = 2, 3$. The work of the authors of this paper together with P. Imkeller on the use of paradifferential calculus [GIP15] and the work of Kupiainen based on renormalization group (RG) techniques [Kup16, KM17]

opened alternative ways to tackle singular SPDEs. All these approaches have in common that they control the a priori ill-defined nonlinearities in the equation using pathwise (deterministic) arguments, and it was not clear if a probabilistic understanding of such singular SPDEs is possible at all. Our uniqueness proof for the stationary martingale solution to the KPZ equation is a first indication that it is possible and is a problem worth investigating closer—even if our method of proof does not extend to other equations because we extensively use the specific structure of the KPZ equation, including the facts that its invariant measure is Gaussian, that its nonlinearity is antisymmetric, and most restrictively that it can be mapped to the SHE through the Cole–Hopf transform.

From the point of view of the weak KPZ universality conjecture, the pathwise approach is difficult to use and, for now, there are only a few convergence results using either regularity structures, paracontrolled distributions, or RG techniques; see [HQ15, HS15, GP17, Hos16]. The martingale approach has the advantage that it is easy to implement, especially starting from discrete particle systems which often do not have the semilinear structure that is at the base of the pathwise theories.

The main limitation of the martingale approach to the SBE/KPZ equation is that currently it works only at stationarity. Using tools from the theory of hydrodynamic limits, it seems possible to extend the results to initial conditions with small relative entropy with respect to the stationary measure. However, this has not been done yet and dealing with even more singular initial conditions is a completely open problem. On the other hand, with energy solutions it is relatively easy to work on the real line, while in the pathwise approach this requires dealing with weighted function spaces, and the question of uniqueness seems still not clear.

To summarize, the main contribution of the present paper is a proof of uniqueness of energy solutions (in the refined formulation of Jara and Gubinelli [GJ13]) on the real line and on the torus. We start in section 2 by introducing the notion of solution and the space of trajectories where solutions live. Subsequently, we discuss in section 3 several key estimates available in this space, estimates which allow us to control a large class of additive functionals. After these preliminaries we show in section 4 how to implement the Cole–Hopf transformation at the level of energy solutions and, by a careful control of some error terms, how to establish the Itô formula which proves the mapping from the SBE to the SHE. Using the uniqueness for the SHE, we conclude the uniqueness of energy solutions. In Appendix A we add some detail on how to modify the proof to deal with the case of periodic boundary conditions. Appendix B contains an auxiliary moment bound for the Cole-Hopf transformation of the stochastic Burgers equation.

Notation. We use the notation $a \lesssim b$ if there exists a constant $c > 0$, independent of the variables under consideration, such that $a \leqslant c \cdot b$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. The Schwartz space on $\mathbb{R}^d$ is denoted with $\mathscr{S}(\mathbb{R}^d)$, and its dual $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions. The notation $\mathscr{D}'(\mathbb{R}^d)$ refers to the distributions on $\mathbb{R}^d$, the dual space of $C_c^\infty(\mathbb{R}^d)$. The Fourier transform of $u \in \mathscr{S}'(\mathbb{R}^d)$ is denoted with $\hat{u} = \mathscr{F}u$, and we use the normalization $\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi \cdot x} u(x) \mathrm{d}x$. For $\alpha \in \mathbb{R}$, we use the following slightly unusual (but of course equivalent to the usual) norm for the space $H^\alpha(\mathbb{R}^d)$

$$H^\alpha\left(\mathbb{R}^d\right) := \left\{ u \in \mathscr{S}'\left(\mathbb{R}^d\right) : \|u\|_{H^\alpha(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \left(1 + |2\pi\xi|^2\right)^\alpha < \infty \right\}.$$

Throughout this paper we work with the quadratic variation in the sense of Russo and Vallois [RV07]: A real-valued stochastic process $(X_t)_{t\geqslant 0}$ has quadratic variation $([X]_t)_{t\geqslant 0}$ if

$$[X]_t = \lim_{\varepsilon \to 0} \int_0^t \frac{1}{\varepsilon}(X_{s+\varepsilon} - X_s)^2 \mathrm{d}s,$$

where the convergence is uniform on compacts in probability. If X is a continuous semimartingale, then $[X]$ is its semimartingale quadratic variation. Despite the fact that we deal with continuous processes, we use the notation $[\cdot]$ for the quadratic variation because $\langle \cdot, \cdot \rangle$ will be reserved for the inner products in various Hilbert spaces.

2. Controlled processes and energy solutions

2.1. **Burgers equation.** In this section we follow Gonçalves and Jara [GJ14] and Gubinelli and Jara [GJ13] in defining stationary energy solutions to the SBE $u \colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$,

$$(5) \qquad \mathrm{d}u_t = \nu \Delta u_t \mathrm{d}t + \lambda \partial_x u_t^2 \mathrm{d}t + \sqrt{D}\partial_x \mathrm{d}W_t,$$

where $\lambda \in \mathbb{R}$, $\nu, \sqrt{D} > 0$, and $\partial_t W$ is a space-time white noise. Recall that from a probabilistic point of view the key difficulty in making sense of (5) is that we expect the law of (a multiple of) the white noise on $\mathbb{R}$ to be invariant under the dynamics, but the square of the white noise can only be defined as a Hida distribution and not as a random variable. To overcome this problem, we first introduce a class of processes u which at fixed times are distributed as the white noise but for which the nonlinear term $\partial_x u^2$ is defined as a space-time distribution. In this class of processes it then makes sense to look for solutions of the Burgers equation (5).

If $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ is a filtered probability space, then for an adapted process W with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ we say that $\partial_t W$ is a *space-time white noise* on that space if for all $\varphi \in \mathscr{S}(\mathbb{R})$ the process $(W_t(\varphi))_{t\geqslant 0}$ is a Brownian motion in the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ with variance $\mathbb{E}[W_t(\varphi)^2] = t\|\varphi\|_{L^2(\mathbb{R})}^2$ for all $t \geqslant 0$. A *(space) white noise with variance* σ^2 is a random variable η with values in $\mathscr{S}'(\mathbb{R})$ such that $(\eta(\varphi))_{\varphi \in \mathscr{S}(\mathbb{R})}$ is a centered Gaussian process with covariance $\mathbb{E}[\eta(\varphi)\eta(\psi)] = \sigma^2 \langle \varphi, \psi \rangle_{L^2(\mathbb{R})}$. If $\sigma = 1$, we simply call η a white noise. Throughout, we write μ for the law of the white noise on $\mathscr{S}'(\mathbb{R})$.

Definition 2.1 (Controlled process). Let $\nu, D > 0$, let $\partial_t W$ be a space-time white noise on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and let η be an $\mathcal{F}_0$-measurable space white noise. Denote with $\mathcal{Q}_{\nu,D}(W, \eta)$ the space of pairs $(u, \mathcal{A})$ of adapted stochastic processes with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ such that

    i) $u_0 = \sqrt{D/(2\nu)}\eta$ and the law of u_t is that of a white noise with variance $D/(2\nu)$ for all $t \geqslant 0$;

    ii) for any test function $\varphi \in \mathscr{S}(\mathbb{R})$, the process $t \mapsto \mathcal{A}_t(\varphi)$ is almost surely of zero quadratic variation, satisfies $\mathcal{A}_0(\varphi) = 0$, and the pair $(u(\varphi), \mathcal{A}(\varphi))$ solves the equation

$$(6) \qquad u_t(\varphi) = u_0(\varphi) + \nu \int_0^t u_s(\Delta\varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) - \sqrt{D}W_t(\partial_x\varphi), \qquad t \geqslant 0;$$

iii) for any $T > 0$ the time-reversed processes $\hat{u}_t = u_{T-t}$, $\hat{\mathcal{A}}_t = -(\mathcal{A}_T - \mathcal{A}_{T-t})$ satisfy

$$\hat{u}_t(\varphi) = \hat{u}_0(\varphi) + \nu \int_0^t \hat{u}_s(\Delta\varphi)\mathrm{d}s + \hat{\mathcal{A}}_t(\varphi) - \sqrt{D}\hat{W}_t(\partial_x\varphi), \qquad t \in [0,T], \varphi \in \mathscr{S}(\mathbb{R}),$$

where $\partial_t \hat{W}$ is a space-time white noise in the filtration generated by $(\hat{u}, \hat{\mathcal{A}})$.

If there exist a space-time white noise $\partial_t W$ and a white noise η such that $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}(W, \eta)$, then we simply write $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}$. For $\nu = 1$ and $D = 2$, we omit the parameters in the notation and write $(u, \mathcal{A}) \in \mathcal{Q}(W, \eta)$, respectively $(u, \mathcal{A}) \in \mathcal{Q}$.

We will see that $\mathcal{Q}_{\nu,D}(W, \eta)$ contains the probabilistically strong solution to (5), while $\mathcal{Q}_{\nu,D}$ is the space in which to look for probabilistically weak solutions.

Controlled processes were first introduced in [GJ13] on the circle, and the definition on the real line is essentially the same. For $\mathcal{A} = 0$ the process $(X, 0) \in \mathcal{Q}_{\nu,D}$ is the stationary Ornstein–Uhlenbeck process. It is the unique-in-law solution to the SPDE

$$\mathrm{d}X_t = \nu\Delta X_t\mathrm{d}t + \sqrt{D}\partial_x\mathrm{d}W_t$$

with initial condition $u_0 \sim \sqrt{D/(2\nu)}\eta$. Allowing $\mathcal{A} \neq 0$ has the intuitive meaning of considering perturbations of the Ornstein–Uhlenbeck process with antisymmetric drifts of zero quadratic variation. In this sense we say that a couple $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}$ is a process *controlled* by the Ornstein–Uhlenbeck process.

As we will see below, for controlled processes we are able to construct some interesting additive functionals. In particular, for any controlled process, the Burgers drift makes sense as a space-time distribution:

Proposition 2.2. *Let $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}$, let $\rho \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$, and write $\rho^N := N\rho(N\cdot)$ for $N \in \mathbb{N}$. Then for all $\varphi \in \mathscr{S}(\mathbb{R})$, the process*

$$\int_0^t (u_s * \rho^N)^2(-\partial_x\varphi)\mathrm{d}s, \qquad t \geqslant 0,$$

converges uniformly on compacts in probability to a limiting process that we denote with

$$\int_0^t \partial_x u_s^2\mathrm{d}s(\varphi), \qquad t \geqslant 0.$$

As the notation suggests, this limit does not depend on the function ρ.

The proof will be given in section 3.3. Note that the convolution $u_s * \rho^N$ is well-defined for $\rho^N \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and not only for $\rho^N \in \mathscr{S}(\mathbb{R})$ because at fixed times u_s is a white noise.

Now we can define what it means for a controlled process to solve the SBE.

Definition 2.3. Let $\partial_t W$ be a space-time white noise on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, let η be an $\mathcal{F}_0$-measurable space white noise, and let $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}(W, \eta)$. Then u is called a *strong stationary solution* to the SBE

$$(7) \qquad \mathrm{d}u_t = \nu\Delta u_t\mathrm{d}t + \lambda\partial_x u_t^2\mathrm{d}t + \sqrt{D}\partial_x\mathrm{d}W_t, \qquad u_0 = \sqrt{\frac{D}{2\nu}}\eta,$$

if $\mathcal{A}(\varphi) = \lambda\int_0^{\cdot} \partial_x u_s^2\mathrm{d}s(\varphi)$ for all $\varphi \in \mathscr{S}(\mathbb{R})$. If $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}$ and $\mathcal{A}(\varphi) = \lambda\int_0^{\cdot} \partial_x u_s^2\mathrm{d}s(\varphi)$ for all $\varphi \in \mathscr{S}(\mathbb{R})$, then u is called an *energy solution* to (7).

The notion of energy solutions was introduced in a slightly weaker formulation by Gonçalves and Jara in [GJ14], and the formulation here is due to [GJ13]. Note that strong stationary solutions correspond to probabilistically strong solutions, while energy solutions are probabilistically weak in the sense that we do not fix the probability space and the noise driving the equation. Our main result is that strong and weak uniqueness hold for our solutions.

Before we state this precisely, let us introduce some notation. Given $\sigma \in C_c^\infty(\mathbb{R})$ with $\sigma \geqslant 0$ and $\int_\mathbb{R} \sigma(x)\mathrm{d}x = 1$, we write $Z^{(\sigma)}$ for the unique solution to the linear multiplicative SHE

$$(8) \qquad \mathrm{d}Z_t^{(\sigma)} = \nu\Delta Z_t^{(\sigma)}\mathrm{d}t + \frac{\lambda\sqrt{D}}{\nu}Z_t^{(\sigma)}\mathrm{d}W_t, \qquad Z_0^{(\sigma)}(x) = e^{\frac{\lambda}{\nu}\sqrt{\frac{D}{2\nu}}\eta(\Theta_x^{(\sigma)})},$$

where $\Theta_x^{(\sigma)} = \mathbb{1}_{(-\infty,x]} - \int_.^\infty \sigma(y)\mathrm{d}y$. If $\tilde{\sigma}$ is another function of this form, then $\eta(\Theta_x^{(\tilde{\sigma})}) = \eta(\Theta_x^{(\sigma)}) + \chi$, where χ is an $\mathcal{F}_0$-measurable random variable that is independent of x, and in particular $Z_t^{(\tilde{\sigma})}(x) = e^{\frac{\lambda}{\nu}\sqrt{\frac{D}{2\nu}}\chi}Z_t^{(\sigma)}(x)$ for all $t \geq 0$ and $x \in \mathbb{R}$, which implies $\partial_x \log Z^{(\tilde{\sigma})} = \partial_x \log Z^{(\sigma)}$ for the distributional derivative ∂_x.

Theorem 2.4. *Let $\partial_t W$ be a space-time white noise on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and let η be an $\mathcal{F}_0$-measurable space white noise. Then the strong stationary u solution to*

$$(9) \qquad \mathrm{d}u_t = \nu\Delta u_t\mathrm{d}t + \lambda\partial_x u_t^2\mathrm{d}t + \sqrt{D}\partial_x\mathrm{d}W_t, \qquad u_0 = \sqrt{\frac{D}{2\nu}}\eta,$$

is unique up to indistinguishability. Moreover, for $\lambda \neq 0$, we have $u = \frac{\nu}{\lambda}\partial_x \log Z^{(\sigma)}$, where the derivative is taken in the distributional sense, and $Z^{(\sigma)}$ is the unique solution to the linear multiplicative SHE (8) for an arbitrary $\sigma \in C_c^\infty(\mathbb{R})$ with $\sigma \geqslant 0$ and $\int_\mathbb{R} \sigma(x)\mathrm{d}x = 1$. Consequently, any two energy solutions of (9) have the same law.

The proof will be given in section 4.1.

Remark 2.5. As far as we are aware, this is the first time that existence and uniqueness for an intrinsic notion of solution to the Burgers equation on the real line is obtained. The results of [Hai13, GP17, KM17] are restricted to the circle $\mathbb{T}$. In [HL15] the linear multiplicative heat equation is solved on the real line using regularity structures, and by the strong maximum principle of [CGF17] the solution is strictly positive, so in particular its logarithm is well-defined. Then it should be possible to show that the derivative of the logarithm is a modeled distribution and solves the SBE in the sense of regularity structures. However, it is not at all obvious if and in which sense this solution is unique.

Remark 2.6 (Reduction to standard parameters). To simplify notation, we will assume from now on that $\nu = 1$ and $D = 2$ which can be achieved by a simple transformation. Indeed, it is easy to see that $(u, \mathcal{A}) \in \mathcal{Q}_{\nu,D}(W, \eta)$ if and only if $(u^{\nu,D}, \mathcal{A}^{\nu,D}) \in \mathcal{Q}(W^{\nu,D}, \eta)$, where

$$u_t^{\nu,D}(\varphi) := \sqrt{\frac{2\nu}{D}}u_{t/\nu}(\varphi), \quad \mathcal{A}_t^{\nu,D}(\varphi) := \sqrt{\frac{2\nu}{D}}\mathcal{A}_{t/\nu}(\varphi), \quad W_t^{\nu,D}(\varphi) := \sqrt{\nu}W_{t/\nu}(\varphi),$$

and $\partial_t W^{\nu,D}$ is a space-time white noise. Moreover, u is a strong stationary solution to (9) if and only if $u^{\nu,D}$ is a strong stationary solution to

$$\mathrm{d}u_t^{\nu,D} = \Delta u_t^{\nu,D}\mathrm{d}t + \lambda\sqrt{\frac{D}{2\nu^3}}\partial_x(u_t^{\nu,D})^2\mathrm{d}t + \sqrt{2}\partial_x\mathrm{d}W_t^{\nu,D}, \qquad u_0^{\nu,D} = \eta.$$

By also rescaling the space variable, it would be possible to set $\lambda = 1$ in (5). However, in the periodic setting discussed below, the rescaling of the space variable would change the size of the torus on which the solution lives, so we prefer to work with general λ instead.

Remark 2.7 (Martingale problem). Energy solutions can be understood as solutions to a martingale problem. Indeed, given a pair of stochastic processes $(u, \mathcal{A})$ with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$, we need to check the following criteria to verify that u is the unique-in-law energy solution to (9):

i) The law of u_t is that of the white noise with variance $D/(2\nu)$ for all $t \geqslant 0$.

ii) For any test function $\varphi \in \mathscr{S}(\mathbb{R})$, the process $t \mapsto \mathcal{A}_t(\varphi)$ is almost surely of zero quadratic variation, satisfies $\mathcal{A}_0(\varphi) = 0$, and the pair $(u(\varphi), \mathcal{A}(\varphi))$ solves the equation

$$u_t(\varphi) = u_0(\varphi) + \nu\int_0^t u_s(\Delta\varphi)\mathrm{d}s + \mathcal{A}_t(\varphi) + M_t(\varphi), \qquad t \geqslant 0,$$

where $M(\varphi)$ is a continuous martingale in the filtration generated by $(u, \mathcal{A})$, such that $M_0(\varphi) = 0$ and $M(\varphi)$ has quadratic variation $[M(\varphi)]_t = D\|\partial_x\varphi\|_{L^2}^2 t$.

iii) For any $T > 0$ the time-reversed processes $\hat{u}_t = u_{T-t}$, $\hat{\mathcal{A}}_t = -(\mathcal{A}_T - \mathcal{A}_{T-t})$ satisfy

$$\hat{u}_t(\varphi) = \hat{u}_0(\varphi) + \nu\int_0^t \hat{u}_s(\Delta\varphi)\mathrm{d}s + \hat{\mathcal{A}}_t(\varphi) + \hat{M}_t(\varphi), \qquad t \in [0, T],$$

where $\hat{M}(\varphi)$ is a continuous martingale in the filtration generated by $(\hat{u}, \hat{\mathcal{A}})$, such that $\hat{M}_0(\varphi) = 0$ and $\hat{M}(\varphi)$ has quadratic variation $[\hat{M}(\varphi)]_t = D\|\partial_x\varphi\|_{L^2}^2 t$.

iv) There exists $\rho \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$ and with $\rho^N := N\rho(N\cdot)$ we have

$$\mathcal{A}_t(\varphi) = \lambda\lim_{N\to\infty}\int_0^t (u_s * \rho^N)^2(-\partial_x\varphi)\mathrm{d}s$$

for all $t \geq 0$ and $\varphi \in \mathscr{S}(\mathbb{R})$.

Remark 2.8 (Different notions of energy solutions). In [GJ14] a slightly weaker notion of energy solution was introduced. Roughly speaking, Gonçalves and Jara only make assumptions i), ii), and iv) of Remark 2.7 but do not consider the time reversal condition iii). They then show that a wide class of weakly asymmetric exclusion type processes have subsequential scaling limits that satisfy conditions i), ii), and iv). We do not know whether this weaker formulation still identifies the subsequential limits uniquely. However, the proof in [GJ14] actually shows that any subsequential scaling limit of their particle system also satisfies iii), even if Gonçalves and Jara do not mention this explicitly. Indeed, the time-reversal condition iii) is satisfied for the particle system, and it trivially carries over to the

limit. Therefore, the combination of [GJ14] with our uniqueness result proves the weak KPZ universality conjecture for a wide class of particle systems.

Remark 2.9 (Relaxations). It suffices to assume $(u, \mathcal{A}) \in C(\mathbb{R}_+, \mathscr{D}'(\mathbb{R}))$ and to verify the conditions of Remark 2.7 for $\varphi \in C_c^\infty(\mathbb{R})$ instead of $\varphi \in \mathscr{S}(\mathbb{R})$. For our proof of uniqueness we do need to handle test functions in $\mathscr{S}(\mathbb{R})$, but since all terms in the decomposition

$$u_t(\varphi) = u_0(\varphi) + \nu \int_0^t u_s(\Delta\varphi)\mathrm{d}s + \lambda \int_0^t \partial_x u_s^{\diamond 2}\mathrm{d}s(\varphi) + \sqrt{D}W_t(\varphi)$$

come with good continuity estimates, it follows from the above conditions that u and $\mathcal{A}$ have trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ and satisfy the same equation also for test functions in $\mathscr{S}(\mathbb{R})$. It is even possible to only assume that $(u(\varphi), \mathcal{A}(\varphi))$ is a family of continuous adapted stochastic processes indexed by $\varphi \in C_c^\infty(\mathbb{R})$ such that the conditions i)–iv) in Remark 2.7 hold up to a null set that may depend on φ. In that case we can find a version $(\tilde{u}, \tilde{\mathcal{A}})$ with values in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ such that $\mathbb{P}(u(\varphi) = \tilde{u}(\varphi), \mathcal{A}(\varphi) = \tilde{\mathcal{A}}(\varphi)) = 1$ for all $\varphi \in C_c^\infty(\mathbb{R})$. These relaxations may come in handy when proving the convergence of fluctuations of microscopic systems to the Burgers equation.

