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LANGLANDS CORRESPONDENCE FOR ISOCRYSTALS AND THE EXISTENCE OF CRYSTALLINE COMPANIONS FOR CURVES

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Contents

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Introduction

The Weil conjectures were finally proven by P. Deligne in the 1970s, culminating in the theory of weights for ℓ-adic cohomology in his celebrated paper [De1]. In that paper, Deligne made the following conjecture on the existence of *compatible systems*:

Conjecture ([De1, 1.2.10]). *Soient X normal connexe de type fini sur $\mathbb{F}_p$, et $\mathscr{F}$ un faisceau lisse irréducible dont le déterminant est défini par un caractère d'ordre fini du groupe fondamental.*

(ii) *Il existe un corps de nombres $E \subset \overline{\mathbb{Q}}_\ell$ tel que le polynôme* $\det(1 - F_x t, \mathscr{F})$ *pour $x \in |X|$, soit à coefficients dans E.*

(v) *Pour E convenable* (*peut-être plus grand qu'en* (ii))*, et chaque place non archimédienne λ première à p, il existe un E_λ-faisceau compatible à $\mathscr{F}$* (*mêmes valeurs propres des Frobenius*)*.*

(vi) *Pour λ divisant p, on espère des petits camarades cristallins.*

Part (vi) is written vaguely because a good theory of p-adic cohomology was not available at the time Deligne conjectured it. R. Crew made this conjecture more precise in [Cr, 4.13] after P. Berthelot's foundational works in p-adic cohomology theory. This conjecture has been one of the driving forces in developing a p-adic cohomology theory over fields of positive characteristic parallel to the ℓ-adic cohomology theory (e.g., introduction of [Ch]).

When X is a curve, all parts of the the conjecture except for (vi) are consequences of the Langlands correspondence, which was proven by V. Drinfeld in the rank 2 case and by L. Lafforgue in the higher rank case. Moreover, Deligne and Drinfeld proved all parts of the conjecture except for (vi) for any smooth scheme X as a consequence of the Langlands correspondence. In this paper, we prove part (vi) of the conjecture when X is a curve. In fact, we prove a stronger result: a correspondence between irreducible overconvergent F-isocrystals with finite determinant on an open dense subscheme of X and cuspidal automorphic representations of the function field of X with finite central character (see Theorem 4.2.2). Finally, in Theorem 4.4.5, we prove the converse of Deligne's conjecture when X is smooth using the techniques of Deligne and Drinfeld in [EK] and [Dr] assuming the Bertini-type conjecture in 4.4.2: for any overconvergent F-isocrystal over a smooth scheme, there exists an ℓ-adic companion for any $\ell \neq p$.

Our strategy of proof is similar to the ℓ-adic case. First, we apply the *product formula for epsilon factors*, which was proven in the p-adic setting by Abe and A. Marmora in [AM]. By using Deligne's *principe de récurrence*, the product formula for epsilon factors reduces one to associating an isocrystal to a cuspidal automorphic representation (cf. [A2]). Finally, we use the moduli spaces of *shtukas* to establish the Langlands correspondence for isocrystals as was done by Drinfeld and Lafforgue in the ℓ-adic case. In order to apply the methods of Drinfeld and Lafforgue, we construct a six functor formalism for suitable p-adic cohomology theory for certain algebraic stacks.

Before explaining our construction, let us review the history of attempts to construct a six functor formalism in the p-adic setting. For a more detailed overview of the history, we refer the reader to [I], [Ke3]. The first p-adic cohomology defined for an arbitrary separated scheme of finite type over a perfect field of characteristic p was proposed by Berthelot during the 1980s (called rigid cohomology). He also defined a theory of coefficients for rigid cohomology, called overconvergent F-isocrystals: these can be seen as a p-adic analogue of vector bundles with an integrable connection. One can see from this analogy that it is not reasonable to expect a six functor formalism in the style of A. Grothendieck in the framework of overconvergent F-isocrystals. In order to remedy this, and in the analogy with the complex situation, Berthelot introduced the theory of arithmetic $\mathscr{D}$-modules. We refer to [Ber4] for a beautiful survey by the founder himself. In the p-adic case, the theory is very complicated since we need to deal with differential operators of infinite order unlike the complex situation. As a result, many of the foundational properties had been left as conjectures. Among these conjectures, the most important one concerns the preservation of finiteness properties of the arithmetic $\mathscr{D}$-modules under various cohomological operations. A big step toward this problem was the introduction of overholonomic modules by D. Caro, which potentially bypasses Berthelot's original strategy to construct the six functor formalism. His work was successful in proving stability for most of the standard cohomological operations, but the finiteness of overconvergent F-isocrystals was still unresolved. A breakthrough was achieved by K. S. Kedlaya in his resolution of Shiho's conjecture, or the proof of the semistable reduction theorem [Ke5]. This extremely powerful theorem enabled us to answer many tough questions in the theory of arithmetic $\mathscr{D}$-modules: a finiteness result by Caro and N. Tsuzuki (cf. [CT]), an analogue of Weil II by Abe and Caro (cf. [AC1]), and many more. Even though we do not explicitly use Kedlaya's theorem in the proof of the Langlands correspondence for isocrystals, the main theorem of this paper can be seen as another application of Kedlaya's result. In previous works ([CT, AC1] etc.), Kedlaya's result was used to develop a theory of arithmetic $\mathscr{D}$-modules for "realizable schemes". We refer to §1.1 for a detailed overview. In particular, quasi-projective schemes are included in this framework. However, to construct the isocrystals corresponding to cuspidal automorphic representations, the category of quasi-projective schemes is too restricted. A large part of this paper is devoted to constructing a theory of arithmetic $\mathscr{D}$-modules for *admissible stacks*. In particular, we end our search, since Monsky and Washnitzer, for a p-adic six functor formalism for separated schemes of finite type over a perfect field.

Our construction of a six functor formalism is more or less formal: making full use of the existence of the formalism in a local situation, we glue. Even though we do not axiomatize, it can be carried out for any cohomology theory over a field admitting a reasonable six functor formalism locally. First, let us explain the construction in the case of schemes. As we have already mentioned, for *realizable schemes* (e.g., quasi-projective schemes) we already have the formalism thanks to works of Caro and others. For a realizable scheme X, we denote by $D^{\mathrm{b}}_{\mathrm{hol}}(X)$ the associated triangulated category with t-structure, and by $\mathrm{Hol}(X)$ its heart. The category $\mathrm{Hol}(X)$ is analogous to the category of perverse sheaves in the philosophy of the Riemann–Hilbert correspondence. When X is a scheme of finite type over k, we are able to take a finite open covering $\{U_i\}$ by realizable schemes, and define $\mathrm{Hol}(X)$ by gluing $\bigl\{\mathrm{Hol}(U_i)\bigr\}$. The first difficulty is to define the derived category. A starting point of our construction is an analogy with Beilinson's equivalence proven in [AC2]

$$D^{\mathrm{b}}(\mathrm{Hol}(X)) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{hol}}(X),$$

where X is a realizable scheme. This equivalence suggests defining the derived category naively by $D^{\mathrm{b}}(\mathrm{Hol}(X))$ for general scheme X. The next problem is to construct the cohomological functors. To do this, we construct cohomological functors for finite morphisms and projections separately and combine these for the general case. Let us explain the method in the easier case where $f\colon X \to Y$ is a finite morphism between realizable schemes. In that case the pushforward f_+ is an exact functor, so we can define $f_+\colon \mathrm{Hol}(X) \to \mathrm{Hol}(Y)$ by gluing, and we consider the associated derived functor to get the functor between derived categories. The definition of $f^!$ is more technical. When the morphism f is between realizable schemes, by using some general nonsense, we are able to show that $f^!$ is the right derived functor of $\mathscr{H}^0 f^!$ using the fact that $(f_+, f^!)$ is an adjoint pair and f_+ is exact. Thus, for a general finite morphism f, we define $\mathscr{H}^0 f^!$ by gluing, and we define the functor between derived categories by taking the right derived functor. Since the category $\mathrm{Hol}(X)$ does *not* possess enough injectives, we need techniques of Ind-categories to overcome this deficit. Even though the construction is more involved, we can define the cohomological functors for projections $X \times Y \to Y$ using similar ideas. Since any morphism between separated schemes of finite type can be factored into a closed immersion and a projection, we may define the cohomological operations in complete generality by composition.

For an algebraic stack $\mathfrak{X}$, we use a simplicial technique to construct the derived category $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$: take a presentation $X \to \mathfrak{X}$, and consider the simplicial algebraic space $X_\bullet := \mathrm{cosk}_0(X \to \mathfrak{X})$. The derived category of $\mathfrak{X}$ should coincide with that of $X_\bullet$ with suitable conditions on the cohomology. Since $\mathscr{D}$-modules behave like perverse sheaves, there are minor differences with the analogous construction in the ℓ-adic setting (cf. [LO]). However, the construction is mostly parallel. Now, we would like to construct the cohomological operations for algebraic stacks in a manner similarly to that for schemes described above. However, in general, morphisms between algebraic stacks cannot be written as a composition of finite morphisms and projections. In this paper, we restrict our attention to admissible stacks, i.e., algebraic stacks whose diagonal morphisms are finite.

This formalism is especially used to show the ℓ-independence of the traces of actions of correspondences on cohomology groups. This is then used to calculate the traces of elements of certain Hecke algebra acting of cohomology groups of the moduli spaces of shtukas. We refer to §4.2 for a more detailed explanation of the proof of the main theorem.

Let us give an overview of the organization of this paper. We begin with collecting known results concerning arithmetic $\mathscr{D}$-modules in §1.1, and the subsection contains few new facts. In §1.2, we show some elementary properties of Ind-categories. In §1.3, we introduce a t-structure corresponding to constructible sheaves in the spirit of the Riemann–Hilbert correspondences. This t-structure is useful when we construct various types of trace maps. In arithmetic $\mathscr{D}$-module theory, the coefficient categories are K-additive where K is a complete discrete valuation field whose residue field is k. However, for the Langlands correspondence it is convenient to work with $\overline{\mathbb{Q}}_p$-coefficients. Passing from K-coefficients to $\overline{K}$-coefficients is rather formal, and some generality is developed in §1.4. We conclude the first section in §1.5 by constructing the trace maps for flat morphisms in the style of [SGA4]. This foundational property had been lacking in the theory of arithmetic $\mathscr{D}$-modules, and it plays an important role in the proof of the ℓ-independence type theorem, which is the main theme in §3.

In §2, we develop a theory for algebraic stacks. Most of the properties used in this section are formal in the six functor formalism, and almost no knowledge of arithmetic $\mathscr{D}$-modules is required. In §2.1 we define the triangulated category of holonomic complexes for algebraic stacks. Some cohomological operations for algebraic stacks are introduced in §2.2. In §2.3, we restrict our attention to so-called *admissible stacks*. Any morphism between admissible stacks can be factorized into morphisms which have already been treated in §2.2, and our construction of six functor formalism for these types of stacks is completed. We show basic properties of the operations in this subsection. The final subsection §2.4 is complementary, and we collect some facts which are needed in the proof of the Langlands correspondence.

In §3, we show an ℓ-independence type theorem of the trace of the action of a correspondence on cohomology groups. With the trace formalism developed in §1.5, even though there are some differences since we are dealing with algebraic stacks, our task is to translate the proof of [KS] in our language.

In the final section, §4, we show the Langlands correspondence. In order to be friendly to readers who are only interested in §4, the section begins with reviewing p-adic theory as well as recalling some notation of this paper. We state the main theorem and explain the idea of the proof in the second subsection. The actual proof is written in the third subsection, and we conclude the paper with some well-known applications.

Conventions and notation.

0.0.1. In this paper, we usually use Roman fonts (e.g., X) for schemes, script fonts (e.g., $\mathscr{X}$) for formal schemes, and Gothic fonts (e.g., $\mathfrak{X}$) for algebraic stacks. When we write $(-)^{(\prime)}$ it means "$(-)$ (resp. $(-)'$)". Throughout this paper, we fix a prime number p. When a discrete valuation field K is fixed and its residue field is finite, we often denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of K. Throughout this paper, we fix a universe $\mathbb{U}$.

0.0.2. For the terminologies of algebraic stacks, we follow [LM]. Especially, any scheme, algebraic space, or algebraic stack is assumed quasi-separated. For an algebraic stack $\mathfrak{X}$, we denote by $\mathfrak{X}_{\mathrm{sm}}$ the category of affine schemes over $\mathfrak{X}$ such that the structural morphism $X \to \mathfrak{X}$ is smooth. Morphisms between $X, Y \in \mathfrak{X}_{\mathrm{sm}}$ are smooth morphisms $X \to Y$ over $\mathfrak{X}$. Recall that a *presentation* of $\mathfrak{X}$ is a smooth surjective morphism $\mathcal{X} \to \mathfrak{X}$ from an algebraic space $\mathcal{X}$. Finite morphisms or universal homeomorphisms between algebraic stacks are always assumed representable.

0.0.3. When $d \geq 0$ is an integer, smooth morphisms between algebraic stacks of relative dimension d are understood to be equidimensional. Let $P\colon \mathcal{X} \to \mathfrak{X}$ be a smooth morphism from an algebraic space to an algebraic stack. Then the continuous function $\dim(P)\colon \mathcal{X} \to \mathbb{N}$ is defined in [LM, (11.14)]. This function is called the *relative dimension of* P and is sometimes denoted by $d_{\mathcal{X}/\mathfrak{X}}$.

0.0.4. Let X be a topological space, and let $\pi_0(X)$ be the set of connected components of X. Let $d\colon \pi_0(X) \to \mathbb{Z}$ be a map. For any connected component Y of X, assume that a category $\mathcal{C}_Y$ endowed with an autofunctor $T_Y\colon \mathcal{C}_Y \xrightarrow{\sim} \mathcal{C}_Y$ is attached, and $\mathcal{C}_X \cong \prod_{Y \in \pi_0(X)} \mathcal{C}_Y$ via which T_X is identified with $(T_Y)_{Y \in \pi_0(X)}$. For $M \in \mathcal{C}_X$, we define $T^d(M)$ as follows: Let $M = (M_Y)_{Y \in \pi_0(X)} \in \prod \mathcal{C}_Y$. Then

$T^d(M) := (T^{d(Y)} M_Y)_Y$. We may take the autofunctor T to be the shift functor or the Tate twist functor. When $T = [1]$ (resp. $T = (1)$), the functor T^d is denoted by $[d]$ (resp. (d)).

1. Preliminaries

1.1. **Review of arithmetic $\mathscr{D}$-modules.** Let us briefly recall the status of the theory of arithmetic $\mathscr{D}$-modules. Let s be a positive integer, and put $q := p^s$. Let R be a complete discrete valuation ring whose residue field, which is assumed to be a perfect field of characteristic p, is denoted by k. Put $K := \mathrm{Frac}(R)$. We moreover assume that the s-th absolute Frobenius homomorphism $\sigma\colon k \xrightarrow{\sim} k$ sending x to x^q lifts to an automorphism $R \xrightarrow{\sim} R$ also denoted by σ.

1.1.1. **Definition** ([AC1, 1.1.3])**.** A scheme over k is said to be *realizable* if it can be embedded into a proper smooth formal scheme over $\mathrm{Spf}(R)$. We denote by $\mathrm{Real}(k/R)$ the full subcategory of the category of k-schemes $\mathrm{Sch}(k)$ consisting of realizable schemes.

For a realizable scheme X, the triangulated category of *holonomic complexes* $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$, endowed with a t-structure, is defined. Let us recall the construction. Let $\mathscr{P}$ be a proper smooth formal scheme over $\mathrm{Spf}(R)$. Then the category of overholonomic $\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}}$-modules (without Frobenius structure) is defined by Caro in [Ca2]. We denote by $\mathrm{Hol}(\mathscr{P})$ its thick full subcategory generated by overholonomic $\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}}$-modules which can be endowed with s'-th Frobenius structure for some positive integer s' divisible by s (but we do not consider the Frobenius structure). The objects of $\mathrm{Hol}(\mathscr{P})$ are called *holonomic modules*. By definition, $D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}})$ is the full subcategory of $D^{\mathrm{b}}(\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}})$ whose cohomology complexes are holonomic. Of course, this subcategory is triangulated by [KSc, 13.2.7].

Now, let $X \hookrightarrow \mathscr{P}$ be an embedding into a proper smooth formal scheme, whose existence is assured since X is a realizable scheme. Then $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ is the subcategory of $D^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{P})$ which is supported on X. This category does not depend on the choice of the embedding up to canonical equivalence, and it is well-defined. Moreover, the t-structure is compatible with this equivalence (cf. [AC1, 1.2.8]). The heart of the triangulated category is denoted by $\mathrm{Hol}(X/K)$. For further details of this category, one can refer to [AC1, §1.1, §1.2] and [AC2, §1]. In [AC2], $\mathrm{Hol}(X/K)$ and $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ are denoted by $\mathrm{Hol}_F(X/K)$ and $D^{\mathrm{b}}_{\mathrm{hol},F}(X/K)$, respectively.

Remark. In [AC2], the category $\mathrm{Hol}_F(X/K)$ is introduced using the category of "overholonomic modules *after any base change*", whereas, here, we simply used the category of overholonomic modules to define $\mathrm{Hol}(X/K)$. Since overholonomic modules with Frobenius structure are overholonomic modules after any base change by [AC2, 1.2], the categories $\mathrm{Hol}_F(X/K)$ in [AC2] and $\mathrm{Hol}(X/K)$ defined above are the same. However, to prove that $\mathrm{Hol}(X)$ is a noetherian category (cf. [AC2, 1.5]), it is convenient to work in the category of overholonomic modules after any base change.

1.1.2. **Remark.** Lifting $R \xrightarrow{\sim} R$ of the Frobenius automorphism of k is not unique in general. Let $\sigma'\colon R \xrightarrow{\sim} R$ be another lifting. Let $\mathscr{X}$ be a smooth formal scheme over R. We denote by $\mathscr{X}^{\sigma^{(\prime)}} := \mathscr{X} \otimes_{R\nearrow\sigma^{(\prime)}} R$. Locally on $\mathscr{X}$, we have the following commutative diagram over $\mathrm{Spf}(R)$:

$$\begin{tikzcd} & \mathscr{X}^{\sigma} \arrow[d, "G", "\sim"'] \\ \mathscr{X} \arrow[ur, "F"] \arrow[r, "F'"'] & \mathscr{X}^{\sigma'}, \end{tikzcd}$$

where F and F' denote liftings of the relative s-th Frobenius. For a $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$-module $\mathscr{M}$, we have

$$F^*(\mathscr{M}^{\sigma}) \cong F^*(G^*\mathscr{M}^{\sigma'}) \cong F'^*(\mathscr{M}^{\sigma'}),$$

where $\mathscr{M}^{\sigma^{(\prime)}}$ denotes the $\mathscr{D}^{\dagger}_{\mathscr{X}^{\sigma^{(\prime)}},\mathbb{Q}}$-module defined by changing base using $\sigma^{(\prime)}$. This shows that endowing $\mathscr{M}$ with a Frobenius structure with respect to σ is equivalent to endowing $\mathscr{M}$ with a Frobenius structure with respect to σ'. Thus, our category $\mathrm{Hol}(X/K)$ does not depend on the choice of σ. However, the category of modules with Frobenius structure *does* depend on the choice. For example, assume k is algebraically closed, and consider σ and σ'. Put

$$K_0^{(\prime)} := \bigl\{x \in K \mid \sigma^{(\prime)}(x) = x\bigr\}.$$

We may take σ, σ' so that K_0 and K_0' are not the same. Consider the unit object K in $\mathrm{Hol}(\mathrm{Spf}(R))$. Endow it with a Frobenius structure $\Phi^{(\prime)}$ with respect to $\sigma^{(\prime)}$. Then

$$\mathrm{Hom}_{F^{(\prime)}\text{-}\mathrm{Hol}(X/K)}\bigl((K,\Phi^{(\prime)}),(K,\Phi^{(\prime)})\bigr) \cong K_0^{(\prime)}.$$

Thus, we do not have an equivalence of categories between F-$\mathrm{Hol}(X/K)$ and F'-$\mathrm{Hol}(X/K)$ compatible with the forgetful functors to $\mathrm{Hol}(X/K)$.

1.1.3. The six functors have already been defined for realizable schemes. For details one can refer to [AC1], [AC2]. For the convenience of the reader, we collect known results. Let $f\colon X \to Y$ be a morphism in $\mathrm{Real}(k/R)$. Then we have the triangulated functors

$$f_!, f_+\colon D^{\mathrm{b}}_{\mathrm{hol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{hol}}(Y/K), \qquad f^!, f^+\colon D^{\mathrm{b}}_{\mathrm{hol}}(Y/K) \to D^{\mathrm{b}}_{\mathrm{hol}}(X/K).$$

These functors satisfy the following fundamental properties of six functor formalism:

(1) $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ is a closed symmetric monoidal category, namely it is equipped with the tensor product $\otimes$ and the unit object K_X with which it forms a symmetric monoidal category (sometimes called commutative tensor category as in [KSc, 4.2.16]), and $\otimes$ has the left adjoint functor $\mathcal{H}om$. The adjoint functor $\mathcal{H}om$ is denoted sometimes by $\mathcal{H}om_X$ for clarification, and it is called the *internal hom* (cf. [AC1, 1.1.6, Appendix]).

(2) f^+ is monoidal, namely it commutes with $\otimes$ and preserves the unit object.

(3) Given composable morphisms f and g, there exists a canonical isomorphism $(f \circ g)^+ \cong g^+ \circ f^+$. We associate $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ to $X \in \mathrm{Real}(k/R)$, and with this pullback and the canonical isomorphisms for compositions, we have the fibered category over $\mathrm{Real}(k/R)$. Moreover, we have a similar fibered category for $f^!$ as well (cf. [AC1, 1.3.14], checking of the category being fibered readily follows from the construction of the functor).

(4) (f^+, f_+) and $(f_!, f^!)$ are adjoint pairs (cf. [AC1, 1.3.14 (viii)]).

(5) We have a morphism of functors $f_! \to f_+$ compatible with transitivity isomorphisms of composition. This morphism is an isomorphism when f is proper (cf. [AC1, 1.3.7, 1.3.14 (vi)]).

(6) When j is an open immersion, there exists the isomorphism $j^+ \xrightarrow{\sim} j^!$ compatible with the transition isomorphism of the composition of two open immersions (cf. [V2, II.3.5]).

(7) Let Vec_K be the abelian category of K-vector spaces, and we denote by $D^{\mathrm{b}}_{\mathrm{fin}}(\mathrm{Vec}_K)$ the derived category consisting of bounded complexes of K-vector spaces whose cohomologies are finite dimensional. There exists a canonical equivalence of monoidal categories $\mathbb{R}\Gamma\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathrm{Spec}(k)/K) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{fin}}(\mathrm{Vec}_K)$. For $X \to \mathrm{Spec}(k)$ in $\mathrm{Real}(k/R)$, we put $\mathbb{R}\mathrm{Hom}(-,-) := \mathbb{R}\Gamma \circ f_+ \circ \mathcal{H}om(-,-)$. Note that we have an isomorphism $\mathbb{R}^i\mathrm{Hom}(\mathscr{F},\mathscr{G}) \cong \mathrm{Hom}(\mathscr{F},\mathscr{G}[i])$ for $\mathscr{F},\mathscr{G} \in D^{\mathrm{b}}_{\mathrm{hol}}(X)$.

