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SIMPLE GROUPS OF MORLEY RANK 3 ARE ALGEBRAIC

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1. Introduction

Model theory is a branch of mathematical logic concerned with the study of classes of mathematical structures by considering first-order sentences and formulas. There are numerous interactions between model theory and other areas of mathematics, such as number theory, sometimes these interactions are spectacular, such as Pila's work on the André–Oort Conjecture [15] or Hrushovski's proof of the function field Mordell–Lang conjecture [9]. In particular model theory of abelian groups is used in the latter, and a result of Wagner on abelian groups *of finite Morley rank* is used in a recent paper concerning the Mordell–Lang conjecture [3].

Morley rank is a model-theoretical notion of dimension. It generalizes the dimension of an algebraic variety (when the ground field is algebraically closed). In this paper, we are concerned with *groups* of finite Morley rank. The main example of such a group is an algebraic group defined over an algebraically closed field in the field language (Zilber [20]). Independently, in the late 1970s Gregory Cherlin [6, §6] and Boris Zilber [20] formulated the following algebraicity conjecture.

Conjecture 1.1 (Cherlin–Zilber conjecture or algebraicity conjecture)**.** *An infinite simple group of finite Morley rank is algebraic over an algebraically closed field.*

This is the main conjecture on groups of finite Morley rank, and it is still open. Most studies on groups of finite Morley rank focus on this conjecture.

In the 1980s, Borovik proposed to attack the algebraicity conjecture by transferring methods from the classification of the finite simple groups, and to analyze a minimal counterexample from its involutions. Borovik's program has been very effective for several important classes of groups of finite Morley rank, including locally finite groups [18]. Its main success is the main theorem of [1] which ensures that any simple group of finite Morley rank with an infinite abelian subgroup of exponent 2 satisfies the Cherlin–Zilber conjecture.

However, despite numerous papers on the subject, the Cherlin–Zilber conjecture is still open. In this paper we show that any simple group of Morley rank 3 is algebraic over an algebraically closed field. Due to the absence of any internal group theoretic structure allowing local analysis, one resorts to a more geometrical analysis. We note that such an analysis is also encountered in the Borovik program, but it is often associated there with the geometry of involutions or other aspects of local analysis.

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As a matter of fact, in [6] the algebraicity conjecture was formulated as a result of local analysis of simple groups of Morley rank 3. The main result of [6] can be summarized as follows, where a *bad group* is a nonsolvable group of Morley rank 3 containing no definable subgroup of Morley rank 2.

Fact 1.2 (Cherlin [6])**.** Let G be an infinite simple group of Morley rank at most 3. Then G has Morley rank 3, and one of the following two assertions is satisfied:

- there is an algebraically closed field K such that $G \simeq \mathrm{PSL}_2(K)$,
- G is a bad group.

Thus bad groups have become a major obstacle to the Cherlin–Zilber conjecture. These groups have been studied in [6], [13], and [14], whose results are summarized in Facts 2.3 and 2.4, respectively. Later, it was shown that no bad group is existentially closed [11] or linear [12]. However, these groups appeared very resistant, and only sparse supplementary information was known on them.

Furthermore, Nesin has shown in [14] that a bad group acts on a natural geometry, which is not very far from being a non-Desarguesian projective plane of Morley rank 2. However, Baldwin [2] discovered non-Desarguesian projective planes of Morley rank 2. Thus, the question of the existence, or not, of a bad group was still fully open. In this paper, we show that bad groups do not exist.

Main Theorem 1.3. *There is no bad group* (*in the sense of Cherlin*).

Note that other more general notions of bad groups have been introduced independently by Corredor [7] and by Borovik and Poizat [4], where a *bad group* is defined to be a nonsolvable connected group of finite Morley rank all of whose proper connected definable subgroups are nilpotent. Such a bad group has similar properties to original bad groups. Moreover, Jaligot later introduced a more general notion of bad groups [10] and obtained similar results. However, we recall that, in this paper, a *bad group* is defined as a nonsolvable group of Morley rank 3 containing no definable subgroup of Morley rank 2.

Our proof of Main Theorem 1.3 goes as follows. First we note that it is sufficient to study *simple* bad groups since for any bad group G, the quotient group $G/Z(G)$ is a *simple* bad group by [13, §4, Introduction].

Then we fix a simple bad group G, and we introduce a notion of line as a coset of a Borel subgroup of G (Definition 3.1). In §3 we study their behavior, mainly in regards with conjugacy classes of elements of G.

In §4 we propose a definition of a plane (Definition 4.1). This section is dedicated to proving that G contains a plane (Theorem 4.14). This result is the key point of our demonstration. Roughly speaking, we show that for each nontrivial element g of G such that $g = [u, v]$ for $(u, v) \in G \times G$, the union of the preimages of g, by maps of the form $\mathrm{ad}_v : G \to G$ defined by $\mathrm{ad}_v(x) = [x, v]$, is almost a plane, and from this, we obtain a plane.

In last section, §5, we try to show that our notions of lines and planes provide a structure of projective space over the group G. Indeed, such a structure would provide a division ring (see [8, p. 124, Theorem 7.15]), and probably it would be easy to conclude. However, a contradiction occurs along the way, and achieves our proof.

The other simple groups of dimension 3.

- If G is a nonbad simple group of Morley rank 3, then G is isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K (Fact 1.2). As in §3 we may define a *line* in G to be a coset of a connected subgroup of dimension 1, and we may define a plane as in §4. It is possible to show that two sorts of planes occur: the cosets of Borel subgroups; and the subsets of the form aJ where J is defined to be
  - the set of involutions when the characteristic c of K is not 2,
  - the set of involutions and the identity element when $c = 2$.

  The plane J is normalized by G, and there is no such a plane in a bad group (Lemma 5.12). Another important difference between G and a bad group is the presence of a Weyl group. Indeed, the first lemma of this paper is not verified in G (Lemma 3.2), because we have $jT = Tj$ for any torus T and any involution $j \in N_G(T) \setminus T$.
- The group $\mathrm{SO}_3(\mathbb{R})$ is not of finite Morley rank and is not even stable [13]. However, our definitions of lines and planes naturally extend to $\mathrm{SO}_3(\mathbb{R})$. Then, as above, the set J of involutions in $\mathrm{SO}_3(\mathbb{R})$ forms a plane, and the presence of a Weyl group is again a major difference between $\mathrm{SO}_3(\mathbb{R})$ and bad groups. Moreover, we note that the plane J has a structure of projective plane, whereas this is false in $\mathrm{PSL}_2(K)$ [5, Fact 8.15].

Note. In very recent preprints [17, 19], by analyzing the present paper, Poizat and Wagner generalize our main result to other groups, and they eliminate other groups of finite Morley rank.

2. Background material

A thorough analysis of groups of finite Morley rank can be found in [5] and [1]. In this section we recall some definitions and known results.

2.1. **Borovik–Poizat axioms.** Let $(G, \cdot, ^{-1}, 1, \ldots)$ be a group equipped with additional structure. This group G is said to be *ranked* if there is a function "rk" which assigns to each nonempty definable set S an integer, its "dimension" $\mathrm{rk}\,(S)$, and which satisfies the following axioms for every definable sets A and B.

Definition: For any integer n, $\mathrm{rk}\,(A) > n$ if and only if A contains an infinite family of disjoint definable subsets A_i of rank n.

Definability: For any uniformly definable family $\{A_b \ : \ b \in B\}$ of definable sets and for any $n \in \mathbb{N}$, the set $\{b \in B \ : \ \mathrm{rk}\,(A_b) = n\}$ is also definable.

Finite Bounds: For any uniformly definable family $\mathfrak{F}$ of finite subsets of A, the sizes of the sets in $\mathfrak{F}$ are bounded.

