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MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS

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Contents

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Introduction

The purpose of this paper is to provide a monoidal categorification of the quantum cluster algebra structure on the unipotent quantum coordinate ring $A_q(\mathfrak{n}(w))$, which is associated with a symmetric Kac–Moody algebra $\mathfrak{g}$ and a Weyl group element w.

The notion of cluster algebras was introduced by Fomin and Zelevinsky in [6] for studying total positivity and upper global bases. Since their introduction, a lot of connections and applications have been discovered in various fields of mathematics including representation theory, Teichmüller theory, tropical geometry, integrable systems, and Poisson geometry.

A cluster algebra is a $\mathbb{Z}$-subalgebra of a rational function field given by a set of generators, called the *cluster variables*. These generators are grouped into overlapping subsets, called the *clusters*, and the clusters are defined inductively by a procedure called *mutation* from the *initial cluster* $\{X_i\}_{1\le i\le r}$, which is controlled by an exchange matrix $\widetilde{B}$. We call a monomial of cluster variables in each cluster *a cluster monomial.*

Fomin and Zelevinsky proved that every cluster variable is a Laurent polynomial of the initial cluster $\{X_i\}_{1\le i\le r}$, and they conjectured that this Laurent polynomial has positive coefficients [6]. This *positivity conjecture* was proved by Lee and Schiffler in the *skew-symmetric* cluster algebra case in [30]. The *linearly independence conjecture* on cluster monomials was proved in the skew-symmetric cluster algebra case in [4].

The notion of quantum cluster algebras, introduced by Berenstein and Zelevinsky in [3], can be considered as a q-analogue of cluster algebras. The commutation relation among the cluster variables is determined by a skew-symmetric matrix L. As in the cluster algebra case, every cluster variable belongs to $\mathbb{Z}[q^{\pm 1/2}][X_i^{\pm 1}]_{1\le i\le r}$ [3] and is expected to be an element of $\mathbb{Z}_{\ge 0}[q^{\pm 1/2}][X_i^{\pm 1}]_{1\le i\le r}$, which is referred to as the *quantum positivity conjecture* (cf. [5, Conjecture 4.7]). In [24], Kimura and Qin proved the quantum positivity conjecture for quantum cluster algebras containing *acyclic* seed and specific coefficients.

The *unipotent quantum coordinate rings* $A_q(\mathfrak{n})$ and $A_q(\mathfrak{n}(w))$ are examples of quantum cluster algebras arising from Lie theory. The algebra $A_q(\mathfrak{n})$ is a q-deformation of the coordinate ring $\mathbb{C}[N]$ of the unipotent subgroup and is isomorphic to the negative half $U_q^-(\mathfrak{g})$ of the quantum group as $\mathbb{Q}(q)$-algebras. The algebra $A_q(\mathfrak{n}(w))$ is a $\mathbb{Q}(q)$-subalgebra of $A_q(\mathfrak{n})$ generated by a set of the *dual Poincaré–Birkhoff–Witt (PBW) basis elements* associated with a Weyl group element w. The unipotent quantum coordinate ring $A_q(\mathfrak{n})$ has a very interesting basis, the so-called *upper global basis* (dual canonical basis) $\mathbf{B}^{\mathrm{up}}$, which is dual to the lower global basis (canonical basis) [16, 31]. The upper global basis has been studied emphasizing its multiplicative structure. For example, Berenstein and Zelevinsky [2] conjectured that, in the case $\mathfrak{g}$ is of type A_n, the product b_1b_2 of two elements b_1 and b_2 in $\mathbf{B}^{\mathrm{up}}$ is again an element of $\mathbf{B}^{\mathrm{up}}$ up to a multiple of a power of q if and only if they are q-commuting; i.e., $b_1b_2 = q^m b_2 b_1$ for some $m \in \mathbb{Z}$. This conjecture turned out to be not true in general, because Leclerc [29] found examples of an *imaginary* element $b \in \mathbf{B}^{\mathrm{up}}$ such that b^2 does not belong to $\mathbf{B}^{\mathrm{up}}$. Nevertheless, the idea of considering subsets of $\mathbf{B}^{\mathrm{up}}$ whose elements are q-commuting with each other and studying the relations between those subsets has survived, and it became one of the motivations of the study of (quantum) cluster algebras.

In a series of papers [8, 9, 11], Geiß, Leclerc, and Schröer showed that the unipotent quantum coordinate ring $A_q(\mathfrak{n}(w))$ has a skew-symmetric quantum cluster algebra structure whose initial cluster consists of the so-called *unipotent quantum minors*. In [23], Kimura proved that $A_q(\mathfrak{n}(w))$ is *compatible* with the upper global basis $\mathbf{B}^{\mathrm{up}}$ of $A_q(\mathfrak{n})$; i.e., the set $\mathbf{B}^{\mathrm{up}}(w) := A_q(\mathfrak{n}(w)) \cap \mathbf{B}^{\mathrm{up}}$ is a basis of $A_q(\mathfrak{n}(w))$. Thus, with a result of [4], one can expect that every cluster monomial of $A_q(\mathfrak{n}(w))$ is contained in the upper global basis $\mathbf{B}^{\mathrm{up}}(w)$, which is named the *quantization conjecture* by Kimura [23].

Conjecture ([11, Conjecture 12.9], [23, Conjecture 1.1(2)]). *When $\mathfrak{g}$ is a symmetric Kac–Moody algebra, every quantum cluster monomial in $A_{q^{1/2}}(\mathfrak{n}(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(\mathfrak{n}(w))$ belongs to the upper global basis $\mathbf{B}^{\mathrm{up}}$ up to a power of $q^{1/2}$.*

It can be regarded as a reformulation of Berenstein–Zelevinsky's ideas on the multiplicative properties of $\mathbf{B}^{\mathrm{up}}$. There are some partial results of this conjecture. It is proved for $\mathfrak{g} = A_2$, A_3, A_4 and $A_q(\mathfrak{n}(w)) = A_q(\mathfrak{n})$ in [2] and [7, Section 12]. When $\mathfrak{g} = A_1^{(1)}$, A_n and w is a square of a Coxeter element, it is shown in [26] and [27] that the cluster variables belong to the upper global basis. When $\mathfrak{g}$ is symmetric and w is a square of a Coxeter element, the conjecture is proved in [24]. Notably, Qin provided recently a proof of the conjecture for a large class with a condition on the Weyl group element w [37]. Note that Nakajima proposed a geometric approach of this conjecture via quiver varieties [35].

In this paper, we prove the above conjecture completely by showing that there exists a *monoidal categorification of* $A_{q^{1/2}}(\mathfrak{n}(w))$.

In [12], Hernandez and Leclerc introduced the notion of *monoidal categorification* of cluster algebras. A simple object S of a monoidal category $\mathcal{C}$ is *real* if $S \otimes S$ is simple, and it is *prime* if there exists no nontrivial factorization $S \simeq S_1 \otimes S_2$. They say that $\mathcal{C}$ is a monoidal categorification of a cluster algebra A if the Grothendieck ring of $\mathcal{C}$ is isomorphic to A and if

(M1) the cluster monomials of A are the classes of real simple objects of $\mathcal{C}$,

(M2) the cluster variables of A are the classes of real simple prime objects of $\mathcal{C}$.

(Note that the above version is weaker than the original definition of the monoidal categorification in [12].) They proved that certain categories of modules over symmetric quantum affine algebras $U'_q(\mathfrak{g})$ give monoidal categorifications of some cluster algebras. Nakajima extended this result to the cases of the cluster algebras of types A, D, E [36] (see also [13]). It is worthwhile to remark that once a cluster algebra A has a monoidal categorification, the positivity of cluster variables of A and the linear independence of cluster monomials of A follow (see [12, Proposition 2.2]).

In this paper, we refine Hernandez–Leclerc's notion of monoidal categorifications including the quantum cluster algebra case. Let us briefly explain it. Let $\mathcal{C}$ be an abelian monoidal category equipped with an auto-equivalence q and a tensor product which is compatible with a decomposition $\mathcal{C} = \bigoplus_{\beta \in \mathsf{Q}} \mathcal{C}_\beta$. Fix a finite index set $J = J_{\mathrm{ex}} \sqcup J_{\mathrm{fr}}$ with a decomposition into the exchangeable part and the frozen part. Let $\mathscr{S}$ be a quadruple $(\{M_i\}_{i\in J}, L, \widetilde{B}, D)$ of a family of simple objects $\{M_i\}_{i\in J}$ in $\mathscr{C}$, an integer-valued skew-symmetric $J \times J$-matrix $L = (\lambda_{i,j})$, an integer-valued $J \times J_{\mathrm{ex}}$-matrix $\widetilde{B} = (b_{i,j})$ with a skew-symmetric principal part, and a family of elements $D = \{d_i\}_{i\in J}$ in Q. If this datum satisfies the conditions in Definition 6.2.1 below, then it is called a *quantum monoidal seed* in $\mathcal{C}$. For each $k \in J_{\mathrm{ex}}$, we have mutations $\mu_k(L)$, $\mu_k(\widetilde{B})$, and $\mu_k(D)$ of L, $\widetilde{B}$, and D, respectively. We say that a quantum monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, L, \widetilde{B}, D)$ *admits a mutation in direction* $k \in J_{\mathrm{ex}}$ if there exists a simple object $M'_k \in \mathcal{C}_{\mu_k(D)_k}$ which fits into two short exact sequences (0.2) below in $\mathcal{C}$ *reflecting* the mutation rule in quantum cluster algebras, and thus obtained quadruple $\mu_k(\mathscr{S}) := (\{M_i\}_{i\neq k} \cup \{M'_k\}, \mu_k(L), \mu_k(\widetilde{B}), \mu_k(D))$ is again a quantum monoidal seed in $\mathcal{C}$. We call $\mu_k(\mathscr{S})$ the mutation of $\mathscr{S}$ in direction $k \in J_{\mathrm{ex}}$.

Now the category $\mathcal{C}$ is called a *monoidal categorification of a quantum cluster algebra A over* $\mathbb{Z}[q^{\pm 1/2}]$ if

(0.1)

(i) the Grothendieck ring $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ is isomorphic to A,

(ii) there exists a quantum monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, L, \widetilde{B}, D)$ in $\mathcal{C}$ such that $[\mathscr{S}] := (\{q^{m_i}[M_i]\}_{i\in J}, L, \widetilde{B})$ is a quantum seed of A for some $m_i \in \frac{1}{2}\mathbb{Z}$,

(iii) $\mathscr{S}$ admits successive mutations in all directions in J_{ex}.

The existence of monoidal category $\mathcal{C}$ which provides a monoidal categorification of quantum cluster algebra A implies the following:

(QM1) Every quantum cluster monomial corresponds to the isomorphism class of a real simple object of $\mathcal{C}$. In particular, the set of quantum cluster monomials is $\mathbb{Z}[q^{\pm 1/2}]$-linearly independent.

(QM2) The quantum positivity conjecture holds for A.

In the case of unipotent quantum coordinate ring $A_q(\mathfrak{n})$, there is a natural candidate for monoidal categorification, the category of finite-dimensional graded modules over a *Khovanov–Lauda–Rouquier algebras* ([21,22], [38]). The Khovanov–Lauda–Rouquier algebras (abbreviated by KLR algebras), introduced by Khovanov–Lauda [21, 22] and Rouquier [38] independently, are a family of $\mathbb{Z}$-graded algebras which categorifies the negative half $U_q^-(\mathfrak{g})$ of a *symmetrizable* quantum group $U_q(\mathfrak{g})$. More precisely, there exists a family of algebras $\{R(-\beta)\}_{\beta \in \mathsf{Q}^-}$ such that the Grothendieck ring of R-gmod $:= \bigoplus_{\beta \in \mathsf{Q}^-} R(-\beta)$-gmod, the direct sum of the categories of finite-dimensional graded $R(-\beta)$-modules, is isomorphic to the integral form $A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm 1}]}$ of $A_q(\mathfrak{n}) \simeq U_q^-(\mathfrak{g})$. Here the tensor functor $\otimes$ of the monoidal category R-gmod is given by the convolution product $\circ$, and the action of q is given by the grading shift functor. In [39,40], Varagnolo–Vasserot and Rouquier proved that the upper global basis $\mathbf{B}^{\mathrm{up}}$ of $A_q(\mathfrak{n})$ corresponds to the set of the isomorphism classes of all *self-dual* simple modules of R-gmod under the assumption that R is associated with a *symmetric* quantum group $U_q(\mathfrak{g})$ and the base field is of characteristic 0.

Combining works of [11,23,40], the unipotent quantum coordinate ring $A_q(\mathfrak{n}(w))$ associated with a symmetric quantum group $U_q(\mathfrak{g})$ and a Weyl group element w is isomorphic to the Grothendieck group of a monoidal abelian full subcategory $\mathcal{C}_w$ of R-gmod whose base field $\mathbf{k}$ is of characteristic 0, satisfying the following properties: (i) $\mathcal{C}_w$ is stable under extensions and grading shift functor, (ii) the composition factors of $M \in \mathcal{C}_w$ are contained in $\mathbf{B}^{\mathrm{up}}(w)$ (see Definition 11.2.1). In particular, the first condition in (0.1) holds. However, it is not evident that the second and the third conditions in (0.1) on quantum monoidal seeds are satisfied. The purpose of this paper is to ensure that those conditions hold in $\mathcal{C}_w$.

In order to establish it, in the first part of the paper, we start with a continuation of the work of [15] about the convolution products, heads, and socles of graded modules over symmetric KLR algebras. One of the main results in [15] is that the convolution product $M \circ N$ of a real simple $R(\beta)$-module M and a simple $R(\gamma)$-module N has a unique simple quotient and a unique simple submodule. Moreover, if $M \circ N \simeq N \circ M$ up to a grading shift, then $M \circ N$ is simple. In such a case we say that M and N *commute*. The main tool of [15] was the R-matrix $\mathbf{r}_{M,N}$, constructed in [14], which is a homogeneous homomorphism from $M \circ N$ to $N \circ M$ of degree $\Lambda(M,N)$. In this work, we define some integers encoding necessary information on $M \circ N$,

$$\widetilde{\Lambda}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + (\beta,\gamma)\big), \quad \mathfrak{d}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + \Lambda(N,M)\big),$$

and study the representation theoretic meaning of the integers $\Lambda(M,N)$, $\widetilde{\Lambda}(M,N)$, and $\mathfrak{d}(M,N)$.

We then prove Leclerc's first conjecture [29] on the multiplicative structure of elements in $\mathbf{B}^{\mathrm{up}}$, when the generalized Cartan matrix is symmetric (Theorem 4.1.1 and Theorem 4.2.1). Theorem 4.2.1 is due to McNamara [34, Lemma 7.5], and the authors thank him for informing us of his result.

We say that $b \in \mathbf{B}^{\mathrm{up}}$ is *real* if $b^2 \in q^{\mathbb{Z}}\, \mathbf{B}^{\mathrm{up}} := \bigsqcup_{n \in \mathbb{Z}} q^n \mathbf{B}^{\mathrm{up}}$.

Theorem ([29, Conjecture 1]). *Let b_1 and b_2 be elements in $\mathbf{B}^{\mathrm{up}}$ such that one of them is real and $b_1 b_2 \notin q^{\mathbb{Z}}\mathbf{B}^{\mathrm{up}}$. Then the expansion of $b_1 b_2$ with respect to $\mathbf{B}^{\mathrm{up}}$ is of the form*

$$b_1 b_2 = q^m b' + q^s b'' + \sum_{c \neq b', b''} \gamma^c_{b_1, b_2}(q) c,$$

where $b' \neq b''$, $m, s \in \mathbb{Z}$, $m < s$, *and*

$$\gamma^c_{b_1,b_2}(q) \in q^{m+1}\mathbb{Z}[q] \cap q^{s-1}\mathbb{Z}[q^{-1}].$$

More precisely, we prove that $q^m b'$ and $q^s b''$ correspond to the simple head and the simple socle of $M \circ N$, respectively, when b_1 corresponds to a simple module M and b_2 corresponds to a simple module N.

Next, we move to provide an algebraic framework for monoidal categorification of quantum cluster algebras. In order to simplify the conditions of quantum monoidal seeds and their mutations, we introduce the notion of *admissible pairs* in $\mathcal{C}_w$. A pair $(\{M_i\}_{i\in J}, \widetilde{B})$ is called admissible in $\mathcal{C}_w$ if (i) $\{M_i\}_{i\in J}$ is a commuting family of self-dual real simple objects of $\mathcal{C}_w$, (ii) $\widetilde{B}$ is an integer-valued $J \times J_{\mathrm{ex}}$-matrix with a skew-symmetric principal part, and (iii) for each $k \in J$, there exists a self-dual simple object M'_k in $\mathcal{C}_w$ such that M'_k commutes with M_i for all $i \in J \setminus \{k\}$ and there are exact sequences in $\mathcal{C}_w$

$$\begin{aligned} &0 \to q \bigodot_{b_{i,k}>0} M_i^{\odot b_{i,k}} \to q^{\widetilde{\Lambda}(M_k,M'_k)} M_k \circ M'_k \to \bigodot_{b_{i,k}<0} M_i^{\odot(-b_{i,k})} \to 0, \\ &0 \to q \bigodot_{b_{i,k}<0} M_i^{\odot(-b_{i,k})} \to q^{\widetilde{\Lambda}(M'_k,M_k)} M'_k \circ M_k \to \bigodot_{b_{i,k}>0} M_i^{\odot b_{i,k}} \to 0, \end{aligned} \tag{0.2}$$

where $\widetilde{\Lambda}(M_k, M'_k)$ and $\widetilde{\Lambda}(M'_k, M_k)$ are prescribed integers and $\bigodot$ is a convolution product up to a power of q.

For an admissible pair $(\{M_i\}_{i\in J}, \widetilde{B})$, let $\Lambda = (\Lambda_{i,j})_{i,j\in J}$ be the skew-symmetric matrix where $\Lambda_{i,j}$ is the homogeneous degree of $\mathbf{r}_{M_i,M_j}$, the R-matrix between M_i and M_j, and let $D = \{d_i\}_{i\in J}$ be the family of elements in Q given by $M_i \in R(-d_i)$-gmod.

Then, together with the result of [11], our main theorem in the first part of the paper reads as follows.

Main Theorem 1 (Theorem 7.1.3 and Corollary 7.1.4). *If there exists an admissible pair* $(\{M_i\}_{i\in K}, \widetilde{B})$ *in* $\mathcal{C}_w$ *such that* $[\mathscr{S}] := \big(\{q^{-(\mathrm{wt}(M_i),\mathrm{wt}(M_i))/4}[M_i]\}_{i\in J}, -\Lambda, \widetilde{B}, D\big)$ *is an initial seed of* $A_{q^{1/2}}(\mathfrak{n}(w))$*, then* $\mathcal{C}_w$ *is a monoidal categorification of* $A_{q^{1/2}}(\mathfrak{n}(w))$*.*

The second part of this paper (Sections 8–11) is mainly devoted to showing that there exists an admissible pair in $\mathcal{C}_w$ for every symmetric Kac–Moody algebra $\mathfrak{g}$ and its Weyl group element w. In [11], Geiß, Leclerc, and Schröer provided an initial quantum seed in $A_q(\mathfrak{n}(w))$ whose quantum cluster variables are unipotent quantum minors. The unipotent quantum minors are elements in $A_q(\mathfrak{n})$, which are regarded as a q-analogue of a generalization of the minors of upper triangular matrices. In particular, they are elements in $\mathbf{B}^{\mathrm{up}}$. We define the *determinantial module* $\mathsf{M}(\mu, \zeta)$ to be the simple module in R-gmod corresponding to the unipotent quantum minor $\mathrm{D}(\mu, \zeta)$ under the isomorphism $A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm 1}]} \simeq K(R\text{-gmod})$. Here (μ, ζ) is a pair of elements in the weight lattice of $\mathfrak{g}$ satisfying certain conditions.

Our main theorem of the second part is as follows.

Main Theorem 2 (Theorem 11.2.2). *Let* $(\{D(k, 0)\}_{1\le k\le r}, \widetilde{B}, L)$ *be the initial quantum seed of* $A_q(\mathfrak{n}(w))$ *in* [11] *with respect to a reduced expression* $\widetilde{w} = s_{i_r} \cdots s_{i_1}$

of w*. Let* $\mathsf{M}(k,0) := \mathsf{M}(s_{i_1} \cdots s_{i_k}\varpi_{i_k}, \varpi_{i_k})$ *be the determinantial module corresponding to the unipotent quantum minor* $D(k,0)$*. Then the pair*

$$(\{\mathsf{M}(k,0)\}_{1\le k\le r}, \widetilde{B})$$

is admissible in $\mathcal{C}_w$*.*

Combining these theorems, the category $\mathcal{C}_w$ gives a monoidal categorification of the quantum cluster algebra $A_q(\mathfrak{n}(w))$. If we take the base field of the symmetric KLR algebra to be of characteristic 0, these theorems, along with Theorem 2.1.4 due to [39, 40], imply the quantization conjecture.

The most essential condition for an admissible pair is that there exists the *first mutation* $\mathsf{M}(k,0)'$ in the exact sequences (0.2) for each $k \in J_{\rm ex}$. To establish this, we investigate the properties of determinantial modules and those of their convolution products. Note that a unipotent quantum minor is the image of a global basis element of the *quantum coordinate ring* $A_q(\mathfrak{g})$ under a natural projection $A_q(\mathfrak{g}) \to A_q(\mathfrak{n})$. Since there exists a bicrystal embedding from the crystal basis $B(A_q(\mathfrak{g}))$ of $A_q(\mathfrak{g})$ to the crystal basis $B(\widetilde{U}_q(\mathfrak{g}))$ of the *modified quantum groups* $\widetilde{U}_q(\mathfrak{g})$, this investigation amounts to the study of the interplay among the crystal and global bases of $A_q(\mathfrak{g})$, $\widetilde{U}_q(\mathfrak{g})$, and $A_q(\mathfrak{n})$. Hence we start the second part of the paper with the studies of those algebras and their crystal/global bases along the line of the works in [17–19].

Next, we recall the (unipotent) quantum minors and the *T-system*, an equation consisting of three terms in products of unipotent quantum minors studied in [3, 11]. A detailed study of the relation between $A_q(\mathfrak{g})$, $\widetilde{U}_q(\mathfrak{g})$, and $A_q(\mathfrak{n})$ and their global bases enables us to establish several equations involving unipotent quantum minors in the algebra $A_q(\mathfrak{n})$. The upshot is that those equations can be translated into exact sequences in the category R-gmod involving convolution products of determinantial modules via the categorification of $U_q^-(\mathfrak{g})$. It enables us to show that the pair $(\{\mathsf{M}(k,0)\}_{1\le k\le r}, \widetilde{B})$ is admissible.

The paper is organized as follows. In Section 1, we briefly review basic materials on quantum group $U_q(\mathfrak{g})$ and KLR algebra R. In Section 2, we continue the study in [15] of the R-matrices between R-modules. In Section 3, we derive certain properties of $\widetilde{\Lambda}(M,N)$ and $\mathfrak{d}(M,N)$. In Section 4, we prove the first conjecture of Leclerc in [29]. In Section 5, we recall the definition of quantum cluster algebras. In Section 6, we give the definitions of a monoidal seed, a quantum monoidal seed, a monoidal categorification of a cluster algebra, and a monoidal categorification of a quantum cluster algebra. In Section 7, we prove Main Theorem 1. In Section 8, we review the algebras $A_q(\mathfrak{g})$, $\widetilde{U}_q(\mathfrak{g})$, and $A_q(\mathfrak{n})$, and study the relations among them. In Section 9, we study the properties of quantum minors including T-systems and generalized T-systems. In Section 10, we study the determinantial modules over KLR algebras. Finally, in Section 11, we establish Main Theorem 2.

1. Quantum groups and global bases

In this section, we briefly recall the quantum groups and the crystal and global bases theory for $U_q(\mathfrak{g})$. We refer to [16, 17, 20] for materials in this subsection.

1.1. **Quantum groups.** Let I be an index set. A *Cartan datum* is a quintuple $(A, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ consisting of

- (i) an integer-valued matrix $A = (a_{ij})_{i,j\in I}$, called the *symmetrizable generalized Cartan matrix*, which satisfies
  - (a) $a_{ii} = 2$ $(i \in I)$,
  - (b) $a_{ij} \le 0$ $(i \neq j)$,
  - (c) there exists a diagonal matrix $D = \mathrm{diag}(\mathsf{s}_i \mid i \in I)$ such that DA is symmetric, and s_i are relatively prime positive integers,
- (ii) a free abelian group P, called the *weight lattice*,
- (iii) $\Pi = \{\alpha_i \in \mathsf{P} \mid \ i \in I\}$, called the set of *simple roots*,
- (iv) $\mathsf{P}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathsf{P}, \mathbb{Z})$, called the *co-weight lattice*,
- (v) $\Pi^\vee = \{h_i \mid i \in I\} \subset \mathsf{P}^\vee$, called the set of *simple coroots*, satisfying the following properties:
  - (1) $\langle h_i, \alpha_j\rangle = a_{ij}$ for all $i, j \in I$,
  - (2) Π is linearly independent over $\mathbb{Q}$,
  - (3) for each $i \in I$, there exists $\varpi_i \in \mathsf{P}$ such that $\langle h_j, \varpi_i\rangle = \delta_{ij}$ for all $j \in I$.

  We call ϖ_i the *fundamental weights.*

The free abelian group $\mathsf{Q} := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is called the *root lattice*. Set $\mathsf{Q}^+ = \sum_{i\in I} \mathbb{Z}_{\ge 0}$ $\alpha_i \subset \mathsf{Q}$ and $\mathsf{Q}^- = \sum_{i\in I} \mathbb{Z}_{\le 0}\alpha_i \subset \mathsf{Q}$. For $\beta = \sum_{i\in I} m_i\alpha_i \in \mathsf{Q}$, we set $|\beta| = \sum_{i\in I} |m_i|$.

Set $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} \mathsf{P}^\vee$. Then there exists a symmetric bilinear form $(\ \ ,\ \ )$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \alpha_j) = \mathsf{s}_i a_{ij} \quad (i, j \in I) \quad \text{and } \langle h_i, \lambda\rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \text{ for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

The *Weyl group* of $\mathfrak{g}$ is the group of linear transformations on $\mathfrak{h}^*$ generated by s_i $(i \in I)$, where

$$s_i(\lambda) := \lambda - \langle h_i, \lambda\rangle\alpha_i \quad \text{for} \quad \lambda \in \mathfrak{h}^*,\ i \in I.$$

Let q be an indeterminate. For each $i \in I$, set $q_i = q^{\mathsf{s}_i}$.

Definition 1.1.1. *The* quantum group *associated with a Cartan datum* $(A, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ *is the algebra* $U_q(\mathfrak{g})$ *over* $\mathbb{Q}(q)$ *generated by* e_i, f_i $(i \in I)$ *and* q^h $(h \in \mathsf{P}^\vee)$ *satisfying the following relations:*

$$\begin{aligned}
&q^0 = 1,\ q^h q^{h'} = q^{h+h'} \quad \textit{for } h, h' \in \mathsf{P}^\vee,\\
&q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i \quad \textit{for } h \in \mathsf{P}^\vee, i \in I,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \quad \textit{where } t_i = q^{\mathsf{s}_i h_i},\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i e_i^{1-a_{ij}-r} e_j e_i^r = 0 \quad \textit{if } i \neq j,\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_i f_i^{1-a_{ij}-r} f_j f_i^r = 0 \quad \textit{if } i \neq j.
\end{aligned}$$

Here, we set $[n]_i = \dfrac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, $[n]_i! = \prod_{k=1}^n [k]_i$, and $\begin{bmatrix} m \\ n \end{bmatrix}_i = \dfrac{[m]_i!}{[m-n]_i![n]_i!}$ for $i \in I$ and $m, n \in \mathbb{Z}_{\ge 0}$ such that $m \ge n$.

Let $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i's (resp. f_i's), and let $U_q^0(\mathfrak{g})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h ($h \in \mathsf{P}^\vee$). Then we have the *triangular decomposition*

$$U_q(\mathfrak{g}) \simeq U_q^-(\mathfrak{g}) \otimes U_q^0(\mathfrak{g}) \otimes U_q^+(\mathfrak{g}),$$

and the *weight space decomposition*

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathsf{Q}} U_q(\mathfrak{g})_\beta,$$

where $U_q(\mathfrak{g})_\beta := \big\{x \in U_q(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h,\beta\rangle} x \text{ for any } h \in \mathsf{P}\big\}$.

There are $\mathbb{Q}(q)$-algebra antiautomorphisms φ and * of $U_q(\mathfrak{g})$ given as follows:

$$\varphi(e_i) = f_i, \quad \varphi(f_i) = e_i, \quad \varphi(q^h) = q^h,$$
$$e_i^* = e_i, \qquad f_i^* = f_i, \qquad (q^h)^* = q^{-h}.$$

There is also a $\mathbb{Q}$-algebra automorphism $^-$ of $U_q(\mathfrak{g})$ given by

$$\overline{e}_i = e_i, \quad \overline{f}_i = f_i, \quad \overline{q^h} = q^{-h}, \quad \overline{q} = q^{-1}.$$

We define the divided powers by

$$e_i^{(n)} = e_i^n/[n]_i!, \quad f_i^{(n)} = f_i^n/[n]_i! \ \ (n \in \mathbb{Z}_{\ge 0}).$$

Let us denote by $U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$, q^h, and $\displaystyle\prod_{k=1}^{n} \frac{\{q^{1-k}q^h\}}{[k]}$ ($i \in I,\ n \in \mathbb{Z}_{\ge 0},\ h \in \mathsf{P}^\vee$), where $\{x\} := (x - x^{-1})/(q - q^{-1})$. Let us also denote by $U_q^-(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}$ the $\mathbb{Z}[q^{\pm1}]$-subalgebra of $U_q^-(\mathfrak{g})$ generated by $f_i^{(n)}$ ($i \in I$, $n \in \mathbb{Z}_{\ge 0}$), and by $U_q^+(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}$ the $\mathbb{Z}[q^{\pm1}]$-subalgebra of $U_q^+(\mathfrak{g})$ generated by $e_i^{(n)}$ ($i \in I$, $n \in \mathbb{Z}_{\ge 0}$).

1.2. **Integrable representations.** A $U_q(\mathfrak{g})$-module M is called *integrable* if $M = \bigoplus_{\eta \in \mathsf{P}} M_\eta$ where $M_\eta := \{m \in M \mid q^h m = q^{\langle h,\eta\rangle} m\}$, $\dim M_\eta < \infty$, and the actions of e_i and f_i on M are locally nilpotent for all $i \in I$. We denote by $\mathscr{O}_{\rm int}(\mathfrak{g})$ the category of integrable left $U_q(\mathfrak{g})$-modules M satisfying that there exist finitely many weights $\lambda_1, \ldots, \lambda_m$ such that $\mathrm{wt}(M) \subset \cup_j(\lambda_j + \mathsf{Q}^-)$. The category $\mathscr{O}_{\rm int}(\mathfrak{g})$ is semisimple with its simple objects being isomorphic to the highest weight modules $V(\lambda)$ with highest weight vector u_λ of highest weight $\lambda \in \mathsf{P}^+ := \{\mu \in \mathsf{P} \mid \langle h_i, \mu\rangle \ge 0 \text{ for all } i \in I\}$, the set of dominant integral weights.

For $M \in \mathscr{O}_{\rm int}(\mathfrak{g})$, let us denote by $\mathbf{D}_\varphi M$ the left $U_q(\mathfrak{g})$-module $\bigoplus_{\eta \in \mathsf{P}} \mathrm{Hom}_{\mathbb{Q}(q)}(M_\eta, \mathbb{Q}(q))$ with the action of $U_q(\mathfrak{g})$ given by

$$(a\psi)(m) = \psi(\varphi(a)m) \quad \text{for } \psi \in \mathbf{D}_\varphi M,\ m \in M, \text{ and } a \in U_q(\mathfrak{g}).$$

Then $\mathbf{D}_\varphi M$ belongs to $\mathscr{O}_{\rm int}(\mathfrak{g})$.

For a left $U_q(\mathfrak{g})$-module M, we denote by $M^{\rm r}$ the right $U_q(\mathfrak{g})$-module $\{m^{\rm r} \mid m \in M\}$ with the right action of $U_q(\mathfrak{g})$ given by

$$(m^{\rm r})\, x = (\varphi(x)m)^{\rm r} \text{ for } m \in M \text{ and } x \in U_q(\mathfrak{g}).$$

We denote by $\mathscr{O}^{\rm r}_{\rm int}(\mathfrak{g})$ the category of right integrable $U_q(\mathfrak{g})$-modules $M^{\rm r}$ such that $M \in \mathscr{O}_{\rm int}(\mathfrak{g})$.

There are two comultiplications Δ_+ and Δ_- on $U_q(\mathfrak{g})$ defined as follows:

(1.1)
$$\Delta_+(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta_+(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta_+(q^h) = q^h \otimes q^h,$$

(1.2)
$$\Delta_-(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta_-(f_i) = f_i \otimes 1 + t_i \otimes f_i, \quad \Delta_-(q^h) = q^h \otimes q^h.$$

For two $U_q(\mathfrak{g})$-modules M_1 and M_2, let us denote by $M_1 \otimes_+ M_2$ and $M_1 \otimes_- M_2$ the vector space $M_1 \otimes_{\mathbb{Q}(q)} M_2$ endowed with $U_q(\mathfrak{g})$-module structure induced by the comultiplications Δ_+ and Δ_-, respectively. Then we have

$$\mathbf{D}_\varphi(M_1 \otimes_\pm M_2) \simeq (\mathbf{D}_\varphi M_1) \otimes_\mp (\mathbf{D}_\varphi M_2).$$

For any $i \in I$, there exists a unique $\mathbb{Q}(q)$-linear endomorphism e'_i of $U_q^-(\mathfrak{g})$ such that

$$e'_i(f_j) = \delta_{i,j} \ (j \in I), \quad e'_i(xy) = (e'_i x)y + q_i^{\langle h_i, \beta\rangle} x(e'_i y) \ (x \in U_q^-(\mathfrak{g})_\beta, y \in U_q^-(\mathfrak{g})).$$

The *quantum boson algebra* $B_q(\mathfrak{g})$ is defined as the subalgebra of $\mathrm{End}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}))$ generated by f_i and e'_i ($i \in I$). Then $B_q(\mathfrak{g})$ has a $\mathbb{Q}(q)$-algebra anti-automorphism φ which sends e'_i to f_i and f_i to e'_i. As a $B_q(\mathfrak{g})$-module, $U_q^-(\mathfrak{g})$ is simple.

The simple $U_q(\mathfrak{g})$-module $V(\lambda)$ and the simple $B_q(\mathfrak{g})$-module $U_q^-(\mathfrak{g})$ have a unique non-degenerate symmetric bilinear form $(\ ,\ )$ such that

$$(u_\lambda, u_\lambda) = 1 \text{ and } (xu, v) = (u, \varphi(x)v) \text{ for } u, v \in V(\lambda) \text{ and } x \in U_q(\mathfrak{g}),$$
$$(\mathbf{1}, \mathbf{1}) = 1 \text{ and } (xu, v) = (u, \varphi(x)v) \text{ for } u, v \in U_q^-(\mathfrak{g}) \text{ and } x \in B_q(\mathfrak{g}).$$

Note that $(\ ,\ )$ induces the non-degenerate bilinear form

$$\langle \cdot, \cdot \rangle \colon V(\lambda)^{\mathrm{r}} \times V(\lambda) \to \mathbb{Q}(q)$$

given by $\langle u^{\mathrm{r}}, v\rangle = (u, v)$, by which $\mathbf{D}_\varphi V(\lambda)$ is canonically isomorphic to $V(\lambda)$.

1.3. **Crystal bases and global bases.** For a subring A of $\mathbb{Q}(q)$, we say that L is an *A-lattice* of a $\mathbb{Q}(q)$-vector space V if L is a free A-submodule of V such that $V = \mathbb{Q}(q) \otimes_A L$.

Let us denote by $\mathbf{A}_0$ (resp. $\mathbf{A}_\infty$) the ring of rational functions in $\mathbb{Q}(q)$ which are regular at $q = 0$ (resp. $q = \infty$). Set $\mathbf{A} := \mathbb{Q}[q^{\pm 1}]$.

Let M be a $U_q(\mathfrak{g})$-module in $\mathscr{O}_{\mathrm{int}}(\mathfrak{g})$. Then, for each $i \in I$, any $u \in M$ can be uniquely written as

$$u = \sum_{n \geq 0} f_i^{(n)} u_n \quad \text{with } e_i u_n = 0.$$

We define the *lower Kashiwara operators* by

$$\tilde{e}_i^{\mathrm{low}}(u) = \sum_{n \geq 1} f_i^{(n-1)} u_n \quad \text{and} \quad \tilde{f}_i^{\mathrm{low}}(u) = \sum_{n \geq 0} f_i^{(n+1)} u_n,$$

and the *upper Kashiwara operators* by

$$\tilde{e}_i^{\mathrm{up}}(u) = \tilde{e}_i^{\mathrm{low}} q_i^{-1} t_i^{-1} u \quad \text{and} \quad \tilde{f}_i^{\mathrm{up}}(u) = \tilde{f}_i^{\mathrm{low}} q_i^{-1} t_i u.$$

Similarly, for each $i \in I$, any element $x \in U_q^-(\mathfrak{g})$ can be written uniquely as

$$x = \sum_{n \geq 0} f_i^{(n)} x_n \quad \text{with } e'_i x_n = 0.$$

We define the *Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ on $U_q^-(\mathfrak{g})$ by

$$\tilde{e}_i x = \sum_{n\geq 1} f_i^{(n-1)} x_n, \quad \tilde{f}_i x = \sum_{n \geq 0} f_i^{(n+1)} x_n.$$

We say that an $\mathbf{A}_0$-lattice L of M is a lower (resp. upper) crystal lattice of M if $L = \bigoplus_{\eta \in \mathsf{P}} L_\eta$, where $L_\eta = L \cap M_\eta$ and it is invariant by the lower (resp. upper) Kashiwara operators.

Lemma 1.3.1. *Let L be a lower crystal lattice of $M \in \mathscr{O}_{\mathrm{int}}(\mathfrak{g})$. Then we have*

(i) $\bigoplus_{\lambda \in \mathsf{P}} q^{-(\lambda,\lambda)/2} L_\lambda$ *is an upper crystal lattice of M.*
(ii) $L^\vee := \{\psi \in \mathbf{D}_\varphi M \mid \langle \psi, L\rangle \in \mathbf{A}_0\}$ *is an upper crystal lattice of* $\mathbf{D}_\varphi M$.

Proof. (i) Let ϕ_M be the endomorphism of M given by $\phi_M|_{M_\lambda} = q^{-(\lambda,\lambda)/2}\,\mathrm{id}_{M_\lambda}$. Then we have $\tilde{e}_i^{\mathrm{up}} = \phi_M \circ \tilde{e}_i^{\mathrm{low}} \circ \phi_M^{-1}$ and $\tilde{f}_i^{\mathrm{up}} = \phi_M \circ \tilde{f}_i^{\mathrm{low}} \circ \phi_M^{-1}$.

Item (ii) follows from (3.2.1), (3.2.2) in [17]. Note that the definition of upper Kashiwara operators are slightly different from the ones in [17], but similar properties hold. □

Definition 1.3.2. *A* lower (*resp.* upper) crystal basis *of M consists of a pair (L, B) satisfying the following conditions:*

(i) *L is a lower (resp. upper) crystal lattice of M,*
(ii) *$B = \sqcup_{\eta\in\mathsf{P}} B_\eta$ is a basis of the $\mathbb{Q}$-vector space L/qL, where $B_\eta = B \cap (L_\eta/qL_\eta)$,*
(iii) *the induced maps $\tilde{e}_i$ and $\tilde{f}_i$ on L/qL satisfy*

$$\tilde{e}_i B, \tilde{f}_i B \subset B \sqcup \{0\}, \text{ and } \tilde{f}_i b = b' \text{ if and only if } b = \tilde{e}_i b' \text{ for } b, b' \in B.$$

Here $\tilde{e}_i$ and $\tilde{f}_i$ denote the lower (resp. upper) Kashiwara operators.

For $\lambda \in \mathsf{P}^+$, let u_λ be the highest weight vector of $V(\lambda)$. Let $L^{\mathrm{low}}(\lambda)$ be the $\mathbf{A}_0$-submodule of $V(\lambda)$ generated by $\left\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_l} u_\lambda \mid l \in \mathbb{Z}_{\geq 0},\ i_1,\ldots,i_l \in I\right\}$, and let $B(\lambda)$ be the subset of $L^{\mathrm{low}}(\lambda)/qL^{\mathrm{low}}(\lambda)$ given by

$$B^{\mathrm{low}}(\lambda) = \left\{\tilde{f}_{i_1}\cdots\tilde{f}_{i_l} u_\lambda \mod qL(\lambda) \mid l \in \mathbb{Z}_{\geq 0},\ i_1,\ldots,i_l \in I\right\} \backslash \{0\}.$$

It is shown in [16] that $(L^{\mathrm{low}}(\lambda), B^{\mathrm{low}}(\lambda))$ is a lower crystal basis of $V(\lambda)$. Using the non-degenerate symmetric bilinear form $(\ ,\ )$, $V(\lambda)$ has the upper crystal basis $(L^{\mathrm{up}}(\lambda), B^{\mathrm{up}}(\lambda))$ where

$$L^{\mathrm{up}}(\lambda) := \{u \in V(\lambda) \mid (u, L^{\mathrm{low}}(\lambda)) \subset \mathbf{A}_0\},$$

and $B^{\mathrm{up}}(\lambda) \subset L^{\mathrm{up}}(\lambda)/qL^{\mathrm{up}}(\lambda)$ is the dual basis of $B^{\mathrm{low}}(\lambda)$ with respect to the induced non-degenerate pairing between $L^{\mathrm{up}}(\lambda)/qL^{\mathrm{up}}(\lambda)$ and $L^{\mathrm{low}}(\lambda)/qL^{\mathrm{low}}(\lambda)$.