2.2. **KPZ equation.** Once we understand how to deal with the SBE, it is not difficult to also handle the KPZ equation. Let $\partial_t W$ be a space-time white noise on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$, and let χ be an $\mathcal{F}_0$-measurable random variable with values in $\mathscr{S}'(\mathbb{R})$ such that $\eta := \partial_x\chi$ is a space white noise. Then we denote with $\mathcal{Q}^{\mathrm{KPZ}}(W, \chi)$ the space of pairs $(h, \mathcal{B})$ of adapted stochastic processes with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ which solve for all $\varphi \in \mathscr{S}(\mathbb{R})$ the equation

(10) $$h_t(\varphi) = \chi(\varphi) + \int_0^t h_s(\Delta\varphi)\mathrm{d}s + \mathcal{B}_t(\varphi) + \sqrt{2}W_t(\varphi),$$

and are such that $\mathcal{B}_0(\varphi) = 0$, and for which $u := \partial_x h$, $\mathcal{A} := \partial_x\mathcal{B}$ satisfy $(u, \mathcal{A}) \in \mathcal{Q}(W, \eta)$. Similarly, we write $(h, \mathcal{B}) \in \mathcal{Q}^{\mathrm{KPZ}}$ if there exist (W, χ) as above such that $(h, \mathcal{B}) \in \mathcal{Q}^{\mathrm{KPZ}}(W, \chi)$. In Proposition 3.15 we show in fact the following generalization of Proposition 2.2: If $(u, \mathcal{A}) \in \mathcal{Q}$ and $(\rho^N) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is an approximate identity (i.e., $\sup_N \|\rho^N\|_{L^1(\mathbb{R})} < \infty$, $\widehat{\rho^N}(0) = 1$ for all N, $\lim_{N\to\infty} \widehat{\rho^N}(x) = 1$ for all x), then there exists a process $\int_0^\cdot u_s^{\diamond 2}\mathrm{d}s \in C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$, defined by

$$\left(\int_0^t u_s^{\diamond 2}\mathrm{d}s\right)(\varphi) := \lim_{n\to\infty} \int_0^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s,$$

where the convergence takes place in $L^p(\mathbb{P})$ uniformly on compacts, and the limit does not depend on the approximate identity (ρ^N).

So we call $(h, \mathcal{B}) \in \mathcal{Q}^{\mathrm{KPZ}}(W, \chi)$ a *strong almost-stationary solution* to the KPZ equation

(11) $$\mathrm{d}h_t = \Delta h_t\mathrm{d}t + \lambda((\partial_x h_t)^2 - \infty)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \qquad h_0 = \chi,$$

if $\mathcal{B} = \lambda\int_0^\cdot u_s^{\diamond 2}\mathrm{d}s$. Similarly, we call $(h, \mathcal{B}) \in \mathcal{Q}^{\mathrm{KPZ}}$ an *energy solution* to (11) if $\mathrm{law}(h_0) = \mathrm{law}(\chi)$ and $\mathcal{B} = \lambda\int_0^\cdot u_s^{\diamond 2}\mathrm{d}s$. The terminology "almost-stationary" comes from [GJ14], and it indicates that for fixed $t \geqslant 0$ the process $x \mapsto h_t(x)$ is always a two-sided Brownian motion; however, the distribution of $h_t(0)$ may depend on time. The analogous result to Theorem 2.4 is then the following.

Theorem 2.10. *Let $\partial_t W$ be a space-time white noise on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$, and let χ be an $\mathcal{F}_0$-measurable random variable with values in $\mathscr{S}'(\mathbb{R})$ such that $\partial_x \chi$ is a space white noise. Then the strong almost-stationary solution h to*

$$\text{(12)} \qquad \mathrm{d}h_t = \Delta h_t \mathrm{d}t + \lambda((\partial_x h_t)^2 - \infty)\mathrm{d}t + \sqrt{2}\mathrm{d}W_t, \qquad h_0 = \chi,$$

is unique up to indistinguishability. Moreover, for $\lambda \neq 0$, we have

$$\text{(13)} \qquad\qquad h_t = \lambda^{-1} \log Z_t + \frac{\lambda^3}{12}t, \qquad t \geq 0,$$

where Z is the unique solution to the linear multiplicative SHE

$$\text{(14)} \qquad\qquad \mathrm{d}Z_t = \Delta Z_t \mathrm{d}t + \sqrt{2}\lambda Z_t \mathrm{d}W_t, \qquad Z_0 = e^{\lambda\chi}.$$

Consequently, any two energy solutions of (12) have the same law.

The proof will be given in section 4.1.

Remark 2.11. It is maybe somewhat surprising that the energy solution h to the KPZ equation is not equal to the Cole–Hopf solution $\tilde{h} := \lambda^{-1} \log Z$, but instead we have to add the drift $t\lambda^3/12$ to $\tilde{h}$ to obtain h. Remarkably, this drift often appears in results about the Cole–Hopf solution of the KPZ equation. For example in [BG97, Theorem 2.3] it has to be added to obtain the Cole–Hopf solution as the scaling limit for the fluctuations of the height profile of the weakly asymmetric exclusion process (there the drift is $-t/24$ because Bertini and Giacomin consider different parameters for the equation). The same drift also appears in [ACQ11, Theorem 1.1], in the key formula [SS10, (4.17)], and in [FQ15, Theorem 1.1].

In [GJ14, Theorem 3] it is claimed that the Cole–Hopf solution is an energy solution to the KPZ equation, and as we have seen this is not quite correct. The reason is that the proof in [GJ14] is based on the convergence result of [BG97], but they did not take the drift $-t/24$ into account which Bertini and Giacomin had to add to obtain the Cole–Hopf solution in the limit.

Remark 2.12 (Martingale problem). Given a pair of stochastic processes $(h, \mathcal{B})$ with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ we need to check the following criteria to verify that h is the unique-in-law energy solution to (12):

i) For all $\varphi \in \mathscr{S}(\mathbb{R})$, we have $\mathcal{B}_0(\varphi) = 0$ and

$$h_t(\varphi) = \chi(\varphi) + \int_0^t h_s(\Delta\varphi)\mathrm{d}s + \mathcal{B}_t(\varphi) + W_t(\varphi)$$

for a continuous martingale $W(\varphi)$ starting in 0 and with quadratic variation $[W(\varphi)]_t = 2t\|\varphi\|_{L^2(\mathbb{R})}^2$.

ii) The pair $u := \partial_x h$, $\mathcal{A} := \partial_x \mathcal{B}$ satisfies conditions i), ii), iii) in Remark 2.7.

iii) There exists $\rho \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, such that $\int_{\mathbb{R}} \rho(x)\mathrm{d}x = 1$ and with $\rho^N := N\rho(N\cdot)$, we have

$$\mathcal{B}_t(\varphi) = \lambda \lim_{N \to \infty} \int_0^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s$$

for all $t \geq 0$ and $\varphi \in \mathscr{S}(\mathbb{R})$.

2.3. **The periodic case.** It is also useful to have a theory for the periodic model $u\colon \mathbb{R}_+ \times \mathbb{T} \to \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and

$$(15) \qquad \mathrm{d}u_t = \Delta u_t \mathrm{d}t + \lambda \partial_x u_t^2 \mathrm{d}t + \sqrt{2}\partial_x \mathrm{d}W_t, \qquad u_0 = \eta,$$

for a periodic space-time white noise $\partial_t W$ and a periodic space white noise η. For a process W with trajectories in $C(\mathbb{R}_+, \mathscr{S}'(\mathbb{T}))$, where $\mathscr{S}'(\mathbb{T})$ are the (Schwartz) distributions on the circle, we say that $\partial_t W$ is a periodic space-time white noise if for all $\varphi \in C^\infty(\mathbb{T})$ the process $(W_t(\varphi))_{t\geqslant 0}$ is a Brownian motion with variance $\mathbb{E}[|W_t(\varphi)|^2] = t\|\varphi\|_{L^2(\mathbb{T})}^2$. A periodic space white noise is a centered Gaussian process $(\eta(\varphi))_{\varphi \in C^\infty(\mathbb{T})}$ with trajectories in $\mathscr{S}'(\mathbb{T})$, such that for all $\varphi, \psi \in C^\infty(\mathbb{T})$, we have $\mathbb{E}[\eta(\varphi)\eta(\psi)] = \langle \Pi_0\varphi, \Pi_0\psi\rangle_{L^2(\mathbb{T})}$, where $\Pi_0\varphi = \varphi - \int_\mathbb{T} \varphi(x)\mathrm{d}x$ is the projection of φ onto the mean-zero functions. The reason for setting the zero Fourier mode of η equal to zero is that the SBE is a conservation law and any solution u to (15) satisfies $\widehat{u_t}(0) = \widehat{u_0}(0)$ for all $t \geqslant 0$, and therefore shifting $\hat{\eta}(0)$ simply results in a shift of $\widehat{u_t}(0)$ by the same value, for all $t \geqslant 0$. So for simplicity we assume $\hat{\eta}(0) = 0$. Controlled processes are defined as before, except that now we test against $\varphi \in C^\infty(\mathbb{T})$ and all noises are replaced by their periodic counterparts. Then it is easy to adapt the proof of Proposition 3.15 to show that also in the periodic setting the Burgers drift $\int_0^\cdot \partial_x u_s^2 \mathrm{d}s$ is well-defined; alternatively, see [GJ13, Lemma 1]. Thus, we define strong stationary solutions, respectively energy solutions, to the periodic Burgers equation exactly as in the nonperiodic setting. We then have the analogous uniqueness result to Theorem 2.4:

Theorem 2.13. *Let $\partial_t W$ be a periodic space-time white noise on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, and let η be an $\mathcal{F}_0$-measurable periodic space white noise. Then the strong stationary u solution to*

$$(16) \qquad \mathrm{d}u_t = \Delta u_t \mathrm{d}t + \lambda \partial_x u_t^2 \mathrm{d}t + \sqrt{2}\partial_x \mathrm{d}W_t, \qquad u_0 = \eta,$$

is unique up to indistinguishability. Moreover, for $\lambda \neq 0$, we have $u = \lambda^{-1}\partial_x \log Z$, where the derivative is taken in the distributional sense and Z is the unique solution to the linear multiplicative SHE

$$(17) \qquad \mathrm{d}Z_t = \Delta Z_t \mathrm{d}t + \sqrt{2}\lambda Z_t \mathrm{d}W_t, \qquad Z_0 = e^{\lambda I\eta},$$

for $\mathscr{F}_\mathbb{T}(I\eta)(k) := (2\pi i k)^{-1}\mathscr{F}_\mathbb{T}\eta(k)$, $k \in \mathbb{Z} \setminus \{0\}$, $\mathscr{F}_\mathbb{T}(I\eta)(0) = 0$, and where $\mathscr{F}_\mathbb{T}$ denotes the Fourier transform on $\mathbb{T}$. Consequently, any two energy solutions of (16) have the same law.

We explain in Appendix A how to modify the arguments for the nonperiodic case in order to prove Theorem 2.13.

3. ADDITIVE FUNCTIONALS OF CONTROLLED PROCESSES

3.1. **Itô trick and Kipnis–Varadhan inequality.** Our main method for controlling additive functionals of controlled processes is to write them as a sum of a forward and a backward martingale which enables us to apply martingale inequalities. For that purpose we first introduce some notation. Throughout this section we fix $(u, \mathcal{A}) \in \mathcal{Q}$.

Definition 3.1. The space of *cylinder functions* $\mathscr{C}$ consists of all $F\colon \mathscr{S}'(\mathbb{R}) \to \mathbb{R}$ of the form $F(u) = f(u(\varphi_1), \ldots, u(\varphi_n))$ for some $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n \in \mathscr{S}(\mathbb{R})$ and $f \in C^2(\mathbb{R}^n)$ with polynomial growth of its partial derivatives up to order 2.

For $F \in \mathscr{C}$, we define the action of the *Ornstein–Uhlenbeck generator* L_0 as

$$L_0 F(u) := \sum_{i=1}^{n} \partial_i f(u(\varphi_1), \ldots, u(\varphi_n)) u(\Delta\varphi_i)$$

$$+ \sum_{i,j=1}^{n} \partial_{ij}^2 f(u(\varphi_1), \ldots, u(\varphi_n)) \langle \partial_x \varphi_i, \partial_x \varphi_j \rangle_{L^2(\mathbb{R})}.$$

With the help of Itô's formula it is easy to verify that if $(X, 0) \in \mathcal{Q}$ is the stationary Ornstein–Uhlenbeck process (see the discussion below Definition 2.1) and $F \in \mathscr{C}$, then $F(X_t) - F(X_0) - \int_0^t L_0 F(X_s) \mathrm{d}s$, $t \geqslant 0$, is a martingale and in particular $L_0 F$ is indeed the action of the generator of X on F. We will see in Corollary 3.8 that L_0 can be uniquely extended from $\mathscr{C}$ to a closed unbounded operator on $L^2(\mu)$, also denoted by L_0, so $\mathscr{C}$ is a core for L_0. We also define the Malliavin derivative

$$\mathrm{D}_x F(u) := \sum_{i=1}^{n} \partial_i f(u(\varphi_1), \ldots, u(\varphi_n)) \varphi_i(x), \qquad x \in \mathbb{R},$$

for all $F \in \mathscr{C}$, and since μ is the law of the white noise we are in a standard Gaussian setting and D is closable as an unbounded operator from $L^p(\mu)$ to $L^p(\mu; L^2(\mathbb{R}))$ for all $p \in [1, \infty)$; see for example [Nua06]. Similarly also $F \mapsto \partial_x \mathrm{D}_x F$ is closable from $L^p(\mu)$ to $L^p(\mu; L^2(\mathbb{R}))$ for all $p \in [1, \infty)$, and we denote the domain of the resulting operator by $\mathscr{W}^{1,p}$. Then $\mathscr{W}^{1,p}$ is the completion of $\mathscr{C}$ with respect to the norm $\mathbb{E}[|F|^p]^{1/p} + \mathbb{E}[\|\partial_x \mathrm{D}F\|_{L^2(\mathbb{R})}^p]^{1/p}$. So writing

$$\mathcal{E}(F(u)) := 2\|\partial_x \mathrm{D}_x F(u)\|_{L^2(\mathbb{R})}^2,$$

we have $\mathcal{E}(F(\cdot)) \in L^{p/2}(\mu)$ for all $F \in \mathscr{W}^{1,p}$. Finally, we denote

$$\|F\|_1^2 := \mathbb{E}[\mathcal{E}(F(u_0))].$$

The following martingale or Itô trick is well known for Markov processes; see for example the monograph [KLO12], and in the case of controlled processes on $\mathbb{R}_+ \times \mathbb{T}$ it is due to [GJ13]. The proof is in all cases essentially the same.

Proposition 3.2 (Itô trick)**.** *Let $T > 0$, $p \geqslant 1$, and $F \in L^p([0, T]; \mathscr{W}^{1,p})$. Then we have for all $(u, \mathcal{A}) \in \mathcal{Q}$,*

$$(18) \qquad \mathbb{E}\left[\sup_{t \leqslant T} \left| \int_0^t L_0 F(s, u_s) \mathrm{d}s \right|^p \right] \lesssim \left(\int_0^T \|\mathcal{E}(F(s, \cdot))\|_{L^{p/2}} \mathrm{d}s \right)^{p/2}.$$

For $p = 2$, we get in particular

$$\mathbb{E}\left[\sup_{t \leqslant T} \left| \int_0^t L_0 F(s, u_s) \mathrm{d}s \right|^2 \right] \lesssim \int_0^T \|F(s, \cdot)\|_1^2 \mathrm{d}s.$$

Proof. We first assume that $F(t) \in \mathscr{C}$ for all $t \in [0, T]$ and that $t \mapsto F(t, u) \in C^1(\mathbb{R})$ for all $u \in \mathscr{S}'(\mathbb{R})$. Since $(u_t(\varphi))_{t \geqslant 0}$ is a Dirichlet process for all $\varphi \in \mathscr{S}(\mathbb{R})$ (the sum of a local martingale and a zero quadratic variation process), we can then apply the Itô formula for Dirichlet processes (see [RV07]) to F and obtain for $t \geqslant 0$

$$F(t, u_t) = F(0, u_0) + \int_0^t (\partial_s F(s, u_s) + L_0 F(s, u_s)) \mathrm{d}s + \int_0^t \partial_u F(s, u_s) \mathrm{d}\mathcal{A}_s + M_t^F,$$

for a continuous martingale M^F with $M_0^F = 0$ and quadratic variation $\mathrm{d}[M^F]_s = \mathcal{E}(F(s, u_s))\mathrm{d}s$. Similarly, we get for $\hat{u}_t = u_{T-t}$

$$F(T - T, \hat{u}_T) = F(T - (T - t), \hat{u}_{T-t}) + \int_{T-t}^{T} (\partial_s F(T - s, \hat{u}_s) + L_0 F(T - s, \hat{u}_s))\mathrm{d}s$$

$$+ \int_{T-t}^{T} \partial_u F(T - s, \hat{u}_s)\mathrm{d}\hat{\mathcal{A}}_s + \hat{M}_T^F - \hat{M}_{T-t}^F$$

for a continuous backward martingale $\hat{M}^F$ with $\hat{M}_0^F = 0$ and quadratic variation $\mathrm{d}[\hat{M}^F]_s = \mathcal{E}(F(T - s, \hat{u}_s))\mathrm{d}s$. Adding these two formulas, we get

$$0 = 2 \int_0^t L_0 F(s, u_s)\mathrm{d}s + M_t^F + \hat{M}_T^F - \hat{M}_{T-t}^F,$$

and thus the Burkholder–Davis–Gundy inequality yields

$$\mathbb{E}\left[\sup_{t \leqslant T} \left| \int_0^t L_0 F(s, u_s)\mathrm{d}s \right|^p \right] \leqslant \mathbb{E}\left[\sup_{t \leqslant T} |M_t^F + \hat{M}_T^F - \hat{M}_{T-t}^F|^p \right]$$

$$\lesssim \mathbb{E}[[M^F]_T^{p/2}] + \mathbb{E}[[\hat{M}^F]_T^{p/2}]$$

$$\simeq \mathbb{E}\left[\left(\int_0^T \mathcal{E}(F(s, u_s))\mathrm{d}s \right)^{p/2} \right]$$

$$\leqslant \left(\int_0^T \|\mathcal{E}(F(s, \cdot)\|_{L^{p/2}}\mathrm{d}s \right)^{p/2},$$

where the last step follows from Minkowski's inequality.

For a general $F \in L^p([0, T]; \mathscr{W}^{1,p})$, we first approximate F in $L^p([0, T]; \mathscr{W}^{1,p})$ by a step function that is piecewise constant in time, then we approximate each of the finitely many values that the step function takes by a cylinder function, and finally we mollify the jumps of the new step function. In that way our bound extends to all of $L^p([0, T]; \mathscr{W}^{1,p})$. □

Remark 3.3. The right-hand side of (18) does not involve the $L^p([0, T]; L^p(\mu))$-norm of F, and indeed it is possible to extend the result to the following space. Identify all $F, \tilde{F} \in \mathscr{C}$ with $\mathbb{E}[\mathcal{E}(F(u_0) - \tilde{F}(u_0))^{p/2}]^{1/p} = 0$, and write $\dot{\mathscr{W}}^{1,p}$ for the completion of the resulting equivalence classes with respect to the norm $F \mapsto \mathbb{E}[\mathcal{E}(F(u_0))^{p/2}]^{1/p}$. Then (18) holds for all $F \in L^p([0, T]; \dot{\mathscr{W}}^{1,p})$ provided that the integral on the left-hand side in (18) makes sense. But we will not need this.