These axioms were introduced in [16], where it is shown that the groups as above satisfy a fourth axiom, namely the *additivity axiom*, and they are precisely the groups of finite Morley rank. Moreover, the function rk assigns to each definable set its Morley rank. In this paper, as in [5] and [1], the Morley rank will be denoted by rk.

2.2. **Morley degree.** A nonempty definable set A is said to have *Morley degree* 1 if for any definable subset B of A, either $\mathrm{rk}\, B < \mathrm{rk}\, A$ or $\mathrm{rk}\,(A \setminus B) < \mathrm{rk}\, A$. The set A is said to have *Morley degree* d if A is the disjoint union of d definable sets of Morley degree 1 and Morley rank $\mathrm{rk}\, A$.

Fact 2.1.

- Every nonempty definable set has a unique degree [5, Lemmas 4.12 and 4.14].
- Let X and Y be definable subsets of Morley degree d and d', respectively. Then $X \times Y$ has Morley degree dd' [5, Proposition 4.2].
- A group of finite Morley rank has Morley degree 1 if and only if it is *connected*, namely it has no proper definable subgroup of finite index [6, §2.2].

Moreover, the following elementary result will be useful for us.

Fact 2.2. Let $f : E \to F$ be a definable map. If the set E has Morley degree 1 and $r = \mathrm{rk}\, f^{-1}(y)$ is constant for $y \in F$, then the Morley degree of F is 1.

Proof. Let B be a definable subset of F of Morley rank $\mathrm{rk}\, F$. We show that $\mathrm{rk}\,(F \setminus B) < \mathrm{rk}\, F$. By the additivity axiom, we have $\mathrm{rk}\, E = r + \mathrm{rk}\, F$ and

$$\mathrm{rk}\, f^{-1}(B) = r + \mathrm{rk}\, B = r + \mathrm{rk}\, F = \mathrm{rk}\, E.$$

Since E has Morley degree 1, we obtain $\mathrm{rk}\, f^{-1}(F \setminus B) = \mathrm{rk}\,(E \setminus f^{-1}(B)) < \mathrm{rk}\, E$, and by the additivity axiom again

$$\mathrm{rk}\,(F \setminus B) = \mathrm{rk}\, f^{-1}(F \setminus B) - r < \mathrm{rk}\, E - r = \mathrm{rk}\, F,$$

so F has Morley degree 1. □

2.3. **Bad groups.** The main properties of bad groups are summarized in the following facts, where a *Borel subgroup* of a bad group G is defined to be an infinite definable proper subgroup of G.

Fact 2.3 ([6, §5.2] and [13])**.** Let G be a simple bad group, and let B be a Borel subgroup of G.

(1) $B = C_G(b)$ for any nontrivial element b of B.
(2) B is connected, abelian, self-normalizing, and of Morley rank 1.
(3) $C_G(x)$ is a Borel subgroup for each nontrivial element x of G.
(4) If A is another Borel subgroup of G, then A is conjugate with B, and either $A = B$ or $A \cap B = \{1\}$.
(5) $G = \bigcup_{g \in G} B^g$.
(6) G has no involution.

Fact 2.4 ([14, Lemma 18])**.** Let A and B be two distinct Borel subgroups of a simple bad group G. Then $\mathrm{rk}\,(ABA) = 3$, $\mathrm{rk}\,(AB) = 2$, and AB has Morley degree 1.

The following result, due to Delahan and Nesin, was proved for a more general notion of bad groups and is used in our final argument.

Fact 2.5 ([5, Proposition 13.4])**.** A simple bad group G cannot have an involutive definable automorphism.

3. Lines

In this paper G denotes a fixed simple bad group. We fix a Borel subgroup B of G, and we denote by $\mathscr{B}$ the set of Borel subgroups of G.

In this section, we define a *line* of G, and we provide their basic properties. We note that, by conjugation of Borel subgroups (Fact 2.3(4)), any Borel subgroup is a *line* in the following sense.

Definition 3.1. A *line* of G is a subset of the form uBv for two elements u and v of G.

We denote by Λ the set of lines of G.

We note that, by Fact 2.3(2), each line has Morley rank 1 and Morley degree 1.

Lemma 3.2. *Let uBv and rBs be two lines. Then $uBv = rBs$ if and only if $uB = rB$ and $Bv = Bs$.*

Proof. We may assume that $uBv = rBs$. Then we have

$$B = u^{-1}rBsv^{-1} = u^{-1}rsv^{-1}B^{sv^{-1}},$$

so $u^{-1}rsv^{-1} \in B^{sv^{-1}}$ and $B = B^{sv^{-1}}$. Now sv^{-1} belongs to B since B is self-normalizing by Fact 2.3. Hence we obtain $Bv = Bs$, and the equality $uB = rB$ follows from $uBv = rBs$. □

By the above lemma, the set Λ identifies with $(G/B)_l \times (G/B)_r$, where $(G/B)_l$ (resp. $(G/B)_r$) denotes the set of left cosets (resp. right cosets) of B in G. Then Λ is a definable set. Moreover, since G is connected of Morley rank 3 and B has Morley rank 1, the Morley rank of Λ is 4 and its Morley degree is 1. In particular, Λ is a uniformly definable family.

Lemma 3.3. *Two distinct elements x and y of G lie in one and only one line $l(x, y)$. Moreover, the map $l : \{(x, y) \in G \times G \mid x \neq y\} \to \Lambda$ is definable.*

Proof. By Fact 2.3(5), there exists $v \in G$ such that $y^{-1}x$ belongs to B^v. Then x and y lie in uBv for $u = yv^{-1}$.

Now, if rBs is a line containing x and y, then we find two elements b_1 and b_2 of B such that $x = rb_1s$ and $y = rb_2s$. Thus $y^{-1}x = s^{-1}b_2^{-1}b_1s$ is a nontrivial element of B^s. But $y^{-1}x$ belongs to B^v by the choice of v, hence we have $B^s = B^v$ (Fact 2.3(4)). Since B is self-normalizing, sv^{-1} belongs to B, and we obtain $Bs = Bv$, so there exists $b \in B$ such that $s = bv$. This implies that $u = yv^{-1} = (rb_2s)(s^{-1}b) = rb_2b$ belongs to rB, and $rBs = uBv$ is the unique line containing x and y.

Moreover, since Λ is a uniformly definable family, the set $\{(x, y) \in G \times G \mid x \neq y\} \times \Lambda$ is definable, and

$$\Gamma = \{((x, y), uBv) \in (G \times G) \times \Lambda \mid x \neq y,\ x \in uBv,\ y \in uBv\}$$

is a definable subset of it. But Γ is precisely the graph of the map l, hence l is definable. □

Lemma 3.4. *If $uBv = (uBv)^g$ for $uBv \in \Lambda \setminus \mathscr{B}$ and $g \in G$, then $g = 1$.*

Proof. We have $uBv = g^{-1}uBvg$, so $uB = g^{-1}uB$ and $Bv = Bvg$ by Lemma 3.2, and g belongs to the Borel subgroups $B^{u^{-1}}$ and B^v. If g is nontrivial, then $B^{u^{-1}} = B^v$ (Fact 2.3(4)), and vu belongs to $N_G(B) = B$. Consequently, u belongs to $v^{-1}B$, and we obtain $uBv = B^v$, contradicting $uBv \notin \mathscr{B}$. Thus $g = 1$. □

Definition 3.5. For each $g \in G$ and each definable subset X of G, we consider the following subsets of Λ:

$$\mathscr{L}(g, X) = \{l(g, x) \in \Lambda \mid x \in X \setminus \{g\}\},$$

$$\Lambda_X = \{\lambda \in \Lambda \mid \lambda \cap X \text{ is infinite}\}.$$

Since the map l is definable (Lemma 3.3), the set $\mathscr{L}(g, X)$ is definable for each $g \in G$ and each definable subset X of G. Moreover, by the Definablity axiom, the set $\Lambda_X = \{\lambda \in \Lambda \mid \mathrm{rk}\,(\lambda \cap X) = 1\}$ is definable too.