An (abstract) *crystal* is a set B together with maps

$$\mathrm{wt}\colon B \to \mathsf{P},\ \ \varepsilon_i, \varphi_i\colon B \to \mathbb{Z} \sqcup \{\infty\} \text{ and } \tilde{e}_i, \tilde{f}_i\colon B \to B \sqcup \{0\} \text{ for } i \in I,$$

such that

(C1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for any i,
(C2) if $b \in B$ satisfies $\tilde{e}_i(b) \neq 0$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1,\ \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1,\ \mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i,$$

(C3) if $b \in B$ satisfies $\tilde{f}_i(b) \neq 0$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1,\ \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1,\ \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i,$$

(C4) for $b, b' \in B$, $b' = \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$,

(C5) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

Recall that, with the notions of *morphism* and *tensor product rule* of crystals, the category of crystals becomes a monoidal category [19]. If (L, B) is a crystal basis of M, then B is an abstract crystal. Since $B^{\mathrm{low}}(\lambda) \simeq B^{\mathrm{up}}(\lambda)$, we drop the superscripts for simplicity.

Let V be a $\mathbb{Q}(q)$-vector space, and let L_0 be an $\mathbf{A}_0$-lattice of V, L_∞ an $\mathbf{A}_\infty$-lattice of V, and $V_\mathbf{A}$ an $\mathbf{A}$-lattice of V. We say that the triple $(V_\mathbf{A}, L_0, L_\infty)$ is *balanced* if the following canonical map is a $\mathbb{Q}$-linear isomorphism:

$$E := V_\mathbf{A} \cap L_0 \cap L_\infty \xrightarrow{\sim} L_0/qL_0.$$

The inverse of the above isomorphism $G\colon L_0/qL_0 \xrightarrow{\sim} E$ is called the *globalizing map.* If $(V_\mathbf{A}, L_0, L_\infty)$ is balanced, then we have

$$\mathbb{Q}(q) \underset{\mathbb{Q}}{\otimes} E \xrightarrow{\sim} V,\ \mathbf{A} \underset{\mathbb{Q}}{\otimes} E \xrightarrow{\sim} V_\mathbf{A},\ \mathbf{A}_0 \underset{\mathbb{Q}}{\otimes} E \xrightarrow{\sim} L_0, \text{ and } \mathbf{A}_\infty \underset{\mathbb{Q}}{\otimes} E \xrightarrow{\sim} L_\infty.$$

Hence, if B is a basis of L_0/qL_0, then $G(B)$ is a basis of V, $V_\mathbf{A}$, L_0, and L_∞. We call $G(B)$ a *global basis.*

We define the two $\mathbf{A}$-lattices of $V(\lambda)$ by

$$V^{\mathrm{low}}(\lambda)_\mathbf{A} := \big(\mathbb{Q} \otimes U_q^-(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}\big) u_\lambda \quad \text{and}$$
$$V^{\mathrm{up}}(\lambda)_\mathbf{A} := \big\{u \in V(\lambda) \mid \big(u, V^{\mathrm{low}}(\lambda)_\mathbf{A}\big) \subset \mathbf{A}\big\}.$$

Recall that there is a $\mathbb{Q}$-linear automorphism—on $V(\lambda)$ defined by

$$\overline{Pu_\lambda} = \overline{P}u_\lambda, \quad \text{for } \ P \in U_q(\mathfrak{g}).$$

Then $\big(V^{\mathrm{low}}(\lambda)_\mathbf{A},\ L^{\mathrm{low}}(\lambda),\ \overline{L^{\mathrm{low}}(\lambda)}\big)$ and $\big(V^{\mathrm{up}}(\lambda)_\mathbf{A},\ L^{\mathrm{up}}(\lambda),\ \overline{L^{\mathrm{up}}(\lambda)}\big)$ are balanced. Let us denote by G^{low}_λ and G^{up}_λ the associated globalizing maps, respectively. (If there is no danger of confusion, we simply denote them G^{low} and G^{up}, respectively.) Then the sets

$$\mathbf{B}^{\mathrm{low}}(\lambda) := \{G^{\mathrm{low}}_\lambda(b) \mid b \in B^{\mathrm{low}}(\lambda)\} \quad \text{and} \quad \mathbf{B}^{\mathrm{up}}(\lambda) := \{G^{\mathrm{up}}_\lambda(b) \mid b \in B^{\mathrm{up}}(\lambda)\}$$

form $\mathbb{Z}[q^{\pm 1}]$-bases of

$$V^{\mathrm{low}}(\lambda)_{\mathbb{Z}[q^{\pm 1}]} := U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} u_\lambda \quad \text{and}$$
$$V^{\mathrm{up}}(\lambda)_{\mathbb{Z}[q^{\pm 1}]} := \big\{u \in V(\lambda) \mid \big(u, V^{\mathrm{low}}(\lambda)_{\mathbb{Z}[q^{\pm 1}]}\big) \subset \mathbb{Z}[q^{\pm 1}]\big\},$$

respectively. They are called the *lower global basis* and the *upper global basis* of $V(\lambda)$.

Set

$$L(\infty) := \sum_{l \in \mathbb{Z}_{\geq 0},\, i_1, \ldots, i_l \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot \mathbf{1} \subset U_q^-(\mathfrak{g}) \quad \text{and}$$

$$B(\infty) := \Big\{\tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \cdot \mathbf{1} \mod qL(\infty) \mid l \in \mathbb{Z}_{\geq 0}, i_1, \ldots, i_l \in I\Big\} \subset L(\infty)/qL(\infty).$$

Then $(L(\infty), B(\infty))$ is a lower crystal basis of the simple $B_q(\mathfrak{g})$-module $U_q^-(\mathfrak{g})$ and the triple $(\mathbb{Q} \otimes U_q^-(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}, L(\infty), \overline{L(\infty)})$ is balanced. Let us denote the globalizing map by G^{low}. Then the set

$$\mathbf{B}^{\mathrm{low}}(U_q^-(\mathfrak{g})) := \{G^{\mathrm{low}}(b) \mid b \in B(\infty)\}$$

forms a $\mathbb{Z}[q^{\pm 1}]$-basis of $U_q^-(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}$ and is called the *lower global basis* of $U_q^-(\mathfrak{g})$.

Let us denote by

$$\mathbf{B}^{\mathrm{up}}(U_q^-(\mathfrak{g})) := \{G^{\mathrm{up}}(b) \mid b \in B(\infty)\} \tag{1.3}$$

the dual basis of $\mathbf{B}^{\mathrm{low}}(U_q^-(\mathfrak{g}))$ with respect to $(\ ,\ )$. Then it is a $\mathbb{Z}[q^{\pm 1}]$-basis of

$$U_q^-(\mathfrak{g})^\vee_{\mathbb{Z}[q^{\pm 1}]} := \{x \in U_q^-(\mathfrak{g}) \mid (x, U_q^-(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}) \subset \mathbb{Z}[q^{\pm 1}]\}$$

and called the *upper global basis* of $U_q^-(\mathfrak{g})$. Note that $U_q^-(\mathfrak{g})^\vee_{\mathbb{Z}[q^{\pm 1}]}$ has a $\mathbb{Z}[q^{\pm 1}]$-algebra structure as a subalgebra of $U_q^-(\mathfrak{g})$ (see also Section 8.2).

2. KLR ALGEBRAS AND R-MATRICES

2.1. **KLR algebras.** We recall the definition of Khovanov–Lauda–Rouquier algebra or quiver Hecke algebra (hereafter, we abbreviate it as KLR algebra) associated with a given Cartan datum $(A, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$.

Let $\mathbf{k}$ be a base field. For $i, j \in I$ such that $i \neq j$, set

$$S_{i,j} = \left\{(p,q) \in \mathbb{Z}^2_{\geq 0} \mid (\alpha_i, \alpha_i)p + (\alpha_j, \alpha_j)q = -2(\alpha_i, \alpha_j)\right\}.$$

Let us take a family of polynomials $(Q_{ij})_{i,j \in I}$ in $\mathbf{k}[u,v]$ which are of the form

$$Q_{ij}(u,v) = \begin{cases} 0 & \text{if } i = j, \\ \sum\limits_{(p,q) \in S_{i,j}} t_{i,j;p,q} u^p v^q & \text{if } i \neq j \end{cases} \tag{2.1}$$

with $t_{i,j;p,q} \in \mathbf{k}$ such that $Q_{i,j}(u,v) = Q_{j,i}(v,u)$ and $t_{i,j:-a_{ij},0} \in \mathbf{k}^\times$.

We denote by $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ the symmetric group on n letters, where $s_i := (i, i+1)$ is the transposition of i and $i+1$. Then $\mathfrak{S}_n$ acts on I^n by place permutations.

For $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathsf{Q}^+$ such that $|\beta| = n$, we set

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Definition 2.1.1. *For $\beta \in \mathsf{Q}^+$ with $|\beta| = n$, the* KLR algebra $R(\beta)$ *at β associated with a Cartan datum $(A, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$ and a matrix $(Q_{ij})_{i,j \in I}$ is the algebra over $\mathbf{k}$ generated by the elements $\{e(\nu)\}_{\nu \in I^\beta}$, $\{x_k\}_{1 \leq k \leq n}$, $\{\tau_m\}_{1 \leq m \leq n-1}$ satisfying the* *following defining relations:*

$$
\begin{aligned}
&e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1,\\
&x_k x_m = x_m x_k, \quad x_k e(\nu) = e(\nu)x_k,\\
&\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k\tau_m = \tau_m\tau_k \quad \text{if } |k-m|>1,\\
&\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1})e(\nu),\\
&(\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } m=k, \nu_k=\nu_{k+1},\\ e(\nu) & \text{if } m=k+1, \nu_k=\nu_{k+1},\\ 0 & \text{otherwise},\end{cases}\\
&(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu)\\
&= \begin{cases} \dfrac{Q_{\nu_k,\nu_{k+1}}(x_k,x_{k+1}) - Q_{\nu_k,\nu_{k+1}}(x_{k+2},x_{k+1})}{x_k - x_{k+2}}e(\nu) & \text{if } \nu_k=\nu_{k+2},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

The above relations are homogeneous provided that

$$\deg e(\nu) = 0, \quad \deg\, x_k e(\nu) = (\alpha_{\nu_k},\alpha_{\nu_k}), \quad \deg\, \tau_l e(\nu) = -(\alpha_{\nu_l},\alpha_{\nu_{l+1}}),$$

and hence $R(\beta)$ is a $\mathbb{Z}$-graded algebra.

For a graded $R(\beta)$-module $M = \bigoplus_{k\in\mathbb{Z}} M_k$, we define $qM = \bigoplus_{k\in\mathbb{Z}}(qM)_k$, where

$$(qM)_k = M_{k-1} \ (k\in\mathbb{Z}).$$

We call q the *grading shift functor* on the category of graded $R(\beta)$-modules.

If M is an $R(\beta)$-module, then we set $\mathrm{wt}(M) = -\beta \in \mathsf{Q}^-$ and call it the *weight* of M.

We denote by $R(\beta)$-Mod the category of $R(\beta)$-modules, and by $R(\beta)$-mod the full subcategory of $R(\beta)$-Mod consisting of modules M such that M are finite-dimensional over $\mathbf{k}$, and the actions of the x_k's on M are nilpotent.

Similarly, we denote by $R(\beta)$-gMod and by $R(\beta)$-gmod the category of graded $R(\beta)$-modules and the category of graded $R(\beta)$-modules which are finite-dimensional over $\mathbf{k}$, respectively. We set

$$R\text{-gmod} = \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)\text{-gmod} \quad \text{and} \quad R\text{-mod} = \bigoplus_{\beta\in\mathsf{Q}^+} R(\beta)\text{-mod}.$$

For $\beta,\gamma \in \mathsf{Q}^+$ with $|\beta| = m$, $|\gamma| = n$, set

$$e(\beta,\gamma) = \sum_{\substack{\nu\in I^{\beta+\gamma},\\ (\nu_1,\ldots,\nu_m)\in I^\beta,\\ (\nu_{m+1},\ldots,\nu_{m+n})\in I^\gamma}} e(\nu) \in R(\beta+\gamma).$$

Then $e(\beta,\gamma)$ is an idempotent. Let

$$R(\beta)\otimes R(\gamma) \to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$$

be the $\mathbf{k}$-algebra homomorphism given by $e(\mu)\otimes e(\nu) \mapsto e(\mu * \nu)$ ($\mu \in I^\beta$ and $\nu \in I^\gamma$) $x_k\otimes 1 \mapsto x_k e(\beta,\gamma)$ $(1\le k\le m)$, $1\otimes x_k \mapsto x_{m+k}e(\beta,\gamma)$ $(1\le k\le n)$, $\tau_k\otimes 1 \mapsto \tau_k e(\beta,\gamma)$ $(1\le k< m)$, and $1\otimes\tau_k \mapsto \tau_{m+k}e(\beta,\gamma)$ $(1\le k<n)$. Here $\mu*\nu$ is the concatenation of μ and ν; i.e., $\mu*\nu = (\mu_1,\ldots,\mu_m,\nu_1,\ldots,\nu_n)$.

For an $R(\beta)$-module M and an $R(\gamma)$-module N, we define the *convolution product* $M \circ N$ by

$$M \circ N = R(\beta+\gamma)e(\beta,\gamma) \underset{R(\beta)\otimes R(\gamma)}{\otimes} (M \otimes N).$$

For $M \in R(\beta)$-mod, the dual space

$$M^* := \mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$$

admits an $R(\beta)$-module structure via

$$(r \cdot f)(u) := f(\psi(r)u) \quad (r \in R(\beta),\ u \in M),$$

where ψ denotes the $\mathbf{k}$-algebra anti-involution on $R(\beta)$ which fixes the generators $e(\nu)$, x_m, and τ_k for $\nu \in I^\beta, 1 \le m \le |\beta|$, and $1 \le k < |\beta|$.

It is known that (see [28, Theorem 2.2 (2)])

$$(M_1 \circ M_2)^* \simeq q^{(\beta,\gamma)}(M_2^* \circ M_1^*)$$

for any $M_1 \in R(\beta)$-gmod and $M_2 \in R(\gamma)$-gmod.

A simple module M in R-gmod is called *self-dual* if $M^* \simeq M$. Every simple module is isomorphic to a grading shift of a self-dual simple module [21, Section 3.2]. Note also that we have $\mathrm{End}_{R(\beta)} M \simeq \mathbf{k}$ for every simple module M in $R(\beta)$-gmod [21, Corollary 3.19].

Let us denote by $K(R\text{-gmod})$ the Grothendieck group of R-gmod. Then, $K(R\text{-gmod})$ is an algebra over $\mathbb{Z}[q^{\pm 1}]$ with the multiplication induced by the convolution product and the $\mathbb{Z}[q^{\pm 1}]$-action induced by the grading shift functor q.

In [21, 38], it is shown that a KLR algebra *categorifies* the negative half of the corresponding quantum group. More precisely, we have the following theorem.

Theorem 2.1.2 ([21, 38])**.** *For a given Cartan datum* $(A, \mathsf{P}, \Pi, \mathsf{P}^\vee, \Pi^\vee)$*, we take a parameter matrix* $(Q_{ij})_{i,j\in J}$ *satisfying the conditions in* (2.1)*, and let* $U_q(\mathfrak{g})$ *and* $R(\beta)$ *be the associated quantum group and the KLR algebras, respectively. Then there exists a* $\mathbb{Z}[q^{\pm 1}]$*-algebra isomorphism*

$$U_q^-(\mathfrak{g})^\vee_{\mathbb{Z}[q^{\pm 1}]} \simeq K(R\text{-gmod}). \tag{2.2}$$

KLR algebras also categorify the upper global bases.

Definition 2.1.3. *We say that a KLR algebra R is* symmetric *if $Q_{i,j}(u,v)$ is a polynomial in $u - v$ for all $i, j \in I$.*

In particular, the corresponding generalized Cartan matrix A is symmetric. In symmetric case, we assume $(\alpha_i, \alpha_i) = 2$ for $i \in I$.

Theorem 2.1.4 ([39, 40])**.** *Assume that the KLR algebra R is symmetric and the base field* $\mathbf{k}$ *is of characteristic* 0*. Then under the isomorphism* (2.2) *in* Theorem 2.1.2*, the upper global basis corresponds to the set of the isomorphism classes of self-dual simple R-modules.*

2.2. **R-matrices for KLR algebras.**

For $|\beta| = n$ and $1 \le a < n$, we define $\varphi_a \in R(\beta)$ by

$$\varphi_a e(\nu) = \begin{cases} \big(\tau_a x_a - x_a \tau_a\big)e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ \tau_a e(\nu) & \text{otherwise.} \end{cases}$$

They are called the *intertwiners*. Since $\{\varphi_a\}_{1\le a<n}$ satisfies the braid relation, $\varphi_w := \varphi_{i_1} \cdots \varphi_{i_\ell}$ does not depend on the choice of reduced expression $w = s_{i_1} \cdots s_{i_\ell}$.

For $m, n \in \mathbb{Z}_{\geq 0}$, let us denote by $w[m,n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k+n & \text{if } 1 \leq k \leq m, \\ k-m & \text{if } m < k \leq m+n. \end{cases}$$

Let $\beta, \gamma \in \mathsf{Q}^+$ with $|\beta| = m$, $|\gamma| = n$, and let M be an $R(\beta)$-module and N an $R(\gamma)$-module. Then the map $M \otimes N \to N \circ M$ given by $u \otimes v \longmapsto \varphi_{w[n,m]}(v \otimes u)$ is $R(\beta) \otimes R(\gamma)$-linear, and hence it extends to an $R(\beta+\gamma)$-module homomorphism

$$R_{M,N} \colon M \circ N \longrightarrow N \circ M.$$

Assume that the KLR algebra $R(\beta)$ is symmetric. Let z be an indeterminate which is homogeneous of degree 2, and let ψ_z be the graded algebra homomorphism

$$\psi_z \colon R(\beta) \to \mathbf{k}[z] \otimes R(\beta)$$

given by

$$\psi_z(x_k) = x_k + z, \quad \psi_z(\tau_k) = \tau_k, \quad \psi_z(e(\nu)) = e(\nu).$$

For an $R(\beta)$-module M, we denote by M_z the $\big(\mathbf{k}[z] \otimes R(\beta)\big)$-module $\mathbf{k}[z] \otimes M$ with the action of $R(\beta)$ twisted by ψ_z. Namely,

$$\begin{aligned} e(\nu)(a \otimes u) &= a \otimes e(\nu)u, \\ x_k(a \otimes u) &= (za) \otimes u + a \otimes (x_k u), \\ \tau_k(a \otimes u) &= a \otimes (\tau_k u) \end{aligned}$$

for $\nu \in I^\beta$, $a \in \mathbf{k}[z]$, and $u \in M$. Note that the multiplication by z on $\mathbf{k}[z]$ induces an $R(\beta)$-module endomorphism on M_z. For $u \in M$, we sometimes denote by u_z the corresponding element $1 \otimes u$ of the $R(\beta)$-module M_z.

For a non-zero $M \in R(\beta)$-mod and a non-zero $N \in R(\gamma)$-mod,

(2.3) let s be the order of zero of $R_{M_z,N} \colon M_z \circ N \longrightarrow N \circ M_z$; i.e., the largest non-negative integer such that the image of $R_{M_z,N}$ is contained in $z^s(N \circ M_z)$.

Note that such an s exists because $R_{M_z,N}$ does not vanish [14, Proposition 1.4.4 (iii)]. We denote by $R^{\mathrm{ren}}_{M_z,N}$ the morphism $z^{-s}R_{M_z,N}$.

Definition 2.2.1. *Assume that $R(\beta)$ is symmetric. For a non-zero $M \in R(\beta)$*-mod *and a non-zero $N \in R(\gamma)$*-mod*, let s be an integer as in* (2.3)*. We define*

$$\mathbf{r}_{M,N} \colon M \circ N \to N \circ M$$

by

$$\mathbf{r}_{M,N} = R^{\mathrm{ren}}_{M_z,N}|_{z=0},$$

and call it the renormalized R-matrix.

By the definition, the renormalized R-matrix $\mathbf{r}_{M,N}$ never vanishes.

We define also

$$\mathbf{r}_{N,M} \colon N \circ M \to M \circ N$$

by

$$\mathbf{r}_{N,M} = \big((-z)^{-t}R_{N,M_z}\big)|_{z=0},$$

where t is the order of zero of R_{N,M_z}.

If $R(\beta)$ and $R(\gamma)$ are symmetric, then s coincides with the order of zero of R_{M,N_z}, and $\big(z^{-s}R_{M_z,N}\big)|_{z=0} = \big((-z)^{-s}R_{M,N_z}\big)|_{z=0}$ (see [15, (1.11)]).

By the construction, if the composition $(N_1\circ\mathbf{r}_{M,N_2})\circ(\mathbf{r}_{M,N_1}\circ N_2)$ for $M, N_1, N_2 \in R$-mod does not vanish, then it is equal to $\mathbf{r}_{M,N_1\circ N_2}$.

Definition 2.2.2. *A simple $R(\beta)$-module M is called* real *if $M \circ M$ is simple.*

The following lemma was used significantly in [15].

Lemma 2.2.3 ([15, Lemma 3.1])**.** *Let $\beta_k \in \mathsf{Q}^+$ and $M_k \in R(\beta_k)$-mod $(k = 1, 2, 3)$. Let X be an $R(\beta_1 + \beta_2)$-submodule of $M_1 \circ M_2$ and Y an $R(\beta_2 + \beta_3)$-submodule of $M_2 \circ M_3$ such that $X \circ M_3 \subset M_1 \circ Y$ as submodules of $M_1 \circ M_2 \circ M_3$. Then there exists an $R(\beta_2)$-submodule N of M_2 such that $X \subset M_1 \circ N$ and $N \circ M_3 \subset Y$.*

One of the main results in [15] is the following theorem.

Theorem 2.2.4 ([15, Theorem 3.2])**.** *Let $\beta, \gamma \in \mathsf{Q}^+$ and assume that $R(\beta)$ is symmetric. Let M be a real simple module in $R(\beta)$-mod and N a simple module in $R(\gamma)$-mod. Then*

(i) *$M \circ N$ and $N \circ M$ have simple socles and simple heads.*
(ii) *Moreover, $\mathrm{Im}(\mathbf{r}_{M,N})$ is equal to the head of $M \circ N$ and socle of $N \circ M$. Similarly, $\mathrm{Im}(\mathbf{r}_{N,M})$ is equal to the head of $N \circ M$ and socle of $M \circ N$.*

We will use the following convention frequently.

Definition 2.2.5. *For simple R-modules M and N, we denote by $M \nabla N$ the head of $M \circ N$ and by $M \Delta N$ the socle of $M \circ N$.*

3. SIMPLICITY OF HEADS AND SOCLES OF CONVOLUTION PRODUCTS

In this section, we assume that $R(\beta)$ is symmetric for any $\beta \in \mathsf{Q}^+$; i.e., $Q_{ij}(u, v)$ is a function in $u - v$ for any $i, j \in I$.

We also work always in the category of graded modules. For the sake of simplicity, *we simply say that M is an R-module instead of saying that M is a graded $R(\beta)$-module for $\beta \in \mathsf{Q}^+$.* We also sometimes ignore grading shifts if there is no danger of confusion. Hence, for R-modules M and N, *we sometimes say that $f : M \to N$ is a homomorphism if $f : q^a M \to N$ is a morphism in R-gmod for some $a \in \mathbb{Z}$.* If we want to emphasize that $f : q^a M \to N$ is a morphism in R-gmod, we say so.

3.1. **Homogeneous degrees of R-matrices.**

Definition 3.1.1. *For non-zero $M, N \in R$-gmod, we denote by $\Lambda(M, N)$ the homogeneous degree of the R-matrix $\mathbf{r}_{M,N}$.*

Hence

$$R^{\mathrm{ren}}_{M_z,N} : M_z \circ N \to q^{-\Lambda(M,N)} N \circ M_z \quad \text{and}$$
$$\mathbf{r}_{M,N} : M \circ N \to q^{-\Lambda(M,N)} N \circ M$$

are morphisms in R-gMod and in R-gmod, respectively.

Lemma 3.1.2. *For non-zero R-modules M and N, we have*

$$\Lambda(M, N) \equiv \big(\mathrm{wt}(M), \mathrm{wt}(N)\big) \mod 2.$$

Proof. Set $\beta := -\mathrm{wt}(M)$ and $\gamma := -\mathrm{wt}(N)$. By [14, (1.3.3)], the homogeneous degree of $R_{M_z,N}$ is $-(\beta, \gamma) + 2(\beta, \gamma)_{\mathrm{n}}$, where $(\,\bullet\,,\,\bullet\,)_{\mathrm{n}}$ is the symmetric bilinear form on Q given by $(\alpha_i, \alpha_j)_{\mathrm{n}} = \delta_{ij}$. Hence $R^{\mathrm{ren}}_{M_z,N} = z^{-s} R_{M_z,N}$ has degree $-(\beta, \gamma) + 2(\beta, \gamma)_{\mathrm{n}} - 2s$. ☐

Definition 3.1.3. *For non-zero R-modules M and N, we set*

$$\widetilde{\Lambda}(M,N) := \frac{1}{2}\big(\Lambda(M,N) + (\mathrm{wt}(M), \mathrm{wt}(N))\big) \in \mathbb{Z}.$$

Lemma 3.1.4. *Let M and N be self-dual simple modules. If one of them is real, then*

$$q^{\widetilde{\Lambda}(M,N)} M \nabla N$$

is a self-dual simple module.

Proof. Set $\beta = \mathrm{wt}(M)$ and $\gamma = \mathrm{wt}(N)$. Set $M \nabla N = q^c L$ for some self-dual simple module L and some $c \in \mathbb{Z}$. Then we have

$$M \circ N \twoheadrightarrow q^c L \rightarrowtail q^{-\Lambda(M,N)} N \circ M,$$

since $M \nabla N = \mathrm{Im}\,\mathbf{r}_{M,N}$. Taking dual, we obtain

$$q^{\Lambda(M,N)+(\beta,\gamma)} M \circ N \twoheadrightarrow q^{-c} L \rightarrowtail q^{(\beta,\gamma)} N \circ M.$$

In particular, $q^{-c-\Lambda(M,N)-(\beta,\gamma)} L$ is a simple quotient of $M \circ N$. Hence we have $c = -c - \Lambda(M,N) - (\beta,\gamma)$, which implies $c = -\widetilde{\Lambda}(M,N)$. ☐

Lemma 3.1.5. (i) *Let M_k be non-zero modules $(k = 1,2,3)$, and let $\varphi_1 : L \to M_1 \circ M_2$ and $\varphi_2 : M_2 \circ M_3 \to L'$ be non-zero homomorphisms. Assume further that M_2 is a simple module. Then the composition*

$$L \circ M_3 \xrightarrow{\varphi_1 \circ M_3} M_1 \circ M_2 \circ M_3 \xrightarrow{M_1 \circ \varphi_2} M_1 \circ L'$$

does not vanish.

(ii) *Let M be a simple module, and let N_1, N_2 be non-zero modules. Then the composition*

$$M \circ N_1 \circ N_2 \xrightarrow{\mathbf{r}_{M,N_1} \circ N_2} N_1 \circ M \circ N_2 \xrightarrow{N_1 \circ\, \mathbf{r}_{M,N_2}} N_1 \circ N_2 \circ M$$

coincides with $\mathbf{r}_{M,N_1 \circ N_2}$, and the composition

$$N_1 \circ N_2 \circ M \xrightarrow{N_1 \circ\, \mathbf{r}_{N_2,M}} N_1 \circ M \circ N_2 \xrightarrow{\mathbf{r}_{N_1,M} \circ N_2} M \circ N_1 \circ N_2$$

coincides with $\mathbf{r}_{N_1 \circ N_2, M}$.

In particular, we have

$$\Lambda(M, N_1 \circ N_2) = \Lambda(M, N_1) + \Lambda(M, N_2)$$

and

$$\Lambda(N_1 \circ N_2, M) = \Lambda(N_1, M) + \Lambda(N_2, M).$$

Proof. (i) Assume that the composition vanishes. Then we have $\mathrm{Im}\varphi_1 \circ M_3 \subset M_1 \circ \mathrm{Ker}\,\varphi_2$. By Lemma 2.2.3, there is a submodule N of M_2 such that $\mathrm{Im}\varphi_1 \subset M_1 \circ N$ and $N \circ M_3 \subset \mathrm{Ker}\,\varphi_2$. The first inclusion implies that $N \neq 0$ since φ_1 is non-zero, and the second implies $N \neq M_2$ since φ_2 is non-zero. It contradicts the simplicity of M_2.

(ii) It is enough to show that the compositions $(N_1 \circ \mathbf{r}_{M,N_2}) \circ (\mathbf{r}_{M,N_1} \circ N_2)$ and $(\mathbf{r}_{N_1,M} \circ N_2) \circ (N_1 \circ \mathbf{r}_{N_2,M})$ do not vanish, but these immediately follow from (i). ☐

3.2. Properties of $\widetilde{\Lambda}(M,N)$ and $\mathfrak{d}(M,N)$.

Lemma 3.2.1. *Let M and N be simple R-modules. Then we have*

(i) $\Lambda(M,N)+\Lambda(N,M)\in 2\mathbb{Z}_{\ge 0}$.
(ii) *If $\Lambda(M,N)+\Lambda(N,M)=2m$ for some $m\in\mathbb{Z}_{\ge 0}$, then*

$$R^{\mathrm{ren}}_{M_z,N}\circ R^{\mathrm{ren}}_{N,M_z}=z^m\,\mathrm{id}_{N\circ M_z}\quad\text{and}\quad R^{\mathrm{ren}}_{N,M_z}\circ R^{\mathrm{ren}}_{M_z,N}=z^m\,\mathrm{id}_{M_z\circ N}$$

up to constant multiples.

Proof. By [14, Proposition 1.6.2], the morphism

$$R^{\mathrm{ren}}_{N,M_z}\circ R^{\mathrm{ren}}_{M_z,N}:M_z\circ N\to M_z\circ N$$

is equal to $f(z)\,\mathrm{id}_{M_z\circ N}$ for some $0\ne f(z)\in\mathbf{k}[z]$. Since $R^{\mathrm{ren}}_{N,M_z}\circ R^{\mathrm{ren}}_{M_z,N}$ is homogeneous of degree $\Lambda(M,N)+\Lambda(N,M)$, we have $f(z)=cz^{\frac{1}{2}(\Lambda(M,N)+\Lambda(N,M))}$ for some $c\in\mathbf{k}^\times$. ☐

Definition 3.2.2. *For non-zero modules M and N, we set*

$$\mathfrak{d}(M,N)=\frac{1}{2}\big(\Lambda(M,N)+\Lambda(N,M)\big).$$

Note that if M and N are simple modules, then we have $\mathfrak{d}(M,N)\in\mathbb{Z}_{\ge 0}$. Note also that if M,N_1,N_2 are simple modules, then we have $\mathfrak{d}(M,N_1\circ N_2)=\mathfrak{d}(M,N_1)+\mathfrak{d}(M,N_2)$ by Lemma 3.1.5 (ii).

Lemma 3.2.3 ([15]). *Let M,N be simple modules and assume that one of them is real. Then the following conditions are equivalent:*

(i) $\mathfrak{d}(M,N)=0$.
(ii) $\mathbf{r}_{M,N}$ *and* $\mathbf{r}_{N,M}$ *are inverse to each other up to a constant multiple.*
(iii) $M\circ N$ *and* $N\circ M$ *are isomorphic up to a grading shift.*
(iv) $M\nabla N$ *and* $N\nabla M$ *are isomorphic up to a grading shift.*
(v) $M\circ N$ *is simple.*

Proof. By specializing the equations in Lemma 3.2.1 (ii) at $z=0$, we obtain that $\mathfrak{d}(M,N)=0$ if and only if $\mathbf{r}_{M,N}\circ\mathbf{r}_{N,M}=\mathrm{id}_{N\circ M}$ and $\mathbf{r}_{N,M}\circ\mathbf{r}_{M,N}=\mathrm{id}_{M\circ N}$ up to non-zero constant multiples. Hence the conditions (i) and (ii) are equivalent.

The conditions (ii), (iii), (iv), and (v) are equivalent by [15, Theorem 3.2, Proposition 3.8, and Corollary 3.9]. ☐

Definition 3.2.4. *Let M,N be simple modules.*

(i) *We say that M and N* commute *if $\mathfrak{d}(M,N)=0$.*
(ii) *We say that M and N are* simply linked *if $\mathfrak{d}(M,N)=1$.*

Proposition 3.2.5. *Let $M_1,\dots,M_r$ be a commuting family of real simple modules. Then the convolution product*

$$M_1\circ\cdots\circ M_r$$

is a real simple module.

Proof. We shall first show the simplicity of the convolutions. By induction on r, we may assume that $M_2 \circ \cdots \circ M_r$ is a simple module. Then we have

$$\mathfrak{d}(M_1, M_2 \circ \cdots \circ M_r) = \sum_{s=2}^{r} \mathfrak{d}(M_1, M_s) = 0$$

so that $M_1 \circ \cdots \circ M_r$ is simple by Lemma 3.2.3.

Since $(M_1 \circ \cdots \circ M_r) \circ (M_1 \circ \cdots \circ M_r)$ is also simple, $M_1 \circ \cdots \circ M_r$ is real. $\square$

Definition 3.2.6. *Let $M_1, \ldots, M_m$ be real simple modules. Assume that they commute with each other. We set*

$$\begin{aligned} M_1 \odot M_2 &:= q^{\widetilde{\Lambda}(M_1,M_2)} M_1 \circ M_2, \\ \bigodot_{1 \le k \le m} M_k &:= (\cdots (M_1 \odot M_2) \cdots) \odot M_{m-1}) \odot M_m \\ &\simeq q^{\sum_{1 \le i < j \le m} \widetilde{\Lambda}(M_i,M_j)} M_1 \circ \cdots \circ M_m. \end{aligned}$$

It is invariant under the permutations of $M_1, \ldots, M_m$.

Lemma 3.2.7. *Let $M_1, \ldots, M_m$ be real simple modules commuting with each other. Then for any $\sigma \in \mathfrak{S}_m$, we have*

$$\bigodot_{1 \le k \le m} M_k \simeq \bigodot_{1 \le k \le m} M_{\sigma(k)} \quad \textit{in } R\text{-gmod}.$$

Moreover, if the M_k's are self-dual, then so is $\odot_{1 \le k \le m} M_k$.

Proof. It follows from Lemma 3.1.4 and $q^{\widetilde{\Lambda}(M_i,M_j)} M_i \circ M_j \simeq q^{\widetilde{\Lambda}(M_j,M_i)} M_j \circ M_i$. $\square$

Proposition 3.2.8. *Let $f : N_1 \to N_2$ be a morphism between non-zero R-modules N_1, N_2, and let M be a non-zero R-module.*

(i) *If $\Lambda(M, N_1) = \Lambda(M, N_2)$, then the following diagram is commutative:*

$$\begin{tikzcd} M \circ N_1 \arrow[r, "\mathbf{r}_{M,N_1}"] \arrow[d, "M \circ f"'] & N_1 \circ M \arrow[d, "f \circ M"] \\ M \circ N_2 \arrow[r, "\mathbf{r}_{M,N_2}"] & N_2 \circ M. \end{tikzcd}$$

(ii) *If $\Lambda(M, N_1) < \Lambda(M, N_2)$, then the composition*

$$M \circ N_1 \xrightarrow{M \circ f} M \circ N_2 \xrightarrow{\mathbf{r}_{M,N_2}} N_2 \circ M$$

vanishes.

(iii) *If $\Lambda(M, N_1) > \Lambda(M, N_2)$, then the composition*

$$M \circ N_1 \xrightarrow{\mathbf{r}_{M,N_1}} N_1 \circ M \xrightarrow{f \circ M} N_2 \circ M$$

vanishes.

(iv) *If f is surjective, then we have*

$$\Lambda(M, N_1) \ge \Lambda(M, N_2) \quad \textit{and} \quad \Lambda(N_1, M) \ge \Lambda(N_2, M).$$

If f is injective, then we have

$$\Lambda(M, N_1) \le \Lambda(M, N_2) \quad \textit{and} \quad \Lambda(N_1, M) \le \Lambda(N_2, M).$$

Proof. Let s_i be the order of zero of R_{M_z,N_i} for $i = 1, 2$. Then we have $\Lambda(M, N_1) - \Lambda(M, N_2) = 2(s_2 - s_1)$.

Set $m := \min\{s_1, s_2\}$. Then the following diagram is commutative:

$$\begin{tikzcd}
M_z \circ N_1 \arrow[r, "z^{-m} R_{M_z,N_1}"] \arrow[d, "M_z \circ f"'] & N_1 \circ M_z \arrow[d, "f \circ M_z"] \\
M_z \circ N_2 \arrow[r, "z^{-m} R_{M_z,N_2}"] & N_2 \circ M_z.
\end{tikzcd}$$

(i) If $s_1 = s_2$, then by specializing $z = 0$ in the above diagram, we obtain the commutativity of the diagram in (i).

(ii) If $s_1 > s_2$, then we have

$$z^{-m} R_{M_z,N_1} = z^{s_1 - m}\big(z^{-s_1} R_{M_z,N_1}\big)$$

so that $z^{-m} R_{M_z,N_1}|_{z=0}$ vanishes. Hence we have

$$\mathbf{r}_{M,N_2} \circ (M \circ f) = z^{-m} R_{M_z,N_2}|_{z=0} \circ (M \circ f) = 0,$$

as desired. In particular, f is not surjective.

(iii) Similarly, if $s_1 < s_2$, then we have $(f \circ M) \circ \mathbf{r}_{M,N_1} = 0$, and f is not injective.

(iv) The statements for $\Lambda(M, N_1)$ and $\Lambda(M, N_2)$ follow from (ii) and (iii). The other statements can be shown in a similar way. ☐

Proposition 3.2.9. *Let M and N be simple modules. We assume that one of them is real. Then we have*

$$\mathrm{Hom}_{R\text{-mod}}(M \circ N, N \circ M) = \mathbf{k}\,\mathbf{r}_{M,N}.$$

Proof. Since the other case can be proved similarly, we assume that M is real. Let $f\colon M \circ N \to N \circ M$ be a morphism. Note that we have $\mathbf{r}_{M,\,M\circ N} = M \circ \mathbf{r}_{M,N}$ and $\mathbf{r}_{M,\,N\circ M} = \mathbf{r}_{M,N} \circ M$ by Lemma 3.1.5 (ii) and by the fact that $\mathbf{r}_{M,M} = \mathrm{id}_{M\circ M}$ up to a constant multiple. Thus, by Proposition 3.2.8, we have a commutative diagram (up to a constant multiple)

$$\begin{tikzcd}
M \circ M \circ N \arrow[r, "M\circ \mathbf{r}_{M,N}"] \arrow[d, "M\circ f"'] & M \circ N \circ M \arrow[d, "f\circ M"] \\
M \circ N \circ M \arrow[r, "\mathbf{r}_{M,N}\circ M"] & N \circ M \circ M.
\end{tikzcd}$$

Hence we have

$$M \circ \mathrm{Im}(\mathbf{r}_{M,N}) \subset f^{-1}\big(\mathrm{Im}(\mathbf{r}_{M,N})\big) \circ M.$$

Hence there exists a submodule K of N such that $\mathrm{Im}(\mathbf{r}_{M,N}) \subset K \circ M$ and $M \circ K \subset f^{-1}\big(\mathrm{Im}(\mathbf{r}_{M,N})\big)$ by Lemma 2.2.3. Since $K \neq 0$, we have $K = N$. Hence $f(M \circ N) \subset \mathrm{Im}(\mathbf{r}_{M,N})$, which means that f factors as $M \circ N \to \mathrm{soc}(N \circ M) \rightarrowtail N \circ M$. It remains to remark that $\mathrm{Hom}_{R\text{-mod}}\big(M \circ N, \mathrm{soc}(N \circ M)\big) = \mathbf{k}\,\mathbf{r}_{M,N}$. ☐

Proposition 3.2.10. *Let L, M, and N be simple modules. Then we have*

$$\begin{aligned}&\Lambda(L,S) \le \Lambda(L,M) + \Lambda(L,N),\ \ \Lambda(S,L) \le \Lambda(M,L) + \Lambda(N,L),\ \ \textit{and}\\ &\mathfrak{d}(S,L) \le \mathfrak{d}(M,L) + \mathfrak{d}(N,L)\end{aligned} \tag{3.1}$$

for any subquotient S of $M\circ N$. Moreover, when L is real, the following conditions are equivalent:

(i) *L commutes with M and N.*
(ii) *Any simple subquotient S of $M\circ N$ commutes with L and satisfies $\Lambda(L,S) = \Lambda(L,M)+\Lambda(L,N)$.*
(iii) *Any simple subquotient S of $M\circ N$ commutes with L and satisfies $\Lambda(S,L) = \Lambda(M,L)+\Lambda(N,L)$.*

Proof. The inequalities (3.1) are consequences of Proposition 3.2.8. Let us show the equivalence of (i)–(iii).