(8) Consider the following cartesian diagram of schemes:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y. \end{tikzcd} \tag{1.1.3.1}$$

Assume that the schemes are realizable. Then we have a canonical isomorphism $g^+ f_! \cong f'_! g'^+$ compatible with compositions. When f is proper (resp. open immersion), this isomorphism is the base change homomorphism defined by the adjointness of (f^+, f_+) (resp. $(f_!, f^!)$) via the isomorphism of (5) (resp. (6) (cf. [AC1, 1.3.14 (vii)]).

(9) We have a canonical isomorphism $f_!\mathscr{F} \otimes \mathscr{G} \cong f_!(\mathscr{F} \otimes f^+\mathscr{G})$ (cf. [AC1, Appendix]).

(10) Let i be a closed immersion in $\mathrm{Real}(k/K)$, and let j be the open immersion defined by the complement. Then we have a canonical distinguished triangle of functors
$$j_! j^! \to \mathrm{id} \to i_+ i^+ \xrightarrow{+1},$$
where the first and second morphisms are adjunction morphisms.

Before recalling several more properties, let us show the following lemma:

Lemma. *Let $\iota\colon X \to X'$ be a universal homeomorphism in* $\mathrm{Real}(k/R)$*. Then the adjoint pair $(\iota_+, \iota^!)$ induces an equivalence between $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ and $D^{\mathrm{b}}_{\mathrm{hol}}(X'/K)$, and we have a canonical isomorphism $\iota^+ \cong \iota^!$. Moreover, assume we are given the following commutative diagram where ι and ι' are universal homeomorphisms:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota"'] & Y \arrow[d, "\iota'"] \\ X' \arrow[r, "f'"'] & Y'. \end{tikzcd}$$

Then $f^{(\prime)}_+$, $f^{(\prime)}_!$, $f^{(\prime)+}$, and $f^{(\prime)!}$ commute canonically with $\iota^{(\prime)+} \cong \iota^{(\prime)!}$.

Proof. The first equivalence is nothing but [AC1, 1.3.12]. Since $\iota_! \xrightarrow{\sim} \iota_+$ by (5) above, we have $\iota^+ \cong (\iota_!)^{-1} \cong (\iota_+)^{-1} \cong \iota^!$. Commutation results follows by transitivity of pushforwards and pullbacks. $\square$

This result can be applied in particular when f is the relative Frobenius morphism (see the remark below). We need some more properties, which may not be regarded as standard properties of six functor formalism:

(11) For X in $\mathrm{Real}(k/R)$, let $X^\sigma := X \otimes_{k\nearrow\sigma} k$. Then we have a pullback $\sigma^*\colon D^{\mathrm{b}}_{\mathrm{hol}}(X/K) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{hol}}(X^\sigma/K)$, which is exact, and all the cohomological functors commute canonically with this pullback. (This follows easily from the definition of the cohomological functors; see also [Ber3, 4.5].)

Now, for a separated scheme of finite type over k, we denote by $\mathrm{Isoc}^\dagger(X/K)$ the thick full subcategory of the category of overconvergent isocrystals on X generated by those which can be endowed with the s'-th Frobenius structure for some $s|s'$. We caution that our notation is slightly different from the standard one as in [Ber1, 2.3.6]. In fact, by his notation, Berthelot simply means the category of overconvergent isocrystals, and Frobenius structure does not play any role in his definition.

(12) Let X be a realizable scheme such that X_{red} is a smooth realizable scheme of dimension $d\colon \pi_0(X) \to \mathbb{N}$ (cf. 0.0.3). Then there exists a fully faithful functor $\mathrm{sp}_+\colon \mathrm{Isoc}^\dagger(X/K) \to \mathrm{Hol}(X/K)$ called the *specialization functor*. We denote the essential image[1] of sp_+ shifted by $-d$ by $\mathrm{Sm}(X/K) \subset \mathrm{Hol}(X/K)[-d] \subset D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ (cf. [Ca5, 4.2.2]).

Remark. (i) Let $F\colon X \to X$ be the s-th absolute Frobenius endomorphism. Combining (11) and the lemma above applied to the s-th relative Frobenius morphism $F_{X/k}\colon X \to X^\sigma$, we get an equivalence of categories

$$F^* := F^+_{X/k} \circ \sigma^*\colon D^{\mathrm{b}}_{\mathrm{hol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{hol}}(X/K).$$

This pullback is nothing but the one used in [Ber3, Definition 4.5]. For a cohomological functor $C\colon D^{\mathrm{b}}_{\mathrm{hol}}(X/K) \to D^{\mathrm{b}}_{\mathrm{hol}}(Y/K)$, we say that C *commutes with a Frobenius pullback* if there exists a *canonical* isomorphism $C \circ F^* \cong F^* \circ C$.

(ii) Let $F\colon X \to X'$ be the relative Frobenius morphism. For $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(X')$, the lemma above yields a canonical isomorphism $\alpha\colon F_+F^!\mathscr{M} \xrightarrow{\sim} \mathscr{M}$. On the other hand, when X and X' can be lifted to smooth formal schemes $\mathscr{X}$, $\mathscr{X}'$, [Ber3, 4.2.4] gives us another isomorphism $\beta\colon F_+F^!\mathscr{M} \cong \mathscr{M}$, since $F^\flat\mathscr{D}^\dagger_{\mathscr{X}'} \cong F^*\mathscr{D}^\dagger_{\mathscr{X}'} \otimes \omega_{\mathscr{X}/\mathscr{X}'}$ by [Ber3, 2.4.4]. These two isomorphisms coincide. Indeed, to see this, it suffices to check the coincidence for $\mathscr{M} = \mathscr{O}_{\mathscr{X}'}$. By left-to-right conversion, it suffices to check for $\mathscr{M} = \omega_{\mathscr{X}'}$ and F_+, $F^!$ the corresponding functors for right modules. The homomorphism α is nothing but the adjunction map, and this is by definition Virrion's trace map $\mathrm{Tr}^{\mathrm{Vir}}\colon F_*\big(\omega_{\mathscr{X}} \otimes_{\mathscr{D}^\dagger_{\mathscr{X}}} F^*\mathscr{D}^\dagger_{\mathscr{X}'}\big) \to \omega_{\mathscr{X}'}$ defined in [V1, III.7.1]. By [V1, III.5.4], this map can be characterized as the unique map γ such that the following diagram commutes:

$$\begin{tikzcd} F_*\omega_{\mathscr{X}} \arrow[r, two heads] \arrow[d, "\mathrm{Tr}_F"'] & F_*\big(\omega_{\mathscr{X}} \otimes_{\mathscr{D}^\dagger_{\mathscr{X}}} F^*\mathscr{D}^\dagger_{\mathscr{X}'}\big), \arrow[dl, "\gamma"] \\ \omega_{\mathscr{X}'} & \end{tikzcd}$$

where Tr_F is the homomorphism induced by the trace map of Hartshorne [Ber3, 2.4.2]. Thus, it suffices to check that the diagram is commutative when $\gamma = \beta$. By construction of the equivalence of [Ber3, 4.2.4], β is defined by taking limit to [Ber3, (2.5.6.2)]. It is not hard to check the commutativity using [A1, 1.5] and the description of Garnier [A1, (2.2.1)] of the isomorphism [Ber3, 2.5.2]. The details are left to the reader.

[1] In [Ca5], the essential image is denoted by $\mathrm{Isoc}^{\dagger\dagger}(X/K)$.

(iii) The identification of (ii) shows that the commutation isomorphisms $F^! \circ f_+ \cong f_+ \circ F^!$, defined by the lemma and by [Ber4, 4.3.9], coincide.

1.1.4. We also have the duality formalism. Let $p\colon X \to \mathrm{Spec}(k)$ be the structural morphism of a realizable scheme. We put $K_X^\omega := p^!(K)$ and call it the *dualizing complex*. We put $\mathbb{D}_X := \mathcal{H}om(-, K_X^\omega)$ and call it the *dual functor*. For $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$, we have

$$(\star)\qquad \begin{aligned}\mathrm{Hom}(\mathbb{D}_X\mathscr{F}, \mathbb{D}_X\mathscr{F}) &\cong \mathrm{Hom}(\mathbb{D}_X\mathscr{F} \otimes \mathscr{F}, K_X^\omega)\\ &\cong \mathrm{Hom}(\mathscr{F} \otimes \mathbb{D}_X\mathscr{F}, K_X^\omega) \cong \mathrm{Hom}(\mathscr{F}, \mathbb{D}_X\mathbb{D}_X\mathscr{F}).\end{aligned}$$

The identity in the abelian group on the left-hand side induces a homomorphism $\mathscr{F} \to \mathbb{D}_X\mathbb{D}_X\mathscr{F}$.

Lemma. *The induced homomorphism of functors* $\mathrm{id} \to \mathbb{D}_X \circ \mathbb{D}_X$ *is an isomorphism.*

Proof. We already know that $\mathrm{id} \cong \mathbb{D}_X \circ \mathbb{D}_X$ by [V2, II.3.5], even though the isomorphism may not be equal to the one in the claim. Let us show that the given homomorphism in the lemma is actually an isomorphism using this Virrion's isomorphism. By *dévissage*, it suffices to check the equivalence for holonomic modules. We recall that objects of $\mathrm{Hol}(X/K)$ have finite length by [AC2, 1.5]. Thus, we only need to show the lemma for irreducible modules $\mathscr{F}$. By Virrion's result, $\mathbb{D}_X\mathbb{D}_X(\mathscr{F})$ is irreducible as well, and it remains to show that the homomorphism is not 0. If this were 0, the corresponding element of the left-hand side of $(\star)$ should also be 0, which is a contradiction. $\square$

This isomorphism induces a canonical isomorphism $\mathcal{H}om(\mathscr{F}, \mathscr{G}) \cong \mathbb{D}_X\big(\mathscr{F} \otimes \mathbb{D}_X(\mathscr{G})\big)$ for $\mathscr{F}, \mathscr{G}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. This can be seen by a similar argument to [Ha, V.2.6].

1.1.5. Let $f\colon X \to Y$ be a morphism between realizable schemes, and $\mathscr{F}, \mathscr{F}' \in D^{\mathrm{b}}_{\mathrm{hol}}(X)$. Since f^+ is monoidal, we have the canonical isomorphism $f^+\big((-) \otimes (-)\big) \cong f^+(-) \otimes f^+(-)$. By taking the adjoint, we have the homomorphism $f_+(-) \otimes f_+(-) \to f_+\big((-) \otimes (-)\big)$. This induces a homomorphism

$$f_+\,\mathcal{H}om_X(\mathscr{F}, \mathscr{F}') \otimes f_+(\mathscr{F}) \to f_+\big(\mathcal{H}om_X(\mathscr{F}, \mathscr{F}') \otimes \mathscr{F}\big) \to f_+(\mathscr{F}'),$$

where the second homomorphism is the evaluation map. Taking the adjoint, we have the homomorphism

$$f_+\,\mathcal{H}om_X(\mathscr{F}, \mathscr{F}') \to \mathcal{H}om_Y\big(f_+(\mathscr{F}), f_+(\mathscr{F}')\big).$$

Now, let $\mathscr{G} \in D^{\mathrm{b}}_{\mathrm{hol}}(Y)$. When f is *proper*, 1.1.3 (5) and the adjointness of the pair $(f_!, f^!)$ induce a homomorphism

$$(\star)\qquad f_+\,\mathcal{H}om_X\big(\mathscr{F}, f^!\mathscr{G}\big) \to \mathcal{H}om_Y\big(f_+\mathscr{F}, \mathscr{G}\big).$$

Proposition. *This homomorphism is an isomorphism.*

Proof. For an open subscheme U of X, we denote by $j_U\colon U \hookrightarrow X$ the open immersion. Let p_Z be the structural morphism of a scheme Z, and let $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(X)$. Assume that $p_{U+}(j_U^+\mathscr{M}) = 0$ for any U. Then $\mathscr{M} = 0$. Indeed, for any closed point $i_x\colon x \hookrightarrow X$, put $U_x := X \setminus \{x\}$, and we have the localization triangle

$$i_x^!(\mathscr{M}) \to p_{X+}(\mathscr{M}) \to p_{U_x+}j_{U_x}^+(\mathscr{M}) \xrightarrow{+1}.$$

By assumption, we have $i_x^!(\mathscr{M})=0$. By [AC1, 1.3.11], $\mathscr{M}=0$.

By the construction, the homomorphism in question is compatible with restriction to open subschemes. Thus, the observation above reduces to checking that the induced homomorphism $p_{Y+}f_+\,\mathcal{H}om_X\bigl(\mathscr{F},f^!\mathscr{G}\bigr)\to p_{Y+}\,\mathcal{H}om_Y\bigl(f_+\mathscr{F},\mathscr{G}\bigr)$ is isomorphic. Via the isomorphism 1.1.3 (7), this homomorphism is nothing but the canonical homomorphism $\mathrm{Hom}_X\bigl(\mathscr{F},f^!\mathscr{G}\bigr)\to\mathrm{Hom}_Y\bigl(f_+\mathscr{F},\mathscr{G}\bigr)$ of the adjoint pair $(f_+,f^!)$. □

Let $j\colon U\to X$ be an open immersion. Then there exists a unique homomorphism

$$(\star\star)\qquad \mathcal{H}om\bigl(j_!\mathscr{F},\mathscr{G}\bigr)\to j_+\,\mathcal{H}om\bigl(\mathscr{F},j^!\mathscr{G}\bigr)$$

such that its restriction to U is the identity. This is an isomorphism since we know that these two objects are isomorphic abstractly by [AC1, A.7]. Let $f\colon X\to Y$ be a morphism between realizable schemes. This morphism factorizes in $\mathrm{Real}(k/R)$ as $X\xrightarrow{j}\overline{X}\xrightarrow{\overline{f}}Y$, where j is an open immersion and $\overline{f}$ is proper. We define

$$f_+\,\mathcal{H}om_X\bigl(\mathscr{F},f^!\mathscr{G}\bigr)\xrightarrow[(\star)]{\sim}j_+\,\mathcal{H}om\bigl(\overline{f}_+\mathscr{F},j^!\mathscr{G}\bigr)\xleftarrow[(\star\star)]{\sim}\mathcal{H}om_Y\bigl(f_!\mathscr{F},\mathscr{G}\bigr).$$

It is standard to check that this does not depend on the choice of the factorization. Using this isomorphism, we may prove the following two more isomorphisms, which we record here for future use:

$$\mathcal{H}om(\mathscr{F},f_+\mathscr{G})\cong f_+\,\mathcal{H}om(f^+\mathscr{F},\mathscr{G}),\qquad f^!\,\mathcal{H}om(\mathscr{F},\mathscr{G})\cong\mathcal{H}om(f^+\mathscr{F},f^!\mathscr{G}).$$

1.1.6. Consider the cartesian diagram (1.1.3.1). We assume that the schemes are realizable. Then we define the base change homomorphism $g'^+\circ f^!\to f'^!\circ g^+$ to be the adjunction of the following composition:

$$f'_!\circ g'^+\circ f^!\xleftarrow{\sim}g^+\circ f_!\circ f^!\xrightarrow{\mathrm{adj}_{f'}}g^+.$$

By definition, the following diagram is commutative, which we will use later:

$$\begin{tikzcd}
g^+f_!f^! \arrow[r, "\sim"] \arrow[d, "\mathrm{adj}_f"'] & f'_!g'^+f^! \arrow[r] & f'_!f'^!g^+ \arrow[d, "\mathrm{adj}_{f'}"] \\
g^+ \arrow[rr, equal] & & g^+.
\end{tikzcd}$$

1.1.7. For realizable schemes X_1, X_2 and $\mathscr{M}_1\in D^{\mathrm{b}}_{\mathrm{hol}}(X_1)$, $\mathscr{M}_2\in D^{\mathrm{b}}_{\mathrm{hol}}(X_2)$, we put $\mathscr{M}_1\boxtimes\mathscr{M}_2:=p_1^+(\mathscr{M}_1)\otimes p_2^+(\mathscr{M}_2)$, where $p_i\colon X_1\times X_2\to X_i$ denotes the i-th projection. This functor is called the *exterior tensor product*.

Now, let $f^{(\prime)}\colon X^{(\prime)}\to Y^{(\prime)}$ be a morphism of realizable schemes, and take an object $\mathscr{M}^{(\prime)}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(X^{(\prime)}/K)$. We have the canonical isomorphism

$$(f\times f')^+\bigl((-)\boxtimes(-)\bigr)\cong f^+(-)\boxtimes f'^+(-)$$

since f^+ and f'^+ are monoidal. By taking the adjoint, we have a homomorphism

$$f_+(\mathscr{M})\boxtimes f'_+(\mathscr{M}')\to(f\times f')_+\bigl(\mathscr{M}\boxtimes\mathscr{M}'\bigr).$$

Proposition. *This homomorphism is an isomorphism.*

Proof. When f and f' are immersions, the proposition is essentially contained in the proof of [AC1, 1.3.3 (i)]. Thus, we may assume f and f' to be smooth proper and $X^{(\prime)}$, $Y^{(\prime)}$ can be lifted to proper smooth formal schemes $\mathscr{X}^{(\prime)}$, $\mathscr{Y}^{(\prime)}$. In this situation, we have the canonical isomorphism

$$f^!(-)\boxtimes f'^!(-)\cong(f\times f')^!\big((-)\boxtimes(-)\big),$$

and by taking the adjunction of [V1], we have the homomorphism

$$\rho\colon (f\times f')_+\big(\mathscr{M}\boxtimes\mathscr{M}'\big)\to f_+(\mathscr{M})\boxtimes f'_+(\mathscr{M}').$$

The homomorphism in the statement is the dual of this homomorphism. Thus, it suffices to show that ρ is an isomorphism. Let $f^{(\prime)}_n\colon X^{(\prime)}_n\to Y^{(\prime)}_n$ be a smooth morphism of relative dimension $d^{(\prime)}$ between proper smooth schemes over R/π^{n+1} where π is a uniformizer of R. For perfect $\mathscr{D}^{(m)}$-complexes on $X^{(\prime)}_n$ and $Y^{(\prime)}_n$, we have the homomorphism $\rho_n\colon (f_n\times f'_n)_+\big((-)\boxtimes(-)\big)\to f_{n+}(-)\boxtimes f'_{n+}(-)$ by similar construction to ρ, and it suffices to show that ρ_n is an isomorphism since $D^{\mathrm{b}}_{\mathrm{coh}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}})=D_{\mathrm{perf}}(\widehat{\mathscr{D}}^{(m)}_{\mathscr{X}})$ by [Ber3, 4.4.8]. By [Ha, VII.4.1], the following diagram commutes:

$$\begin{tikzcd}
R^df_*(\omega_{X_n/Y_n})\boxtimes R^{d'}f'_*(\omega_{X'_n/Y'_n}) \arrow[r] \arrow[d] & \mathcal{O}_{Y_n}\boxtimes\mathcal{O}_{Y'_n} \arrow[d, "\sim"] \\
R^{d+d'}(f_n\times f'_n)_*(\omega_{X_n\times X'_n/Y_n\times Y'_n}) \arrow[r] & \mathcal{O}_{Y_n\times Y'_n},
\end{tikzcd}$$

where the vertical homomorphisms are trace maps, and ω denotes the canonical bundle sheaf. The commutativity shows that ρ_n is nothing but the homomorphism induced by the isomorphism

$$\mathscr{D}^{(m)}_{Y_n\times Y'_n\leftarrow X_n\times X'_n}\cong\mathscr{D}^{(m)}_{Y_n\leftarrow X_n}\boxtimes\mathscr{D}^{(m)}_{Y'_n\leftarrow X'_n}$$

(cf. [A1, Lemma 4.5 (ii)]), and we get the proposition by using the Künneth formula for quasi-coherent sheaves. $\square$

1.1.8. Finally, we recall the following result:

Theorem ([AC2]). *Let X be a realizable scheme. Then the canonical functor $D^{\mathrm{b}}(\mathrm{Hol}(X/K))\to D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ induces an equivalence of triangulated categories.*

In the current formalism, the cycle class map is missing. We shall construct trace maps and a cycle class formalism in the coming subsections, which are important tools in showing the ℓ-independence type result.

1.2. **Ind-categories.**

1.2.1. **Lemma.** *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, and assume $\mathcal{A}$ has enough injective objects. Let $F\colon\mathcal{A}\to\mathcal{B}$ be a left exact functor, and assume that we have an adjoint pair (G,F) such that G is* exact. *Then for $M\in D^+(\mathcal{A})$ and $N\in D(\mathcal{B})$, we have*

$$\mathrm{Hom}_{D(\mathcal{A})}(G(N),M)\cong\mathrm{Hom}_{D(\mathcal{B})}(N,\mathbb{R}F(M)).$$

Proof. By the exactness of G, F sends injective objects to injective objects. Thus, for a complex of injective objects $I^\bullet\in C^+(\mathcal{A})$ and a complex $N^\bullet\in C(\mathcal{B})$, it suffices to show that

$$\mathrm{Hom}_{K(\mathcal{A})}(G(N^\bullet),I^\bullet)\cong\mathrm{Hom}_{K(\mathcal{B})}(N^\bullet,F(I^\bullet)).$$

By the adjointness, we have $\mathrm{Hom}^\bullet(G(N^\bullet), I^\bullet) \cong \mathrm{Hom}^\bullet(N^\bullet, F(I^\bullet))$ in $C(\mathrm{Ab})$ where $\mathrm{Hom}^\bullet$ is the functor defined in [Ha, I.6]. Since $\mathrm{Hom}_{K(\mathcal{A})} = \mathscr{H}^0\mathrm{Hom}^\bullet$, we get the isomorphism. □

Remark. The proof also shows that if moreover $\mathcal{B}$ has enough injectives, we have

$$\mathbb{R}\mathrm{Hom}_{D(\mathcal{A})}\big(G(N), M\big) \cong \mathbb{R}\mathrm{Hom}_{D(\mathcal{B})}\big(N, \mathbb{R}F(M)\big).$$

1.2.2. Let us collect some facts on Ind-categories. Let $\mathcal{A}$ be a category. Let $\mathcal{A}^\wedge$ be the category of presheaves on $\mathcal{A}$, and let $\mathrm{h}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}^\wedge$ be the canonical embedding. Then $\mathrm{Ind}(\mathcal{A})$ is the full subcategory of $\mathcal{A}^\wedge$ consisting of objects which can be written as a filtrant small inductive limit of the image of $\mathrm{h}_{\mathcal{A}}$. By definition, $\mathrm{h}_{\mathcal{A}}$ induces a functor $\iota_{\mathcal{A}} \colon \mathcal{A} \to \mathrm{Ind}(\mathcal{A})$. We sometimes abbreviate this as ι. Since $\mathrm{h}_{\mathcal{A}}$ is fully faithful by the Yoneda lemma [KSc, 1.4.4], $\iota_{\mathcal{A}}$ is fully faithful as well. For details see [KSc, §6].