Remark 3.4. If in the setting of Proposition 3.2 $F(s)$ has a finite chaos expansion of length n for all $s \in [0, T]$ (see section 3.2 for the definition), then also $\mathcal{E}(F(s))$ has a chaos expansion of length $2n - 2$ and therefore Gaussian hypercontractivity shows that for all $p \geqslant 1$,

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left| \int_0^t L_0 F(s, u_s)\mathrm{d}s \right|^p \right] \lesssim \left(\int_0^T \|F(s, \cdot)\|_1^2 \mathrm{d}s \right)^{p/2}.$$

The bound in Proposition 3.2 allows us to control $\int_0^\cdot F(s, u_s)\mathrm{d}s$, provided that we are able to solve the Poisson equation

$$L_0 G(s) = F(s)$$

for all $s \in [0, T]$. Note that this is an infinite-dimensional PDE which a priori is difficult to solve, but that we only need to consider it in $L^2(\mu)$ which has a lot of structure as a Gaussian Hilbert space. We will discuss this further in section 3.2. Nonetheless, we will encounter situations where we are unable to solve the Poisson equation explicitly, and in that case we rely on the method of Kipnis and Varadhan allowing us to bound $\int_0^\cdot F(s, u_s)\mathrm{d}s$ in terms of a certain variational norm of F. We define for $F \in L^2(\mu)$

$$\|F\|_{-1}^2 := \sup_{G \in \mathscr{C}} \{2\mathbb{E}[F(u_0)G(u_0)] - \|G\|_1^2\},$$

and we write $F \in \dot{\mathscr{H}}^{-1}$ if the right-hand side is finite. For details on $\|\cdot\|_{-1}$ and $\dot{\mathscr{H}}^{-1}$, see [KLO12, Chapter 2.2]. Here we just remark that for $F \in L^2(\mu)$ and $G \in \mathscr{C}$, we have

$$2\mathbb{E}[F(u_0)G(u_0)] - \|G\|_1^2 \leqslant 2\Big(\sup_{\substack{H \in \mathscr{C} \\ \|H\|_1 = 1}} \mathbb{E}[F(u_0)H(u_0)]\Big)\|G\|_1 - \|G\|_1^2$$

$$\leqslant \Big(\sup_{\substack{H \in \mathscr{C} \\ \|H\|_1 = 1}} \mathbb{E}[F(u_0)H(u_0)]\Big)^2,$$

and on the other hand we have $2\lambda\mathbb{E}[F(u_0)H(u_0)] \leqslant \|F\|_{-1}^2 + \lambda^2$ for all $H \in \mathscr{C}$ with $\|H\|_1 = 1$ and all $\lambda > 0$, and therefore

$$\mathbb{E}[F(u_0)H(u_0)] \leqslant \inf_{\lambda > 0} \Big\{ \frac{\|F\|_{-1}^2}{2\lambda} + \frac{\lambda}{2} \Big\} = \|F\|_{-1},$$

which proves that

$$(19) \qquad \|F\|_{-1} = \sup_{\substack{H \in \mathscr{C} \\ \|H\|_1 = 1}} \mathbb{E}[F(u_0)H(u_0)].$$

We will need a slightly refined version of the Kipnis–Varadhan inequality which also controls the p-variation. Recall that for $p \geqslant 1$, the p-variation of $f \colon [0, T] \to \mathbb{R}$ is defined as

$$\|f\|_{p-\mathrm{var};[0,T]}^p := \sup \Big\{ \sum_{k=0}^{n-1} |f(t_{k+1}) - f(t_k)|^p : n \in \mathbb{N}, 0 = t_0 < \cdots < t_n = T \Big\}.$$

Corollary 3.5 (Kipnis–Varadhan inequality)**.** *Let $T > 0$ and $F \in L^2([0, T], \dot{\mathscr{H}}^{-1} \cap L^2(\mu))$, and let $(u, \mathcal{A}) \in \mathcal{Q}$ be a controlled process. Then for all $p > 2$,*

$$\mathbb{E}\left[\left\|\int_0^\cdot F(s, u_s)\mathrm{d}s\right\|_{p-\mathrm{var};[0,T]}^2\right] + \mathbb{E}\left[\sup_{t \leqslant T}\left|\int_0^t F(s, u_s)\mathrm{d}s\right|^2\right] \lesssim \int_0^T \|F(s, \cdot)\|_{-1}^2\mathrm{d}s.$$

Proof. The nonreversible version of the Kipnis–Varadhan inequality is due to [Wu99], and our proof is essentially the same as in [KLO12, FQ15]. But we are not aware of any reference for the statement about the p-variation. Note that since the integral vanishes in zero, its supremum norm can be controlled by its p-variation. Let $H \in L^2([0, T]; \mathscr{W}^{1,2})$, and decompose

$$(20) \qquad \int_0^\cdot F(s, u_s)\mathrm{d}s = \int_0^\cdot L_0H(s, u_s)\mathrm{d}s + \int_0^\cdot (F - L_0H)(s, u_s)\mathrm{d}s.$$

For the first term on the right-hand side, we apply the same martingale decomposition as in the proof of the Itô trick to get $\int_0^t L_0 H(s, u_s) \mathrm{d}s = -1/2(M_t^H + \hat{M}_T^H - \hat{M}_{T-t}^H)$. By [Lep76, Proposition 2] we can therefore control the p-variation by

$$\mathbb{E}\left[\left\|\int_0^\cdot L_0 H(s, u_s) \mathrm{d}s\right\|_{p-\mathrm{var};[0,T]}^2\right] \lesssim \mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t L_0 H(s, u_s)\mathrm{d}s\right|^2\right]$$

$$\lesssim \int_0^T \|H(s, \cdot)\|_1^2 \mathrm{d}s,$$

where the second inequality follows from Proposition 3.2. For the second term on the right-hand side of (20), we get

$$\mathbb{E}\left[\left\|\int_0^\cdot (F - L_0 H)(s, u_s)\mathrm{d}s\right\|_{p-\mathrm{var};[0,T]}^2\right] \leqslant \mathbb{E}\left[\left\|\int_0^\cdot (F - L_0 H)(s, u_s)\mathrm{d}s\right\|_{1-\mathrm{var};[0,T]}^2\right]$$

$$\leqslant \mathbb{E}\left[\left(\int_0^T |(F - L_0 H)(s, u_s)|\mathrm{d}s\right)^2\right]$$

$$\leqslant T\int_0^T \|(F - L_0 H)(s)\|_{L^2(\mu)}^2 \mathrm{d}s,$$

and therefore overall

$$\mathbb{E}\left[\left\|\int_0^\cdot F(s, u_s)\mathrm{d}s\right\|_{p-\mathrm{var};[0,T]}^2\right] \lesssim \int_0^T (\|H(s, \cdot)\|_1^2 + T\|(F - L_0 H)(s)\|_{L^2(\mu)}^2)\mathrm{d}s.$$

Now take $H_\lambda(s)$ as the solution to the resolvent equation $(\lambda - L_0)H_\lambda(s) = -F(s)$. Note that unlike the Poisson equation, the resolvent equation is always solvable and $H_\lambda(s) = -\int_0^\infty e^{-\lambda r} P_r^{\mathrm{OU}} F(s)\mathrm{d}r$, where P^{OU} is the semigroup generated by L_0. Then $(F - L_0 H_\lambda)(s) = -\lambda H_\lambda(s)$, and by Lemma 3.9 we have $\|H_\lambda(s)\|_1^2 = 2\langle H_\lambda(s), (-L_0)H_\lambda(s)\rangle_{L^2(\mu)}$, which yields

$$\lambda\|H_\lambda(s)\|_{L^2(\mu)}^2 + \frac{1}{2}\|H_\lambda(s)\|_1^2 = \langle H_\lambda(s), (\lambda - L_0)H_\lambda(s)\rangle_{L^2(\mu)}$$

$$= \langle H_\lambda(s), -F(s)\rangle_{L^2(\mu)} \leqslant \|H_\lambda(s)\|_1 \|F(s)\|_{-1},$$

from which we get $\|H_\lambda(s)\|_1 \leqslant 2\|F(s)\|_{-1}$, and then $\|H_\lambda(s)\|_{L^2(\mu)}^2 \leqslant \lambda^{-1} 2\|F(s)\|_{-1}^2$. Therefore,

$$\|H_\lambda(s)\|_1^2 + T\|(F - L_0 H_\lambda)(s)\|_{L^2(\mu)}^2 = \|H_\lambda(s)\|_1^2 + T\|\lambda H_\lambda(s)\|_{L^2(\mu)}^2$$

$$\lesssim \|F(s)\|_{-1}^2 + \lambda T\|F(s)\|_{-1}^2,$$

and now it suffices to send $\lambda \to 0$. $\square$

3.2. **Gaussian analysis.** To turn the Itô trick or the Kipnis–Varadhan inequality into a useful bound, we must be able either to solve the Poisson equation $L_0 G = F$ for a given F or to control the variational norm appearing in the Kipnis–Varadhan inequality. Here we discuss how to exploit the Gaussian structure of $L^2(\mu)$ in order to do so. For details on Gaussian Hilbert spaces we refer to [Jan97, Nua06]. Since $L^2(\mu)$ is a Gaussian Hilbert space, we have the orthogonal decomposition

$$L^2(\mu) = \bigoplus_{n\geqslant 0} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the closure in $L^2(\mu)$ of the span of all random variables of the form $u \mapsto H_n(u(\varphi))$, with $H_n(x) = e^{x^2/2}(-1)^n\partial_x^n e^{-x^2/2}$ being the nth Hermite polynomial and where $\varphi \in \mathscr{S}(\mathbb{R})$ with $\|\varphi\|_{L^2(\mathbb{R})} = 1$. The space $\mathcal{H}_n$ is called the nth *homogeneous chaos*, and $\bigoplus_{k=0}^n \mathcal{H}_k$ is the nth *inhomogeneous chaos*. Also, μ is the law of the white noise on $\mathbb{R}$, and therefore we can identify

$$\mathcal{H}_n = \{W_n(f_n) : f_n \in L^2_s(\mathbb{R}^n)\},$$

where $W_n(f_n)$ is the multiple Wiener–Itô integral of $f_n \in L^2_s(\mathbb{R}^n)$, that is

$$W_n(f_n) := \int_{\mathbb{R}^n} f(y_1, \ldots, y_n) W(\mathrm{d}y_1 \cdots \mathrm{d}y_n).$$

Here $L^2_s(\mathbb{R}^n)$ are the equivalence classes of $L^2(\mathbb{R}^n)$ that are induced by the seminorm

$$\|f\|_{L^2_s(\mathbb{R}^n)} := \|\tilde{f}\|_{L^2_s(\mathbb{R}^n)}, \qquad \tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

where $\mathcal{S}_n$ denotes the set of permutations of $\{1, \ldots, n\}$. Of course, $\|\cdot\|_{L^2_s(\mathbb{R}^n)}$ is a norm on $L^2_s(\mathbb{R}^n)$, and we usually identify an equivalence class in $L^2_s(\mathbb{R}^n)$ with its symmetric representative. The link between the multiple stochastic integrals W_n and the Malliavin derivative D is explained in the following partial integration-by-parts rule, which will be used for some explicit computations below.

Lemma 3.6. *Let $f \in L^2_s(\mathbb{R}^n)$, and let F be Malliavin differentiable in $L^2(\mu)$. Then*

$$\mathbb{E}[W_n(f)F] = \int_{\mathbb{R}} \mathbb{E}[W_{n-1}(f(y, \cdot))\mathrm{D}_y F]\mathrm{d}y.$$

Proof. The proof is virtually the same as for [Nua06, Lemma 1.2.1]. Since the span of functions of the form $f = f_1 \otimes \cdots \otimes f_n$ is dense in $L^2_s(\mathbb{R}^n)$, it suffices to argue for such f. By polarization it suffices to consider $f_1 = \cdots = f_n$ with $\|f_1\|_{L^2(\mathbb{R})} = 1$, for which $W_n(f) = H_n(W_1(f_1))$ for the nth Hermite polynomial H_n. By another approximation argument we may suppose that $F = \Phi(W_1(f_1), W_1(\varphi_1), \ldots, W_1(\varphi_m))$ for orthonormal $\varphi_1, \ldots, \varphi_m \in L^2(\mathbb{R})$ that are also orthogonal to f_1 and for $\Phi \in C_c^\infty(\mathbb{R}^{m+1})$. So if ν_{m+1} denotes the $(m+1)$-dimensional standard normal distribution, then

$$\mathbb{E}[W_n(f)F] = \mathbb{E}[H_n(W_1(f_1))\Phi(W_1(f_1), W_1(\varphi_1), \ldots, W_1(\varphi_m))]$$

$$= \int_{\mathbb{R}^{m+1}} H_n(x_1)\Phi(x_1, x_2, \ldots, x_{m+1})\nu_{m+1}(\mathrm{d}x_1, \ldots, \mathrm{d}x_{m+1})$$

$$= \int_{\mathbb{R}^{m+1}} H_{n-1}(x_1)\partial_{x_1}\Phi(x_1, x_2, \ldots, x_{m+1})\nu_{m+1}(\mathrm{d}x_1, \ldots, \mathrm{d}x_{m+1})$$

$$= \mathbb{E}\left[H_{n-1}(W_1(f_1)) \int_{\mathbb{R}} \mathrm{D}_y F f_1(y)\mathrm{d}y\right] = \int_{\mathbb{R}} \mathbb{E}[W_{n-1}(f(y, \cdot))\mathrm{D}_y F]\mathrm{d}y,$$

which concludes the proof. $\square$

Recall that so far we defined the operator L_0 acting on cylinder functions. If we consider a cylinder function $F \in \mathcal{H}_n$ for some given n, then the action of L_0 is particularly simple.

Lemma 3.7. *Let $n \geqslant 0$ and $F \in \mathscr{C}$ be such that in $L^2(\mu)$ we have $F = W_n(f_n)$ for $f_n \in H_s^2(\mathbb{R}^n) \subset L_s^2(\mathbb{R}^n)$, the twice weakly differentiable symmetric functions from $\mathbb{R}^n$ to $\mathbb{R}$ that satisfy*

$$\|f_n\|_{H_s^2(\mathbb{R}^n)}^2 \simeq \|f_n\|_{L_s^2(\mathbb{R}^n)}^2 + \|\Delta f_n\|_{L_s^2(\mathbb{R}^n)}^2 < \infty.$$

Then

$$L_0 F = L_0 W_n(f_n) = W_n(\Delta f_n) \quad in \ L^2(\mu).$$

Proof. Consider first a functional $G \in \mathscr{C}$ of the form $G(u) = H_n(u(\varphi))$, where $\varphi \in \mathscr{S}(\mathbb{R})$ with $\|\varphi\|_{L^2(\mathbb{R})} = 1$. In that case

$$(21) \qquad L_0 G(u) = H_n'(u(\varphi))u(\Delta\varphi) + H_n''(u(\varphi))\langle \partial_x \varphi, \partial_x \varphi \rangle_{L^2(\mathbb{R})}$$
$$= nH_{n-1}(u(\varphi))u(\Delta\varphi) - n(n-1)H_{n-2}(u(\varphi))\langle \varphi, \Delta\varphi \rangle_{L^2(\mathbb{R})},$$

where in the second step we used that $H_k' = kH_{k-1}$ for $k \geqslant 1$ and $H_0' = 0$. Now we use that $\|\varphi\|_{L^2(\mathbb{R})} = 1$ to rewrite $H_k(W_1(\varphi)) = W_k(\varphi^{\otimes k})$ (see [Nua06, Proposition 1.1.4]) and note the additional factor $k!$ in our definition of H_k compared to the one in [Nua06]. Thus, we can apply [Nua06, Proposition 1.1.2] to compute the first term on the right-hand side:

$$nH_{n-1}(u(\varphi))u(\Delta\varphi) = nW_{n-1}(\varphi^{\otimes(n-1)})W_1(\Delta\varphi)$$
$$= nW_n(\varphi^{\otimes(n-1)} \otimes \Delta\varphi) + n(n-1)W_{n-2}(\varphi^{\otimes(n-2)})\langle \varphi, \Delta\varphi \rangle_{L^2(\mathbb{R})}$$
$$= W_n(\Delta\varphi^{\otimes n}) + n(n-1)H_{n-2}(u(\varphi))\langle \varphi, \Delta\varphi \rangle_{L^2(\mathbb{R})},$$

where in the first term on the right-hand side Δ denotes the Laplacian on $\mathbb{R}^n$. Plugging this back into (21), we obtain $L_0 W_n(\varphi^{\otimes n}) = W_n(\Delta\varphi^{\otimes n})$. By polarization this extends to $W_n(\varphi_1 \otimes \cdots \otimes \varphi_n)$, and then to general $\varphi \in H_s^2(\mathbb{R}^n)$ by taking the closure of the span of functions of the form $\varphi_1 \otimes \cdots \otimes \varphi_n$ with $\varphi_i \in \mathscr{S}(\mathbb{R})$. $\qquad\square$

Corollary 3.8. *The operator L_0 is closable in $L^2(\mu)$, and the domain of its closure, still denoted with L_0, is*

$$\mathrm{dom}(L_0) = \Big\{ F = \sum_{n \geqslant 0} W_n(f_n) : f_n \in H_s^2(\mathbb{R}^n) \forall n \in \mathbb{N} \text{ and } \sum_{n \geqslant 0} n! \|f_n\|_{H_s^2(\mathbb{R}^n)}^2 < \infty \Big\}.$$

For $F \in \mathrm{dom}(L_0)$, we have

$$(22) \qquad\qquad\qquad L_0 F = \sum_{n \geqslant 0} W_n(\Delta f_n).$$

Proof. Let $F(u) = f(u(\varphi_1), \ldots, u(\varphi_n)) \in \mathscr{C}$ be a cylinder function with chaos expansion $F = \sum_{n \geqslant 0} W_n(f_n)$. By a standard approximation argument it follows that $L_0 F = \sum_{n \geqslant 0} W_n(\Delta f_n)$. But then [Nua06, formula (iii) on p. 9] yields

$$\mathbb{E}[|F|^2] = \sum_{n \geqslant 0} n! \|f_n\|_{L_s^2(\mathbb{R}^n)}^2, \qquad \mathbb{E}[|L_0 F|^2] = \sum_{n \geqslant 0} n! \|\Delta f_n\|_{L_s^2(\mathbb{R}^n)}^2,$$

from which our claim readily follows because the Laplace operator on $L_s^2(\mathbb{R}^n)$ is a closed operator with domain $H_s^2(\mathbb{R}^n)$. $\qquad\square$

Before we continue, let us link the $\| \cdot \|_1$-norm defined in section 3.1 with the operator L_0.

Lemma 3.9. *For $F \in \mathscr{C}$, we have*

$$\|F\|_1^2 = \mathbb{E}[\mathcal{E}(F(u_0))] = 2\mathbb{E}[F(u_0)(-L_0 F)(u_0)].$$

Proof. See [GP15, section 2.4] for a proof in the periodic case which works also in our setting. □

Next, we define two auxiliary Hilbert spaces that will be useful in controlling additive functionals of controlled processes.

Definition 3.10. We identify all $F, \tilde{F} \in \mathscr{C}$ with $\|F - \tilde{F}\|_1 = 0$, and we write $\dot{\mathscr{H}}^1$ for the completion of the equivalence classes of $\mathscr{C}$ with respect to $\|\cdot\|_1$.

Similarly, we identify $F, \tilde{F} \in L^2(\mu)$ with $\|F\|_{-1} + \|\tilde{F}\|_{-1} < \infty$ if $\|F - \tilde{F}\|_{-1} = 0$ and the space $\dot{\mathscr{H}}^{-1}$ is defined as the completion of the equivalence classes with respect to $\|\cdot\|_{-1}$.

Definition 3.11. Recall that for $r \in \mathbb{R}$ and $n \in \mathbb{N}$, the homogeneous Sobolev space $\dot{H}^r(\mathbb{R}^n)$ is constructed in the following way: We set for $f \in \mathscr{S}(\mathbb{R})$

$$\|f\|^2_{\dot{H}^r(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\hat{f}(z)|^2 |2\pi z|^{2r} \mathrm{d}z \in [0, \infty],$$

and we consider only those f with $\|f\|_{\dot{H}^r(\mathbb{R}^n)} < \infty$, where we identify f and g if $\|f - g\|_{\dot{H}^r(\mathbb{R}^n)} = 0$. The space $\dot{H}^r(\mathbb{R}^n)$ is then the completion of the equivalence classes with respect to $\|\cdot\|_{\dot{H}^r(\mathbb{R}^n)}$.

We write $\dot{H}^r_s(\mathbb{R}^n)$ for the space that is obtained if we perform the same construction replacing $\|\cdot\|_{\dot{H}^r(\mathbb{R}^n)}$ by

$$\|f\|^2_{\dot{H}^r_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\widehat{\tilde{f}}(z)|^2 |2\pi z|^{2r} \mathrm{d}z, \quad \tilde{f}(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Remark 3.12. By construction, $\dot{H}^r_s(\mathbb{R}^n)$ is always a Hilbert space. For $r < n/2$ there is an explicit description (see [BCD11, Propositions 1.34 and 1.35]) for the nonsymmetric case:

$$\dot{H}^r_s(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'_s(\mathbb{R}^n) : \hat{f} \in L^1_{\mathrm{loc}} \text{ and } \int_{\mathbb{R}^n} |\hat{f}(z)|^2 |2\pi z|^{2r} \mathrm{d}z < \infty \right\}.$$

Here we write $\mathscr{S}'_s(\mathbb{R}^n)$ for those tempered distributions u with $u(\varphi) = u(\tilde{\varphi})$ for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, where $\tilde{\varphi}$ is the symmetrization of φ.

Lemma 3.13. *For $f_n \in \mathscr{S}_s(\mathbb{R}^n)$ and $n \geqslant 1$, we have*

$$\|W_n(f_n)\|^2_1 = n! \|f_n\|^2_{\dot{H}^1_s(\mathbb{R}^n)}, \qquad \|W_n(f_n)\|^2_{-1} = n! \|f_n\|^2_{\dot{H}^{-1}_s(\mathbb{R}^n)}.$$

Proof. For the $\dot{\mathscr{H}}^1$-norm it suffices to note that

$$\mathbb{E}_\mu[W_n(f_n)(-L_0)W_n(f_n)] = n! \|\nabla f_n\|^2_{L^2_s(\mathbb{R}^n)} = n! \|f_n\|^2_{\dot{H}^1_s(\mathbb{R}^n)},$$

where the last equality follows from Plancherel's formula. For the $\dot{\mathscr{H}}^{-1}$-norm let us consider a test function $G = \sum_{m \geqslant 0} W_m(g_m) \in \mathscr{C}$. Then

$$\mathbb{E}[W_n(f_n)(u_0)G(u_0)] = n! \langle f_n, g_n \rangle_{L^2_s(\mathbb{R}^n)},$$

and Plancherel's formula and then the Cauchy–Schwarz inequality give

$$\langle f_n, g_n\rangle_{L_s^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \widehat{f_n}(z)(\hat{g}_n(z))^* \mathrm{d}z$$

$$\leqslant \left(\int_{\mathbb{R}^n} |\widehat{f_n}(z)|^2 |2\pi z|^{-2} \mathrm{d}z\right)^{1/2} \left(\int_{\mathbb{R}^n} |\widehat{g_n}(z)|^2 |2\pi z|^2 \mathrm{d}z\right)^{1/2}$$

$$= \|f_n\|_{\dot{H}_s^{-1}(\mathbb{R}^n)} \|g_n\|_{\dot{H}_s^1(\mathbb{R}^n)},$$

from which (19) shows that $\|W_n(f_n)\|_{-1}^2 \leqslant n! \|f_n\|_{\dot{H}_s^{-1}(\mathbb{R}^n)}^2$. To see the converse inequality, let $f_n \in \dot{H}_s^{-1}(\mathbb{R}^n)$. Then we have for $\hat{g}(z) := (\widehat{f_n}(z))^* |2\pi z|^{-2}/\|f_n\|_{\dot{H}_s^{-1}}$,

$$\int_{\mathbb{R}^n} \widehat{f_n}(z)\hat{g}(z)\mathrm{d}z = \frac{\|f_n\|_{\dot{H}_s^{-1}}^2}{\|f_n\|_{\dot{H}_s^{-1}}} = \|f_n\|_{\dot{H}_s^{-1}},$$

and $\|g\|_{\dot{H}_s^1} = 1$. Of course $W_n(g)$ may not be in $\mathscr{C}$, but we can approximate it by functions in $\mathscr{C}$, and this concludes the proof. $\qquad\square$

Corollary 3.14. *We have an explicit representation of $\dot{\mathscr{H}}^1$ and $\dot{\mathscr{H}}^{-1}$ via*

$$\dot{\mathscr{H}}^1 = \left\{\sum_{n\geqslant 1} W_n(f_n) : f_n \in \dot{H}_s^1(\mathbb{R}^n), \ \sum_n n! \|f_n\|_{\dot{H}_s^1(\mathbb{R}^n)}^2 < \infty\right\},$$

$$\dot{\mathscr{H}}^{-1} = \left\{\sum_{n\geqslant 1} W_n(f_n) : f_n \in \dot{H}_s^{-1}(\mathbb{R}^n), \ \sum_n n! \|f_n\|_{\dot{H}_s^{-1}(\mathbb{R}^n)}^2 < \infty\right\}.$$

Moreover, there is a unique extension of L_0 from $\mathrm{dom}(L_0) \subset L^2(\mu)$ to $\dot{\mathscr{H}}^1$ for which L_0 is an isometry from $\dot{\mathscr{H}}^1$ to $\dot{\mathscr{H}}^{-1}$.