Lemma 3.6. *Let $\lambda_1, \ldots, \lambda_n$ be n lines. Then $\lambda_1 \cup \cdots \cup \lambda_n$ is a definable set of Morley rank 1 and Morley degree n.*

Proof. For each i, the set $A_i = \lambda_i \cap (\bigcup_{j \neq i} \lambda_j)$ has at most $n-1$ elements by Lemma 3.3, and $\lambda_1 \cup \cdots \cup \lambda_n$ is the disjoint union of $\lambda_1 \setminus A_1, \ldots, \lambda_n \setminus A_n, \bigcup_{i=1}^n A_i$. Since each line λ_i has Morley rank 1 and Morley degree 1 (Fact 2.3(2)), the result follows. □

Lemma 3.7. *If Λ_0 is a definable subset of Λ, then $\bigcup \Lambda_0$ is a definable subset of G. Moreover, if Λ_0 is infinite, then $\bigcup \Lambda_0$ has Morley rank at least 2.*

Proof. Since Λ_0 is a definable subset of the uniformly definable family Λ, the set $\bigcup \Lambda_0 = \{x \in G \mid \exists \lambda \in \Lambda_0,\ x \in \lambda\}$ is definable. Moreover, if Λ_0 is infinite, then $\bigcup \Lambda_0$ has Morley rank at least 2 by Lemma 3.6. □

Corollary 3.8. *The subset $\bigcup \Lambda_X$ of G is definable for each definable subset X of G.*

4. Planes

Our original aim, before we reached a contradiction, was to find a definable structure of projective space on our bad group G. In this section, we introduce a notion of planes, and we show that G has such a plane (Theorem 4.14). We fix a definable subset X of G, of Morley rank 2.

Definition 4.1. The definable subset X of G is said to be a *plane* if it satisfies $\widetilde{X} = X$ where

$$\widetilde{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g, X)) = 1\}.$$

Lemma 4.2. *The set $\widetilde{X}$ is a definable subset of $\bigcup \Lambda_X$.*

Proof. If $g \in G$ does not belong to $\bigcup \Lambda_X$, then $l(g, x) \cap X$ is finite for each $x \in X$, and since X has Morley rank 2, the set $\mathscr{L}(g, X)$ has Morley rank 2, so $g \notin \widetilde{X}$. Thus $\widetilde{X}$ is contained in $\bigcup \Lambda_X$.

We show that $\widetilde{X}$ is definable. We consider the set

$$A = \{(g, \lambda) \in G \times \Lambda \mid g \in \lambda,\ \exists x \in X \setminus \{g\},\ x \in \lambda\}$$

and the map $f : A \to G$ defined by $f(g, \lambda) = g$. We note that, since Λ is a uniformly definable family, A is definable, and f is definable too. Moreover, the preimage by f of each $g \in G$ is $f^{-1}(g) = \{g\} \times \mathscr{L}(g, X)$, and we have $\mathrm{rk}\,(f^{-1}(g)) = \mathrm{rk}\,\mathscr{L}(g, X)$. Consequently, we obtain $\widetilde{X} = \{g \in G \mid \mathrm{rk}\,(f^{-1}(g)) = 1\}$, and $\widetilde{X}$ is definable. □

Lemma 4.3. *The Morley ranks of* Λ_X *and of* $\bigcup \Lambda_X$ *are at most* 2. *Moreover,* Λ_X *is infinite if and only if* $\mathrm{rk} \bigcup \Lambda_X = 2$.

Proof. We consider the surjective definable map

$$l_0 : (X \times X) \cap l^{-1}(\Lambda_X) \to \Lambda_X$$

defined by $l_0(x, y) = l(x, y)$. For each $\lambda \in \Lambda_X$, we have $l_0^{-1}(\lambda) = \{(x, y) \in (\lambda \cap X) \times (\lambda \cap X) \mid x \neq y\}$, and since $\mathrm{rk}\,\lambda = 1$, we obtain $\mathrm{rk}\,(\lambda \cap X) = 1$ and $\mathrm{rk}\, l_0^{-1}(\lambda) = 2$. But we have

$$\mathrm{rk}\,((X \times X) \cap l^{-1}(\Lambda_X)) \leq \mathrm{rk}\,(X \times X) = 2\mathrm{rk}\, X = 4,$$

hence $\mathrm{rk}\,\Lambda_X$ is at most $4 - 2 = 2$.

We show that $\mathrm{rk} \bigcup \Lambda_X \leq 2$. We consider the definable set

$$A = \{(x, \lambda) \in G \times \Lambda_X \mid x \in \lambda \setminus X\}$$

and the definable map $l_1 : A \to \Lambda_X$ defined by $l_1(x, \lambda) = \lambda$. For each $\lambda \in \Lambda_X$, we have $\mathrm{rk}\,\lambda = 1 = \mathrm{rk}\,(\lambda \cap X)$, so, since each line has Morley degree 1, the preimage $l_1^{-1}(\lambda)$ is finite. Consequently, we obtain $\mathrm{rk}\, A \leq \mathrm{rk}\,\Lambda_X \leq 2$. But the definable map $l_2 : A \to (\bigcup \Lambda_X) \setminus X$, defined by $l_2(x, \lambda) = x$, is surjective, hence the Morley rank of $(\bigcup \Lambda_X) \setminus X$ is at most $\mathrm{rk}\, A \leq 2$. Since X has Morley rank 2, we obtain $\mathrm{rk} \bigcup \Lambda_X \leq 2$.

Now it follows from Lemmas 3.6 and 3.7 that Λ_X is infinite if and only if $\mathrm{rk} \bigcup \Lambda_X = 2$. □

Proposition 4.4. *For each* $g \in \widetilde{X}$, *we have* $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1$.

Moreover, if X *has Morley degree* 1, *then* $\widetilde{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1\}$ *and* $G \setminus \widetilde{X} = \{g \in G \mid \mathscr{L}(g) \cap \Lambda_X \text{ is finite}\}$.

Proof. First we note that $\mathscr{L}(g) \cap \Lambda_X = \mathscr{L}(g, X) \cap \Lambda_X$ for any $g \in G$. For each $g \in G$, we consider the definable map $l_g : X \setminus \{g\} \to \mathscr{L}(g, X)$ defined by $l_g(x) = l(g, x)$. In particular, the preimage $l_g^{-1}(\lambda)$ of each $\lambda \in \mathscr{L}(g, X)$ is $(\lambda \cap X) \setminus \{g\}$.

We show that $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) \leq 1$ for each $g \in G$. We may assume that Λ_X is infinite. Then, by Lemma 4.3, the set $\bigcup \Lambda_X$ has Morley rank 2. Let $g \in G$ and $u_g : \bigcup(\mathscr{L}(g) \cap \Lambda_X) \setminus \{g\} \to \mathscr{L}(g) \cap \Lambda_X$ be the map defined by $u_g(x) = l(g, x)$. Since each line has Morley rank 1, the preimage of each element of $\mathscr{L}(g) \cap \Lambda_X$ has Morley rank 1. Consequently, we have

$$\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = \mathrm{rk} \bigcup (\mathscr{L}(g) \cap \Lambda_X) - 1 \leq \mathrm{rk} \bigcup \Lambda_X - 1 = 1.$$

Let $g \in \widetilde{X}$. We show that $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1$. For each $\lambda \in \mathscr{L}(g, X) \setminus \Lambda_X$, the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is finite, and since $g \in \widetilde{X}$, we have $\mathrm{rk}\,\mathscr{L}(g, X) = 1$. Consequently, $l_g^{-1}(\mathscr{L}(g, X) \setminus \Lambda_X)$ has Morley rank at most 1, and $l_g^{-1}(\mathscr{L}(g, X) \cap \Lambda_X)$ has Morley rank $\mathrm{rk}\, X = 2$. But the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is infinite of Morley rank 1 for each $\lambda \in \mathscr{L}(g, X) \cap \Lambda_X$. Hence we obtain $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 2 - 1 = 1$.