Now, we assume that $\mathcal{A}$ is an abelian category. We have the following properties:

(1) The category $\mathrm{Ind}(\mathcal{A})$ is abelian, and the functor $\iota_{\mathcal{A}}$ is exact. Moreover, $\mathrm{Ind}(\mathcal{A})$ admits small inductive limits, and small filtrant inductive limits are exact (cf. [KSc, 8.6.5]).
(2) Assume $\mathcal{A}$ to be essentially small. Then $\mathrm{Ind}(\mathcal{A})$ is a Grothendieck category, and in particular, it possesses enough injectives and admits small projective limits (cf. [KSc, 8.6.5, 9.6.2, 8.3.27]).
(3) The category $\mathcal{A}$ is a thick subcategory of $\mathrm{Ind}(\mathcal{A})$ by [KSc, 8.6.11]. This in particular shows that any direct factor of objects of $\mathcal{A}$ is in $\mathcal{A}$, since a direct factor is the kernel of a projector.
(4) Let $X_\bullet \colon I \to \mathcal{A}$ be an inductive system. Since $\iota_{\mathcal{A}}$ is fully faithful, if $\varinjlim \iota_{\mathcal{A}}(X_i)$ is in the essential image of $\iota_{\mathcal{A}}$, then $\varinjlim X_i$ exists in $\mathcal{A}$ and $\varinjlim \iota_{\mathcal{A}}(X_i) \xrightarrow{\sim} \iota_{\mathcal{A}}(\varinjlim X_i)$.

Now, let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then it extends uniquely to an additive functor $IF \colon \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ such that IF commutes with arbitrary small filtrant inductive limits by [KSc, 6.1.9]. Since a small direct sum can be written as a filtrant inductive limit of finite sums, IF commutes with small direct sums as well. We have the following additional properties:

(5) If F is left (resp. right) exact, IF is also (cf. [KSc, 8.6.8]).
(6) Let $G \colon \mathcal{B} \to \mathcal{C}$ be another additive functor between abelian categories. Then $I(G \circ F) \cong IG \circ IF$ (cf. [KSc, 6.1.11]).

If there is nothing to be confused, by abuse of notation, we denote IF simply by F.

Remark. In general $\iota_{\mathcal{A}}$ does not commute with inductive limits (cf. [KSc, 6.1.20]), and in [KSc], inductive limits in $\mathrm{Ind}(\mathcal{A})$ are distinguished by using "$\varinjlim$". In this paper, we simply denote this limit by $\varinjlim$ if no confusion can arise, and when we use inductive limits, it is understood to be taken in $\mathrm{Ind}(\mathcal{A})$, not in $\mathcal{A}$, unless otherwise stated.

1.2.3. **Lemma.** *Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, and assume that $\mathcal{B}$ admits small filtrant inductive limits. Then the restriction functor yields an equivalence $\mathrm{Fct}^{\mathrm{il,add}}(\mathrm{Ind}(\mathcal{A}), \mathcal{B}) \xrightarrow{\sim} \mathrm{Fct}^{\mathrm{add}}(\mathcal{A}, \mathcal{B})$, where the target (resp. source) is the category of additive functors (resp. of additive functors which commute with small filtrant inductive limits).*

Proof. This is a reorganization of [KSc, 6.3.2]; see also [SGA4, I, 8.7.3]. The quasi-inverse is the functor sending F to $\sigma_{\mathcal{B}} \circ IF$, where $\sigma_{\mathcal{B}} \colon \mathrm{Ind}(\mathcal{B}) \to \mathcal{B}$ is the functor taking the inductive limit (cf. [KSc, 6.3.1]). $\square$

1.2.4. Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, and assume moreover that $\mathcal{A}$ is a *noetherian category* (i.e., an essentially small category whose objects are noetherian, cf. [Ga, II.4]). Let $f_* \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. Recall that $\mathrm{Ind}(\mathcal{A})$ has enough injectives (cf. 1.2.2 (2)). Thus, If_* can be derived to get $\mathbb{R}f_* \colon D^+(\mathrm{Ind}(\mathcal{A})) \to D^+(\mathrm{Ind}(\mathcal{B}))$, by abuse of notation. We also recall that the canonical functor $\iota_{\mathcal{A}} \colon D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}_{\mathcal{A}}(\mathrm{Ind}(\mathcal{A}))$ gives an equivalence by [KSc, 15.3.1], and the same for $\mathcal{B}$.

Lemma. *Let $f_* \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor as above. Then for any integer $i \geq 0$, $\mathbb{R}^i f_* \colon \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ commutes with an arbitrary small filtrant inductive limit.*

Proof. Since we are assuming $\mathcal{A}$ to be noetherian, $\mathrm{Ind}(\mathcal{A})$ is a locally noetherian category (cf. [Ga, II.4]), and by [Ga, II.4, Cor 1 of Thm 1], small filtrant inductive limits of injective objects in $\mathrm{Ind}(\mathcal{A})$ remain to be injective. Thus, we may apply [KSc, 15.3.3] to conclude the proof. $\square$

1.2.5. Recall that a δ-functor $\{f^i\}$ between abelian categories is called the *right satellite of* f^0 if $f^i = 0$ for $i < 0$, and it is universal among them (cf. [Gr, 2.2]).

Lemma. *The composition functor $\{\mathbb{R}^i f_* \circ \iota\} \colon \mathcal{A} \to \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is the right satellite functor of $If_* \circ \iota_{\mathcal{A}} \cong \iota_{\mathcal{B}} \circ f_*$.*

Proof. Since $\{\mathbb{R}^i f_* \circ \iota\}$ is a right δ-functor, it remains to show that it is universal. Let $\{G^i\} \colon \mathcal{A} \to \mathrm{Ind}(\mathcal{B})$ be a right δ-functor with a morphism of functors $\iota_{\mathcal{B}} \circ f_* \to G^0$. By Lemma 1.2.3, $\{G^i\}$ extends uniquely to a collection of functors $\{\widetilde{G}^i\} \colon \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$. Then $\{\widetilde{G}^i\}$ is a right δ-functor as well by [KSc, 8.6.6], with a morphism $If_* \to \widetilde{G}^0$. By the universal property of $\{\mathbb{R}^i f_*\}$, we get a morphism $\{\mathbb{R}^i f_*\} \to \{\widetilde{G}^i\}$, which induces the morphism $\varphi \colon \{\mathbb{R}^i f_* \circ \iota\} \to \{G^i\}$. Now, any morphism $\mathbb{R}^i f_* \circ \iota \to G^i$ extends uniquely to $\mathbb{R}^i f_* \to \widetilde{G}^i$ by Lemmas 1.2.4 and 1.2.3. Using [KSc, 8.6.6] again, $\{\mathbb{R}^i f_*\} \to \{\widetilde{G}^i\}$ is a morphism of δ-functors if φ is. Thus the uniqueness of φ follows, and we conclude that the δ-functor in question is universal. $\square$

1.2.6. **Lemma.** *Let $f^* \colon \mathcal{B} \to \mathcal{A}$ be an exact functor such that (f^*, f_*) is an adjoint pair. Assume we are given a functor $f_+ \colon D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{B})$ such that (f^*, f_+) is an adjoint pair. Then $f_+ \cong \mathbb{R}f_* \circ \iota$ on $D^{\mathrm{b}}(\mathcal{A})$.*

Proof. First, let us show that for $X \in \mathcal{A}$, $\mathscr{H}^i f_+(X) = 0$ for $i < 0$. If $f_+(X) \neq 0$, by the boundedness condition, there exists an integer d such that $\mathscr{H}^d f_+(X) \neq 0$ and $\mathscr{H}^i f_+(X) = 0$ for $i < d$. Assume $d < 0$. Then for any $Y \in \mathcal{B}$, we have

$$\mathrm{Hom}_{\mathcal{B}}\bigl(Y, \mathscr{H}^d f_+(X)\bigr) \cong \mathrm{Hom}_{D(\mathcal{B})}\bigl(Y, f_+(X)[d]\bigr) \cong \mathrm{Hom}_{D(\mathcal{A})}\bigl(f^*(Y), X[d]\bigr) = 0,$$

where the last equality holds since $\mathscr{H}^i(X[d]) = 0$ for $i \leq 0$. This contradicts the assumption, and thus, $\mathscr{H}^i f_+(X) = 0$ for $i < 0$. In the same way, we get that $(f^*, \mathscr{H}^0 f_+)$ is an adjoint pair, and in particular, $\mathscr{H}^0 f_+ \cong f_*$.

This shows that the collection of functors $\{\mathscr{H}^i f_+\}$ is a (right) δ-functor. Since $\{\mathscr{H}^i f_+\}$ is a δ-functor from $\mathcal{A}$ to $\mathcal{B}$ with the isomorphism $f_* \cong \mathscr{H}^0 f_+$, Lemma

1.2.5 yields a homomorphism $\{\mathbb{R}^i f_* \circ \iota\} \to \{\mathscr{H}^i f_+\}$ of δ-functors. For $X \in \mathcal{A}$, we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathrm{Ind}(\mathcal{B}))}\bigl(f_+(X),\mathbb{R}f_*(X)\bigr) &\cong \mathrm{Hom}_{D(\mathrm{Ind}(\mathcal{A}))}\bigl(f^*f_+(X),X\bigr)\\ &\cong \mathrm{Hom}_{D(\mathcal{A})}\bigl(f^*f_+(X),X\bigr) \cong \mathrm{Hom}_{D(\mathcal{B})}\bigl(f_+(X),f_+(X)\bigr),\end{aligned}$$

where we used the canonical equivalence $D^{\mathrm{b}}(\mathcal{A}) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathcal{A}}(\mathrm{Ind}(\mathcal{A}))$ recalled in 1.2.4. Thus, the identity of $f_+(X)$ defines a homomorphism $\rho\colon f_+(X) \to \mathbb{R}f_*(X)$, which induces the isomorphism on $\mathscr{H}^0$. By the universal property of a satellite functor, the composition $\{\mathbb{R}^i f_*(X)\} \to \{\mathscr{H}^i f_+(X)\} \xrightarrow{\rho} \{\mathbb{R}^i f_*(X)\}$ is the identity, which shows that $\mathbb{R}^i f_*(X)$ is a direct factor of $\mathscr{H}^i f_+(X)$. This shows that $\mathbb{R}^i f_*(X)$ is in $\mathcal{B}$ by 1.2.2 (3) and $\mathbb{R}^i f_*(X) = 0$ for $i \gg 0$, which means that $\mathbb{R}f_*(X)$ is in $D^{\mathrm{b}}_{\mathcal{B}}(\mathrm{Ind}(\mathcal{B})) \xleftarrow{\sim} D^{\mathrm{b}}(\mathcal{B})$. Thus $\mathbb{R}f_*$ induces a functor from $D^{\mathrm{b}}(\mathcal{A})$ to $D^{\mathrm{b}}(\mathcal{B})$. For any $Y \in D^{\mathrm{b}}(\mathcal{B})$, we have

$$\mathrm{Hom}_{D(\mathcal{B})}(Y,\mathbb{R}f_*(X)) \cong \mathrm{Hom}_{D(\mathcal{A})}(f^*(Y),X) \cong \mathrm{Hom}_{D(\mathcal{B})}(Y,f_+(X)).$$

Thus $\mathbb{R}f_*(X) \xrightarrow{\sim} f_+(X)$ by the Yoneda lemma, as required. □

1.2.7. Now, let us apply the preceding general results to the theory of arithmetic $\mathscr{D}$-modules. We use the notation of §1.1. First, we need:

Lemma. *For a realizable scheme X, the category* $\mathrm{Hol}(X/K)$ *is noetherian and artinian.*

Proof. Let X be a realizable variety. Then the category $\mathrm{Hol}(X/K)$ is essentially small. Indeed, to check this, it suffices to show that for a smooth formal scheme $\mathscr{X}$, the category of coherent $\mathscr{D}^\dagger_{\mathscr{X},\mathbb{Q}}$-modules is essentially small. The verification is standard. Now, any object of $\mathrm{Hol}(X/K)$ is noetherian and artinian by [AC2, 1.5]. □

Definition. For a realizable scheme X, we put $M(X/K) := \mathrm{Ind}(\mathrm{Hol}(X/K))$. This is a Grothendieck category by 1.2.2 (2) and the lemma above.

1.2.8. Let $\phi\colon X \to Y$ be a smooth morphism equidimensional of relative dimension d between realizable schemes. We have the following functors via the equivalence of Theorem 1.1.8 compatible with Frobenius pullbacks:

$$\phi_+[-d]\colon D^{\mathrm{b}}(\mathrm{Hol}(X/K)) \rightleftarrows D^{\mathrm{b}}(\mathrm{Hol}(Y/K))\colon \phi^+[d].$$

Lemma. *We have an adjoint pair $(\phi^+[d], \phi_+[-d])$, and $\phi^+[d]$ is exact. The adjunction map is compatible with Frobenius pullbacks.*

Proof. Since (ϕ^+, ϕ_+) is an adjoint pair, the adjointness follows. The exactness is by [AC1, 1.3.2 (i)]. □

We put $\phi_* := \mathscr{H}^0(\phi_+[-d])$, $\phi^* := \mathscr{H}^0(\phi^+[d])$. We have the right derived functor $\mathbb{R}\phi_*\colon D^+(M(X/K)) \to D^+(M(Y/K))$. By Lemma 1.2.6 together with the lemma above, $\phi_+[-d]$ is the right derived functor of ϕ_*, namely, $\phi_+[-d] \cong \mathbb{R}\phi_*$ on $D^{\mathrm{b}}(\mathrm{Hol}(X/K))$, which is a full subcategory of $D^+(M(X/K))$.

Now, let $\phi\colon X \to Y$ be a smooth morphism which may not be equidimensional. Then there exists a decomposition $X = \coprod X_i$ where X_i is an open subscheme of X such that the induced morphism $\phi_i\colon X_i \to Y$ is equidimensional. We put $\phi_* := \sum \phi_{i*}$ and $\phi^* := \sum \phi_i^*$. Note that when ϕ is an open immersion, then we have $\mathbb{R}\phi_* \cong \phi_+$.

1.2.9. Let $f\colon X \to Y$ be a finite morphism between realizable schemes. Consider the functors

$$f_! \xrightarrow{\sim} f_+ \colon D^{\mathrm{b}}(\mathrm{Hol}(X/K)) \rightleftarrows D^{\mathrm{b}}(\mathrm{Hol}(Y/K)) \colon f^!.$$

We have the following:

Lemma. *The functors $f_! \xrightarrow{\sim} f_+$ are exact and $(f_+, f^!)$ is an adjoint pair compatible with Frobenius pullbacks.*

Proof. The exactness is by [AC1, 1.3.13], and the other claims follow by 1.1.3. □

Now, we have the associated right derived functor $\mathbb{R}(\mathscr{H}^0 f^!)\colon D^+(M(X/K)) \to D^+(M(Y/K))$. By Lemma 1.2.6 together with the lemma above, we have $f^! \cong \mathbb{R}(\mathscr{H}^0 f^!)$ on $D^{\mathrm{b}}(\mathrm{Hol}(Y/K))$.

1.2.10. Let X, Y be realizable schemes, and consider the projections $p\colon X \times Y \to Y$, $q\colon X \times Y \to X$. Let $\mathscr{A}$ be an object in $\mathrm{Hol}(X/K)$. We have the functors:

$$\begin{aligned} p_{\mathscr{A}+}(-) := p_+ \mathcal{H}om(q^+\mathscr{A}, -) \colon D^{\mathrm{b}}&(\mathrm{Hol}(X \times Y/K)) \\ &\rightleftarrows D^{\mathrm{b}}(\mathrm{Hol}(Y/K)) \colon \mathscr{A} \boxtimes (-) =: p^+_{\mathscr{A}}. \end{aligned}$$

Now, assume that $\mathscr{A}$ is endowed with Frobenius structure $\mathscr{A} \xrightarrow{\sim} F^*\mathscr{A}$. Then we have an isomorphism of functors $F^* \circ p_{\mathscr{A}+} \cong p_{\mathscr{A}+} \circ F^*$, and $F^* \circ p^+_{\mathscr{A}} \cong p^+_{\mathscr{A}} \circ F^*$. Thus, $p_{\mathscr{A}+}$ and $p^+_{\mathscr{A}}$ are compatible with Frobenius pullbacks. We have:

Lemma. *The functor $p^+_{\mathscr{A}}$ is exact, and $(p^+_{\mathscr{A}}, p_{\mathscr{A}+})$ is an adjoint pair. Moreover, if $\mathscr{A}$ is endowed with Frobenius structure, the pair is compatible with Frobenius pullbacks.*

Proof. The exactness of $p^+_{\mathscr{A}}$ follows from [AC1, 1.3.3 (ii)]. By definition (cf. [AC1, 1.1.8 (i)]), we have $q^+\mathscr{A} \otimes p^+(-) \cong \mathscr{A} \boxtimes (-)$. Thus, we get

$$\begin{aligned} \mathrm{Hom}_{X\times Y}\bigl(p^+_{\mathscr{A}}(-), -\bigr) &\cong \mathrm{Hom}_{X\times Y}\bigl(q^+\mathscr{A} \otimes p^+(-), -\bigr) \\ &\cong \mathrm{Hom}_{X\times Y}\bigl(p^+(-), \mathcal{H}om(q^+\mathscr{A}, -)\bigr) \\ &\cong \mathrm{Hom}_{Y}\bigl(-, p_+ \mathcal{H}om(q^+\mathscr{A}, -)\bigr), \end{aligned}$$

where the second and the last isomorphism holds by the adjunction properties (cf. 1.1.3). □

We put $p_{\mathscr{A}*} := \mathscr{H}^0 p_{\mathscr{A}+}$, $p^*_{\mathscr{A}} := \mathscr{H}^0 p^+_{\mathscr{A}}$. Once again, we get $p_{\mathscr{A}+} \cong \mathbb{R}p_{\mathscr{A}*}$ on $D^{\mathrm{b}}(\mathrm{Hol}(X \times Y/K))$.

1.2.11. **Lemma.** *Let X be a realizable scheme, let $j\colon U \hookrightarrow X$ be an open immersion, and let $i\colon Z \hookrightarrow X$ be its complement. For an injective object $\mathscr{I}$ in $M(X/K)$, we have an exact sequence*

$$0 \to \mathscr{H}^0 i_+ i^!(\mathscr{I}) \to \mathscr{I} \to \mathscr{H}^0 j_+ j^+(\mathscr{I}) \to 0.$$

Proof. Since $\mathscr{H}^0 i_+ i^!$ is a left exact functor, we may take its right derived functor, and this is denoted by $\mathbb{R}(\mathscr{H}^0 i_+ i^!)$. Let us put

$$F := \mathrm{Coker}\bigl(\mathrm{id} \to \mathscr{H}^0(j_+ j^+)\bigr) \colon \mathrm{Hol}(X/K) \to \mathrm{Hol}(X/K).$$

We show that $\mathbb{R}^1(\mathscr{H}^0 i_+ i^!)(\mathscr{M}) \cong IF(\mathscr{M})$ for $\mathscr{M} \in M(X/K)$. Since $\mathscr{H}^0 i^!$ is left exact and i_+ is exact, we have $\mathbb{R}^1(\mathscr{H}^0 i_+ i^!) \cong i_+ \mathbb{R}^1(\mathscr{H}^0 i^!)$. When $\mathscr{M} \in \mathrm{Hol}(X/K)$, we have the isomorphism by Lemma 1.2.9 and the localization triangle. Lemma 1.2.7 and Lemma 1.2.4 show that the functor $\mathbb{R}^1(\mathscr{H}^0 i_+ i^!)$ commutes with small filtrant inductive limits. Thus, by Lemma 1.2.3, this isomorphism uniquely extends to the isomorphism we want. This shows that for an injective object $\mathscr{I}$, $IF(\mathscr{I})=0$, and we get the short exact sequence in the statement of the lemma. $\square$

1.3. **Constructible t-structures.** We need to introduce a t-structure on the triangulated category $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ whose heart corresponds to the category of *constructible sheaves* in the philosophy of the Riemann–Hilbert correspondence. We keep the notation from §1.1.

1.3.1. Let X be a realizable scheme. For $\mathscr{M}\in\mathrm{Hol}(X/K)$, we define the *support*, denoted by $\mathrm{Supp}(\mathscr{M})$, to be the smallest closed subset $Z\subset X$ such that $\mathscr{M}$ is 0 if we pull back to $X\setminus Z$. When X_{red} is smooth of dimension d, we say that a complex $\mathscr{M}\in D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ is *smooth* if $\mathscr{H}^i(\mathscr{M})[-d]$ is in $\mathrm{Sm}(X/K)$ for any i (cf. 1.1.3 (12)).

Now, we define the following two full subcategories of $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$:

- ${}^{\mathrm{c}}D^{\geq 0}$ consists of complexes $\mathscr{M}$ such that $\dim\bigl(\mathrm{Supp}(\mathscr{H}^n(\mathscr{M}))\bigr)\leq n$ for any $n\geq 0$, and $\mathscr{H}^n(\mathscr{M})=0$ for any $n<0$.
- ${}^{\mathrm{c}}D^{\leq 0}$ consists of complexes $\mathscr{M}$ such that $\mathscr{H}^k i^+_W(\mathscr{M})=0$ for any closed subscheme $i_W\colon W\hookrightarrow X$ and $k>\dim(W)$.

We note that the extension property holds, namely, for a triangle $\mathscr{M}'\to\mathscr{M}\to\mathscr{M}''\xrightarrow{+1}$, if $\mathscr{M}'$ and $\mathscr{M}''$ are in ${}^{\mathrm{c}}D^{\star}(X)$ ($\star\in\{\geq 0,\leq 0\}$), then $\mathscr{M}$ is also.

Example. Let X be a smooth curve. Then ${}^{\mathrm{c}}D^{\geq 0}$ consists of complexes $\mathscr{M}$ such that $\mathscr{H}^i(\mathscr{M})=0$ for $i<0$, and $\mathscr{H}^0(\mathscr{M})$ is supported on a finite union of points. The category ${}^{\mathrm{c}}D^{\leq 0}$ consists of complexes $\mathscr{N}$ such that $\mathscr{H}^i(\mathscr{N})=0$ for $i>1$, and $\mathscr{H}^0 i^+_x\mathscr{H}^1(\mathscr{M})=0$ for any closed point x. For example, $i_{x+}(K)$ and K_X ($\cong\mathrm{sp}_+(\mathcal{O}_{X,\mathbb{Q}})[-1]$ where $\mathcal{O}_{X,\mathbb{Q}}$ denotes the constant overconvergent isocrystal) are in both ${}^{\mathrm{c}}D^{\geq 0}$ and ${}^{\mathrm{c}}D^{\leq 0}$. For a smooth realizable scheme X, any object of $\mathrm{Sm}(X/K)$ (cf. 1.1.3 (12)) is in both $D^{\geq 0}$ and $D^{\leq 0}$. This can be checked by the right exactness of i^+ (cf. [AC1, 1.3.2 (ii)]).

1.3.2. **Lemma.** *Let $i\colon Z\hookrightarrow X$ be a closed immersion, and let $j\colon U\hookrightarrow X$ be its complement. Then i^+, $j_!$, i_+, j^+ all preserve both ${}^{\mathrm{c}}D^{\geq 0}$ and ${}^{\mathrm{c}}D^{\leq 0}$.*

Proof. Since $i_!\cong i_+$ and j^+ are exact by [AC1, 1.3.2], the verification is easy. Let us show the preservation for i^+. Since the verification is Zariski local with respect to X, we may assume that X is affine. Then the verification is reduced to the case where Z is defined by a function $f\in\mathcal{O}_X$. In this case, we know that for any $\mathscr{M}\in\mathrm{Hol}(X)$, $\mathscr{H}^k i^+\mathscr{M}=0$ for $i\neq 0,-1$.