Proof. We only have to prove the statement about the extension of L_0. Since $(1 + |2\pi z|^2)^2 > |2\pi z|^2$ for all $z \in \mathbb{R}^n$, we have $H_s^2(\mathbb{R}^n) \subset \dot{H}_s^1(\mathbb{R}^n)$ and $\|f_n\|_{\dot{H}_s^1(\mathbb{R}^n)}^2 \leqslant \|f_n\|_{H_s^2(\mathbb{R}^n)}^2$ for all $f_n \in H_s^2(\mathbb{R}^n)$. Thus,

$$\mathrm{dom}_0(L_0) := \{F \in \mathrm{dom}(L_0) : \mathbb{E}[F] = 0\} \subset \dot{\mathscr{H}}^1.$$

For $F = \sum_{n\geqslant 1} W_n(f_n) \in \mathrm{dom}_0(L_0)$, we have

$$\|L_0 F\|_{-1}^2 = \sum_{n\geqslant 1} n! \|\Delta f_n\|_{\dot{H}_s^{-1}(\mathbb{R}^n)}^2,$$

and

$$\|\Delta f_n\|_{\dot{H}_s^{-1}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} (|2\pi z|^2 |\widehat{f_n}(z)|)^2 |2\pi z|^{-2} \mathrm{d}z = \|f_n\|_{\dot{H}_s^1(\mathbb{R}^n)}^2,$$

which proves that $\|L_0 F\|_{-1}^2 = \|F\|_1^2$ and thus that L_0 is an isometry from $\mathrm{dom}_0(L_0) \subset \dot{\mathscr{H}}^1$ to $\dot{\mathscr{H}}^{-1}$ which can be uniquely extended to all of $\dot{\mathscr{H}}^1$ because $\mathrm{dom}_0(L_0)$ is dense in $\dot{\mathscr{H}}^1$. $\qquad\square$

3.3. **The Burgers and KPZ nonlinearity.** With the tools we have at hand, it is now straightforward to construct the KPZ nonlinearity (and in particular the Burgers nonlinearity) for all controlled processes.

Proposition 3.15. *Let $T > 0$, $p \geq 1$, let $\rho, \varphi \in \mathscr{S}(\mathbb{R})$, and let $(u, \mathcal{A}) \in \mathcal{Q}$. Then*

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t [(u_s * \rho)^2 - \|\rho\|_{L^2(\mathbb{R})}^2](\varphi) \mathrm{d}s \right|^p \right]$$

$$(23) \qquad\qquad \lesssim (T^{p/2} \vee T^p)(\|\rho\|_{L^1(\mathbb{R})}^2 \|\varphi\|_{H^{-1/2}(\mathbb{R})})^p.$$

Therefore, the integral $\int_0^\cdot [(u_s * \rho)^2 - \|\rho\|_{L^2(\mathbb{R})}](\varphi)\mathrm{d}s$ is well-defined also for $\rho \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Moreover, there exists a unique process $\int_0^\cdot u_s^{\diamond 2}\mathrm{d}s \in C(\mathbb{R}_+, \mathscr{S}'(\mathbb{R}))$ such that for every approximate identity $(\rho^N) \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (i.e., $\sup_N \|\rho^N\|_{L^1(\mathbb{R})} < \infty$, $\widehat{\rho^N}(0) = 1$ for all N, $\lim_{N\to\infty} \widehat{\rho^N}(x) = 1$ for all x), we have

$$\left(\int_0^t u_s^{\diamond 2}\mathrm{d}s\right)(\varphi) := \lim_{n\to\infty} \int_0^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s, \qquad t \geq 0, \varphi \in \mathscr{S}(\mathbb{R}),$$

where the convergence is in $L^p(\mathbb{P})$, uniformly on compact subsets of $\mathbb{R}_+$. If $\rho \in L^1(\mathbb{R})$ is such that $\hat{\rho}(0) = 1$, $x\rho \in L^1(\mathbb{R})$ and $\rho^N := N\rho(N\cdot)$, then

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t u_s^{\diamond 2}\mathrm{d}s(\varphi) - \int_0^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s \right|^p \right]$$

$$(24) \qquad\qquad \lesssim (T^{p/2} \vee T^p)N^{-p/2}\{(1 + \|\rho\|_{L^1(\mathbb{R})})^{3/2}\|x\rho\|_{L^1(\mathbb{R})}^{1/2}\|\varphi\|_{L^2(\mathbb{R})}\}^p.$$

Proof. Let us set

$$F(u) := (u * \rho)^2(\varphi) - \|\rho\|_{L^2(\mathbb{R})}^2 \int \varphi(x)\mathrm{d}x$$

$$= \int_{\mathbb{R}} (u(\rho(x - \cdot))u(\rho(x - \cdot)) - \|\rho\|_{L^2(\mathbb{R})}^2)\varphi(x)\mathrm{d}x$$

$$= \int_{\mathbb{R}} W_2(\rho(x - \cdot)^{\otimes 2})\varphi(x)\mathrm{d}x = W_2\left(\int_{\mathbb{R}} \rho(x - \cdot)^{\otimes 2}\varphi(x)\mathrm{d}x\right),$$

where in the last step we applied the stochastic Fubini theorem. Due to infrared problems, it seems impossible to directly solve the Poisson equation $L_0 G = F$, so instead we consider H with $(1 - L_0)H = F$ which means that $H = W_2(h)$ for

$$(1 - \Delta)h(y_1, y_2) = \int_{\mathbb{R}} \rho(x - y_1)\rho(x - y_2)\varphi(x)\mathrm{d}x$$

or in Fourier variables

$$\hat{h}(\xi_1, \xi_2) = \frac{\hat{\rho}(-\xi_1)\hat{\rho}(-\xi_2)\hat{\varphi}(\xi_1 + \xi_2)}{1 + |2\pi\xi|^2}.$$

Then we have

$$\mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t F(u_s)\mathrm{d}s \right|^p \right] = \mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t (1 - L_0)H(u_s)\mathrm{d}s \right|^p \right]$$

$$(25) \qquad\qquad \lesssim \mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t H(u_s)\mathrm{d}s \right|^p \right] + \mathbb{E}\left[\sup_{t \leq T} \left| \int_0^t L_0 H(u_s)\mathrm{d}s \right|^p \right].$$

For the first term on the right-hand side, we further apply Gaussian hypercontractivity to estimate

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t H(u_s)\mathrm{d}s\right|^p\right] \leqslant T^p\mathbb{E}[|H(u_0)|^p] \simeq T^p\mathbb{E}[|H(u_0)|^2]^{p/2} \simeq T^p\|h\|_{L^2(\mathbb{R})}^p$$

$$\simeq T^p\left(\int_{\mathbb{R}^2}\frac{|\hat{\rho}(-\xi_1)|^2|\hat{\rho}(-\xi_2)|^2|\hat{\varphi}(\xi_1+\xi_2)|^2}{(1+|2\pi\xi|^2)^2}\mathrm{d}\xi\right)^{p/2}$$

$$\leqslant T^p\|\rho\|_{L^1(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}^2}\frac{|\hat{\varphi}(\xi_1+\xi_2)|^2}{(1+|\xi|^2)^2}\mathrm{d}\xi\right)^{p/2}$$

$$\lesssim T^p\|\rho\|_{L^1(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}}\frac{|\hat{\varphi}(\xi_1)|^2}{(1+|\xi_1|^2)^{3/2}}\mathrm{d}\xi_1\right)^{p/2}$$

$$\simeq T^p\|\rho\|_{L^1(\mathbb{R})}^{2p}\|\varphi\|_{H^{-3/2}(\mathbb{R})}^p,$$

where we used the completion of the square to compute

$$\int_{\mathbb{R}}\frac{1}{(1+|\xi_1-\xi_2|^2+|\xi_2|^2)^2}\mathrm{d}\xi_2 = \int_{\mathbb{R}}\frac{1}{(1+2|\xi_2-\xi_1/2|^2+|\xi_1|^2/2)^2}\mathrm{d}\xi_2$$

$$= \int_{\mathbb{R}}\frac{1}{(1+2|\xi_2|^2+|\xi_1|^2/2)^2}\mathrm{d}\xi_2 \lesssim \frac{1}{(1+|\xi_1|^2)^{3/2}}.$$

By Proposition 3.2 together with Remark 3.4 and Lemma 3.13, the second term on the right-hand side of (25) is bounded by

$$\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t L_0 H(u_s)\mathrm{d}s\right|^p\right] \lesssim T^{p/2}\|h\|_{\dot{H}_s^1(\mathbb{R}^2)}^p$$

$$= T^{p/2}\left(\int_{\mathbb{R}^2}\left|\frac{\hat{\rho}(-\xi_1)\hat{\rho}(-\xi_2)\hat{\varphi}(\xi_1+\xi_2)}{1+|2\pi\xi|^2}\right|^2|2\pi\xi|^2\mathrm{d}\xi\right)^{p/2}$$

$$\leqslant T^{p/2}\|\rho\|_{L^1(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}^2}\frac{|\hat{\varphi}(\xi_1+\xi_2)|^2}{1+|\xi|^2}\mathrm{d}\xi\right)^{p/2}$$

$$\lesssim T^{p/2}\|\rho\|_{L^1(\mathbb{R})}^{2p}\left(\int_{\mathbb{R}}\frac{|\hat{\varphi}(\xi_1)|^2}{(1+|\xi_1|^2)^{1/2}}\mathrm{d}\xi_1\right)^{p/2}$$

$$\simeq T^{p/2}\|\rho\|_{L^1(\mathbb{R})}^{2p}\|\varphi\|_{H^{-1/2}(\mathbb{R})}^p.$$

Since $\|\varphi\|_{H^{-1/2}} \geqslant \|\varphi\|_{H^{-3/2}}$, the claimed estimate (23) follows.

If (ρ^N) is an approximate identity, then $\widehat{\rho^N}(\xi_1)\widehat{\rho^N}(\xi_2)$ converges to 1 for all ξ_1, ξ_2, and since $\|\rho^N\|_{L^1}$ and thus $\|\widehat{\rho^N}\|_{L^\infty}$ is uniformly bounded the convergence of $\int_0^t[(u_s*\rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s$ follows from the above arguments together with dominated convergence theorem. If $(\tilde{\rho}^N)$ is another approximate identity, then $|\widehat{\tilde{\rho}^N}(x) - \widehat{\rho^N}(x)|$ converges pointwise to 0 and is uniformly bounded, so the independence of the limit from the approximate identity follows once more from the dominated convergence theorem.

Finally, for $F^N(u) := u_s^{\diamond 2}(\varphi) - [(u*\rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)$, we can solve the Poisson equation directly (strictly speaking, we would have to first approximate $u_s^{\diamond 2}$ by $(u_s*\rho^M)^2 - \|\rho^M\|_{L^2(\mathbb{R})}^2$, but for simplicity we argue already in the limit

$M \to \infty$). We get $F^N = L_0 H^N$ for $H^N = W_2(h^N)$ with

$$\widehat{h^N}(\xi_1, \xi_2) = \frac{(1 - \widehat{\rho^N}(-\xi_1))\hat{\varphi}(\xi_1 + \xi_2)}{|2\pi\xi|^2} + \frac{\widehat{\rho^N}(-\xi_1)(1 - \widehat{\rho^N}(-\xi_2))\hat{\varphi}(\xi_1 + \xi_2)}{|2\pi\xi|^2}.$$

Let us concentrate on the first term, the second one being essentially of the same form (start by bounding $|\widehat{\rho^N}(-\xi_1)| \leqslant \|\rho^N\|_{L^1} \leqslant (1 + \|\rho^N\|_{L^1})$ in that case):

$$\left\| \mathscr{F}^{-1}\left(\frac{(1 - \widehat{\rho^N}(-\xi_1))\hat{\varphi}(\xi_1 + \xi_2)}{|2\pi\xi|^2} \right) \right\|_{\dot{H}^1_s(\mathbb{R}^2)}^2$$

$$\leqslant \left\| \mathscr{F}^{-1}\left(\frac{(1 - \widehat{\rho^N}(-\xi_1))\hat{\varphi}(\xi_1 + \xi_2)}{|2\pi\xi|^2} \right) \right\|_{\dot{H}^1(\mathbb{R}^2)}^2$$

$$= \int_{\mathbb{R}^2} \left| \frac{(1 - \widehat{\rho^N}(-\xi_1))\hat{\varphi}(\xi_1 + \xi_2)}{|2\pi\xi|^2} \right|^2 |2\pi\xi|^2 \mathrm{d}\xi$$

$$\leqslant \int_{\mathbb{R}^2} \min\{\|\partial_x \widehat{\rho^N}\|_{L^\infty}|\xi_1|, (1 + \|\rho^N\|_{L^1})\}^2 \frac{|\hat{\varphi}(\xi_1 + \xi_2)|^2}{|\xi_1|^2} \mathrm{d}\xi$$

$$\leqslant 2 \int_0^{(1 + \|\rho^N\|_{L^1})\|\partial_x \widehat{\rho^N}\|_{L^\infty}^{-1}} \|\partial_x \widehat{\rho^N}\|_{L^\infty}^2 \mathrm{d}\xi_1 \|\varphi\|_{L^2}^2$$

$$\quad + 2 \int_{(1 + \|\rho^N\|_{L^1})\|\partial_x \hat{\rho}_n\|_{L^\infty}^{-1}}^{\infty} \frac{(1 + \|\rho^N\|_{L^1})^2}{|\xi_1|^2} \mathrm{d}\xi_1 \|\varphi\|_{L^2}^2$$

$$\simeq (1 + \|\rho^N\|_{L^1})\|\partial_x \widehat{\rho^N}\|_{L^\infty} \|\varphi\|_{L^2(\mathbb{R})}^2 \lesssim (1 + \|\rho^N\|_{L^1})\|x\rho^N\|_{L^1}\|\varphi\|_{L^2}^2.$$

Now $\|\rho^N\|_{L^1} = \|\rho\|_{L^1}$ and $\|x\rho^N\|_{L^1} = N^{-1}\|x\rho\|_{L^1}$, and therefore our claim follows from Proposition 3.2 together with Remark 3.4 and Lemma 3.13.   □

Proposition 2.2 about the Burgers drift follows by setting

$$\int_0^t \partial_x u_s^2 \mathrm{d}s(\varphi) := \int_0^t u_s^{\diamond 2} \mathrm{d}s(-\partial_x \varphi).$$

Remark 3.16. For the Ornstein–Uhlenbeck process $(X, 0) \in \mathcal{Q}$, one can check that the process $\int_0^{\cdot} X_s^{\diamond 2} \mathrm{d}s$ has regularity $C([0, T], C^{1-}_{\mathrm{loc}}(\mathbb{R}))$. But given the bounds of Proposition 3.15, we cannot even evaluate $\int_0^{\cdot} X_s^{\diamond 2} \mathrm{d}s$ in a point, because the Dirac delta just fails to be in $H^{-1/2}(\mathbb{R})$. The reason is that the martingale argument on which our proof is based gives us at least regularity $1/2-$ in time, and this prevents us from getting better space regularity. On the other hand we are able to increase the time regularity by an interpolation argument as shown in the following corollary which will be useful for controlling certain Young integrals below.

Corollary 3.17. *For all $\varphi \in \mathscr{S}(\mathbb{R})$ and all $T > 0$, the process $\left(\left(\int_0^{\cdot} u_s^{\diamond 2}\mathrm{d}s\right)(\varphi)\right)_{t \in [0,T]}$ is almost surely in $C^{3/4-}([0, T], \mathbb{R})$, and we have for all $\rho \in \mathscr{S}(\mathbb{R})$ with $\hat{\rho}(0) = 1$ and for all $\alpha < 3/4$*

$$\lim_{n \to \infty} \left\| \int_0^{\cdot} u_s^{\diamond 2}\mathrm{d}s(\varphi) - \int_0^{\cdot} [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s \right\|_{C^\alpha([0,T],\mathbb{R})} = 0,$$

where the convergence is in L^p for all $p > 0$ and we write again $\rho^N := N\rho(N\cdot)$.

Proof. Proposition 3.15 yields for all N, $0 \leqslant s < t$, and $p \in [1, \infty)$

$$(26) \quad \mathbb{E}\left[\left|\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi) - \int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s\right|^p\right] \lesssim |t - s|^{p/2}N^{-p/2},$$

and a direct estimate gives

$$\mathbb{E}\left[\left|\int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2(\mathbb{R})}^2](\varphi)\mathrm{d}s\right|^p\right]$$

$$(27) \qquad \lesssim |t - s|^p \mathbb{E}[|((u_0 * \rho^N)^2 - \|\rho^N\|_{L^2}^2)(\varphi)|^2]^{p/2}.$$

The expectation on the right-hand side is

$$\mathbb{E}[|[(u_0 * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)|^2] = \mathbb{E}\left[\left|\int_{\mathbb{R}} W_2(\rho^N(x - \cdot)^{\otimes 2})(u_0)\varphi(x)\mathrm{d}x\right|^2\right]$$

$$= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}x' \mathbb{E}[W_2(\rho^N(x - \cdot)^{\otimes 2})(u_0)W_2(\rho^N(x' - \cdot)^{\otimes 2})(u_0)]\varphi(x)\varphi(x')$$

$$\simeq \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}x' \int_{\mathbb{R}^2} \mathrm{d}y_1\mathrm{d}y_2\rho^N(x - y_1)\rho^N(x - y_2)\rho^N(x' - y_1)\rho^N(x' - y_2)\varphi(x)\varphi(x')$$

$$= \int_{\mathbb{R}} \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}x'|(\rho^N * \rho^N)(x - x')|^2\varphi(x)\varphi(x') \leqslant \|(\rho^N * \rho^N)^2 * \varphi\|_{L^2}\|\varphi\|_{L^2}$$

$$\leqslant \|(\rho^N * \rho^N)^2\|_{L^1}\|\varphi\|_{L^2}^2 = \|(\rho * \rho)^N\|_{L^2}^2\|\varphi\|_{L^2}^2 = N\|\rho * \rho\|_{L^2}^2\|\varphi\|_{L^2}^2.$$

Plugging this into (27), we get for all N and $0 \leqslant s < t$

$$(28) \qquad \mathbb{E}\left[\left|\int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)\mathrm{d}s\right|^p\right] \lesssim_{\rho,\varphi} |t - s|^p N^{p/2}.$$

Now if $0 \leqslant s < t \leqslant s + 1$, we apply (26) and (28) with $N \simeq |t - s|^{-1/2}$, and we get $\mathbb{E}\left[\left|\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi)\right|^p\right] \lesssim |t - s|^{3p/4}$, from which Kolmogorov's continuity criterion yields the local Hölder-continuity of order $3/4-$. Moreover, for $N > |t - s|^{-1/2}$, equation (26) gives for all $\lambda \in [0, 1]$

$$\mathbb{E}\left[\left|\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi) - \int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)\mathrm{d}s\right|^p\right] \lesssim |t - s|^{p(1/2 + (1-\lambda)/4)}N^{-\lambda p/2},$$

while for $N \leqslant |t - s|^{-1/2}$, we get

$$\mathbb{E}\left[\left|\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi) - \int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)\mathrm{d}s\right|^p\right]$$

$$\lesssim \{|t - s|^{p/2}N^{-p/2}\}^\lambda$$

$$\times \left\{\mathbb{E}\left[\left|\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi)\right|^p\right] + \mathbb{E}\left[\left|\int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)\mathrm{d}s\right|^p\right]\right\}^{1-\lambda}$$

$$\lesssim \{|t - s|^{p/2}N^{-p/2}\}^\lambda \times \{|t - s|^{3p/4} + |t - s|^p N^{p/2}\}^{1-\lambda}$$

$$\leqslant \{|t - s|^{p/2}N^{-p/2}\}^\lambda \times \{|t - s|^{3p/4}\}^{1-\lambda},$$

so that choosing $\lambda > 0$ small and applying once more Kolmogorov's continuity criterion, we get the convergence of $\int_s^t [(u_s * \rho^N)^2 - \|\rho^N\|_{L^2}^2](\varphi)\mathrm{d}s$ to $\int_s^t u_s^{\diamond 2}\mathrm{d}s(\varphi)$ in $L^p(\Omega; C^\alpha([0, T], \mathbb{R}))$ for all $\alpha < 3/4$ and all $p > 0$. $\square$

4. Proof of the main results

4.1. **Mapping to the stochastic heat equation.** Let $\partial_t W$ be a space-time white noise on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$, and let η be an $\mathcal{F}_0$-measurable space white noise. Let $(u, \mathcal{A}) \in \mathcal{Q}(W, \eta)$ be a strong stationary solution to the SBE

$$(29) \qquad \mathrm{d}u_t = \Delta u_t \mathrm{d}t + \lambda \partial_x u_t^2 \mathrm{d}t + \sqrt{2}\partial_x \mathrm{d}W_t, \qquad u_0 = \eta.$$

Our aim is to show that u is unique up to indistinguishability, that is to prove the first part of Theorem 2.4. The case $\lambda = 0$ is well understood, so from now on let $\lambda \neq 0$. The basic strategy is to integrate u in the space variable and then to exponentiate the integral and to show that the resulting process solves (a variant of) the linear SHE. However, it is not immediately obvious how to perform the integration in such a way that we obtain a useful integral process. Note that any integral h of u is determined uniquely by its derivative u and the value $h(\sigma)$ for a test function σ with $\int_{\mathbb{R}} \sigma(x)\mathrm{d}x = 1$. So the idea, inspired by [FQ15], is to fix one such test function and to consider the integral h with $h_t(\sigma) \equiv 0$ for all $t \geqslant 0$.