Let $M\circ N = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0$ be a Jordan–Hölder series of $M\circ N$. Then the renormalized R-matrix $R^{\mathrm{ren}}_{L_z,M\circ N} = (M\circ R^{\mathrm{ren}}_{L_z,N})\circ(R^{\mathrm{ren}}_{L_z,M}\circ N)\colon L_z\circ M\circ N\to M\circ N\circ L_z$ is homogeneous of degree $\Lambda(L,M)+\Lambda(L,N)$, and it sends $L_z\circ K_k$ to $K_k\circ L_z$ for any $k\in\mathbb{Z}$. Hence $f := \mathbf{r}_{L,M\circ N} = R^{\mathrm{ren}}_{L_z,M\circ N}|_{z=0}$ sends $L\circ K_k$ to $K_k\circ L$.

First assume (i). Then f is an isomorphism. Hence $f|_{L\circ K_k}\colon L\circ K_k\to K_k\circ L$ is injective. By comparing their dimension, $f|_{L\circ K_k}$ is an isomorphism, Hence $f|_{L\circ(K_k/K_{k+1})}$ is an isomorphism of homogeneous degree $\Lambda(L,M)+\Lambda(L,N)$. Hence we obtain (ii).

Conversely, assume (ii). Then, $R^{\mathrm{ren}}_{L_z,M\circ N}|_{L_z\circ(K_k/K_{k+1})}$ and $R^{\mathrm{ren}}_{L_z,K_k/K_{k+1}}$ have the same homogeneous degree, and hence they should coincide. It implies that $f|_{L\circ(K_k/K_{k+1})} = \mathbf{r}_{L,K_k/K_{k+1}}$ is an isomorphism for any k. Therefore $f = (M\circ\mathbf{r}_{L,N})\circ(\mathbf{r}_{L,M}\circ N)$ is an isomorphism, which implies that $\mathbf{r}_{L,N}$ and $\mathbf{r}_{L,M}$ are isomorphisms. Thus we obtain (i).

Similarly, (i) and (iii) are equivalent. □

Lemma 3.2.11. *Let L, M, and N be simple modules. We assume that L is real and commutes with M. Then the diagram*

$$\begin{tikzcd}
L\circ(M\circ N) \arrow[r, "\mathbf{r}_{L,M\circ N}"] \arrow[d] & (M\circ N)\circ L \arrow[d] \\
L\circ(M\nabla N) \arrow[r, "\mathbf{r}_{L,M\nabla N}"] & (M\nabla N)\circ L
\end{tikzcd}$$

commutes.

Proof. Otherwise the composition

$$\begin{tikzcd}
L\circ M\circ N \arrow[r, "\sim", "\mathbf{r}_{L,M}\circ N"'] & M\circ L\circ N \arrow[r, "M\circ\mathbf{r}_{L,N}"'] & M\circ N\circ L \arrow[r] & (M\nabla N)\circ L
\end{tikzcd}$$

vanishes by Proposition 3.2.8. Hence we have

$$M\circ \mathrm{Im}(\mathbf{r}_{L,N}) \subset \mathrm{Ker}(M\circ N\to M\nabla N)\circ L.$$

Hence, by Lemma 2.2.3, there exists a submodule K of N such that

$$\mathrm{Im}(\mathbf{r}_{L,N}) \subset K\circ L \text{ and } M\circ K\subset \mathrm{Ker}(M\circ N\to M\nabla N).$$

The first inclusion implies $K\neq 0$ and the second implies $K\neq N$, which contradicts the simplicity of N. □

The following lemma can be proved similarly.

Lemma 3.2.12. *Let L, M, and N be simple modules. We assume that L is real and commutes with N. Then the diagram*

$$\begin{tikzcd}
(M \circ N) \circ L \arrow[r, "\mathbf{r}_{M\circ N,L}"] \arrow[d] & L \circ (M \circ N) \arrow[d] \\
(M \nabla N) \circ L \arrow[r, "\mathbf{r}_{M\nabla N,L}"] & L \circ (M \nabla N)
\end{tikzcd}$$

commutes.

The following proposition follows from Lemma 3.2.11 and Lemma 3.2.12.

Proposition 3.2.13. *Let L, M, and N be simple modules. Assume that L is real. Then we have the following:*

(i) *If L and M commute, then*

$$\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N).$$

(ii) *If L and N commute, then*

$$\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L).$$

Proposition 3.2.14. *Let M be a real simple module, and let N be a module with a simple socle. If the following diagram*

$$\begin{tikzcd}
\mathrm{soc}(N) \circ M \arrow[r, "\mathbf{r}_{\mathrm{soc}(N),M}"] \arrow[d, tail] & M \circ \mathrm{soc}(N) \arrow[d, tail] \\
N \circ M \arrow[r, "\mathbf{r}_{N,M}"] & M \circ N
\end{tikzcd}$$

commutes up to a non-zero constant multiple, then $\mathrm{soc}\big(M \circ \mathrm{soc}(N)\big)$ *is equal to the socle of $M \circ N$. In particular, $M \circ N$ has a simple socle.*

Proof. Let S be an arbitrary simple submodule of $M \circ N$. Then we have the following commutative diagram:

$$\begin{tikzcd}
S \circ M_z \arrow[r, "R_{S\circ M_z}"] \arrow[d, tail] & M_z \circ S \arrow[d, tail] \\
M \circ N \circ M_z \arrow[r, "R_{M\circ N, M_z}"] & M \circ M \circ N.
\end{tikzcd}$$

By multiplying z^{-m}, where m is the order of zero of $R_{M\circ N,M}$, and specializing at $z = 0$, we have a commutative diagram (up to a constant multiple)

$$\begin{tikzcd}
S \circ M \arrow[r] \arrow[d, tail] & M \circ S \arrow[d, tail] \\
M \circ N \circ M \arrow[r, "M\circ \mathbf{r}_{N,M}"] & M \circ M \circ N.
\end{tikzcd}$$

Here, we use the fact that $\mathbf{r}_{M\circ N,M} = (\mathbf{r}_{M,M} \circ N) \circ (M \circ \mathbf{r}_{N,M})$ from Lemma 3.1.5 and the fact that $\mathbf{r}_{M,M}$ is equal to $\mathrm{id}_{M\circ M}$ up to a non-zero constant multiple, because M is a real simple module.

It follows that $S \circ M \subset M \circ (\mathbf{r}_{N,M})^{-1}(S)$. Hence there exists a submodule K of N such that $S \subset M \circ K$ and $K \circ M \subset (\mathbf{r}_{N,M})^{-1}(S)$ by Lemma 2.2.3. Hence $K \neq 0$ and $\operatorname{soc}(N) \subset K$ by the assumption. Hence $\mathbf{r}_{N,M}\big(\operatorname{soc}(N) \circ M\big) \subset \mathbf{r}_{N,M}\big(K \circ M\big) \subset S$. Since $\mathbf{r}_{N,M}\big(\operatorname{soc}(N) \circ M\big)$ is non-zero by the assumption, we have $\mathbf{r}_{N,M}\big(\operatorname{soc}(N) \circ M\big) = S$. Thus we obtain the desired result. ☐

The following is a dual form of the preceding proposition.

Proposition 3.2.15. *Let M be a real simple module. Let N be a module with a simple head. If the following diagram*

$$\begin{tikzcd}
M \circ N \arrow[r, "\mathbf{r}_{M,N}"] \arrow[d, two heads] & N \circ M \arrow[d, two heads] \\
M \circ \operatorname{hd}(N) \arrow[r, "\mathbf{r}_{M,\operatorname{hd}(N)}"] & \operatorname{hd}(N) \circ M
\end{tikzcd}$$

commutes up to a non-zero constant multiple, then $M \nabla \operatorname{hd}(N)$ is equal to the simple head of $M \circ N$.

Proposition 3.2.16. *Let L, M, and N be simple modules. We assume that L is real and one of M and N is real.*

- (i) *If $\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N)$, then $L \circ M \circ N$ has a simple head and $N \circ M \circ L$ has a simple socle.*
- (ii) *If $\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L)$, then $M \circ N \circ L$ has a simple head and $L \circ N \circ M$ has a simple socle.*
- (iii) *If $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$, then $L \circ M \circ N$ and $M \circ N \circ L$ have simple heads, and $N \circ M \circ L$ and $L \circ N \circ M$ have simple socles.*

Proof. (i) Denote $k = \Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(M, N)$ and $m = \Lambda(M, N)$. Then the diagram

$$\begin{tikzcd}
L \circ M \circ N \arrow[r, "\mathbf{r}_{L,M\circ N}"] \arrow[d, two heads] & q^{-k} M \circ N \circ L \arrow[d, two heads] \\
L \circ (M \nabla N) \arrow[r, "\mathbf{r}_{L,M\nabla N}"] \arrow[d, tail] & q^{-k} (M \nabla N) \circ L \arrow[d, tail] \\
q^{-m} L \circ N \circ M \arrow[r, "\mathbf{r}_{L,N\circ M}"] & q^{-k-m} N \circ M \circ L
\end{tikzcd}$$

commutes. Hence Proposition 3.2.14 and Proposition 3.2.15 imply that $L \circ M \circ N$ has a simple head and $N \circ M \circ L$ has a simple socle. Item (ii) is proved similarly.
(iii) If $\mathfrak{d}(L, M \nabla N) = \mathfrak{d}(L, M) + \mathfrak{d}(L, N)$, then we have $\Lambda(L, M \nabla N) = \Lambda(L, M) + \Lambda(L, N)$ and $\Lambda(M \nabla N, L) = \Lambda(M, L) + \Lambda(N, L)$ by Proposition 3.2.8. Thus the statements in (iii) follow from (i) and (ii). ☐

Proposition 3.2.17. *Let M and N be simple modules. Assume that one of them is real and $\mathfrak{d}(M, N) = 1$. Then we have an exact sequence*

$$0 \to M \,\Delta\, N \to M \circ N \to M \nabla N \to 0.$$

In particular, $M \circ N$ has length 2.

Proof. In the course of the proof, we ignore the grading.

Set $X = M_z \circ N$ and $Y = N \circ M_z$. By $R^{\mathrm{ren}}_{N,M_z} \colon Y \rightarrowtail X$ let us regard Y as a submodule of X. By the condition, we have $R^{\mathrm{ren}}_{N,M_z} \circ R^{\mathrm{ren}}_{M_z,N} = z\,\mathrm{id}_X$ up to a constant multiple (see Lemma 3.2.1 (ii)), and hence we have

$$zX \subset Y \subset X.$$

We have an exact sequence

$$0 \longrightarrow \frac{Y}{zX} \longrightarrow \frac{X}{zX} \longrightarrow \frac{X}{Y} \longrightarrow 0.$$

Since

$$M \circ N \simeq \frac{X}{zX} \twoheadrightarrow \frac{X}{Y} \rightarrowtail \frac{z^{-1}Y}{Y} \simeq N \circ M,$$

we have $\dfrac{X}{Y} \simeq M \mathbin{\nabla} N$ by Proposition 3.2.9. Similarly,

$$N \circ M \simeq \frac{Y}{zY} \twoheadrightarrow \frac{Y}{zX} \rightarrowtail \frac{X}{zX} \simeq M \circ N$$

implies that $\dfrac{Y}{zX} \simeq M \mathbin{\Delta} N$ by Proposition 3.2.9. □

Lemma 3.2.18. *Let M and N be simple modules. Assume that one of them is real. If there is an exact sequence*

$$0 \to q^m X \longrightarrow M \circ N \longrightarrow q^n Y \longrightarrow 0$$

for self-dual simple modules X, Y and integers m, n, then we have

$$\mathfrak{d}(M,N) = m - n.$$

Proof. We may assume that M and N are self-dual without loss of generality. Then we have $n = -\widetilde{\Lambda}(M,N)$. Since

$$q^m X \simeq q^{\Lambda(N,M)} N \mathbin{\nabla} M \simeq q^{\Lambda(N,M)-\widetilde{\Lambda}(N,M)} \big(q^{\widetilde{\Lambda}(N,M)} N \mathbin{\nabla} M\big),$$

we have $m = \Lambda(N,M) - \widetilde{\Lambda}(N,M)$. Thus we obtain

$$m - n = \Lambda(N,M) - \widetilde{\Lambda}(N,M) + \widetilde{\Lambda}(M,N) = \mathfrak{d}(M,N).$$

□

Lemma 3.2.19. *Let M and N be simple modules. Assume that one of them is real. If the equation*

$$[M][N] = q^m[X] + q^n[Y]$$

holds in $K(R\text{-}\mathrm{gmod})$ for self-dual simple modules X, Y and integers m, n such that $m \geq n$, then we have

(i) $\mathfrak{d}(M,N) = m - n > 0$,

(ii) *there exists an exact sequence*

$$0 \longrightarrow q^m X \longrightarrow M \circ N \longrightarrow q^n Y \longrightarrow 0,$$

(iii) *$q^m X$ is the socle of $M \circ N$ and $q^n Y$ is the head of $M \circ N$.*

Proof. First note that $\mathfrak{d}(M,N) > 0$ since $M \circ N$ is not simple. By the assumption, there exists either an exact sequence

$$0 \longrightarrow q^m X \longrightarrow M \circ N \longrightarrow q^n Y \longrightarrow 0,$$

or

$$0 \longrightarrow q^n Y \longrightarrow M \circ N \longrightarrow q^m X \longrightarrow 0.$$

The second sequence cannot exist by Lemma 3.2.18 because $\mathfrak{d}(M,N) = n - m \le 0$. Hence the first sequence exists, and the assertion (iii) follows from Theorem 2.2.4. □

Proposition 3.2.20. *Let X, Y, M, and N be simple R-modules. Assume that there is an exact sequence*

$$0 \to X \to M \circ N \to Y \to 0,$$

$X \circ N$ and $Y \circ N$ are simple, and $X \circ N \not\simeq Y \circ N$ are ungraded modules. Then N is a real simple module.

Proof. Assume that N is not real. Then $N \circ N$ is reducible, and we have $\mathbf{r}_{N,N} \neq c\,\mathrm{id}_{N \circ N}$ for any $c \in \mathbf{k}$ by [15, Corollary 3.3]. Note that $N \circ N$ is of length 2, because $M \circ N \circ N$ is of length 2.

Let S be a simple submodule of $N \circ N$. Consider an exact sequence

$$0 \longrightarrow X \circ N \longrightarrow M \circ N \circ N \longrightarrow Y \circ N \longrightarrow 0.$$

Then we have

$$(X \circ N) \cap (M \circ S) = 0. \tag{3.2}$$

Indeed, if $(X \circ N) \subset (M \circ S)$, then there exists a submodule Z of N such that $X \subset M \circ Z$ and $Z \circ N \subset S$ by [15, Lemma 3.1]. It contradicts the simplicity of N. Thus (3.2) holds.

Note that (3.2) implies

$$M \circ S \simeq Y \circ N$$

since $Y \circ N$ is simple.

(a) Assume first that $N \circ N$ is semisimple so that $N \circ N = S \oplus S'$ for some simple submodule S' of $N \circ N$. Then $M \circ S \simeq Y \circ N \simeq M \circ S'$. Hence $M \circ S \simeq X \circ N \simeq M \circ S'$. Therefore we obtain $X \circ N \simeq Y \circ N$, which is a contradiction.

(b) Assume that $N \circ N$ is not semisimple so that S is a unique non-zero proper submodule of $N \circ N$ and $(N \circ N)/S$ is a unique non-zero proper quotient of $N \circ N$. Without loss of generality, we may assume that $\mathbf{k}$ is algebraically closed [21, Corollary 3.19]. Let $x \in \mathbf{k}$ be an eigenvalue of $\mathbf{r}_{N,N}$. Since $\mathbf{r}_{N,N} \notin \mathbf{k}\,\mathrm{id}_{N \circ N}$, we have $0 \subsetneq \mathrm{Im}(\mathbf{r}_{N,N} - x\,\mathrm{id}_{N \circ N}) \subsetneq N \circ N$. It follows that

$$S = \mathrm{Im}(\mathbf{r}_{N,N} - x\,\mathrm{id}_{N \circ N}) \simeq (N \circ N)/S,$$

and hence we have an exact sequence

$$0 \longrightarrow M \circ S \longrightarrow M \circ N \circ N \longrightarrow M \circ \big((N \circ N)/S\big) \longrightarrow 0.$$

Since $M \circ N \circ N$ is of length 2, we have

$$X \circ N \simeq M \circ S \simeq M \circ \big((N \circ N)/S\big) \simeq Y \circ N,$$

which is a contradiction. □

Corollary 3.2.21. *Let X, Y, N be simple R-modules, and let M be a real simple R-module. If we have an exact sequence*

$$0 \to X \to M \circ N \to Y \to 0$$

and if $X \circ N$ and $Y \circ N$ are simple, then N is a real simple module.

Proof. Since M is real and $M \circ N$ is not simple, X is not isomorphic to Y as an ungraded module by Lemma 3.2.3 (iv). It follows that $X \circ N$ is not isomorphic to $Y \circ N$, because $K(R\text{-gmod})$ is a domain so that $[X \circ N] = q^m[Y \circ N]$ for some $m \in \mathbb{Z}$ implies $[X] = q^m[Y]$. Now the assertion follows from Proposition 3.2.20. ☐

Lemma 3.2.22. *Let $\{M_i\}_{1 \le i \le n}$ and $\{N_i\}_{1 \le i \le n}$ be a pair of commuting families of real simple modules. We assume that*

(a) *$\{M_i \nabla N_i\}_{1 \le i \le n}$ is a commuting family of real simple modules,*
(b) *$M_i \nabla N_i$ commutes with N_j for any $1 \le i, j \le n$.*

Then we have

$$(\circ_{1 \le i \le n} M_i) \nabla (\circ_{1 \le j \le n} N_j) \simeq \circ_{1 \le i \le n} (M_i \nabla N_i) \quad \textit{up to a grading shift.}$$

Proof. Since $\circ_{1 \le i \le n}(M_i \nabla N_i)$ is simple, it is enough to give an epimorphism $(\circ_{1 \le i \le n} M_i) \circ (\circ_{1 \le j \le n} N_j) \twoheadrightarrow \circ_{1 \le i \le n}(M_i \nabla N_i)$. We shall show it by induction on n. For $n > 0$, we have

$$\begin{aligned}
&(\circ_{1 \le i \le n} M_i) \circ (\circ_{1 \le j \le n} N_j) \simeq (\circ_{1 \le i \le n-1} M_i) \circ M_n \circ N_n \circ (\circ_{1 \le j \le n-1} N_j) \\
&\twoheadrightarrow (\circ_{1 \le i \le n-1} M_i) \circ (M_n \nabla N_n) \circ (\circ_{1 \le j \le n-1} N_j) \\
&\simeq (\circ_{1 \le i \le n-1} M_i) \circ (\circ_{1 \le j \le n-1} N_j) \circ (M_n \nabla N_n) \\
&\twoheadrightarrow (\circ_{1 \le i \le n-1} (M_i \nabla N_i)) \circ (M_n \nabla N_n),
\end{aligned}$$

as desired. ☐

4. LECLERC'S CONJECTURE

In this section, R is assumed to be a symmetric KLR algebra over a base field $\mathbf{k}$.

4.1. **Leclerc's conjecture.**

The following theorem is a part of Leclerc's conjecture stated in the Introduction.

Theorem 4.1.1. *Let M and N be simple modules. We assume that M is real. Then we have the equalities in the Grothendieck group $K(R\text{-gmod})$ as follows:*

(i) $[M \circ N] = [M \nabla N] + \sum_k [S_k]$
with simple modules S_k such that $\Lambda(M, S_k) < \Lambda(M, M \nabla N) = \Lambda(M, N)$,
(ii) $[M \circ N] = [M \Delta N] + \sum_k [S_k]$
with simple modules S_k such that $\Lambda(S_k, M) < \Lambda(M \Delta N, M) = \Lambda(N, M)$,
(iii) $[N \circ M] = [N \nabla M] + \sum_k [S_k]$
with simple modules S_k such that $\Lambda(S_k, M) < \Lambda(N \nabla M, M) = \Lambda(N, M)$,
(iv) $[N \circ M] = [N \Delta M] + \sum_k [S_k]$
with simple modules S_k such that $\Lambda(M, S_k) < \Lambda(M, N \Delta M) = \Lambda(M, N)$.

In particular, $M \nabla N$ as well as $M \Delta N$ appears only once in the Jordan–Hölder series of $M \circ N$ in R-mod.

The following result is an immediate consequence of this theorem.

Corollary 4.1.2. *Let M and N be simple modules. We assume that one of them is real. Assume that M and N do not commute, Then we have the equality in the Grothendieck group $K(R\text{-gmod})$*

$$[M \circ N] = [M \nabla N] + [M \Delta N] + \sum_k [S_k]$$

with simple modules S_k. Moreover we have the following:

(i) *If M is real, then we have $\Lambda(M, M \Delta N) < \Lambda(M,N)$, $\Lambda(M \nabla N, M) < \Lambda(N,M)$ and $\Lambda(M, S_k) < \Lambda(M,N)$, $\Lambda(S_k, M) < \Lambda(N,M)$.*

(ii) *If N is real, then we have $\Lambda(N, M \nabla N) < \Lambda(N,M)$, $\Lambda(M \Delta N, N) < \Lambda(M,N)$ and $\Lambda(N, S_k) < \Lambda(N,M)$, $\Lambda(S_k, N) < \Lambda(M,N)$.*

Proof of Theorem 4.1.1. We shall prove only (i). The other statements are proved similarly.

$$M \circ N = K_0 \supset K_1 \supset \cdots \supset K_\ell \supset K_{\ell+1} = 0.$$

Then we have $K_0/K_1 \simeq M \nabla N$. Let us consider the renormalized R-matrix $R^{\mathrm{ren}}_{M_z, M\circ N} = (M \circ R^{\mathrm{ren}}_{M_z,N}) \circ (R^{\mathrm{ren}}_{M_z,M} \circ N)$

$$M_z \circ M \circ N \xrightarrow{R^{\mathrm{ren}}_{M_z,M} \circ N} M \circ M_z \circ N \xrightarrow{M \circ R^{\mathrm{ren}}_{M_z,N}} M \circ N \circ M_z.$$

Then $R^{\mathrm{ren}}_{M_z, M\circ N}$ sends $M_z \circ K_k$ to $K_k \circ M_z$ for any k. Hence evaluating the above diagram at $z = 0$, we obtain

$$\begin{tikzcd}
M \circ M \circ N \arrow[r, "M \circ \mathbf{r}_{M,N}"] & M \circ N \circ M \\
M \circ K_1 \arrow[u, hook] \arrow[r] & K_1 \circ M . \arrow[u, hook]
\end{tikzcd}$$

Since $\mathrm{Im}(\mathbf{r}_{M,N} \colon M \circ N \to N \circ M) \simeq (M \circ N)/K_1$, we have $\mathbf{r}_{M,N}(K_1) = 0$. Hence, $R^{\mathrm{ren}}_{M_z, M\circ N}$ sends $M_z \circ K_1$ to $(K_1 \circ M_z) \cap z\big((M \circ N) \circ M_z\big) = z(K_1 \circ M_z)$. Thus $z^{-1} R^{\mathrm{ren}}_{M_z, M\circ N}|_{M_z \circ K_1}$ is well defined. Then it sends $M_z \circ K_k$ to $K_k \circ M_z$ for $k \geq 1$. Thus we obtain an R-matrix

$$z^{-1} R^{\mathrm{ren}}_{M_z, M\circ N}|_{M_z \circ (K_k/K_{k+1})} \colon M_z \circ (K_k/K_{k+1}) \to (K_k/K_{k+1}) \circ M_z \quad \text{for } 1 \leq k \leq \ell.$$

Hence we have

$$R^{\mathrm{ren}}_{M_z, K_k/K_{k+1}} = z^{-s_k} z^{-1} R^{\mathrm{ren}}_{M_z, M\circ N}|_{M_z \circ (K_k/K_{k+1})}$$

for some $s_k \in \mathbb{Z}_{\geq 0}$. Since the homogeneous degree of $R^{\mathrm{ren}}_{M_z, M\circ N}$ is $\Lambda(M, M \circ N) = \Lambda(M,N)$, we obtain

$$\Lambda(M, K_k/K_{k+1}) = \Lambda(M,N) - 2(1 + s_k) < \Lambda(M,N).$$

□

Recall that the isomorphism classes of self-dual simple modules in R-gmod are parametrized by the crystal basis $B(\infty)$ [28]. The following theorem is an application of the above theorem.

Theorem 4.1.3. *Let* ϕ *be an element of the Grothendieck group* $K(R\text{-gmod})$ *given by*

$$\phi = \sum_{b \in B(\infty)} a_b [L_b],$$

where L_b *is the self-dual simple module corresponding to* $b \in B(\infty)$ *and* $a_b \in \mathbb{Z}[q^{\pm 1}]$. *Let* A *be a real simple module in* R-gmod. *Assume that we have an equality*

$$\phi[A] = q^l [A] \phi$$

in $K(R\text{-gmod})$ *for some* $l \in \mathbb{Z}$. *Then* A *commutes with* L_b *and*

$$l = \Lambda(A, L_b)$$

for every $b \in B(\infty)$ *such that* $a_b \neq 0$.

Proof. Note that we have

$$\phi[A] = \sum_b a_b [L_b \circ A] = \sum_b a_b ([L_b \nabla A] + \sum_k [S_{b,k}]) \quad \text{and}$$
$$q^l [A] \phi = q^l \sum_b a_b [A \circ L_b] = q^l \sum_b a_b (q^{\Lambda(L_b, A)} [L_b \nabla A] + \sum_k [S^{b,k}]),$$

for some simple modules $S_{b,k}$ and $S^{b,k}$ satisfying

$$\Lambda(S_{b,k}, A) < \Lambda(L_b, A) \quad \text{and} \quad \Lambda(S^{b,k}, A) < \Lambda(L_b, A)$$

by Theorem 4.1.1.

We may assume that $\{b \in B(\infty) \mid a_b \neq 0\} \neq \emptyset$. Set

$$t := \max \{\Lambda(L_b, A) \mid a_b \neq 0\}.$$

By taking the classes of self-dual simple modules S with $\Lambda(S, A) = t$ in the expansions of $\phi[A]$ and $q^l[A]\phi$, we obtain

$$\sum_{\Lambda(L_b, A) = t} a_b [L_b \nabla A] = \sum_{\Lambda(L_b, A) = t} q^l a_b q^{\Lambda(L_b, A)} [L_b \nabla A].$$

In particular, we have $t = -l$.

Set

$$t' := \max \{\Lambda(A, L_b) \mid a_b \neq 0\}.$$

Then, by a similar argument we have $t' = l$.

It follows that

$$0 = t + t' \geq \Lambda(L_b, A) + \Lambda(A, L_b) \geq 0$$

for every b such that $a_b \neq 0$. Hence A and L_b commute.

Since

$$\sum a_b \, q^{\Lambda(A, L_b)} [A \circ L_b] = \sum a_b [L_b \circ A] = \phi[A] = q^l [A] \phi = q^l \sum a_b [A \circ L_b],$$

we have

$$l = \Lambda(A, L_b)$$

for any b such that $a_b \neq 0$, as desired. □

Corollary 4.1.4. *Let M and N be simple modules. Assume that one of them is real. If $[M]$ and $[N]$ q-commute (i.e., $[M][N] = q^n[N][M]$ for some $n \in \mathbb{Z}$), then M and N commute. In particular, $M \circ N$ is simple.*

The following corollary is an immediate consequence of the corollary above and Theorem 2.1.4.

Corollary 4.1.5. *Assume that the generalized Cartan matrix A is symmetric and that $b_1, b_2 \in B(\infty)$ satisfy the following conditions*:

- (i) *one of $G^{\mathrm{up}}(b_1)^2$ and $G^{\mathrm{up}}(b_2)^2$ is a member of the upper global basis up to a power of q,*
- (ii) *$G^{\mathrm{up}}(b_1)$ and $G^{\mathrm{up}}(b_2)$ q-commute.*

Then their product $G^{\mathrm{up}}(b_1)G^{\mathrm{up}}(b_2)$ is a member of the upper global basis of $U_q^-(\mathfrak{g})$ up to a power of q.

4.2. **Geometric results.** The result of this subsection (Theorem 4.2.1) was explained to us by Peter McNamara. It will be used in the proof of the crucial result Theorem 10.3.1. *In this subsection, we assume further that the base field* $\mathbf{k}$ *is of characteristic* 0.

Theorem 4.2.1 ([34, Lemma 7.5])**.** *Assume that the base field $\mathbf{k}$ is of characteristic 0. Assume that $M \in R$-gmod has a head q^cH with a self-dual simple module H and $c \in \mathbb{Z}$. Then we have the equality in the Grothendieck group $K(R$-gmod$)$*

$$[M] = q^c[H] + \sum_k q^{c_k}[S_k]$$

with self-dual simple modules S_k and $c_k > c$.

By duality, we obtain the following corollary.

Corollary 4.2.2. *Assume that the base field $\mathbf{k}$ is a field of characteristic 0. Assume that $M \in R$-gmod has a socle q^cS with a self-dual simple module S and $c \in \mathbb{Z}$. Then we have the equality in $K(R$-gmod$)$*

$$[M] = q^c[S] + \sum_k q^{c_k}[S_k]$$

with self-dual simple modules S_k and $c_k < c$.

Applying this theorem to convolution products, we obtain the following corollary.

Corollary 4.2.3. *Assume that the base field $\mathbf{k}$ is of characteristic 0. Let M and N be simple modules. We assume that one of them is real. Then we have the equalities in $K(R$-gmod$)$ as follows:*

- (i) $[M \circ N] = [M \nabla N] + \sum_k q^{c_k}[S_k]$
  with self-dual simple modules S_k and
  $$c_k > -\widetilde{\Lambda}(M,N) = \big(-\Lambda(M,N) - (\mathrm{wt}(M), \mathrm{wt}(N))\big)/2.$$
- (ii) $[M \circ N] = [M \,\Delta\, N] + \sum_k q^{c_k}[S_k]$
  with self-dual simple modules S_k and $c_k < \big(\Lambda(N,M) - (\mathrm{wt}(N), \mathrm{wt}(M))\big)/2$.

Note that $q^{\widetilde{\Lambda}(M,N)} M \nabla N$ is self-dual by Lemma 3.1.4.

Theorem 4.1.1 and Theorem 4.2.1 solve affirmatively Conjecture 1 of Leclerc [29] in the symmetric generalized Cartan matrix case, as stated in the Introduction. More precisely, let R be a symmetric KLR algebra over a base field $\mathbf{k}$ of characteristic 0, and let M and N be simple modules over R. Assume further that M is real. Then by Theorem 4.1.1 $M \nabla N$ and $M \Delta N$ appear exactly once in a Jordan–Hölder series of $M \circ N$. Write $M \nabla N = q^m H$ and $M \Delta N = q^s S$ for some self-dual simple modules H, S, and $m, s \in \mathbb{Z}$. By Theorem 4.2.1, we have

$$[M \circ N] = q^m[H] + q^s[S] + \sum_k q^{c_k}[S_k],$$

where S_k are self-dual simple modules, and $m < c_k < s$ for all k. Collecting the terms, we obtain

$$[M \circ N] = q^m[H] + q^s[S] + \sum_{L \not\simeq H,S} \gamma^L_{M,N}(q)[L],$$

with

$$\gamma^L_{M,N}(q) \in q^{m+1}\mathbb{Z}[q] \cap q^{s-1}\mathbb{Z}[q^{-1}],$$

which proves Leclerc's first conjecture via Theorem 2.1.4.

We obtain the following result which is a generalization of Lemma 3.2.18 in the characteristic-zero case.

Corollary 4.2.4. *Assume that the base field* $\mathbf{k}$ *is of characteristic* 0. *Let* M *and* N *be simple modules. We assume that one of them is real. Write*

$$[M \circ N] = \sum_{k=1}^{n} q^{c_k}[S_k]$$

with self-dual simple modules S_k *and* $c_k \in \mathbb{Z}$. *Then we have*

$$\max\{c_k \mid 1 \le k \le n\} - \min\{c_k \mid 1 \le k \le n\} = \mathfrak{d}(M,N).$$

4.3. **Proof of Theorem 4.2.1.** Recall that the graded algebra $R(\beta)$ $(\beta \in \mathsf{Q}^+)$ is geometrically realized as follows [40]. There exist a reductive group G and a G-equivariant projective morphism $f\colon X \to Y$ from a smooth algebraic G-variety X to an affine G-variety Y defined over the complex number field $\mathbb{C}$ such that

$$R(\beta) \simeq \widetilde{\mathrm{End}}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(\mathrm{R}f_*(\mathbf{k}_X[\dim X])) \quad \text{as a graded } \mathbf{k}\text{-algebra}.$$

Here, $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)$ denotes the G- equivariant derived category of the G-variety Y with coefficient $\mathbf{k}$, and $\widetilde{\mathrm{End}}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(K) = \widetilde{\mathrm{Hom}}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(K,K)$ with

$$\widetilde{\mathrm{Hom}}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(K,K') := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(K,K'[n]).$$

We denote by $\mathbf{k}_X[\dim X]$ the direct sum of the constant sheaves on each connected component of X, all of which are shifted by their dimensions. By the decomposition theorem [1], we have a decomposition

$$\mathrm{R}f_*(\mathbf{k}_X[\dim X]) \simeq \bigoplus_{a\in J} E_a \otimes \mathcal{F}_a,$$

where $\{\mathcal{F}_a\}_{a\in J}$ is a finite family of simple perverse sheaves on Y and E_a is a non-zero finite-dimensional graded $\mathbf{k}$-vector space such that

$$H^k(E_a) \simeq H^{-k}(E_a) \quad \text{for any } k \in \mathbb{Z}. \tag{4.1}$$

The last fact (4.1) follows from the hard Lefschetz theorem [1].

Set $A_{a,b} = \widetilde{\mathrm{Hom}}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(\mathbf{k}_Y)}(\mathcal{F}_b, \mathcal{F}_a)$. Then we have the multiplication morphisms

$$A_{a,b} \otimes A_{b,c} \to A_{a,c}$$

so that

$$A := \bigoplus_{a,b\in J} A_{a,b}$$

has a structure of $\mathbb{Z}$-graded algebra such that

$$A_{\le 0} := \bigoplus_{n\le 0} A_n = A_0 \simeq \mathbf{k}^J.$$

Hence the family of the isomorphism classes of simple objects (up to a grading shift) in A-gmod is $\{\mathbf{k}_a\}_{a\in J}$. Here, $\mathbf{k}_a$ is the module obtained by the algebra homomorphism $A \to A_{\le 0} \simeq \mathbf{k}^J \to \mathbf{k}$, where the last arrow is the ath projection. Hence we have

$$K(A\text{-gmod}) \simeq \bigoplus_{a\in J} \mathbb{Z}[q^{\pm 1}][\mathbf{k}_a].$$

On the other hand, we have

$$R(\beta) \simeq \bigoplus_{a,b\in J} E_a \otimes A_{a,b} \otimes E_b^*.$$

Set

$$L := \bigoplus_{a,b\in J} E_a \otimes A_{a,b}.$$

Then, L is endowed with a natural structure of $(\bigoplus_{a,b\in J} E_a \otimes A_{a,b} \otimes E_b^*, A)$-bimodule. It is well known that the functor $M \mapsto L \otimes_A M$ gives a graded Morita-equivalence

$$\Phi\colon A\text{-gmod} \xrightarrow{\sim} R(\beta)\text{-gmod}.$$

Note that $\Phi(\mathbf{k}_a) \simeq E_a$ and $\{E_a\}_{a\in J}$ is the set of isomorphism classes of self-dual simple graded $R(\beta)$-modules by (4.1).

By the above observation, in order to prove the theorem, it is enough to show the corresponding statement for the graded ring A, which is obvious.

5. Quantum cluster algebras

In this section we recall the definition of skew-symmetric quantum cluster algebras following [3] and [11, Section 8].

5.1. **Quantum seeds.**

Fix a finite index set $J = J_{\rm ex} \sqcup J_{\rm fr}$ with the decomposition into the set $J_{\rm ex}$ of exchangeable indices and the set $J_{\rm fr}$ of frozen indices. Let $L = (\lambda_{ij})_{i,j\in J}$ be a skew-symmetric integer-valued $J \times J$-matrix.

Definition 5.1.1. *We define $\mathscr{P}(L)$ as the $\mathbb{Z}[q^{\pm 1/2}]$-algebra generated by a family of elements $\{X_i\}_{i\in J}$ with the defining relations*

$$X_i X_j = q^{\lambda_{ij}} X_j X_i \quad (i, j \in J).$$

We denote by $\mathscr{F}(L)$ the skew field of fractions of $\mathscr{P}(L)$.

For $\mathbf{a} = (a_i)_{i\in J} \in \mathbb{Z}^J$, we define the element $X^{\mathbf{a}}$ of $\mathscr{F}(L)$ as

$$X^{\mathbf{a}} := q^{1/2\sum_{i>j} a_i a_j \lambda_{ij}} \overrightarrow{\prod}_{i\in J} X_i^{a_i}.$$

Here we take a total order $<$ on the set J and $\overrightarrow{\prod}_{i\in J} X_i^{a_i} = X_{i_1}^{a_{i_1}} \cdots X_{i_r}^{a_{i_r}}$ where $J = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$. Note that $X^{\mathbf{a}}$ does not depend on the choice of a total order of J.

We have

$$X^{\mathbf{a}}X^{\mathbf{b}} = q^{1/2\sum_{i,j\in J} a_i b_j \lambda_{ij}} X^{\mathbf{a}+\mathbf{b}}. \tag{5.1}$$

If $\mathbf{a} \in \mathbb{Z}^J_{\geq 0}$, then $X^{\mathbf{a}}$ belongs to $\mathscr{P}(L)$.

It is well known that $\{X^{\mathbf{a}}\}_{\mathbf{a}\in\mathbb{Z}^J_{\geq 0}}$ is a basis of $\mathscr{P}(L)$ as a $\mathbb{Z}[q^{\pm 1/2}]$-module.

Let A be a $\mathbb{Z}[q^{\pm 1/2}]$-algebra. We say that a family $\{x_i\}_{i\in J}$ of elements of A is *L-commuting* if it satisfies $x_i x_j = q^{\lambda_{ij}} x_j x_i$ for any $i, j \in J$. In such a case we can define $x^{\mathbf{a}}$ for any $\mathbf{a} \in \mathbb{Z}^J_{\geq 0}$. We say that an L-commuting family $\{x_i\}_{i\in J}$ is *algebraically independent* if the algebra map $\mathscr{P}(L) \to A$ given by $X_i \mapsto x_i$ is injective.

Let $\widetilde{B} = (b_{ij})_{(i,j)\in J\times J_{\mathrm{ex}}}$ be an integer-valued $J \times J_{\mathrm{ex}}$-matrix. We assume that the *principal part* $B := (b_{ij})_{i,j\in J_{\mathrm{ex}}}$ of $\widetilde{B}$ is skew-symmetric.

To the matrix $\widetilde{B}$ we can associate the quiver $Q_{\widetilde{B}}$ without loops, 2-cycles, and arrows between frozen vertices such that its vertices are labeled by J and the arrows are given by

(5.2)

$$b_{ij} = (\text{the number of arrows from } i \text{ to } j) - (\text{the number of arrows from } j \text{ to } i).$$

Here we extend the $J \times J_{\mathrm{ex}}$-matrix $\widetilde{B}$ to the skew-symmetric $J \times J$-matrix $\widetilde{B}' = (b_{ij})_{i,j\in J}$ by setting $b_{ij} = 0$ for $i, j \in J_{\mathrm{fr}}$.

Conversely, whenever we have a quiver with vertices labeled by J and without loops, 2-cycles, and arrows between frozen vertices, we can associate a $J \times J_{\mathrm{ex}}$-matrix $\widetilde{B}$ by (5.2).

We say that the pair $(L, \widetilde{B})$ is *compatible* if there exists a positive integer d such that

$$\sum_{k\in J} \lambda_{ik} b_{kj} = \delta_{ij} d \quad (i \in J,\ j \in J_{\mathrm{ex}}). \tag{5.3}$$

Let $(L, \widetilde{B})$ be a compatible pair and A a $\mathbb{Z}[q^{\pm 1/2}]$-algebra. We say that $\mathscr{S} = (\{x_i\}_{i\in J}, L, \widetilde{B})$ is a *quantum seed* in A if $\{x_i\}_{i\in J}$ is an algebraically independent L-commuting family of elements of A.

The set $\{x_i\}_{i\in J}$ is called the *cluster* of $\mathscr{S}$ and its elements the *cluster variables.* The cluster variables x_i $(i \in J_{\mathrm{fr}})$ are called the *frozen variables.* The elements $x^{\mathbf{a}}$ $(\mathbf{a} \in \mathbb{Z}^J_{\geq 0})$ are called the *quantum cluster monomials.*

5.2. **Mutation.**

For $k \in J_{\mathrm{ex}}$, we define a $J \times J$-matrix $E = (e_{ij})_{i,j\in J}$ and a $J_{\mathrm{ex}} \times J_{\mathrm{ex}}$-matrix $F = (f_{ij})_{i,j\in J_{\mathrm{ex}}}$ as follows:

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, -b_{ik}) & \text{if } i \neq j = k, \end{cases} \qquad f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } i = j = k, \\ \max(0, b_{kj}) & \text{if } i = k \neq j. \end{cases}$$

The *mutation* $\mu_k(L, \widetilde{B}) := (\mu_k(L), \mu_k(\widetilde{B}))$ *of a compatible pair* $(L, \widetilde{B})$ *in direction* k is given by

$$\mu_k(L) := (E^T)\, L\, E, \qquad \mu_k(\widetilde{B}) := E\, \widetilde{B}\, F.$$

Then the pair $(\mu_k(L), \mu_k(\widetilde{B}))$ is also compatible with the same integer d as in the case of $(L, \widetilde{B})$ [3].