Now we assume that X has Morley degree 1. Let $g \in G$ such that

$$\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 1.$$

We show that $g \in \widetilde{X}$. Since the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is infinite of Morley rank 1 for each $\lambda \in \mathscr{L}(g, X) \cap \Lambda_X$, the set $l_g^{-1}(\mathscr{L}(g, X) \cap \Lambda_X)$ has Morley rank

$$1 + \mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 2 = \mathrm{rk}\, X.$$

Then, since X has Morley degree 1, the preimage of $\mathscr{L}(g,X)\setminus\Lambda_X$ has Morley rank at most 1. Moreover, for each $\lambda\in\mathscr{L}(g,X)\setminus\Lambda_X$, the preimage $l_g^{-1}(\lambda)=(\lambda\cap X)\setminus\{g\}$ is finite and nonempty, so we obtain

$$\operatorname{rk}(\mathscr{L}(g,X)\setminus\Lambda_X)=\operatorname{rk} l_g^{-1}(\mathscr{L}(g,X)\setminus\Lambda_X)\leq 1.$$

This shows that $\operatorname{rk}\mathscr{L}(g,X)=1$ and $g\in\widetilde{X}$.

Furthermore, since $\operatorname{rk}(\mathscr{L}(g,X)\cap\Lambda_X)\leq 1$ for each $g\in G$, we obtain $G\setminus\widetilde{X}=\{g\in G\mid \mathscr{L}(g)\cap\Lambda_X \text{ is finite}\}$, as desired. □

Corollary 4.5. *We have* $\operatorname{rk}(\widetilde{X}\setminus X)\leq 1$.

Proof. We remember that $\widetilde{X}$ is definable by Lemma 4.2, so the sets $Y=\widetilde{X}\setminus X$ and $A=\{(y,\lambda)\in Y\times\Lambda_X\mid y\in\lambda\}$ are definable too. Let $l_Y:A\to Y$ and $l_D:A\to\Lambda_X$ be the definable maps defined by $l_Y(y,\lambda)=y$ and $l_D(y,\lambda)=\lambda$, respectively. On the one hand, for each $\lambda\in\Lambda_X$, the set $\lambda\cap X$ is infinite, and since λ has Morley rank 1 and Morley degree 1 (Fact 2.3(2)), the set $\lambda\cap Y$ is finite and $l_D^{-1}(\lambda)$ has Morley rank at most 0. This implies $\operatorname{rk}A\leq\operatorname{rk}\Lambda_X\leq 2$ (Lemma 4.3). On the other hand, for each $y\in Y$, we have $\operatorname{rk}(\mathscr{L}(y)\cap\Lambda_X)=1$ by Proposition 4.4, so $l_Y^{-1}(y)$ has Morley rank 1, and we obtain $\operatorname{rk}A=1+\operatorname{rk}Y$. Consequently, the Morley rank of Y is at most 1. □

Lemma 4.6. *For each $g\in G$, the set $\mathscr{L}(g,X)$ is infinite.*

Proof. Indeed, $\bigcup\mathscr{L}(g,X)$ is definable (Lemma 3.7) and contains X. Since $\operatorname{rk}X=2$, we obtain $\operatorname{rk}(\bigcup\mathscr{L}(g,X))\geq 2$, and $\mathscr{L}(g,X)$ is infinite (Lemma 3.6). □

Corollary 4.7. *If the Morley degree of X is not* 1*, then* $\operatorname{rk}\widetilde{X}<2$. *In particular, any plane has Morley degree* 1.

Proof. Let n be the Morley degree of X, and let $X_1,\dots,X_n$ be n definable subsets of X of Morley rank 2 and Morley degree 1 such that X is the disjoint union of $X_1,\dots,X_n$. For each $g\in\widetilde{X}$, we have $\operatorname{rk}\mathscr{L}(g,X)=1$, so we obtain $\operatorname{rk}\mathscr{L}(g,X_i)\leq 1$ for each i, and $g\in\widetilde{X_i}$ for each i by Lemma 4.6. Thus $\widetilde{X}$ is contained in $\widetilde{X_1}\cap\widetilde{X_2}$. Since $X_1\cap X_2=\emptyset$, the set $\widetilde{X}$ is contained in $(X_1\cap Y_2)\cup(Y_1\cap X_2)\cup(Y_1\cap Y_2)$, where $Y_1=\widetilde{X_1}\setminus X_1$ and $Y_2=\widetilde{X_2}\setminus X_2$. Since Y_1 and Y_2 have Morley rank at most 1 by Corollary 4.5, we obtain $\operatorname{rk}\widetilde{X}<2$. □

Lemma 4.8. *We assume that X has Morley degree* 1 *and that Y is another definable subset of G of Morley rank* 2 *and Morley degree* 1*. If $X\cap Y$ has Morley rank* 2*, then $\widetilde{X}=\widetilde{Y}$.*

Proof. Let $g\in G$. If g belongs to $\widetilde{X\cap Y}$, then we have $\operatorname{rk}\mathscr{L}(g,X\cap Y)=1$. Since X has Morley degree 1 and $X\cap Y$ has Morley rank 2, the set $X\setminus Y$ has Morley rank at most 1, and the set $\mathscr{L}(g,X\setminus Y)$ has Morley rank at most 1. Thus $\mathscr{L}(g,X)$ has Morley rank 1, and g belongs to $\widetilde{X}$.

Conversely, if $g\in\widetilde{X}$, then $\mathscr{L}(g,X)$ has Morley rank 1, so $\mathscr{L}(g,X\cap Y)\subseteq\mathscr{L}(g,X)$ has Morley rank at most 1. Then Lemma 4.6 gives $g\in\widetilde{X\cap Y}$. This shows that $\widetilde{X\cap Y}=\widetilde{X}$. In the same way, we obtain $\widetilde{X\cap Y}=\widetilde{Y}$, so $\widetilde{X}=\widetilde{Y}$. □

For each $a\in G$, let $\mathscr{L}(a)=\mathscr{L}(a,G)$ be the (definable) set of lines containing a. Moreover, we note that $\mathscr{L}(1)=\mathscr{B}$.

Lemma 4.9. *Let* Λ_0 *be a definable subset of* Λ. *If* $\mathrm{rk} \bigcup \Lambda_0 = 2$, *then we have* $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) \leq 1$ *for each* $g \in G$.

Moreover, if further $\mathrm{rk}\,\Lambda_0 = 2$, *then the set* $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ *has Morley rank* 2.

Proof. We show that $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) \leq 1$ for each $g \in G$. Let $g \in G$ and $l_g : \bigcup(\mathscr{L}(g) \cap \Lambda_0) \setminus \{g\} \to \mathscr{L}(g) \cap \Lambda_0$ be the map defined by $l_g(x) = l(g, x)$. Since each line has Morley rank 1, the preimage of each element of $\mathscr{L}(g) \cap \Lambda_0$ has Morley rank 1. Consequently, we have

$$\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = \mathrm{rk} \bigcup(\mathscr{L}(g) \cap \Lambda_0) - 1 \leq \mathrm{rk} \bigcup \Lambda_0 - 1 = 1,$$

as desired.