More concretely, we take $\sigma \in C_c^\infty(\mathbb{R})$ with $\sigma \geqslant 0$ and $\int_{\mathbb{R}} \sigma(x)\mathrm{d}x = 1$ and consider the function

$$\Theta_x(z) := \int_{\mathbb{R}} \Theta_{x,y}(z)\sigma(y)\mathrm{d}y := \int_{\mathbb{R}} (\mathbb{1}_{y \leqslant z \leqslant x} - \mathbb{1}_{x < z < y})\sigma(y)\mathrm{d}y$$

$$(30) \qquad = \mathbb{1}_{z \leqslant x} - \int_{\mathbb{R}} (\mathbb{1}_{y > z}\mathbb{1}_{z \leqslant x} + \mathbb{1}_{x < z}\mathbb{1}_{z < y})\sigma(y)\mathrm{d}y = \mathbb{1}_{(-\infty, x]}(z) - \int_z^\infty \sigma(y)\mathrm{d}y,$$

which satisfies for any $\varphi \in \mathscr{S}(\mathbb{R})$

$$\partial_x \langle \Theta_x, \varphi \rangle_{L^2(\mathbb{R})} = \partial_x \left(\int_{-\infty}^x \varphi(z)\mathrm{d}z - \int_{\mathbb{R}} \varphi(z) \int_z^\infty \sigma(y)\mathrm{d}y\mathrm{d}z \right) = \varphi(x)$$

and

$$\int_{\mathbb{R}} \langle \Theta_x, \varphi \rangle_{L^2(\mathbb{R})}\sigma(x)\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (\mathbb{1}_{y \leqslant z \leqslant x} - \mathbb{1}_{x < z < y})\sigma(y)\sigma(x)\mathrm{d}y\mathrm{d}x\varphi(z)\mathrm{d}z = 0,$$

so $x \mapsto \langle \Theta_x, \varphi \rangle_{L^2(\mathbb{R})}$ is the unique integral of φ which vanishes when tested against σ. Moreover,

$$\langle \Theta_x, \partial_z \varphi \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} (\mathbb{1}_{y \leqslant x}(\varphi(x) - \varphi(y)) - \mathbb{1}_{x < y}(\varphi(y) - \varphi(x)))\sigma(y)\mathrm{d}y$$

$$(31) \qquad\qquad = \varphi(x) - \langle \varphi, \sigma \rangle_{L^2(\mathbb{R})},$$

and in particular

$$(32) \qquad \Delta_x \langle \Theta_x, \varphi \rangle_{L^2(\mathbb{R})} = \partial_x \varphi(x) = \langle \Theta_x, \Delta \varphi \rangle_{L^2(\mathbb{R})} + \langle \partial_z \varphi, \sigma \rangle_{L_z^2(\mathbb{R})},$$

where the notation $L_z^2(\mathbb{R})$ means that the $L^2(\mathbb{R})$-norm is taken in the variable z. Now let $\rho \in \mathscr{S}(\mathbb{R})$ be an even function with $\hat{\rho} \in C_c^\infty(\mathbb{R})$ and such that $\hat{\rho} \equiv 1$ on a neighborhood of 0. We write $\rho^L(x) := L\rho(Lx)$, and by our assumptions on ρ there exists for every $L \in \mathbb{N}$ an $M \in \mathbb{N}$ such that $\rho^N * \rho^L = \rho^L$ for all $N \geqslant M$. This will turn out to be convenient later. We also write $\rho_x^L(z) := \rho^L(x - z)$ and $\Theta_x^L := \rho^L * \Theta_x \in \mathscr{S}(\mathbb{R})$, and we define

$$u_t^L(x) := (u_t * \rho^L)(x) = u_t(\rho_x^L), \qquad h_t^L(x) := \int_{\mathbb{R}} \Theta_x(z)u_t^L(z)\mathrm{d}z, \qquad x \in \mathbb{R}.$$

Using that ρ^L is even, we get $h_t^L(x) = u_t(\Theta_x^L)$. So since u is a strong stationary solution to (29), we have

$$\mathrm{d}h_t^L(x) = u_t(\Delta_z \Theta_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(\Theta_x^L) + \sqrt{2}\mathrm{d}W_t(-\partial_z \Theta_x^L), \quad \mathrm{d}[h^L(x)]_t = 2\|\partial_z \Theta_x^L\|_{L^2(\mathbb{R})}^2 \mathrm{d}t,$$

and (31) yields
(33)
$$\partial_z \Theta_x^L(z) = \int_{\mathbb{R}} \Theta_x(z')\partial_z \rho^L(z-z')\mathrm{d}z' = -\rho_z^L(x) + \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})} = -\rho_x^L(z) + (\rho^L \ast \sigma)(z).$$

Now set $\phi_t^L(x) := e^{\lambda h_t^L(x)}$. Then the Itô formula for Dirichlet processes of [RV07] gives

$$\mathrm{d}\phi_t^L(x) = \phi_t^L(x)\left(\lambda \mathrm{d}h_t^L(x) + \frac{1}{2}\lambda^2 \mathrm{d}[h^L(x)]_t\right)$$

$$= \lambda \phi_t^L(x)\Big(u_t^L(\Delta_z \Theta_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(\Theta_x^L) + \sqrt{2}\mathrm{d}W_t(-\partial_z \Theta_x^L)$$

$$+ \lambda\|\rho_x^L - \rho^L \ast \sigma\|_{L^2(\mathbb{R})}^2 \mathrm{d}t\Big),$$

and from (32) we get

$$\Delta_z \Theta_x^L(z) = \langle \Theta_x, \Delta_z \rho_z^L \rangle_{L^2(\mathbb{R})} = \langle \Theta_x, \Delta_{z'} \rho_z^L \rangle_{L^2_{z'}(\mathbb{R})}$$

$$= \Delta_x \langle \Theta_x, \rho_z^L \rangle_{L^2(\mathbb{R})} - \langle \partial_{z'} \rho_z^L, \sigma \rangle_{L^2_{z'}(\mathbb{R})}$$

$$= \Delta_x \Theta_x^L(z) + \partial_z(\rho^L \ast \sigma)(z).$$

Therefore, $u_t^L(\Delta_z \Theta_x^L) = \Delta_x h_t^L(x) + u_t(\rho^L \ast \partial_z \sigma)$. Since moreover

$$\lambda \phi_t^L(x)\Delta_x h_t^L(x) = \Delta_x \phi_t^L(x) - \lambda^2 \phi_t^L(x)(\partial_x h_t^L(x))^2,$$

we obtain

$$\mathrm{d}\phi_t^L(x) = \Delta_x \phi_t^L(x)\mathrm{d}t + \sqrt{2}\lambda \phi_t^L(x)\mathrm{d}W_t(-\partial_z \Theta_x^L)$$

$$+ \lambda^2 \phi_t^L(x)(\lambda^{-1}\mathrm{d}\mathcal{A}_t(\Theta_x^L) + (\lambda^{-1}u_t(\rho^L \ast \partial_z \sigma) - (u_t^L(x))^2 + \|\rho_x^L - \rho^L \ast \sigma\|_{L^2(\mathbb{R})}^2)\mathrm{d}t).$$

Expanding the $L^2(\mathbb{R})$-inner product and noting that $W(-\partial_z \Theta_x^L) = W(\rho_x^L) - W(\rho^L \ast \sigma)$ by (33), we deduce that

$$\mathrm{d}\phi_t^L(x) = \Delta_x \phi_t^L(x)\mathrm{d}t + \sqrt{2}\lambda \phi_t^L(x)\mathrm{d}W_t(\rho_x^L) + \lambda^2 \mathrm{d}R_t^L(x) + \lambda^2 K_x^L \phi^L(x)\mathrm{d}t$$

(34) $$+ \lambda^2 \phi_t^L(x)\mathrm{d}Q_t^L - 2\lambda^2 \phi_t^L(x)\langle \rho_x^L, \rho^L \ast \sigma \rangle_{L^2(\mathbb{R})}\mathrm{d}t - \sqrt{2}\lambda \phi_t^L(x)\mathrm{d}W_t(\rho^L \ast \sigma),$$

where we introduced the processes

(35) $$R_t^L(x) := \int_0^t \phi_s^L(x)\{\lambda^{-1}\mathrm{d}\mathcal{A}_s(\Theta_x^L) - ((u_s^L(x))^2 - \langle (u_s^L)^2, \sigma \rangle_{L^2(\mathbb{R})})\mathrm{d}s - K_x^L \mathrm{d}s\}$$

for the deterministic function

(36) $$K_x^L := -((\rho^L \ast \rho^L \ast \Theta_x)^2(x) - \langle (\rho^L \ast \rho^L \ast \Theta_x)^2, \sigma \rangle_{L^2(\mathbb{R})}),$$

and

(37) $$Q_t^L := \int_0^t \{-\langle (u_s^L)^2 - \|\rho^L\|_{L^2(\mathbb{R})}^2, \sigma \rangle_{L^2(\mathbb{R})} + \|\rho^L \ast \sigma\|_{L^2(\mathbb{R})}^2 + \lambda^{-1}u_s(\rho^L \ast \partial_z \sigma)\}\mathrm{d}s.$$

Integrating (34) against $\varphi \in C_c^\infty(\mathbb{R})$, we get

$$d\phi_t^L(\varphi) = \phi_t^L(\Delta\varphi)dt + \sqrt{2}\lambda \int_{\mathbb{R}} \phi_t^L(x)\varphi(x)dW_t(\rho_x^L)dx + \lambda^2 dR_t^L(\varphi) + \lambda^2\phi_t^L(K^L\varphi)$$

$$+ \lambda^2\phi_t^L(\varphi)dQ_t^L - 2\lambda^2 \int_{\mathbb{R}} \phi_t^L(x)\varphi(x)\langle\rho_x^L, \rho^L * \sigma\rangle_{L^2(\mathbb{R})}dxdt$$

(38) $$- \sqrt{2}\lambda\phi_t^L(\varphi)dW_t(\rho^L * \sigma).$$

In section 4.2 we will prove the following three lemmas.

Lemma 4.1. *We have for all $T > 0$, $p > 2$, and all $\varphi \in C_c^\infty(\mathbb{R})$*

$$\lim_{L\to\infty} (\mathbb{E}[\|R^L(\varphi)\|_{p-\text{var};[0,T]}^2] + \mathbb{E}[\sup_{t\leqslant T}|R_t^L(\varphi)|^2]) = 0.$$

Lemma 4.2. *The deterministic function K^L converges to $\lambda^2/12$ as $L \to \infty$ and is uniformly bounded in the sense that $\sup_{L\in\mathbb{N}, x\in\mathbb{R}}|K_x^L| < \infty$.*

Lemma 4.3. *For all $T > 0$, the process $(Q_t^L)_{t\in[0,T]}$ converges in probability in $C^{3/4-}([0,T], \mathbb{R})$ to the zero quadratic variation process*

$$Q_t := -\int_0^t u_s^{\diamond 2}ds(\sigma) + \|\sigma\|_{L^2(\mathbb{R})}^2 t + \lambda^{-1}\int_0^t u_s(\partial_z\sigma)ds, \qquad t \in [0,T].$$

With the help of these results it is easy to prove our main theorem.

Proof of Theorem 2.4. Consider the expansion (38) of $\phi^L(\varphi)$. Denoting $\phi_t(\varphi) := \lim_{L\to\infty} \phi_t^L(\varphi)$, the stochastic integrals converge to

$$\sqrt{2}\lambda \int_{[0,t]\times\mathbb{R}} \phi_s(x)\varphi(x)dW_s(x)dx - \sqrt{2}\lambda \int_0^t \phi_s(\varphi)dW_s(\sigma)$$

by the stochastic dominated convergence theorem; see [RY99, Proposition IV.2.13] for a formulation in the finite-dimensional setting whose proof carries over without problems to our situation. Lemma 4.1 shows that the p-variation of $R^L(\varphi)$ converges to zero in $L^2(\mathbb{P})$ whenever $p > 2$. Combining this with Lemmas 4.2 and 4.3, the p-variation of $\phi^L(\varphi)$ stays uniformly bounded in L, and therefore we can use once more that Lemma 4.3 gives us local convergence in $C^{3/4-}$ for Q^L to obtain that $\int_0^{\cdot} \phi_s^L(\varphi)dQ_s^L$ converges as a Young integral to $\int_0^{\cdot} \phi_s(\varphi)dQ_s$. In conclusion, we get

$$\phi_t(\varphi) = \langle e^{\eta(\Theta_\cdot)}, \varphi\rangle_{L^2} + \int_0^t \phi_s(\Delta\varphi)ds + \sqrt{2}\lambda \int_{[0,t]\times\mathbb{R}} \phi_s(x)\varphi(x)dW_s(x)dx$$

$$+ \frac{\lambda^4}{12}\int_0^t \phi_s(\varphi)ds + \lambda^2\int_0^t \phi_s(\varphi)dQ_s - 2\lambda^2\int_0^t \phi_s(\varphi\sigma)ds$$

$$- \sqrt{2}\lambda\int_0^t \phi_s(\varphi)dW_s(\sigma).$$

So let us define

$$X_t := \sqrt{2}\lambda W_t(\sigma) + \left(-\frac{\lambda^4}{12} + \lambda^2\|\sigma\|_{L^2}^2\right)t - \lambda^2 Q_t, \quad Z_t(x) := e^{X_t}\phi_t(x), \quad t \geqslant 0,$$

for which $Z_0(\varphi) = \phi_0(\varphi)$ and

$$
\begin{aligned}
\mathrm{d}Z_t(\varphi) &= e^{X_t}\mathrm{d}\phi_t(\varphi) + Z_t(\varphi)\mathrm{d}X_t + \frac{1}{2}Z_t(\varphi)\mathrm{d}[X]_t + \mathrm{d}[\phi(\varphi), e^X]_t \\
&= Z_t(\Delta\varphi)\mathrm{d}t + \sqrt{2}\lambda \int_{\mathbb{R}} Z_t(x)\varphi(x)\mathrm{d}W_t(x)\mathrm{d}x + \frac{\lambda^4}{12}Z_t(\varphi)\mathrm{d}t + \lambda^2 Z_t(\varphi)\mathrm{d}Q_t \\
&\quad - 2\lambda^2 Z_t(\varphi\sigma)\mathrm{d}t - \sqrt{2}\lambda Z_t(\varphi)\mathrm{d}W_t(\sigma) \\
&\quad + \sqrt{2}\lambda Z_t(\varphi)\mathrm{d}W_t(\sigma) + Z_t(\varphi)\left(-\frac{\lambda^4}{12} + \lambda^2\|\sigma\|_{L^2}^2\right)\mathrm{d}t - \lambda^2 Z_t(\varphi)\mathrm{d}Q_t \\
&\quad + \lambda^2 Z_t(\varphi)\|\sigma\|_{L^2}^2\mathrm{d}t \\
&\quad + 2\lambda^2\langle\phi_t\varphi, e^{X_t}\sigma\rangle_{L^2}\mathrm{d}t - 2\lambda^2 Z_t(\varphi)\|\sigma\|_{L^2}^2\mathrm{d}t \\
(39)\qquad &= Z_t(\Delta\varphi)\mathrm{d}t + \sqrt{2}\lambda \int_{\mathbb{R}} Z_t(x)\varphi(x)\mathrm{d}W_t(x)\mathrm{d}x.
\end{aligned}
$$

This shows that Z is a weak solution to the SHE. By Lemma B.1 in Appendix B there exist $c, T > 0$ such that $\sup_{x\in\mathbb{R}, t\in[0,T]} e^{-c|x|}\mathbb{E}[Z_t(x)^2] < \infty$. This allows us to extend (39) on $[0,T]$ to $\varphi \in C^2$ which together with their derivatives up to order 2 decay superexponentially. We can also extend (39) to time-dependent $\varphi \in C^{1,2}([0,T]\times\mathbb{R})$ which are such that

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}} e^{a|x|}\{|\partial_t\varphi(t,x)| + |\varphi(t,x)| + |\partial_x\varphi(t,x)| + |\Delta\varphi(t,x)|\} < \infty
$$

for all $a > 0$, for which we get for $t \in [0,T]$

$$
\begin{aligned}
Z_t(\varphi(t)) = Z_0(\varphi(0)) &+ \int_0^t Z_s(\partial_s\varphi(s) + \Delta\varphi(s))\mathrm{d}s \\
&+ \sqrt{2}\lambda \int_{[0,t]\times\mathbb{R}} Z_s(x)\varphi(s,x)\mathrm{d}W_s(x)\mathrm{d}x;
\end{aligned}
$$

see, e.g., [Wal86, Exercise 3.1]. We apply this with $\varphi(s,y) = p(t+\varepsilon-s, x-y)$ for $t \in [0,T]$ and $x \in \mathbb{R}$ fixed and where p is the heat kernel generated by Δ. Then $(\partial_s + \Delta)\varphi \equiv 0$, and

$$
Z_t(p(\varepsilon, x-\cdot)) = Z_0(p(t+\varepsilon, x-\cdot)) + \sqrt{2}\lambda \int_{[0,t]\times\mathbb{R}} Z_s(y)p(t+\varepsilon-s, x-y)\mathrm{d}W_s(y)\mathrm{d}y.
$$

Since $Z_r = \phi_r e^{X_r}$ and e^{X_r} does not depend on the space variable x, it is easy to see that Z_t and Z_0 have modifications which are continuous in x. Therefore, as $\varepsilon \to 0$, the term on the left-hand side converges to $Z_t(x)$, and the first term on the right-hand side converges to $Z_0(p(t, x-\cdot))$. For the stochastic integral we have

$$
\begin{aligned}
\mathbb{E}&\left[\left|\int_{[0,t]\times\mathbb{R}} Z_s(y)(p(t+\varepsilon-s, x-y) - p(t-s, x-y))\mathrm{d}W_s(y)\mathrm{d}y\right|^2\right] \\
&= \int_{[0,t]\times\mathbb{R}} \mathbb{E}[Z_s(y)^2](p(t+\varepsilon-s, x-y) - p(t-s, x-y))^2\mathrm{d}s\mathrm{d}y \\
&\leqslant \int_{[0,t]\times\mathbb{R}} e^{c|y|}(p(t+\varepsilon-s, x-y) - p(t-s, x-y))^2\mathrm{d}s\mathrm{d}y \\
&\qquad\qquad \times \sup_{y\in\mathbb{R}, t\in[0,T]} e^{-c|y|}\mathbb{E}[Z_t(y)^2].
\end{aligned}
$$

By the dominated convergence theorem, the right-hand side converges to zero, and therefore $(Z_t)_{t \in [0,T]}$ is the unique mild solution to the multiplicative SHE as defined in Walsh [Wal86, Theorem 3.2]. Walsh considers the equation on $[0, L]$ with Neumann boundary conditions, but as [Qua14, Theorem 2.4] shows, the arguments carry over to $\mathbb{R}$ as long as we have at most exponential growth of $\mathbb{E}[Z_t(x)^2]$ in x.

So far we have only showed uniqueness on a small time interval, to extend this to, say, $[0, 2T]$, we need a uniform L^2-bound of the type $\mathbb{E}[Z_t(x)^2] \lesssim e^{c|x|}$ for all $t \in [0, 2T]$. Lemma B.1 gives us such a bound if we know that $\mathbb{E}[Z_t(x)^{10}] \lesssim e^{c'|x|}$ for $t \in [0, T]$ and some $c' > 0$. But now we know that $(Z_t)_{t \in [0,T]}$ is the unique mild solution to the SHE, and for $t = 0$ and $p > 0$ we have for some $c_p > 0$

$$\mathbb{E}[Z_0(x)^p] = \mathbb{E}[e^{\lambda p u_0(\Theta_x)}] = e^{\frac{1}{2}\lambda^2 p^2 \|\Theta_x\|_{L^2}^2} \lesssim e^{c_p|x|}.$$

Therefore, the Burkholder–Davis–Gundy inequality together with the arguments from the proof of [Qua14, Lemma 2.3] show that $\mathbb{E}[Z_t(x)^p] \lesssim e^{c_p'|x|}$ for $t \in [0, T]$ and some $c_p' > 0$. Now the uniqueness extends to $[0, 2T]$, and of course we can iterate the argument to see that $(Z_t)_{t \geq 0}$ is the unique mild solution to the SHE on $[0, \infty)$.

But we know that

$$u = \lim_{L \to \infty} u^L = \lim_{L \to \infty} \partial_x h^L = \lim_{L \to \infty} \partial_x \lambda^{-1} \log \phi^L = \partial_x \lambda^{-1} \log \phi,$$

where the derivative is taken in the distributional sense. Since for fixed $t \geq 0$ we have $Z_t(x) = \phi_t(x) e^{X_t}$ and X_t does not depend on the space variable, we get

$$\partial_x(\log Z_t(x)) = \partial_x(\log \phi_t(x) + X_t) = \partial_x \log \phi_t(x),$$

and therefore the strong stationary solution u is unique up to indistinguishability.