Note that for each $k \in J_{\mathrm{ex}}$, we have

$$\mu_k(\widetilde{B})_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + (-1)^{\delta(b_{ik}<0)}\max(b_{ik}b_{kj}, 0) & \text{otherwise,} \end{cases} \tag{5.4}$$

and

$$\mu_k(L)_{ij} = \begin{cases} 0 & \text{if } i = j \\ -\lambda_{kj} + \sum_{t\in J} \max(0, -b_{tk})\lambda_{tj} & \text{if } i = k,\ j \neq k, \\ -\lambda_{ik} + \sum_{t\in J} \max(0, -b_{tk})\lambda_{it} & \text{if } i \neq k,\ j = k, \\ \lambda_{ij} & \text{otherwise.} \end{cases}$$

Note also that we have

$$\sum_{t\in J} \max(0, -b_{tk})\lambda_{it} = \sum_{t\in J} \max(0, b_{tk})\lambda_{it}$$

for $i \in J$ with $i \neq k$, since $(L, \widetilde{B})$ is compatible.

We define

$$a'_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, b_{ik}) & \text{if } i \neq k, \end{cases} \qquad a''_i = \begin{cases} -1 & \text{if } i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq k, \end{cases} \tag{5.5}$$

and set $\mathbf{a}' := (a'_i)_{i\in J}$ and $\mathbf{a}'' := (a''_i)_{i\in J}$.

Let A be a $\mathbb{Z}[q^{\pm 1/2}]$-algebra contained in a skew field K. Let $\mathscr{S} = (\{x_i\}_{i\in J}, L, \widetilde{B})$ be a quantum seed in A. Define the elements $\mu_k(x)_i$ of K by

$$\mu_k(x)_i := \begin{cases} x^{\mathbf{a}'} + x^{\mathbf{a}''}, & \text{if } i = k, \\ x_i & \text{if } i \neq k. \end{cases} \tag{5.6}$$

Then $\{\mu_k(x)_i\}$ is an algebraically independent $\mu_k(L)$-commuting family in K. We call

$$\mu_k(\mathscr{S}) := \big(\{\mu_k(x)_i\}_{i\in J}, \mu_k(L), \mu_k(\widetilde{B})\big)$$

the *mutation of $\mathscr{S}$ in direction k*. It becomes a new quantum seed in K.

Definition 5.2.1. *Let $\mathscr{S} = (\{x_i\}_{i\in J}, L, \widetilde{B})$ be a quantum seed in A. The* quantum cluster algebra *$\mathscr{A}_{q^{1/2}}(\mathscr{S})$ associated to the quantum seed $\mathscr{S}$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of the skew field K generated by all the quantum cluster variables in the quantum seeds obtained from $\mathscr{S}$ by any sequence of mutations.*

We call $\mathscr{S}$ the *initial quantum seed* of the quantum cluster algebra $\mathscr{A}_{q^{1/2}}(\mathscr{S})$.

6. MONOIDAL CATEGORIFICATION OF CLUSTER ALGEBRAS

Throughout this section, fix $J = J_{\mathrm{ex}} \sqcup J_{\mathrm{fr}}$ and a base field $\mathbf{k}$.

Let $\mathcal{C}$ be a $\mathbf{k}$-linear abelian monoidal category. For the definition of monoidal category, see, for example, [14, Appendix A.1]. Note that in [14], it was called the *tensor category.* A $\mathbf{k}$-linear abelian monoidal category is a $\mathbf{k}$-linear monoidal category such that it is abelian and the tensor functor $\otimes$ is $\mathbf{k}$-bilinear and exact.

We assume further the following conditions on $\mathcal{C}$:

$$(6.1) \qquad \begin{cases} \text{(i) Any object of } \mathcal{C} \text{ has a finite length,} \\ \text{(ii) } \mathbf{k} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(S,S) \text{ for any simple object } S \text{ of } \mathcal{C}. \end{cases}$$

A simple object M in $\mathcal{C}$ is called *real* if $M \otimes M$ is simple.

6.1. **Ungraded cases.**

Definition 6.1.1. *Let $\mathscr{S} = (\{M_i\}_{i\in J}, \widetilde{B})$ be a pair of a family $\{M_i\}_{i\in J}$ of simple objects in $\mathcal{C}$ and an integer-valued $J \times J_{\mathrm{ex}}$-matrix $\widetilde{B} = (b_{ij})_{(i,j)\in J\times J_{\mathrm{ex}}}$ whose principal part is skew-symmetric. We call $\mathscr{S}$ a* monoidal seed in $\mathcal{C}$ *if*

(i) *$M_i \otimes M_j \simeq M_j \otimes M_i$ for any $i, j \in J$,*

(ii) *$\bigotimes_{i\in J} M_i^{\otimes a_i}$ is simple for any $(a_i)_{i\in J} \in \mathbb{Z}_{\geq 0}^J$.*

Definition 6.1.2. *For $k \in J_{\mathrm{ex}}$, we say that a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, \widetilde{B})$* admits a mutation in direction k *if there exists a simple object $M_k' \in \mathcal{C}$ such that*

(i) *there exist exact sequences in $\mathcal{C}$,*

$$0 \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to M_k \otimes M_k' \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to 0,$$
$$0 \to \bigotimes_{b_{ik}<0} M_i^{\otimes(-b_{ik})} \to M_k' \otimes M_k \to \bigotimes_{b_{ik}>0} M_i^{\otimes b_{ik}} \to 0;$$

(ii) *the pair $\mu_k(\mathscr{S}) := (\{M_i\}_{i\neq k} \cup \{M_k'\}, \mu_k(\widetilde{B}))$ is a monoidal seed in $\mathcal{C}$.*

Recall that a cluster algebra A with an initial seed $(\{x_i\}_{i\in J}, \widetilde{B})$ is the $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_i | i \in J)$ generated by all the cluster variables in the seeds obtained from $(\{x_i\}_{i\in J}, \widetilde{B})$ by any sequence of mutations. Here, the mutation x_k' of a cluster variable x_k ($k \in J_{\mathrm{ex}}$) is given by

$$x_k' = \frac{\prod_{b_{ik}\geq 0} x_i^{b_{ik}} + \prod_{b_{ik}\leq 0} x_i^{-b_{ik}}}{x_k},$$

and the mutation of $\widetilde{B}$ is given in (5.4).

Definition 6.1.3. *A $\mathbf{k}$-linear abelian monoidal category $\mathcal{C}$ satisfying* (6.1) *is called a* monoidal categorification of a cluster algebra A if

(i) the Grothendieck ring $K(\mathcal{C})$ is isomorphic to A,

(ii) there exists a monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, \widetilde{B})$ in $\mathcal{C}$ such that $[\mathscr{S}] := (\{[M_i]\}_{i\in J}, \widetilde{B})$ is the initial seed of A and $\mathscr{S}$ admits successive mutations in all directions.

Note that if $\mathcal{C}$ is a monoidal categorification of A, then every seed in A is of the form $(\{[M_i]\}_{i\in J}, \widetilde{B})$ for some monoidal seed $(\{M_i\}_{i\in J}, \widetilde{B})$ in $\mathcal{C}$. In particular, all the cluster monomials in A are the classes of real simple objects in $\mathcal{C}$.

6.2. **Graded cases.** Let Q be a free abelian group equipped with a symmetric bilinear form

$$(\ ,\ ) : \mathsf{Q} \times \mathsf{Q} \to \mathbb{Z} \quad \text{such that } (\beta, \beta) \in 2\mathbb{Z} \text{ for all } \beta \in \mathsf{Q}.$$

We consider a **k**-linear abelian monoidal category $\mathcal{C}$ satisfying (6.1) and the following conditions:

(6.2)

- (i) We have a direct sum decomposition $\mathcal{C} = \bigoplus_{\beta \in \mathsf{Q}} \mathcal{C}_\beta$ such that the tensor product $\otimes$ sends $\mathcal{C}_\beta \times \mathcal{C}_\gamma$ to $\mathcal{C}_{\beta+\gamma}$ for every $\beta, \gamma \in \mathsf{Q}$.
- (ii) There exists an object $Q \in \mathcal{C}_0$ satisfying
  - (a) there is an isomorphism
  $$R_Q(X) : Q \otimes X \xrightarrow{\sim} X \otimes Q$$
  functorial in $X \in \mathcal{C}$ such that
  $$\begin{tikzcd} Q \otimes X \otimes Y \arrow[r, "R_Q(X)"'] \arrow[rr, bend left, "R_Q(X \otimes Y)"] & X \otimes Q \otimes Y \arrow[r, "R_Q(Y)"'] & X \otimes Y \otimes Q \end{tikzcd}$$
  commutes for any $X, Y \in \mathcal{C}$;
  - (b) the functor $X \mapsto Q \otimes X$ is an equivalence of categories.
- (iii) for any M, $N \in \mathcal{C}$, we have $\mathrm{Hom}_{\mathcal{C}}(M, Q^{\otimes n} \otimes N) = 0$ except finitely many integers n.

We denote by q the auto-equivalence $Q \otimes \bullet$ of $\mathcal{C}$, and call it the *grading shift functor*.

In such a case the Grothendieck group $K(\mathcal{C})$ is a Q-graded $\mathbb{Z}[q^{\pm 1}]$-algebra: $K(\mathcal{C}) = \bigoplus_{\beta \in \mathsf{Q}} K(\mathcal{C})_\beta$ where $K(\mathcal{C})_\beta = K(\mathcal{C}_\beta)$. Moreover, we have

$$K(\mathcal{C}) = \bigoplus_{S} \mathbb{Z}[q^{\pm 1}][S],$$

where S ranges over equivalence classes of simple modules. Here, two simple modules S and S' are equivalent if $q^n S \simeq S'$ for some $n \in \mathbb{Z}$.

For $M \in \mathcal{C}_\beta$, we sometimes write $\beta = \mathrm{wt}(M)$ and call it the *weight* of M. Similarly, for $x \in \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C}_\beta)$, we write $\beta = \mathrm{wt}(x)$ and call it the *weight* of x.

Definition 6.2.1. *We call a quadruple* $\mathscr{S} = (\{M_i\}_{i \in J}, L, \widetilde{B}, D)$ *a* quantum monoidal seed in $\mathcal{C}$ *if it satisfies the following conditions*:

- (i) $\widetilde{B} = (b_{ij})_{i \in J, j \in J_{\mathrm{ex}}}$ *is an integer-valued* $J \times J_{\mathrm{ex}}$*-matrix whose principal part is skew-symmetric,*
- (ii) $L = (\lambda_{ij})_{i,j \in J}$ *is an integer-valued skew-symmetric* $J \times J$*-matrix,*
- (iii) $D = \{d_i\}_{i \in J}$ *is a family of elements in* Q*,*
- (iv) $\{M_i\}_{i \in J}$ *is a family of simple objects such that* $M_i \in \mathcal{C}_{d_i}$ *for any* $i \in J$*,*
- (v) $M_i \otimes M_j \simeq q^{\lambda_{ij}} M_j \otimes M_i$ *for all* $i, j \in J$*,*
- (vi) $M_{i_1} \otimes \cdots \otimes M_{i_t}$ *is simple for any sequence* $(i_1, \ldots, i_t)$ *in* J*,*
- (vii) *The pair* $(L, \widetilde{B})$ *is compatible in the sense of* (5.3) *with* $d = 2$*,*
- (viii) $\lambda_{ij} - (d_i, d_j) \in 2\mathbb{Z}$ *for all* $i, j \in J$*,*
- (ix) $\sum_{i \in J} b_{ik} d_i = 0$ *for all* $k \in J_{\mathrm{ex}}$.

Let $\mathscr{S}=(\{M_i\}_{i\in J},L,\widetilde{B},D)$ be a quantum monoidal seed. For any $X\in\mathcal{C}_\beta$ and $Y\in\mathcal{C}_\gamma$ such that $X\otimes Y\simeq q^cY\otimes X$ and $c+(\beta,\gamma)\in 2\mathbb{Z}$, we set

$$\widetilde{\Lambda}(X,Y)=\frac{1}{2}\big(-c+(\beta,\gamma)\big)\in\mathbb{Z}$$

and

$$X\odot Y:=q^{\widetilde{\Lambda}(X,Y)}X\otimes Y\simeq q^{\widetilde{\Lambda}(Y,X)}Y\otimes X.$$

Then $X\odot Y\simeq Y\odot X$. For any sequence $(i_1,\dots,i_\ell)$ in J, we define

$$\bigodot_{s=1}^{\ell}M_{i_s}:=(\cdots((M_{i_1}\odot M_{i_2})\odot M_{i_3})\cdots)\odot M_{i_\ell}.$$

Then we have

$$\bigodot_{s=1}^{\ell}M_{i_s}=q^{\frac{1}{2}\sum_{1\le u<v\le\ell}(-\lambda_{i_ui_v}+(d_{i_u},d_{i_v}))}M_{i_1}\otimes\cdots\otimes M_{i_\ell}.$$

For any $w\in\mathfrak{S}_\ell$, we have

$$\bigodot_{s=1}^{\ell}M_{i_{w(s)}}\simeq\bigodot_{s=1}^{\ell}M_{i_s}.$$

Hence for any subset A of J and any set of non-negative integers $\{m_a\}_{a\in A}$, we can define $\bigodot_{a\in A}M_a^{\odot m_a}$.

For $(a_i)_{i\in J}\in\mathbb{Z}_{\ge0}^J$ and $(b_i)_{i\in J}\in\mathbb{Z}_{\ge0}^J$, we have

$$\big(\bigodot_{i\in J}M_i^{\odot a_i}\big)\odot\big(\bigodot_{i\in J}M_i^{\odot b_i}\big)\simeq\bigodot_{i\in J}M_i^{\odot(a_i+b_i)}.$$

Let $\mathscr{S}=(\{M_i\}_{i\in J},L,\widetilde{B},D)$ be a quantum monoidal seed. When the L-commuting family $\{[M_i]\}_{i\in J}$ of elements of $\mathbb{Z}[q^{\pm1/2}]\otimes_{\mathbb{Z}[q^{\pm1}]}K(\mathcal{C})$ is algebraically independent, we shall define a quantum seed $[\mathscr{S}]$ in $\mathbb{Z}[q^{\pm1/2}]\otimes_{\mathbb{Z}[q^{\pm1}]}K(\mathcal{C})$ by

$$[\mathscr{S}]=(\{q^{-(d_i,d_i)/4}[M_i]\}_{i\in J},L,\widetilde{B}).$$

Set

$$X_i:=q^{-(d_i,d_i)/4}[M_i].$$

Then for any $\mathbf{a}=(a_i)_{i\in J}\in\mathbb{Z}_{\ge0}^J$, we have

$$X^{\mathbf{a}}=q^{-(\mu,\mu)/4}[\bigodot_{i\in J}M_i^{\odot a_i}],$$

where $\mu=\mathrm{wt}(\bigodot_{i\in J}M_i^{\odot a_i})=\mathrm{wt}(X^{\mathbf{a}})=\sum_{i\in J}a_id_i$.

For a given $k\in J_{\mathrm{ex}}$, we define the *mutation* $\mu_k(D)\in\mathsf{Q}^J$ *of* D *in direction* k *with respect to* $\widetilde{B}$ by

$$\mu_k(D)_i=d_i\ (i\ne k),\quad \mu_k(D)_k=-d_k+\sum_{b_{ik}>0}b_{ik}d_i.$$

Note that

$$\mu_k(\mu_k(D))=D.$$

Note also that $(\mu_k(L),\mu_k(\widetilde{B}),\mu_k(D))$ satisfies conditions (viii) and (ix) in Definition 6.2.1 for any $k\in J_{\mathrm{ex}}$.

We have the following lemma.

Lemma 6.2.2. *Set $X'_k = \mu_k(X)_k$, the mutation of X_k as in* (5.6). *Set $\zeta = \mathrm{wt}(X'_k) = -d_k + \sum_{b_{ik}>0} b_{ik}d_i$. Then we have*

$$q^{m_k}[M_k]q^{(\zeta,\zeta)/4}X'_k = q[\bigodot_{b_{ik}>0} M_i^{\odot b_{ik}}] + [\bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}],$$
$$q^{m'_k}q^{(\zeta,\zeta)/4}X'_k[M_k] = [\bigodot_{b_{ik}>0} M_i^{\odot b_{ik}}] + q[\bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}],$$

where

$$\begin{cases} m_k = \dfrac{1}{2}(d_k,\zeta) + \dfrac{1}{2}\displaystyle\sum_{b_{ik}<0} \lambda_{ki}b_{ik}, \\ m'_k = \dfrac{1}{2}(d_k,\zeta) + \dfrac{1}{2}\displaystyle\sum_{b_{ik}>0} \lambda_{ki}b_{ik}. \end{cases} \tag{6.3}$$

Proof. By (5.1), we have

$$X_kX^{\mathbf{a}} = q^{\frac{1}{2}\sum_{i\in J} a_i\lambda_{ki}}X^{\mathbf{e}_k+\mathbf{a}} \quad \text{for } \mathbf{a} = (a_i)_{i\in J} \in \mathbb{Z}^J \text{ and } (\mathbf{e}_k)_i = \delta_{ik}\ (i\in J).$$

Let $\mathbf{a}'$ and $\mathbf{a}''$ be as in (5.5). Because

$$\sum_{i\in J} a'_i\lambda_{ki} - \sum_{i\in J} a''_i\lambda_{ki} = \sum_{b_{ik}>0} b_{ik}\lambda_{ki} - \sum_{b_{ik}<0}(-b_{ik})\lambda_{ki} = \sum_{i\in J} b_{ik}\lambda_{ki} = 2,$$

we have

$$X_kX'_k = X_k(X^{\mathbf{a}'} + X^{\mathbf{a}''}) = q^{\frac{1}{2}\sum_i a''_i\lambda_{ki}}(qX^{\mathbf{e_k}+\mathbf{a}'} + X^{\mathbf{e_k}+\mathbf{a}''}).$$

Note that $\mathrm{wt}(X^{\mathbf{e_k}+\mathbf{a}'}) = \mathrm{wt}(X^{\mathbf{e_k}+\mathbf{a}''}) = d_k + \zeta$. It follows that

$$\begin{aligned} m_k &= -\frac{1}{4}((d_k,d_k)+(\zeta,\zeta)) - \frac{1}{2}\sum_{i\in J} a''_i\lambda_{ki} + \frac{1}{4}(\zeta+d_k,\zeta+d_k) \\ &= \frac{1}{2}(d_k,\zeta) + \frac{1}{2}\sum_{b_{ik}<0} b_{ik}\lambda_{ki}. \end{aligned}$$

One can calculate m'_k in a similar way. ☐

Definition 6.2.3. *We say that a quantum monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, L, \widetilde{B}, D)$* admits a mutation in direction $k \in J_{\mathrm{ex}}$ *if there exists a simple object $M'_k \in \mathcal{C}_{\mu_k(D)_k}$ such that*

(i) *there exist exact sequences in $\mathcal{C}$,*

$$0 \to q\bigodot_{b_{ik}>0} M_i^{\odot b_{ik}} \to q^{m_k}M_k \otimes M'_k \to \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})} \to 0, \tag{6.4}$$

$$0 \to q\bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})} \to q^{m'_k}M'_k \otimes M_k \to \bigodot_{b_{ik}>0} M_i^{\odot b_{ik}} \to 0, \tag{6.5}$$

where m_k and m'_k are as in (6.3).

(ii) *$\mu_k(\mathscr{S}) := \big(\{M_i\}_{i\neq k} \sqcup \{M'_k\}, \mu_k(L), \mu_k(\widetilde{B}), \mu_k(D)\big)$ is a quantum monoidal seed in $\mathcal{C}$.*

We call $\mu_k(\mathscr{S})$ the mutation *of $\mathscr{S}$ in direction k.*

By Lemma 6.2.2, the following lemma is obvious.

Lemma 6.2.4. *Let $\mathscr{S} = (\{M_i\}_{i\in J}, L, \widetilde{B}, D)$ be a quantum monoidal seed which admits a mutation in direction $k \in J_{\mathrm{ex}}$. Then we have*

$$[\mu_k(\mathscr{S})] = \mu_k([\mathscr{S}]).$$

Definition 6.2.5. *Assume that a* $\mathbf{k}$*-linear abelian monoidal category* $\mathcal{C}$ *satisfies* (6.1) *and* (6.2). *The category* $\mathcal{C}$ *is called a* monoidal categorification of a quantum cluster algebra A over $\mathbb{Z}[q^{\pm 1/2}]$ if

(i) the Grothendieck ring $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ is isomorphic to A,
(ii) there exists a quantum monoidal seed $\mathscr{S} = (\{M_i\}_{i \in J}, L, \widetilde{B}, D)$ in $\mathcal{C}$ such that $[\mathscr{S}] := (\{q^{-(d_i,d_i)/4}[M_i]\}_{i\in J}, L, \widetilde{B})$ is a quantum seed of A,
(iii) $\mathscr{S}$ admits successive mutations in all the directions.

Note that if $\mathcal{C}$ is a monoidal categorification of a quantum cluster algebra A, then any quantum seed in A obtained by a sequence of mutations from the initial quantum seed is of the form $(\{q^{-(d_i,d_i)/4}[M_i]\}_{i\in J}, L, \widetilde{B})$ for some quantum monoidal seed $(\{M_i\}_{i\in J}, L, \widetilde{B}, D)$. In particular, all the quantum cluster monomials in A are the classes of real simple objects in $\mathcal{C}$ up to a power of $q^{1/2}$.

7. Monoidal categorification via modules over KLR algebras

7.1. **Admissible pair.**

In this section, we assume that *R is a symmetric KLR algebra over a base field* $\mathbf{k}$.

From now on, we focus on the case when $\mathcal{C}$ is a full subcategory of R-gmod stable under taking convolution products, subquotients, extensions, and grading shift. In particular, we have

$$\mathcal{C} = \bigoplus_{\beta \in \mathsf{Q}^-} \mathcal{C}_\beta, \quad \text{where } \mathcal{C}_\beta := \mathcal{C} \cap R(-\beta)\text{-gmod},$$

and we have the grading shift functor q on $\mathcal{C}$. Hence we have

$$K(\mathcal{C}_\beta) \subset U_q^-(\mathfrak{g})_\beta,$$

and $K(\mathcal{C})$ has a $\mathbb{Z}[q^{\pm 1}]$-basis consisting of the isomorphism classes of self-dual simple modules.

Definition 7.1.1. *A pair* $(\{M_i\}_{i\in J}, \widetilde{B})$ *is called* admissible *if*

(i) *$\{M_i\}_{i\in J}$ is a family of real simple self-dual objects of $\mathcal{C}$ which commute with each other,*
(ii) *$\widetilde{B}$ is an integer-valued $J \times J_{\rm ex}$-matrix with a skew-symmetric principal part,*
(iii) *for each $k \in J_{\rm ex}$, there exists a self-dual simple object M'_k of $\mathcal{C}$ such that there is an exact sequence in $\mathcal{C}$*

$$0 \to q \bigodot_{b_{ik}>0} M_i^{\odot b_{ik}} \to q^{\widetilde{\Lambda}(M_k, M'_k)} M_k \circ M'_k \to \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})} \to 0, \tag{7.1}$$

and M'_k commutes with M_i for any $i \neq k$.

Note that M'_k is uniquely determined by k and $(\{M_i\}_{i\in J}, \widetilde{B})$. Indeed, it follows from $q^{\widetilde{\Lambda}(M_k,M'_k)} M_k \nabla M'_k \simeq \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}$ and [15, Corollary 3.7]. Note also that if there is an epimorphism $q^m M_k \circ M'_k \twoheadrightarrow \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}$ for some $m \in \mathbb{Z}$, then m should coincide with $\widetilde{\Lambda}(M_k, M'_k)$ by Lemma 3.1.4 and Lemma 3.2.7.

For an admissible pair $(\{M_i\}_{i\in J}, \widetilde{B})$, let $\Lambda = (\Lambda_{ij})_{i,j\in J}$ be the skew-symmetric matrix given by $\Lambda_{ij} = \Lambda(M_i, M_j)$. and let $D = \{d_i\}_{i\in J}$ be the family of elements of Q^- given by $d_i = \mathrm{wt}(M_i)$.

Now we can simplify the conditions in Definition 6.2.1 and Definition 6.2.3 as follows.

Proposition 7.1.2. *Let* $(\{M_i\}_{i\in J}, \widetilde{B})$ *be an admissible pair in* $\mathcal{C}$*, and let* M'_k $(k \in J_{\mathrm{ex}})$ *be as in* Definition 7.1.1. *Then we have the following properties:*

(a) *The quadruple* $\mathscr{S} := (\{M_i\}_{i\in J}, -\Lambda, \widetilde{B}, D)$ *is a quantum monoidal seed in* $\mathcal{C}$.
(b) *The self-dual simple object* M'_k *is real for every* $k \in J_{\mathrm{ex}}$.
(c) *The quantum monoidal seed* $\mathscr{S}$ *admits a mutation in each direction* $k \in J_{\mathrm{ex}}$.
(d) M_k *and* M'_k *are simply linked for any* $k \in J_{\mathrm{ex}}$ *(i.e.,* $\mathfrak{d}(M_k, M'_k) = 1$*).*
(e) *For any* $j \in J$ *and* $k \in J_{\mathrm{ex}}$*, we have*

$$\begin{aligned}\Lambda(M_j, M'_k) &= -\Lambda(M_j, M_k) - \textstyle\sum_{b_{ik}<0} \Lambda(M_j, M_i)b_{ik},\\ \Lambda(M'_k, M_j) &= -\Lambda(M_k, M_j) + \textstyle\sum_{b_{ik}>0} \Lambda(M_i, M_j)b_{ik}.\end{aligned} \tag{7.2}$$

Proof. Item (d) follows from the exact sequence (7.1) and Lemma 3.2.18.

Item (b) follows from the exact sequence (7.1) by applying Corollary 3.2.21 to the case

$$M = M_k,\ N = M'_k,\ X = q \bigodot_{b_{ik}>0} M_i^{\odot b_{ik}},\ \text{and } Y = \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}.$$

Item (e) follows from

$$\begin{aligned}\Lambda(M_j, M_k) + \Lambda(M_j, M'_k) &= \Lambda(M_j, M_k \nabla M'_k) = \Lambda\big(M_j, \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})}\big)\\ &= \sum_{b_{ik}<0} \Lambda(M_j, M_i)(-b_{ik})\end{aligned}$$

and

$$\begin{aligned}\Lambda(M_k, M_j) + \Lambda(M'_k, M_j) &= \Lambda(M'_k \nabla M_k, M_j) = \Lambda\big(\bigodot_{b_{ik}>0} M_i^{\odot b_{ik}}, M_j\big)\\ &= \sum_{b_{ik}>0} \Lambda(M_i, M_j)b_{ik}.\end{aligned}$$

Let us show (a). The conditions (i)–(v) in Definition 6.2.1 are satisfied by the construction. The condition (vi) follows from Proposition 3.2.5 and the fact that M_i is real simple for every $i \in J$. The condition (viii) is nothing but Lemma 3.1.2. The condition (ix) follows easily from the fact that the weights of the first and the last terms in the exact sequence (7.1) coincide.

Let us show the condition (vii) in Definition 6.2.1. By (7.2) and (d) of this proposition, we have

$$\begin{aligned}2\delta_{jk} = 2\,\mathfrak{d}(M_j, M'_k) &= -2\,\mathfrak{d}(M_j, M_k) - \sum_{b_{ik}<0} \Lambda(M_j, M_i)b_{ik} + \sum_{b_{ik}>0} \Lambda(M_i, M_j)b_{ik}\\ &= -\sum_{b_{ik}<0} \Lambda(M_j, M_i)b_{ik} - \sum_{b_{ik}>0} \Lambda(M_j, M_i)b_{ik} = -\sum_{i\in J} \Lambda(M_j, M_i)b_{ik}\end{aligned}$$

for $k \in J_{\mathrm{ex}}$ and $j \in J$. Thus we have shown that $\mathscr{S}$ is a quantum monoidal seed in $\mathcal{C}$.

Let us show (c). Let $k \in J_{\mathrm{ex}}$. The exact sequence (6.4) follows from (7.1) and the equality

$$\widetilde{\Lambda}(M_k, M_k') = \frac{1}{2}\big((\mathrm{wt}(M_k, M_k') - \sum_{b_{ik}<0} \Lambda(M_k, M_i) b_{ik}\big) = m_k, \tag{7.3}$$

which is an immediate consequence of (7.2).

Similarly, taking the dual of the exact sequence (7.1), we obtain an exact sequence

$$0 \to \bigodot_{b_{ik}<0} M_i^{\odot(-b_{ik})} \to q^{-\widetilde{\Lambda}(M_k, M_k') + (\mathrm{wt}\, M_k, \mathrm{wt}\, M_k')} M_k' \circ M_k \to q^{-1} \bigodot_{b_{ik}>0} M_i^{\odot b_{ik}} \to 0,$$

which gives the exact sequence (6.5).

It remains to prove that $\mu_k(\mathscr{S}) := (\{M_i\}_{i \neq k} \cup \{M_k'\}, \mu_k(-\Lambda), \mu_k(\widetilde{B}), \mu_k(D))$ is a quantum monoidal seed in $\mathcal{C}$ for any $k \in J_{\mathrm{ex}}$.

We see easily that $\mu_k(\mathscr{S})$ satisfies the conditions (i)–(iv) and (vii)–(ix) in Definition 6.2.1.

For the condition (v), it is enough to show that for $i, j \in J$ we have

$$\mu_k(-\Lambda)_{ij} = -\Lambda(\mu_k(M)_i, \mu_k(M)_j),$$

where $\mu_k(M)_i = M_i$ for $i \neq k$ and $\mu_k(M)_k = M_k'$. In the case $i \neq k$ and $j \neq k$, we have

$$\mu_k(-\Lambda)_{ij} = -\Lambda(M_i, M_j) = -\Lambda(\mu_k(M_i), \mu_k(M_j)).$$

The other cases follow from (7.2).

The condition (vi) in Definition 6.2.1 for $\mu_k(\mathscr{S})$ follows from Proposition 3.2.5 and the fact that $\{\mu_k(M)_i\}_{i \in J}$ is a commuting family of real simple modules. ☐

Now we are ready to give one of our main theorems.

Theorem 7.1.3. *Let* $(\{M_i\}_{i \in J}, \widetilde{B})$ *be an admissible pair in* $\mathcal{C}$ *and set*

$$\mathscr{S} = (\{M_i\}_{i \in J}, -\Lambda, \widetilde{B}, D)$$

as in Proposition 7.1.2. *We set* $[\mathscr{S}] := (\{q^{-\frac{1}{4}(\mathrm{wt}(M_i), \mathrm{wt}(M_i))}[M_i]\}_{i \in J}, -\Lambda, \widetilde{B}, D)$. *We assume further that*

(7.4) *The* $\mathbb{Q}(q^{1/2})$*-algebra* $\mathbb{Q}(q^{1/2}) \underset{\mathbb{Z}[q^{\pm 1}]}{\otimes} K(\mathcal{C})$ *is isomorphic to* $\mathbb{Q}(q^{1/2}) \underset{\mathbb{Z}[q^{\pm 1}]}{\otimes} \mathscr{A}_{q^{1/2}}([\mathscr{S}])$.

Then, for each $x \in J_{\mathrm{ex}}$, *the pair* $\big(\{\mu_x(M)_i\}_{i \in J}, \mu_x(\widetilde{B})\big)$ *is admissible in* $\mathcal{C}$.

Proof. In Proposition 7.1.2 (b), we have already shown that the condition (i) in Definition 7.1.1 holds for $(\{\mu_x(M)_i\}_{i \in J}, \mu_x(\widetilde{B}))$. The condition (ii) is clear from the definition. Let us show (iii). Set $N_i := \mu_x(M)_i$ and $b'_{ij} := \mu_x(\widetilde{B})_{ij}$ for $i \in J$ and $j \in J_{\mathrm{ex}}$. It is enough to show that, for any $y \in J_{\mathrm{ex}}$, there exists a self-dual simple module $M_y'' \in \mathcal{C}$ such that there is a short exact sequence

(7.5)

$$0 \longrightarrow q \bigodot_{b'_{iy}>0} N_i^{\odot b'_{iy}} \longrightarrow q^{\widetilde{\Lambda}(N_y, M_y'')} N_y \circ M_y'' \longrightarrow \bigodot_{b'_{iy}<0} N_i^{\odot(-b'_{iy})} \longrightarrow 0$$

and

$$\mathfrak{d}(N_i, M''_y) = 0 \quad \text{for } i \neq y.$$

If $x = y$, then $b'_{iy} = -b_{ix}$, and hence $M''_y = M_x$ satisfies the desired condition.

Assume that $x \neq y$ and $b_{xy} = 0$. Then $b'_{iy} = b_{iy}$ for any i and $N_i = M_i$ for any $i \neq x$. Hence $M''_y = \mu_y(M)_y$ satisfies the desired condition.

We will show the assertion in the case $b_{xy} > 0$. We omit the proof of the case $b_{xy} < 0$ because it can be shown in a similar way.

Recall that we have

$$b'_{iy} = \begin{cases} b_{iy} + b_{ix}b_{xy} & \text{if } b_{ix} > 0, \\ b_{iy} & \text{if } b_{ix} \le 0 \end{cases} \tag{7.6}$$

for $i \in J$ different from x and y.

Set

$$\begin{aligned}
&M'_x := \mu_x(M)_x, \qquad M'_y := \mu_y(M)_y, \\
&C := \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}, \quad S := \bigodot_{b_{ix}<0,\ i\neq y} M_i^{\odot -b_{ix}}, \\
&P := \bigodot_{b_{iy}>0, i\neq x} M_i^{\odot b_{iy}}, \quad Q := \bigodot_{b'_{iy}<0,\ i\neq x} M_i^{\odot -b'_{iy}}, \\
&A := \bigodot_{b'_{iy}\le 0,\ b_{ix}>0} M_i^{\odot b_{ix}b_{xy}} \odot \bigodot_{b_{iy}<0,\ b'_{iy}>0,\ b_{ix}>0} M_i^{\odot -b_{iy}} \\
&\quad \simeq \bigodot_{b_{iy}<0,\ b_{ix}>0} M_i^{\odot \min(b_{ix}b_{xy}, -b_{iy})}, \\
&B := \bigodot_{b_{iy}\ge 0,\ b_{ix}>0} M_i^{\odot b_{ix}b_{xy}} \odot \bigodot_{b'_{iy}>0,\ b_{iy}<0,\ b_{ix}>0} M_i^{\odot b'_{iy}}.
\end{aligned}$$

Then using (7.6) repeatedly, we have

$$Q \odot A \simeq \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}}, \quad A \odot B \simeq C^{\odot b_{xy}}, \quad \text{and} \quad B \odot P \simeq \odot \bigodot_{b'_{iy}>0} M_i^{\odot b'_{iy}}.$$

Set

$$L := (M'_x)^{\odot b_{xy}}, \quad V := M_x^{\odot b_{xy}},$$

and set

$$X := \bigodot_{b_{iy}>0} M_i^{\odot b_{iy}} \simeq M_x^{\odot b_{xy}} \odot P = V \odot P, \quad Y := \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}} \simeq Q \odot A.$$

Then (7.6) is read as

$$0 \longrightarrow q(B \odot P) \longrightarrow q^{\widetilde{\Lambda}(M_y, M''_y)} M_y \circ M''_y \longrightarrow L \odot Q \longrightarrow 0. \tag{7.7}$$

Note that we have

$$0 \to qC \to q^{\widetilde{\Lambda}(M_x, M'_x)} M_x \circ M'_x \to M_y^{\odot b_{xy}} \odot S \to 0, \tag{7.8}$$

$$0 \to qX \to q^{\widetilde{\Lambda}(M_y, M'_y)} M_y \circ M'_y \to Y \to 0. \tag{7.9}$$

Taking the convolution products of $L = (M'_x)^{\odot b_{xy}}$ and (7.9), we obtain

$$\begin{tikzcd} 0 \arrow[r] & qL \circ X \arrow[r] & q^{\widetilde{\Lambda}(M_y, M'_y)} L \circ (M_y \circ M'_y) \arrow[r] & L \circ Y \arrow[r] & 0, \\ 0 \arrow[r] & qX \circ L \arrow[r] & q^{\widetilde{\Lambda}(M_y, M'_y)} (M_y \circ M'_y) \circ L \arrow[r] & Y \circ L \arrow[r] & 0. \end{tikzcd}$$

Since L commutes with M_y, we have

$$\begin{aligned} &\Lambda(L, Y) = \Lambda(L, M_y \nabla M'_y) \\ &= \Lambda(L, M_y) + \Lambda(L, M'_y) = \Lambda(L, M_y \circ M'_y). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\Lambda(M'_x, X) - \Lambda(M'_x, Y) \\ &= \Lambda(M'_x, \bigodot_{b_{iy}>0} M_i^{\odot b_{iy}}) - \Lambda(M'_x, \bigodot_{b_{iy}<0} M_i^{\odot -b_{iy}}) \\ &= \sum_{b_{iy}>0} \Lambda(M'_x, M_i) b_{iy} - \sum_{b_{iy}<0} \Lambda(M'_x, M_i)(-b_{iy}) \\ &= \sum_{i \in J} \Lambda(M'_x, M_i) b_{iy} = \sum_{i \neq x} \Lambda(M'_x, M_i) b_{iy} + \Lambda(M'_x, M_x) b_{xy} \\ &= \sum_{i \neq x} \Lambda(M'_x, M_i)(b'_{iy} - \delta(b_{ix} > 0) b_{ix} b_{xy}) + \Lambda(M'_x, M_x) b_{xy} \\ &= \sum_{i \neq x} \Lambda(M'_x, M_i) b'_{iy} - \sum_{b_{ix}>0} \Lambda(M'_x, M_i) b_{ix} b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\ &\underset{(a)}{=} 0 - \Lambda(M'_x, \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}) b_{xy} + \Lambda(M'_x, M_x) b_{xy} \\ &= \big(-\Lambda(M'_x, \bigodot_{b_{ix}>0} M_i^{\odot b_{ix}}) + \Lambda(M'_x, M_x)\big) b_{xy} \\ &= (-\Lambda(M'_x, M'_x \nabla M_x) + \Lambda(M'_x, M_x)) b_{xy} \\ &= (-\Lambda(M'_x, M'_x) - \Lambda(M'_x, M_x) + \Lambda(M'_x, M_x)) b_{xy} = 0. \end{aligned}$$

Note that we used the compatibility of the pair $\big((-\Lambda(\mu_x(M_i), \mu_x(M_j)))_{i,j \in J}, \mu_x(\widetilde{B})\big)$ when we derive the equality (a).