We suppose further that $\mathrm{rk}\,\Lambda_0 = 2$, and we show that $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ has Morley rank 2. Let $U = \bigcup \Lambda_0$, $A = \{(u, \lambda) \in U \times \Lambda_0 \mid u \in \lambda\}$, and let $f : A \to \Lambda_0$ be the map defined by $f(u, \lambda) = \lambda$. Then A and f are definable, and the preimage $f^{-1}(\lambda)$ of each $\lambda \in \Lambda_0$ has Morley rank $\mathrm{rk}\,\lambda = 1$, so $\mathrm{rk}\,A = 1 + \mathrm{rk}\,\Lambda_0 = 3$. Now let $h : A \to U$ be the map defined by $h(u, \lambda) = u$. It is a definable map, and the preimage $h^{-1}(u)$ of each $u \in U$ has Morley rank of either 0 or 1 by the previous paragraph.

But the preimage of $U_0 = \{u \in U \mid \mathrm{rk}\,h^{-1}(u) = 0\}$ has Morley rank

$$\mathrm{rk}\,h^{-1}(U_0) = \mathrm{rk}\,U_0 \leq \mathrm{rk}\,U = 2 < \mathrm{rk}\,A,$$

so the preimage of $U_1 = \{u \in U \mid \mathrm{rk}\,h^{-1}(u) = 1\}$ has Morley rank 3. Hence we obtain $\mathrm{rk}\,U_1 = 3 - 1 = 2$. Moreover, we note that

$$U_1 = \{u \in U \mid \mathrm{rk}\,(\mathscr{L}(u) \cap \Lambda_0) = 1\} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\},$$

so $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ has Morley rank 2. □

Proposition 4.10. *Let* X *be a definable subset of* G *of Morley rank* 2 *and Morley degree* 1 . *Then* $\mathrm{rk}\,\widetilde{X} = 2$ *if and only if* Λ_X *has Morley rank* 2.

In this case, Λ_X *and* $\widetilde{X}$ *have Morley degree* 1, *and* $\widetilde{X}$ *contains a generic definable subset of* X.

Proof. We consider the definable set $A = \{(x, \lambda) \in \widetilde{X} \times \Lambda_X \mid x \in \lambda\}$ and the definable maps $l_1 : A \to \widetilde{X}$ and $l_2 : A \to \Lambda_X$ defined by $l_1(x, \lambda) = x$ and $l_2(x, \lambda) = \lambda$, respectively. By Proposition 4.4, the preimage $l_1^{-1}(g)$ of each element g of $\widetilde{X}$ has Morley rank 1, so $\mathrm{rk}\,A = 1 + \mathrm{rk}\,\widetilde{X}$. Moreover, the preimage $l_2^{-1}(\lambda)$ of each $\lambda \in \Lambda_X$ has Morley rank at most 1, so $\mathrm{rk}\,A \leq 1 + \mathrm{rk}\,\Lambda_X$. Then we obtain $\mathrm{rk}\,\widetilde{X} \leq \mathrm{rk}\,\Lambda_X$. In particular, it follows from Lemma 4.3 that if $\mathrm{rk}\,\widetilde{X} = 2$, then $\mathrm{rk}\,\Lambda_X = 2$. Hence we may assume that $\mathrm{rk}\,\Lambda_X = 2$.

At this stage, Lemma 4.3 gives $\mathrm{rk} \bigcup \Lambda_X = 2$, and by Lemma 4.9 and Proposition 4.4, we obtain $\mathrm{rk}\,\widetilde{X} = 2$. Moreover, it follows from Corollary 4.5 that $\widetilde{X}$ has Morley degree 1 and that $X \cap \widetilde{X}$ is a generic definable subset of X contained in $\widetilde{X}$.

We show that the Morley degree of Λ_X is 1. Let $l_0 : \{(x, y) \in X \times X \mid x \neq y\} \to \Lambda$ be the definable map defined by $l_0(x, y) = l(x, y)$. Since the Morley degree of X is 1, that of $\{(x, y) \in X \times X \mid x \neq y\}$ is 1 too. For each $\lambda \in \Lambda_X$, we have $\mathrm{rk}\,l_0^{-1}(\lambda) = \mathrm{rk}\,((\lambda \cap X) \times (\lambda \cap X)) = 2$. Since $\mathrm{rk}\,\Lambda_X = 2$, we obtain

$$\mathrm{rk}\,l_0^{-1}(\Lambda_X) = 2 + \mathrm{rk}\,\Lambda_X = 4 = \mathrm{rk}\,\{(x, y) \in X \times X \mid x \neq y\},$$

and since the Morley degree of $\{(x,y) \in X \times X \mid x \neq y\}$ is 1, the Morley degree of $l_0^{-1}(\Lambda_X)$ is 1 too. Now the Morley degree of Λ_X is 1 by Fact 2.2. □

Lemma 4.11. *Let g be a nontrivial element such that $g = [u,v]$ for $(u,v) \in G \times G$. Then we have $\{x \in G \mid [x,v] = g\} = C_G(v)u$ and $\{y \in G \mid [u,y] = g\} = C_G(u)v$. In particular, they are two lines and have Morley rank 1 and Morley degree 1.*

Proof. The equalities are obvious. Moreover, by Fact 2.3 the sets $C_G(v)u$ and $C_G(u)v$ are two lines, and they have Morley rank 1 and Morley degree 1. □

Lemma 4.12. *For each $a \in G$, the set $a^G \cap B$ has exactly one element.*

Proof. We may assume $a \neq 1$. By Fact 2.3(5), there is $g \in G$ such that a^g belongs to B. If $a^h \in B$ for $h \in G$, then a is a nontrivial element of $B^{g^{-1}} \cap B^{h^{-1}}$. By Fact 2.3(4), we obtain $B^{g^{-1}} = B^{h^{-1}}$, and $h^{-1}g$ belongs to $N_G(B) = B$. But B is abelian (Fact 2.3(2)), so $h^{-1}g$ centralizes a^h, and $a^h = (a^h)^{h^{-1}g} = a^g$. Hence $a^G \cap B = \{a^g\}$. □

The following result isolates a step of the proof of Theorem 4.14. Its proof and that of Theorem 4.14 were originally a lot more complicated, and Bruno Poizat provided a simplification.

For each $g \in G$, we consider the following definable subset of G:

$$X(g) = \{x \in G \mid \exists y \in G,\ [x,y] = g\}.$$

Proposition 4.13. *For each nontrivial element g of G, the set $X(g)$ has Morley rank at most 2.*

Proof. We assume toward a contradiction that $X(g)$ has Morley rank 3. Then the Morley rank of $X(g^z)$ is 3 for each $z \in G$. We recall that, by Fact 2.3, the conjugacy class g^G of g has Morley rank $\operatorname{rk} g^G = \operatorname{rk} G - \operatorname{rk} C_G(g) = 2$.

We consider $V = \{(x,y) \in G \times G \mid [x,y] \in g^G\}$ and the definable surjective map $f : V \to g^G$ defined by $f(x,y) = [x,y]$. For each $z \in G$, we have

$$f^{-1}(g^z) = \{(x,y) \in G \times G \mid [x,y] = g^z\},$$

and by Lemma 4.11 this set has Morley rank $\operatorname{rk} f^{-1}(g^z) = \operatorname{rk} X(g^z) + 1 = 3 + 1 = 4$, so $\operatorname{rk} V = 4 + \operatorname{rk} g^G = 6$. Since $G \times G$ is a connected group of Morley rank 6, the set V is a definable generic subset of $G \times G$, and there is $(x,y) \in V$ such that (y,x) belongs to V. Thus $[x,y] \in g^G$ and its inverse $[y,x] \in g^G$ are conjugate, and they are equal by Lemma 4.12, contradicting that G has no involution (Fact 2.3(6)). □

Theorem 4.14. *There is a plane in G.*

Proof. It is sufficient to show that there is a definable subset X of G satisfying the following properties:

(1) its Morley rank is 2 and its Morley degree is 1;
(2) Λ_X has Morley rank 2.