The uniqueness in law of energy solutions follows in the same way from the weak uniqueness of Z. □

Proof of Theorem 2.10. Let $(h, \mathcal{B}) \in \mathcal{Q}^{\mathrm{KPZ}}$ be a strong almost-stationary solution to the KPZ equation

$$dh_t = \Delta h_t dt + \lambda((\partial_x h_t)^2 - \infty) dt + \sqrt{2} dW_t, \qquad h_0 = \chi.$$

Since by definition of the pair $(u, \mathcal{A}) \in \mathcal{Q}(W, \eta)$, we have $u = \partial_x h$ and $\partial_x u(\Theta_x) = u(x)$, we get

$$(40) \qquad\qquad h_t(x) = u_t(\Theta_x) + h_t(\sigma), \qquad t \geq 0, x \in \mathbb{R}.$$

Again, by definition, u is a strong stationary solution to the SBE. So in the proof of Theorem 2.4 we showed that

$$(41) \qquad\qquad u_t(\Theta_x) = \lambda^{-1} \log \phi_t(x) = \lambda^{-1}(\log \tilde{Z}_t(x) - X_t),$$

where $\tilde{Z}$ solves the linear multiplicative heat equation with initial condition $e^{\lambda u_0(\Theta_\cdot)}$, and X was defined by $X_0 = 0$ and

$$dX_t = \sqrt{2}\lambda dW_t(\sigma) + \left(-\frac{\lambda^4}{12} + \lambda^2\|\sigma\|_{L^2}^2 - \lambda^2(-u_t^{\diamond 2}(\sigma) + \|\sigma\|_{L^2}^2 + \lambda^{-1}u_t(\partial_z \sigma))\right) dt$$

$$= \sqrt{2}\lambda dW_t(\sigma) - \frac{\lambda^4}{12}dt + \lambda^2 u_t^{\diamond 2}(\sigma)dt - \lambda u_t(\partial_z \sigma)dt$$

$$= \lambda(h_t(\Delta\sigma)dt + \lambda u_t^{\diamond 2}(\sigma)dt + \sqrt{2}dW_t(\sigma)) - \frac{\lambda^4}{12}dt = \lambda dh_t(\sigma) - \frac{\lambda^4}{12}dt.$$

This shows that $\lambda^{-1}X_t = h_t(\sigma) - h_0(\sigma) - \frac{\lambda^3}{12}t$, and therefore we get with (40) and (41)

$$h_t(x) = \lambda^{-1}(\log \tilde{Z}_t(x) - X_t) + h_t(\sigma) = \lambda^{-1}\log \tilde{Z}_t(x) + h_0(\sigma) + \frac{\lambda^3}{12}t.$$

Finally, $\tilde{Z}_t = e^{-\lambda\chi(\sigma)}Z_t$, where Z solves (14), the linear multiplicative heat equation with initial condition $e^{\lambda\chi}$, and this concludes the proof of the strong uniqueness for strong almost-stationary solutions. The weak uniqueness of energy solutions follows from the weak uniqueness of Z. $\qquad\square$

4.2. Convergence of the remainder terms.
We now proceed to prove Lemmas 4.1–4.3 on the convergence of R^L, K^L, and Q^L, respectively.

4.2.1. *Proof of Lemmas* 4.1 *and* 4.2. To treat R^L, we introduce the auxiliary process

$$R_t^{L,N}(x) := \int_0^t \phi_s^L(x)\{(-(\rho^N * u_s)^2(\partial_z\Theta_x^L)) - ((u_s^L(x))^2 - \langle(u_s^L)^2, \sigma\rangle_{L^2}) - K_x^{L,N}\}\mathrm{d}s$$

for

$$(42) \qquad K_x^{L,N} := \lambda^2 \int_{\mathbb{R}^2} g_x^{L,N}(y_1, y_2)\Theta_x^L(y_1)\Theta_x^L(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

We will show in Lemma 4.5 that $K_x^{L,N}$ converges to K_x^L for all $x \in \mathbb{R}$ and is uniformly bounded in $N \in \mathbb{N}$ and $x \in \mathbb{R}$. Since by assumption $\mathcal{A} = \int_0^\cdot \partial_x u_s^2 \mathrm{d}s$ and $\partial_z\Theta_x^L \in \mathscr{S}(\mathbb{R})$ is a nice test function, we get from Corollary 3.17 that $R_t^{L,N}(\varphi)$ converges to $R_t^L(\varphi)$ in $L^2(\mathbb{P})$.

We also define

$$r^{L,N}(u_s, x) := \phi_s^L(x)((-(\rho^N * u_s)^2(\partial_z\Theta_x^L)) - ((u_s^L(x))^2 - \langle(u_s^L)^2, \sigma\rangle_{L^2}) - K_x^{L,N})$$

so that $R_t^{L,N}(x) = \int_0^t r^{L,N}(u_s, x)\mathrm{d}s$. Using Corollary 3.5, we can estimate for $\varphi \in C_c^\infty(\mathbb{R})$

$$(43) \qquad \mathbb{E}[\|R^{L,N}(\varphi)\|_{p-\mathrm{var};[0,T]}^2] + \mathbb{E}[\sup_{0\leqslant t\leqslant T}|R_t^{L,N}(\varphi)|^2] \lesssim T\|r^{L,N}(\cdot, \varphi)\|_{-1}^2,$$

where we recall that

$$\|r^{L,N}(\cdot, \varphi)\|_{-1}^2 = \sup_{F\in\mathscr{C}}\{2\mathbb{E}[r^{L,N}(u_0, \varphi)F(u_0)] - \|F\|_1^2\},$$

where $\mathscr{C}$ are the cylinder functions and $\|F\|_1^2 = 2\mathbb{E}[\|\partial_x\mathrm{D}F(u_0)\|_{L^2(\mathbb{R})}^2]$ in terms of the Malliavin derivative D associated to the measure μ. We prove below that we can choose $K_x^{L,N}$ so that $\mathbb{E}[r^{L,N}(u_t, x)] = \mathbb{E}[r^{L,N}(u_0, x)] = 0$ for all $x \in \mathbb{R}$. This is necessary for $\|r^{L,N}(\cdot, \varphi)\|_{-1}$ to be finite for all $\varphi \in C_c^\infty(\mathbb{R})$. At this point everything boils down to controlling $\mathbb{E}[r^{L,N}(u_0, \varphi)F(u_0)]$ and to showing that it goes to zero as first $N \to \infty$ and then $L \to \infty$.

Observe that the random variable $(-(\rho^N * u_0)^2(\partial_z\Theta_x^L)) - ((u_0^L(x))^2 - \langle(u_0^L)^2, \sigma\rangle_{L^2})$ is an element of the second homogeneous chaos of u_0. Let us compute its kernel.

From (31) we get

$$
- (\rho^N * u_0)^2 (\partial_z \Theta_x^L) = \int_{\mathbb{R}^2} \left[-\int_{\mathbb{R}} \mathrm{d}z \partial_z \Theta_x^L(z) \rho^N(z - y_1) \rho^N(z - y_2) \right] W(\mathrm{d}y_1 \mathrm{d}y_2)
$$
$$
- \int_{\mathbb{R}} \|\rho^N\|_{L^2(\mathbb{R})}^2 \partial_z \Theta_x^L(z) \mathrm{d}z
$$
$$
= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}z' \Theta_x(z') \partial_{z'} \rho^L(z - z') \rho_z^N(y_1) \rho_z^N(y_2) \right] W(\mathrm{d}y_1 \mathrm{d}y_2) - 0
$$
$$
= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \mathrm{d}z (\rho^L(z - x) - \langle \rho^L(z - \cdot), \sigma \rangle_{L^2(\mathbb{R})}) \rho_z^N(y_1) \rho_z^N(y_2) \right] W(\mathrm{d}y_1 \mathrm{d}y_2)
$$
$$
= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}} \mathrm{d}z (\rho_x^L(z) - \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})}) \rho_z^N(y_1) \rho_z^N(y_2) \right] W(\mathrm{d}y_1 \mathrm{d}y_2),
$$

and furthermore

$$
(u_0^L(x))^2 - \langle (u_0^L)^2, \sigma \rangle_{L^2} = \int_{\mathbb{R}^2} [\rho_x^L(y_1) \rho_x^L(y_2) - \langle \rho_{y_1}^L \rho_{y_2}^L, \sigma \rangle_{L^2(\mathbb{R})}] W(\mathrm{d}y_1 \mathrm{d}y_2)
$$
$$
+ \|\rho^L\|_{L^2(\mathbb{R})}^2 (1 - \int_{\mathbb{R}} \sigma(x) \mathrm{d}x)
$$
$$
= \int_{\mathbb{R}^2} [\rho_x^L(y_1) \rho_x^L(y_2) - \langle \rho_{y_1}^L \rho_{y_2}^L, \sigma \rangle_{L^2(\mathbb{R})}] W(\mathrm{d}y_1 \mathrm{d}y_2).
$$

Therefore, let

$$
g_x^{L,N}(y_1, y_2) := \int_{\mathbb{R}} \mathrm{d}z (\rho_x^L(z) - \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})}) \rho_z^N(y_1) \rho_z^N(y_2)
$$
(44)
$$
- [\rho_x^L(y_1) \rho_x^L(y_2) - \langle \rho_{y_1}^L \rho_{y_2}^L, \sigma \rangle_{L^2(\mathbb{R})}],
$$

so that

$$
W_2(g_x^{L,N}) = (-(\rho^N * u_0)^2 (\partial_z \Theta_x^L)) - ((u_0^L(x))^2 - \langle (u_0^L)^2, \sigma \rangle_{L^2}).
$$

We also let $W_1(g_x^{L,N}(y_1, \cdot)) := \int_{\mathbb{R}} g_x^{L,N}(y_1, y_2) W(\mathrm{d}y_2)$. Using the partial integration by parts derived in Lemma 3.6, we are able to bound $\|r^{L,N}(\cdot, \varphi)\|_{-1}$ by a constant:

Lemma 4.4. *Setting* $K_x^{L,N} := \lambda^2 \int_{\mathbb{R}^2} g_x^{L,N}(y_1, y_2) \Theta_x^L(y_1) \Theta_x^L(y_2) \mathrm{d}y_1 \mathrm{d}y_2$, *we have for all* $F \in \mathscr{C}$

(45)
$$
|\mathbb{E}[r^{L,N}(u_0, \varphi) F(u_0)]| \lesssim \|F\|_1 (A_1^{L,N} + C_1^{L,N}),
$$

and in particular $\|r^{L,N}(\cdot, \varphi)\|_{-1} \lesssim A_1^{L,N} + C_1^{L,N}$, *where*

(46)
$$
A_1^{L,N} := \mathbb{E} \left[\left\| \int_{\mathbb{R}} \varphi(x) W_1(g_x^{L,N}(y_1, \cdot)) \diamond \phi_0^L(x) \mathrm{d}x \right\|_{\dot{H}_{y_1}^{-1}(\mathbb{R})}^2 \right]^{1/2}
$$

and

(47)
$$
C_1^{L,N} := \mathbb{E} \left[\left\| \int_{\mathbb{R}} \varphi(x) \phi_0^L(x) \int_{\mathbb{R}} g_x^{L,N}(y_1, y_2) \Theta_x^L(y_1) \mathrm{d}y_1 \mathrm{d}x \right\|_{\dot{H}_{y_2}^{-1}(\mathbb{R})}^2 \right]^{1/2}.
$$

Here the notation $\dot{H}_y^\alpha(\mathbb{R})$ *means that the norm is taken in the y-variable and*

$$
W_1(g_x^{L,N}(y_1, \cdot)) \diamond \phi_0^L(x) := W_1(g_x^{L,N}(y_1, \cdot)) \phi_0^L(x) - \int_{\mathbb{R}} g_x^{L,N}(y_1, y_2) \mathrm{D}_{y_2} \phi_0^L(x) \mathrm{d}y_2
$$

is a partial Wick contraction in the sense that $\mathbb{E}[W_1(g_x^{L,N}(y_1, \cdot)) \diamond \phi_0^L(x)] = 0$.

Proof. Consider

$$(48) \qquad \mathbb{E}[r^{L,N}(u_0,\varphi)F(u_0)] = \int_{\mathbb{R}} \varphi(x)\mathbb{E}[(W_2(g_x^{L,N}) - K_x^{L,N})\phi_0^L(x)F(u_0)]\mathrm{d}x.$$

Partially integrating by parts $W_2(g_x^{L,N})$, we have

$$\mathbb{E}[W_2(g_x^{L,N})\phi_0^L(x)F(u_0)] = \int_{\mathbb{R}} \mathbb{E}[W_1(g_x^{L,N}(y_1,\cdot))\mathrm{D}_{y_1}[\phi_0^L(x)F(u_0)]]\mathrm{d}y_1$$

$$= \int_{\mathbb{R}} \mathbb{E}[W_1(g_x^{L,N}(y_1,\cdot))\phi_0^L(x)\mathrm{D}_{y_1}F(u_0)]\mathrm{d}y_1$$

$$+ \int_{\mathbb{R}} \mathbb{E}[W_1(g_x^{L,N}(y_1,\cdot))\mathrm{D}_{y_1}(\phi_0^L(x))F(u_0)]\mathrm{d}y_1.$$

The second term on the right-hand side can be integrated by parts again to obtain

$$\int_{\mathbb{R}} \mathbb{E}[W_1(g_x^{L,N}(y_1,\cdot))\mathrm{D}_{y_1}(\phi_0^L(x))F(u_0)]\mathrm{d}y_1$$

$$= \int_{\mathbb{R}^2} g_x^{L,N}(y_1,y_2)\mathbb{E}[\mathrm{D}_{y_1,y_2}^2(\phi_0^L(x))F(u_0)]\mathrm{d}y_1\mathrm{d}y_2$$

$$+ \int_{\mathbb{R}^2} g_x^{L,N}(y_1,y_2)\mathbb{E}[\mathrm{D}_{y_1}(\phi_0^L(x))\mathrm{D}_{y_2}F(u_0)]\mathrm{d}y_1\mathrm{d}y_2,$$

while the first term can be written as

$$\int_{\mathbb{R}} \mathbb{E}[W_1(g_x^{L,N}(y_1,\cdot))\phi_0^L(x)\mathrm{D}_{y_1}F(u_0)]\mathrm{d}y_1$$

$$= \int_{\mathbb{R}} \mathbb{E}[(W_1(g_x^{L,N}(y_1,\cdot)) \diamond \phi_0^L(x))\mathrm{D}_{y_1}F(u_0)]\mathrm{d}y_1$$

$$+ \int_{\mathbb{R}^2} g_x^{L,N}(y_1,y_2)\mathbb{E}[\mathrm{D}_{y_2}(\phi_0^L(x))\mathrm{D}_{y_1}F(u_0)]\mathrm{d}y_1\mathrm{d}y_2.$$

In conclusion, we have the decomposition

$$(49) \qquad \mathbb{E}[r^{L,N}(u_0,\varphi)F(u_0)] = A^{L,N} + B^{L,N} + C^{L,N}$$

with

$$A^{L,N} := \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} \mathbb{E}[(W_1(g_x^{L,N}(y_1,\cdot)) \diamond \phi_0^L(x))\mathrm{D}_{y_1}F(u_0)]\mathrm{d}y_1\mathrm{d}x,$$

$$B^{L,N} := \int_{\mathbb{R}} \mathrm{d}x\varphi(x)$$

$$\times \left[\int_{\mathbb{R}^2} g_x^{L,N}(y_1,y_2)\mathbb{E}\left[\mathrm{D}_{y_1,y_2}^2(\phi_0^L(x))F(u_0)\right]\mathrm{d}y_1\mathrm{d}y_2 - K_x^{L,N}\mathbb{E}[\phi_0^L(x)F(u_0)]\right],$$

and

$$C^{L,N} := 2\int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}^2} g_x^{L,N}(y_1,y_2)\mathbb{E}[\mathrm{D}_{y_1}(\phi_0^L(x))\mathrm{D}_{y_2}F(u_0)]\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}x.$$

So it suffices to bound the three terms $A^{L,N}$, $B^{L,N}$, $C^{L,N}$ independently. In order to proceed, observe that

$$\mathrm{D}_{y_1}\phi_0^L(x) = \mathrm{D}_{y_1}(e^{\lambda u_0(\Theta_x^L)}) = \lambda\phi_0^L(x)\Theta_x^L(y_1),$$

$$\mathrm{D}_{y_1,y_2}^2\phi_0^L(x) = \lambda^2\phi_0^L(x)\Theta_x^L(y_1)\Theta_x^L(y_2),$$

so that by definition of $K_x^{L,N}$

$$B^{L,N}$$

$$= \int_{\mathbb{R}} \varphi(x) \left[\int_{\mathbb{R}^2} g_x^{L,N}(y_1, y_2) \lambda^2 \Theta_x^L(y_1) \Theta_x^L(y_2) \mathrm{d}y_1 \mathrm{d}y_2 - K_x^{L,N} \right] \mathbb{E}[\phi_0^L(x) F(u_0)] \mathrm{d}x$$

$$= \int_{\mathbb{R}} \varphi(x) [K_x^{L,N} - K_x^{L,N}] \mathbb{E}[\phi_0^L(x) F(u_0)] \mathrm{d}x = 0$$

and

$$C^{L,N} = 2 \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}^2} g_x^{L,N}(y_1, y_2) \lambda \Theta_x^L(y_1) \mathbb{E}[\phi_0^L(x) \mathrm{D}_{y_2} F(u_0)] \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}x.$$

Using the duality of $\dot{H}^1(\mathbb{R})$ and $\dot{H}^{-1}(\mathbb{R})$ and the Cauchy–Schwarz inequality, we bound further

$$(A^{L,N})^2 \leqslant \mathbb{E} \left[\left\| \int_{\mathbb{R}} \varphi(x) (W_1(g_x^{L,N}(y_1, \cdot)) \diamond \phi_0^L(x)) \mathrm{d}x \right\|_{\dot{H}_{y_1}^{-1}(\mathbb{R})}^2 \right] \mathbb{E}[\|\mathrm{D}_{y_1} F(u_0)\|_{\dot{H}_{y_1}^1(\mathbb{R})}^2]$$

$$= (A_1^{L,N})^2 \|F\|_1^2,$$

where $A_1^{L,N}$ is the constant defined in (46). Similarly, we obtain

$$(C^{L,N})^2$$

$$\lesssim \mathbb{E} \left[\left\| \int_{\mathbb{R}} \varphi(x) \phi_0^L(x) \int_{\mathbb{R}} g_x^{L,N}(y_1, y_2) \Theta_x^L(y_1) \mathrm{d}y_1 \mathrm{d}x \right\|_{\dot{H}_{y_2}^{-1}(\mathbb{R})} \right]^2 \mathbb{E}[\|\mathrm{D}_{y_2} F(u_0)\|_{\dot{H}_{y_2}^1(\mathbb{R})}^2]$$

$$= (C_1^{L,N})^2 \|F\|_1^2,$$

where $C_1^{L,N}$ is the constant in (47). This proves (45). $\square$

So to control $R^{L,N}$, it remains to show that the two constants $A_1^{L,N}$ and $C_1^{L,N}$ vanish in the limit $N, L \to \infty$. Before doing so, let us prove Lemma 4.2. More precisely, we show the following refined version:

Lemma 4.5. *We have* $\sup_{L,N,x} |K_x^{L,N}| < \infty$ *and* $K_x^L = \lim_{N \to \infty} K_x^{L,N}$ *as well as* $\lim_{L \to \infty} K_x^L = \lambda^2/12$ *for all* $x \in \mathbb{R}$.

Proof. Recall that $\Theta_x^L = \Theta_x * \rho^L$, so

$$\lambda^{-2} K_x^{L,N} = \int_{\mathbb{R}^2} g_x^{L,N}(y_1, y_2) \Theta_x^L(y_1) \Theta_x^L(y_2) \mathrm{d}y_1 \mathrm{d}y_2$$

$$= \int_{\mathbb{R}} (\rho_x^L(z) - \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})}) (\rho^N * \rho^L * \Theta_x)^2(z) \, \mathrm{d}z$$

$$(50) \qquad\qquad - ((\rho^L * \rho^L * \Theta_x)^2(x) - \langle (\rho^L * \rho^L * \Theta_x)^2, \sigma \rangle_{L^2(\mathbb{R})}).$$

By (33) we know that for $N \to \infty$ the first term on the right-hand side converges to

$$\int_{\mathbb{R}} (\rho_x^L(z) - \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})}) (\rho^L * \Theta_x)(z)^2 \mathrm{d}z = - \int_{\mathbb{R}} \partial_z(\rho^L * \Theta_x)(z) (\rho^L * \Theta_x)^2(z) \mathrm{d}z$$

$$= -\frac{1}{3} \int_{\mathbb{R}} \partial_z(\rho^L * \Theta_x)^3(z) \mathrm{d}z = 0,$$

where in the last step we used that $\rho^L * \Theta_x \in \mathscr{S}(\mathbb{R})$. Moreover,

$$
\left| \int_{\mathbb{R}} (\rho_x^L(z) - \langle \rho_z^L, \sigma \rangle_{L^2(\mathbb{R})})(\rho^N * \rho^L * \Theta_x)^2(z)\mathrm{d}z \right|
$$

$$
= |(\rho^L * (\rho^N * \rho^L * \Theta_x)^2)(x) - \langle \rho^L * (\rho^N * \rho^L * \Theta_x)^2, \sigma \rangle_{L^2(\mathbb{R})}|
$$

$$
\leqslant \|\rho^L\|_{L^1(\mathbb{R})} \|(\rho^N * \rho^L * \Theta_x)^2\|_{L^\infty(\mathbb{R})}(1 + \|\sigma\|_{L^1(\mathbb{R})})
$$

$$
\leqslant \|\Theta_x\|_{L^\infty(\mathbb{R})}^2(1 + \|\sigma\|_{L^1(\mathbb{R})}) \lesssim 1,
$$

and by similar arguments also the second term on the right-hand side of (50) stays bounded in L, N, x. Recalling that $\Theta_x(z) = \mathbb{1}_{(-\infty,x]}(z) - \int_z^\infty \sigma(y)\mathrm{d}y$, we get by symmetry of $\rho * \rho$

$$
\lim_{L\to\infty} (\rho^L * \rho^L * \Theta_x)(x) = \lim_{L\to\infty} (\rho * \rho)^L * \Theta_x(x) = \frac{1}{2} - \int_x^\infty \sigma(y)\mathrm{d}y
$$

as well as $\lim_{L\to\infty} (\rho^L * \rho^L * \Theta_x) = \mathbb{1}_{(-\infty,x]} - \int_\cdot^\infty \sigma(y)\mathrm{d}y$ in $L^p(\mathbb{R})$ for any $p \in [1,\infty)$. In particular,

$$
\lim_{L\to\infty} ((\rho^L * \rho^L * \Theta_x)^2(x) - \langle (\rho^L * \rho^L * \Theta_x)^2, \sigma \rangle_{L^2(\mathbb{R})})
$$

$$
= \frac{1}{4} - \int_x^\infty \sigma(y)\mathrm{d}y + \left(\int_x^\infty \sigma(y)\mathrm{d}y \right)^2
$$

(51)
$$
- \int_{\mathbb{R}} \left(\mathbb{1}_{(-\infty,x]}(z) - \int_z^\infty \sigma(y)\mathrm{d}y \right)^2 \sigma(z)\mathrm{d}z.
$$