Since $L = (M'_x)^{\odot b_{xy}}$, the equality $\Lambda(M'_x, X) = \Lambda(M'_x, Y)$ implies

$$\Lambda(L, X) = \Lambda(L, Y) = \Lambda(L, M_y \circ M'_y).$$

Hence the following diagram is commutative by Proposition 3.2.8 (i):

$$\begin{tikzcd} 0 \arrow[r] & qL \circ X \arrow[r] \arrow[d, "\mathbf{r}_{L,X}"] & q^{\widetilde{\Lambda}(M_y, M'_y)} L \circ (M_y \circ M'_y) \arrow[r] \arrow[d, "\mathbf{r}_{L, M_y \circ M'_y}"] & L \circ Y \arrow[r] \arrow[d, "\mathbf{r}_{L,Y}"', "\wr"] & 0 \\ 0 \arrow[r] & q^{d+1} X \circ L \arrow[r] & q^{d + \widetilde{\Lambda}(M_y, M'_y)} (M_y \circ M'_y) \circ L \arrow[r] & q^d Y \circ L \arrow[r] & 0, \end{tikzcd}$$

where $d = -\Lambda(L, X) = -\Lambda(L, M_y \circ M'_y) = -\Lambda(L, Y)$. Note that since $L = (M'_x)^{\odot b_{xy}}$ commutes with Q and A, $\mathbf{r}_{L,Y}$ is an isomorphism. Hence we have

$$\operatorname{Im}(\mathbf{r}_{L,Y}) \simeq L \circ Y.$$

Therefore we obtain an exact sequence

$$0 \longrightarrow \mathrm{Im}(\mathbf{r}_{L,X}) \longrightarrow \mathrm{Im}(\mathbf{r}_{L,M_y \circ M'_y}) \longrightarrow L \circ Y \longrightarrow 0. \tag{7.10}$$

On the other hand, $\mathbf{r}_{L,M_y \circ M'_y}$ decomposes (up to a grading shift) by Lemma 3.1.5 as follows:

$$\begin{tikzcd}
L \circ M_y \circ M'_y \arrow[r, "\sim", "\mathbf{r}_{L,M_y} \circ M'_y"'] \arrow[rr, bend left, "\mathbf{r}_{L,M_y \circ M'_y}"] & M_y \circ L \circ M'_y \arrow[r, "M_y \circ \mathbf{r}_{L,M'_y}"'] & M_y \circ M'_y \circ L.
\end{tikzcd}$$

Since $L = (M'_x)^{\odot b_{xy}}$ commutes with M_y, the homomorphisms $\mathbf{r}_{L,M_y} \circ M'_y$ is an isomorphism, and hence we have

$$\mathrm{Im}(\mathbf{r}_{L,M_y \circ M'_y}) \simeq M_y \circ (L \nabla M'_y) \quad \text{up to a grading shift.}$$

Similarly, $\mathbf{r}_{L,X}$ decomposes (up to a grading shift) as follows:

$$\begin{tikzcd}
L \circ V \circ P \arrow[r, "\mathbf{r}_{L,V} \circ P"'] \arrow[rr, bend left, "\mathbf{r}_{L,X}"] & V \circ L \circ P \arrow[r, "\sim", "V \circ \mathbf{r}_{L,P}"'] & V \circ P \circ L.
\end{tikzcd}$$

Since L commutes with P, the homomorphism $V \circ \mathbf{r}_{L,P}$ is an isomorphism, and hence we have

$$\mathrm{Im}(\mathbf{r}_{L,X}) \simeq (L \nabla V) \circ P \simeq \big((M'_x)^{\circ b_{xy}} \nabla M_x^{\circ b_{xy}}\big) \circ P \quad \text{up to a grading shift.}$$

On the other hand, Lemma 3.2.22 implies that

$$(M'_x)^{\circ b_{xy}} \nabla M_x^{\circ b_{xy}} \simeq (M'_x \nabla M_x)^{\circ b_{xy}} \simeq C^{\circ b_{xy}} \simeq B \odot A,$$

and hence we obtain

$$\mathrm{Im}(\mathbf{r}_{L,X}) \simeq (B \odot P) \odot A \quad \text{up to a grading shift.}$$

Thus the exact sequence (7.10) becomes the exact sequence in $\mathcal{C}$,

$$0 \longrightarrow q^m (B \odot P) \odot A \longrightarrow q^n M_y \circ (L \nabla M'_y) \longrightarrow (L \odot Q) \odot A \longrightarrow 0 \tag{7.11}$$

for some $m, n \in \mathbb{Z}$. Since $(L \odot Q) \odot A$ is self-dual, $n = \widetilde{\Lambda}(M_y, L \nabla M'_y)$. On the other hand, by Proposition 3.2.13 (i) and Proposition 7.1.2 (d), we have

$$\mathfrak{d}(M_y, L \nabla M'_y) \le \mathfrak{d}(M_y, L) + \mathfrak{d}(M_y, M'_y) = 1.$$

By the exact sequence (7.11), $M_y \circ (L \nabla M'_y)$ is not simple, and we conclude

$$\mathfrak{d}(M_y, L \nabla M'_y) = 1.$$

Then Lemma 3.2.18 implies that $m = 1$. Thus we obtain an exact sequence in $\mathcal{C}$,

$$0 \to q(B \odot P) \odot A \to q^{\widetilde{\Lambda}(M_y, L \nabla M'_y)} M_y \circ (L \nabla M'_y) \to (L \odot Q) \odot A \to 0. \tag{7.12}$$

Now we shall rewrite (7.12) by using $\bullet \circ A$ instead of $\bullet \odot A$. We have

$$\begin{aligned}
\widetilde{\Lambda}(B, A) + \widetilde{\Lambda}(A, A) &= b_{xy}\widetilde{\Lambda}(C, A) = b_{xy}\widetilde{\Lambda}(M'_x \nabla M_x, A) \\
&= b_{xy}\widetilde{\Lambda}(M'_x, A) + b_{xy}\widetilde{\Lambda}(M_x, A) = \widetilde{\Lambda}(L, A) + b_{xy}\widetilde{\Lambda}(M_x, A).
\end{aligned}$$

On the other hand, the exact sequence (7.9) gives

$$\begin{aligned}b_{xy}\widetilde{\Lambda}(M_x, A) + \widetilde{\Lambda}(P, A) &= \widetilde{\Lambda}(X, A) = \widetilde{\Lambda}(M'_y \nabla M_y, A)\\ &= \widetilde{\Lambda}(M'_y, A) + \widetilde{\Lambda}(M_y, A) = \widetilde{\Lambda}(M_y \nabla M'_y, A) = \widetilde{\Lambda}(Y, A)\\ &= \widetilde{\Lambda}(Q, A) + \widetilde{\Lambda}(A, A).\end{aligned}$$

It follows that

$$\begin{aligned}\widetilde{\Lambda}(B \circ P, A) &= \widetilde{\Lambda}(B, A) + \widetilde{\Lambda}(P, A)\\ &= \big(\widetilde{\Lambda}(L, A) + b_{xy}\widetilde{\Lambda}(M_x, A) - \widetilde{\Lambda}(A, A)\big)\\ &\qquad +\big(\widetilde{\Lambda}(Q, A) + \widetilde{\Lambda}(A, A) - b_{xy}\widetilde{\Lambda}(M_x, A)\big)\\ &= \widetilde{\Lambda}(L, A) + \widetilde{\Lambda}(Q, A) = \widetilde{\Lambda}(L \circ Q, A).\end{aligned}$$

Hence we have

$$0 \longrightarrow q(B \odot P) \circ A \longrightarrow q^c M_y \circ (L \nabla M'_y) \longrightarrow (L \odot Q) \circ A \longrightarrow 0,$$

where $c = \widetilde{\Lambda}(M_y, L \nabla M'_y) - \widetilde{\Lambda}(B \odot P, A)$ by Lemma 3.1.4.

Thus we obtain the identity in $K(R\text{-gmod})$,

$$q^c[M_y][L \nabla M'_y] = \big(q[B \odot P] + [L \odot Q]\big)[A].$$

On the other hand, the hypothesis (7.4) implies that there exists $\phi \in \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm1}]} K(\mathcal{C})$ corresponding to $\mu_y\mu_x([M])$ so that it satisfies

$$[M_y]\phi = q[B \odot P] + [L \odot Q] \tag{7.13}$$

and

$$\phi[\mu_x(M)_i] = q^{\lambda'_{yi}}[\mu_x(M)_i]\phi \quad \text{for } i \neq y, \tag{7.14}$$

where $\mu_y\mu_x(-\Lambda) = (\lambda'_{ij})_{i,j\in J}$.

Hence, in $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm1}]} K(\mathcal{C})$, we have

$$[M_y]\phi[A] = \big(q[B \odot P] + [L \odot Q]\big)[A] = q^c[M_y][L \nabla M'_y].$$

Since $\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm1}]} K(\mathcal{C})$ is a domain, we conclude that

$$\phi[A] = q^c[L \nabla M'_y].$$

On the other hand, (7.14) implies

$$\phi[A] = q^l[A]\phi \quad \text{for some } l \in \mathbb{Z}.$$

Hence, Theorem 4.1.3 implies that, when we write

$$\phi = \sum_{b\in B(\infty)} a_b[L_b] \quad \text{for some } a_b \in \mathbb{Q}(q^{1/2}),$$

we have

$$L_b \circ A \simeq q^l A \circ L_b \quad \text{whenever} \quad a_b \neq 0.$$

In particular, each module $L_b \circ A$ with $a_b \neq 0$ is simple because A is a real simple module. Thus we obtain

$$q^c[L \nabla M'_y] = \phi[A] = \sum_{b\in B(\infty)} a_b[L_b \circ A].$$

Since $L \nabla M'_y$ is simple, there exists b_0 such that $L_{b_0} \circ A$ is isomorphic to $L \nabla M'_y$ up to a grading shift, and $a_b = 0$ for $b \neq b_0$. Set $M''_y := L_{b_0}$. Then we conclude that $\phi[A] = q^m[M''_y \circ A] = q^m[M''_y][A]$ so that

$$\phi = q^m[M''_y] \quad \text{for some} \quad m \in \mathbb{Z}.$$

We emphasize that M''_y is a self-dual simple module in R-gmod which satisfies that $M''_y \circ A \simeq L \nabla M_y$ up to a grading shift.

Now (7.13) implies

$$q^m[M_y \circ M''_y] = q[B \odot P] + [L \odot Q].$$

Hence there exists an exact sequence

$$0 \longrightarrow W \longrightarrow q^m\ M_y \circ M''_y \longrightarrow Z \longrightarrow 0,$$

where $W = qB \odot P$ and $Z = L \odot Q$ or $W = L \odot Q$ and $Z = qB \odot P$. By Lemma 3.2.18, the second case does not occur, and we have an exact sequence

$$0 \longrightarrow qB \odot P \longrightarrow q^m\ M_y \circ M''_y \longrightarrow L \odot Q \longrightarrow 0.$$

Since M_y, M''_y, and $L \odot Q$ are self-dual, we have $m = \widetilde{\Lambda}(M_y, M''_y)$, and we obtain the desired short exact sequence (7.7).

Since ϕ commutes with $[\mu_x(M)_i]$ up to a power of q in $K(\mathcal{C})$, and $\mu_x(M)_i$ is real simple, M''_y commutes with $\mu_x(M)_i$ for $i \neq y$, by Corollary 4.1.4. □

Corollary 7.1.4. *Let $(\{M_i\}_{i\in J}, \widetilde{B})$ be an admissible pair in $\mathcal{C}$. Under the assumption* (7.4)*, $\mathcal{C}$ is a monoidal categorification of the quantum cluster algebra $\mathscr{A}_{q^{1/2}}([\mathscr{S}])$. Furthermore, the following statements hold*:

(i) *The quantum monoidal seed $\mathscr{S} = (\{M_i\}_{i\in J}, -\Lambda, \widetilde{B}, D)$ admits successive mutations in all directions.*

(ii) *Any cluster monomial in $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ is the isomorphism class of a real simple object in $\mathcal{C}$ up to a power of $q^{1/2}$.*

(iii) *Any cluster monomial in $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm 1}]} K(\mathcal{C})$ is a Laurent polynomial of the initial cluster variables with a coefficient in $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$.*

Proof. Items (i) and (ii) are straightforward.

Let us show (iii). Let x be a cluster monomial. By the Laurent phenomenon [3], we can write

$$xX^{\mathbf{c}} = \sum_{\mathbf{a}\in\mathbb{Z}^J_{\geq 0}} c_{\mathbf{a}} X^{\mathbf{a}},$$

where $X = (X_i)_{i\in J}$ is the initial cluster, $\mathbf{c} \in \mathbb{Z}^J_{\geq 0}$, and $c_{\mathbf{a}} \in \mathbb{Q}(q^{\pm 1/2})$. Since x and $X^{\mathbf{c}}$ are the isomorphism classes of simple modules up to a power of $q^{1/2}$, their product $xX^{\mathbf{c}}$ can be written as a linear combination of the isomorphism classes of simple modules with coefficients in $\mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$. Since every $X^{\mathbf{a}}$ is the isomorphism class of a simple module up to a power of $q^{1/2}$, we have $c_{\mathbf{a}} \in \mathbb{Z}_{\geq 0}[q^{\pm 1/2}]$. □

8. QUANTUM COORDINATE RINGS AND MODIFIED QUANTIZED ENVELOPING ALGEBRAS

8.1. **Quantum coordinate ring.** Let $U_q(\mathfrak{g})^*$ be $\mathrm{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$. Then the comultiplication Δ_+ (see (1.1)) induces the multiplication μ on $U_q(\mathfrak{g})^*$ as follows:

$$\mu\colon U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^* \to (U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}))^* \xrightarrow{(\Delta_+)^*} U_q(\mathfrak{g})^*.$$

Later on, it will be convenient to use Sweedler's notation $\Delta_+(x) = x_{(1)} \otimes x_{(2)}$. With this notation,

$$\big(fg\big)(x) = f(x_{(1)})\, g(x_{(2)}) \quad \text{for } f, g \in U_q(\mathfrak{g})^* \text{ and } x \in U_q(\mathfrak{g}).$$

The $U_q(\mathfrak{g})$-bimodule structure on $U_q(\mathfrak{g})$ induces a $U_q(\mathfrak{g})$-bimodule structure on $U_q(\mathfrak{g})^*$. Namely,

$$(x \cdot f)(v) = f(vx) \quad \text{and} \quad (f \cdot x)(v) = f(xv) \quad \text{for } f \in U_q(\mathfrak{g})^* \text{ and } x, v \in U_q(\mathfrak{g}).$$

Then the multiplication μ is a morphism of a $U_q(\mathfrak{g})$-bimodule, where $U_q(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*$ has the structure of a $U_q(\mathfrak{g})$-bimodule via Δ_+. That is, for $f, g \in U_q(\mathfrak{g})^*$ and $x, y \in U_q(\mathfrak{g})$, we have

$$x(fg)y = (x_{(1)} f y_{(1)})(x_{(2)} g y_{(2)}),$$

where $\Delta_+(x) = x_{(1)} \otimes x_{(2)}$ and $\Delta_+(y) = y_{(1)} \otimes y_{(2)}$.

Definition 8.1.1. *We define the* quantum coordinate ring $A_q(\mathfrak{g})$ *as follows:*

$$A_q(\mathfrak{g}) = \{u \in U_q(\mathfrak{g})^* \mid \ U_q(\mathfrak{g})u \textit{ belongs to } \mathcal{O}_{\mathrm{int}}(\mathfrak{g}) \textit{ and } uU_q(\mathfrak{g}) \textit{ belongs to } \mathcal{O}^{\mathrm{r}}_{\mathrm{int}}(\mathfrak{g})\}.$$

Then, $A_q(\mathfrak{g})$ is a subring of $U_q(\mathfrak{g})^*$ because (i) μ is $U_q(\mathfrak{g})$-bilinear, and (ii) $\mathcal{O}_{\mathrm{int}}(\mathfrak{g})$ and $\mathcal{O}^{\mathrm{r}}_{\mathrm{int}}(\mathfrak{g})$ are closed under the tensor product.

We have the weight decomposition $A_q(\mathfrak{g}) = \bigoplus_{\eta,\zeta \in \mathsf{P}} A_q(\mathfrak{g})_{\eta,\zeta}$, where

$$A_q(\mathfrak{g})_{\eta,\zeta} := \{\psi \in A_q(\mathfrak{g}) \mid q^{h_{\mathrm{l}}} \cdot \psi \cdot q^{h_{\mathrm{r}}} = q^{\langle h_{\mathrm{l}}, \eta\rangle + \langle h_{\mathrm{r}}, \zeta\rangle}\psi \text{ for } h_{\mathrm{l}}, h_{\mathrm{r}} \in \mathsf{P}^\vee\},$$

For $\psi \in A_q(\mathfrak{g})_{\eta,\zeta}$, we write

$$\mathrm{wt}_{\mathrm{l}}(\psi) = \eta \quad \text{and} \quad \mathrm{wt}_{\mathrm{r}}(\psi) = \zeta.$$

For any $V \in \mathcal{O}_{\mathrm{int}}(\mathfrak{g})$, we have the $U_q(\mathfrak{g})$-bilinear homomorphism

$$\Phi_V \colon V \otimes (\mathbf{D}_\varphi V)^{\mathrm{r}} \to A_q(\mathfrak{g})$$

given by

$$\Phi_V(v \otimes \psi^{\mathrm{r}})(a) = \langle \psi^{\mathrm{r}}, av\rangle = \langle \psi^{\mathrm{r}} a, v\rangle \quad \text{for } v \in V,\ \psi \in \mathbf{D}_\varphi V \text{ and } a \in U_q(\mathfrak{g}).$$

Proposition 8.1.2 ([17, Proposition 7.2.2])**.** *We have an isomorphism* Φ *of* $U_q(\mathfrak{g})$-*bimodules*

$$\Phi \colon \bigoplus_{\lambda \in \mathsf{P}^+} V(\lambda) \underset{\mathbb{Q}(q)}{\otimes} V(\lambda)^{\mathrm{r}} \xrightarrow{\sim} A_q(\mathfrak{g}) \tag{8.1}$$

given by $\Phi|_{V(\lambda) \otimes_{\mathbb{Q}(q)} V(\lambda)^{\mathrm{r}}} = \Phi_\lambda := \Phi_{V(\lambda)}$. *Namely,*

$$\Phi(u \otimes v^{\mathrm{r}})(x) = \langle v^{\mathrm{r}}, xu\rangle = \langle v^{\mathrm{r}}x, u\rangle = (v, xu) \textit{ for any } v, u \in V(\lambda) \textit{ and } x \in U_q(\mathfrak{g}).$$

We introduce the crystal basis $\big(L^{\mathrm{up}}(A_q(\mathfrak{g})), B(A_q(\mathfrak{g}))\big)$ of $A_q(\mathfrak{g})$ as the images by Φ of

$$\bigoplus_{\lambda \in \mathsf{P}^+} L^{\mathrm{up}}(\lambda) \otimes L^{\mathrm{up}}(\lambda)^{\mathrm{r}} \text{ and } \bigsqcup_{\lambda \in \mathsf{P}^+} B(\lambda) \otimes B(\lambda)^{\mathrm{r}}.$$

Hence it is a crystal base with respect to the left action of $U_q(\mathfrak{g})$ and also the right action of $U_q(\mathfrak{g})$. We sometimes write by e_i^* and f_i^* the operators of $A_q(\mathfrak{g})$ obtained by the right actions of e_i and f_i.

We define the $\mathbb{Z}[q^{\pm 1}]$-form of $A_q(\mathfrak{g})$ by

$$A_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]} := \big\{\psi \in A_q(\mathfrak{g}) \mid \langle \psi,\, U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm 1}]}\rangle \subset \mathbb{Z}[q^{\pm 1}]\big\}.$$

We define the bar-involution $-$ of $A_q(\mathfrak{g})$ by

$$\overline{\psi}(x) = \overline{\psi(\overline{x})} \quad \text{for } \psi \in A_q(\mathfrak{g}),\ x \in U_q(\mathfrak{g}).$$

Note that the bar-involution is not a ring homomorphism but it satisfies

$$\overline{\psi\,\theta} = q^{(\mathrm{wt_l}(\psi),\mathrm{wt_l}(\theta))-(\mathrm{wt_r}(\psi),\mathrm{wt_r}(\theta))}\,\overline{\theta}\,\overline{\psi} \quad \text{for any } \psi,\ \theta \in A_q(\mathfrak{g}).$$

Since we do not use this formula and it is proved similarly to Proposition 8.1.4 below, we omit its proof.

The triple $\big(\mathbb{Q}\otimes A_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]},\ L^{\mathrm{up}}(A_q(\mathfrak{g})),\ \overline{L^{\mathrm{up}}(A_q(\mathfrak{g}))}\big)$ is balanced [17, Theorem 1], and hence there exists an upper global basis of $A_q(\mathfrak{g})$,

$$\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{g})) := \{G^{\mathrm{up}}(b) \mid b \in B^{\mathrm{up}}(A_q(\mathfrak{g}))\}.$$

For $\lambda \in \mathsf{P}^+$ and $\mu \in W\lambda$, we denote by u_μ the unique member of the upper global basis of $V(\lambda)$ with weight μ. It is also a member of the lower global basis.

Proposition 8.1.3. *Let $\lambda \in \mathsf{P}^+$, $w \in W$, and $b \in B(\lambda)$. Then, $\Phi(G^{\mathrm{up}}(b)\otimes u^{\mathrm{r}}_{w\lambda})$ is a member of the upper global basis of $A_q(\mathfrak{g})$.*

Proof. The element $\psi := \Phi(G^{\mathrm{up}}(b)\otimes u^{\mathrm{r}}_{w\lambda})$ is bar-invariant and a member of crystal basis modulo $qL^{\mathrm{up}}(A_q(\mathfrak{g}))$. For any $P \in U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}$,

$$\langle \psi, P\rangle = \big(u_{w\lambda}, PG^{\mathrm{up}}(b)\big)$$

belongs to $\mathbb{Z}[q^{\pm1}]$ because $PG^{\mathrm{up}}(b) \in V^{\mathrm{up}}(\lambda)_{\mathbb{Z}[q^{\pm1}]}$ and $u_{w\lambda} \in V^{\mathrm{low}}(\lambda)_{\mathbb{Z}[q^{\pm1}]}$. Hence ψ belongs to $A_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}$. ☐

The $\mathbb{Q}(q)$-algebra anti-automorphism φ of $U_q(\mathfrak{g})$ induces a $\mathbb{Q}(q)$-linear automorphism φ^* of $A_q(\mathfrak{g})$ by

$$\big(\varphi^*\psi\big)(x) = \psi\big(\varphi(x)\big) \quad \text{for any } x \in U_q(\mathfrak{g}).$$

We have

$$\varphi^*\big(\Phi(u\otimes v^{\mathrm{r}})\big) = \Phi(v\otimes u^{\mathrm{r}}),$$

and

$$\mathrm{wt_l}(\varphi^*\psi) = \mathrm{wt_r}(\psi) \quad \text{and} \quad \mathrm{wt_r}(\varphi^*\psi) = \mathrm{wt_l}(\psi).$$

It is obvious that φ^* preserves $A_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}$, $L^{\mathrm{up}}(A_q(\mathfrak{g}))$, and $\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{g}))$.

Proposition 8.1.4.

$$\varphi^*(\psi\theta) = q^{(\mathrm{wt_r}(\psi),\mathrm{wt_r}(\theta))-(\mathrm{wt_l}(\psi),\mathrm{wt_l}(\theta))}(\varphi^*\psi)(\varphi^*\theta).$$

In order to prove this proposition, we prepare a sublemma.

Let ξ be the $\mathbb{Q}(q)$-algebra automorphism of $U_q(\mathfrak{g})$ given by

$$\xi(e_i) = q_i^{-1}t_ie_i, \qquad \xi(f_i) = q_if_it_i^{-1}, \qquad \xi(q^h) = q^h.$$

We can easily see

$$(\xi\otimes\xi)\circ\Delta_+ = \Delta_-\circ\xi.$$

Let ξ^* be the automorphism of $A_q(\mathfrak{g})$ given by

$$(\xi^*\psi)(x) = \psi(\xi(x)) \quad \text{for } \psi \in A_q(\mathfrak{g}) \text{ and } x \in U_q(\mathfrak{g}).$$

Sublemma 8.1.5. *We have*

$$\xi^*(\psi) = q^{A(\mathrm{wt_l}(\psi),\mathrm{wt_r}(\psi))}\psi,$$

where $A(\lambda,\mu) = \dfrac{1}{2}\big((\mu,\mu)-(\lambda,\lambda)\big)$.

Proof. Let us show that, for each x, the following equality,

$$\psi(\xi(x)) = q^{A(\mathrm{wt}_l(\psi),\mathrm{wt}_r(\psi))}\psi(x), \tag{8.2}$$

holds for any ψ.

The equality (8.2) is obviously true for $x = q^h$. If (8.2) is true for x, then

$$\begin{aligned}\xi^*(\psi)(xe_i) &= \psi\big(\xi(xe_i)\big) = \psi\big(\xi(x)e_it_i\big)q_i \\ &= q^{(\alpha_i,\mathrm{wt}_l(\psi))+(\alpha_i,\alpha_i)/2}\psi(\xi(x)e_i) \\ &= q^{(\alpha_i,\mathrm{wt}_l(\psi))+(\alpha_i,\alpha_i)/2}\big(\xi^*(e_i\psi)\big)(x) \\ &= q^{(\alpha_i,\mathrm{wt}_l(\psi))+(\alpha_i,\alpha_i)/2+A(\mathrm{wt}_l(\psi)+\alpha_i,\mathrm{wt}_r(\psi))}(e_i\psi)(x).\end{aligned}$$

Since $\|\lambda+\alpha_i\|^2 = \|\lambda\|^2 + 2(\alpha_i,\lambda) + \|\alpha_i\|^2$, (8.2) holds for xe_i. Similarly if (8.2) holds for x, then it holds for xf_i. ☐

Proof of Proposition 8.1.4. We have

$$(\varphi\circ\varphi)\circ\Delta_- = \Delta_+\circ\varphi.$$

Hence, we have

$$\begin{aligned}\langle\varphi^*(\psi\theta),x\rangle &= \langle\psi\theta,\varphi(x)\rangle \\ &= \langle\psi\otimes\theta,\Delta_+(\varphi(x))\rangle \\ &= \langle\psi\otimes\theta,(\varphi\otimes\varphi)\circ\Delta_-(x)\rangle \\ &= \langle\varphi^*(\psi)\otimes\varphi^*(\theta),\ \Delta_-(x)\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}\langle\xi^*(\varphi^*(\psi\theta)),x\rangle &= \langle\varphi^*(\psi\theta),\xi(x)\rangle = \langle\varphi^*(\psi)\otimes\varphi^*(\theta),\Delta_-(\xi(x))\rangle \\ &= \langle\varphi^*(\psi)\otimes\varphi^*(\theta),(\xi\otimes\xi)\circ\Delta_+x\rangle \\ &= \langle\xi^*\varphi^*(\psi)\otimes\xi^*\varphi^*(\theta),\Delta_+x\rangle \\ &= \langle\big(\xi^*\varphi^*(\psi)\big)\big(\xi^*\varphi^*(\theta)\big),x\rangle \\ &= q^{A(\mathrm{wt}_r(\psi),\mathrm{wt}_l(\psi))+A(\mathrm{wt}_r(\theta),\mathrm{wt}_l(\theta))}\langle(\varphi^*\psi)\,(\varphi^*\theta),\ x\rangle.\end{aligned}$$

Therefore we obtain

$$\varphi^*(\psi\theta) = q^c(\varphi^*\psi)\,(\varphi^*\theta)$$

with

$$\begin{aligned}c = {} & A(\mathrm{wt}_r(\psi),\mathrm{wt}_l(\psi)) + A(\mathrm{wt}_r(\theta),\mathrm{wt}_l(\theta)) \\ & -A(\mathrm{wt}_r(\psi)+\mathrm{wt}_r(\theta),\mathrm{wt}_l(\psi)+\mathrm{wt}_l(\theta)) \\ = {} & (\mathrm{wt}_r(\psi),\mathrm{wt}_r(\theta)) - (\mathrm{wt}_l(\psi),\mathrm{wt}_l(\theta)). \quad \square\end{aligned}$$

8.2. **Unipotent quantum coordinate ring.** Let us endow $U_q^+(\mathfrak{g})\otimes U_q^+(\mathfrak{g})$ with the algebra structure defined by

$$(x_1\otimes x_2)\cdot(y_1\otimes y_2) = q^{-(\mathrm{wt}(x_2),\mathrm{wt}(y_1))}(x_1y_1\otimes x_2y_2).$$

Let $\Delta_{\mathfrak{n}}$ be the algebra homomorphism $U_q^+(\mathfrak{g})\to U_q^+(\mathfrak{g})\otimes U_q^+(\mathfrak{g})$ given by

$$\Delta_{\mathfrak{n}}(e_i) = e_i\otimes 1 + 1\otimes e_i.$$

Set

$$A_q(\mathfrak{n}) = \bigoplus_{\beta\in\mathsf{Q}^-} A_q(\mathfrak{n})_\beta \qquad \text{where } A_q(\mathfrak{n})_\beta := (U_q^+(\mathfrak{g})_{-\beta})^*.$$

Defining the bilinear form $\langle\ \cdot\ ,\ \cdot\ \rangle\colon (A_q(\mathfrak{n})\otimes A_q(\mathfrak{n}))\times(U_q^+(\mathfrak{g})\otimes U_q^+(\mathfrak{g}))\to\mathbb{Q}(q)$ by

$$\langle\psi\otimes\theta, x\otimes y\rangle=\theta(x)\psi(y),$$

we get an algebra structure on $A_q(\mathfrak{n})$ given by

$$(\psi\cdot\theta)(x)=\langle\psi\otimes\theta,\Delta_{\mathfrak{n}}(x)\rangle=\theta(x_{(1)})\psi(x_{(2)}),$$

where $\Delta_{\mathfrak{n}}(x)=x_{(1)}\otimes x_{(2)}$.

Since $U_q^+(\mathfrak{g})$ has a $U_q^+(\mathfrak{g})$-bimodule structure, so does $A_q(\mathfrak{n})$.

We define the $\mathbb{Z}[q^{\pm1}]$-form of $A_q(\mathfrak{n})$ by

$$A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm1}]}=\left\{\psi\in A_q(\mathfrak{n})\mid\psi\big(U_q^+(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}\big)\subset\mathbb{Z}[q^{\pm1}]\right\},$$

and define the bar-involution $-$ on $A_q(\mathfrak{n})$ by

$$\overline{\psi}(x)=\overline{\psi(\overline{x})}\quad\text{for }\psi\in A_q(\mathfrak{n})\text{ and }x\in U_q^+(\mathfrak{g}).$$

Note that the bar-involution is not a ring homomorphism but it satisfies

$$\overline{\psi\ \theta}\ =q^{(\mathrm{wt}(\psi),\mathrm{wt}(\theta))}\ \overline{\theta}\ \overline{\psi}\quad\text{for any }\psi,\,\theta\in A_q(\mathfrak{n}).$$

For $i\in I$, we denote by e_i^* the right action of e_i on $A_q(\mathfrak{n})$.

Lemma 8.2.1. *For $u,v\in A_q(\mathfrak{n})$, we have q-boson relations*

$$e_i(uv)=(e_iu)v+q^{(\alpha_i,\mathrm{wt}(u))}u(e_iv)\quad\textit{and}\quad e_i^*(uv)=u(e_i^*v)+q^{(\alpha_i,\mathrm{wt}(v))}(e_i^*u)v.$$

Proof.

$$\langle e_i(uv),x\rangle=\langle uv,xe_i\rangle=\langle u\otimes v,\ \Delta_{\mathfrak{n}}(xe_i)\rangle.$$

If we set $\Delta_{\mathfrak{n}}x=x_{(1)}\otimes x_{(2)}$, then we have

$$\Delta_{\mathfrak{n}}(xe_i)=(x_{(1)}\otimes x_{(2)})(e_i\otimes1+1\otimes e_i)=q^{-(\alpha_i,\mathrm{wt}(x_{(2)}))}(x_{(1)}e_i)\otimes x_{(2)}+x_{(1)}\otimes(x_{(2)}e_i).$$

Hence, we have

$$\begin{aligned}\langle u\otimes v,\ \Delta_{\mathfrak{n}}(xe_i)\rangle&=q^{-(\alpha_i,\mathrm{wt}(x_{(2)}))}u(x_{(2)})v(x_{(1)}e_i)+u(x_{(2)}e_i)v(x_{(1)})\\&=q^{(\alpha_i,\mathrm{wt}(u))}u(x_{(2)})\cdot(e_iv)(x_{(1)})+(e_iu)(x_{(2)})\cdot v(x_{(1)})\\&=\langle q^{(\alpha_i,\mathrm{wt}(u))}u\otimes(e_iv)+(e_iu)\otimes v,\ \Delta_{\mathfrak{n}}x\rangle.\end{aligned}$$

The second identity follows in a similar way. □

We define the map $\iota\colon U_q^-(\mathfrak{g})\to A_q(\mathfrak{n})$ by

$$\langle\iota(u),x\rangle=(u,\varphi(x))\qquad\text{for any }u\in U_q^-(\mathfrak{g})\text{ and }x\in U_q^+(\mathfrak{g}).$$

Since $(\ ,\ )$ is a non-degenerate bilinear form on $U_q^-(\mathfrak{g})$, ι is injective. The relation

$$\langle\iota(e_i'u),x\rangle=(e_i'u,\varphi(x))=(u,f_i\varphi(x))=(u,\varphi(xe_i))=\langle\iota(u),xe_i\rangle=\langle e_i\iota(u),x\rangle$$

implies that

$$\iota(e_i'u)=e_i\iota(u).$$

Lemma 8.2.2. *ι is an algebra isomorphism.*

Proof. The map ι is an algebra homomorphism because e_i' and e_i both satisfy the same q-boson relation. □

Hence, the algebra $A_q(\mathfrak{n})$ has an upper crystal basis $(L^{\mathrm{up}}(A_q(\mathfrak{n})), B(A_q(\mathfrak{n})))$ such that $B(A_q(\mathfrak{n})) \simeq B(\infty)$. Furthermore, $A_q(\mathfrak{n})$ has an upper global basis

$$\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{n})) = \{G^{\mathrm{up}}(b)\}_{\ b\in B(A_q(\mathfrak{n}))}$$

induced by the balanced triple $\big(\mathbb{Q}\otimes A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm1}]}, L^{\mathrm{up}}(A_q(\mathfrak{n})), \overline{L^{\mathrm{up}}(A_q(\mathfrak{n}))}\big)$ (see (1.3)).

There exists an injective map

$$\bar{\iota}_\lambda\colon B(\lambda) \to B(\infty)$$

induced by the $U_q^+(\mathfrak{g})$-linear homomorphism $\iota_\lambda\colon V(\lambda) \to A_q(\mathfrak{n})$ given by

$$v \longmapsto \big(U_q^+(\mathfrak{g}) \ni a \mapsto (av, u_\lambda)\big).$$

The map $\bar{\iota}_\lambda$ commutes with $\tilde{e}_i$. We have

$$G_\lambda^{\mathrm{low}}(b) = G^{\mathrm{low}}(\bar{\iota}_\lambda(b))u_\lambda \quad \text{and} \quad \iota_\lambda G_\lambda^{\mathrm{up}}(b) = G^{\mathrm{up}}(\bar{\iota}_\lambda(b)) \quad \text{for any } b\in B(\lambda).$$

Remark 8.2.3. Note that the multiplication on $A_q(\mathfrak{n})$ given in [11] is different from ours. Indeed, by denoting the product of ψ and ϕ in [11, Section 4.2] by $\psi\cdot\phi$, for $x \in U_q^+(\mathfrak{g})$, we have

$$(\psi\cdot\phi)(x) = \psi(x^{(1)})\phi(x^{(2)}),$$

where $\Delta_+(x) = x^{(1)}q^{h_{(1)}} \otimes x^{(2)}q^{h_{(2)}}$ for $x^{(1)}, x^{(2)} \in U_q^+(\mathfrak{g})$, $h_{(1)}, h_{(2)} \in \mathsf{P}^\vee$. By Lemma 8.5.3 below, we have

$$\begin{aligned}(\psi\cdot\phi)(x) \quad &= \psi\big(q^{(\mathrm{wt}(x_{(1)}),\mathrm{wt}(x_{(2)}))}(x_{(2)})\big)\phi\big(x_{(1)}\big)\\ &= q^{(\mathrm{wt}(x_{(1)}),\mathrm{wt}(x_{(2)}))}\psi(x_{(2)})\phi(x_{(1)}) = q^{(\mathrm{wt}(\psi),\mathrm{wt}(\phi))}(\psi\phi)(x)\end{aligned}$$

for $x \in U_q^+(\mathfrak{g})$, where $\Delta_{\mathfrak{n}}(x) = x_{(1)}\otimes x_{(2)}$. In particular, we have a $\mathbb{Q}(q)$-algebra isomorphism from $(A_q(\mathfrak{n}),\cdot)$ to $A_q(\mathfrak{n})$ given by

$$x \mapsto q^{-\frac{1}{2}(\beta,\beta)}x \quad \text{for } x\in A_q(\mathfrak{n})_\beta. \tag{8.3}$$

Note also that the bar-involution $-$ is a ring anti-isomorphism between $A_q(\mathfrak{n})$ and $(A_q(\mathfrak{n}),\cdot)$.

8.3. **Modified quantum enveloping algebra.** For the materials in this subsection we refer the reader to [19, 32]. We denote by $\mathrm{Mod}(\mathfrak{g},\mathsf{P})$ the category of left $U_q(\mathfrak{g})$-modules with the weight space decomposition. Let (forget) be the functor from $\mathrm{Mod}(\mathfrak{g},\mathsf{P})$ to the category of vector spaces over $\mathbb{Q}(q)$, forgetting the $U_q(\mathfrak{g})$-module structure.

Let us denote by $\mathscr{R}$ the endomorphism ring of (forget). Note that $\mathscr{R}$ contains $U_q(\mathfrak{g})$. For $\eta\in\mathsf{P}$, let $a_\eta\in\mathscr{R}$ denote the projector $M\to M_\eta$ to the weight space of weight η. Then the defining relation of a_η (as a left $U_q(\mathfrak{g})$-module) is

$$q^h a_\eta = q^{\langle h,\eta\rangle}a_\eta.$$

We have

$$a_\eta a_\zeta = \delta_{\eta,\zeta}a_\eta, \quad a_\eta P = Pa_{\eta-\xi} \quad \text{for } \xi\in\mathsf{Q} \text{ and } P\in U_q(\mathfrak{g})_\xi.$$

Then $\mathscr{R}$ is isomorphic to $\prod_{\eta\in\mathsf{P}} U_q(\mathfrak{g})a_\eta$. We set

$$\widetilde{U}_q(\mathfrak{g}) := \bigoplus_{\eta\in\mathsf{P}} U_q(\mathfrak{g})a_\eta \subset \mathscr{R}.$$

Then $\widetilde{U}_q(\mathfrak{g})$ is a subalgebra of $\mathscr{R}$. We call it the *modified quantum enveloping algebra.* Note that any $U_q(\mathfrak{g})$-module in $\mathrm{Mod}(\mathfrak{g},\mathsf{P})$ has a natural $\widetilde{U}_q(\mathfrak{g})$-module structure.

The (anti-)automorphisms $*$, φ, and $^{-}$ of $U_q(\mathfrak{g})$ extend to the ones of $\widetilde{U}_q(\mathfrak{g})$ by

$$a_\eta^* = a_{-\eta}, \quad \varphi(a_\eta) = a_\eta, \quad \overline{a}_\eta = a_\eta.$$

For a dominant integral weight $\lambda \in \mathsf{P}^+$, let us denote by $V(\lambda)$ (resp. $V(-\lambda)$) the irreducible module with highest (resp. lowest) weight λ (resp. $-\lambda$). Let u_λ (resp. $u_{-\lambda}$) be the highest (resp. lowest) weight vector.

For $\lambda \in \mathsf{P}^+$, $\mu \in \mathsf{P}^- := -\mathsf{P}^+$, we set

$$V(\lambda,\mu) := V(\lambda) \otimes_- V(\mu).$$

Then $V(\lambda,\mu)$ is generated by $u_\lambda \otimes_- u_\mu$ as a $U_q(\mathfrak{g})$-module, and the defining relation of $u_\lambda \otimes_- u_\mu$ is

$$\begin{aligned}&q^h(u_\lambda \otimes_- u_\mu) = q^{\langle h,\lambda+\mu\rangle}(u_\lambda \otimes_- u_\mu),\\ &e_i^{1-\langle h_i,\mu\rangle}(u_\lambda \otimes_- u_\mu) = 0, \quad f_i^{1+\langle h_i,\lambda\rangle}(u_\lambda \otimes_- u_\mu) = 0.\end{aligned}$$

Let us define the $\mathbb{Q}$-linear automorphism $^{-}$ of $V(\lambda,\mu)$ by

$$\overline{P(u_\lambda \otimes_- u_\mu)} = \overline{P}(u_\lambda \otimes_- u_\mu).$$

We set

(i) $L^{\mathrm{low}}(\lambda,\mu) := L^{\mathrm{low}}(\lambda) \otimes_{\mathbf{A}_0} L^{\mathrm{low}}(\mu)$,
(ii) $V(\lambda,\mu)_{\mathbb{Z}[q^{\pm1}]} := V(\lambda)_{\mathbb{Z}[q^{\pm1}]} \otimes_{\mathbb{Z}[q^{\pm1}]} V(\mu)_{\mathbb{Z}[q^{\pm1}]}$,
(iii) $B(\lambda,\mu) := B(\lambda) \otimes B(\mu)$.

Proposition 8.3.1 ([32]). *$(L^{\mathrm{low}}(\lambda,\mu), B(\lambda,\mu))$ is a lower crystal basis of $V(\lambda,\mu)$. Furthermore, $\big(\mathbb{Q} \otimes V(\lambda,\mu)_{\mathbb{Z}[q^{\pm1}]},\, L^{\mathrm{low}}(\lambda,\mu),\, \overline{L^{\mathrm{low}}(\lambda,\mu)}\big)$ is balanced, and there exists a lower global basis $\mathbf{B}^{\mathrm{low}}(V(\lambda,\mu))$ obtained from the lower crystal basis $(L^{\mathrm{low}}(\lambda,\mu), B(\lambda,\mu))$.*

Theorem 8.3.2 ([32]). *The algebra $\widetilde{U}_q(\mathfrak{g})$ has a lower crystal basis $(L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})), B(\widetilde{U}_q(\mathfrak{g})))$ satisfying the following properties:*

(i) *$L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda\in\mathsf{P}} L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_\lambda)$ and $B(\widetilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda\in\mathsf{P}} B(\widetilde{U}_q(\mathfrak{g})a_\lambda)$, where*
  - *$L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_\lambda) = L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})) \cap U_q(\mathfrak{g})a_\lambda$ and*
  - *$B(\widetilde{U}_q(\mathfrak{g})a_\lambda) = B(\widetilde{U}_q(\mathfrak{g})) \cap \big(L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_\lambda)/qL^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_\lambda)\big)$.*

(ii) *Set $\widetilde{U}_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]} := \bigoplus_{\eta\in\mathsf{P}} U_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]}a_\eta$. Then $\big(\mathbb{Q}\otimes\widetilde{U}_q(\mathfrak{g})_{\mathbb{Z}[q^{\pm1}]},\, L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})),\, \overline{L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g}))}\big)$ is balanced, and $\widetilde{U}_q(\mathfrak{g})$ has the lower global basis $\mathbf{B}^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})) := \{G^{\mathrm{low}}(b) \mid b \in B(\widetilde{U}_q(\mathfrak{g}))\}$.*

(iii) *For any $\lambda \in \mathsf{P}^+$ and $\mu \in \mathsf{P}^-$, let*

$$\Psi_{\lambda,\mu}\colon U_q(\mathfrak{g})a_{\lambda+\mu} \to V(\lambda,\mu)$$

be the $U_q(\mathfrak{g})$-linear map $a_{\lambda+\mu} \longmapsto u_\lambda \otimes u_\mu$. Then we have

$$\Psi_{\lambda,\mu}\big(L(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu})\big) = L^{\mathrm{low}}(\lambda,\mu).$$

(iv) *Let $\overline{\Psi}_{\lambda,\mu}$ be the induced homomorphism*

$$L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu})/qL^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu}) \longrightarrow L^{\mathrm{low}}(\lambda,\mu)/qL^{\mathrm{low}}(\lambda,\mu).$$

Then we have

(a) $\{b \in B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu}) \mid \overline{\Psi}_{\lambda,\mu} b \neq 0\} \xrightarrow{\sim} B(\lambda,\mu)$,

(b) $\Psi_{\lambda,\mu}\big(G^{\mathrm{low}}(b)\big) = G^{\mathrm{low}}(\overline{\Psi}_{\lambda,\mu}(b))$ *for any* $b \in B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu})$.