Indeed, by Proposition 4.10, for such a subset X, the set $\widetilde{X}$ has Morley rank 2 and Morley degree 1, and it contains a generic definable subset of X. At this stage, Lemma 4.8 shows that $Y = \widetilde{X}$ is a plane.

We fix a nontrivial element g such that $g = [u, v]$ for $(u, v) \in G \times G$.

1. For each $x \in X(g)$, there are infinitely many lines containing x and contained in $X(g)$.

Since x belongs to $X(g)$, there is $y \in G$ such that $[x, y] = g$. We note that, since g is nontrivial, x and y are nontrivial, and we have $C_G(x) \neq C_G(y)$. In particular $C_G(x)y$ is a line, and it does not contain 1. Thus, for each $c \in C_G(x)$, the set $l_c = C_G(cy)x$ is a line, and by Lemma 4.11 we have $[r, cy] = [x, cy] = [x, y] = g$ for each $r \in l_c$. So l_c is a line containing x and contained in $X(g)$.

If $l_c = l_d$ for two elements c and d of $C_G(x)$, then we have $C_G(cy) = C_G(dy)$, and $C_G(cy)$ is a line containing cy and dy. But $C_G(x)y$ is another line containing cy and dy, and we have $C_G(x)y \neq C_G(cy)$ because $C_G(x)y$ does not contain 1. Hence Lemma 3.3 gives $c = d$, and $\{l_c \in \Lambda \mid c \in C_G(x)\}$ is an infinite family of lines containing x and contained in $X(g)$.

2. $\operatorname{rk} X(g) = 2$.

By part 1 the set $X(g)$ contains infinitely many lines, so it has Morley rank at least 2 (Lemma 3.6) and by Proposition 4.13 it has Morley rank 2.

3. $\operatorname{rk} \Lambda_{X(g)} = 2$.

By Lemma 4.3 the set $\Lambda_{X(g)}$ has Morley rank at most 2. Since $X(g)$ is infinite by part 2, for each positive integer n we can find n distinct elements $x_1, \dots, x_n$ in $X(g)$. By part 1 the set $\Lambda_i = \{\lambda \in \Lambda_{X(g)} \mid x_i \in \lambda\}$ is infinite for each i. We may assume that its Morley rank is 1 for each i. Then, since there are finitely many lines containing two distinct elements among $x_1, \dots, x_n$ (Lemma 3.3), the union $\bigcup_{i=1}^n \Lambda_i$ has Morley rank 1 and Morley degree at least n. This implies that $\Lambda_{X(g)}$ does not have Morley rank 1, so $\operatorname{rk} \Lambda_{X(g)} = 2$.

4. Conclusion.

By part 2 the set $X(g)$ has Morley rank 2. Let d be its Morley degree. Then $X(g)$ is the disjoint union of definable subsets $X_1, \dots, X_d$ of Morley rank 2 and Morley degree 1.

For each element λ of $\Lambda_{X(g)}$, since $\lambda \cap X(g)$ is infinite and since λ has Morley rank 1 and Morley degree 1, there is a unique $i \in \{1, \dots, d\}$ such that $\lambda \cap X_i$ is infinite, that is $\lambda \in \Lambda_{X_i}$. Thus, each $\lambda \in \Lambda_{X(g)}$ belongs to a unique definable set Λ_{X_i} for $i \in \{1, \dots, d\}$. Hence $\Lambda_{X(g)}$ is the disjoint union of $\Lambda_{X_1}, \dots, \Lambda_{X_d}$, and there exists $i \in \{1, \dots, d\}$ such that $\operatorname{rk} \Lambda_{X_i} = 2$. Now the set X_i satisfies the conditions (1) and (2) of the beginning of our proof, so $\widetilde{X_i}$ is a plane. □

5. A projective space?

In this section we analyze planes. We remember that by Theorem 4.14 the group G has a plane, and that by Corollary 4.7 any plane has Morley degree 1. The initial goal of this section was to show that if X and Y are two distinct planes, then $\Lambda_X \cap \Lambda_Y$ has a unique element. However, along the way, we obtain our final contradiction.

Definition 5.1. For each line λ we consider the following subset of Λ:

$$\mathscr{L}(\lambda) = \{m \in \Lambda \mid \lambda \cap m \text{ is not empty}\}.$$

Lemma 5.2. *For any line λ the set $\mathscr{L}(\lambda)$ is definable and it has Morley rank* 3 *and Morley degree* 1.

Proof. We consider the definable map $f : \lambda \times (G \setminus \lambda) \to \mathscr{L}(\lambda) \setminus \{\lambda\}$ defined by $f(x, g) = l(x, g)$. By Lemma 3.3 for each $m \in \mathscr{L}(\lambda) \setminus \{\lambda\}$, there is a unique element x in $\lambda \cap m$. Moreover, for any $g \in G \setminus \lambda$, we have $f(x, g) = m$ if and only if $g \in m \setminus \{x\}$. Consequently, we have $\mathrm{rk}\, f^{-1}(m) = \mathrm{rk}\, m = 1$, and

$$\mathrm{rk}\, \mathscr{L}(\lambda) = \mathrm{rk}\, (\lambda \times (G \setminus \lambda)) - 1 = 3.$$

Furthermore, since λ and G have Morley degree 1, the Morley degree of $\lambda \times G$ and $\lambda \times (G \setminus \lambda)$ is 1, and the Morley degree of $\mathscr{L}(\lambda) \setminus \{\lambda\}$ and $\mathscr{L}(\lambda)$ is 1 too (Fact 2.2). □

Lemma 5.3. *Let X be a plane, and let $\lambda \in \Lambda_X$. Then $\mathscr{L}(\lambda) \cap \Lambda_X$ has Morley rank* 2.

Proof. Since λ belongs to Λ_X, the set $\lambda \cap X$ is infinite, and since λ is a line, we have $\mathrm{rk}\, (\lambda \cap X) = 1$. We consider the definable set

$$\mathscr{A} = \{(x, m) \in (\lambda \cap X) \times \Lambda_X \mid m \neq \lambda,\ x \in m\},$$

and the definable maps $p : \mathscr{A} \to \lambda \cap X$ and $q : \mathscr{A} \to \Lambda_X$ are defined by $p(x, m) = x$ and $q(x, m) = m$, respectively. By Proposition 4.4, the set $p^{-1}(x)$ has Morley rank 1 for each $x \in \lambda \cap X$, so $\mathrm{rk}\, \mathscr{A} = 1 + \mathrm{rk}\, (\lambda \cap X) = 2$.

Moreover, each $m \in \Lambda_X \setminus \{\lambda\}$ contains at most one element of λ (Lemma 3.3), so q is an injective map and its image has Morley rank $\mathrm{rk}\, \mathscr{A} = 2$. But the image of q is contained in $(\mathscr{L}(\lambda) \cap \Lambda_X) \setminus \{\lambda\}$, and we have $\mathrm{rk}\, \Lambda_X \leq 2$ (Lemma 4.3), hence $\mathscr{L}(\lambda) \cap \Lambda_X$ has Morley rank 2. □

Lemma 5.4. *Let λ_1 and λ_2 be two distinct lines. Then $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank* 2 *and Morley degree* 1.