For the last term on the right-hand side we further get

$$
\int_{\mathbb{R}} \left(\mathbb{1}_{(-\infty,x]}(z) - \int_z^\infty \sigma(y)\mathrm{d}y \right)^2 \sigma(z)\mathrm{d}z
$$

$$
= \int_{-\infty}^x \sigma(z)\mathrm{d}z - 2 \int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}z \mathbb{1}_{z\leqslant x} \mathbb{1}_{y\geqslant z} \sigma(y)\sigma(z)
$$

$$
+ \int_{\mathbb{R}^3} \mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2 \mathbb{1}_{y_1\geqslant z} \mathbb{1}_{y_2\geqslant z} \sigma(y_1)\sigma(y_2)\sigma(z)
$$

$$
= 1 - \int_x^\infty \sigma(z)\mathrm{d}z - 2 \int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}z \mathbb{1}_{z\leqslant x} \mathbb{1}_{y\geqslant z} \sigma(y)\sigma(z)
$$

$$
+ \int_{\mathbb{R}^3} \mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2 \mathbb{1}_{y_1\geqslant z} \mathbb{1}_{y_2\geqslant z} \sigma(y_1)\sigma(y_2)\sigma(z),
$$

and the three-dimensional integral takes the value

$$
\int_{\mathbb{R}^3} \mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2 \mathbb{1}_{y_1\geqslant z} \mathbb{1}_{y_2\geqslant z} \sigma(y_1)\sigma(y_2)\sigma(z)
$$

$$
= \int_{\mathbb{R}^3} \mathrm{d}z\mathrm{d}y_1\mathrm{d}y_2 \mathbb{1}_{z=\min\{y_1,y_2,z\}} \sigma(y_1)\sigma(y_2)\sigma(z) = \frac{1}{3}
$$

by symmetry of the variables z, y_1, y_2. To compute the two-dimensional integral, observe first that

$$
(\mathbb{1}_{z\leqslant x, z\leqslant y} + \mathbb{1}_{y\leqslant x, y<z}) = \mathbb{1}_{z\leqslant y} - \mathbb{1}_{x<z\leqslant y} + \mathbb{1}_{y<z} - \mathbb{1}_{x<y<z} = 1 - \mathbb{1}_{x<y} \mathbb{1}_{x<z},
$$

and integrating this against $\sigma(y)\sigma(z)\mathrm{d}y\mathrm{d}z$ and using the symmetry in (y, z), we get

$$
\int_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}z \mathbb{1}_{z\leqslant x} \mathbb{1}_{y\geqslant z} \sigma(y)\sigma(z) = \frac{1}{2} \left(1 - \left(\int_x^\infty \sigma(z)\mathrm{d}z \right)^2 \right).
$$

Plugging all this back into (51), we have

$$\lim_{L \to \infty} ((\rho^L * \rho^L * \Theta_x)^2(x) - \langle(\rho^L * \rho^L * \Theta_x)^2, \sigma\rangle_{L^2(\mathbb{R})})$$

$$= \frac{1}{4} - \int_x^\infty \sigma(y)\mathrm{d}y + \left(\int_x^\infty \sigma(y)\mathrm{d}y\right)^2$$

$$- \left(1 - \int_x^\infty \sigma(z)\mathrm{d}z - \left(1 - \left(\int_x^\infty \sigma(z)\mathrm{d}z\right)^2\right) + \frac{1}{3}\right)$$

$$= \frac{1}{4} - \frac{1}{3} = -\frac{1}{12},$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following computation will be useful for controlling both $A_1^{L,N}$ and $C_1^{L,N}$, which is why we outsource it in a separate lemma.

Lemma 4.6. *Define the kernel*

$$G_{x,x'}^{L,N}(y_1) := \int_{\mathbb{R}} \Theta_{x'}^L(y_2) g_x^{L,N}(y_1, y_2)\mathrm{d}y_2.$$

Then there exists $C > 0$ such that for all $L \in \mathbb{N}$ there is $M(L) \in \mathbb{N}$ with

$$\sup_{x,x' \in \mathbb{R}, N \geqslant M(L)} \|G_{x,x'}^{L,N}\|_{\dot{H}^{-1}(\mathbb{R})} \leqslant CL^{-1/2}.$$

Proof. We argue by duality. For $f \in C_c^\infty$, we have

(52)
$$\langle G_{x,x'}^{L,N}, f\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \mathrm{d}z(\rho_x^L(z) - \langle\rho_z^L, \sigma\rangle_{L^2(\mathbb{R})})(\rho^N * f)(z)(\rho^N * \Theta_{x'}^L)(z)$$

$$- \{(\rho^L * f)(x)(\rho^L * \Theta_{x'}^L)(x) - \langle(\rho^L * f)(\rho^L * \Theta_{x'}^L), \sigma\rangle_{L^2(\mathbb{R})}\}$$

$$= (\rho^L * ((\rho^N * f)(\rho^N * \Theta_{x'}^L)))(x)$$

$$- \int_{\mathbb{R}} \mathrm{d}z\sigma(z)(\rho^L * ((\rho^N * f)(\rho^N * \Theta_{x'}^L)))(z)$$

$$- \left\{(\rho^L * f)(x)(\rho^L * \Theta_{x'}^L)(x) - \int_{\mathbb{R}} \mathrm{d}z\sigma(z)(\rho^L * f)(z)(\rho^L * \Theta_{x'}^L)(z)\right\}$$

$$= \mathbb{E}[f(x - R_1^L - R_2^N)\Theta_{x'}^L(x - R_1^L - R_3^N) - f(Z - R_1^L - R_2^N)\Theta_{x'}^L(Z - R_1^L - R_3^N)]$$

$$- \mathbb{E}[f(x - R_1^L)\Theta_{x'}^L(x - R_4^L) - f(Z - R_1^L)\Theta_{x'}^L(Z - R_4^L)],$$

where the random variables $(R_1^L, R_2^N, R_3^N, R_4^L, Z)$ are independent and $R_i^L \sim \rho^L$, $R_i^N \sim \rho^N$, and $Z \sim \sigma$ (note that ρ^L, ρ^N, σ are all probability densities). The observation that we can simplify the notation in this way is taken from [FQ15]. Note that by assumption $\rho^L * \rho^N = \rho^L$ for sufficiently large N, and therefore

$$\mathbb{E}[\Theta_{x'}^L(x - R_1^L - R_3^N) - \Theta_{x'}^L(x - R_4^L)] = ((\rho^L * \rho^N) * \Theta_{x'}^L)(x) - (\rho^L * \Theta_{x'}^L)(x) = 0.$$

Similarly $\mathbb{E}[f(Z)(\Theta_{x'}^L(Z - R_1^L - R_3^N) - \Theta_{x'}^L(Z - R_4^L))] = 0$ by the independence of Z and R_i^L, R_i^N, hence we can regroup

$$\langle G_{x,x'}^{L,N}, f\rangle_{L^2(\mathbb{R})}$$
$$= \mathbb{E}[(f(x - R_1^L - R_2^N) - f(x))(\Theta_{x'}^L(x - R_1^L - R_3^N) - \Theta_{x'}^L(x - R_4^L))]$$
$$+ \mathbb{E}[(f(x - R_1^L - R_2^N) - f(x - R_1^L))\Theta_{x'}^L(x - R_4^L)]$$
$$- \mathbb{E}[(f(Z - R_1^L - R_2^N) - f(Z))(\Theta_{x'}^L(Z - R_1^L - R_3^N) - \Theta_{x'}^L(Z - R_4^L))]$$
$$(53) \qquad - \mathbb{E}[(f(Z - R_1^L - R_2^N) - f(Z - R_1^L))\Theta_{x'}^L(Z - R_4^L)].$$

Let us estimate for example the most complicated term

$$(54)$$
$$|\mathbb{E}[(f(Z - R_1^L - R_2^N) - f(Z))(\Theta_{x'}^L(Z - R_1^L - R_3^N) - \Theta_{x'}^L(Z - R_4^L))]|$$
$$= \left|\mathbb{E}\left[\left(\int_Z^{Z - R_1^L - R_2^N} f'(y)\mathrm{d}y\right)(\Theta_{x'}^L(Z - R_1^L - R_3^N) - \Theta_{x'}^L(Z - R_4^L))\right]\right|$$
$$\leqslant 2\|f\|_{\dot{H}^1(\mathbb{R})}\|\Theta_{x'}^L\|_{L^\infty(\mathbb{R})}|\mathbb{E}[|R_1^L + R_2^N|^{1/2}]| \lesssim \|f\|_{\dot{H}^1(\mathbb{R})}(L^{-1/2} + N^{-1/2}).$$

The other terms can be controlled using the same arguments, and therefore we get

$$(55) \qquad \langle G_{x,x'}^{L,N}, f\rangle_{L^2(\mathbb{R})} \lesssim \|f\|_{\dot{H}^1(\mathbb{R})}(L^{-1/2} + N^{-1/2}) \lesssim \|f\|_{\dot{H}^1(\mathbb{R})}L^{-1/2},$$

which yields $\|G_{x,x'}^{L,N}\|_{\dot{H}^{-1}(\mathbb{R})} \lesssim L^{-1/2}$ by the density of $C_c^\infty(\mathbb{R})$ in $\dot{H}^1(\mathbb{R})$. $\qquad\square$

Lemma 4.7. *We have* $\lim_{L\to\infty}\limsup_{N\to\infty} A_1^{L,N} = 0$.

Proof. We expand the squared $\dot{H}^{-1}$-norm as

$$(A_1^{L,N})^2 = \mathbb{E}\left[\left\|\int_\mathbb{R} \varphi(x)W_1(g_x^{L,N}(y_1, \cdot)) \diamond \phi_0^L(x)\mathrm{d}x\right\|_{\dot{H}_{y_1}^{-1}(\mathbb{R})}^2\right]$$
$$= \int_{\mathbb{R}^2} \varphi(x)\varphi(x')\mathbb{E}\left[\langle\phi_0^L(x)\diamond W_1(g_x^{L,N}(y_1, \cdot)), \phi_0^L(x')\diamond W_1(g_{x'}^{L,N}(y_1, \cdot))\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\right]\mathrm{d}x\mathrm{d}x'.$$

Integrating by parts the W_1 terms and taking into account the cancellations due to the partial Wick contractions, we get

$$\mathbb{E}\left[\langle\phi_0^L(x)\diamond W_1(g_x^{L,N}(y_1, \cdot)), \phi_0^L(x')\diamond W_1(g_{x'}^{L,N}(y_1, \cdot))\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\right]$$
$$= \mathbb{E}[\phi_0^L(x)\phi_0^L(x')]\int_\mathbb{R}\langle g_x^{L,N}(y_1, y_2), g_{x'}^{L,N}(y_1, y_2)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2$$
$$+ \int_{\mathbb{R}^2}\mathbb{E}[(\mathrm{D}_{y_3}\phi_0^L(x))(\mathrm{D}_{y_2}\phi_0^L(x'))]\langle g_x^{L,N}(y_1, y_2), g_{x'}^{L,N}(y_1, y_3)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2\mathrm{d}y_3.$$

The second term can be written as

$$\int_{\mathbb{R}^2} \mathbb{E}[(D_{y_3}\phi_0^L(x))(D_{y_2}\phi_0^L(x'))]\langle g_x^{L,N}(y_1,y_2), g_{x'}^{L,N}(y_1,y_3)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2\mathrm{d}y_3$$

$$= \lambda^2 \mathbb{E}[\phi_0^L(x)\phi_0^L(x')]\int_{\mathbb{R}^2} \Theta_x^L(y_3)\Theta_{x'}^L(y_2)\langle g_x^{L,N}(y_1,y_2), g_{x'}^{L,N}(y_1,y_3)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2\mathrm{d}y_3,$$

so letting $V^L(x,x') := \lambda^2|\varphi(x)\varphi(x')\mathbb{E}[\phi_0^L(x)\phi_0^L(x')]|$, we have

$$(A_1^{L,N})^2 = \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' V^L(x,x') \int_{\mathbb{R}} \langle g_x^{L,N}(y_1,y_2), g_{x'}^{L,N}(y_1,y_2)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2$$

$$+ \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' V^L(x,x')$$

$$\times \int_{\mathbb{R}^2} \Theta_x^L(y_3)\Theta_{x'}^L(y_2)\langle g_x^{L,N}(y_1,y_2), g_{x'}^{L,N}(y_1,y_3)\rangle_{\dot{H}_{y_1}^{-1}(\mathbb{R})}\mathrm{d}y_2\mathrm{d}y_3$$

(56) $=: A_{1,1} + A_{1,2}.$

Let us consider first $A_{1,2} = \int_{\mathbb{R}^2} V^L(x,x')\langle G_{x,x'}^{L,N}, G_{x',x}^{L,N}\rangle_{\dot{H}^{-1}(\mathbb{R})}\mathrm{d}x\mathrm{d}x'$, which according to Lemma 4.6 can be bounded by

$$A_{1,2} \leqslant \int_{\mathbb{R}^2} V^L(x,x')\|G_{x,x'}^{L,N}\|_{\dot{H}^{-1}(\mathbb{R})}\|G_{x',x}^{L,N}\|_{\dot{H}^{-1}(\mathbb{R})}\mathrm{d}x\mathrm{d}x'$$

$$\lesssim L^{-1}\int_{\mathbb{R}^2} V^L(x,x')\mathrm{d}x\mathrm{d}x' \lesssim L^{-1}$$

for all large N. We continue by estimating the term $A_{1,1}$ in (56) which is bounded by

$$(57) \quad A_{1,1} \leqslant \int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' V^L(x,x') \int_{\mathbb{R}} \|g_x^{L,N}(\cdot,y_2)\|_{\dot{H}^{-1}(\mathbb{R})}\|g_{x'}^{L,N}(\cdot,y_2)\|_{\dot{H}^{-1}(\mathbb{R})}\mathrm{d}y_2.$$

To treat the $\dot{H}^{-1}(\mathbb{R})$-norms, we argue again by duality, as in the proof of Lemma 4.6. Therefore, let $f \in C_c^\infty(\mathbb{R})$, and consider

(58)
$$\langle g_x^{L,N}(\cdot,y_2), f\rangle_{L^2(\mathbb{R})}$$

$$= \int_{\mathbb{R}} \mathrm{d}z(\rho_x^L(z) - \langle \rho_z^L, \sigma\rangle_{L^2(\mathbb{R})})(\rho^N * f)(z)\rho_z^N(y_2)$$

$$- ((\rho^L * f)(x)\rho_x^L(y_2) - \langle(\rho^L * f)\rho_{y_2}^L, \sigma\rangle_{L^2(\mathbb{R})})$$

$$= \mathbb{E}[f(x - R_1^L - R_2^N)\rho^N(x - y_2 - R_1^L) - f(y_2 - R_1^N - R_2^N)\sigma(y_2 - R_1^N - R_3^L)]$$

$$- \mathbb{E}[f(x - R_1^L)\rho^L(x - y_2) - f(y_2 - R_1^L - R_2^L)\sigma(y_2 - R_1^L)],$$

where R_i^L, R_i^N, Z are independent random variables as above. Now observe that by our assumptions on ρ

$$\mathbb{E}[\rho^N(x - y_2 - R_1^L) - \rho^L(x - y_2)] = (\rho^L * \rho^N)(x - y_2) - \rho^L(x - y_2) = 0$$

if N is large enough, and similarly $\mathbb{E}[\sigma(y_2 - R_1^N - R_3^L)] = \mathbb{E}[\sigma(y_2 - R_1^L)]$. So for large N we can decompose the expectations in (58) as

$$
\begin{aligned}
\langle g_x^{L,N}(\cdot, y_2), f\rangle_{L^2(\mathbb{R})} \\
= \mathbb{E}[(f(x - R_1^L - R_2^N) - f(x - R_1^L))\rho^N(x - y_2 - R_1^L)] \\
+ \mathbb{E}[(f(x - R_1^L) - f(x))(\rho^N(x - y_2 - R_1^L) - \rho^L(x - y_2))] \\
- \mathbb{E}[(f(y_2 - R_1^N - R_2^N) - f(y_2 - R_1^L - R_2^L))\sigma(y_2 - R_1^N - R_3^L)] \\
- \mathbb{E}[(f(y_2 - R_1^L - R_2^L) - f(y_2))(\sigma(y_2 - R_1^N - R_3^L) - \sigma(y_2 - R_1^L))].
\end{aligned}
$$

Bounding each term individually as in the proof of Lemma 4.6, we get

$$
\begin{aligned}
\langle g_x^{L,N}(\cdot, y_2), f\rangle_{L^2(\mathbb{R})} \leqslant \|f\|_{\dot{H}^1(\mathbb{R})} \times \Big(\mathbb{E}[|R_2^N|^{1/2}\rho^N(x - y_2 - R_1^L)] \\
+ \mathbb{E}[|R_1^L|^{1/2}(\rho^N(x - y_2 - R_1^L) + \rho^L(x - y_2))] \\
+ \mathbb{E}[|R_1^N + R_2^N + R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^N - R_3^L)] \\
+ \mathbb{E}[|R_1^L + R_2^L|^{1/2}(\sigma(y_2 - R_1^N - R_3^L) + \sigma(y_2 - R_1^L))] \Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|g_x^{L,N}(\cdot, y_2)\|_{\dot{H}^{-1}(\mathbb{R})} \\
\leqslant \mathbb{E}[|R_2^N|^{1/2}\rho^N(x - y_2 - R_1^L)] + \mathbb{E}[|R_1^L|^{1/2}(\rho^N(x - y_2 - R_1^L) + \rho^L(x - y_2))] \\
+ \mathbb{E}[|R_1^N + R_2^N + R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^N - R_3^L)] \\
+ \mathbb{E}[|R_1^L + R_2^L|^{1/2}(\sigma(y_2 - R_1^N - R_3^L) + \sigma(y_2 - R_1^L))].
\end{aligned}
$$

By the same computation we get a similar bound for $\|g_{x'}^{L,N}(\cdot, y_2)\|_{\dot{H}^{-1}(\mathbb{R})}$, and plugging these back into (57), we generate a number of products between different expectations. Let us treat three prototypical cases: Writing $V^L(x) := |\lambda\varphi(x)|\mathbb{E}[|\phi_0^L(x)|^2]^{1/2}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}x' |V^L(x,x')| \int_{\mathbb{R}} \mathbb{E}[|R_1^L|^{1/2}\rho^N(x - y_2 - R_1^L)]\mathbb{E}[|R_1^L|^{1/2}\rho^N(x' - y_2 - R_1^L)]\mathrm{d}y_2 \\
\leqslant \int_{\mathbb{R}^2} \mathrm{d}x \mathrm{d}x' V^L(x)V^L(x') \\
\times \int_{\mathbb{R}} \mathbb{E}[|R_1^L|^{1/2}\rho^N(x - y_2 - R_1^L)]\mathbb{E}[|R_1^L|^{1/2}\rho^N(x' - y_2 - R_1^L)]\mathrm{d}y_2 \\
= \int_{\mathbb{R}} \mathbb{E}[|R_1^L|^{1/2}(\rho^N * V^L)(y_2 + R_1^L)]\mathbb{E}[|R_1^L|^{1/2}(\rho^N * V^L)(y_2 + R_1^L)]\mathrm{d}y_2 \\
= \mathbb{E}\left[|R_1^L|^{1/2}|R_2^L|^{1/2} \int_{\mathbb{R}} (\rho^N * V^L)(y_2 + R_1^L)(\rho^N * V^L)(y_2 + R_2^L)\mathrm{d}y_2\right] \\
\leqslant \mathbb{E}[|R_1^L|^{1/2}|R_2^L|^{1/2}\|(\rho^N * V^L)\|_{L^2(\mathbb{R})}^2] \lesssim L^{-1}(\|\rho^N\|_{L^1(\mathbb{R})}\|V^L\|_{L^2(\mathbb{R})})^2 \lesssim L^{-1},
\end{aligned}
$$

where we introduced a new independent copy R_2^L of R_1^L; this is a trick that we will apply several times in the following. Another situation occurs if only one of the

two expectations depends on x (respectively x'), for example,

$$
\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' V^L(x) V^L(x')
$$

$$
\times \int_{\mathbb{R}} \mathbb{E}[|R_2^N|^{1/2}\rho^N(x - y_2 - R_1^L)]\mathbb{E}[|R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^N - R_3^L)]\mathrm{d}y_2
$$

$$
= \|V^L\|_{L^1(\mathbb{R})}
$$

$$
\times \int_{\mathbb{R}} \mathbb{E}[|R_2^N|^{1/2}(\rho^N * V^L)(y_2 + R_1^L)]\mathbb{E}[|R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^N - R_3^L)]\mathrm{d}y_2
$$

$$
\simeq \mathbb{E}\left[|R_2^N|^{1/2}|R_4^L + R_5^L|^{1/2} \int_{\mathbb{R}} (\rho^N * V^L)(y_2 + R_1^L)\sigma(y_2 - R_6^N - R_7^L)\mathrm{d}y_2\right]
$$

$$
\leqslant \mathbb{E}[|R_2^N|^{1/2}|R_4^L + R_5^L|^{1/2}\|\rho^N * V^L\|_{L^2(\mathbb{R})}\|\sigma\|_{L^2(\mathbb{R})}]
$$

$$
\lesssim N^{-1/2}L^{-1/2}\|\rho^N\|_{L^1(\mathbb{R})}\|V^L\|_{L^2(\mathbb{R})}\|\sigma\|_{L^2(\mathbb{R})} \lesssim L^{-1}.
$$

Finally, we have to handle the case where none of the expectations depend on x or x', for example,

$$
\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}x' V^L(x) V^L(x')
$$

$$
\times \int_{\mathbb{R}} \mathbb{E}[|R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^N - R_3^L)]\mathbb{E}[|R_1^L + R_2^L|^{1/2}\sigma(y_2 - R_1^L)]\mathrm{d}y_2
$$

$$
= \|V^L\|_{L^1(\mathbb{R})}^2 \mathbb{E}\left[|R_1^L + R_2^L|^{1/2}|R_4^L + R_5^L|^{1/2} \int_{\mathbb{R}} \sigma(y_2 - R_1^N - R_3^L)\sigma(y_2 - R_4^L)\mathrm{d}y_2\right]
$$

$$
\lesssim \mathbb{E}[|R_1^L + R_2^L|^{1/2}|R_4^L + R_5^L|^{1/2}\|\sigma\|_{L^2(\mathbb{R})}^2] \lesssim L^{-1}.
$$

In conclusion $A_{1,1}$ also vanishes as first $N \to \infty$ and then $L \to \infty$, and this concludes the proof. $\qquad\square$

Lemma 4.8. *We have* $\lim_{L\to\infty} \lim_{N\to\infty} C_1^{L,N} = 0$.