Since i^+_W is right exact by [AC1, 1.3.2 (ii)], the preservation for ${}^{\mathrm{c}}D^{\leq 0}$ is easy. Let us show the preservation for ${}^{\mathrm{c}}D^{\geq 0}$. By the extension property, it suffices to check for $\mathscr{M}$ of the form $\mathscr{M}=\mathscr{N}[-n]$ such that $\mathscr{N}\in\mathrm{Hol}(X)$ and $\dim\bigl(\mathrm{Supp}(\mathscr{N})\bigr)\leq n$. By using the extension property again, this is reduced even to the case where $\mathscr{N}$ is irreducible. In particular, we may assume that the support of $\mathscr{N}$ is irreducible. In this case, we have two possibilities: $\mathrm{Supp}(\mathscr{N})\subset Z$ or $\mathrm{Supp}(\mathscr{N})\not\subset Z$. When $\mathrm{Supp}(\mathscr{N})\subset Z$, we get $\mathscr{H}^{-1}i^+(\mathscr{M})=0$, and the other case follows since $\dim\bigl(\mathrm{Supp}(\mathscr{N})\cap Z\bigr)<\dim\bigl(\mathrm{Supp}(\mathscr{N})\bigr)$.

Let us show the lemma for $j_!$ by using the induction on the dimension of X. When j is affine, the claim follows easily since $j_!$ is exact by [AC1, 1.3.13]. In general, take $\mathscr{M}\in{}^{\mathrm{c}}D^{\star}(U)$. Let $j'\colon V\hookrightarrow U$ be an affine open dense subscheme, and let i' be the closed immersion into U defined by the complement. Consider the triangle $j'_! j'^+ \mathscr{M} \to \mathscr{M} \to i'_+ i'^+ \mathscr{M} \xrightarrow{+1}$. Since $j \circ j'$ is affine, $j_! j'_! j'^+ \mathscr{M}$ is in ${}^{\mathrm{c}}D^\star(X)$, and $j_! i'_+ i'^+ \mathscr{M}$ is in ${}^{\mathrm{c}}D^\star(X)$ as well by the induction hypothesis together with the lemma for i'^+ we have already treated. Using the extension property, we conclude. □

1.3.3. **Proposition.** *The categories ${}^{\mathrm{c}}D^{\geq 0}$ and ${}^{\mathrm{c}}D^{\leq 0}$ define a t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$.*

Proof. We put $D(X) := D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. Let U be an open subset of X, and let Z be its complement. Put $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$. For $\star \in \{\geq 0, \leq 0\}$, $\mathscr{M}$ is in ${}^{\mathrm{c}}D^\star(X)$ if and only if $i^+(\mathscr{M})$ and $j^+(\mathscr{M})$ are in ${}^{\mathrm{c}}D^\star(Z)$ and ${}^{\mathrm{c}}D^\star(U)$, respectively. This follows by the extension property and Lemma 1.3.2.

Now, we proceed as [KW, p.143]. We use the induction on the dimension of X. We may assume X to be reduced by Lemma 1.1.3. It suffices to check, for a smooth open affine subscheme $j\colon U \hookrightarrow X$ equidimensional of dimension $\dim(X)$, that the restriction of ${}^{\mathrm{c}}D^{\geq 0}$ and ${}^{\mathrm{c}}D^{\leq 0}$ to the subcategory

$$T(X,U) := \bigl\{\mathscr{E} \in D(X) \mid \mathscr{H}^i(j^+\mathscr{E}) \text{ is smooth on } U \text{ for any } i\bigr\}$$

defines a t-structure, since $\bigcup_U T(X,U) = D(X)$. Let $i\colon Z \hookrightarrow X$ be the complement of U. By the observation above, $\mathscr{M} \in T(X,U)$ is in ${}^{\mathrm{c}}D^\star(X)$ if and only if $j^+(\mathscr{M})$ and $i^+(\mathscr{M})$ are in ${}^{\mathrm{c}}D^\star(U)$ and ${}^{\mathrm{c}}D^\star(Z)$, respectively. We note that ${}^{\mathrm{c}}D^\star(Z)$ defines a t-structure on $D(Z)$ by induction hypothesis.

Let us check the axioms of the t-structure [BBD, 1.3.1]. Axiom (ii) is obvious, and axiom (iii) can be shown by a similar argument to [KW, pp.140, 141] using the t-structures of $D(Z)$. Let us check (i). By *dévissage* using the localization triangle $j_! j^+ \to \mathrm{id} \to i_+ i^+ \xrightarrow{+1}$ twice and by the induction hypothesis, we only have to show that $\mathrm{Hom}(i_+\mathscr{B}, j_!\mathscr{C}) = 0$ when $\mathscr{B} \in {}^{\mathrm{c}}D^{<0}(Z)$ and $\mathscr{C} \in {}^{\mathrm{c}}D^{\geq 0}(U)$ such that $\mathscr{H}^i(\mathscr{C})$ is smooth for any i. Then, $\mathscr{H}^{k-1}(i_+\mathscr{B}) = 0$ for $k > \dim(Z)$. On the other hand, $j_!$ is exact by [AC1, 1.3.13] since j is affine, and thus $\mathscr{H}^k(j_!\mathscr{C}) = 0$ for $k < \dim(U)$ by the smoothness of $\mathscr{H}^i(\mathscr{C})$, so the claim follows. □

Definition. The t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$ defined in the proposition is called the *constructible t-structure*, and briefly, *c-t-structure*. The heart of the t-structure is denoted by $\mathrm{Con}(X/K)$, and it is called the category of *constructible modules*. The cohomology functor for this t-structure is denoted by ${}^{\mathrm{c}}\mathscr{H}^*$.

Remark. Our constructible t-structure can be regarded as a generalization of *perverse t-structure* introduced in [Le] and also as a p-adic analogue of the t-structure defined in [Ka].

1.3.4. **Lemma.** *Let $f\colon X \to Y$ be a morphism between realizable schemes.*

(i) *The functor f^+ is c-t-exact, and f_+ is left c-t-exact. Moreover, the pair $({}^{\mathrm{c}}\mathscr{H}^0 f^+, {}^{\mathrm{c}}\mathscr{H}^0 f_+)$ is an adjoint pair.*

(ii) *When $f =: i$ is a closed immersion, i_+ is c-t-exact and ${}^{\mathrm{c}}\mathscr{H}^0 i^!$ is left c-t-exact. Moreover, $(i_+, {}^{\mathrm{c}}\mathscr{H}^0 i^!)$ is an adjoint pair.*

(iii) *When $f =: j$ is an open immersion, $j_!$ is c-t-exact, and $(j_!, j^+)$ is an adjoint pair.*

Proof. Claims (ii) and (iii) are nothing but Lemma 1.3.2, and we reproduced these for the record.

Let us show (i). We only need to show the exactness of f^+. The verification is Zariski local, so we may assume X and Y to be realizable. Thus we can take the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, hook, "i'"] \arrow[d, "f"'] & P \arrow[d, "\tilde{f}"] \\ Y \arrow[r, hook, "i"'] & Q, \end{tikzcd}$$

where horizontal morphisms are closed immersions, Q is smooth, and $\tilde{f}$ is smooth. By (ii), which we have already verified, it suffices to show the claim for $(i \circ f)^+$. By Lemma 1.3.2, we already know that i'^+ is c-t-exact. Thus, it remains to show that $\tilde{f}^+$ is c-t-exact, which we can check easily using [AC1, 1.3.2 (i)]. $\square$

1.3.5. **Lemma.** *Let X be an irreducible realizable scheme. Let $\mathscr{M}$ be a constructible module on X such that* $\mathrm{Supp}(\mathscr{M}) = X$. *Then there exists an open dense subscheme $j\colon U \hookrightarrow X$ such that $j^+\mathscr{M}$ is in* $\mathrm{Sm}(U/K)$. *The rank of $j^+\mathscr{M}$ is called the* generic rank of $\mathscr{M}$.

Proof. For any complex in $D^{\mathrm{b}}_{\mathrm{hol}}(X)$, there exists an open dense subscheme $j\colon U \hookrightarrow X$ such that the cohomology modules of $j^+\mathscr{M}$ is smooth. $\square$

1.3.6. **Lemma.** *Let X be a realizable scheme. Then the category* $\mathrm{Con}(X/K)$ *is noetherian.*

Proof. Since $\mathrm{Hol}(X/K)$ is essentially small by Lemma 1.2.7, so is $D^{\mathrm{b}}(\mathrm{Hol}(X))$. Since $\mathrm{Con}(X)$ is a full subcategory, it is also essentially small.

Let us show that the category is noetherian. It suffices to show the claim for each irreducible component of X, so we may assume X to be irreducible. Assume X is smooth, and let $\mathscr{M}$ be a smooth constructible module on X. We claim that for any submodule $\mathscr{N}$ of $\mathscr{M}$, there exists an open dense subscheme U such that $\mathscr{N}$ is a nonzero smooth constructible module on U. Assume the contrary. Then there exists a nowhere dense closed subset $i\colon Z \hookrightarrow X$ such that we have the nonzero homomorphism $i_+i^+(\mathscr{N}) \to \mathscr{M}$. Shrinking X if necessary, we may assume that both X and Z are smooth and $i^+\mathscr{N}$ is smooth on Z. Taking the adjoint, we get a nonzero homomorphism $i^+\mathscr{N} \to i^!\mathscr{M}$. By [A1, 5.6], we have $i^!\mathscr{M} \cong i^+\mathscr{M}(-d)[-2d]$ where d is the codimension of Z in X, which is impossible.

We use noetherian induction on the support of $\mathscr{M}$. We may assume that X is reduced. Moreover, we may assume $\mathrm{Supp}(\mathscr{M}) = X$, otherwise, we can conclude by the induction hypothesis. Let $\mathscr{M}$ be a constructible module, and let $\{\mathscr{M}_i\}_{i\in\mathbb{N}}$ be an ascending chain of submodules of $\mathscr{M}$. There exists N such that the generic rank (cf. Lemma 1.3.5) of $\mathscr{M}_i$ is the same for any $i \geq N$. Since it suffices to show that the ascending chain $\{\mathscr{M}_i/\mathscr{M}_N\}_{i\geq N}$ is stationary in $\mathscr{M}/\mathscr{M}_N$, we may assume that $\mathrm{Supp}(\mathscr{M}_i) \subset \mathrm{Supp}(\mathscr{M})$ is nowhere dense. Let U be an open dense smooth subscheme of X such that $\mathscr{M}$ is smooth on U. By what we have shown, $\mathscr{M}_i$ is 0 on U. Let $i\colon Z \hookrightarrow X$ be the complement. Then by the c-t-exactness of i_+ and i^+ and the induction hypothesis, $\mathscr{M}_i \cong i_+i^+\mathscr{M}_i$ is stationary in $i_+i^+\mathscr{M}$, as required. $\square$

Remark. Contrary to $\mathrm{Hol}(X/K)$, $\mathrm{Con}(X/K)$ is not artinian. Indeed, let X be a smooth realizable scheme, and take a descending sequence of open subschemes

$X \supset U_1 \supsetneq U_2 \supsetneq \cdots$. Denote by $j_i \colon U_i \hookrightarrow X$ the inclusion. For any $\mathscr{M} \in \mathrm{Sm}(X/K)$, we have $\mathscr{M} \supset j_{1!}j_1^+(\mathscr{M}) \supsetneq j_{2!}j_2^+(\mathscr{M}) \supsetneq \cdots$, and the claim follows.

1.3.7. **Lemma.** *Let X be a realizable scheme. For a closed point $x \in X$, denote by $i_x \colon \{x\} \to X$ the closed immersion.*

(i) *For $\mathscr{F} \in \mathrm{Con}(X)$, $\mathscr{F} = 0$ if and only if $i_x^+(\mathscr{F}) = 0$ for any closed point x. In particular, a homomorphism ϕ in $\mathrm{Con}(X)$ is 0 if and only if $i_x^+(\phi) = 0$ for any closed point x.*

(ii) *Let $f \colon s' \to s$ be a morphism of points (i.e., connected schemes of dimension 0 of finite type over k). Then f^+ is faithful and conservative.*

Proof. For (i), use [AC1, 1.3.11], and (ii) is left to the reader. □

1.3.8. **Lemma.** *Let $f \colon X \to Y$ be a morphism of realizable schemes such that for any $y \in Y$, the dimension of the fiber $f^{-1}(y)$ is $\leq d$. Then for any $\mathscr{M} \in \mathrm{Con}(X)$, ${}^{\mathrm{c}}\mathscr{H}^i f_!(\mathscr{M}) = 0$ for $i > 2d$ and $i < 0$.*

Proof. By Lemma 1.3.7, it suffices to show that for any closed point $y \in Y$, $i_y^+\,{}^{\mathrm{c}}\mathscr{H}^i f_!(\mathscr{M}) = 0$ for $i \notin [0, 2d]$. By the c-t-exactness of i_y^+ and base change, it is reduced to showing that ${}^{\mathrm{c}}\mathscr{H}^i f_{y!}(\mathscr{M}) = 0$ for $i \notin [0, 2d]$ where $f_y \colon X \times_Y \{y\} \to \{y\}$. Since over a point, c-t-structure and the usual t-structure coincide, it remains to show that if X is a realizable scheme of dimension d, and $\mathscr{M} \in \mathrm{Con}(X)$, then $\mathscr{H}^i f_!\mathscr{M} = 0$ for $i \notin [0, 2d]$. We use the induction on the dimension of X. When the dimension of X is 0, then the verification is easy. This in particular implies that $f_!$ is c-t-exact when f is quasi-finite. Let us assume that the lemma holds for $d < N$. By Lemma 1.1.3, we may assume that X is reduced. By the c-t-exactness of Lemma 1.3.4 and the induction hypothesis, we may shrink X by its open dense subscheme. Thus, we may assume that there exists a finite morphism $g \colon X \to \mathbb{A}^N$. Since $g_!\mathscr{M}$ is constructible by the c-t-exactness of $g_!$ that we have already verified, it suffices to check the claim for $X = \mathbb{A}^N$. Shrinking X further, we may assume that there exists a divisor Z of $P := \mathbb{P}^N$ such that $X = P \setminus Z$ and $\mathscr{M} = \mathscr{N}[-N]$ where $\mathscr{N} \in \mathrm{Hol}(X)$ by Lemma 1.3.5. Then the lemma follows by the definition of $f_!$ as well as [Hu, 5.4.1]. □

1.3.9. **Lemma.** *Let $\mathscr{M} \in \mathrm{Con}(X)$, and let $\{u_i \colon U_i \hookrightarrow X\}$ be a finite open covering of X. Let $u_{ij} \colon U_i \cap U_j \hookrightarrow X$ be the immersion. Then the following sequence is exact in $\mathrm{Con}(X)$:*

$$\bigoplus_{i,j} u_{ij!}u_{ij}^+\mathscr{M} \to \bigoplus_i u_{i!}u_i^+\mathscr{M} \to \mathscr{M} \to 0.$$

Proof. To check the exactness, it suffices to check it after taking i_x^+ for each closed point $x \in X$ by Lemma 1.3.7. By the commutativity of i_x^+ and $u_{\star!}$, the verification is just a combinatorial problem. □

1.3.10. We use the category $\mathrm{Ind}(\mathrm{Con}(X))$ later. Let us prepare some properties of this category. Let $f \colon X \to Y$ be a morphism between realizable schemes. Since f^+ is c-t-exact, we have a functor $f^+ \colon \mathrm{Ind}(\mathrm{Con}(Y)) \to \mathrm{Ind}(\mathrm{Con}(X))$.

Lemma. *We use the notation of Lemma 1.3.7.*

(i) *Let $\mathscr{F} \in \mathrm{Ind}(\mathrm{Con}(X))$. Assume that $i_x^+\mathscr{F} = 0$ for any closed point x of X. Then $\mathscr{F} = 0$.*

(ii) *Let* $g\colon s' \to s$ *be a morphism of points. Then for* $\mathscr{F} \in \mathrm{Ind}(\mathrm{Con}(s))$, $\mathscr{F} = 0$ *if and only if* $g^+(\mathscr{F}) = 0$. *In particular, a homomorphism* ϕ *in* $\mathrm{Ind}(\mathrm{Con}(s))$ *is an isomorphism if and only if* $g^+(\phi)$ *is also.*

Proof. Let us show (i). Write $\mathscr{F} = \varinjlim_{i\in I} \mathscr{F}_i$ where I is a small filtrant category and $\mathscr{F}_i \in \mathrm{Con}(X)$. Fix $i \in I$, and let $\mathscr{E}_j := \mathrm{Ker}(\mathscr{F}_i \to \mathscr{F}_j)$. Since $\mathrm{Con}(X)$ is noetherian by Lemma 1.3.6, there exists $j_0 \in I$ such that $\mathscr{E}_{j_0} = \mathscr{E}_j$ for any $j \geq j_0$. Now, we have $i^+_x(\mathscr{F}_i/\mathscr{E}_{j_0}) = i^+_x \varinjlim_j (\mathscr{F}_i/\mathscr{E}_j) \hookrightarrow i^+_x\mathscr{F} = 0$, thus $\mathscr{E}_{j_0} = \mathscr{F}_i$ by Lemma 1.3.7. This shows that the homomorphism $\mathscr{F}_i \to \mathscr{F}_{j_0}$ is 0, and the claim follows.

Let us show (ii). Let $\mathscr{F} = \varinjlim \mathscr{F}_i$ where I is a small filtrant and $\mathscr{F}_i \in \mathrm{Con}(s)$. For each $i \in I$, there exists $j \in I$ such that $g^+\mathscr{F}_i \to g^+\mathscr{F}_j$ is 0. Thus, by Lemma 1.3.7, we get that $\mathscr{F}_i \to \mathscr{F}_j$ is 0 as well. □

1.4. **Extension of scalars and Frobenius structures.** So far, the coefficient categories we have treated (e.g., $\mathrm{Hol}(X/K)$ or $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$) are K-additive. For the Langlands correspondence, we need to consider L-coefficients with Frobenius structure where L is an algebraic field extension of K. We introduce such categories in this subsection when the extension is finite. Scalar extended categories of isocrystals have already been introduced in [AM, 7.3], and the idea of our construction is essentially the same, but we hope that the usability is improved.

Extension of scalars.

1.4.1. Let K be an arbitrary field, and let $\mathcal{A}$ be a K-additive category. We take a finite field extension L of K. We define the category $\mathcal{A}_L$ as follows. The objects consist of pairs (X,ρ) such that $X \in \mathrm{Ob}(\mathcal{A})$, and a K-algebra homomorphism $\rho\colon L \to \mathrm{End}(X)$, called the *$L$-structure* (cf. [DM, after Remark 3.10]). The morphisms are morphisms in $\mathcal{A}$ compatible with L-structures, or more precisely,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}_L}&\big((X,\rho),(X',\rho')\big)\\ &= \big\{f \in \mathrm{Hom}_{\mathcal{A}}(X,X') \mid f\circ\rho(x) = \rho'(x)\circ f \text{ for any } x \in L\big\}.\end{aligned}$$

We have the forgetful functor $\mathrm{for}_L\colon \mathcal{A}_L \to \mathcal{A}$. Let $X \in \mathcal{A}$. Then we define $X \otimes_K L \in \mathcal{A}_L$ as follows: Take a basis $x_1,\dots,x_d$ of L over K. Then $X\otimes_K L := (\bigoplus_{i=1}^d X\otimes x_i, \rho')$ such that for $x \in L$, write $x\cdot x_k = \sum a_i x_i$ with $a_i \in K$, and $\rho'(x)|_{X\otimes x_k} := \sum \rho(a_i)\otimes x_i$, where $\rho(a_i)$ denotes the structural action of $a_i \in K$ on X. We can check easily that this does not depend on the choice of the basis of L up to canonical isomorphism. We denote by $\iota_L := (-)\otimes_K L\colon \mathcal{A} \to \mathcal{A}_L$. Moreover, if $\mathcal{A}$ is abelian, then $\mathcal{A}_L$ is abelian as well, and ι_L is exact.

For $X \in \mathcal{A}$ and $Y \in \mathcal{A}_L$, we have

$$\mathrm{Hom}_{\mathcal{A}}(X, \mathrm{for}_L(Y)) \cong \mathrm{Hom}_{\mathcal{A}_L}(\iota_L(X), Y),$$

in other words, we have an adjoint pair $(\iota_L, \mathrm{for}_L)$. Thus, if $\mathcal{A}$ is an abelian category, the functor for_L sends injective objects in $\mathcal{A}_L$ to injective objects in $\mathcal{A}$.

Let $f\colon \mathcal{A} \to \mathcal{B}$ be a K-additive functor between K-additive categories. Then there exists a unique functor $f_L\colon \mathcal{A}_L \to \mathcal{B}_L$ which is compatible with both ι_L and for_L. If $\mathcal{A}$ and $\mathcal{B}$ are abelian, f_L is left (resp. right) exact, if f is also.

1.4.2. Let $\mathcal{A}$ be a K-additive category. Let $X, Y \in \mathcal{A}_L$. On the abelian group $\mathrm{Hom}(\mathrm{for}_L(X), \mathrm{for}_L(Y))$, we endow with $L \otimes_K L$-module structure as follows: we define the left L-structure by $(a \cdot \phi)(x) := a(\phi(x))$ and the right L-structure by $(\phi \cdot a)(x) = \phi(ax)$. For $a \in K$, both L-structures are compatible, and we get the $L \otimes_K L$-module structure. This $L \otimes_K L$-module is denoted by $\mathrm{Hom}_K(X, Y)$. By definition, we have

$$\mathrm{Hom}(X, Y) \cong \bigl\{\phi \in \mathrm{Hom}_K(X, Y) \mid (a \otimes 1)\phi = (1 \otimes a)\phi = 0 \text{ for any } a \in L\bigr\}.$$

Note that if L/K is a separable extension, then $L \otimes_K L$ is a product of fields, and any $L \otimes_K L$-module is flat.