Proof. Let $A = \{(x, y) \in \lambda_1 \times \lambda_2 \mid x \notin \lambda_1,\ y \notin \lambda_2\}$, and let $f : A \to (\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)) \setminus \{\lambda_1, \lambda_2\}$ be the map defined by $f(x, y) = l(x, y)$. This map is definable and bijective by Lemma 3.3. Since λ_1 and λ_2 are two lines, the sets $\lambda_1 \times \lambda_2$ and A have Morley rank 2 and Morley degree 1, and since f is a definable bijection, $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank 2 and Morley degree 1. □

Proposition 5.5. *If X and Y are two distinct planes, then $\Lambda_X \cap \Lambda_Y$ has at most one element.*

Proof. Suppose toward a contradiction that λ_1 and λ_2 are two distinct elements of $\Lambda_X \cap \Lambda_Y$. By Lemma 5.3, the sets $\mathscr{L}(\lambda_1) \cap \Lambda_X$ and $\mathscr{L}(\lambda_2) \cap \Lambda_X$ have Morley rank 2. But Λ_X has Morley rank 2 and Morley degree 1 by Proposition 4.10, hence $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2) \cap \Lambda_X$ has Morley rank 2. In the same way, $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2) \cap \Lambda_Y$ has Morley rank 2. Thus, since $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank 2 and Morley degree 1 (Lemma 5.4), the set $\Lambda_X \cap \Lambda_Y$ has Morley rank 2.

Since $\Lambda_X \cap \Lambda_Y$ is infinite, the set $U = \bigcup(\Lambda_X \cap \Lambda_Y)$ has Morley rank at least 2 by Lemma 3.7, and since U is contained in $\bigcup \Lambda_X$, its Morley rank is exactly 2 (Lemma 4.3). Now the set $Z = \{g \in G \mid \mathrm{rk}\, (\mathscr{L}(g) \cap \Lambda_X \cap \Lambda_Y) = 1\}$ has Morley rank 2 by Lemma 4.9. But Proposition 4.4 says that Z is contained in $X \cap Y$, hence $X \cap Y$ has Morley rank 2 and Lemma 4.8 gives $X = Y$, a contradiction. □

From now on, we try to show that the set $\Lambda_X \cap \Lambda_Y$ has exactly one element. However, the final contradiction will appear earlier.

Corollary 5.6. *Let X be a plane, and let $(a, b) \in G \times G$. Then the following assertions are equivalent:*

- $aXb = X$.
- $a\Lambda_X b = \Lambda_X$.
- $aXb \cap X$ *has Morley rank* 2.

Proof. We note that aXb is a plane and that $a\Lambda_X b = \Lambda_{aXb}$. If $aXb \cap X$ has Morley rank 2, then $aXb = X$ by Lemma 4.8, and if $aXb = X$, then we have $a\Lambda_X b = \Lambda_{aXb} = \Lambda_X$. Moreover, if $a\Lambda_X b = \Lambda_X$, then we have $\Lambda_{aXb} = \Lambda_X$ and $aXb = X$ by Proposition 5.5, so $aXb \cap X = X$ has Morley rank 2. □

By Fact 2.4, if A is a Borel subgroup distinct from B, then $\mathrm{rk}\,(ABA) = 3$. The following result is slightly more general, and its proof is different.

We recall that if a group H of finite Morley rank acts definably on a set E, then the *stabilizer* of any definable subset F of E is defined to be

$$\mathrm{Stab}F = \{h \in H \mid \mathrm{rk}\,((h \cdot F)\Delta F) < \mathrm{rk}\,(F)\},$$

where Δ stands for the symmetric difference. It is a definable subgroup of H by [5, Lemma 5.11].

Lemma 5.7. *Let A and C be two Borel subgroups distinct from B. Then* $\mathrm{rk}\,(ABC) = 3$.

Proof. We consider the action of G on itself by left multiplication. Then we have $b \cdot BC = BC$ for each $b \in B$, so B is contained in $\mathrm{Stab}(BC)$.

We assume toward a contradiction that C is contained in $\mathrm{Stab}(BC)$. Since BC has Morley rank 2 and Morley degree 1 (Fact 2.4), we have $\mathrm{rk}\,(cBC \setminus BC) \leq 1$ for each $c \in C$, and since $\mathrm{rk}\,C = 1$, we obtain $\mathrm{rk}\,(CBC \setminus BC) \leq 2$ and $\mathrm{rk}\,(CBC) = 2$, contradicting Fact 2.4. Consequently, C is not contained in $\mathrm{Stab}(BC)$, and since $\mathrm{Stab}(BC)$ contains B, Fact 2.3 implies that $\mathrm{Stab}(BC) = B$.

We assume toward a contradiction that $\mathrm{rk}\,(ABC) \neq 3$. Since $\mathrm{rk}\,(BC) = 2$, we have $\mathrm{rk}\,(ABC) = 2$ and ABC is a disjoint union of finitely many definable subsets $E_1, \ldots, E_k$ of Morley rank 2 and Morley degree 1. For each $a \in A$, the set aBC has Morley rank $\mathrm{rk}\,(BC) = 2$ and Morley degree 1, so there exists a unique $i \in \{1, \ldots, k\}$ such that $\mathrm{rk}\,(aBC \cap E_i) = 2$. Since A is infinite, there are $i \in \{1, \ldots, k\}$ and two distinct elements a and a' of A such that $\mathrm{rk}\,(aBC \cap E_i) = \mathrm{rk}\,(a'BC \cap E_i) = 2$. Since E_i has Morley degree 1, the Morley rank of $aBC \cap a'BC$ is 2, and we obtain $\mathrm{rk}\,(a'^{-1}aBC \cap BC) = 2$. But BC has Morley degree 1, hence $a'^{-1}a$ belongs to $\mathrm{Stab}(BC) = B$. Thus $a'^{-1}a$ belongs to $A \cap B = \{1\}$ (Fact 2.3(4)), contradicting that a and a' are distinct. So we have $\mathrm{rk}\,(ABC) = 3$, as desired. □

Corollary 5.8. *Let A and C be two distinct Borel subgroups. Then* $\mathrm{rk}\,(BA \cap BC) = 1$.

Proof. We may assume $A \neq B$ and $C \neq B$. By Fact 2.4, we have

$$1 = \mathrm{rk}\,B \leq \mathrm{rk}\,(BA \cap BC) \leq \mathrm{rk}\,(BA) = 2.$$

We assume toward a contradiction that $\text{rk}\,(BA \cap BC) = 2$. Since BC has Morley rank 2 and Morley degree 1 (Fact 2.4), the set $E = BC \setminus BA$ has Morley rank at most 1. Consequently, EA has Morley rank at most $\text{rk}\,E + \text{rk}\,A = 2$, and since $(BA \cap BC)A \subseteq BA$ has Morley rank 2, we obtain $\text{rk}\,(BCA) = \text{rk}\,(EA \cup (BA \cap BC)A) = 2$, contradicting that BCA has Morley rank 3 (Lemma 5.7). □

Lemma 5.9. *For any plane X, we have $BX \neq X$ and $XB \neq X$.*

Proof. We assume toward a contradiction that $BX = X$ for a plane X. Let $x \in X$. Since X is a plane, Proposition 4.4 gives $\text{rk}\,(\mathscr{L}(x, X) \cap \Lambda_X) = 1$, so $\mathscr{L}(x, X) \cap \Lambda_X$ is infinite. But each line containing x has the form $B^u x$ for $u \in G$, hence there exist $u \notin B$ and $v \notin B$ such that $B^u \neq B^v$, and such that $B^u x$ and $B^v x$ belong to $\mathscr{L}(x, X) \cap \Lambda_X$. In particular, there is a cofinite subset S of B such that $S^u x$ and $S^v x$ are contained in X.