(v) $B(\widetilde{U}_q(\mathfrak{g}))$ *has a structure of crystal such that the injective map induced by* (iv) (a)

$$B(\lambda,\mu) \to B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+\mu}) \subset B(\widetilde{U}_q(\mathfrak{g}))$$

is a strict embedding of crystals for any $\lambda \in \mathsf{P}^+$ *and* $\mu \in \mathsf{P}^-$.

For $\lambda \in \mathsf{P}$, take any $\zeta \in \mathsf{P}^+$ and $\eta \in \mathsf{P}^-$ such that $\lambda = \zeta + \eta$. Then $B(\zeta) \otimes B(\eta)$ is embedded into $B(\widetilde{U}_q(\mathfrak{g})a_\lambda)$.

For $\mu \in \mathsf{P}$, let $T_\mu = \{t_\mu\}$ be the crystal with

$$\mathrm{wt}(t_\mu) = \mu, \quad \varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty, \quad \tilde{e}_i(t_\mu) = \tilde{f}_i(t_\mu) = 0.$$

Since we have

$$B(\zeta) \hookrightarrow B(\infty) \otimes T_\zeta,\ B(\eta) \hookrightarrow T_\eta \otimes B(-\infty),\ \text{and}\ T_\zeta \otimes T_\eta \simeq T_{\zeta+\eta},$$

$B(\zeta) \otimes B(\eta)$ is embedded into the crystal $B(\infty) \otimes T_\lambda \otimes B(-\infty)$. Taking $\zeta \to \infty$ and $\eta \to -\infty$, we have

Lemma 8.3.3 ([19])**.** *For any $\lambda \in \mathsf{P}$, we have a canonical crystal isomorphism*

$$B(\widetilde{U}_q(\mathfrak{g})a_\lambda) \simeq B(\infty) \otimes T_\lambda \otimes B(-\infty).$$

Hence we identify

$$B(\widetilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in \mathsf{P}} B(\infty) \otimes T_\lambda \otimes B(-\infty).$$

For $\xi \in \mathsf{Q}_-$ and $\eta \in \mathsf{Q}_+$, we shall denote by

$$U_q^-(\mathfrak{g})_{>\xi} := \bigoplus_{\xi' \in \mathsf{Q}_- \cap (\xi + \mathsf{Q}_+) \setminus \{\xi\}} U_q^-(\mathfrak{g})_{\xi'}, \qquad U_q^+(\mathfrak{g})_{<\eta} := \bigoplus_{\eta' \in \mathsf{Q}_+ \cap (\eta + \mathsf{Q}_-) \setminus \{\eta\}} U_q^+(\mathfrak{g})_{\eta'}.$$

Then for any $\lambda \in \mathsf{P}$, $b_- \in B(\infty)_\xi$, and $b_+ \in B(-\infty)_\eta$, we have

$$(8.4) \quad G^{\mathrm{low}}(b_- \otimes t_\lambda \otimes b_+) - G^{\mathrm{low}}(b_-)G^{\mathrm{low}}(b_+)a_\lambda \in U_q^-(\mathfrak{g})_{>\xi}U_q^+(\mathfrak{g})_{<\eta}a_\lambda$$

[19, (3.1.1)]. In particular, we have

$$G^{\mathrm{low}}(b_\infty \otimes t_\lambda \otimes b_+) = G^{\mathrm{low}}(b_+)a_\lambda \quad \text{and}\ G^{\mathrm{low}}(b_- \otimes t_\lambda \otimes b_{-\infty}) = G^{\mathrm{low}}(b_-)a_\lambda.$$

Theorem 8.3.4 ([19])**.**

(i) $L^{\mathrm{low}}(\widetilde{U}_q(\mathfrak{g}))$ *is invariant under the anti-automorphisms* $*$ *and* φ.

(ii) $B(\widetilde{U}_q(\mathfrak{g}))^* = \varphi(B(\widetilde{U}_q(\mathfrak{g}))) = B(\widetilde{U}_q(\mathfrak{g}))$.

(iii) $\big(G^{\mathrm{low}}(b)\big)^* = G^{\mathrm{low}}(b^*)$ *and* $\varphi(G^{\mathrm{low}}(b)) = G^{\mathrm{low}}(\varphi(b))$ *for* $b \in B(\widetilde{U}_q(\mathfrak{g}))$.

Corollary 8.3.5 ([19])**.** *For $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, we have*

(1) $(b_1 \otimes t_\mu \otimes b_2)^* = b_1^* \otimes t_{-\mu-\mathrm{wt}(b_1)-\mathrm{wt}(b_2)} \otimes b_2^*$.

(2) $\varphi(b_1 \otimes t_\mu \otimes b_2) = \varphi(b_2) \otimes t_{\mu+\mathrm{wt}(b_1)+\mathrm{wt}(b_2)} \otimes \varphi(b_1)$.

We define, for $b \in B$ with $B = B(\widetilde{U}_q(\mathfrak{g})), B(\infty)$, or $B(-\infty)$,

$$\begin{aligned}&\varepsilon_i^*(b) = \varepsilon_i(b^*),\ \varphi_i^*(b) = \varphi_i(b^*),\ \mathrm{wt}^*(b) = \mathrm{wt}(b^*),\\ &\tilde{e}_i^*(b) = \tilde{e}_i(b^*)^*,\ \text{and}\ \tilde{f}_i^*(b) = \tilde{f}_i(b^*)^*.\end{aligned}$$

This defines another crystal structure on $\widetilde{U}_q(\mathfrak{g})$: For $b_1 \in B(\infty)$, $b_2 \in B(-\infty)$, and $\eta \in \mathsf{P}$, we have

$$\begin{aligned}\varepsilon_i^*(b_1 \otimes t_\eta \otimes b_2) &= \max(\varepsilon_i^*(b_1), \varphi_i^*(b_2) + \langle h_i, \eta\rangle),\\ \varphi_i^*(b_1 \otimes t_\eta \otimes b_2) &= \max(\varepsilon_i^*(b_1) - \langle h_i, \eta\rangle, \varphi_i^*(b_2)),\\ &= \varepsilon_i^*(b_1 \otimes t_\eta \otimes b_2) + \langle h_i, \mathrm{wt}^*(b_1 \otimes t_\eta \otimes b_2)\rangle,\\ \mathrm{wt}^*(b_1 \otimes t_\eta \otimes b_2) &= -\eta,\\ \tilde{e}_i^*(b_1 \otimes t_\eta \otimes b_2) &= \begin{cases}(\tilde{e}_i^* b_1) \otimes t_{\eta-\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) \ge \varphi_i^*(b_2) + \langle h_i, \eta\rangle,\\ b_1 \otimes t_{\eta-\alpha_i} \otimes (\tilde{e}_i^* b_2) & \text{if } \varepsilon_i^*(b_1) < \varphi_i^*(b_2) + \langle h_i, \eta\rangle,\end{cases}\\ \tilde{f}_i^*(b_1 \otimes t_\eta \otimes b_2) &= \begin{cases}(\tilde{f}_i^* b_1) \otimes t_{\eta+\alpha_i} \otimes b_2 & \text{if } \varepsilon_i^*(b_1) > \varphi_i^*(b_2) + \langle h_i, \eta\rangle,\\ b_1 \otimes t_{\eta+\alpha_i} \otimes (\tilde{f}_i^* b_2) & \text{if } \varepsilon_i^*(b_1) \le \varphi_i^*(b_2) + \langle h_i, \eta\rangle.\end{cases}\end{aligned}$$

In particular, we have

$$\tilde{e}_i \circ \varphi = \varphi \circ \tilde{f}_i^* \quad \text{and} \quad \tilde{f}_i \circ \varphi = \varphi \circ \tilde{e}_i^* \quad \text{for every } i \in I.$$

8.4. **Relationship of $A_q(\mathfrak{g})$ and $\widetilde{U}_q(\mathfrak{g})$.** There exists a canonical pairing $A_q(\mathfrak{g}) \times \widetilde{U}_q(\mathfrak{g}) \to \mathbb{Q}(q)$ by

$$\langle \psi, x a_\mu\rangle = \delta_{\mu, \mathrm{wt}_\mathrm{l}(\psi)} \psi(x) \quad \text{for any } \psi \in A_q(\mathfrak{g}),\ x \in U_q(\mathfrak{g}), \text{ and } \mu \in \mathsf{P}.$$

Theorem 8.4.1 ([19]). *There exists a bi-crystal embedding*

$$\bar{\iota}_\mathfrak{g} \colon B(A_q(\mathfrak{g})) \longrightarrow B(\widetilde{U}_q(\mathfrak{g}))$$

which satisfies

$$\langle G^{\mathrm{up}}(b), \varphi(G^{\mathrm{low}}(b'))\rangle = \delta_{\bar{\iota}_\mathfrak{g}(b), b'}$$

for any $b \in B(A_q(\mathfrak{g}))$ and $b' \in B(\widetilde{U}_q(\mathfrak{g}))$.

8.5. **Relationship of $A_q(\mathfrak{g})$ and $A_q(\mathfrak{n})$.**

Definition 8.5.1. *Let $p_\mathfrak{n} \colon A_q(\mathfrak{g}) \to A_q(\mathfrak{n})$ be the homomorphism induced by $U_q^+(\mathfrak{g}) \to U_q(\mathfrak{g})$,*

$$\langle p_\mathfrak{n}(\psi), x\rangle = \psi(x) \quad \textit{for any } x \in U_q^+(\mathfrak{g}).$$

Then we have

$$\mathrm{wt}(p_\mathfrak{n}(\psi)) = \mathrm{wt}_\mathrm{l}(\psi) - \mathrm{wt}_\mathrm{r}(\psi).$$

It is obvious that $p_\mathfrak{n}$ sends all $\Phi(u_{w\lambda} \otimes u^\mathrm{r}_{w\lambda})$ ($\lambda \in \mathsf{P}^+$ and $w \in W$) to 1. Note that $\bar{\iota}_\mathfrak{g}(u_{w\lambda} \otimes u^\mathrm{r}_{w\lambda}) = b_\infty \otimes t_{w\lambda} \otimes b_{-\infty} \in B(\widetilde{U}_q(\mathfrak{g}))$.

Proposition 8.5.2. *For $b \in B(A_q(\mathfrak{g}))$, set*

$$\bar{\iota}_\mathfrak{g}(b) = b_1 \otimes t_\zeta \otimes b_2 \in B(\infty) \otimes T_\zeta \otimes B(-\infty) \subset B(\widetilde{U}_q(\mathfrak{g}))$$

($\zeta \in \mathsf{P}$). *Then we have*

$$p_\mathfrak{n}(G^{\mathrm{up}}(b)) = \delta_{b_2, b_{-\infty}} G^{\mathrm{up}}(b_1).$$

Proof. Set $\eta := \mathrm{wt}(b_1) + \zeta + \mathrm{wt}(b_2) = \mathrm{wt}_{\mathrm{l}}(b)$. Then for any $\tilde{b} \in B(\infty)$, we have

$$\begin{aligned}
&\big\langle p_{\mathfrak{n}}(G^{\mathrm{up}}(b)), \varphi(G^{\mathrm{low}}(\tilde{b}))\big\rangle = \big\langle G^{\mathrm{up}}(b), G^{\mathrm{low}}(\varphi(\tilde{b}))a_\eta\big\rangle \\
&\quad = \big\langle G^{\mathrm{up}}(b), G^{\mathrm{low}}(b_\infty \otimes t_\eta \otimes \varphi(\tilde{b}))\big\rangle = \big\langle G^{\mathrm{up}}(b), \varphi(G^{\mathrm{low}}(\tilde{b} \otimes t_{\eta-\mathrm{wt}(\tilde{b})} \otimes b_{-\infty}))\big\rangle \\
&\quad = \delta(\bar{\iota}_{\mathfrak{g}}(b) = \tilde{b} \otimes t_{\eta-\mathrm{wt}(\tilde{b})} \otimes b_{-\infty}) = \delta(b_2 = b_{-\infty}, b_1 = \tilde{b}). \qquad \square
\end{aligned}$$

Hence the map $p_{\mathfrak{n}}$ sends the upper global basis of $A_q(\mathfrak{g})$ to the upper global basis of $A_q(\mathfrak{n})$ or zero. Thus we have a map

$$\overline{p}_{\mathfrak{n}} \colon B(A_q(\mathfrak{g})) \to B(A_q(\mathfrak{n})) \bigsqcup \{0\}.$$

Although the map $p_{\mathfrak{n}}$ is not an algebra homomorphism, it preserves the multiplications up to a power of q, as we will see below.

Lemma 8.5.3. *For $x \in U_q^+(\mathfrak{g})$, if $\Delta_{\mathfrak{n}}(x) = x_{(1)} \otimes x_{(2)}$, then*

$$\Delta_+(x) = q^{\mathrm{wt}(x_{(1)})} x_{(2)} \otimes x_{(1)}. \tag{8.5}$$

Proof. Assume that (8.5) holds for $x \in U_q^+(\mathfrak{g})$. Note that

$$\Delta_{\mathfrak{n}}(e_i x) = (e_i \otimes 1 + 1 \otimes e_i)(x_{(1)} \otimes x_{(2)}) = e_i x_{(1)} \otimes x_{(2)} + q^{-(\alpha_i, \mathrm{wt}(x_{(1)}))} x_{(1)} \otimes (e_i x_{(2)}).$$

On the other hand, we have

$$\begin{aligned}
\Delta_+(e_i x) &= (e_i \otimes 1 + q^{\alpha_i} \otimes e_i)(q^{\mathrm{wt}(x_{(1)})} x_{(2)} \otimes x_{(1)}) \\
&= (e_i q^{\mathrm{wt}(x_1)}) x_{(2)} \otimes x_{(1)} + (q^{\alpha_i + \mathrm{wt}(x_{(1)})} x_{(2)}) \otimes (e_i x_{(1)}) \\
&= q^{-(\alpha_i, \mathrm{wt}(x_{(1)}))} (q^{\mathrm{wt}(x_{(1)})} e_i x_{(2)}) \otimes x_{(1)} + (q^{\mathrm{wt}(e_i x_{(1)})} x_{(2)}) \otimes (e_i x_{(1)}).
\end{aligned}$$

Hence (8.5) holds for $e_i x$. $\square$

Proposition 8.5.4. *For $\psi, \theta \in A_q(\mathfrak{g})$, we have*

$$p_{\mathfrak{n}}(\psi\theta) = q^{(\mathrm{wt}_{\mathrm{r}}(\psi), \mathrm{wt}_{\mathrm{r}}(\theta) - \mathrm{wt}_{\mathrm{l}}(\theta))} p_{\mathfrak{n}}(\psi) p_{\mathfrak{n}}(\theta).$$

Proof. For $x \in U_q^+(\mathfrak{g})$, set $\Delta_{\mathfrak{n}}(x) = x_{(1)} \otimes x_{(2)}$. Then, we have

$$\begin{aligned}
\langle p_{\mathfrak{n}}(\psi\theta), x\rangle &= \langle \psi\theta, x\rangle = \langle \psi \otimes \theta, q^{\mathrm{wt}(x_{(1)})} x_{(2)} \otimes x_{(1)}\rangle = \langle \psi, q^{\mathrm{wt}(x_{(1)})} x_{(2)}\rangle \langle \theta, x_{(1)}\rangle \\
&= q^{(\mathrm{wt}_{\mathrm{r}}(\psi), \mathrm{wt}(x_{(1)}))} \langle \psi, x_{(2)}\rangle \langle \theta, x_{(1)}\rangle \\
&= q^{(\mathrm{wt}_{\mathrm{r}}(\psi), \mathrm{wt}(x_{(1)}))} \langle p_{\mathfrak{n}}(\psi), x_{(2)}\rangle \langle p_{\mathfrak{n}}(\theta), x_{(1)}\rangle \\
&\underset{\text{(a)}}{=} q^{(\mathrm{wt}_{\mathrm{r}}(\psi), \mathrm{wt}_{\mathrm{r}}(\theta) - \mathrm{wt}_{\mathrm{l}}(\theta))} \langle p_{\mathfrak{n}}(\psi) \otimes p_{\mathfrak{n}}(\theta), \Delta_{\mathfrak{n}}(x)\rangle \\
&= q^{(\mathrm{wt}_{\mathrm{r}}(\psi), \mathrm{wt}_{\mathrm{r}}(\theta) - \mathrm{wt}_{\mathrm{l}}(\theta))} \langle p_{\mathfrak{n}}(\psi) p_{\mathfrak{n}}(\theta), x\rangle.
\end{aligned}$$

Here, we used $\mathrm{wt}(x_{(1)}) = -\mathrm{wt}\big(p_{\mathfrak{n}}(\theta)\big)$ in (a). $\square$

8.6. Global basis of $\widetilde{U}_q(\mathfrak{g})$ and tensor products of $U_q(\mathfrak{g})$-modules in $\mathscr{O}_{\mathrm{int}}(\mathfrak{g})$.

Let V be an integrable $U_q(\mathfrak{g})$-module with a bar-involution $-$; that is, there is a $\mathbb{Q}$-linear automorphism $-$ satisfying $\overline{Pv} = \overline{P}\,\overline{v}$ for all $P \in U_q(\mathfrak{g})$ and for all $v \in V$. Then, for any $\lambda \in \mathsf{P}^+$, there exists a unique bar-involution $-$ on $V(\lambda) \otimes_- V$ satisfying

$$\overline{(u_\lambda \otimes_- v)} = u_\lambda \otimes_- \overline{v} \text{ for any } v \in V.$$

Indeed, there exists $\Xi \in \mathbf{1} + \prod_{\beta \in \mathsf{Q}_+ \setminus \{0\}} U_q^+(\mathfrak{g})_\beta \otimes U_q^-(\mathfrak{g})_{-\beta}$, which defines a bar-involution by setting

$$\overline{u \otimes_- v} := \Xi\big(\overline{u} \otimes_- \overline{v}\big)$$

(see [33, Chapter 4]). Assume that V has a lower crystal basis $\big(L(V), B(V)\big)$ and an $\mathbf{A}$-form $V_\mathbf{A}$ such that $\big(V_\mathbf{A}, L(V), \overline{L(V)}\big)$ is balanced. Then we have

Proposition 8.6.1. *The triple* $\Big(V(\lambda)_\mathbf{A} \otimes_\mathbf{A} V_\mathbf{A}, L(\lambda) \otimes_{\mathbf{A}_0} L(V), \overline{L(\lambda) \otimes_{\mathbf{A}_0} L(V)}\Big)$ *in* $V(\lambda) \otimes_- V$ *is balanced.*

Note that $u_\lambda \otimes_- G^{\mathrm{low}}(b)$ is a lower global basis for any $b \in B(V)$, i.e., $G^{\mathrm{low}}(u_\lambda \otimes b) = u_\lambda \otimes_- G^{\mathrm{low}}(b)$.

In particular, it applies to $V(\lambda) \otimes_- V(\mu)$. Moreover, we have the following proposition.

Proposition 8.6.2. *Let* $\lambda, \mu \in \mathsf{P}^+$ *and* $w \in W$. *Then for any* $b \in B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+w\mu})$, $G^{\mathrm{low}}(b)(u_\lambda \otimes_- u_{w\mu})$ *vanishes or is a member of the lower global basis of* $V(\lambda) \otimes_- V(\mu)$.

Hence we have a crystal morphism

$$\pi_{\lambda,w\mu} \colon B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+w\mu}) \to B(\lambda) \otimes B(\mu) \tag{8.6}$$

by $G^{\mathrm{low}}(b)(u_\lambda \otimes_- u_{w\mu}) = G^{\mathrm{low}}(\pi_{\lambda,w\mu}(b))$.

Similarly, we have a bar-involution $-$ on $V \otimes_+ V(\lambda)$ such that

$$\overline{(v \otimes_+ u_\lambda)} = \overline{v} \otimes_+ u_\lambda \text{ for any } v \in V.$$

Hence if V has an upper crystal basis $(L^{\mathrm{up}}(V), B(V))$ and an $\mathbf{A}$-form $V_\mathbf{A}$ such that $\big(V_\mathbf{A}, L^{\mathrm{up}}(V), \overline{L^{\mathrm{up}}(V)}\big)$ is balanced, then $V \otimes_+ V(\lambda)$ has an upper global basis. Note that $G^{\mathrm{up}}(b) \otimes_+ u_\lambda$ is a member of the upper global basis for $b \in B(V)$.

In particular for $\lambda, \mu \in \mathsf{P}$, $V(\lambda) \otimes_- V(\mu)$ has a lower global basis and $V(\lambda) \otimes_+ V(\mu)$ has an upper global basis.

The bilinear form

$$(\,\bullet\,,\,\bullet\,)\colon \Big(V(\lambda) \otimes_- V(\mu)\Big) \times \Big(V(\lambda) \otimes_+ V(\mu)\Big) \to \mathbf{k}$$

defined by $(u \otimes_- v, u' \otimes_+ v') = (u, u')(v, v')$, $u, u' \in V(\lambda)$, $v, v' \in V(\mu)$ satisfies

$$(ax, y) = (x, \varphi(a)y) \text{ for any } x \in V(\lambda) \otimes_- V(\mu),\ y \in V(\lambda) \otimes_+ V(\mu),\ a \in U_q(\mathfrak{g}).$$

With respect to this bilinear form, the lower global basis of $V(\lambda) \otimes_- V(\mu)$ and the upper global basis of $V(\lambda) \otimes_+ V(\mu)$ are the dual bases of each other.

9. Quantum minors and T-systems

9.1. **Quantum minors.** Using the isomorphism Φ in (8.1), for each $\lambda \in \mathsf{P}^+$ and $\mu, \zeta \in W\lambda$, we define the elements

$$\Delta(\mu, \zeta) := \Phi(u_\mu \otimes u^{\mathrm{r}}_\zeta) \in A_q(\mathfrak{g})$$

and

$$\mathrm{D}(\mu, \zeta) := p_\mathfrak{n}(\Delta(\mu, \zeta)) \in A_q(\mathfrak{n}).$$

The element $\Delta(\mu, \zeta)$ is called a (*generalized*) *quantum minor* and $\mathrm{D}(\mu, \zeta)$ is called *a unipotent quantum minor.*

Lemma 9.1.1. $\Delta(\mu, \zeta)$ *is a member of the upper global basis of* $A_q(\mathfrak{g})$. *Moreover,* $\mathrm{D}(\mu, \zeta)$ *is either a member of the upper global basis of* $A_q(\mathfrak{n})$ *or zero.*

Proof. Our assertions follow from Proposition 8.1.3 and Proposition 8.5.2. □

Lemma 9.1.2 ([3, (9.13)]). *For $u, v \in W$ and $\lambda, \mu \in \mathsf{P}^+$, we have*

$$\Delta(u\lambda, v\lambda)\Delta(u\mu, v\mu) = \Delta\big(u(\lambda+\mu), v(\lambda+\mu)\big).$$

By Proposition 8.5.4, we have the following corollary.

Corollary 9.1.3. *For $u, v \in W$ and $\lambda, \mu \in \mathsf{P}^+$, we have*

$$\mathrm{D}(u\lambda, v\lambda)\mathrm{D}(u\mu, v\mu) = q^{-(v\lambda, v\mu - u\mu)}\mathrm{D}\big(u(\lambda+\mu), v(\lambda+\mu)\big).$$

Note that

$$\mathrm{D}(\mu, \mu) = 1 \quad \text{for } \mu \in W\lambda\ .$$

Then $\mathrm{D}(\mu, \zeta) \neq 0$ if and only if $\mu \preceq \zeta$. Recall that for μ, ζ in the same W-orbit, we say that $\mu \preceq \zeta$ if there exists a sequence $\{\beta_k\}_{1\le k\le l}$ of positive real roots such that, defining $\lambda_0 = \zeta$, $\lambda_k = s_{\beta_k}\lambda_{k-1}$ $(1 \le k \le l)$, we have $(\beta_k, \lambda_{k-1}) \ge 0$ and $\lambda_l = \mu$.

More precisely, we have the following lemma.

Lemma 9.1.4. *Let $\lambda \in \mathsf{P}^+$ and $\mu, \zeta \in W\lambda$. Then the following conditions are equivalent:*

- (i) $\mathrm{D}(\mu, \zeta)$ *is an element of the upper global basis of* $A_q(\mathfrak{n})$,
- (ii) $\mathrm{D}(\mu, \zeta) \neq 0$,
- (iii) $u_\mu \in U_q^-(\mathfrak{g})u_\zeta$,
- (iv) $u_\zeta \in U_q^+(\mathfrak{g})u_\mu$,
- (v) $\mu \preceq \zeta$,
- (vi) *for any $w \in W$ such that $\mu = w\lambda$, there exists $u \le w$ (in the Bruhat order) such that $\zeta = u\lambda$,*
- (vii) *there exist $u, v \in W$ such that $\mu = w\lambda$, $\zeta = u\lambda$, and $u \le w$.*

Proof. (i) and (ii) are equivalent by Lemma 9.1.1. The equivalence of (ii), (iii), and (iv) is obvious. The equivalence of (v), (vi), and (vii) is well known. The equivalence of (iv) and (vi) is proved in [18]. ☐

For any $u \in A_q(\mathfrak{n}) \setminus \{0\}$ and $i \in I$, we set

$$\varepsilon_i(u) := \max\{n \in \mathbb{Z}_{\ge 0} \mid e_i^n u \neq 0\}\,,$$
$$\varepsilon_i^*(u) := \max\{n \in \mathbb{Z}_{\ge 0} \mid {e_i^*}^n u \neq 0\}\,.$$

Then for any $b \in B(A_q(\mathfrak{n}))$, we have

$$\varepsilon_i(G^{\mathrm{up}}(b)) = \varepsilon_i(b) \text{ and } \varepsilon_i^*(G^{\mathrm{up}}(b)) = \varepsilon_i^*(b).$$

Lemma 9.1.5. *Let $\lambda \in \mathsf{P}^+$, $\mu, \zeta \in W\lambda$ such that $\mu \preceq \zeta$ and $i \in I$.*

- (i) *If $n := \langle h_i, \mu\rangle \ge 0$, then*

$$\varepsilon_i(\mathrm{D}(\mu, \zeta)) = 0 \quad \textit{and} \quad e_i^{(n)}\mathrm{D}(s_i\mu, \zeta) = \mathrm{D}(\mu, \zeta).$$

- (ii) *If $\langle h_i, \mu\rangle \le 0$ and $s_i\mu \preceq \zeta$, then $\varepsilon_i(\mathrm{D}(\mu, \zeta)) = -\langle h_i, \mu\rangle$.*
- (iii) *If $m := -\langle h_i, \zeta\rangle \ge 0$, then*

$$\varepsilon_i^*(\mathrm{D}(\mu, \zeta)) = 0 \quad \textit{and} \quad e_i^{*(m)}\mathrm{D}(\mu, s_i\zeta) = \mathrm{D}(\mu, \zeta).$$

- (iv) *If $\langle h_i, \zeta\rangle \ge 0$ and $\mu \preceq s_i\zeta$, then $\varepsilon_i^*(\mathrm{D}(\mu, \zeta)) = \langle h_i, \zeta\rangle$.*

Proof. We have $\varepsilon_i\big(\Delta(\mu,\zeta)\big) = \max(-\langle h_i,\mu\rangle, 0)$ and $\varepsilon_i^*\big(\Delta(\mu,\zeta)\big) = \max(\langle h_i,\zeta\rangle, 0)$. Moreover, $p_{\mathfrak{n}}$ commutes with $e_i^{(n)}$ and $e_i^{*(n)}$.

Let us show (ii). Set $\ell = -\langle h_i,\mu\rangle$. Then we have $e_i^{\ell+1}\Delta(\mu,\zeta) = 0$, which implies $e_i^{\ell+1}\mathrm{D}(\mu,\zeta) = 0$. Hence $\varepsilon_i(\mathrm{D}(\mu,\zeta)) \le \ell$. We have

$$e_i^{(\ell)}\Delta(\mu,\zeta) = \Delta(s_i\mu,\zeta).$$

Hence we have $e_i^{(\ell)}\mathrm{D}(\mu,\zeta) = \mathrm{D}(s_i\mu,\zeta)$. By the assumption $s_i\mu \preceq \zeta$, $\mathrm{D}(s_i\mu,\zeta)$ does not vanish. Hence we have $\varepsilon_i(\mathrm{D}(\mu,\zeta)) \ge \ell$.

The other statements can be proved similarly. ☐

Proposition 9.1.6 ([3, (10.2)]). *Let $\lambda,\mu \in \mathsf{P}^+$ and $s,t,s',t' \in W$ such that $\ell(s's) = \ell(s') + \ell(s)$ and $\ell(t't) = \ell(t') + \ell(t)$. Then we have*

(i) $\Delta(s's\lambda, t'\lambda)\Delta(s'\mu, t't\mu) = q^{(s\lambda,\mu)-(\lambda,t\mu)}\Delta(s'\mu, t't\mu)\Delta(s's\lambda, t'\lambda)$.

(ii) *If we assume further that $s's\lambda \preceq t'\lambda$ and $s'\mu \preceq t't\mu$, then we have*

$$\mathrm{D}(s's\lambda, t'\lambda)\mathrm{D}(s'\mu, t't\mu) = q^{(s's\lambda+t'\lambda,\ s'\mu-t't\mu)}\mathrm{D}(s'\mu, t't\mu)\mathrm{D}(s's\lambda, t'\lambda), \tag{9.1}$$

or equivalently

$$q^{(t'\lambda,\ t't\mu-s'\mu)}\mathrm{D}(s's\lambda, t'\lambda)\mathrm{D}(s'\mu, t't\mu) = q^{(s'\mu-t't\mu,\ s's\lambda)}\mathrm{D}(s'\mu, t't\mu)\mathrm{D}(s's\lambda, t'\lambda). \tag{9.2}$$

Note that (ii) follows from Proposition 8.5.4 and (i). Note also that both sides of (9.2) are bar-invariant, and hence they are members of the upper global basis as seen by Corollary 4.1.5.

Proposition 9.1.7. *For $\lambda,\mu \in \mathsf{P}^+$ and $s,t \in W$, set $\bar{\iota}_{\mathfrak{g}}\big(u_{s\lambda}\otimes(u_\lambda)^{\mathrm{r}}\big) = b_-\otimes t_\lambda\otimes b_{-\infty}$ and $\bar{\iota}_{\mathfrak{g}}\big(u_\mu\otimes(u_{t\mu})^{\mathrm{r}}\big) = b_\infty\otimes t_{t\mu}\otimes b_+$ with $b_\mp \in B(\pm\infty)$. Then we have*

$$\Delta(s\lambda,\lambda)\Delta(\mu,t\mu) = G^{\mathrm{up}}\big(\bar{\iota}_{\mathfrak{g}}{}^{-1}(b_-\otimes t_{\lambda+t\mu}\otimes b_+)\big).$$

Proof. Recall that there is a pairing $(\,\bullet\,,\,\bullet\,) : \big(V(\lambda)\otimes_- V(\mu)\big)\times\big(V(\lambda)\otimes_+ V(\mu)\big) \to \mathbb{Q}(q)$ defined by $(u\otimes_- v, u'\otimes_+ v') = (u,u')(v,v')$. It satisfies

$$\big(P(u\otimes_- v), u'\otimes_+ v'\big) = \big(u\otimes_- v, \varphi(P)(u'\otimes_+ v')\big) \quad \text{for any } P \in U_q(\mathfrak{g}).$$

For $u,u' \in V(\lambda)$ and $v,v' \in V(\mu)$, we have

$$\begin{aligned}\langle \Phi(u\otimes u'^{\mathrm{r}})\Phi(v\otimes v'^{\mathrm{r}}), P\rangle &= \big(u'\otimes_- v', P(u\otimes_+ v)\big)\\ &= \big(\varphi(P)(u'\otimes_- v'), u\otimes_+ v\big).\end{aligned}$$

Hence for $P \in U_q(\mathfrak{g})$, we have

$$\langle \Delta(s\lambda,\lambda)\Delta(\mu,t\mu), Pa_\zeta\rangle = \delta(\zeta = s\lambda+\mu)\big(\varphi(P)(u_\lambda\otimes_- u_{t\mu}), u_{s\lambda}\otimes_+ u_\mu\big).$$

If $Pa_\zeta = G^{\mathrm{low}}(\varphi(b))$ for $b \in B(\widetilde{U}_q(\mathfrak{g}))$, then we have

$$\langle \Delta(s\lambda,\lambda)\Delta(\mu,t\mu), \varphi(G^{\mathrm{low}}(b))\rangle = \delta(\zeta = s\lambda+\mu)\big(G^{\mathrm{low}}(b)(u_\lambda\otimes_- u_{t\mu}), u_{s\lambda}\otimes_+ u_\mu\big).$$

The element $G^{\mathrm{low}}(b)(u_\lambda\otimes_- u_{t\mu})$ vanishes or is a global basis of $V(\lambda)\otimes_- V(\mu)$ by Proposition 8.6.2. Since $u_{s\lambda}\otimes_+ u_\mu$ is a member of the upper global basis of $V(\lambda)\otimes_+ V(\mu)$, we have

$$\langle \Delta(s\lambda,\lambda)\Delta(\mu,t\mu), \varphi(G^{\mathrm{low}}(b))\rangle = \delta(\zeta = s\lambda+\mu)\delta\big(\pi_{\lambda,t\mu}(b) = u_{s\lambda}\otimes u_\mu\big).$$

Here $\pi_{\lambda,t\mu}\colon B(\widetilde{U}_q(\mathfrak{g})a_{\lambda+t\mu}) \to B(\lambda)\otimes B(\mu)$ is the crystal morphism given in (8.6).

Hence we obtain

$$\Delta(s\lambda,\lambda)\Delta(\mu,t\mu)=G^{\mathrm{up}}(\bar{\iota}_{\mathfrak{g}}{}^{-1}(b)),$$

where $b\in B(\widetilde{U}_q(\mathfrak{g}))$ is a unique element such that

$$\big(G^{\mathrm{low}}(b)(u_\lambda\otimes_- u_{s\mu}),u_{s\lambda}\otimes_+ u_\mu\big)=\ 1$$

. On the other hand, we have $G^{\mathrm{low}}(b_+)u_{t\mu}=u_\mu$ and $G^{\mathrm{low}}(b_-)u_\lambda=u_{s\lambda}$. The last equality implies $\varphi(G^{\mathrm{low}}(b_-))u_{s\lambda}=u_\lambda$ because

$$(\varphi(G^{\mathrm{low}}(b_-))u_{s\lambda},u_\lambda)=(u_{s\lambda},G^{\mathrm{low}}(b_-)u_\lambda)=(u_{s\lambda},u_{s\lambda})=1.$$

As seen in (8.4), we have

$$G^{\mathrm{low}}(b_-)G^{\mathrm{low}}(b_+)a_{\lambda+t\mu}-G^{\mathrm{low}}(b_-\otimes t_{\lambda+t\mu}\otimes b_+)\in U_q^-(\mathfrak{g})_{>s\lambda-\lambda}U_q^+(\mathfrak{g})_{<\mu-t\mu}a_{\lambda+t\mu}.$$

Hence we obtain

$$\begin{aligned}\big(G^{\mathrm{low}}&(b_-\otimes t_{\lambda+t\mu}\otimes b_+)(u_\lambda\otimes_- u_{t\mu}),\ u_{s\lambda}\otimes_+ u_\mu\big)\\&=\big(G^{\mathrm{low}}(b_-)G^{\mathrm{low}}(b_+)(u_\lambda\otimes_- u_{t\mu}),\ u_{s\lambda}\otimes_+ u_\mu\big)\\&\qquad=\big(G^{\mathrm{low}}(b_+)(u_\lambda\otimes_- u_{t\mu}),\ \varphi(G^{\mathrm{low}}(b_-))(u_{s\lambda}\otimes_+ u_\mu)\big)=1.\end{aligned}$$

In the last equality, we used $G^{\mathrm{low}}(b_+)(u_\lambda\otimes_- u_{t\mu})=u_\lambda\otimes_-(G^{\mathrm{low}}(b_+)u_{t\mu})=u_\lambda\otimes_- u_\mu$ and $\varphi(G^{\mathrm{low}}(b_-))(u_{s\lambda}\otimes_+ u_\mu)=\big(\varphi(G^{\mathrm{low}}(b_-))u_{s\lambda}\big)\otimes_+ u_\mu=u_\lambda\otimes_+ u_\mu$.

Hence we conclude that $b=b_-\otimes t_{\lambda+t\mu}\otimes b_+$. □

Let

$$\iota_{\lambda,\mu}\colon V(\lambda+\mu)\rightarrowtail V(\lambda)\otimes V(\mu)$$

be the canonical embedding and

$$\bar{\iota}_{\lambda,\mu}\colon B(\lambda+\mu)\rightarrowtail B(\lambda)\otimes B(\mu)$$

the induced crystal embedding.

Lemma 9.1.8. *For $\lambda,\mu\in\mathsf{P}^+$ and $x,y\in W$ such that $x\ge y$, we have*

$$u_{x\lambda}\otimes u_{y\mu}\in\bar{\iota}_{\lambda,\mu}(B(\lambda+\mu))\subset B(\lambda)\otimes B(\mu).$$

Proof. Let us show by induction on $\ell(x)$ the length of x in W. We may assume that $x\ne 1$. Then there exists $i\in I$ such that $s_ix<x$. If $s_iy<y$, then $s_ix\ge s_iy$ and $\tilde{e}_i^{\max}(u_{x\lambda}\otimes u_{y\mu})=u_{s_ix\lambda}\otimes u_{s_iy\mu}$. If $s_iy>y$, then $s_ix\ge y$ and $\tilde{e}_i^{\max}(u_{x\lambda}\otimes u_{y\mu})=u_{s_ix\lambda}\otimes u_{y\mu}$. In both cases, $u_{x\lambda}\otimes u_{y\mu}$ is connected with an element of $\bar{\iota}_{\lambda,\mu}(B(\lambda+\mu))$. □

Lemma 9.1.9. *For $\lambda,\mu\in\mathsf{P}^+$ and $w\in W$, we have*

$$\Delta(w\lambda,\lambda)\Delta(\mu,\mu)=G^{\mathrm{up}}\big(\bar{\iota}_{\lambda,\mu}^{-1}(u_{w\lambda}\otimes u_\mu)\otimes u_{\lambda+\mu}{}^{\mathrm{r}}\big).$$

Proof. We have

$$\begin{aligned}\bar{\iota}_{\mathfrak{g}}(u_{w\lambda}\otimes u_\lambda^{\mathrm{r}})&=b_{w\lambda}\otimes t_\lambda\otimes b_{-\infty},\\ \bar{\iota}_{\mathfrak{g}}(u_\mu\otimes u_\mu^{\mathrm{r}})&=b_\infty\otimes t_\mu\otimes b_{-\infty},\end{aligned}$$

where $b_{w\lambda}:=\bar{\iota}_\lambda(u_{w\lambda})$. Hence Proposition 9.1.7 implies that

$$\Delta(w\lambda,\lambda)\Delta(\mu,\mu)=G^{\mathrm{up}}\big(\bar{\iota}_{\mathfrak{g}}{}^{-1}(b_{w\lambda}\otimes t_{\lambda+\mu}\otimes b_{-\infty})\big).$$

Then, $\bar{\iota}_{\mathfrak{g}}\big(\bar{\iota}_{\lambda,\mu}^{-1}(u_{w\lambda}\otimes u_\mu)\otimes u_{\lambda+\mu}{}^{\mathrm{r}}\big)=b_{w\lambda}\otimes t_{\lambda+\mu}\otimes b_{-\infty}$ gives the desired result. □

9.2. T**-system.** In this subsection, we recall the T*-system* among the (unipotent) quantum minors for later use (see [25] for T-system).

Proposition 9.2.1 ([11, Proposition 3.2])**.** *Assume that the Kac–Moody algebra* $\mathfrak{g}$ *is of symmetric type. Assume that* $u, v \in W$ *and* $i \in I$ *satisfy* $u < us_i$ *and* $v < vs_i$*. Then*

$$\begin{aligned}&\Delta(us_i\varpi_i, vs_i\varpi_i)\Delta(u\varpi_i, v\varpi_i) = q^{-1}\Delta(us_i\varpi_i, v\varpi_i)\Delta(u\varpi_i, vs_i\varpi_i) + \Delta(u\lambda, v\lambda),\\ &\Delta(u\varpi_i, v\varpi_i)\Delta(us_i\varpi_i, vs_i\varpi_i) = q\Delta(u\varpi_i\ , vs_i\varpi_i)\Delta(us_i\varpi_i, v\varpi_i) + \Delta(u\lambda, v\lambda),\end{aligned}$$

and

$$\begin{aligned}&q^{(vs_i\varpi_i, v\varpi_i - u\varpi_i)}\mathrm{D}(us_i\varpi_i, vs_i\varpi_i)\mathrm{D}(u\varpi_i, v\varpi_i)\\ &\quad = q^{-1+(v\varpi_i, vs_i\varpi_i - u\varpi_i)}\mathrm{D}(us_i\varpi_i, v\varpi_i)\mathrm{D}(u\varpi_i, vs_i\varpi_i) + \mathrm{D}(u\lambda, v\lambda)\\ &\quad = q^{-1+(vs_i\varpi_i, v\varpi_i - us_i\varpi_i)}\mathrm{D}(u\varpi_i, vs_i\varpi_i)\mathrm{D}(us_i\varpi_i, v\varpi_i) + \mathrm{D}(u\lambda, v\lambda),\\ &q^{(v\varpi_i, vs_i\varpi_i - us_i\varpi_i)}\mathrm{D}(u\varpi_i, v\varpi_i)\mathrm{D}(us_i\varpi_i, vs_i\varpi_i)\\ &\quad = q^{1+(vs_i\varpi_i, v\varpi_i - us_i\varpi_i)}\mathrm{D}(u\varpi_i, vs_i\varpi_i)\mathrm{D}(us_i\varpi_i, v\varpi_i) + \mathrm{D}(u\lambda, v\lambda)\\ &\quad = q^{1+(v\varpi_i, vs_i\varpi_i - u\varpi_i)}\mathrm{D}(us_i\varpi_i, v\varpi_i)\mathrm{D}(u\varpi_i, vs_i\varpi_i) + \mathrm{D}(u\lambda, v\lambda),\end{aligned}$$

where $\lambda = s_i\varpi_i + \varpi_i$*.*

Note that the difference of λ and $-\sum_{j\neq i} a_{j,i}\varpi_j$ are W-invariant. Hence we have $\mathrm{D}(u\lambda, v\lambda) = \prod_{j\neq i}\mathrm{D}(u\varpi_j, v\varpi_j)^{-a_{j,i}}$ from Corollary 9.1.3, by disregarding a power of q.