Lemma. *Let L/K be a separable extension, and let M be an $L \otimes_K L$-module. Put $I := \mathrm{Ker}(L \otimes_K L \to L)$. Then we have a canonical isomorphism*

$$M_0 := \bigl\{m \in M \mid a \cdot m = 0 \text{ for any } a \in I\bigr\} \xrightarrow{\sim} M/IM.$$

Proof. This follows from the following more general fact: Let $i\colon Z \hookrightarrow X$ be a closed immersion of schemes, and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the composition $\underline{\Gamma}_Z(\mathcal{M}) \to \mathcal{M} \to i_* i^*(\mathcal{M})$ is an isomorphism if $X = Z \sqcup (X \setminus Z)$ as schemes. □

Corollary. *Let L/K be a separable extension. Then, for $X, Y \in \mathcal{A}_L$, we have a canonical isomorphism*

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} L \otimes_{L \otimes_K L} \mathrm{Hom}_K(X, Y).$$

1.4.3. Now, let $\mathcal{A}$ be a K-abelian category. Assume L/K is a separable extension, and consider the derived category $D(\mathcal{A}_L)$. We have the following functor

$$\mathrm{Hom}^\bullet_K\colon C(\mathcal{A}_L)^\circ \times C(\mathcal{A}_L) \to C(L \otimes_K L), \quad (X^\bullet, Y^\bullet) \mapsto \prod_{i \in \mathbb{Z}} \mathrm{Hom}_K\bigl(X^i, Y^{i+\bullet}\bigr),$$

and the differential is defined as in [Ha, I.6]. Since for_L sends injective objects to injective objects, we can take the associated derive functor and get $\mathbb{R}\mathrm{Hom}^\bullet_K(-,-)$ as in [Ha, II.3]. We have the following:

Lemma. *Let $X \in D(\mathcal{A}_L)$, $Y \in D^+(\mathcal{A}_L)$. Then we have an isomorphism*

$$\mathbb{R}\mathrm{Hom}(X, Y) \xrightarrow{\sim} L \otimes_{L \otimes_K L} \mathbb{R}\mathrm{Hom}^\bullet_K(X, Y).$$

Proof. Use Corollary 1.4.2. □

Corollary. *The functor $D^{\mathrm{b}}(\mathcal{A}_L) \to D^{\mathrm{b}}(\mathcal{A})_L$ is fully faithful.*

Proof. For $X \in D(\mathcal{A}_L)$, let us denote by X' the image in $D(\mathcal{A})_L$. We have

$$\mathbb{R}^i\mathrm{Hom}^\bullet_K(X, Y) \cong \mathrm{Hom}_K(X', Y'[i])$$

as $L \otimes_K L$-modules. This shows that the functor is fully faithful by the lemma above and Corollary 1.4.2. □

Remark. This corollary shows that if $X, Y \in D^{\mathrm{b}}(\mathcal{A}_L)$ are isomorphic in $D^{\mathrm{b}}(\mathcal{A})_L$, then they are isomorphic in $D^{\mathrm{b}}(\mathcal{A}_L)$. For example, assume we are given two K-additive functors $F, G\colon D^{\mathrm{b}}(\mathcal{A}) \to D^{\mathrm{b}}(\mathcal{B})$ and a K-additive morphism $\alpha\colon F \to G$ of functors. Furthermore, assume that these functors have L-additive liftings $\widetilde{F}, \widetilde{G}\colon D^{\mathrm{b}}(\mathcal{A}_L) \to D^{\mathrm{b}}(\mathcal{B}_L)$. Then the full faithfulness implies that α can be lifted to $\widetilde{\alpha}\colon \widetilde{F} \to \widetilde{G}$ and that $\widetilde{\alpha}$ is an isomorphism if α is also.

1.4.4. **Lemma.** *Let $\mathcal{A}$ be a K-additive noetherian category. Then we have a canonical equivalence* $\mathrm{Ind}(\mathcal{A}_L) \xrightarrow{\sim} \mathrm{Ind}(\mathcal{A})_L$.

Proof. It is easy to check that it is fully faithful. Let (X, ρ) be an object in $\mathrm{Ind}(\mathcal{A})_L$. We may write $X = \varinjlim_{i \in I} X_i$ where $X_i \in \mathcal{A}$ and $X_i \subset X$ by [De2, 4.2.1 (ii)]. Let $x \in L$ such that $K[x] = L$, and let $[L : K] =: d$. Put $X'_i := \sum_{j=0}^{d-1} \rho(x^j)(X_i)$ where the sum is taken in X. Then X'_i is stable under the action of L, and it defines an object in $\mathcal{A}_L$. The limit $\varinjlim_{i \in I}(X'_i, \rho)$ is sent to (X, ρ). □

1.4.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a K-additive functor between K-abelian categories. Assume that F is left exact and that $\mathcal{A}_L$ has enough injectives. Note that $\mathcal{A}$ also has enough injectives since $\mathrm{id} \hookrightarrow \mathrm{for}_L \circ \iota_L$ and for_L preserves injective objects. Since, again, for_L preserves injective objects and commutes with F, the functors $\mathbb{R}F$ and for_L commute. Moreover, $\mathbb{R}F$ and ι_L commute. Indeed, since F and ι_L commute, it suffices to show that for an injective object I in $\mathcal{A}$, $\mathbb{R}^i F(\iota_L(I)) = 0$ for $i > 0$. For this, it suffices to show that $\mathrm{for}_L \circ (\mathbb{R}^i F) \circ \iota_L(I) = 0$. We have

$$\mathrm{for}_L \circ (\mathbb{R}^i F) \circ \iota_L(I) \cong (\mathbb{R}^i F) \circ \mathrm{for}_L \circ \iota_L(I) = 0,$$

where the second equality holds by the fact that $\mathrm{for}_L \circ \iota_L(I)$ is a finite direct sum of copies of I and thus injective.

Frobenius structure.

1.4.6. Now, let us consider the *Frobenius structure*. We fix an automorphism $\sigma\colon K \to K$, and put $K_0 := K^{\sigma=1}$. Let $\mathcal{A}$ be a K-additive category, and let $F^*\colon \mathcal{A} \to \mathcal{A}$ be a σ-semilinear functor; namely, for $X, Y \in \mathcal{A}$ the homomorphism $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(F^*X, F^*Y)$ is σ-semilinear. We define the category F-$\mathcal{A}$ to be the category of pairs (X', Φ) such that $X' \in \mathrm{Ob}(\mathcal{A})$, and an isomorphism $\Phi\colon F^*X' \xrightarrow{\sim} X'$ called the *Frobenius structure*.[2] Morphisms in F-$\mathcal{A}$ are morphisms in $\mathcal{A}$ respecting Φ. Then the category F-$\mathcal{A}$ is K_0-additive.

There exists the forgetful functor

$$\mathrm{for}_F\colon F\text{-}\mathcal{A} \to \mathcal{A}; \quad (X', \Phi) \mapsto X'.$$

This functor is faithful. For X, Y in F-$\mathcal{A}$, we have a K_0-linear endomorphism

$$\begin{aligned} F\colon \mathrm{Hom}\bigl(\mathrm{for}_F(X), \mathrm{for}_F(Y)\bigr) &\to \mathrm{Hom}\bigl(\mathrm{for}_F(F^*X), \mathrm{for}_F(F^*Y)\bigr) \\ &\cong \mathrm{Hom}\bigl(\mathrm{for}_F(X), \mathrm{for}_F(Y)\bigr), \end{aligned} \tag{1.4.6.1}$$

where the last isomorphism is induced by the Frobenius structures of X and Y.

Now, assume that $\mathcal{A}$ is abelian. Then F-$\mathcal{A}$ is abelian as well. Indeed, assume we are given a morphism $f\colon X \to Y$ in F-$\mathcal{A}$. Then the Frobenius structure on X induces a Frobenius structure on $\mathrm{Ker}(\mathrm{for}_F(f))$, which is the kernel of f. This construction shows that $\mathrm{for}_F(\mathrm{Ker}(f)) \cong \mathrm{for}_F(\mathrm{Ker}(f))$. Replacing Ker by Coker, we get the same result. Thus, we get the claim.

[2] In [Ber3, 4.5.1], Frobenius structure is defined to be an isomorphism with the opposite direction $\Psi\colon X' \xrightarrow{\sim} F^*X'$. To be compatible with that of rigid cohomology, we choose the other convention. See footnote (1) in 2.7 of [A1].

The construction shows that for_F is an exact functor, and the following diagram is commutative:

$$\begin{tikzcd} D(F\text{-}\mathcal{A}) \arrow[r, "\mathscr{H}^i"] \arrow[d, "\mathrm{for}_F"'] & F\text{-}\mathcal{A} \arrow[d, "\mathrm{for}_F"] \\ D(\mathcal{A}) \arrow[r, "\mathscr{H}^i"'] & \mathcal{A}. \end{tikzcd}$$

Moreover, if $\mathrm{for}_F(X) = 0$, then $X = 0$. This implies that a sequence C in F-$\mathcal{A}$ is exact if and only if the sequence $\mathrm{for}_F(C)$ is exact.

Finally, let $(\mathcal{A}, F)$ and $(\mathcal{B}, G)$ be K-additive categories with a semilinear endofunctor. Assume we are given a functor $f\colon \mathcal{A} \to \mathcal{B}$ and an equivalence $f \circ F \cong G \circ f$. Then we have a canonical functor $\widetilde{f}\colon F\text{-}\mathcal{A} \to G\text{-}\mathcal{B}$ such that $f \circ \mathrm{for}_F \cong \mathrm{for}_G \circ \widetilde{f}$.

1.4.7. Now, assume further that F^* is an equivalence of categories and that $\mathcal{A}$ is a Grothendieck category. We define

$$(-)_F\colon \mathcal{A} \to F\text{-}\mathcal{A};\ X \mapsto X_F := \bigoplus_{n\in\mathbb{Z}} (F^*)^n X.$$

Then it can be checked easily that $((-)_F, \mathrm{for}_F)$ is an adjoint pair. Furthermore, $(-)_F$ is exact, since the functor $\mathrm{for}_F \circ (-)_F$ is exact. Thus for_F sends injective objects to injective objects. Filtrant inductive limits are representable in F-$\mathcal{A}$ and commute with for_F. Let G be a generator of $\mathcal{A}$. Then G_F is a generator of F-$\mathcal{A}$. Indeed, assume given two morphisms $f, g\colon X \to Y$ in F-$\mathcal{A}$. Then there exists $\phi\colon G \to \mathrm{for}_F(X)$ such that $\phi \circ \mathrm{for}_F(f) \neq \phi \circ \mathrm{for}_F(g)$. By taking the adjoint, we have $\phi_F\colon G_F \to X$. Then $\phi_F \circ f \neq \phi_F \circ g$ as required. This shows that F-$\mathcal{A}$ is a Grothendieck category as well.

In the following, we often assume:

(*) F^* is an equivalence, and $\mathcal{A}$ is a noetherian category.

This assumption implies that $\mathrm{Ind}(\mathcal{A})$ is a Grothendieck category endowed with semilinear autofunctor F^*. Thus by the result above, the category F-$\mathrm{Ind}(\mathcal{A})$ is a Grothendieck category.

1.4.8. We retain the assumption (*) in 1.4.7. Take X, Y in F-$\mathrm{Ind}(\mathcal{A})$. Then the homomorphism F in (1.4.6.1) is an isomorphism, and $\mathrm{Hom}\big(\mathrm{for}_F(X), \mathrm{for}_F(Y)\big)$ is a $K_0[F^{\pm1}]$-module. Here the $K_0[F^{\pm1}]$-module structure is defined so that $F \cdot \varphi := F \circ \varphi \circ F^{-1}$ for $\varphi\colon \mathrm{for}_F(X) \to \mathrm{for}_F(Y)$. This module is denoted by $\mathrm{Hom}_\rho(X, Y)$. On the other hand, for a $K_0[F^{\pm}]$-module M and $X = (X', \Phi) \in F$-$\mathrm{Ind}(\mathcal{A})$, we define $M \otimes_{K_0} X$ in F-$\mathrm{Ind}(\mathcal{A})$ as follows: Write $M = \varinjlim M_i$ as K_0-vector spaces such that M_i is finite dimensional. As an object in $\mathrm{Ind}(\mathcal{A})$, it is $\varinjlim (M_i \otimes_{K_0} X')$. The Frobenius structure is defined as

$$M \otimes_{K_0} X' \xrightarrow[\sim]{(F\cdot)\otimes\Phi} M \otimes_{K_0} F^* X' \cong F^*(M \otimes_{K_0} X'),$$

where the last isomorphism follows since F^* is an exact functor and thus commutes with the functor $M_i \otimes_{K_0}$. The functor $M \otimes_{K_0}$ is exact. Now, for any $K_0[F^{\pm1}]$-module M, we have

$$\mathrm{Hom}_{F\text{-}\mathrm{Ind}(\mathcal{A})}(M \otimes_{K_0} X, Y) \xrightarrow{\sim} \mathrm{Hom}_{K_0[F^{\pm1}]}\big(M, \mathrm{Hom}_\rho(X, Y)\big).$$

This shows that if Y is an injective object, $\mathrm{Hom}_\rho(X, Y)$ is an injective $K_0[F^{\pm1}]$-module.

As in [Ha, I.6], for $X, Y \in C(F\text{-Ind}(\mathcal{A}))$, we define a complex of $K_0[F^{\pm 1}]$-modules $\mathrm{Hom}^\bullet_\rho(X,Y)$. We can take the derived functor, and we get

$$\mathbb{R}\mathrm{Hom}^\bullet_\rho \colon D(F\text{-Ind}(\mathcal{A}))^\circ \times D^+(F\text{-Ind}(\mathcal{A})) \to D(K_0[F^{\pm 1}]).$$

Abusing notation, we write $\mathrm{Hom}_\rho := \mathbb{R}^0\mathrm{Hom}^\bullet_\rho$. Let $\varphi\colon \mathrm{Spec}(K_0[F^{\pm 1}]) \to \mathrm{Spec}(K_0)$ be the canonical morphism. We have the canonical isomorphism $\varphi_*\mathrm{Hom}_\rho(X,Y) \cong \mathrm{Hom}(\mathrm{for}_F(X), \mathrm{for}_F(Y))$ as K_0-vector spaces.

Lemma. *We regard K_0 as a $K_0[F^{\pm}]$-module such that F acts trivially. For $X, Y \in D(F\text{-Ind}(\mathcal{A}))$ such that $X \in D^-$, $Y \in D^+$, we have*

$$\mathbb{R}\mathrm{Hom}_{K_0[F^{\pm 1}]}\big(K_0, \mathbb{R}\mathrm{Hom}^\bullet_\rho(X,Y)\big) \cong \mathbb{R}\mathrm{Hom}(X,Y).$$

Proof. We have a canonical isomorphism

$$\mathrm{Hom}_{K_0[F^{\pm 1}]}\big(K_0, \mathrm{Hom}_\rho(X,Y)\big) \cong \mathrm{Hom}(X,Y).$$

Since the functor $\mathrm{Hom}_\rho(X, -)$ preserves injective objects, we get the lemma. □

For a $K_0[F^{\pm}]$-module M, we put

$$M^F := \mathrm{Hom}_{K_0[F^{\pm}]}(K_0, M), \qquad M_F := \mathrm{Ext}^1_{K_0[F^{\pm}]}(K_0, M).$$

Corollary. *Let $X, Y \in D(F\text{-Ind}(\mathcal{A}))$ such that $X \in D^-$ and $Y \in D^+$. Then there exists the short exact sequence*

$$0 \to \mathrm{Hom}_\rho(X, Y[-1])_F \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}_\rho(X,Y)^F \to 0.$$

1.4.9. Let $\mathcal{A}$ be a K-additive category, and let $F^*\colon \mathcal{A} \to \mathcal{A}$ be a σ-semilinear functor. We fix a finite field extension L and an isomorphism $\sigma_L\colon L \to L$ compatible with σ. Put $L_0 := L^{\sigma=1}$. We define $F^*_L\colon \mathcal{A}_L \to \mathcal{A}_L$ as follows: Let $\rho\colon L \to \mathrm{End}(X)$ be an object of $\mathcal{A}_L$. We have a σ-semilinear homomorphism $F^*(\rho)\colon L \xrightarrow{\rho} \mathrm{End}(X) \to \mathrm{End}(F^*X)$. We put

$$F^*_L(\rho) := F^*(\rho) \circ \sigma_L^{-1} \colon L \to \mathrm{End}(F^*X).$$

This is a homomorphism of K-algebras. We define $F^*_L\colon \mathcal{A}_L \to \mathcal{A}_L$ by sending (X, ρ) to $(F^*X, F^*_L(\rho))$. The L_0-additive category $F_L\text{-}\mathcal{A}_L$ is sometimes denoted by $F\text{-}\mathcal{A}_L$. The K-additive functors $\iota_L\colon \mathcal{A} \to \mathcal{A}_L$ and $\mathrm{for}_L\colon \mathcal{A}_L \to \mathcal{A}$ induce the functors $F\text{-}\mathcal{A} \to F_L\text{-}\mathcal{A}_L$ and $F_L\text{-}\mathcal{A}_L \to F\text{-}\mathcal{A}$ which are denoted abusively by ι_L and for_L, respectively. We can check that $(\iota_L, \mathrm{for}_L)$ is an adjoint pair, and ι_L is exact. In particular, for_L preserves injective objects.

Let $\mathcal{B}$ be another K-additive category endowed with an σ-semilinear endofunctor G^*. Let $f\colon \mathcal{A} \to \mathcal{B}$ be a K-additive functor between K-additive categories compatible with F^* and G^*. Then there exists a unique functor $f_L\colon F\text{-}\mathcal{A}_L \to G\text{-}\mathcal{B}_L$ compatible with for_L and ι_L.

Remark. Assume that σ and σ_L are identity. Then $F\text{-}\mathcal{A}$ is a K-additive category, and it makes sense to consider the category $(F\text{-}\mathcal{A})_L$. We leave the reader to check that there exists a canonical equivalence $(F\text{-}\mathcal{A})_L \cong F\text{-}\mathcal{A}_L$.

Application of the theory.

1.4.10. To work with p-adic cohomology theory, we often need to fix a base as in §1.1. Let R be a complete discrete valuation ring with residue field k which is assumed perfect, and let $K = \mathrm{Frac}(R)$. We assume that there exists a positive integer s such that $\sigma\colon K \xrightarrow{\sim} K$ is the extension of a lifting $R \xrightarrow{\sim} R$ of the s-th absolute Frobenius automorphism of k. Now, we consider the following two types of data.

Geometric case: We fix a finite field extension L of K, and put $\mathfrak{T}_{\emptyset} := (k, R, K, L)$. This is called a *geometric base tuple.* We put $L_0 := L$ in this case.

Arithmetic case: We fix a finite field extension L and an automorphism $\sigma\colon L \to L$ such that $\sigma(K) = K$ and $\sigma|_K$ is a lifting of the s-th Frobenius automorphism of k. We put $\mathfrak{T}_F := (k, R, K, L, s, \sigma)$ and call this an *arithmetic base tuple.* We put $L_0 := L^{\sigma=1}$. We call the geometric base tuple (k, R, K, L) the *associated geometric base tuple.*

By "base tuple", we mean either geometric or arithmetic base tuple. For a geometric base tuple, we sometimes put an index $\cdot_{\emptyset}$, and for arithmetic base tuple, $\cdot_F$.

Definition. Let X be a realizable scheme over k. The category $\mathrm{Hol}(X/K)$ (cf. 1.1.1) is endowed with the (s-th) Frobenius pullback F^* (cf. Remark 1.1.3). Moreover, F^* induces an auto-equivalence of $\mathrm{Hol}(X/K)$. Thus, we can apply the general results and constructions developed in the preceding paragraphs.

Geometric case: Let $\mathfrak{T}_{\emptyset}$ be a geometric base tuple as above. In this case, we put

$$\mathrm{Hol}(X/\mathfrak{T}_{\emptyset}) := \mathrm{Hol}(X/K)_L, \qquad \mathrm{Isoc}^{\dagger}(X/\mathfrak{T}_{\emptyset}) := \mathrm{Isoc}^{\dagger}(X/K)_L,$$
$$M(X/\mathfrak{T}_{\emptyset}) := \mathrm{Ind}(\mathrm{Hol}(X/\mathfrak{T}_{\emptyset})), \qquad D(X/\mathfrak{T}_{\emptyset}) := D(M(X/\mathfrak{T}_{\emptyset})).$$

Arithmetic case: Let $\mathfrak{T}_F$ be an arithmetic base tuple, and let $\mathfrak{T}_{\emptyset}$ be the associated geometric tuple. In this case, we put

$$\mathrm{Hol}(X/\mathfrak{T}_F) := F\text{-}\mathrm{Hol}(X/K)_L, \qquad \mathrm{Isoc}^{\dagger}(X/\mathfrak{T}_F) := F\text{-}\mathrm{Isoc}^{\dagger}(X/K)_L,$$
$$M(X/\mathfrak{T}_F) := F_L\text{-}\mathrm{Ind}(\mathrm{Hol}(X/\mathfrak{T}_{\emptyset})), \qquad D(X/\mathfrak{T}_F) := D(M(X/\mathfrak{T}_F)).$$

If there is nothing to be confused, we sometimes omit the base tuple $/\mathfrak{T}_{\emptyset}$ or $/\mathfrak{T}_F$. We also denote $\mathrm{Hol}(X/\mathfrak{T}_{\emptyset})$ (resp. $\mathrm{Hol}(X/\mathfrak{T}_F)$), etc., by $\mathrm{Hol}(X/L_{\emptyset})$ (resp. $\mathrm{Hol}(X/L_F)$), etc.

Remark. (i) For a realizable scheme over k, recall that $\mathrm{Isoc}^{\dagger}(X/K)$ is slightly smaller than the category of overconvergent isocrystals (cf. 1.1.3). However, our category $\mathrm{Isoc}^{\dagger}(X/K_F)$ coincides with the category of overconvergent F-isocrystals F-$\mathrm{Isoc}^{\dagger}(X/K)$ defined in [Ber1, 2.3.7].

(ii) For a scheme X over a field, let us denote by $D^{\mathrm{b}}_{\mathrm{c}}(X)$ the category of constructible $\mathbb{Q}_{\ell}$-complexes. Let k be a finite field, let $\overline{k}$ be its algebraic closure, and let X be a scheme of finite type over k. Put $\overline{X} := X \otimes_k \overline{k}$. Under the philosophy of the Riemann–Hilbert correspondence, $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_{\emptyset})$ (resp. $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$) plays the role of $D^{\mathrm{b}}_{\mathrm{c}}(\overline{X})$ (resp. $D^{\mathrm{b}}_{\mathrm{c}}(X)$) in p-adic cohomology theory.

Now, note that $D^{\mathrm{b}}_{\mathrm{c}}(X)$ does not depend on the base field k. On the other hand, a priori, $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$ depends on $\mathfrak{T}_F$. However, we show in Corollary 1.4.11 that the category, in fact, does *not* depend on the choice of the base tuple under some conditions, which reinforces the justification of the analogy.

1.4.11. **Lemma.** (i) *Let* $\mathfrak{T}'_{\emptyset} = (k', R', K', K')$ *be a geometric base tuple over a tuple* $\mathfrak{T}_{\emptyset} := (k, R, K, K')$*, namely* K'/K *is a finite extension. Then, there exists a canonical equivalence* $\mathrm{Hol}(X \otimes_k k'/\mathfrak{T}'_{\emptyset}) \xrightarrow{\sim} \mathrm{Hol}(X/\mathfrak{T}_{\emptyset})$.

(ii) *Let* $\mathfrak{T}'_F = (k', R', K', K', s, \sigma)$ *be an arithmetic base tuple over a tuple* $\mathfrak{T}_F := (k, R, K, K', s, \sigma)$*. Then there exists a canonical equivalence* $\mathrm{Hol}(X \otimes_k k'/\mathfrak{T}'_F) \xrightarrow{\sim} \mathrm{Hol}(X/\mathfrak{T}_F)$.