Now, since $BX = X$, the sets $BS^u x$ and $BS^v x$ are contained in X. By Fact 2.4 the set BB^u, and so $Bu^{-1}B$, has Morley rank 2, and since $Bu^{-1}(B \setminus S)$ is a finite union of lines, the set $Bu^{-1}(B \setminus S)$ has Morley rank 1 (Lemma 3.6), and $Bu^{-1}S$ has Morley rank 2. Thus, the sets $BS^u x = Bu^{-1}Sux$ and $BS^v x = Bv^{-1}Svx$ are subsets of X of Morley rank 2, and since the Morley degree of X is 1, the set $BS^u x \cap BS^v x$ has Morley rank 2. This implies that $\text{rk}\,(BB^u \cap BB^v) = 2$, contradicting Corollary 5.8. Now we have $BX \neq X$ and in the same way, we show that $XB \neq X$. □

Corollary 5.10. *For any plane X, the stabilizer of X for the action of G on itself by left multiplication is finite.*

Proof. By Corollary 5.6, we have $\text{Stab}X = \{a \in G \mid aX = X\}$. If $\text{Stab}X$ is infinite, then it contains a Borel subgroup, contradicting Lemma 5.9. □

Proposition 5.11. *Let X be a plane. Then for each plane Y, there exist a unique $a \in G$ and a unique $b \in G$ such that $Y = aX = Xb$.*

Proof. We fix $\alpha \in G$, and we consider the following definable subset of Λ:

$$A = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda \mid \alpha \in \lambda_1 \cap \lambda_2,\ \lambda_1 \neq \lambda_2\}.$$

We show that A has Morley rank 4 and Morley degree 1. Let $U = \{(x, y) \in G \times G \mid y \notin l(x, \alpha)\}$. Then U is a generic definable subset of $G \times G$, and it has Morley rank 6 and Morley degree 1. Let $f : U \to A$ be the definable surjective map defined by $f(x, y) = (l(x, \alpha), l(y, \alpha))$. Since each line has Morley rank 1, the preimage of each $(\lambda_1, \lambda_2) \in A$ has Morley rank $\text{rk}\,\lambda_1 + \text{rk}\,\lambda_2 = 2$, and the set A has Morley rank $\text{rk}\,U - 2 = 4$ and Morley degree 1 (Fact 2.2).

For each plane P, we consider the following definable set:

$$A_P = \{(\lambda_1, \lambda_2) \in \Lambda \times \Lambda \mid \alpha \in \lambda_1 \cap \lambda_2,\ \lambda_1 \neq \lambda_2,\ \exists a \in G,\ a^{-1}\lambda_1 \in \Lambda_P,\ a^{-1}\lambda_2 \in \Lambda_P\}.$$

We show that the set A_X is a generic definable subset of A. Indeed, for each $a \in \alpha X^{-1}$, we have $\alpha \in aX$ and $\text{rk}\,(\mathscr{L}(\alpha) \cap \Lambda_{aX}) = 1$ by Proposition 4.4, so the definable set

$$L_{aX} = \{(\lambda_1, \lambda_2) \in \Lambda_{aX} \times \Lambda_{aX} \mid \alpha \in \lambda_1 \cap \lambda_2,\ \lambda_1 \neq \lambda_2\}$$

has Morley rank 2 $\mathrm{rk}(\mathscr{L}(\alpha) \cap \Lambda_{aX}) = 2$. But αX^{-1} has Morley rank $\mathrm{rk}\, X = 2$, and it follows from Proposition 5.5 that $L_{aX} \cap L_{bX} = \emptyset$ for any two elements a and b of αX^{-1} such that $aX \neq bX$. Moreover, for each $a \in \alpha X^{-1}$, there are finitely many elements $b \in \alpha X^{-1}$ such that $aX = bX$ (Corollary 5.10). Hence the set $A_X = \bigcup_{a \in \alpha X^{-1}} L_{aX}$ has Morley rank $\mathrm{rk}\, \alpha X^{-1} + 2 = 4$, and it is a generic definable subset of A.

In the same way, A_Y is a generic definable subset of A, so there exists $(\lambda_1, \lambda_2) \in A_X \cap A_Y$. Thus there exist two elements u and v of G such that two distinct lines λ_1 and λ_2 belong to $\Lambda_{uX} \cap \Lambda_{vY}$, and we obtain $uX = vY$ by Proposition 5.5, so $Y = aX$ for $a = v^{-1}u$. In the same way, there exists $b \in G$ such that $Y = Xb$.

We show the uniqueness of a and b. Let $S = \{g \in G \mid gX = X\}$. It is a finite subgroup of G by Corollary 5.10. For each $\alpha \in G$, the previous paragraph gives $\beta \in G$ such that $\alpha X = X\beta$. Then, for each $s \in S$, we have $s(\alpha X) = s(X\beta) = X\beta = \alpha X$, and we obtain $s^\alpha X = X$ and $s^\alpha \in S$. Thus any element $\alpha \in G$ normalizes the finite subgroup S, and since G is a simple group, S is trivial. This proves the uniqueness of a, and in the same way we obtain the uniqueness of b. □

By the previous result, the set of planes is $\mathscr{P} = \{aX \mid a \in G\}$, and it identifies with G. Thus, the set of planes is uniformly definable and has Morley rank 3.

Lemma 5.12. *There exists $a \in G$ such that $X^a \neq X$.*

Proof. We assume toward a contradiction that $X^a = X$ for each $a \in G$. Then for each $uBv \in \Lambda_X$ and each $a \in G$, we have

$$(uBv)^a \in \Lambda_X^a = \Lambda_{X^a} = \Lambda_X.$$

Since $\mathrm{rk}\, \Lambda_X = 2$ (Proposition 4.10), the line uBv is a Borel subgroup (Lemma 3.4), and by conjugacy of Borel subgroups we obtain $\Lambda_X = \mathscr{B}$. Now we have $\bigcup \Lambda_X = G$, so $\mathrm{rk} \bigcup \Lambda_X = 3$, contradicting Lemma 4.3. □

From now on, we are ready for the final contradiction. Initially, it was more complicated, but Poizat proposed a simplification by introducing the inverted plane.

Proof. First we note that for each plane Y, the set $y^{-1}Y$ is a plane containing 1, and the set Y^{-1} is a plane too. We fix a plane X containing 1. By Proposition 5.11, there is a bijective map $\mu : G \to G$ defined by $xX = X\mu(x)$, and μ is definable since the set $\mathscr{P}$ of planes is uniformly definable. Moreover, for each $(a, b) \in G \times G$, we have $X\mu(ab) = abX = aX\mu(b) = X\mu(a)\mu(b)$, so μ is an automorphism of G.

Since $X = \widetilde{X}$ contains 1, there are infinitely many Borel subgroups in Λ_X. Let B_1 and B_2 be two distinct Borel subgroups belonging to Λ_X. Then B_1 and B_2 belong to $\Lambda_{X^{-1}}$ too, and we have $X = X^{-1}$ by Proposition 5.5. In the same way, since the plane $x^{-1}X$ contains 1 for each $x \in X$, we have $x^{-1}X = (x^{-1}X)^{-1} = X^{-1}x = Xx$ for each $x \in X$. Thus $\mu(x^{-1}) = x$ for each $x \in X$, and since $X = X^{-1}$, we obtain $\mu^2(x) = x$ for each $x \in X$.

But X is a definable subset of G of Morley rank 2, hence G is generated by X, and μ is an involutive automorphism of G. Thus μ is the identity map by Fact 2.5, contradicting Lemma 5.12. □

Remark 5.13. After Lemma 5.12 we were ready for a new step to provide a structure of projective space over G, which was the initial goal of our section. Indeed, in the first version of this paper, we have shown that, if X and Y are two distinct planes, then $\Lambda_X \cap \Lambda_Y$ has a unique element.

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