9.3. **Revisit of crystal bases and global bases.** In order to prove Theorem 9.3.3 below, we first investigate the upper crystal lattice of $\mathbf{D}_\varphi V$ induced by an upper crystal lattice of $V \in \mathscr{O}_{\mathrm{int}}(\mathfrak{g})$.

Let V be a $U_q(\mathfrak{g})$-module in $\mathscr{O}_{\mathrm{int}}(\mathfrak{g})$. Let L^{up} be an upper crystal lattice of V. Then we have (see Lemma 1.3.1)

$$\bigoplus_{\xi\in\mathsf{P}} q^{(\xi,\xi)/2}(L^{\mathrm{up}})_\xi \text{ is a lower crystal lattice of } V.$$

Recall that, for $\lambda \in \mathsf{P}^+$, the upper crystal lattice $L^{\mathrm{up}}(\lambda)$ and the lower crystal lattice $L^{\mathrm{low}}(\lambda)$ of $V(\lambda)$ are related by

$$L^{\mathrm{up}}(\lambda) = \bigoplus_{\xi\in\mathsf{P}} q^{((\lambda,\lambda)-(\xi,\xi))/2}L^{\mathrm{low}}(\lambda)_\xi \subset L^{\mathrm{low}}(\lambda). \tag{9.3}$$

Write

$$V \simeq \bigoplus_{\lambda\in\mathsf{P}^+} E_\lambda \otimes V(\lambda)$$

with finite-dimensional $\mathbb{Q}(q)$-vector spaces E_λ. Accordingly, we have a canonical decomposition

$$L^{\mathrm{up}} \simeq \bigoplus_{\lambda\in\mathsf{P}^+} C_\lambda \otimes_{\mathbf{A}_0} L^{\mathrm{up}}(\lambda),$$

where $C_\lambda \subset E_\lambda$ is an $\mathbf{A}_0$-lattice of E_λ.

On the other hand, we have

$$\mathbf{D}_\varphi V \simeq \bigoplus_{\lambda\in\mathsf{P}^+} E_\lambda^* \otimes V(\lambda).$$

Note that we have

$$\Phi_V\left((a\otimes u)\otimes(b\otimes v)^{\mathrm{r}}\right)=\langle a,b\rangle\Phi_\lambda(u\otimes v^{\mathrm{r}})\quad\text{for } u,v\in V(\lambda)\text{ and } a\in E_\lambda,\ b\in E_\lambda^*.$$

We define the induced upper crystal lattice $\mathbf{D}_\varphi L^{\mathrm{up}}$ of $\mathbf{D}_\varphi V$ by

$$\mathbf{D}_\varphi L^{\mathrm{up}}:=\bigoplus_{\lambda\in\mathsf{P}^+}C_\lambda^\vee\otimes_{\mathbf{A}_0}L^{\mathrm{up}}(\lambda)\subset\mathbf{D}_\varphi V,$$

where $C_\lambda^\vee:=\{u\in E_\lambda^*\mid\langle u,C_\lambda\rangle\subset\mathbf{A}_0\}$. Then we have

$$\Phi_V\left(L^{\mathrm{up}}\otimes(\mathbf{D}_\varphi L^{\mathrm{up}})^{\mathrm{r}}\right)\subset L^{\mathrm{up}}(A_q(\mathfrak{g})).$$

Indeed, we have

$$\mathbf{D}_\varphi L^{\mathrm{up}}=\left\{u\in\mathbf{D}_\varphi V\mid\Phi_V(L^{\mathrm{up}}\otimes u^{\mathrm{r}})\subset L^{\mathrm{up}}(A_q(\mathfrak{g}))\right\}.$$

Since $(L^{\mathrm{up}}(\lambda))^\vee=L^{\mathrm{low}}(\lambda)$, we have

$$(L^{\mathrm{up}})^\vee=\bigoplus_{\lambda\in\mathsf{P}^+}C_\lambda^\vee\otimes_{\mathbf{A}_0}L^{\mathrm{low}}(\lambda).$$

The properties $L^{\mathrm{up}}(\lambda)\subset L^{\mathrm{low}}(\lambda)$ and $L^{\mathrm{up}}(\lambda)_\lambda=L^{\mathrm{low}}(\lambda)_\lambda$ imply the following lemma.

Lemma 9.3.1. *$\mathbf{D}_\varphi L^{\mathrm{up}}$ is the largest upper crystal lattice of $\mathbf{D}_\varphi V$ contained in the lower crystal lattice $(L^{\mathrm{up}})^\vee$.*

Let $\lambda,\mu\in\mathsf{P}^+$. Then $\bigl(L^{\mathrm{up}}(\lambda)\otimes_+L^{\mathrm{up}}(\mu)\bigr)^\vee=L^{\mathrm{low}}(\lambda)\otimes_-L^{\mathrm{low}}(\mu)$ is a lower crystal lattice of $\mathbf{D}_\varphi\bigl(V(\lambda)\otimes_+V(\mu)\bigr)\simeq V(\lambda)\otimes_-V(\mu)$. Let $\Xi_{\lambda,\mu}\colon V(\lambda)\otimes_+V(\mu)\xrightarrow{\sim}V(\lambda)\otimes_-V(\mu)\simeq\mathbf{D}_\varphi\bigl(V(\lambda)\otimes_+V(\mu)\bigr)$ be the $U_q(\mathfrak{g})$-module isomorphism defined by

$$\Xi_{\lambda,\mu}(u\otimes_+v)=q^{(\lambda,\mu)-(\xi,\eta)}\bigl(u\otimes_-v\bigr)\quad\text{for } u\in V(\lambda)_\xi\text{ and } v\in V(\mu)_\eta.$$

Then

$$\begin{aligned}\widetilde{L}\;&:=\;\Xi_{\lambda,\mu}\bigl(L^{\mathrm{up}}(\lambda)\otimes_+L^{\mathrm{up}}(\mu)\bigr)\\&=\;\bigoplus_{\xi,\eta\in\mathsf{P}}q^{(\lambda,\mu)-(\xi,\eta)}L^{\mathrm{up}}(\lambda)_\xi\otimes_-L^{\mathrm{up}}(\mu)_\eta\end{aligned}$$

is an upper crystal lattice of $V(\lambda)\otimes_-V(\mu)$. Since we have $(\lambda,\mu)-(\xi,\eta)\geq0$ for any $\xi\in\mathrm{wt}\bigl(V(\lambda)\bigr)$ and $\eta\in\mathrm{wt}\bigl(V(\mu)\bigr)$, Lemma 9.3.1 implies that

$$\widetilde{L}\subset\mathbf{D}_\varphi\bigl(L^{\mathrm{up}}(\lambda)\otimes_+L^{\mathrm{up}}(\mu)\bigr).\tag{9.4}$$

Lemma 9.3.2. *Let $\lambda,\mu\in\mathsf{P}^+$ and $x_1,x_2,y_1,y_2\in W$ such that $x_k\geq y_k$ $(k=1,2)$. Then we have*

$$\begin{aligned}&q^{(\lambda,\mu)-(x_2\lambda,y_2\mu)}\Delta(x_1\lambda,x_2\lambda)\Delta(y_1\mu,y_2\mu)\\&\qquad\equiv G^{\mathrm{up}}(\bar\iota_{\lambda,\mu}^{-1}(u_{x_1\lambda}\otimes u_{y_1\mu})\otimes\bar\iota_{\lambda,\mu}^{-1}(u_{x_2\lambda}\otimes u_{y_2\mu})^{\mathrm{r}})\mod qL^{\mathrm{up}}(A_q(\mathfrak{g})).\end{aligned}\tag{9.5}$$

Proof. By the definition, we have

$$\Delta(x_1\lambda,x_2\lambda)\Delta(y_1\mu,y_2\mu)=\Phi_{V(\lambda)\otimes_+V(\mu)}\bigl((u_{x_1\lambda}\otimes_+u_{y_1\mu})\otimes(u_{x_2\lambda}\otimes_-u_{y_2\mu})^{\mathrm{r}}\bigr).$$

Hence we have

$$\begin{aligned}&q^{(\lambda,\mu)-(x_2\lambda,y_2\mu)}\Delta(x_1\lambda,x_2\lambda)\Delta(y_1\mu,y_2\mu)\\&\quad=\Phi_{V(\lambda)\otimes_+V(\mu)}\bigl((u_{x_1\lambda}\otimes_+u_{y_1\mu})\otimes q^{(\lambda,\mu)-(x_2\lambda,y_2\mu)}(u_{x_2\lambda}\otimes_-u_{y_2\mu})^{\mathrm{r}}\bigr)\\&\qquad=\Phi_{V(\lambda)\otimes_+V(\mu)}\bigl((u_{x_1\lambda}\otimes_+u_{y_1\mu})\otimes\bigl(\Xi_{\lambda,\mu}(u_{x_2\lambda}\otimes_+u_{y_2\mu})\bigr)^{\mathrm{r}}\bigr).\end{aligned}$$

The right-hand side of (9.5) can be calculated as follows. Let us take $v_k \in L^{\mathrm{up}}(\lambda+\mu)$ such that $\iota_{\lambda,\mu}(v_k) - u_{x_k\lambda} \otimes_+ u_{y_k\mu} \in qL^{\mathrm{up}}(\lambda) \otimes_+ L^{\mathrm{up}}(\mu)$ for $k = 1, 2$. Here $\iota_{\lambda,\mu} \colon V(\lambda+\mu) \to V(\lambda) \otimes_+ V(\mu)$ denotes the canonical $U_q(\mathfrak{g})$-module homomorphism and such a v_k exists by Lemma 9.1.8.

Then we have

$$\begin{aligned} G^{\mathrm{up}}\Big(&\overline{\iota}_{\lambda,\mu}^{-1}(u_{x_1\lambda} \otimes u_{y_1\mu}) \otimes \big(\overline{\iota}_{\lambda,\mu}^{-1}(u_{x_2\lambda} \otimes u_{y_2\mu})\big)^{\mathrm{r}}\Big) \\ &\equiv \Phi_{\lambda+\mu}(v_1 \otimes v_2^{\mathrm{r}}) \mod qL^{\mathrm{up}}(A_q(\mathfrak{g})) \\ &= \Phi_{V(\lambda)\otimes_+ V(\mu)}\big(\iota_{\lambda,\mu}(v_1) \otimes \big(\Xi_{\lambda,\mu}\iota_{\lambda,\mu}(v_2)\big)^{\mathrm{r}}\big). \end{aligned}$$

The last equality follows from $(v_2, u) = (\Xi_{\lambda,\mu}\iota_{\lambda,\mu}(v_2), \iota_{\lambda,\mu}(u))$ for all $u \in V(\lambda+\mu)$.

On the other hand, we have

$$\iota_{\lambda,\mu}(v_1) \equiv u_{x_1\lambda} \otimes_+ u_{y_1\mu} \mod qL^{\mathrm{up}}(\lambda) \otimes_+ L^{\mathrm{up}}(\mu)$$

and

$$\Xi_{\lambda,\mu}\big(\iota_{\lambda,\mu}(v_2)\big) \equiv \Xi_{\lambda,\mu}(u_{x_2\lambda} \otimes_+ u_{y_2\mu}) \mod q\widetilde{L}.$$

Hence

$$\begin{aligned} &\Phi_{V(\lambda)\otimes_+ V(\mu)}\big((u_{x_1\lambda} \otimes_+ u_{y_1\mu}) \otimes \Xi_{\lambda,\mu}(u_{x_2\lambda} \otimes_+ u_{y_2\mu})^{\mathrm{r}}\big) \\ &\qquad \equiv \Phi_{V(\lambda)\otimes_+ V(\mu)}\big((\iota_{\lambda,\mu}(v_1) \otimes (\Xi_{\lambda,\mu}\iota_{\lambda,\mu}(v_2))^{\mathrm{r}}\big) \mod qL^{\mathrm{up}}(A_q(\mathfrak{g})) \end{aligned}$$

by (9.4), as desired. ☐

Theorem 9.3.3. *Let $\lambda \in \mathsf{P}^+$ and $x, y \in W$ such that $x \geq y$. Then we have*

$$\mathrm{D}(x\lambda, y\lambda)\mathrm{D}(y\lambda, \lambda) \equiv \mathrm{D}(x\lambda, \lambda) \mod qL^{\mathrm{up}}(A_q(\mathfrak{n})).$$

Proof. Applying $p_{\mathfrak{n}}$ to (9.5), we have

$$\begin{aligned} &\mathrm{D}(x\lambda, y\lambda)\mathrm{D}(y\lambda, \lambda) \\ &\qquad \equiv p_{\mathfrak{n}}\Big(G^{\mathrm{up}}(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x\lambda} \otimes u_{y\lambda}) \otimes \overline{\iota}_{\lambda,\lambda}^{-1}(u_{y\lambda} \otimes u_{\lambda})^{\mathrm{r}})\Big) \mod qL^{\mathrm{up}}(A_q(\mathfrak{n})). \end{aligned}$$

Hence the desired result follows from Proposition 8.5.2, Proposition 8.5.4, and Lemma 9.3.4 below. ☐

Lemma 9.3.4. *Let $\lambda \in \mathsf{P}^+$ and $x, y \in W$ such that $x \geq y$. Then we have*

$$\overline{\iota}_{\mathfrak{g}}\Big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x\lambda} \otimes u_{y\lambda}) \otimes \big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{y\lambda} \otimes u_{\lambda})\big)^{\mathrm{r}}\Big) = \overline{\iota}_{\lambda}(u_{x\lambda}) \otimes t_{y\lambda+\lambda} \otimes b_{-\infty}.$$

Proof. We shall argue by induction on $\ell(x)$. We set $b_{x\lambda} = \overline{\iota}_{\lambda}(u_{x\lambda})$. Since the case $x = 1$ is obvious, assume that $x \neq 1$. Take $i \in I$ such that $x' := s_i x < x$.

(a) First assume that $s_i y > y$. Then we have $y \leq x'$. Hence by the induction hypothesis,

$$\overline{\iota}_{\mathfrak{g}}\Big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x'\lambda} \otimes u_{y\lambda}) \otimes \big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{y\lambda} \otimes u_{\lambda})\big)^{\mathrm{r}}\Big) = b_{x'\lambda} \otimes t_{y\lambda+\lambda} \otimes b_{-\infty}. \tag{9.6}$$

We have $\varphi_i(u_{x'\lambda}) = \langle h_i, x'\lambda\rangle$ and $\varphi_i(b_{x'\lambda} \otimes t_{y\lambda+\lambda} \otimes b_{-\infty}) = \varphi_i(b_{x'\lambda} \otimes t_{y\lambda+\lambda}) = \langle h_i, x'\lambda\rangle + \langle h_i, y\lambda\rangle \geq \langle h_i, x'\lambda\rangle$. Hence, applying $\tilde{f}_i^{\langle h_i, x'\lambda\rangle}$ to (9.6), we obtain

$$\overline{\iota}_{\mathfrak{g}}\Big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x\lambda} \otimes u_{y\lambda}) \otimes \big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{y\lambda} \otimes u_{\lambda})\big)^{\mathrm{r}}\Big) = b_{x\lambda} \otimes t_{y\lambda+\lambda} \otimes b_{-\infty}.$$

(b) Assume that $y' := s_i y < y$. Then we have $y' \leq x'$, and the induction hypothesis implies that

$$\overline{\iota}_{\mathfrak{g}}\Big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x'\lambda} \otimes u_{y'\lambda}) \otimes \big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{y'\lambda} \otimes u_{\lambda})\big)^{\mathrm{r}}\Big) = b_{x'\lambda} \otimes t_{y'\lambda+\lambda} \otimes b_{-\infty}.$$

Apply $\tilde{e}_i^{*\ \langle h_i y'\lambda\rangle} \tilde{f}_i^{\langle h_i, x'\lambda + y'\lambda\rangle}$ to both sides. Then the left-hand side yields

$$\overline{\iota}_{\mathfrak{g}}\Big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{x\lambda}\otimes u_{y\lambda})\otimes\big(\overline{\iota}_{\lambda,\lambda}^{-1}(u_{y\lambda}\otimes u_{\lambda})\big)^{\mathrm{r}}\Big).$$

Since $\varphi_i(b_{x'\lambda}\otimes t_{y'\lambda+\lambda}) = \langle h_i, x'\lambda\rangle + \langle h_i, y'\lambda + \lambda\rangle \geq \langle h_i, x'\lambda + y'\lambda\rangle$, the right-hand side yields

$$\begin{aligned}
&\tilde{e}_i^{*\ \langle h_i, y'\lambda\rangle} \tilde{f}_i^{\langle h_i, x'\lambda + y'\lambda\rangle}\big(b_{x'\lambda}\otimes t_{y'\lambda+\lambda}\otimes b_{-\infty}\big)\\
&\quad = \tilde{e}_i^{*\ \langle h_i, y'\lambda\rangle}\big((\tilde{f}_i^{\langle h_i, x'\lambda+y'\lambda\rangle} b_{x'\lambda})\otimes t_{y'\lambda+\lambda}\otimes b_{-\infty}\big)\\
&\quad = \tilde{e}_i^{*\ \langle h_i, y'\lambda\rangle}\big((\tilde{f}_i^{\langle h_i, y'\lambda\rangle} b_{x\lambda})\otimes t_{y'\lambda+\lambda}\otimes b_{-\infty}\big).
\end{aligned}$$

Since $\varepsilon_i^*(b_{x\lambda}) = -\varphi_i(b_{x\lambda}) = \langle h_i, \lambda\rangle$ and $\tilde{f}_i^{\langle h_i, y'\lambda\rangle} b_{x\lambda} = \tilde{f}_i^{*\ \langle h_i, y'\lambda\rangle} b_{x\lambda}$, we have

$$\tilde{e}_i^{*\ \langle h_i, y'\lambda\rangle}\big((\tilde{f}_i^{\langle h_i, y'\lambda\rangle} b_{x\lambda})\otimes t_{y'\lambda+\lambda}\otimes b_{-\infty}\big) = b_{x\lambda}\otimes t_{y\lambda+\lambda}\otimes b_{-\infty}. \qquad \square$$

9.4. **Generalized T-system.** The T-system in Section 9.2 can be interpreted as a system of equations among the three products of elements in $\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{g}))$ or $\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{n}))$. In this subsection, we introduce another among the three products of elements in $\mathbf{B}^{\mathrm{up}}(A_q(\mathfrak{g}))$, called *a generalized T-system.*

Proposition 9.4.1. *Let $\mu \in W\varpi_i$, and set $b = \overline{\iota}_{\varpi_i}(u_\mu) \in B(\infty)$. Then we have*

(9.7)
$$\begin{aligned}
\Delta(\mu, s_i\varpi_i)\Delta(\varpi_i, \varpi_i) = q_i^{-1}G^{\mathrm{up}}\left(\overline{\iota}_{\varpi_i,\varpi_i}^{-1}(u_\mu\otimes u_{\varpi_i})\otimes\left(\overline{\iota}_{\varpi_i,\varpi_i}^{-1}(u_{s_i\varpi_i}\otimes u_{\varpi_i})\right)^{\mathrm{r}}\right)\\
+ G^{\mathrm{up}}\left(\overline{\iota}_{\varpi_i+s_i\varpi_i}^{-1}(\widetilde{e_i^*b})\otimes u_{\varpi_i+s_i\varpi_i}^{\mathrm{r}}\right).
\end{aligned}$$

Note that if $\mu = \varpi_i$, then $b = 1$ and the last term in (9.7) vanishes. If $\mu \neq \varpi_i$, then $\varepsilon_i^*(b) = 1$ and $\overline{\iota}_{\varpi_i+s_i\varpi_i}^{-1}(\tilde{e}_i^*b) \in B(\varpi_i + s_i\varpi_i)$, $u_\mu\otimes u_{\varpi_i} \in \overline{\iota}_{\varpi_i,\varpi_i}B(2\varpi_i)$.

Proof. In the sequel, we omit $\overline{\iota}_{\varpi_i,\varpi_i}^{-1}$ for the sake of simplicity. Set

$$u = \Delta(\mu, s_i\varpi_i)\Delta(\varpi_i, \varpi_i) - q_i^{-1}G^{\mathrm{up}}\left((u_\mu\otimes u_{\varpi_i})\otimes(u_{s_i\varpi_i}\otimes u_{\varpi_i})^{\mathrm{r}}\right).$$

Then $\mathrm{wt}_{\mathrm{r}}(u) = \lambda := \varpi_i + s_i\varpi_i$.

It is obvious that we have $uf_j = 0$ for $j \neq i$. Since $\tilde{e}_i(u_{s_i\varpi_i}\otimes u_{\varpi_i}) = u_{\varpi_i}\otimes u_{\varpi_i}$, we have

$$\begin{aligned}
G^{\mathrm{up}}\left((u_\mu\otimes u_{\varpi_i})\otimes(u_{s_i\varpi_i}\otimes u_{\varpi_i})^{\mathrm{r}}\right)f_i &= G^{\mathrm{up}}\left((u_\mu\otimes u_{\varpi_i})\otimes(u_{\varpi_i}\otimes u_{\varpi_i})^{\mathrm{r}}\right)\\
&= \Delta(\mu, \varpi_i)\Delta(\varpi_i, \varpi_i)\\
&= G^{\mathrm{up}}(u_\mu\otimes u_{\varpi_i}^{\mathrm{r}})G^{\mathrm{up}}(u_{\varpi_i}\otimes u_{\varpi_i}^{\mathrm{r}}).
\end{aligned}$$

Here the second equality follows from Lemma 9.1.9 and the third follows from Proposition 8.1.3. On the other hand, we have

$$\begin{aligned}
\left(\Delta(\mu, s_i\varpi_i)\Delta(\varpi_i, \varpi_i)\right)f_i &= \left(\Delta(\mu, s_i\varpi_i)f_i\right)\left(\Delta(\varpi_i, \varpi_i)t_i^{-1}\right)\\
&= q_i^{-1}\Delta(\mu, \varpi_i)\Delta(\varpi_i, \varpi_i).
\end{aligned}$$

Hence we have $uf_i = 0$. Thus, u is a lowest weight vector of weight λ with respect to the right action of $U_q(\mathfrak{g})$. Therefore there exists some $v \in V(\lambda)$ such that

$$u = \Phi(v\otimes u_\lambda^{\mathrm{r}}).$$

Hence we have $p_{\mathfrak{n}}(u) = \iota_\lambda(v) \in A_q(\mathfrak{n})$. On the other hand, we have

$$\begin{aligned} p_{\mathfrak{n}}\left(\Delta(\mu, s_i\varpi_i)\Delta(\varpi_i, \varpi_i)\right) &= p_{\mathfrak{n}}\left(\Delta(\mu, s_i\varpi_i)\right) p_{\mathfrak{n}}\left(\Delta(\varpi_i, \varpi_i)\right) \\ &= \mathrm{D}(\mu, s_i\varpi_i) = G^{\mathrm{up}}(\tilde{e}_i^* b) \\ &= \iota_\lambda\left(G_\lambda^{\mathrm{up}}\big(\overline{\iota}_\lambda^{-1}(\tilde{e}_i^* b)\big)\right). \end{aligned}$$

Note that since $\varepsilon_i^*(\tilde{e}_i^* b) = 0$ and $\varepsilon_j^*(\tilde{e}_i^* b) \le -\langle h_j, \alpha_i\rangle$ for $j \neq i$, we have $\tilde{e}_i^* b \in \overline{\iota}_\lambda(B(\lambda))$.

Hence in order to prove our assertion, it is enough to show that

$$p_{\mathfrak{n}}\left(G^{\mathrm{up}}\big((u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}}\big)\right) = 0.$$

This follows from Proposition 8.5.2 and

$$\overline{\iota}_{\mathfrak{g}}\left((u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}}\right) = b \otimes t_\lambda \otimes \tilde{e}_i b_{-\infty}. \tag{9.8}$$

Let us prove (9.8). Since

$$(u_\mu \otimes u_{\varpi_i}) \otimes (u_{s_i\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}} = \tilde{e}_i^*\big((u_\mu \otimes u_{\varpi_i}) \otimes (u_{\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}}\big),$$

the left-hand side of (9.8) is equal to

$$\tilde{e}_i^*\left(\overline{\iota}_{\mathfrak{g}}\big((u_\mu \otimes u_{\varpi_i}) \otimes (u_{\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}}\big)\right) = \tilde{e}_i^*(b \otimes t_{2\varpi_i} \otimes b_{-\infty}).$$

Since $\varepsilon_i^*(b) = 1 < \langle h_i, 2\varpi_i\rangle = 2$, we obtain

$$\tilde{e}_i^*\left(b \otimes t_{2\varpi_i} \otimes b_{-\infty}\right) = b \otimes t_{2\varpi_i - \alpha_i} \otimes \tilde{e}_i^* b_{-\infty} = b \otimes t_\lambda \otimes \tilde{e}_i b_{-\infty}. \qquad \square$$

10. KLR ALGEBRAS AND THEIR MODULES

10.1. **Chevalley and Kashiwara operators.** Let us recall the definition of several functors on modules over KLR algebras which are used to categorify $U_q^-(\mathfrak{g})^\vee_{\mathbb{Z}[q^{\pm 1}]}$.

Definition 10.1.1. *Let $\beta \in \mathsf{Q}^+$.*

(i) *For $i \in I$ and $1 \le a \le |\beta|$, set*

$$e_a(i) = \sum_{\nu \in I^\beta, \nu_a = i} e(\nu) \in R(\beta).$$

(ii) *We take conventions*

$$E_i M = e_1(i) M,$$
$$E_i^* M = e_{|\beta|}(i) M,$$

which are functors from $R(\beta)$-gmod to $R(\beta - \alpha_i)$-gmod.

(iii) *For a simple module M, we set*

$$\begin{aligned} &\varepsilon_i(M) = \max\left\{n \in \mathbb{Z}_{\ge 0} \mid E_i^n M \neq 0\right\}, \\ &\varepsilon_i^*(M) = \max\left\{n \in \mathbb{Z}_{\ge 0} \mid E_i^{*\,n} M \neq 0\right\}, \\ &\widetilde{F}_i M = q_i^{\varepsilon_i(M)} L(i) \nabla M, \\ &\widetilde{F}_i^* M = q_i^{\varepsilon_i^*(M)} M \nabla L(i), \\ &\widetilde{E}_i M = q_i^{1-\varepsilon_i(M)} \mathrm{soc}(E_i M) \simeq q_i^{\varepsilon_i(M)-1} \mathrm{hd}(E_i M), \\ &\widetilde{E}_i^* M = q_i^{1-\varepsilon_i^*(M)} \mathrm{soc}(E_i^* M) \simeq q_i^{\varepsilon_i^*(M)-1} \mathrm{hd}(E_i^* M), \\ &\tilde{E}_i^{\max} M = \tilde{E}_i^{\varepsilon_i(M)} M \quad \text{and} \quad \tilde{E}_i^{*\,\max} M = \tilde{E}_i^{*\,\varepsilon_i^*(M)} M. \end{aligned}$$

(iv) *For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}$, we set*

$$L(i^n) = q_i^{n(n-1)/2} \underbrace{L(i) \circ \cdots \circ L(i)}_{n}.$$

Here $L(i)$ denotes the $R(\alpha_i)$-module $R(\alpha_i)/R(\alpha_i)x_1$. Then $L(i^n)$ is a self-dual real simple $R(n\alpha_i)$-module.

Note that, under the isomorphism in Theorem 2.1.2, the functors E_i and E_i^* correspond to the linear operators e_i and e_i^* on $A_q(\mathfrak{n})_{\mathbb{Z}[q^{\pm 1}]} = \iota(U_q^-(\mathfrak{g})^\vee_{\mathbb{Z}[q^{\pm 1}]}) \subset A_q(\mathfrak{n})$, respectively. Note also that, for a simple $R(\beta)$-module S, we have $\tilde{E}_i\widetilde{F}_iS \simeq S$, and $\widetilde{F}_i\tilde{E}_iS \simeq S$ if $\varepsilon_i(M) > 0$.

In the course of proving the following propositions, we use the following notations:

$$\overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2}) := \frac{Q_{i,j}(x_a, x_{a+1}) - Q_{i,j}(x_{a+2}, x_{a+1})}{x_a - x_{a+2}}. \tag{10.1}$$

Then we have

$$\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a = \sum_{i,j \in I} \overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2})e_a(i)e_{a+1}(j)e_{a+2}(i).$$

Proposition 10.1.2. *Let $\beta \in \mathsf{Q}^+$ with $n = |\beta|$. Assume that an $R(\beta)$-module M satisfies $E_iM = 0$. Then the left $R(\alpha_i)$-module homomorphism $R(\alpha_i) \otimes M \longrightarrow q^{(\alpha_i,\beta)}M \circ R(\alpha_i)$ given by*

$$e(i) \otimes u \longmapsto \tau_1 \cdots \tau_n(u \otimes e(i)) \tag{10.2}$$

extends uniquely to an $(R(\alpha_i + \beta), R(\alpha_i))$-bilinear homomorphism

$$R(\alpha_i) \circ M \longrightarrow q^{(\alpha_i,\beta)}M \circ R(\alpha_i). \tag{10.3}$$

Proof. (i) First note that, for $1 \leq a \leq n$,

(10.4)
$$\tau_1 \cdots \tau_{a-1}e_a(i)\tau_{a+1} \cdots \tau_n\big(u \otimes e(i)\big) = \tau_{a+1} \cdots \tau_n\big(e_1(i)\tau_1 \cdots \tau_{a-1}(u \otimes e(i))\big) = 0$$

since $E_iM = 0$.

(ii) In order to see that (10.3) is a well-defined $R(\alpha_i + \beta)$-linear homomorphism, it is enough to show that (10.2) is $R(\beta)$-linear.

(a) Commutation with $x_a \in R(\beta)$ $(1 \leq a \leq n)$: We have

$$\begin{aligned} x_{a+1}\tau_1 \cdots \tau_n\big(u \otimes e(i)\big) &= \tau_1 \cdots \tau_{a-1}x_{a+1}\tau_a \cdots \tau_n\big(u \otimes e(i)\big) \\ &= \tau_1 \cdots \tau_{a-1}\big(\tau_a x_a + e_a(i)\big)\tau_{a+1} \cdots \tau_n\big(u \otimes e(i)\big) \\ &= \tau_1 \cdots \tau_n x_a\big(u \otimes e(i)\big) \end{aligned}$$

by (10.4).

(b) Commutation with $\tau_a \in R(\beta)$ $(1 \le a < n)$: We have

$$\begin{aligned}
&\tau_{a+1}\tau_1 \cdots \tau_n\big(u \otimes e(i)\big)\\
&\quad = \tau_1 \cdots \tau_{a-1}(\tau_{a+1}\tau_a\tau_{a+1})\tau_{a+2} \cdots \tau_n\big(u \otimes e(i)\big)\\
&\quad = \tau_1 \cdots \tau_{a-1}\big(\tau_a\tau_{a+1}\tau_a + \sum_j \overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2})e_a(i)e_{a+1}(j)\big)\tau_{a+2} \cdots \tau_n\big(u \otimes e(i)\big)\\
&\quad = \tau_1 \cdots \tau_n\tau_a\big(u \otimes e(i)\big)\\
&\qquad + \sum_j \tau_1 \cdots \tau_{a-1}\overline{Q}_{i,j}(x_a, x_{a+1}, x_{a+2})e_a(i)e_{a+1}(j)\tau_{a+2} \cdots \tau_n\big(u \otimes e(i)\big).
\end{aligned}$$

The last term vanishes because $E_iM = 0$ implies

$$\begin{aligned}
\tau_1 \cdots \tau_{a-1}&f(x_a, x_{a+1})g(x_{a+2})e_a(i)\tau_{a+2} \cdots \tau_n\big(u \otimes e(i)\big)\\
&= g(x_{a+2})\tau_{a+2} \cdots \tau_n e_1(i)\tau_1 \cdots \tau_{a-1}f(x_a, x_{a+1})\big(u \otimes e(i)\big) = 0
\end{aligned}$$

for any polynomial $f(x_a, x_{a+1})$ and $g(x_{a+2})$.

(iii) Now let us show that (10.3) is right $R(\alpha_i)$-linear. By (10.4), we have

$$\begin{aligned}
\tau_1 \cdots \tau_{a-1}x_a\tau_a \cdots \tau_n\big(u \otimes e(i)\big) &= \tau_1 \cdots \tau_{a-1}\big(\tau_a x_{a+1} - e_a(i)\big)\tau_{a+1} \cdots \tau_n\big(u \otimes e(i)\big)\\
&= \tau_1 \cdots \tau_a x_{a+1}\tau_{a+1} \cdots \tau_n\big(u \otimes e(i)\big)
\end{aligned}$$

for $1 \le a \le n$. Therefore we have

$$x_1\tau_1 \cdots \tau_n\big(u \otimes e(i)\big) = \tau_1 \cdots \tau_n x_{n+1}\big(u \otimes e(i)\big) = \tau_1 \cdots \tau_n\big(u \otimes e(i)x_1\big). \qquad \square$$

Recall that for $m, n \in \mathbb{Z}_{\ge 0}$, we denote by $w[m,n]$ the element of $\mathfrak{S}_{m+n}$ defined by

$$w[m,n](k) = \begin{cases} k+n & \text{if } 1 \le k \le m,\\ k-m & \text{if } m < k \le m+n. \end{cases} \tag{10.5}$$

Set $\tau_{w[m,n]} := \tau_{i_1} \cdots \tau_{i_r}$, where $s_{i_1} \cdots s_{i_r}$ is a reduced expression of $w[m,n]$. Note that $\tau_{w[m,n]}$ does not depend on the choice of reduced expression [14, Corollary 1.4.3].

Proposition 10.1.3. *Let $M \in R(\beta)$-gmod and $N \in R(\gamma)$-gmod, and set $m = |\beta|$ and $n = |\gamma|$. If $E_iM = 0$ for any $i \in \operatorname{supp}(\gamma)$, then*

$$v \otimes u \longmapsto \tau_{w[m,n]}(u \otimes v)$$

gives a well-defined $R(\beta+\gamma)$-linear homomorphism $N \circ M \longrightarrow q^{(\beta,\gamma)} M \circ N$.

Proof. The proceeding proposition implies that

$$v \otimes u \longmapsto \tau_{w[m,n]}(u \otimes v) \quad \text{for } u \in M, v \in R(\gamma)$$

gives a well-defined $R(\beta+\gamma)$-linear homomorphism $R(\gamma) \circ M \to M \circ R(\gamma)$. Hence it is enough to show that it is right $R(\gamma)$-linear. Since we know that it commutes with the right multiplication of x_k, it is enough to show that it commutes with the right multiplication of τ_k. For this, we may assume that $n = 2$ and $k = 1$. Set $\gamma = \alpha_i + \alpha_j$.

Thus we have reduced the problem to the equality

$$\tau_1(\tau_2\tau_1) \cdots (\tau_{m+1}\tau_m)\big(u \otimes e(i) \otimes e(j)\big) = (\tau_2\tau_1) \cdots (\tau_{m+1}\tau_m)\tau_{m+1}\big(u \otimes e(i) \otimes e(j)\big)$$

for $u \in M$, which is a consequence of

$$\begin{aligned}&(\tau_2\tau_1)\cdots(\tau_a\tau_{a-1})\tau_a(\tau_{a+1}\tau_a)\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\\ &\qquad = (\tau_2\tau_1)\cdots(\tau_{a+1}\tau_a)\tau_{a+1}(\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\end{aligned}$$

for $1 \le a \le m$. Note that

$$\begin{aligned}&\tau_a(\tau_{a+1}\tau_a)\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\\ &\qquad = \tau_a(\tau_{a+1}\tau_a)e_{a+1}(i)e_{a+2}(j)(\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\end{aligned}$$

and

$$\begin{aligned}&\tau_a(\tau_{a+1}\tau_a)e_{a+1}(i)e_{a+2}(j)\\ &\qquad = (\tau_{a+1}\tau_a)\tau_{a+1}e_{a+1}(i)e_{a+2}(j) - \overline{Q}_{ji}(x_a, x_{a+1}, x_{a+2})e_a(j)e_{a+1}(i)e_{a+2}(j).\end{aligned}$$

Hence it is enough to show

$$\begin{aligned}&(\tau_2\tau_1)\cdots(\tau_a\tau_{a-1})\overline{Q}_{j,i}(x_a, x_{a+1}, x_{a+2})e_a(j)\\ &\qquad (\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big) = 0.\end{aligned}$$

This follows from

$$\begin{aligned}&(\tau_2\tau_1)\cdots(\tau_a\tau_{a-1})f(x_a)g(x_{a+1}, x_{a+2})e_a(j)(\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\\ &\quad = (\tau_2\cdots\tau_a)(\tau_1\cdots\tau_{a-1})f(x_a)g(x_{a+1}, x_{a+2})e_a(j)\\ &\qquad\qquad (\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\big(u\otimes e(i)\otimes e(j)\big)\\ &\quad = (\tau_2\cdots\tau_a)g(x_{a+1}, x_{a+2})(\tau_{a+2}\tau_{a+1})\cdots(\tau_{m+1}\tau_m)\\ &\qquad\qquad e_1(j)(\tau_1\cdots\tau_{a-1})f(x_a)\big(u\otimes e(i)\otimes e(j)\big)\\ &\quad = 0\end{aligned}$$

for $1 \le a \le m$ and $f(x_a) \in \mathbf{k}[x_a]$, $g(x_{a+1}, x_{a+2}) \in \mathbf{k}[x_{a+1}, x_{a+2}]$. ☐

Let $P(i^n)$ be a projective cover of $L(i^n)$. Define the functor

$$E_i^{(n)} : R(\beta)\text{-Mod} \to R(\beta - n\alpha_i)\text{-Mod}$$

by

$$E_i^{(n)}(M) := P(i^n)^{\psi} \underset{R(n\alpha_i)}{\otimes} E_i^n M,$$

where $P(i^n)^{\psi}$ denotes the right $R(n\alpha_i)$-module obtained from the left $R(\beta)$-module $P(i^n)$ via the anti-automorphism ψ. We define the functor $E_i^{*\,(n)}$ in a similar way. Note that

$$E_i^n \simeq [n]_i! E_i^{(n)}.$$

Corollary 10.1.4. *Let R be a symmetric KLR algebra. Let $i \in I$ and M a simple module. Then we have*

$$\begin{aligned}&\widetilde{\Lambda}\big(L(i), M\big) = \varepsilon_i(M),\\ &\Lambda\big(L(i), M\big) = 2\varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle = \varepsilon_i(M) + \varphi_i(M).\end{aligned}$$

Proof. Set $n = \varepsilon_i(M)$ and $M_0 = E_i^{(n)}(M)$. Then the preceding proposition implies $\Lambda(L(i), M_0) = \big(\alpha_i, \mathrm{wt}(M_0)\big)$. Hence we have $\widetilde{\Lambda}(L(i), M_0) = 0$, which implies

$$\widetilde{\Lambda}(L(i), M) = \widetilde{\Lambda}(L(i), L(i^n) \circ M_0) = \widetilde{\Lambda}\big(L(i), L(i^n)\big) + \widetilde{\Lambda}(L(i), M_0) = n. \quad ☐$$

Proposition 10.1.5. *Let M, N be modules and $m, n \in \mathbb{Z}_{\geq 0}$.*

(i) *If $E_i^{m+1}M = 0$ and $E_i^{n+1}N = 0$, then we have*

$$E_i^{(m+n)}(M \circ N) \simeq q^{mn+n\langle h_i, \mathrm{wt}(M)\rangle} E_i^{(m)}M \circ E_i^{(n)}N.$$

(ii) *If $E_i^{*\,m+1}M = 0$ and $E_i^{*\,n+1}N = 0$, then we have*

$$E_i^{*\,(m+n)}(M \circ N) \simeq q^{mn+m\langle h_i, \mathrm{wt}(N)\rangle} E_i^{*\,(m)}M \circ E_i^{*\,(n)}N.$$

Proof. Our assertions follow from the shuffle lemma [21, Lemma 2.20]. □

The following corollaries are immediate consequences of Proposition 10.1.5.

Corollary 10.1.6. *Let $i \in I$, and let M be a real simple module. Then $\tilde{E}_i^{\max}M$ is also real simple.*

Corollary 10.1.7. *Let $i \in I$, and let M be a simple module with $\varepsilon_i(M) = m$. Then we have $\tilde{E}_i^m M \simeq E_i^{(m)}M$.*

Proposition 10.1.8. *Let M and N be simple modules. We assume that one of them is real. If $\varepsilon_i(M \nabla N) = \varepsilon_i(M)$, then we have an isomorphism in R-gmod*

$$\widetilde{E}_i^{\max}(M \nabla N) \simeq (\widetilde{E}_i^{\max}M) \nabla N.$$

Similarly, if $\varepsilon_i^*(N \nabla M) = \varepsilon_i^*(M)$, then we have

$$\widetilde{E}_i^{*\max}(N \nabla M) \simeq (N \nabla \widetilde{E}_i^{*\max}M).$$

Proof. Set $n = \varepsilon_i(M \nabla N) = \varepsilon_i(M)$ and $M_0 = \widetilde{E}_i^{\max}M$. Then M_0 or N is real. Now we have

$$L(i^n) \otimes M_0 \otimes N \rightarrowtail E_i^n(M \nabla N) \simeq L(i^n) \otimes \widetilde{E}_i^{\max}(M \nabla N),$$

which induces a non-zero map $M_0 \otimes N \to \widetilde{E}_i^{\max}(M \nabla N)$. Hence we have a surjective map

$$M_0 \circ N \twoheadrightarrow \widetilde{E}_i^{\max}(M \nabla N).$$

Since M_0 or N is real by Corollary 10.1.6, $M_0 \circ N$ has a simple head and we obtain the desired result. A similar proof works for the second statement. □

10.2. **Determinantial modules and T-system.**

We will use the materials in Section 9 to obtain properties on the determinantial modules.