Proof. We may reduce to the case where X can be lifted to a smooth formal scheme $\mathscr{X}$ over R. Let $\mathscr{X}' := \mathscr{X} \otimes_R R'$. There exists the functor

$$M(\mathscr{D}^{\dagger}_{\mathscr{X}'/R',\mathbb{Q}}) \to M(\mathscr{D}^{\dagger}_{\mathscr{X}/R,\mathbb{Q}})_{K'},$$

where $M(\mathscr{A})$ denotes the category of $\mathscr{A}$-modules. It is straightforward to show that this functor induces an equivalence of categories. By Remark 1.1.2, we get (i). Now, the following diagram is commutative:

$$\begin{array}{ccc} M(\mathscr{D}^{\dagger}_{\mathscr{X}'/R',\mathbb{Q}}) & \xrightarrow{\sim} & M(\mathscr{D}^{\dagger}_{\mathscr{X}/R,\mathbb{Q}})_{K'} \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ M(\mathscr{D}^{\dagger}_{\mathscr{X}'/R',\mathbb{Q}}) & \xrightarrow[\sim]{} & M(\mathscr{D}^{\dagger}_{\mathscr{X}/R,\mathbb{Q}})_{K'}. \end{array}$$

This diagram implies (ii). □

Corollary. *Assume* k *is a finite field with* $q = p^s$*-elements. Let* K' *be a finite extension of* K*, and put* $\mathfrak{T}_F := (k, R, K, K', s, \mathrm{id})$*,* $\mathfrak{T}_{K',F} := (k', R', K', K', s' := [k' : k]\cdot s, \mathrm{id})$*. Let* X *be a scheme over* k'*. Then, we have an equivalence of categories*

$$\mathrm{Hol}(X/\mathfrak{T}_F) \xrightarrow{\sim} \mathrm{Hol}(X/\mathfrak{T}_{K',F}).$$

Proof. When the extension K'/K is totally ramified, the claim follows from (ii) of the lemma. Thus, we may assume that the extension is unramified. In this case, the verification is essentially the same as [De1, 1.1.10], so we only sketch the proof. Since K'/K is assumed unramified, we have $K' \cong K \otimes_{W(k)} W(k')$. As a scheme over k, we have a canonical isomorphism $X \otimes_k k' \cong \coprod_{\sigma \in \mathrm{Gal}(k'/k)} X^{\sigma}$, where each X^{σ} is canonically isomorphic to X, and the Galois action on k' is compatible in an obvious sense. Put $\mathfrak{T}'_F := (k', R', K', K', s, \mathrm{id})$. Then by the lemma, we get $\mathrm{Hol}(X \otimes_k k'/\mathfrak{T}'_F) \xrightarrow{\sim} \mathrm{Hol}(X/\mathfrak{T}_F)$. There exists $\varphi \in \mathrm{Gal}(k'/k)$ such that, by F, each X^{σ} is sent to $X^{\sigma\cdot\varphi}$. Assume we are given $(\mathscr{M}, \Phi) \in \mathrm{Hol}(X/\mathfrak{T}_{K',F})$. For $0 \leq i < [k' : k]$, we put $(F^*)^i(\mathscr{M})$ on X^{φ^i}, which defines $\mathscr{N}$ in $\mathrm{Hol}(X' \otimes_k k'/K')$. The s'-th Frobenius structure Φ defines an s-th Frobenius structure on $\mathscr{N}$, and defines an object of $\mathrm{Hol}(X/\mathfrak{T}_F)$. It is easy to check that this correspondence yields the equivalence of categories. □

1.4.12. **Remark.** (i) Let $\mathfrak{T} := (k, R, K, L)$ and $\mathfrak{T}_0 := (k, W(k), \mathrm{Frac}(W(k)), L)$. Then Lemma 1.4.11 implies that there exists an equivalence $D(X/\mathfrak{T}_0) \cong D(X/\mathfrak{T})$. This implies that the datum K (and R) is unnecessary in defining the category $D(X/L_{\emptyset})$. Similarly, we do not need K in the definition of $D(X/L_F)$.

(ii) In the proof of the Langlands correspondence, it is convenient to work with $\overline{\mathbb{Q}}_p$-coefficient. For this, we use the 2-inductive limit method as in [De1, 1.1.3] to construct the theory. The details will be explained in 2.4.14.

1.4.13. **Definition.** Assume we are in the situation of 1.4.9, and let $X := (X', \Phi)$ be an object of $F\text{-}\mathcal{A}_L$. Assume we are given an arithmetic tuple as in 1.4.10. For an integer n, we define $X(n) := \bigl(X', p^{-sn} \cdot \Phi\bigr)$ and call it the *n-th Tate twist* of X.

1.4.14. Let $\blacktriangle \in \{\emptyset, F\}$. Let $L := \iota_L(K)$ in $\mathrm{Hol}(\mathrm{Spec}(k)/L_\blacktriangle)$. We have the left exact functor

$$\Gamma\colon M(\mathrm{Spec}(k)/L_\blacktriangle) \to \mathrm{Vec}_{L_0}; \quad \mathscr{M} \mapsto \mathrm{Hom}(L, \mathscr{M}).$$

We can take the associated derived functor $\mathbb{R}\Gamma\colon D^+(\mathrm{Spec}(k)/L_\blacktriangle) \to D^+(\mathrm{Vec}_{L_0})$.

1.5. **Trace map.** In order to establish a cycle class formalism, we need the trace map in the style of [SGA4]. In [A1, 5.5], we constructed an isomorphism $f^! \cong f^+(d)[2d]$ for a smooth morphism f of relative dimension d. However, the construction of this homomorphism is ad hoc, and it does not seem to be easy to check the properties that the trace map should satisfy, for example, transitivity. Furthermore, we need the trace maps for flat morphisms to define the cycle class map.

1.5.1. We fix $\blacktriangle \in \{\emptyset, F\}$, and we fix a base tuple $\mathfrak{T} := \mathfrak{T}_\blacktriangle$ using the notation of 1.4.10 in this subsection. Let X be a realizable scheme over k. We only treat the $L = K$ case in this subsection. This will be generalized in Theorem 2.3.34. When $\blacktriangle = \emptyset$, we denote $D^{\mathrm{b}}(X/K_\emptyset)$ simply by $D^{\mathrm{b}}(X)$, in which case the Tate twist (n) is the identity functor. When $\blacktriangle = F$, we denote $D^{\mathrm{b}}(X/K_F)$ by $D^{\mathrm{b}}(X)$. The main result of this subsection is the following theorem on the existence of the trace map:

Theorem. *Let $f\colon X \to Y$ be a morphism between realizable schemes over k. Let $\mathfrak{M}_d$ be the following set of morphisms of realizable schemes, and let $\mathfrak{M} := \bigcup_{d \geq 0} \mathfrak{M}_d$.*

> *There exists an open subscheme $U \subset Y$ such that $X \times_Y U \to U$ is flat of relative dimension d, and for each $x \in Y \setminus U$, the dimension of $f^{-1}(x)$ is $< d$.*

Then there exists a unique homomorphism $\mathrm{Tr}_f\colon f_! f^+ \mathscr{F}(d)[2d] \to \mathscr{F}$ for any $\mathscr{F}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(Y)$, called the trace map*, satisfying the following conditions.*

(Var 1) Tr_f *is functorial with respect to $\mathscr{F}$.*

(Var 2) *Consider the cartesian diagram* (1.1.3.1) *of realizable schemes. Assume $f \in \mathfrak{M}_d$. Then the following diagram is commutative:*

$$\begin{tikzcd} g^+ f_! f^+(d)[2d] \arrow[r, "\sim"] \arrow[d, "g^+\mathrm{Tr}_f"'] & f'_! g'^+ f^+(d)[2d] \arrow[r, equal] & f'_! f'^+ g^+(d)[2d] \arrow[d, "\mathrm{Tr}_{f'}"] \\ g^+ \arrow[rr, equal] & & g^+. \end{tikzcd}$$

(Var 3) *Let* $X \xrightarrow{g} Y \xrightarrow{f} Z$ *be morphisms of realizable schemes such that* $f \in \mathfrak{M}_d$ *and* $g \in \mathfrak{M}_e$. *Then the following diagram is commutative:*

$$\begin{tikzcd} f_! g_! g^+ f^+(d+e)[2(d+e)] \arrow[r, "\mathrm{Tr}_g"] \arrow[d, leftrightarrow, "\sim"'] & f_! f^+(d)[2d] \arrow[d, "\mathrm{Tr}_f"] \\ (f \circ g)_! (f \circ g)^+(d+e)[2(d+e)] \arrow[r, "\mathrm{Tr}_{f\circ g}"'] & \mathrm{id}. \end{tikzcd}$$

(Var 4-I) *Let* $f \in \mathfrak{M}_0$ *be a finite locally free morphism of rank* n. *Then the composition*

$$\mathscr{F} \to f_+ f^+ \mathscr{F} \xleftarrow{\sim} f_! f^+ \mathscr{F} \xrightarrow{\mathrm{Tr}_f} \mathscr{F}$$

is the multiplication by n.

(Var 4-II) *When* X *and* Y *can be lifted to a proper smooth formal scheme and* f *can be lifted to a smooth morphism of relative dimension* 1 *between them, then the trace map is the one defined in* 1.5.11 *below.*

(Var 5) *The following diagram is commutative:*

$$\begin{tikzcd} f_! f^+(\mathscr{F} \otimes \mathscr{G})(d)[2d] \arrow[r, "\sim"] \arrow[d, "\mathrm{Tr}_f"'] & (f_! f^+ \mathscr{F}(d)[2d]) \otimes \mathscr{G} \arrow[d, "\mathrm{Tr}_f \otimes \mathrm{id}"] \\ \mathscr{F} \otimes \mathscr{G} \arrow[r, equal] & \mathscr{F} \otimes \mathscr{G}. \end{tikzcd}$$

where the upper horizontal homomorphism is the projection formula 1.1.3 (9).

1.5.2. Even though there are many technical differences, the idea of the construction of a trace map is essentially the same as that in [SGA4, XVIII]. Let us start to construct the trace map. First, we list direct consequences from the requested properties.

(1) By (Var 5), it suffices to construct the trace map for $\mathscr{F} = K_Y$.
(2) By assumption, for $\mathscr{F} \in \mathrm{Con}(X)$, we have ${}^{\mathrm{c}}\mathscr{H}^i f_! f^+ \mathscr{F} = 0$ for $i > 2d$ by Lemma 1.3.8. Thus, we have an isomorphism $\mathrm{Hom}(f_! f^+ \mathscr{F}(d)[2d], \mathscr{F}) \cong \mathrm{Hom}({}^{\mathrm{c}}\mathscr{H}^{2d} f_! f^+ \mathscr{F}(d), \mathscr{F})$.
(3) Assume that we have already constructed the trace map when Y is a point. Then by (1), (2) above, Lemma 1.3.7 (i) and the base change property, extensions of this trace map to the general situation are unique, if they exist.
(4) Assume $f = f \amalg f' \colon X' \coprod X'' \to Y$, and assume Tr_f and $\mathrm{Tr}_{f'}$ have already been constructed. Then, by the same argument as [SGA4, XVII, 6.2.3.1], $\mathrm{Tr}_f = \mathrm{Tr}_{f'} + \mathrm{Tr}_{f''}$.
(5) If f is a universal homeomorphism, then the canonical homomorphism

$$\alpha \colon \mathscr{F} \to f_+ f^+ \mathscr{F} \xleftarrow{\sim} f_! f^+ \mathscr{F}$$

is an isomorphism by Lemma 1.1.3. By (Var 4-I), we have $\mathrm{Tr}_f := \deg(f) \cdot \alpha^{-1}$.

(6) Consider the cartesian diagram (1.1.3.1) of realizable schemes. Then the compatibility (Var 2) is equivalent to the commutativity of one of the following diagrams:

(1.5.2.1)

$$\begin{tikzcd} g'^{+}f^{+}\mathscr{F}(d)[2d] \arrow[r, "g'^{+}\mathrm{Tr}^{\mathrm{ad}}_{f}"] \arrow[d, "\sim"'] & g'^{+}f^{!}\mathscr{F} \arrow[d] \\ f'^{+}g^{+}\mathscr{F}(d)[2d] \arrow[r, "\mathrm{Tr}^{\mathrm{ad}}_{f'}g^{+}"'] & f'^{!}g^{+}\mathscr{F}, \end{tikzcd} \qquad \begin{tikzcd} f'^{+}g^{!}\mathscr{F}(d)[2d] \arrow[r, "\mathrm{Tr}^{\mathrm{ad}}_{f'}g^{!}"] \arrow[d] & f'^{!}g^{!}\mathscr{F} \arrow[d, "\sim"] \\ g'^{!}f^{+}\mathscr{F}(d)[2d] \arrow[r, "g'^{!}\mathrm{Tr}^{\mathrm{ad}}_{f}"'] & g'^{!}f^{!}\mathscr{F}, \end{tikzcd}$$

where $\mathrm{Tr}^{\mathrm{ad}}$ denotes the adjoint of the trace map, and the vertical arrows are the base change homomorphisms. The verification is standard using the diagram of 1.1.6.

1.5.3. **Lemma.** *Let $f\colon X \to Y$ be a morphism of realizable schemes of relative dimension $\leq d$. Let $\{U_i\}$ be a finite open covering of X, and let $U_{ij} := U_i \cap U_j$. For $\star \in \{i, ij\}$, we put $u_\star\colon U_\star \to X$, and $f_\star\colon U_\star \hookrightarrow X \xrightarrow{f} Y$. Then for $\mathscr{F} \in \mathrm{Con}(X)$, the following sequence is exact:*

$$\bigoplus_{i,j} {}^{\mathrm{c}}\mathscr{H}^{2d} f_{ij!}u_{ij}^{+}\mathscr{F} \rightrightarrows \bigoplus_{i} {}^{\mathrm{c}}\mathscr{H}^{2d} f_{i!}u_{i}^{+}\mathscr{F} \to {}^{\mathrm{c}}\mathscr{H}^{2d} f_{!}\mathscr{F} \to 0.$$

Proof. By Lemma 1.3.8, ${}^{\mathrm{c}}\mathscr{H}^{2d}f_!$ is right exact, and the claim of the lemma follows by applying this functor to the exact sequence in Lemma 1.3.9. □

1.5.4. First, let $\mathfrak{M}_{\mathrm{et}}$ be the set of étale morphisms between realizable schemes. We show the theorem for $\mathfrak{M}_{\mathrm{et}}$ instead of $\mathfrak{M}$. By 1.5.2 (3), combining with 1.5.2 (4), (5), and Lemma 1.3.7 (ii), if the trace maps exist for morphisms in $\mathfrak{M}_{\mathrm{et}}$, then they are unique. We show the following lemma:

Lemma. *For $f \in \mathfrak{M}_{\mathrm{et}}$, there exists a unique trace map $f_!f^+\mathscr{M} \to \mathscr{M}$ for $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(Y)$ satisfying the properties* (Var 1, 2, 3, 4-I, 5) *if we replace $\mathfrak{M}$ by $\mathfrak{M}_{\mathrm{et}}$. Moreover, the homomorphism $f^+(\mathscr{M}) \to f^!(\mathscr{M})$ defined by taking the adjoint is an isomorphism.*

The proof is divided into several parts, and it is given in 1.5.7.

1.5.5. **Lemma** (Smooth base change for open immersion)**.** *Consider the following cartesian diagram:*

$$\begin{tikzcd} U' \arrow[r, "j'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "g"] \arrow[dr, phantom, "\square"] & Z' \arrow[l, "i'"'] \arrow[d, "g''"] \\ U \arrow[r, "j"'] & X & Z. \arrow[l, "i"] \end{tikzcd}$$

Assume g is smooth, j is an open immersion, and i is the closed immersion defined by the complement. Then the base change homomorphisms $g^+j_+(\mathscr{M}) \to j'_+g'^+(\mathscr{M})$ and $g''^+i^!(\mathscr{M}) \to i'^!g^+(\mathscr{M})$ are isomorphisms for any $\mathscr{M}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(U)$.

Proof. By the localization triangle $i_+i^! \to \mathrm{id} \to j_+j^+ \xrightarrow{+1}$, it suffices to show only the first isomorphism. Obviously, if we restrict the base change homomorphism to U', the homomorphism is an isomorphism. Thus, by the localization triangle, it suffices to show that $i'^!g^+j_+\mathscr{M} = 0$. Since the verification is Zariski local with respect to X', we may assume that g is factored into an étale morphism followed by the projection $\mathbb{A}^n_X \to X$. We can treat étale and projection cases separately. Thus, using [EGAIV, 18.4.6], we may assume that there is a smooth morphism $\mathscr{P}' \to \mathscr{P}$ of smooth formal schemes and a closed embedding $X \hookrightarrow \mathscr{P}$ such that $X' \cong X \times_{\mathscr{P}} \mathscr{P}'$. Then we may use [Ber2, 4.3.12] and [A1, Theorem 5.5] to conclude. □

Corollary (Smooth base change). *Consider the diagram of realizable schemes* (1.1.3.1)*. Assume that g is smooth. Then the base change homomorphism $g^+ \circ f_+ \to f'_+ \circ g'^+$ is an isomorphism.*

Proof. We may factor f as $X \xrightarrow{j} \overline{X} \xrightarrow{p} Y$, where j is an open immersion and p is proper. The base change for p is the proper base change theorem (cf. 1.1.3 (8)), and that for j is the lemma above. □

1.5.6. First, suppose that Y is smooth liftable purely of dimension d, and suppose f is affine. In this case let us construct an isomorphism $f^+K_Y \xrightarrow{\sim} f^!K_Y$. By taking the dual, it is equivalent to constructing $f^+K^\omega_Y \xrightarrow{\sim} f^!K^\omega_Y$ (cf. 1.1.4 for $K^\omega_\star$). Let $\mathscr{Y}$ be a smooth lifting of Y. Since the étale site of Y and $\mathscr{Y}$ are equivalent, there exists the following cartesian diagram where $\mathscr{X}$ and $\mathscr{Y}$ are smooth formal schemes and X and Y are special fibers:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \mathscr{X} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r] & \mathscr{Y}. \end{tikzcd}$$

By [Ca3, 4.1.8, 4.1.9, 4.3.5] and [A1, 3.12], we have canonical isomorphisms

$$\mathbb{D}_{\mathscr{X}} \circ \widetilde{f}^! \circ \mathbb{D}_{\mathscr{Y}}(\mathrm{sp}_+(\mathcal{O}_{\mathscr{Y}_K})) \cong \mathrm{sp}_+\big((\widetilde{f}^*(\mathcal{O}^\vee_{\mathscr{Y}_K}(-d)))^\vee(-d)\big) \cong \mathrm{sp}_+(\mathcal{O}_{\mathscr{X}_K}).$$

This gives us a canonical isomorphism

$$\rho_f \colon \widetilde{f}^+\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \xrightarrow{\sim} \mathcal{O}_{\mathscr{X},\mathbb{Q}} \cong \widetilde{f}^!\mathcal{O}_{\mathscr{Y},\mathbb{Q}}.$$

By Kedlaya's full faithfulness [Ke1], ρ_f extends to the desired isomorphism. Taking the adjoint, we get a homomorphism $t_{\mathscr{Y}} \colon K^\omega_Y \to f_+f^!K^\omega_Y$. We need the lemma below to show that $t_{\mathscr{Y}}$, in fact, does not depend on the choice of $\mathscr{Y}$.

Remark. We remark that for an isocrystal M, the following diagram is commutative:

$$\begin{tikzcd} & \mathrm{sp}_+((M^\vee)^\vee) \arrow[d, dash, "\sim"] \\ \mathrm{sp}_+(M) \arrow[ur] \arrow[r] & \mathbb{D} \circ \mathbb{D}(\mathrm{sp}_+(M)). \end{tikzcd}$$

To check this, by definition (cf. [Ca1, 2.2.12]), it suffices to check the commutativity for $M \cong \mathcal{O}_{\mathscr{Y},\mathbb{Q}}$. Then it is reduced to the commutativity of the following diagram of complexes, whose verification is easy:

$$\begin{tikzcd}[column sep=small] & \mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}} \otimes \bigwedge \mathscr{T}_{\mathscr{Y}} \otimes \mathrm{Hom}_{\mathcal{O}}\bigl(\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_{\mathscr{Y}}, \mathcal{O}_{\mathscr{Y}}), \mathcal{O}_{\mathscr{Y}}\bigr) \arrow[d, "\sim"] \\ \mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}} \otimes \bigwedge \mathscr{T}_{\mathscr{Y}} \arrow[ur] \arrow[r] & \mathrm{Hom}_{\mathscr{D}}\bigl(\mathrm{Hom}_{\mathscr{D}}(\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}} \otimes \bigwedge \mathscr{T}_{\mathscr{Y}}, \mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}} \otimes \omega^{-1}_{\mathscr{Y}}), \mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}} \otimes \omega^{-1}_{\mathscr{Y}}\bigr). \end{tikzcd}$$

Let $\mathscr{M}$ be a holonomic module on Y. By definition, there exists a smooth proper formal scheme $\mathscr{P}$ such that $\mathscr{M}$ can be realized as a $\mathscr{D}^{\dagger}_{\mathscr{P},\mathbb{Q}}$-module $\mathscr{M}_{\mathscr{P}}$. There exists an immersion (not necessarily closed) $i\colon \mathscr{Y} \hookrightarrow \mathscr{P}$. Then the $\mathscr{D}^{\dagger}_{\mathscr{Y},\mathbb{Q}}$-module $i^{!}(\mathscr{M}_{\mathscr{P}})$ (which is overholonomic and canonically isomorphic to $i^{+}(\mathscr{M}_{\mathscr{P}})$) is denoted by $\mathscr{M}\|_{\mathscr{Y}}$ for a moment. This module does not depend on the auxiliary choices up to canonical equivalence.

Lemma. *The following diagram is commutative:*

$$\begin{tikzcd} K^{\omega}_{Y}\|_{\mathscr{Y}} \arrow[r, "t_{\mathscr{Y}}"] \arrow[d, equal] & f_{+}f^{!}K^{\omega}_{Y}\|_{\mathscr{Y}} \arrow[d, hook] \\ \mathcal{O}_{\mathscr{Y},\mathbb{Q}} \arrow[r, "\mathrm{adj}_f"'] & \widetilde{f}_{*}\widetilde{f}^{*}\mathcal{O}_{\mathscr{Y},\mathbb{Q}}. \end{tikzcd}$$

Proof. First, let us show the lemma when f is finite étale of rank n. Since $\widetilde{f}$ is finite étale, we can identify $\widetilde{f}_{+}$ and $\widetilde{f}^{!}$ by $\widetilde{f}_{*}$ and $\widetilde{f}^{*}$, respectively, if we consider the underlying $\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$-module structure. In the following, for simplicity, we do not make any difference between $\widetilde{f}$ and f. In this case, the right vertical homomorphism is, in fact, isomorphic. Let $\mathscr{F}$ be an $\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$-module, and let $\iota\colon f_{*}(f^{*}\mathscr{F})^{\vee} \to (f_{*}f^{*}\mathscr{F})^{\vee}$ be the homomorphism sending φ to $\mathrm{Tr}_f \circ \varphi$, where $\mathrm{Tr}_f\colon f_{*}f^{*}\mathcal{O}_{\mathscr{Y}} \to \mathcal{O}_{\mathscr{Y}}$ is the classical trace map. If $\mathscr{F}$ is a locally free $\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$-module, ι is an isomorphism. We have the following diagram, where we omit sp_{+} and the subscripts $\mathbb{Q}$:

$$\begin{tikzcd} \mathcal{O}_{\mathscr{Y}} \arrow[r] \arrow[d] & f_{!}f^{+}\mathcal{O}_{\mathscr{Y}} \arrow[r, "\sim"'] \arrow[d] & f_{+}f^{+}\mathcal{O}_{\mathscr{Y}} \arrow[r, "\rho_f", "\sim"'] \arrow[d] & f_{+}f^{!}\mathcal{O}_{\mathscr{Y}} \arrow[d] \\ (\mathcal{O}^{\vee}_{\mathscr{Y}})^{\vee} \arrow[r, "(\mathrm{Tr}_f)^{\vee}"'] & (f_{*}f^{*}\mathcal{O}^{\vee}_{\mathscr{Y}})^{\vee} \arrow[r, "\sim", "\iota^{-1}"'] & f_{*}(f^{*}\mathcal{O}^{\vee}_{\mathscr{Y}})^{\vee} \arrow[r, "\sim"] & f_{*}f^{*}\mathcal{O}_{\mathscr{Y}}. \end{tikzcd}$$

Here the vertical morphisms are isomorphic. This diagram is commutative. The commutativity of the left and right square immediately follows by definition. To check the commutativity for the middle one, we need to go back to the definition, which is [V1, IV.1.3]. We note that since f is finite étale, the trace map $f_{+}\mathcal{O}_{\mathscr{X},\mathbb{Q}} \to \mathcal{O}_{\mathscr{Y},\mathbb{Q}}$ defined in [V1, III.5.1] is equal to Tr_f via the identification $f_{+}\mathcal{O}_{\mathscr{X},\mathbb{Q}} \cong f_{*}\mathcal{O}_{\mathscr{X},\mathbb{Q}}$. Since the commutativity is standard routine work, we leave the details to the reader. Now, the verification of the lemma in the finite étale case is reduced to showing the composition of the lower row is the adjunction homomorphism. This is easy.