Proof. Recall that

$$
(C_1^{L,N})^2 = \mathbb{E}\left[\left\|\int_{\mathbb{R}} \varphi(x)\phi_0^L(x) \int_{\mathbb{R}} g_x^{L,N}(y_1, y_2)\Theta_x^L(y_1)\mathrm{d}y_1\mathrm{d}x\right\|_{\dot{H}_{y_2}^{-1}(\mathbb{R})}^2\right]
$$

$$
= \int_{\mathbb{R}^2} V^L(x, x')\langle G_{x,x}^{L,N}, G_{x',x'}^{L,N}\rangle_{\dot{H}^{-1}(\mathbb{R})}\mathrm{d}x\mathrm{d}x'.
$$

So by Lemma 4.6 we get directly $(C_1^{L,N})^2 \lesssim L^{-1}$ for all large N, from where the convergence immediately follows. $\qquad\square$

Lemma 4.1 now follows by combining Lemma 4.4, Lemma 4.7, and Lemma 4.8.

4.2.2. *Proof of Lemma* 4.3. Recall that

$$
Q_t^L := \int_0^t \{-\langle(u_s^L)^2 - \|\rho^L\|_{L^2(\mathbb{R})}^2, \sigma\rangle_{L^2(\mathbb{R})} + \|\rho^L * \sigma\|_{L^2(\mathbb{R})}^2 + \lambda^{-1}u_s(\rho^L * \partial_z\sigma)\}\mathrm{d}s.
$$

By Corollary 3.17 the first term on the right-hand side converges in $L^p(\Omega; C^{3/4-}([0,T], \mathbb{R}))$ to $-\int_0^\cdot u_s^{\diamond 2} \mathrm{d}s(\sigma)$ whenever $p \geqslant 1$ and $T > 0$. The convergence of the remaining terms is obvious, and overall we get

$$\lim_{L \to \infty} Q_t^L = Q_t := -\int_0^t u_s^{\diamond 2} \mathrm{d}s(\sigma) + \|\sigma\|_{L^2(\mathbb{R})}^2 t + \lambda^{-1} \int_0^t u_s(\partial_z \sigma) \mathrm{d}s,$$

where the convergence takes place in $L^p(\Omega; C^{3/4-}([0,T], \mathbb{R}))$.

APPENDIX A. THE PERIODIC CASE

For the periodic equation described in section 2.3 most of the analysis works in the same way. The Itô trick and the Kipnis–Varadhan inequality are shown using exactly the same arguments, and also the Gaussian analysis of section 3.2 works completely analogously. We only have to replace all function spaces over $\mathbb{R}^n$ by the corresponding spaces over $\mathbb{T}^n$, say $L^2(\mathbb{R}^n)$ by $L^2(\mathbb{T}^n)$. The construction of the Burgers nonlinearity and the proof of its time-regularity also carry over to the periodic setting, although we have to replace the integrals over $\mathbb{R}^2$ in Fourier space by sums over $\mathbb{Z}^2$. But since those sums can be estimated by the corresponding integrals, we get the same bounds.

The first significant difference is in the construction of the integral. As discussed in section 4.1, any integral Iu of $u \in \mathscr{S}'(\mathbb{R})$ is determined uniquely by its derivative u and the value $Iu(\sigma)$ for some $\sigma \in \mathscr{S}(\mathbb{R})$ with $\int_\mathbb{R} \sigma(x)\mathrm{d}x \neq 0$. The same is true on the circle, and here there is a canonical candidate for the function σ, namely the constant function 1. So let $(u, \mathcal{A}) \in \mathcal{Q}(W, \eta)$ be a pair of controlled processes, where $\partial_t W$ is a periodic space-time white noise and η is a periodic space white noise. Let $\rho \in \mathscr{S}(\mathbb{R})$ be an even function with $\hat{\rho} \in C_c^\infty(\mathbb{R})$ and such that $\hat{\rho} \equiv 1$ on a neighborhood of 0, and define

(59) $$u_t^L := \mathscr{F}_\mathbb{T}^{-1}(\hat{\rho}(L^{-1}\cdot)\mathscr{F}_\mathbb{T} u_t) = \bar{\rho}^L * u_t, \qquad t \geqslant 0,$$

where $\mathscr{F}_\mathbb{T} u(k) := \int_\mathbb{T} e^{2\pi ikx} u(x)\mathrm{d}x$, respectively $\mathscr{F}_\mathbb{T}^{-1}\psi(x) := \sum_{k \in \mathbb{Z}} e^{2\pi ikx}\psi(k)$, denotes the Fourier transform (respectively inverse Fourier transform) on the torus, $u * v(x) := \int_\mathbb{T} u(x - y)v(y)\mathrm{d}y$ is the convolution on the torus, and $\bar{\rho}^L := \sum_{k \in \mathbb{Z}} L\rho(L(\cdot + k))$ is the periodization of $\rho^L := L\rho(L\cdot)$. For the last identity in (59) we applied Poisson summation; see for example [GP15, Lemma 6]. We then integrate u^L by setting

$$h_t^L := \mathscr{F}_\mathbb{T}^{-1}(\mathscr{F}_\mathbb{T}\Theta \mathscr{F}_\mathbb{T}(u_t^L)) = \Theta * u_t^L = (\Theta * \bar{\rho}^L) * u_t =: \Theta^L * u_t, \qquad t \geqslant 0,$$

where $\mathscr{F}_\mathbb{T}\Theta(k) = \mathbb{1}_{k \neq 0}(2\pi ik)^{-1}$, which corresponds to

$$\Theta(x) = \mathbb{1}_{[-\frac{1}{2},0)}(x)(-x - \frac{1}{2}) + \mathbb{1}_{[0,\frac{1}{2})}(x)(-x + \frac{1}{2}) = -x - \frac{1}{2} + \mathbb{1}_{[0,\frac{1}{2})}(x), \quad x \in [-\frac{1}{2}, \frac{1}{2}),$$

or equivalently $\Theta(x) = \frac{1}{2} - x$ for $x \in [0,1)$. From the representation as a Fourier multiplier, it is obvious that $\partial_x(\Theta * u) = \Pi_0 u$, and since we assumed that $u_t^L = \Pi_0 u_t^L$ for all $t \geqslant 0$, we get $\partial_x h^L = u^L$. Writing $(\Theta * \bar{\rho}^L) * u_t(x) = u_t(\Theta_x^L)$ for $\Theta_x^L(y) := (\Theta * \bar{\rho}^L)(x - y)$, we get from the fact that u is a strong stationary solution of the periodic Burgers equation that

$$\mathrm{d}h_t^L(x) = u_t(\Delta_z \Theta_x^L)\mathrm{d}t + \mathrm{d}\mathcal{A}_t(\Theta_x^L) + \sqrt{2}\mathrm{d}W_t(-\partial_z \Theta_x^L)$$
$$= \Delta_x h_t^L(x) + \mathrm{d}\mathcal{A}_t(\Theta_x^L) + \sqrt{2}\mathrm{d}W_t(\partial_x \Theta_x^L)$$

and $\mathrm{d}[h^L(x)]_t = 2\|\partial_x\Theta_x^L\|_{L^2(\mathbb{T})}^2\mathrm{d}t$. From the expression $\Theta(x) = \frac{1}{2}-x$ for $x \in [0,1)$, we see that $\partial_x\Theta_x = \delta - 1$, where δ denotes the Dirac delta, and therefore $\partial_x\Theta_x^L = \bar{\rho}_x^L - 1$ for $\bar{\rho}_x^L(y) = \bar{\rho}^L(x-y)$. So setting $\phi_t^L(x) := e^{\lambda h_t^L(x)}$, we have

$$\mathrm{d}\phi_t^L(x) = \phi_t^L(x)\left(\lambda\mathrm{d}h_t^L(x) + \frac{1}{2}\lambda^2\mathrm{d}[h^L(x)]_t\right)$$

$$= \lambda\phi_t^L(x)\left(\Delta_x h_t^L(x) + \mathrm{d}\mathcal{A}_t(\Theta_x^L) + \sqrt{2}\mathrm{d}W_t(\partial_x\Theta_x^L) + \lambda\|\bar{\rho}_x^L - 1\|_{L^2(\mathbb{T})}^2\mathrm{d}t\right),$$

and since $\lambda\phi_t^L(x)\Delta_x h_t^L(x) = \Delta_x\phi_t^L(x) - \lambda^2\phi_t^L(x)(\partial_x h_t^L(x))^2$, we get

$$\mathrm{d}\phi_t^L(x) = \Delta_x\phi_t^L(x)\mathrm{d}t + \sqrt{2}\lambda\phi_t^L(x)\mathrm{d}W_t(\partial_x\Theta_x^L)$$

$$+ \lambda^2\phi_t^L(x)(\lambda^{-1}\mathrm{d}\mathcal{A}_t(\Theta_x^L) - ((u_t^L(x))^2 - \|\bar{\rho}_x^L - 1\|_{L^2(\mathbb{T})}^2)\mathrm{d}t)$$

$$= \Delta_x\phi_t^L(x)\mathrm{d}t + \sqrt{2}\lambda\phi_t^L(x)\mathrm{d}W_t(\bar{\rho}_x^L) + \lambda^2\mathrm{d}R_t^L(x) + \lambda^2 K^L\phi_t^L(x)\mathrm{d}t$$

$$+ \lambda^2\phi_t^L(x)\mathrm{d}Q_t^L - \sqrt{2}\lambda\phi_t^L(x)\mathrm{d}W_t(1) - 2\lambda^2\phi_t^L(x)\mathrm{d}t,$$

where we expanded the $L^2(\mathbb{T})$-norm and defined

$$R_t^L(x) := \int_0^t \phi_s^L(x)\{\lambda^{-1}\mathrm{d}\mathcal{A}_s(\Theta_x^L) - \Pi_0((u_s^L(x))^2)\mathrm{d}s - K^L\mathrm{d}s\}$$

for the constant $K^L = \lim_{N\to\infty} K_x^{L,N}$ (which is independent of x) with $K_x^{L,N}$ defined in (61), and

$$Q_t^L := \int_0^t \{-\langle(u_s^L)^2 - \|\bar{\rho}^L\|_{L^2(\mathbb{R})}^2, 1\rangle_{L^2(\mathbb{T})} + 1\}\mathrm{d}s.$$

From here on the proof is completely analogous to the nonperiodic setting provided that we establish the following three lemmas.

Lemma A.1. *We have for all $T > 0$, $p > 2$, and all $\varphi \in C^\infty(\mathbb{T})$*

$$\lim_{L\to\infty}\left(\mathbb{E}[\|R^L(\varphi)\|_{p-\mathrm{var};[0,T]}^2] + \mathbb{E}[\sup_{t\leqslant T}|R_t^L(\varphi)|^2]\right) = 0.$$

Lemma A.2. *The constant K^L converges to $\lambda^2/12$ as $L \to \infty$.*

Lemma A.3. *For all $T > 0$ the process $(Q_t^L)_{t\in[0,T]}$ converges in probability in $C^{3/4-}([0,T],\mathbb{R})$ to the zero quadratic variation process*

$$Q_t := -\int_0^t u_s^{\diamond 2}\mathrm{d}s(1) + t, \qquad t \in [0,T].$$

To prove these lemmas, we follow the argumentation in section 4.2. Here the kernel $g_x^{L,N}$ takes the form
(60)
$$g_x^{L,N}(y_1,y_2) := \int_{\mathbb{T}}\mathrm{d}z(\bar{\rho}_x^L(z)-1)\bar{\rho}_z^N(y_1)\bar{\rho}_z^N(y_2) - (\bar{\rho}_x^L(y_1)\bar{\rho}_x^L(y_2) - \langle\bar{\rho}_{y_1}^L\bar{\rho}_{y_2}^L, 1\rangle_{L^2(\mathbb{T})}),$$

and as in section 4.2 we see that we should choose

$$(61)\qquad K_x^{L,N} := \lambda^2\int_{\mathbb{T}^2} g_x^{L,N}(y_1,y_2)\Theta_x^L(y_1)\Theta_x^L(y_2)\mathrm{d}y_1\mathrm{d}y_2.$$

The proof of Lemma A.2 is not a trivial modification of the one of Lemma 4.2, so we provide the required arguments.

Proof of Lemma A.2. Sending $N \to \infty$ and using that $\Theta_x^L(y) = \Theta^L(x - y)$ and $\bar{\rho}_x^L(y) = \bar{\rho}^L(x - y)$, we get

$$\lim_{N \to \infty} K_x^{L,N} = K^L$$

$$:= \lambda^2 \int_{\mathbb{T}} \mathrm{d}z (\bar{\rho}_x^L(z) - 1) \Theta_x^L(z) \Theta_x^L(z) - \lambda^2 (\langle \Theta^L, \bar{\rho}^L \rangle_{L^2(\mathbb{T})}^2 - \langle (\bar{\rho}^L * \Theta^L)_x^2, 1 \rangle_{L^2(\mathbb{T})}),$$

and the first term on the right-hand side is $\lambda^2 (-1/3) \int_{\mathbb{T}} \mathrm{d}z \partial_z (\Theta_x^L(z)^3) = 0$. Moreover, also

$$\langle \Theta^L, \bar{\rho}^L \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} \Theta^L(y) \bar{\rho}^L(y) \mathrm{d}y = \int_{\mathbb{T}} \Theta^L(y) (\bar{\rho}^L(y) - 1) \mathrm{d}y$$

$$= \int_{\mathbb{T}} \frac{1}{2} \partial_y (\Theta^L(y)^2) \mathrm{d}y = 0,$$

and therefore we remain with

$$\lambda^{-2} K^L = \langle (\bar{\rho}^L * \Theta^L)_x^2, 1 \rangle_{L^2(\mathbb{T})} = \|\bar{\rho}^L * \Theta^L\|_{L^2(\mathbb{T})}^2 = \sum_{k \in \mathbb{Z}} |\mathscr{F}_{\mathbb{T}}(\bar{\rho}^L * \Theta^L)(k)|^2$$

$$= \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|\rho(L^{-1}k)|^2}{(2\pi k)^2} = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{|\rho(L^{-1}k)|^2}{k^2}$$

$$\xrightarrow{L \to \infty} \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{12},$$

which concludes the proof. $\qquad \square$

The rest of the proof is completely analogous to the nonperiodic case. Let us just point out that if $f \in C^\infty(T)$, then

$$\int_{\mathbb{T}} (f(x - y) - f(x)) \bar{\rho}^L(y) \mathrm{d}y = \mathbb{E}[f(x - R^L) - f(x)],$$

where R^L is a random variable with density $\bar{\rho}^L$, and that

$$f(x - R_1^L) - f(x) = \int_{x \,(\mathrm{mod}\,1)}^{(x - R_1^L) \,(\mathrm{mod}\,1)} \partial_y f(y) \mathrm{d}y,$$

and therefore the same line of argumentation as in section 4.2 yields Lemmas A.1 and A.3. From here we follow the same steps as in the proof of Theorem 2.4 to establish Theorem 2.13.

APPENDIX B. EXPONENTIAL INTEGRABILITY

Lemma B.1. *Let* $\phi_t(x) = e^{\lambda u_t(\Theta_x)}$ *for* $t \geq 0$ *and* $x \in \mathbb{R}$, *and*

$$Z_t(x) = \phi_t(x) \exp \left(\sqrt{2} \lambda W_t(\sigma) - \frac{\lambda^4}{12} t + \lambda^2 \int_0^t u_s^{\diamond 2} \mathrm{d}s(\sigma) - \lambda \int_0^t u_s(\partial_z \sigma) \mathrm{d}s \right).$$

Then there exist $c, T > 0$ *such that*

$$\sup_{t \in [0,T], x \in \mathbb{R}} e^{-c|x|} \mathbb{E}[Z_t(x)^2] < \infty.$$

Moreover, if $R \geq 0$ and there exists $c' > 0$ with $\sup_{t\in[0,R],x\in\mathbb{R}} e^{-c'|x|}\mathbb{E}[|Z_t(x)|^{10}] < \infty$, then for some $c, T > 0$, independent of R,

$$\sup_{t\in[0,R+T],x\in\mathbb{R}} e^{-c|x|}\mathbb{E}[Z_t(x)^2] < \infty.$$

Proof. Note that

$$\mathbb{E}[|Z_0(x)|^{10}] = \mathbb{E}[e^{10\lambda u_0(\Theta_x)}] = e^{50\lambda^2\|\Theta_x\|_{L^2}^2} \lesssim e^{c|x|}$$

for some $c > 0$, so the second statement is a generalization of the first one. We have

$$Z_{R+t}(x) = Z_R(x)e^{\lambda(u_{R+t}(\Theta_x)-u_R(\Theta_x))}e^{\sqrt{2}\lambda(W_{R+t}(\sigma)-W_R(\sigma))}$$

$$\times e^{-\frac{\lambda^4}{12}t}e^{\lambda^2\int_R^{R+t}u_s^{\diamond 2}\mathrm{d}s(\sigma)}e^{-\lambda\int_R^{R+t}u_s(\partial_z\sigma)\mathrm{d}s},$$

and therefore with Hölder's and Jensen's inequalities

$$\mathbb{E}[Z_{R+t}(x)^2]^5$$
$$\leqslant \mathbb{E}[Z_R(x)^{10}]\mathbb{E}[e^{10\lambda(u_{R+t}(\Theta_x)-u_R(\Theta_x))}]$$
$$\times \mathbb{E}[e^{10\sqrt{2}\lambda W_t(\sigma)}]\mathbb{E}[e^{10\lambda^2\int_R^{R+t}u_s^{\diamond 2}\mathrm{d}s(\sigma)}]\mathbb{E}[e^{-10\lambda\int_R^{R+t}u_s(\partial_z\sigma)\mathrm{d}s}]$$
$$\lesssim e^{c|x|}\mathbb{E}[e^{20u_0(\Theta_x)}]e^{100\lambda^2\|\sigma\|_{L^2}^2}\mathbb{E}[e^{10\lambda^2\int_R^{R+t}u_s^{\diamond 2}\mathrm{d}s(\sigma)}]\frac{1}{t}\int_R^{R+t}\mathbb{E}[e^{-t10\lambda u_s(\partial_z\sigma)}]\mathrm{d}s$$
$$\lesssim e^{c|x|}\mathbb{E}[e^{10\lambda^2\int_R^{R+t}u_s^{\diamond 2}\mathrm{d}s(\sigma)}].$$

It remains to bound the expectation on the right-hand side. To simplify notation, we will write the following computations directly for $\int_R^{R+t} u_s^{\diamond 2}\mathrm{d}s(\sigma)$, but to make them rigorous we should mollify u_s before taking the square and derive bounds that are uniform in the mollification. Let H be the functional from the proof of Proposition 3.15 with $(1-L_0)H(u) = u^{\diamond 2}(\sigma)$. By [GJ13, Lemma 3] we have

$$\mathbb{E}[e^{10\lambda^2\int_R^{R+t}u_s^{\diamond 2}\mathrm{d}s(\sigma)}] \leqslant \mathbb{E}[e^{20\lambda^2\int_R^{R+t}H(u_s)\mathrm{d}s}]^{1/2}\mathbb{E}[e^{20\lambda^2\int_R^{R+t}(-L_0 H)(u_s)\mathrm{d}s}]^{1/2}$$

$$\lesssim \mathbb{E}[e^{20\lambda^2\int_R^{R+t}H(u_s)\mathrm{d}s}]^{1/2}\mathbb{E}[e^{800\lambda^4\int_R^{R+t}\mathcal{E}(H)(u_s)\mathrm{d}s}]^{1/2}$$

$$\leqslant \left(\frac{1}{t}\int_R^{R+t}\mathbb{E}[e^{20\lambda^2 tH(u_s)}]\mathrm{d}s\right)^{1/2}$$

$$\times \left(\frac{1}{t}\int_R^{R+t}\mathbb{E}[e^{800\lambda^4 t\mathcal{E}(H)(u_s)}]\mathrm{d}s\right)^{1/2}$$

$$= \mathbb{E}[e^{20\lambda^2 tH(u_0)}]^{1/2}\mathbb{E}[e^{800\lambda^4 t\mathcal{E}(H)(u_0)}]^{1/2},$$

where the third step follows from Jensen's inequality. Now $H(u_0)$ and $\mathcal{E}(H)(u_0)$ are random variables in the second inhomogeneous Gaussian chaos generated by u_0, and therefore they have small exponential moments. Indeed, if X is a random variable living in a second-order homogeneous Gaussian chaos, then $\mathbb{E}[|X|^n] \leqslant (n-1)^n\|X\|_{L^2}^n$ for all n (see [Nua06, p. 62]), and therefore by Stirling's formula

$$\mathbb{E}[\exp(\kappa|X|)] = \sum_{n=0}^{\infty}\frac{\kappa^n}{n!}\mathbb{E}[|X|^n] \leqslant \sum_{n=0}^{\infty}\frac{\kappa^n}{n!}n^n\|X\|_{L^2}^n$$

$$\simeq \sum_{n=0}^{\infty}\frac{\kappa^n e^n}{n^{n+\frac{1}{2}}}n^n\|X\|_{L^2}^n \leqslant \sum_{n=0}^{\infty}(\kappa e\|X\|_{L^2})^n,$$

which is finite as long as $\kappa < 1/(e\|X\|_{L^2})$. Since variables in the first- and zero-order Gaussian chaos have all exponential moments and we can make t small by choosing T sufficiently small, the proof is complete once we know that $\|H(u_0)\|_{L^2} + \|\mathcal{E}(H)(u_0)\|_{L^2} < \infty$. But this we already showed in the proof of Proposition 3.15. $\square$

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