In the rest of this paper, we assume that *R is symmetric and the base field* **k** *is of characteristic* 0. Under this condition, the family of self-dual simple R-modules corresponds to the upper global basis of $A_q(\mathfrak{n})$ by Theorem 2.1.4.

Let ch be the map from $K(R\text{-gmod})$ to $A_q(\mathfrak{n})$ obtained by composing ι and the isomorphism (2.2) in Theorem 2.1.2.

Definition 10.2.1. *For $\lambda \in \mathsf{P}^+$ and $\mu, \zeta \in W\lambda$ such that $\mu \preceq \zeta$, let $\mathsf{M}(\mu, \zeta)$ be a simple $R(\zeta - \mu)$-module such that* $\mathrm{ch}(\mathsf{M}(\mu, \zeta)) = \mathrm{D}(\mu, \zeta)$.

Since $\mathrm{D}(\mu, \zeta)$ is a member of the upper global basis, such a module exists uniquely due to Theorem 2.1.4. The module $\mathsf{M}(\mu, \zeta)$ is self-dual, and we call it *the determinantial module.*

Lemma 10.2.2. $\mathsf{M}(\mu, \zeta)$ *is a real simple module.*

Proof. It follows from $\mathrm{ch}(\mathsf{M}(\mu,\zeta)\circ\mathsf{M}(\mu,\zeta)) = \mathrm{ch}\big(\mathsf{M}(\mu,\zeta)\big)^2 = q^{-(\zeta,\zeta-\mu)}\mathrm{D}(2\mu,2\zeta)$ which is a member of the upper global basis up to a power of q. Here the last equality follows from Corollary 9.1.3. ☐

Proposition 10.2.3. *Let $\lambda,\mu\in\mathsf{P}^+$, and $s,s',t,t'\in W$ such that $\ell(s's)=\ell(s')+\ell(s)$, $\ell(t't)=\ell(t')+\ell(t)$, $s's\lambda\preceq t'\lambda$, and $s'\mu\preceq t't\mu$. Then*

(i) *$\mathsf{M}(s's\lambda,t'\lambda)$ and $\mathsf{M}(s'\mu,t't\mu)$ commute,*

(ii) $\Lambda\big(\mathsf{M}(s's\lambda,t'\lambda),\mathsf{M}(s'\mu,t't\mu)\big)=(s's\lambda+t'\lambda,\ t't\mu-s'\mu)$,

(iii) $\widetilde{\Lambda}\big(\mathsf{M}(s's\lambda,t'\lambda),\mathsf{M}(s'\mu,t't\mu)\big)=(t'\lambda,\ t't\mu-s'\mu)$,
$\widetilde{\Lambda}\big(\mathsf{M}(s'\mu,t't\mu),\mathsf{M}(s's\lambda,t'\lambda)\big)=(s'\mu-t't\mu,\ s's\lambda)$.

Proof. It is a consequence of Proposition 9.1.6 (ii) and Corollary 4.1.4. ☐

Proposition 10.2.4. *Let $\lambda\in\mathsf{P}^+$, $\mu,\zeta\in W\lambda$ such that $\mu\preceq\zeta$ and $i\in I$.*

(i) *If $n:=\langle h_i,\mu\rangle\ge 0$, then*

$$\varepsilon_i(\mathsf{M}(\mu,\zeta))=0 \quad \textit{and} \quad \mathsf{M}(s_i\mu,\zeta)\simeq\widetilde{F}_i^n\mathsf{M}(\mu,\zeta)\simeq L(i^n)\nabla\mathsf{M}(\mu,\zeta) \textit{ in } R\text{-gmod}.$$

(ii) *If $\langle h_i,\mu\rangle\le 0$ and $s_i\mu\preceq\zeta$, then $\varepsilon_i(\mathsf{M}(\mu,\zeta))=-\langle h_i,\mu\rangle$.*

(iii) *If $m:=-\langle h_i,\zeta\rangle\ge 0$, then*

$$\varepsilon_i^*(\mathsf{M}(\mu,\zeta))=0 \quad \textit{and} \quad \mathsf{M}(\mu,s_i\zeta)\simeq\widetilde{F}_i^{*\,m}\mathsf{M}(\mu,\zeta)\simeq\mathsf{M}(\mu,\zeta)\nabla L(i^m) \quad \textit{in } R\text{-gmod}.$$

(iv) *If $\langle h_i,\zeta\rangle\ge 0$ and $\mu\preceq s_i\zeta$, then $\varepsilon_i^*(\mathsf{M}(\mu,\zeta))=\langle h_i,\zeta\rangle$.*

Proof. It is a consequence of Lemma 9.1.5. ☐

Proposition 10.2.5. *Assume that $u,v\in W$ and $i\in I$ satisfy $u<us_i$ and $v<vs_i\le u$.*

(i) *We have exact sequences*

$$(10.6)\quad \begin{aligned} &0\longrightarrow\mathsf{M}(u\lambda,v\lambda)\longrightarrow q^{(vs_i\varpi_i,v\varpi_i-u\varpi_i)}\mathsf{M}(us_i\varpi_i,vs_i\varpi_i)\circ\mathsf{M}(u\varpi_i,v\varpi_i)\\ &\qquad\longrightarrow q^{-1+(v\varpi_i,vs_i\varpi_i-u\varpi_i)}\mathsf{M}(us_i\varpi_i,v\varpi_i)\circ\mathsf{M}(u\varpi_i,vs_i\varpi_i)\longrightarrow 0,\end{aligned}$$

and

$$(10.7)\quad \begin{aligned} &0\longrightarrow q^{1+(v\varpi_i,vs_i\varpi_i-u\varpi_i)}\mathsf{M}(us_i\varpi_i,v\varpi_i)\circ\mathsf{M}(u\varpi_i,vs_i\varpi_i)\\ &\qquad\longrightarrow q^{(v\varpi_i,vs_i\varpi_i-us_i\varpi_i)}\mathsf{M}(u\varpi_i,v\varpi_i)\circ\mathsf{M}(us_i\varpi_i,vs_i\varpi_i)\longrightarrow\mathsf{M}(u\lambda,v\lambda)\longrightarrow 0,\end{aligned}$$

where $\lambda=s_i\varpi_i+\varpi_i$.

(ii) $\mathfrak{d}\big(\mathsf{M}(u\varpi_i,v\varpi_i),\mathsf{M}(us_i\varpi_i,vs_i\varpi_i)\big)=1$.

Proof. Since the proof of (10.6) is similar, let us only prove (10.7). (Indeed, they are dual to each other.)

Set

$$\begin{aligned} X&=q^{(v\varpi_i,vs_i\varpi_i-u\varpi_i)}\mathsf{M}(us_i\varpi_i,v\varpi_i)\circ\mathsf{M}(u\varpi_i,vs_i\varpi_i),\\ Y&=q^{(v\varpi_i,vs_i\varpi_i-us_i\varpi_i)}\mathsf{M}(u\varpi_i,v\varpi_i)\circ\mathsf{M}(us_i\varpi_i,vs_i\varpi_i),\\ Z&=\mathsf{M}(u\lambda,v\lambda).\end{aligned}$$

Then Proposition 9.2.1 implies that

$$\mathrm{ch}(Y)=\mathrm{ch}(qX)+\mathrm{ch}(Z).$$

Since X and Z are simple and self-dual, our assertion follows from Lemma 3.2.19. ☐

10.3. **Generalized T-system on determinantial module.**

Theorem 10.3.1. *Let $\lambda \in \mathsf{P}^+$ and $\mu_1, \mu_2, \mu_3 \in W\lambda$ such that $\mu_1 \preceq \mu_2 \preceq \mu_3$. Then there exists a canonical epimorphism*

$$\mathsf{M}(\mu_1, \mu_2) \circ \mathsf{M}(\mu_2, \mu_3) \twoheadrightarrow \mathsf{M}(\mu_1, \mu_3),$$

which is equivalent to saying that $\mathsf{M}(\mu_1, \mu_2) \nabla \mathsf{M}(\mu_2, \mu_3) \simeq \mathsf{M}(\mu_1, \mu_3)$.

In particular, we have

$$\widetilde{\Lambda}(\mathsf{M}(\mu_1, \mu_2), \mathsf{M}(\mu_2, \mu_3)) = 0 \quad \text{and} \quad \Lambda(\mathsf{M}(\mu_1, \mu_2), \mathsf{M}(\mu_2, \mu_3)) = -(\mu_1 - \mu_2, \mu_2 - \mu_3).$$

Proof. (a) Our assertion follows from Theorem 9.3.3 and Theorem 4.2.1 when $\mu_3 = \lambda$.

(b) We shall prove the general case by induction on $|\lambda - \mu_3|$. By (a), we may assume that $\mu_3 \neq \lambda$. Then there exists i such that $\langle h_i, \mu_3 \rangle < 0$. The induction hypothesis yields that

$$\mathsf{M}(\mu_1, \mu_2) \nabla \mathsf{M}(\mu_2, s_i\mu_3) \simeq \mathsf{M}(\mu_1, s_i\mu_3).$$

Since $\mu_1 \preceq \mu_2 \preceq \mu_3 \preceq s_i\mu_3$, Proposition 10.2.4 (iv) gives

$$\varepsilon_i^*\big(\mathsf{M}(\mu_2, s_i\mu_3)\big) = \varepsilon_i^*\big(\mathsf{M}(\mu_1, s_i\mu_3)\big) = -\langle h_i, \mu_3 \rangle.$$

Then Proposition 10.1.8 implies that

$$\widetilde{E}_i^{*\max}\big(\mathsf{M}(\mu_1, \mu_2) \nabla \mathsf{M}(\mu_2, s_i\mu_3)\big) \simeq \mathsf{M}(\mu_1, \mu_2) \nabla \big(\widetilde{E}_i^{*\max}\mathsf{M}(\mu_2, s_i\mu_3)\big),$$

from which we obtain

$$\mathsf{M}(\mu_1, \mu_3) \simeq \mathsf{M}(\mu_1, \mu_2) \nabla \mathsf{M}(\mu_2, \mu_3).$$

By Lemma 3.1.4, we have $\widetilde{\Lambda}\big(\mathsf{M}(\mu_1, \mu_2), \mathsf{M}(\mu_2, \mu_3)\big) = 0$. Hence we obtain

$$\Lambda\big(\mathsf{M}(\mu_1, \mu_2), \mathsf{M}(\mu_2, \mu_3)\big) = -\big(\mathrm{wt}(\mathsf{M}(\mu_1, \mu_2), \mathrm{wt}(\mathsf{M}(\mu_2, \mu_3))\big). \qquad \square$$

Proposition 10.3.2. *Let $i \in I$ and $x, y, z \in W$.*

(i) *If $\ell(xy) = \ell(x) + \ell(y)$, $zs_i > z$, $xy \geq zs_i$, and $x \geq z$, then we have*

$$\mathfrak{d}(\mathsf{M}(xy\varpi_i, zs_i\varpi_i), \mathsf{M}(x\varpi_i, z\varpi_i)) \leq 1.$$

(ii) *If $\ell(zy) = \ell(z) + \ell(y)$, $xs_i > x$, $xs_i \geq zy$, and $x \geq z$, then we have*

$$\mathfrak{d}(\mathsf{M}(xs_i\varpi_i, zy\varpi_i), \mathsf{M}(x\varpi_i, z\varpi_i)) \leq 1.$$

Proof. In the course of proof, we omit $\bar{\iota}_{\varpi_i, \varpi_i}^{-1}$ for the sake of simplicity. If $y\varpi_i = \varpi_i$, then the assertion follows from Proposition 10.2.3 (i). Hence we may assume that $y' := ys_i < y$.

Let us show (i). By Proposition 9.4.1, we have

$$\begin{aligned}\Delta(y\varpi_i, s_i\varpi_i)\Delta(\varpi_i, \varpi_i) = q^{-1}G^{\mathrm{up}}\left((u_{y\varpi_i} \otimes u_{\varpi_i}) \otimes (u_{s_i\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}}\right) \\ + G^{\mathrm{up}}(\bar{\iota}_\lambda^{-1}(\tilde{e}_i^* b) \otimes u_\lambda^{\mathrm{r}}),\end{aligned} \tag{10.8}$$

where $\lambda = \varpi_i + s_i\varpi_i$ and $b = \bar{\iota}_{\varpi_i}(u_{y\varpi_i}) \in B(\infty)$. Let $S_{z,\lambda}^*$ be the operator on $A_q(\mathfrak{g})$ given by the application of $e_{j_1}^{(a_1)} \cdots e_{j_t}^{(a_t)}$ from the right, where $z = s_{j_t} \cdots s_{j_1}$ is a reduced expression of z and $a_k = \langle h_{j_k}, s_{j_{k-1}} \cdots s_{j_1}\lambda \rangle$. Then applying $S_{z,\lambda}^*$ to (10.8), we obtain

$$\begin{aligned}\Delta(y\varpi_i, zs_i\varpi_i)\Delta(\varpi_i, z\varpi_i) = q^{-1}G^{\mathrm{up}}\left((u_{y\varpi_i} \otimes u_{\varpi_i}) \otimes (u_{zs_i\varpi_i} \otimes u_{z\varpi_i})^{\mathrm{r}}\right) \\ + G^{\mathrm{up}}(\bar{\iota}_\lambda^{-1}(\tilde{e}_i^* b) \otimes u_{z\lambda}^{\mathrm{r}}).\end{aligned}$$

Recall that $\mu \in \mathsf{P}$ is called *x-dominant* if $c_k \geq 0$. Here $x = s_{i_r} \cdots s_{i_1}$ is a reduced expression of x and $c_k := \langle h_{i_k}, s_{i_{k-1}} \cdots s_{i_1} \mu \rangle$ $(1 \leq k \leq r)$. Recall that an element $v \in A_q(\mathfrak{g})$ with $\mathrm{wt}_{\mathrm{l}}(v) = \mu$ is called *x-highest* if μ is x-dominant and

$$f_{i_k}^{1+c_k} f_{i_{k-1}}^{(c_{k-1})} \cdots f_{i_1}^{(c_1)} v = 0 \text{ for any } k \ (1 \leq k \leq r).$$

If v is x-highest, then v is a linear combination of x-highest $G^{\mathrm{up}}(b)$'s. Moreover, $S_{x,\mu} G^{\mathrm{up}}(b) := f_{i_r}^{(c_r)} \cdots f_{i_1}^{(c_1)} G^{\mathrm{up}}(b)$ is either a member of the upper global basis or zero. Since $\Delta(y\varpi_i, zs_i\varpi_i)\Delta(\varpi_i, z\varpi_i)$ is x-highest of weight $\mu{:=}y\varpi_i{+}\varpi_i$, we obtain

$$\begin{aligned}\Delta(xy\varpi_i, zs_i\varpi_i)\Delta(x\varpi_i, z\varpi_i) = q^{-1} G^{\mathrm{up}} \left((u_{xy\varpi_i} \otimes u_{x\varpi_i}) \otimes (u_{zs_i\varpi_i} \otimes u_{z\varpi_i})^{\mathrm{r}} \right) \\ + S_{x,\mu} G^{\mathrm{up}} (\overline{\iota}_{\lambda}^{-1} (\tilde{e}_i^* b) \otimes u_{z\lambda}^{\mathrm{r}}).\end{aligned}$$

Applying $p_{\mathfrak{n}}$, we obtain

$$\begin{aligned}q^c \mathrm{D}(xy\varpi_i, zs_i\varpi_i)\mathrm{D}(x\varpi_i, z\varpi_i) = q^{-1} p_{\mathfrak{n}} G^{\mathrm{up}} \left((u_{xy\varpi_i} \otimes u_{x\varpi_i}) \otimes (u_{zs_i\varpi_i} \otimes u_{z\varpi_i})^{\mathrm{r}} \right) \\ + p_{\mathfrak{n}} S_{x,\mu} G^{\mathrm{up}} (\overline{\iota}_{\lambda}^{-1} (\tilde{e}_i^* b) \otimes u_{z\lambda}^{\mathrm{r}})\end{aligned}$$

for some integer c. Hence we obtain (i) by Lemma 3.2.19 (i).

(ii) is proved similarly. By applying φ^* to (10.8), we obtain

$$\begin{aligned}& q^{(s_i\varpi_{i\,i}, \varpi_{i\,i}) - (y\varpi_{i\,i}, \varpi_{i\,i})} \Delta(s_i\varpi_i, y\varpi_i)\Delta(\varpi_i, \varpi_i) \\ & \quad = q^{-1} G^{\mathrm{up}} \left((u_{s_i\varpi_i} \otimes u_{\varpi_i}) \otimes (u_{y\varpi_i} \otimes u_{\varpi_i})^{\mathrm{r}} \right) \\ & \qquad + G^{\mathrm{up}} (u_{\lambda} \otimes (\overline{\iota}_{\lambda}^{-1} \tilde{e}_i^* b)^{\mathrm{r}}).\end{aligned}$$

Here we used Proposition 8.1.4. Then the similar arguments as above show (ii). $\square$

Proposition 10.3.3. *Let $x \in W$ such that $xs_i > x$ and $x\varpi_i \neq \varpi_i$. Then we have*

$$\mathfrak{d}(\mathsf{M}(xs_i\varpi_i, x\varpi_i), \mathsf{M}(x\varpi_i, \varpi_i)) = 1.$$

Proof. By Proposition 10.3.2 (ii), we have $\mathfrak{d}(\mathsf{M}(xs_i\varpi_i, x\varpi_i), \mathsf{M}(x\varpi_i, \varpi_i)) \leq 1$. Assuming $\mathfrak{d}(\mathsf{M}(xs_i\varpi_i, x\varpi_i), \mathsf{M}(x\varpi_i, \varpi_i)) = 0$, let us derive a contradiction.

By Theorem 10.3.1 and the assumption, we have

$$\mathsf{M}(xs_i\varpi_i, x\varpi_i) \circ \mathsf{M}(x\varpi_i, \varpi_i) \simeq \mathsf{M}(xs_i\varpi_i, \varpi_i).$$

Hence we have

$$\varepsilon_j^*(\mathsf{M}(xs_i\varpi_i, \varpi_i)) = \varepsilon_j^*(\mathsf{M}(xs_i\varpi_i, x\varpi_i)) + \varepsilon_j^*(\mathsf{M}(x\varpi_i, \varpi_i))$$

for any $j \in I$. Since $xs_i\varpi_i \preceq x\varpi_i \preceq s_i\varpi_i$, Proposition 10.2.4 implies that

$$\varepsilon_j^*(\mathsf{M}(xs_i\varpi_i, \varpi_i)) = \varepsilon_j^*(\mathsf{M}(x\varpi_i, \varpi_i)) = \langle h_j, \varpi_i \rangle.$$

It implies that

$$\varepsilon_j^*(\mathsf{M}(xs_i\varpi_i, x\varpi_i)) = 0 \qquad \text{for any } j \in I.$$

It is a contradiction since $\mathrm{wt}\big(\mathsf{M}(xs_i\varpi_i, x\varpi_i)\big) = xs_i\varpi_i - x\varpi_i$ does not vanish. $\square$

11. Monoidal categorification of $A_q(\mathfrak{n}(w))$

11.1. **Quantum cluster algebra structure on $A_q(\mathfrak{n}(w))$.**

In this subsection, we shall consider the Kac–Moody algebra $\mathfrak{g}$ associated with a symmetric Cartan matrix $\mathsf{A} = (a_{i,j})_{i,j \in I}$. We shall recall briefly the definition of the subalgebra $A_q(\mathfrak{n}(w))$ of $A_q(\mathfrak{g})$ and its quantum cluster algebra structure by using the results of [11] and [23]. Remark that we bring the results in [11] through the isomorphism (8.3).

For a given $w \in W$, fix a reduced expression $\widetilde{w} = s_{i_r} \cdots s_{i_1}$.

For $s \in \{1, \ldots, r\}$ and $j \in I$, we set

$$\begin{aligned} s_+ &:= \min(\{k \mid s < k \le r,\ i_k = i_s\} \cup \{r+1\}), \\ s_- &:= \max(\{k \mid 1 \le k < s,\ i_k = i_s\} \cup \{0\}), \\ s^-(j) &:= \max(\{k \mid 1 \le k < s,\ i_k = j\} \cup \{0\}). \end{aligned}$$

We set

$$u_k := s_{i_1} \cdots s_{i_k} \text{ for } 0 \le k \le r, \tag{11.1}$$

and

$$\lambda_k := u_k \varpi_{i_k} \text{ for } 1 \le k \le r.$$

Note that $\lambda_{k_-} = u_{k-1}\varpi_{i_k}$, if $k_- > 0$. For $0 \le t \le s \le r$, we set

$$\mathrm{D}(s,t) = \begin{cases} \mathrm{D}(\lambda_s, \lambda_t) & \text{if } 0 < t, \\ \mathrm{D}(\lambda_s, \varpi_{i_s}) & \text{if } 0 = t < s \le r, \\ \mathbf{1} & \text{if } t = s = 0. \end{cases}$$

The $\mathbb{Q}(q)$-subalgebra of $A_q(\mathfrak{n})$ generated by $\mathrm{D}(i, i_-)$ $(1 \le i \le r)$ is independent of the choice of a reduced expression of w. We denote it by $A_q(\mathfrak{n}(w))$. Then every $\mathrm{D}(s,t)$ $(0 \le t \le s \le r)$ is contained in $A_q(\mathfrak{n}(w))$ [11, Corollary 12.4]. The set $\mathbf{B}^{\mathrm{up}}\big(A_q(\mathfrak{n}(w))\big) := \mathbf{B}^{\mathrm{up}}\big(A_q(\mathfrak{g})\big) \cap A_q(\mathfrak{n}(w))$ is a basis of $A_q(\mathfrak{n}(w))$ as a $\mathbb{Q}(q)$-vector space [23, Theorem 4.2.5]. We call it the *upper global basis* of $A_q(\mathfrak{n}(w))$. We denote by $A_q(\mathfrak{n}(w))_{\mathbb{Z}[q^{\pm 1}]}$ the $\mathbb{Z}[q^{\pm 1}]$-module generated by $\mathbf{B}^{\mathrm{up}}\big(A_q(\mathfrak{n}(w))\big)$. Then it is a $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $A_q(\mathfrak{n}(w))$ [23, Section 4.7.2]. We set $A_{q^{1/2}}(\mathfrak{n}(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(\mathfrak{n}(w))$.

Let $J = \{1, \ldots, r\}$, $J_{\mathrm{fr}} := \{k \in J \mid k_+ = r+1\}$, and $J_{\mathrm{ex}} := J \setminus J_{\mathrm{fr}}$.

Definition 11.1.1. *We define the quiver Q with the set of vertices Q_0 and the set of arrows Q_1 as follows:*

(Q_0) $Q_0 = J = \{1, \ldots, r\}$,

(Q_1) *There are two types of arrows:*

$$\begin{aligned} &\text{ordinary arrows} \quad : \quad s \xrightarrow{|a_{i_s,i_t}|} t \quad \textit{if } 1 \le s < t < s_+ < t_+ \le r+1, \\ &\text{horizontal arrows} \quad : \quad s \longrightarrow s_- \quad \textit{if } 1 \le s_- < s \le r. \end{aligned}$$

Let $\widetilde{B} = (b_{i,j})$ be the integer-valued $J \times J_{\mathrm{ex}}$-matrix associated to the quiver Q by (5.2).

Lemma 11.1.2. *Assume that $0 \le d \le b \le a \le c \le r$ and*

- *$i_b = i_a$ when $b \ne 0$,*
- *$i_d = i_c$ when $d \ne 0$.*

Then $\mathrm{D}(a,b)$ *and* $\mathrm{D}(c,d)$ *q-commute; that is, there exists* $\lambda \in \mathbb{Z}$ *such that*

$$\mathrm{D}(a,b)\mathrm{D}(c,d) = q^{\lambda}\mathrm{D}(c,d)\mathrm{D}(a,b).$$

Proof. We may assume $a > 0$. Let u_k be as in (11.1). Take $s' = u_a$, $s = u_a^{-1}u_c$, $t' = u_d$, and $t = u_d^{-1}u_b$. Then we have

$$\mathrm{D}(s'\varpi_{i_a}, t't\varpi_{i_a}) = \mathrm{D}(a,b) \quad \text{and} \quad \mathrm{D}(s's\varpi_{i_c}, t'\varpi_{i_c}) = \mathrm{D}(c,d).$$

From Proposition 9.1.6, our assertion follows. ☐

Hence we have an integer-valued skew-symmetric matrix $L = (\lambda_{i,j})_{1\le i,j\le r}$ such that

$$\mathrm{D}(i,0)\mathrm{D}(j,0) = q^{\lambda_{i,j}}\mathrm{D}(j,0)\mathrm{D}(i,0).$$

Proposition 11.1.3 ([11, Proposition 10.1]). *The pair* $(L, \widetilde{B})$ *is compatible with* $d = 2$ *in* (5.3).

Theorem 11.1.4 ([11, Theorem 12.3]). *Let* $\mathscr{A}_{q^{1/2}}([\mathscr{S}])$ *be the quantum cluster algebra associated to the initial quantum seed* $[\mathscr{S}] := (\{q^{-(d_s,d_s)/4}\mathrm{D}(s,0)\}_{1\le s\le r}, L, \widetilde{B})$. *Then we have an isomorphism of* $\mathbb{Q}(q^{1/2})$*-algebras*

$$\mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Z}[q^{\pm 1/2}]} \mathscr{A}_{q^{1/2}}([\mathscr{S}]) \simeq A_{q^{1/2}}(\mathfrak{n}(w)),$$

where $d_s := \lambda_s - \varpi_{i_s} = \mathrm{wt}(D(s,0))$ *and* $A_{q^{1/2}}(\mathfrak{n}(w)) := \mathbb{Q}(q^{1/2}) \otimes_{\mathbb{Q}(q)} A_q(\mathfrak{n}(w))$.

11.2. **Admissible seeds in the monoidal category $\mathcal{C}_w$.** For $0 \le t \le s \le r$, we set $\mathsf{M}(s,t) = \mathsf{M}(\lambda_s, \lambda_t)$. It is a real simple module with $\mathrm{ch}(\mathsf{M}(s,t)) = D(s,t)$.

Definition 11.2.1. *For* $w \in W$, *let* $\mathcal{C}_w$ *be the smallest monoidal abelian full subcategory of* R-gmod *satisfying the following properties:*

- (i) $\mathcal{C}_w$ *is stable under the subquotients, extensions, and grading shifts,*
- (ii) $\mathcal{C}_w$ *contains* $\mathsf{M}(s, s_-)$ *for all* $1 \le s \le \ell(w)$.

Then by [11], $M \in R$-gmod belongs to $\mathcal{C}_w$ if and only if $\mathrm{ch}(M)$ belongs to $A_q(\mathfrak{n}(w))$. Hence we have a $\mathbb{Z}[q^{\pm 1}]$-algebra isomorphism

$$K(\mathcal{C}_w) \simeq A_q(\mathfrak{n}(w))_{\mathbb{Z}[q^{\pm 1}]}.$$

We set

$$\Lambda := (\Lambda(\mathsf{M}(i,0), \mathsf{M}(j,0)))_{1\le i,j\le r} \quad \text{and} \quad D = (d_i)_{1\le i\le r} := (\mathrm{wt}(\mathsf{M}(i,0)))_{1\le i\le r}.$$

Then, by Proposition 10.2.3, $\mathscr{S} := (\{\mathsf{M}(k,0)\}_{1\le k\le r}, -\Lambda, \widetilde{B}, D)$ is a quantum monoidal seed in $\mathcal{C}_w$. We are now ready to state the main theorem in this section.

Theorem 11.2.2. *The pair* $\big(\{\mathsf{M}(k,0)\}_{1\le k\le r}, \widetilde{B}\big)$ *is admissible.*

As we already explained, combined with Theorem 7.1.3 and Corollary 7.1.4, this theorem implies the following theorem.

Theorem 11.2.3. *The category* $\mathcal{C}_w$ *is a monoidal categorification of the quantum cluster algebra* $A_{q^{1/2}}(\mathfrak{n}(w))$.

In the course of proving Theorem 11.2.2, *we omit grading shifts* if there is no danger of confusion.

We shall start the proof of Theorem 11.2.2 by proving that, for each $s \in J_{\mathrm{ex}}$, there exists a simple module X such that

$$\text{(11.2)} \quad \begin{cases} \text{(a) there exists a surjective homomorphism (up to a grading shift)} \\ \qquad X \circ \mathsf{M}(s,0) \twoheadrightarrow \circ_{t;\, b_{t,s}>0} \mathsf{M}(t,0)^{\circ b_{t,s}}, \\ \text{(b) there exists a surjective homomorphism (up to a grading shift)} \\ \qquad \mathsf{M}(s,0) \circ X \twoheadrightarrow \circ_{t;\ \ b_{t,s}<0} \mathsf{M}(t,0)^{\circ -b_{t,s}}, \\ \text{(c) } \mathfrak{d}(X, \mathsf{M}(s,0)) = 1. \end{cases}$$

We set

$$\begin{aligned} & x := i_s \in I, \\ & I_s := \{ i_k \mid s < k < s_+ \} \subset I \setminus \{x\}, \\ & A := \mathop{\circ}_{t<s<t_+<s_+} \mathsf{M}(t,0)^{\circ |a_{i_s,i_t}|} = \mathop{\circ}_{y \in I_s} \mathsf{M}(s^-(y),0)^{\circ |a_{x,y}|}. \end{aligned}$$

Then A is a real simple module.

Now we claim that the following simple module X satisfies the conditions in (11.2):

$$X := \mathsf{M}(s_+, s) \nabla A.$$

Let us show (11.2) (a). The incoming arrows to s are

- $t \xrightarrow{|a_{x,i_t}|} s$ for $1 \le t < s < t_+ < s_+$,
- $s_+ \longrightarrow s$.

Hence we have

$$\circ_{t;\, b_{t,s}>0} \mathsf{M}(t,0)^{\circ b_{t,s}} \simeq A \circ M(s_+, 0).$$

Then the morphism in (a) is obtained as the composition,

$$\text{(11.3)} \qquad X \circ \mathsf{M}(s,0) \rightarrowtail A \circ \mathsf{M}(s_+, s) \circ \mathsf{M}(s,0) \twoheadrightarrow A \circ \mathsf{M}(s_+, 0).$$

Here the second epimorphism is given in Theorem 10.3.1, and Lemma 3.1.5 asserts that the composition (11.3) is non-zero and hence an epimorphism.

Let us show (11.2) (b). The outgoing arrows from s are

- $s \xrightarrow{|a_{x,i_t}|} t$ for $s < t < s_+ < t_+ \le r+1$.
- $s \longrightarrow s_-$ if $s_- > 0$.

Hence we have

$$\text{(11.4)} \qquad \mathop{\circ}_{t; b_{t,s}<0} \mathsf{M}(t,0)^{\circ - b_{t,s}} \simeq \mathsf{M}(s_-, 0) \circ \left(\mathop{\circ}_{y \in I_s} \mathsf{M}((s_+)^-(y), 0)^{\circ - a_{x,y}} \right).$$

Lemma 11.2.4. *There exists an epimorphism* (*up to a grading*)

$$\Omega : \mathsf{M}(s,0) \circ \mathsf{M}(s_+, s) \circ A \twoheadrightarrow \circ_{t; b_{t,s}<0} \mathsf{M}(t,0)^{\circ - b_{t,s}}.$$

Proof. By the dual of Theorem 10.3.1 and the T-system (10.7) with $i = i_s$, $u = u_{s_+-1}$, and $v = u_{s-1}$, we have morphisms

$$\begin{aligned} & \mathsf{M}(s,0) \rightarrowtail \mathsf{M}(s_-, 0) \circ \mathsf{M}(s, s_-), \\ & \mathsf{M}(s, s_-) \circ \mathsf{M}(s_+, s) \twoheadrightarrow \circ_{y \in I \setminus \{x\}} \mathsf{M}((s_+)^-(y), s^-(y))^{\circ - a_{x,y}} \\ & \qquad\qquad\qquad \simeq \circ_{y \in I_s} \mathsf{M}((s_+)^-(y), s^-(y))^{\circ - a_{x,y}}. \end{aligned}$$

Here the last isomorphism follows from the fact that $(s_+)^-(y) = s^-(y)$ for any $y \notin \{x\} \cup I_s = \{ i_k \mid s \le k < s_+ \}$.

Thus we have a sequence of morphisms

$$\begin{aligned}\mathsf{M}(s,0)\circ\mathsf{M}(s_+,s)\circ A &\overset{\varphi_1}{\rightarrowtail} \mathsf{M}(s_-,0)\circ\mathsf{M}(s,s_-)\circ\mathsf{M}(s_+,s)\circ A\\ &\overset{\varphi_2}{\twoheadrightarrow} \mathsf{M}(s_-,0)\circ\left(\circ_{y\in I_s}\mathsf{M}((s_+)^-(y),s^-(y))^{\circ -a_{x,y}}\right)\circ A.\end{aligned}$$

By Lemma 3.1.5 (i), the composition $\varphi:=\varphi_2\circ\varphi_1$ is non-zero.

Since $A=\circ_{y\in I_s}\mathsf{M}(s^-(y),0)^{\circ -a_{x,y}}$, Theorem 10.3.1 gives the morphisms

$$\begin{aligned}\mathsf{M}(s,0)\circ\mathsf{M}(s_+,s)\circ A &\xrightarrow{\varphi} \mathsf{M}(s_-,0)\circ\left(\circ_{y\in I_s}\mathsf{M}((s_+)^-(y),s^-(y))^{\circ -a_{x,y}}\right)\circ A\\ &\overset{\phi}{\twoheadrightarrow} \mathsf{M}(s_-,0)\circ\left(\circ_{y\in I_s}\mathsf{M}((s_+)^-(y),0)^{\circ -a_{x,y}}\right)\\ &\simeq \circ_{t;b_{t,s}<0}\mathsf{M}(t,0)^{\circ -b_{t,s}}.\end{aligned}$$

Here we have used Lemma 3.2.22 to obtain the morphism ϕ. Note that the module $\circ_{y\in I_s}\mathsf{M}((s_+)^-(y),s^-(y))^{\circ -a_{x,y}}$ is simple. By applying Lemma 3.1.5 once again, $\phi\circ\varphi$ is non-zero, and hence it is an epimorphism. ☐

Lemma 11.2.5. *We have* $\mathfrak{d}(X,\mathsf{M}(s,0))=1$.

Proof. Since A and $\mathsf{M}(s,0)$ commute and $\mathfrak{d}\left(\mathsf{M}(s_+,s),\mathsf{M}(s,0)\right)=1$ by Proposition 10.3.3, we have

$$\mathfrak{d}\left(X,\mathsf{M}(s,0)\right)\le\mathfrak{d}\left(\mathsf{M}(s_+,s),\mathsf{M}(s,0)\right)+\mathfrak{d}\left(A,\mathsf{M}(s,0)\right)\le 1$$

by Proposition 3.2.10 and Lemma 3.2.3. If X and $\mathsf{M}(s,0)$ commute, then (11.2) (a) would imply that $\mathrm{ch}\left(\circ_{t;\ b_{t,s}>0}M(t,0)^{\circ b_{t,s}}\right)$ belongs to $K(R\text{-gmod})\,\mathrm{ch}(\mathsf{M}(s,0))$. It contradicts the result in [10] that all the $\mathrm{ch}(\mathsf{M}(k,0))$'s are prime at $q=1$. ☐

Proposition 11.2.6. *The map* Ω *factors through* $\mathsf{M}(s,0)\circ X$*; that is,*

$$\begin{tikzcd}\mathsf{M}(s,0)\circ\mathsf{M}(s_+,s)\circ A \arrow[rr, two heads, "\Omega"] \arrow[dr, two heads, "\tau"'] & & \circ_{t;b_{t,s}<0}\mathsf{M}(t,0)^{\circ -b_{t,s}}. \\ & \mathsf{M}(s,0)\circ X \arrow[ur, two heads, "\overline{\Omega}"'] & \end{tikzcd}$$

Here τ *is the canonical surjection.*

Proof. We have $1=\mathfrak{d}\left(\mathsf{M}(s,0),\mathsf{M}(s_+,s)\,\nabla\,A\right)$ by Lemma 11.2.5, and

$$\mathfrak{d}\left(\mathsf{M}(s,0),\mathsf{M}(s_+,s)\right)+\mathfrak{d}\left(\mathsf{M}(s,0),A\right)=1$$

by Proposition 10.3.3 with $x=u_{s_+-1}$, $i=i_s$. Hence $\mathsf{M}(s,0)\circ\mathsf{M}(s_+,s)\circ A$ has a simple head by Proposition 3.2.16 (iii). ☐

End of the proof of Theorem 11.2.2. By the above arguments, we have proved the existence of X which satisfies (11.2). By Proposition 3.2.17 and (11.2) (c), $\mathsf{M}(s,0)\circ X$ has composition length 2. Moreover, it has a simple socle and simple head. On the other hand, taking the dual of (11.2) (a), we obtain a monomorphism

$$\bigodot_{t;b_{t,s}>0}\mathsf{M}(t,0)^{\odot b_{t,s}}\rightarrowtail\mathsf{M}(s,0)\circ X$$

in R-mod. Together with (11.2) (b), there exists a short exact sequence in R-gmod:

$$0\to q^c\bigodot_{t;b_{t,s}>0}\mathsf{M}(t,0)^{\odot b_{t,s}}\to q^{\widetilde{\Lambda}(\mathsf{M}(s,0),X)}\mathsf{M}(s,0)\circ X\to\bigodot_{t;b_{t,s}<0}\mathsf{M}(t,0)^{\odot(-b_{t,s})}\to 0$$

for some $c\in\mathbb{Z}$. By Lemma 3.2.18 c must be equal to 1.

It remains to prove that X commutes with $\mathsf{M}(k,0)$ $(k \neq s)$. For any $k \in J$, we have

$$\begin{aligned}\Lambda(\mathsf{M}(k,0),X) &= \Lambda(\mathsf{M}(k,0),\mathsf{M}(s,0)\nabla X) - \Lambda(\mathsf{M}(k,0),\mathsf{M}(s,0))\\ &= \sum_{t;\ b_{t,s}<0} \Lambda(\mathsf{M}(k,0),\mathsf{M}(t,0))(-b_{t,s}) - \Lambda(\mathsf{M}(k,0),\mathsf{M}(s,0))\end{aligned}$$

and

$$\begin{aligned}\Lambda(X,\mathsf{M}(k,0)) &= \Lambda(X\nabla \mathsf{M}(s,0),\mathsf{M}(k,0)) - \Lambda(\mathsf{M}(s,0),\mathsf{M}(k,0))\\ &= \sum_{t;\ b_{t,s}>0} \Lambda(\mathsf{M}(t,0),\mathsf{M}(k,0))b_{t,s} - \Lambda(\mathsf{M}(s,0),\mathsf{M}(k,0)).\end{aligned}$$

Hence we have

$$\begin{aligned}2\mathfrak{d}(\mathsf{M}(k,0),X) &= -2\mathfrak{d}(\mathsf{M}(k,0),\mathsf{M}(s,0)) - \sum_{t;\ b_{t,s}<0} \Lambda(\mathsf{M}(k,0),\mathsf{M}(t,0))b_{t,s}\\ &\qquad - \sum_{t;\ b_{t,s}>0} \Lambda(\mathsf{M}(k,0),\mathsf{M}(t,0))b_{t,s}\\ &= -\sum_{1\le t\le r} \Lambda(\mathsf{M}(k,0),\mathsf{M}(t,0))b_{t,s}\\ &= 2\delta_{k,s}.\end{aligned}$$

We conclude that X commutes with $M(k,0)$ if $k \neq s$. Thus we complete the proof of Theorem 11.2.2. ☐

As a corollary, we prove the following conjecture on the cluster monomials.

Theorem 11.2.7 ([11, Conjecture 12.9], [23, Conjecture 1.1(2)])**.** *Every cluster variable in $A_q(\mathfrak{n}(w))$ is a member of the upper global basis up to a power of $q^{1/2}$.*

Theorem 11.2.2 also implies [11, Conjecture 12.7] in the refined form as follows.

Corollary 11.2.8. $\mathbb{Z}[q^{\pm 1/2}] \otimes_{\mathbb{Z}[q^{\pm1}]} A_q(\mathfrak{n}(w))_{\mathbb{Z}[q^{\pm1}]}$ *has a quantum cluster algebra structure associated with the initial quantum seed*

$$[\mathscr{S}] = (\{q^{-(d_i,d_i)/4}\mathrm{D}(i,0)\}_{1\le i\le r}, L, \widetilde{B});$$

i.e.,

$$\mathbb{Z}[q^{\pm1/2}] \underset{\mathbb{Z}[q^{\pm1}]}{\otimes} A_q(\mathfrak{n}(w))_{\mathbb{Z}[q^{\pm1}]} \simeq \mathscr{A}_{q^{1/2}}([\mathscr{S}]).$$

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