In general, there exists an open dense formal subscheme $j\colon \mathscr{U} \subset \mathscr{Y}$ such that $f'\colon \mathscr{X}' := \mathscr{X} \times_{\mathscr{Y}} \mathscr{U} \to \mathscr{U}$ is finite étale. Put $j'\colon \mathscr{X}' \hookrightarrow \mathscr{X}$. Then by [Ber2, 4.3.10], we have an injection $K^{\omega}_{X} \hookrightarrow j'_{+}K^{\omega}_{X'}$ where X' is the special fiber of $\mathscr{X}'$. Since f

is affine, f_+ is exact, and the homomorphism $f_+f^!K^{\omega}_Y \to j_+f'_+f'^!K^{\omega}_U$ is injective. Consider the following diagram where we omit $\|_{\mathscr{Y}}$:

$$\begin{tikzcd}
 & j_+K^{\omega}_U \arrow[rr] \arrow[dd] & & j_+f'_+f'^!K^{\omega}_U \arrow[dd, "\sim"] \\
K^{\omega}_Y \arrow[ur] \arrow[rr] \arrow[dd, "\sim"'] & & f_+f^!K^{\omega}_Y \arrow[ur, hook] \arrow[dd] & \\
 & j_*\mathcal{O}_{\mathscr{U},\mathbb{Q}} \arrow[rr] & & j_*f'_*f'^*\mathcal{O}_{\mathscr{U},\mathbb{Q}} \\
\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \arrow[ur] \arrow[rr] & & f_*f^*\mathcal{O}_{\mathscr{Y},\mathbb{Q}}. \arrow[ur, hook, "\star"'] &
\end{tikzcd}$$

The diagram is known to be commutative except for the forehead square diagram. By the injectivity of $\star$, the desired commutativity follows by the commutativity of other faces. $\square$

The trace map satisfies the base change property; namely, considering the cartesian diagram of (1.1.3.1) such that Y and Y' possess liftings $\mathscr{Y}$, $\mathscr{Y}'$, then the diagram of (Var 2) is commutative if we replace $\mathrm{Tr}_{f^{(\prime)}}$ by the dual of $t_{\mathscr{Y}^{(\prime)}}$. To check this, it suffices to check the dual of the base change property for $t_{\mathscr{Y}}\colon K^{\omega}_Y \to f_+f^!K^{\omega}_Y$. By using [Ber2, 4.3.10], it suffices to show the base change property after taking $\|_{\mathscr{Y}}$. The lemma above reduces the verification to the base change property for the adjunction homomorphism $\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \to f_*f^*\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$ which follows by the base change property of coherent $\mathcal{O}_{\mathscr{Y},\mathbb{Q}}$-modules. The transitivity can also be checked by a similar argument.

This implies that $t_{\mathscr{Y}}$ does not depend on the choice of $\mathscr{Y}$. Indeed, when Y is a point, all smooth liftings of Y are canonically isomorphic, and $t_{\mathscr{Y}}$ does not depend on the choice. In general, by the base change property and the uniqueness mentioned at the beginning of 1.5.4 shows that $t_{\mathscr{Y}}$ depends only on Y. This justifies denoting the dual of $t_{\mathscr{Y}}$ by $\mathrm{Tr}_f\colon f_!f^+K_Y \to K_Y$.

1.5.7. *Proof of Lemma* 1.5.4. Let us construct the trace map for a general étale morphism. Assume that we have the following cartesian diagram D :

$$\begin{tikzcd}
X \arrow[r, hook, "i'"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \widetilde{X} \arrow[d, "g"] \\
Y \arrow[r, hook, "i"'] & \widetilde{Y},
\end{tikzcd}$$

where $\widetilde{Y}$ is smooth liftable, g is affine étale, and the horizontal morphisms are closed immersions. We have the following uncompleted diagram of solid arrows:

$$\begin{tikzcd}
i'^+g^+K_{\widetilde{Y}} \arrow[r, "\mathrm{Tr}_g", "\sim"'] \arrow[d, "\sim"'] & i'^+g^!K_{\widetilde{Y}} \arrow[d] \\
f^+i^+K_{\widetilde{Y}} \arrow[r, dotted] & f^!i^+K_{\widetilde{Y}}.
\end{tikzcd}$$

The left vertical homomorphism is an isomorphism by transitivity, and the dotted homomorphism is defined so that the diagram commutes. By taking the adjoint, the dotted arrow gives us a homomorphism $\mathrm{Tr}_D\colon f_!f^+K_Y \to K_Y$. Let us check that Tr_D does not depend on the choice of D. If Y is liftable, Tr_D coincides with the trace map Tr_f (for liftable schemes) by the base change property of the trace map for liftable schemes we have already checked. In particular, for a closed point $i_s\colon s \hookrightarrow Y$, $i_s^+\mathrm{Tr}_D$ coincides with the trace map for the liftable schemes $X\times_Y s \to s$. By the uniqueness at the beginning of 1.5.4, Tr_D does not depend on the choice of diagram D, and we are allowed to denote Tr_D by Tr_f. This also shows the base change and transitivity property of Tr_f when the homomorphisms in the diagrams of (Var 2) and (Var 3) are defined.

In general, we can take a diagram D locally on X and Y. By using Lemma 1.5.3, we can glue, and get the desired trace map $f_!f^+K_Y \to K_Y$ similarly to [SGA4, XVIII, 2.9 c)], and check that this is the desired trace map. The details are left to the reader.

Finally, let us show that $f^+(\mathscr{F}) \cong f^!(\mathscr{F})$ for $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{hol}}(Y)$. By (Var 5), the trace map for $\mathscr{F}$ should be

$$f_!f^+\mathscr{F} \cong f_!f^+K_Y \otimes \mathscr{F} \xrightarrow{\mathrm{Tr}_f} \mathscr{F}.$$

This map satisfies (Var 1)–(Var 4) since they hold for $\mathscr{F} = K_Y$. Taking the adjunction, we have a homomorphism $f^+(\mathscr{F}) \to f^!(\mathscr{F})$. When Y is a point, this is easy. Let $s \in Y$ be a closed point, and consider the following diagram:

$$\begin{tikzcd} X_s \arrow[r, "i'_s"] \arrow[d, "f_s"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ s \arrow[r, "i_s"'] & Y. \end{tikzcd}$$

By the compatibility of trace map by base change, the following diagram is commutative:

$$\begin{tikzcd} i'^+_s f^+(\mathscr{F}) \arrow[r] \arrow[d, "\sim"'] & i'^+_s f^!\mathscr{F} \arrow[d] \\ f_s^+ i_s^+(\mathscr{F}) \arrow[r] & f_s^! i_s^+(\mathscr{F}). \end{tikzcd}$$

By (the dual of) smooth base change in Corollary 1.5.5, the right vertical homomorphism is an isomorphism and, by the point case, the bottom horizontal homomorphism is an isomorphism as well. Thus by [AC1, 1.3.11], the claim follows. $\square$

1.5.8. Let us construct the trace map for quasi-finite flat morphism. We follow the construction of [SGA4, XVII, 6.2]. Let $f\colon X \to Y$ be a quasi-finite flat morphism. For an étale morphism $U \to Y$, we consider the category $\Psi_f(U)$ defined as follows: objects consist of collections $(V_i)_{i\in I}$, where I is a pointed finite set with the marked point $0 \in I$ and a decomposition $X \times_Y U = \coprod_{i\in I} V_i$ such that $V_i \to U$ is finite for $i \neq 0$. We denote by $I^* := I \setminus \{0\}$. A morphism from $\varphi = (V_i)_{i\in I}$ to $\varphi' = (V'_i)_{i\in I'}$ is a map $\sigma\colon I \to I'$ such that $\sigma(0) = 0$ and $V_i = \bigcup_{j\in\sigma^{-1}(i)} V'_j$. For a morphism $U \to V$ in Y_{et}, there exists the obvious functor $\Psi_f(V) \to \Psi_f(U)$, and $\Psi_f(U)$ is a fibered category over Y_{et}. This category is denoted by Ψ_f, and an object of the fiber over $U \in Y_{\mathrm{et}}$ is denoted by $\{U; (V_i)_{i\in I}\}$. We refer to [SGA4] for details.

Lemma. *Let $\mathscr{F} \in \mathrm{Con}(Y)$. Then there exists a canonical isomorphism*

$$\tau_f \colon \varinjlim_{\{U;\varphi\}\in\Psi_f} j_{U!}j_U^+\mathscr{F}^{I^*} \xrightarrow{\sim} f_!f^+\mathscr{F},$$

where $j_U \colon U \to Y$ is the étale morphism.

Remark. Before proving the lemma, we remark that the inductive system is *not* filtrant.

Proof. The verification is essentially the same as [SGA4]. Let us construct the homomorphism. Take $\varphi = \{U; (V_i)_{i\in I}\}$. Let $j_i \colon V_i \to X$ be the étale morphism. Since $f_i \colon V_i \to U$ is assumed finite for $i \in I^*$, we have the following homomorphism for $\mathscr{G} \in \mathrm{Con}(U)$:

$$\mathscr{G} \to f_{i+}f_i^+(\mathscr{G}) \xleftarrow{\sim} f_{i!}f_i^+(\mathscr{G}).$$

By using the trace map in Lemma 1.5.4, we get the homomorphism

$$j_{U!}j_U^+(\mathscr{F}) \to j_{U!}f_{i!}f_i^+j_U^+(\mathscr{F}) \cong f_!j_{i!}j_i^+f^+(\mathscr{F}) \xrightarrow{\mathrm{Tr}_{j_i}} f_!f^+(\mathscr{F}),$$

which induces the homomorphism in the statement.

Now, by 1.2.2 (4), it suffices to show that the homomorphism is an isomorphism in $\mathrm{Ind}(\mathrm{Con}(X))$. When f is a universal homeomorphism, the canonical homomorphism $\mathscr{F} \to f_!f^+\mathscr{F}$ is an isomorphism by Lemma 1.1.3. Assume $Y =: s$ is a point. There is a separable extension $s' \to s$ such that $X \times s' \to s'$ is disjoint union of universal homeomorphisms. Thus, the lemma follows by Lemma 1.3.10 (ii).

Let s be a closed point of Y. Put $i_s \colon s \to Y$ to be closed immersion. Since i_s^+ is an exact functor and commutes with direct sum, it commutes with arbitrary inductive limits. Thus, we have

$$i_s^+\bigl(\varinjlim_{\Psi_f} j_{U!}\mathscr{F}^{I^*}\bigr) \cong \varinjlim_{\Psi_f} i_s^+j_{U!}\mathscr{F}^{I^*}.$$

Let $f_s \colon X \times_Y s \to s$. There exists a functor $\Psi_f \to \Psi_{f_s}$. This functor is cofinal by [EGAIV, 18.12.1]. Then by Lemma 1.5.4, $i_s^+\tau_f \cong \tau_{f_s}$, and by the proven case where Y is a point, $i_s^+\tau_f$ is an isomorphism. By Lemma 1.3.10 (i), this implies that τ_f is an isomorphism, as required. □

1.5.9. Let $f \colon X \to Y$ be a quasi-finite flat morphism between realizable schemes. Let us construct the unique trace map $f_!f^+K_Y \to K_Y$ satisfying (Var 1,2,3,4-I). When f is étale, we remark that this trace map coincides with that of Lemma 1.5.4 by uniqueness. The construction is the same as [SGA4, XVII, 6.2], so we only sketch the proof.

Let Ψ'_f be the full subcategory of Ψ_f consisting of $\{U; (V_i)_{i\in I}\}$ such that V_i is locally free of constant rank over U for any $i \neq 0$. This category is cofinal in Ψ_f. For each $\{U; (V_i)_{i\in I}\} \in \Psi'_f$, we have a homomorphism

$$\sum_{i\in I^*} \deg(V_i/U)\cdot \mathrm{Tr}_{j_U} \colon j_{U!}j_U^+(K_Y^{I^*}) \to K_Y.$$

Since the compatibility follows by that of Lemma 1.5.4, this homomorphism induces

$$f_!f^+K_Y \xleftarrow{\sim} \varinjlim_{\varphi\in\Psi'_f} j_{U!}j_U^+K_Y^{I^*} \to K_Y.$$

It is easy to check that this is what we are looking for.

1.5.10. **Lemma.** *Let $f\colon X \to Y$ be the special fiber of a finite étale morphism between smooth formal curves $\widetilde{f}\colon \mathscr{X} \to \mathscr{Y}$. By taking the dual of the trace map, we get $K^{\omega}_Y \to f_+ K^{\omega}_X$. When we restrict this homomorphism to $\mathscr{Y}$ (i.e., taking $\|_{\mathscr{Y}}$ of* 1.5.6*), this dual of trace map is nothing but the homomorphism induced by the adjunction homomorphism $\phi_{\widetilde{f}}\colon \mathcal{O}_{\mathscr{Y},\mathbb{Q}} \to \widetilde{f}_* \mathcal{O}_{\mathscr{X},\mathbb{Q}}$ with the identification $\widetilde{f}_* \mathcal{O}_{\mathscr{X},\mathbb{Q}} \cong \widetilde{f}_+ \mathcal{O}_{\mathscr{X},\mathbb{Q}}$.*

Proof. We may assume $\mathscr{X}$ and $\mathscr{Y}$ to be connected. Let $L_{\mathscr{X}}$, $L_{\mathscr{X}}$ be the largest finite extension of K in $\mathscr{X}$, $\mathscr{Y}$. If $L_{\mathscr{X}} \neq L_{\mathscr{Y}}$, then $\widetilde{f}$ factors as

$$\mathscr{X} \xrightarrow{\alpha} \mathscr{Y} \otimes_{R_{\mathscr{Y}}} R_{\mathscr{X}} \xrightarrow{\beta} \mathscr{Y},$$

where $R_\star$ denotes the ring of integers of $L_\star$, and it suffices to show the lemma for α and β separately. The verification for β is easy. Let us show that for α. In this case, we let $L_{\mathscr{X}} = L_{\mathscr{Y}} =: L$.

Since the formal schemes are curves, by Kedlaya's full faithfulness theorem [Ke1], the homomorphism $\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \to \widetilde{f}_+ \mathcal{O}_{\mathscr{X},\mathbb{Q}}$ induced by $\phi_{\widetilde{f}}$ extends uniquely to the homomorphism $\phi\colon K^{\omega}_Y \to f_+ K^{\omega}_X$. We have

$$\mathrm{Hom}(K^{\omega}_Y, f_+ f^! K^{\omega}_Y) \cong \mathrm{Hom}(f^+ K^{\omega}_Y, f^! K^{\omega}_Y) \sim \mathrm{Hom}(K^{\omega}_X, K^{\omega}_X) \cong L,$$

where $\sim$ is the isomorphism induced by [A1, Theorem 5.5] since $\mathscr{X}$ and $\mathscr{Y}$ are smooth. Thus, there exists $c \in L$ such that $c \cdot \phi = \mathbb{D}(\mathrm{Tr}_f)$. It remains to show that $c = 1$. By definition, the composition $K_Y \to f_! f^+ K_Y \xrightarrow{\mathrm{Tr}_f} K_Y$ is the multiplication by $n := \deg(f)$. Take the dual of this homomorphism, and we get

$$n\colon K^{\omega}_Y \xrightarrow{\mathbb{D}(\mathrm{Tr}_f)} f_+ f^! K^{\omega}_Y \xrightarrow{\mathrm{Tr}^{\mathrm{Vir}}_f} K^{\omega}_Y,$$

where the second homomorphism is the trace map of [V1]. On the other hand, by property of $\mathrm{Tr}^{\mathrm{Vir}}_f$ (cf. [V1, III.5.4]), we get $\mathrm{Tr}^{\mathrm{Vir}}_f \circ \phi = n$. Thus $c = 1$ since $\mathrm{Hom}(K^{\omega}_Y, K^{\omega}_Y) \cong L$. $\square$

1.5.11. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a proper smooth morphism of relative dimension 1 between smooth proper formal schemes. The homomorphism of rings $\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \to f_* \mathcal{O}_{\mathscr{X},\mathbb{Q}}$ induces the homomorphism

$$\mathcal{O}_{\mathscr{Y},\mathbb{Q}} \to \mathbb{R} f_* \big[0 \to \mathcal{O}_{\mathscr{X},\mathbb{Q}} \to \Omega^1_{\mathscr{X}/\mathscr{Y},\mathbb{Q}} \to 0\big] \tag{1.5.11.1}$$

in $D(\mathscr{D}^\dagger_{\mathscr{Y},\mathbb{Q}})$. Since the target of the homomorphism is canonically isomorphic to $f_+ \mathcal{O}_{\mathscr{X},\mathbb{Q}}[-1]$, we have a homomorphism $\alpha_f\colon \mathcal{O}_{\mathscr{Y},\mathbb{Q}}(1)[2] \to f_+ f^! \mathcal{O}_{\mathscr{Y},\mathbb{Q}}$. By [A1, 3.14] and Remark 1.1.3 (iii), this homomorphism is compatible with the Frobenius structure when $\blacktriangle = F$. This trace map only depends on the special fibers because the unit element is sent to the unit element by a ring homomorphism. Thus, we have a homomorphism $K^{\omega}_Y(1)[2] \to f_+ f^! K^{\omega}_Y$. By construction, this homomorphism is compatible with base change; namely, given a morphism of proper smooth formal schemes $g\colon \mathscr{Y}' \to \mathscr{Y}$ such that $d := \dim(\mathscr{Y}') - \dim(\mathscr{Y})$, let $f'\colon \mathscr{X}' := \mathscr{X} \times_{\mathscr{Y}} \mathscr{Y}'$. Then the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{O}_{\mathscr{Y}',\mathbb{Q}}(1)[2] \arrow[r, "\sim"] \arrow[d, "\alpha_{f'}"'] & g^! \mathcal{O}_{\mathscr{Y},\mathbb{Q}}(1)[2-d] \arrow[d, "g^! \alpha_f"] \\
f'_+ f'^! \mathcal{O}_{\mathscr{Y}',\mathbb{Q}} \arrow[r, "\sim"'] & g^! f_+ f^! \mathcal{O}_{\mathscr{Y},\mathbb{Q}}[-d],
\end{tikzcd}$$

where the horizontal homomorphisms are canonical homomorphisms. By taking the dual, we get the *trace map* $\mathrm{Tr}_f\colon f_!f^+K_Y(1)[2] \to K_Y$. This trace map is compatible with pullback g^+ where g is a morphism between liftable proper smooth schemes.

Let us consider the case where $\mathscr{Y}$ is a point. Consider a commutative diagram:

$$\begin{tikzcd} \mathscr{U} \arrow[rr, hook] \arrow[dr, "g"'] & & \mathscr{X} \arrow[dl, "f"] \\ & \mathrm{Spf}(R), & \end{tikzcd}$$

where $\mathscr{U}$ is dense open in $\mathscr{X}$. Put $Z := X \setminus U$ where X, U are the special fibers of $\mathscr{X}$, $\mathscr{U}$, and assume Z is a divisor of X. Then we have an injection

$$\mathscr{H}^{-2}g_+g^!K \cong \mathscr{H}^{-1}f_+\mathcal{O}_{\mathscr{X},\mathbb{Q}}({}^\dagger Z) \hookrightarrow \mathscr{H}^{-1}g_+\mathcal{O}_{\mathscr{U},\mathbb{Q}}. \tag{1.5.11.2}$$

1.5.12. *Proof of Theorem* 1.5.1. Now, we construct the trace map. We need several steps for the construction.

(i) **Absolute curve case**. Let $f\colon X \to \mathrm{Spec}(k)$ be a realizable variety. We put $H^i_{\mathrm{c}}(X) := \mathrm{Hom}\big(K, f_!(K_X)[i]\big)$. Now, assume X to be a curve, and let us construct the trace map. First, let us construct the trace map when X_{red} is smooth and irreducible. Let $\iota\colon X_{\mathrm{red}} \hookrightarrow X$ be the closed immersion, and let $f'\colon \overline{X}_{\mathrm{red}} \to \mathrm{Spec}(k)$ be the smooth compactification of X_{red}. We have already defined $\mathrm{Tr}_{f'}$ in 1.5.11. We define

$$\mathrm{Tr}_f\colon H^2_{\mathrm{c}}(X)(1) \xrightarrow{\lg(\mathcal{O}_{X,\eta})\cdot\iota^*} H^2_{\mathrm{c}}(X_{\mathrm{red}})(1) \xrightarrow{\sim} H^2_{\mathrm{c}}(\overline{X}_{\mathrm{red}})(1) \xrightarrow{\mathrm{Tr}_{f'}} K.$$

In general, we may take an open dense subscheme $U \subset X$ such that U_{red} is smooth. Then we have the canonical isomorphism $H^2_{\mathrm{c}}(U) \xrightarrow{\sim} H^2_{\mathrm{c}}(X)$ by Lemma 1.3.8. Let $U = \coprod_{i\in I} U_i$ be the decomposition into connected components. Then we define

$$\mathrm{Tr}_f\colon H^2_{\mathrm{c}}(X)(1) \xrightarrow{\sim} H^2_{\mathrm{c}}(U)(1) \cong \bigoplus_{i\in I} H^2_{\mathrm{c}}(U_i)(1) \xrightarrow{\sum \mathrm{Tr}_{f|_{U_i}}} K.$$

Lemma. *Let $X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Spec}(k)$ be a morphism of realizable schemes such that f is a quasi-finite flat morphism and g is of relative dimension* 1. *Then we have*

$$\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ g_!(\mathrm{Tr}_f)\colon (g\circ f)_!K_X(1)[2] \to K.$$

Proof. Arguing as [SGA4, XVIII, 1.1.5], we may assume that X and Y are connected smooth affine, and f factors as $X \xrightarrow{F} X' \xrightarrow{f'} Y$, where F is an iterated relative Frobenius endomorphism and f' is a finite étale morphism. It suffices to check the equality for F and f' individually. For F, the claim follows since $\mathrm{Tr}_{g\circ f}$ is compatible with Frobenius structure. It remains to check the lemma when f is finite and étale. Since X and Y are assumed to be smooth and affine, there exist smooth liftings $\mathscr{X} \xrightarrow{\widetilde{f}} \mathscr{Y} \to \mathrm{Spf}(R)$ such that $\widetilde{f}$ is finite flat. In this case, it suffices to check the transitivity after removing the boundary by the injectivity of (1.5.11.2), and the lemma follows by Lemma 1.5.10 and the definition of (1.5.11.1). $\square$

(ii) **Relative affine space case**. Let Y be a realizable scheme, and consider the projection $f\colon X := \mathbb{P}^1_Y \to Y$. There exists a proper smooth formal scheme $\mathscr{P}$

such that $Y \hookrightarrow \mathscr{P}$. Then f can be lifted to the following cartesian diagram:

$$\begin{tikzcd} \mathbb{P}^1_Y \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & \widehat{\mathbb{P}}^1_{\mathscr{P}} \arrow[d, "\widetilde{f}"] \\ Y \arrow[r, hook, "i"'] & \mathscr{P}. \end{tikzcd}$$

We define the trace map

$$\mathrm{Tr}_f \colon f_! f^+ K_Y(1)[2] \cong i^+ \widetilde{f}_! \widetilde{f}^+ K_{\mathscr{P}}(1)[2] \xrightarrow{i^+\mathrm{Tr}_{\widetilde{f}}} i^+ K_{\mathscr{P}} \cong K_Y,$$

where $\mathrm{Tr}_{\widetilde{f}}$ is the one defined in 1.5.11. This map does not depend on the choice of $\mathscr{P}$ by the base change property of $\mathrm{Tr}_{\widetilde{f}}$.

When $f \colon \mathbb{A}^1_X \to X$ is the projection, then we have the factorization $\mathbb{A}^1_X \xrightarrow{j} \mathbb{P}^1_X \xrightarrow{p} X$, and the trace map Tr_f is defined to be the composition $\mathrm{Tr}_p \circ p_!(\mathrm{Tr}_j)$. The base change property can be checked by the base change property for j and p. Now, let $f \colon X := \mathbb{A}^d_Y \to Y$. In this case, we define by iteration as in [SGA4, XVIII, 2.8].

(iii) **Factorization case**. Let $f \colon X \to Y$ be a morphism which possesses a factorization $X \xrightarrow{u} \mathbb{A}^d_Y \xrightarrow{a^d} Y$ such that u is a quasi-finite flat morphism. Then we define $t(f,u) := \mathrm{Tr}_{a^d} \circ a^d_!(\mathrm{Tr}_u)$. We need to check that $t(f,u)$ does not depend on the choice of the factorization. By using the lemma in (i), the verification is the same as [SGA4, XVIII, 2.9 b)].

(iv) **General case**. The construction is the same as [SGA4, 2.9 c), d), e)]. We sketch the construction. When f is a Cohen–Macaulay morphism, then there exists a finite covering of $\{U_i\}$ of X such that the compositions $U_i \to X \to Y$ possess factorizations considered in case (iii). By gluing lemma 1.5.3, we have the trace map in this case. In general, we shrink X suitably, so that f is Cohen–Macaulay. Thus the trace map is constructed, and we conclude the proof of Theorem 1.5.1. □

1.5.13. **Theorem** (Poincaré duality). *Let $X \to Y$ be a smooth morphism of relative dimension d between realizable schemes. Then for $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{hol}}(Y)$, the adjoint of the trace map $\phi_{\mathscr{F}} \colon f^+\mathscr{F}(d)[2d] \to f^!\mathscr{F}$ is an isomorphism.*

Proof. Since the verification is local on X, it suffices to treat the case where f is étale and is the projection $\mathbb{A}^1_Y \to Y$ separately. The étale case has already been treated in Lemma 1.5.4.

Let us treat the projection case. We may shrink Y. Then we can embed Y into a proper smooth formal scheme. By using [A1, Theorem 5.5], we have an isomorphism $f^+\mathscr{F}(d)[2d] \sim f^!\mathscr{F}$, where $\sim$ is the isomorphism induced by [A1] and may not be the same as the one defined by the trace map. It suffices to show the theorem for $\mathscr{F} \in \mathrm{Hol}(Y)$. For this, we may assume $\mathscr{F}$ to be irreducible. Let k' be a finite extension of k. It suffices to show that $\phi_{\mathscr{F}}$ is an isomorphism after pulling back to $X \otimes_k k'$. Thus, we may assume moreover that $\mathscr{F}$ is irreducible also on $X \otimes_k k'$ for any extension k' of k. For a closed point $a \in \mathbb{A}^1$, we denote by $i_a \colon Y \otimes_k k(a) \to \mathbb{A}^1_Y$ the closed immersion defined by a. We claim that $f^+(\mathscr{F})$ is irreducible. Indeed, first, let us assume Y is smooth and $\mathscr{F}$ is smooth. Assume $f^+(\mathscr{F})$ were not irreducible. Then there would exist a smooth object $\mathscr{N} \subset f^+(\mathscr{F})$ and a closed point a of $\mathbb{A}^1(\overline{k})$ such that $i^+_a\mathscr{N}$ and $i^+_a\bigl(f^+(\mathscr{F})/\mathscr{N}\bigr)$ are nonzero. This is a contradiction. In general, $\mathscr{F}$ can be written as $j_{!+}$ of a smooth irreducible object by [AC1, 1.4.9] where j is an open immersion of Y. Since f is smooth, f^+ and $j_{!+}$ commute, and $f^+(\mathscr{F})$ is irreducible.

Now, we know that

$$\mathrm{Hom}(f^+\mathscr{F}(d)[2d], f^!\mathscr{F}) \sim \mathrm{Hom}(f^+\mathscr{F}(d)[2d], f^+\mathscr{F}(d)[2d]).$$

Since $f^+\mathscr{F}$ is irreducible, the Hom group is a division algebra, and it remains to show that $\phi_{\mathscr{F}}$ is not 0. For this, it suffices to check that the trace map $f_!f^+\mathscr{F}(d)[2d] \to \mathscr{F}$ is nonzero. By the base change property of a trace map, we may assume Y to be a point, in which case, the trace map is nonzero by construction. ☐

Corollary. *Let $f\colon X \to Y$ be a flat morphism of relative dimension d between smooth realizable schemes. Then the adjoint of trace map $f^+K_Y(d)[2d] \to f^!K_Y$ is an isomorphism.*

Proof. This follows by the transitivity of the trace map and the Poincaré duality for both X and Y. ☐

1.5.14. Let $i\colon Z \hookrightarrow X$ be a closed immersion of codimension c between smooth realizable schemes. By using the Poincaré duality, we have a canonical isomorphism $i^+K^\omega_X(-c)[-2c] \xrightarrow{\sim} i^!K^\omega_X$. Let us denote by $(-)\widetilde{\otimes}(-) := \mathbb{D}\big(\mathbb{D}(-)\otimes\mathbb{D}(-)\big)$. The projection formula yields the homomorphism $i^+\big(\mathscr{N}\widetilde{\otimes}\mathscr{M}\big) \to i^+(\mathscr{N})\widetilde{\otimes}i^!(\mathscr{M})$ for $\mathscr{M}$, $\mathscr{N}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. Using this homomorphism, we get a homomorphism

$$\begin{aligned} i^+(\mathscr{M})(-c)[-2c] &\cong i^+\big(K^\omega_X\widetilde{\otimes}\mathscr{M}\big)(-c)[-2c] \to i^+(K^\omega_X)(-c)[-2c]\widetilde{\otimes}i^!(\mathscr{M}) \\ &\xrightarrow{\sim} i^!K^\omega_X\widetilde{\otimes}i^!(\mathscr{M}) \cong i^!(\mathscr{M}). \end{aligned} \tag{1.5.14.1}$$

Theorem. *If $\mathscr{M}$ is smooth, then the canonical homomorphism* (1.5.14.1) *is an isomorphism.*

Proof. It suffices to show that when $\mathscr{M}$ is a smooth holonomic module, the canonical homomorphism $i^+\big(\mathscr{N}\widetilde{\otimes}\mathscr{M}\big) \to i^+(\mathscr{N})\widetilde{\otimes}i^!(\mathscr{M})$ is an isomorphism for any $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/K)$. Since i_+ is conservative, it suffices to show that the homomorphism

$$\rho\colon i_+i^+\big(\mathscr{N}\widetilde{\otimes}\mathscr{M}\big) \to i_+\big(i^+(\mathscr{N})\widetilde{\otimes}i^!(\mathscr{M})\big) \cong i_+i^+(\mathscr{N})\widetilde{\otimes}\mathscr{M}$$

is an isomorphism. By definition, this is the unique homomorphism which makes the following diagram commutative:

$$\begin{tikzcd} & i_+i^+\big(\mathscr{N}\widetilde{\otimes}\mathscr{M}\big) \arrow[d, "\rho"] \\ \mathscr{N}\widetilde{\otimes}\mathscr{M} \arrow[ur, "\alpha"] \arrow[r, "\beta"'] & i_+i^+(\mathscr{N})\widetilde{\otimes}\mathscr{M}. \end{tikzcd}$$

where $\alpha := \mathrm{adj}_i$ and $\beta := \mathrm{adj}_i \otimes \mathrm{id}$. Now, since the verification is local, we may assume that Z and X can be lifted to smooth formal schemes $\mathscr{Z}$ and $\mathscr{X}$. It suffices to show the claim after removing the boundaries by [Ber2, 4.3.10]. In this situation, recall that $(-)\widetilde{\otimes}(-) \cong (-)\otimes^\dagger_{\mathcal{O}_{\mathscr{X},\mathbb{Q}}}(-)[-\dim(\mathscr{X})]$ (cf. [AC1, 1.1.6]). Since $\mathscr{M}$ is a coherent $\mathcal{O}_{\mathscr{X},\mathbb{Q}}$-module, we have a canonical isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathscr{D}_{\mathscr{X}}}\big(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}, \mathscr{D}_{\mathscr{X}}\otimes\omega^{-1}_{\mathscr{X}}\big) \cong \mathbb{R}\mathcal{H}om_{\mathscr{D}_{\mathscr{X}}}(\mathscr{N}, \mathscr{D}_{\mathscr{X}}\otimes\omega^{-1}_{\mathscr{X}})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^\vee,$$

where $\mathscr{D}_{\mathscr{X}}$ denotes $\mathscr{D}^{\dagger}_{\mathscr{X},\mathbb{Q}}$. This yields an isomorphism $\gamma\colon \mathbb{D}(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}) \cong \mathbb{D}(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^{\vee}$. Consider the following diagram.

$$
\begin{tikzcd}
\mathbb{D}\bigl(i_+i^+(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M})\bigr) \arrow[r, dash, "\sim"] \arrow[rr, bend left, "\mathbb{D}\alpha"] & i_+i^!\bigl(\mathbb{D}(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M})\bigr) \arrow[r, "\mathrm{adj}_i"'] \arrow[d, dash, "\gamma"', "\sim"] & \mathbb{D}(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}) \arrow[d, dash, "\gamma"', "\sim"] \\
 & i_+i^!\bigl(\mathbb{D}(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^{\vee}\bigr) \arrow[r, "\mathrm{adj}_i"] \arrow[d, dash, "\sim"'] \arrow[dr, phantom, "\heartsuit"] & \mathbb{D}(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^{\vee} \\
\mathbb{D}\bigl(i_+i^+(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}\bigr) \arrow[r, dash, "\gamma", "\sim"'] \arrow[uu, dotted, "\star"] \arrow[d, "\mathbb{D}\beta"'] & i_+i^!\mathbb{D}(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^{\vee} \arrow[ur, bend right, "\mathrm{adj}_i"'] \arrow[d, "\mathrm{adj}_i"] & \\
\mathbb{D}(\mathscr{N}\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}) \arrow[r, dash, "\gamma", "\sim"'] & \mathbb{D}(\mathscr{N})\otimes_{\mathcal{O}_{\mathscr{X}}}\mathscr{M}^{\vee} \arrow[uur, equal, bend right] &
\end{tikzcd}
$$

where adj_i is the homomorphism induced by the adjunction homomorphisms of i. The diagram $\heartsuit$ is commutative by [Ca4, 2.2.7], and the other diagrams formed by solid arrows are commutative as well. We define the isomorphism $\star$ so that the diagram is commutative. By the characterization of ρ, the isomorphism $\star$ is the dual of ρ, which implies that ρ is an isomorphism. $\square$

2. Arithmetic $\mathscr{D}$-modules for algebraic stacks

This section is devoted to constructing a p-adic cohomology theory for algebraic stacks. Even though we do not try to axiomatize, the ideas of this section work also for any rational cohomology theories with standard six functor formalism without essential changes (e.g., algebraic $\mathscr{D}$-module theory, étale cohomology theory with perverse t-structures over a separably closed field with rational coefficients, etc.).

2.0.1. In this section, $\blacktriangle \in \{\emptyset, F\}$ is fixed. Throughout this section, we also fix a base tuple either $\mathfrak{T}_{\emptyset} := (k, R, K, L)$ or $\mathfrak{T}_F := (k, R, K, L, s, \sigma)$ (cf. 1.4.10) depending on which $\blacktriangle$ we take. We often denote $D(X/\mathfrak{T}_{\blacktriangle})$ by $D(X/L_{\blacktriangle})$ or even $D(X/L)$ or $D(X)$ if no confusion may arise. When $\blacktriangle = \emptyset$, Tate twists (n) are understood to be the identity functors as usual. All algebraic stacks are understood to be over k unless otherwise specified.

2.0.2. To construct a theory for general algebraic stacks, we will first need to construct a theory for algebraic spaces. Then using the theory for algebraic spaces, we shall construct a theory for all algebraic stacks. Since the process for generalizing the construction from the case of realizable schemes to algebraic spaces is the same as generalizing from algebraic spaces to algebraic stacks, we shall present the processes at the same time. In order to obtain the theory for general algebraic spaces, the reader should read §2.1 and §2.2 as follows. First, when reading §2.1 and §2.2, replace the word "space" (resp. "good stack", resp. "good presentation") by "quasi-projective scheme" (resp. . . .), the corresponding terminology indicated in the first block of the table below. The reader should then reread §2.1 and §2.2, this time using the second block of the table below instead of the first block.

$\boxed{\text{1st read}}$	space:	quasi-projective scheme
	good stack:	algebraic stack of finite type whose diagonal morphism is quasi-projective
	good presentation:	smooth surjective morphism from a quasi-projective scheme
$\boxed{\text{2nd read}}$	space:	separated algebraic space of finite type
	good stack:	algebraic stack of finite type
	good presentation:	smooth surjective morphism from a separated quasi-compact algebraic space

Note that since good stacks are of finite type, they are in particular quasi-compact. See paragraphs 2.2.26–2.2.28 for additional explanation. Finally, we remark that admissible stacks defined in §2.3 are good stacks in the sense of first read, and a second read is not really necessary if the reader is only interested in six functor formalism for schemes or Deligne–Mumford stacks.

2.1. Definition of the derived category of $\mathscr{D}^\dagger$-modules for stacks.

2.1.1. We first need basic cohomological operations for spaces. Let X be a space over k. In the first read case, we have already defined $M(X/L)$ and $D(X/L)$ in 1.4.10, and in the second read case, see 2.2.26.

Smooth morphism. Let $f\colon X \to Y$ be a smooth morphism between spaces over k. The exact functor $f^*\colon \mathrm{Hol}(Y/K) \to \mathrm{Hol}(X/K)$ (cf. 1.2.8 or 2.2.7) can be extended canonically to $M(Y/L) \to M(X/L)$ by 1.2.2 and 1.4.9, f^* remains to be exact. The derived functor is also denoted by f^*. Similarly, we have the left exact functor $f_*\colon M(X/L) \to M(Y/L)$, and we can take the derived functor $\mathbb{R}f_*\colon D^+(X/L) \to D^+(Y/L)$. By 1.2.1, $(f^*, \mathbb{R}f_*)$ is an adjoint pair. Since for_L and for_F commute with f^* and $\mathbb{R}f_*$ by 1.4.1 and 1.4.6, f^* and $\mathbb{R}f_*$ preserve holonomicity and induce functors between $D^+_{\mathrm{hol}}(X/L)$ and $D^+_{\mathrm{hol}}(Y/L)$.

Finite morphism. Let $f\colon X \to Y$ be a finite morphism between spaces over k. Just as in the smooth morphism case, we can define functors

$$f_+\colon D^\star(X/L) \rightleftarrows D^\star(Y/L)\colon f^!,$$

where $\star = \emptyset$ for f_+ and $+$ for $f^!$. The pair $(f_+, f^!)$ is an adjoint pair. These functors commute with for_L and for_F, and they preserve boundedness and holonomicity.

External tensor product. Let X, Y be spaces over k. Extending the scalar of the external tensor product functor, we have the bifunctor $\boxtimes\colon M(X/L)\times M(Y/L) \to M(X \times Y/L)$, which is exact. Thus, we can take the derived functor. This derived functor preserves boundedness and holonomicity as well.

Dual functor. Let X be a space over k. The dual functor extends canonically to $\mathbb{D}_X\colon \mathrm{Hol}(X/L)^\circ \to \mathrm{Hol}(X/L)$. This induces the functor

$$\mathbb{D}_X\colon D^{\mathrm{b}}(\mathrm{Hol}(X/L))^\circ \to D^{\mathrm{b}}(\mathrm{Hol}(X/L)).$$

Lemma. (i) *Consider the cartesian diagram* (1.1.3.1) *of spaces such that f is smooth and g is finite. Then the base change homomorphisms $f'^* \circ \mathscr{H}^0(g^!) \to \mathscr{H}^0(g'^!) \circ f^*\colon M(Y/L) \to M(X'/L)$ and $f^* \circ g_+ \to g'_+ \circ f'^*\colon M(Y'/L) \to M(X/L)$ are isomorphisms. Moreover, we have $\mathscr{H}^0(g^!) \circ f_* \cong f'_* \circ \mathscr{H}^0(g')^!\colon M(X/L) \to M(Y'/L)$.*

(ii) *For a smooth morphism $f\colon X \to Y$ of spaces of relative dimension d, we have the canonical isomorphism*

$$\bigl(f^* \circ \mathbb{D}_Y\bigr)(d) \cong \mathbb{D}_X \circ f^*\colon \mathrm{Hol}(Y)^\circ \to \mathrm{Hol}(X).$$

Remark. These equalities also hold on the level of $D^{\mathrm{b}}_{\mathrm{hol}}(-/L)$, which we show later.

Proof. In the second read case, this can easily be reduced to the first read case, so we assume we are in the first read case. When $\blacktriangle = \emptyset$ and $L = K$, we only need to show the equality for $\mathrm{Hol}(-/K)$, and these are consequences of Corollary 1.5.5, Theorem 1.5.13, and 1.1.3 (8). Since these equalities are on the level of modules, the results can be extended automatically to $\blacktriangle = F$ and general L. $\square$

2.1.2. Now, let us fix some terminologies on simplicial spaces.

Definition. For an integer $i \geq 0$, let $[i] = \{0, \dots, n\}$ be the ordered set, and put $[-1] := \emptyset$. Let Δ^+ be the category of those objects, and morphisms are increasing *injective* maps.

(i) An *admissible simplicial space* is a contravariant functor $(\Delta^+)^\circ \to \mathrm{Sp}^{\mathrm{sm}}(k)$, where $\mathrm{Sp}^{\mathrm{sm}}(k)$ denotes the category of spaces over k whose morphisms are smooth. Let $X_\bullet$ be an admissible simplicial space. For $i \geq 0$, we often denote $X_\bullet([i])$ by X_i. This can be described as follows:

$$X_\bullet : \Bigl[\, X_0 \leftleftarrows X_1 \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, X_2 \,\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\, \cdots \Bigr].$$

For a space S, let $S_\bullet$ be the constant admissible simplicial space. A morphism of an admissible simplicial space to a space $X_\bullet \to S$ is a morphism $X_\bullet \to S_\bullet$. The morphism is said to be *smooth* if $X_0 \to S$ is.

(ii) A morphism of simplicial spaces $f\colon X_\bullet \to Y_\bullet$ is said to be *cartesian* if for any $\phi\colon [i] \to [j]$, the following diagram is cartesian:

$$\begin{tikzcd} X_j \arrow[r, "f_j"] \arrow[d, "X(\phi)"'] \arrow[dr, phantom, "\square"] & Y_j \arrow[d, "Y(\phi)"] \\ X_i \arrow[r, "f_i"'] & Y_i. \end{tikzcd}$$

(iii) An *admissible double simplicial space* $X_{\bullet\bullet}$ is a functor $((\Delta^+)^2)^\circ \to \mathrm{Sp}^{\mathrm{sm}}(k)$. An admissible simplicial space $S_\bullet$ yields the constant admissible double simplicial space $S_{\bullet\bullet}$ by setting $S_{\bullet i} := S_\bullet$. A morphism from an admissible double simplicial space to an admissible simplicial space $X_{\bullet\bullet} \to S_\bullet$ is defined to be $X_{\bullet\bullet} \to S_{\bullet\bullet}$. This is a collection of morphisms $X_{n\bullet} \to S_n$ satisfying the compatibility conditions.

Remark. Note that we do not consider *degeneracy maps*, only *face maps*. These types of objects are sometimes called strictly simplicial schemes (e.g., [LO]).

2.1.3. **Definition.** Let $\mathfrak{X}$ be an algebraic stack over k, and let $\mathcal{X} \to \mathfrak{X}$ be a presentation (cf. 0.0.2). Put $\mathcal{X}_\bullet := \mathrm{cosk}_0(\mathcal{X} \to \mathfrak{X})$ (i.e., the simplicial space such that $\mathcal{X}_n := \underbrace{\mathcal{X} \times_{\mathfrak{X}} \times \cdots \times_{\mathfrak{X}} \mathcal{X}}_{n}$ and the face morphisms are projections). We say that $\mathcal{X}_\bullet$ is a *simplicial algebraic space presentation of* $\mathfrak{X}$. Let $\mathbf{P}$ be a set of algebraic spaces. A *simplicial* $\mathbf{P}$ *presentation* is a simplicial algebraic space presentation $X_\bullet$

consisting of algebraic spaces belonging to **P** (e.g., simplicial realizable schemes presentation, etc.).

Now, let $\mathfrak{X}$ be a good stack, and let $X \to \mathfrak{X}$ be a good presentation. Since $\mathfrak{X}$ is a good stack, $\mathrm{cosk}_0(X \to \mathfrak{X})$ is an admissible simplicial space. In particular, for any good stack, we may take a simplicial space presentation.

2.1.4. **Definition.** Let $X_\bullet$ be an admissible simplicial space. For a morphism $\phi\colon [i] \to [j]$, since $X(\phi)$ is smooth, the pullback

$$X(\phi)^*\colon M(X_i/L) \to M(X_j/L),$$

which is exact, is defined (cf. 2.1.1). This defines a cofibered category $M(X_\bullet/L)_\bullet$ over Δ^+.

(i) We put $M(X_\bullet/L) := \mathrm{sec}_+(M(X_\bullet/L)_\bullet)$ (see §A.1 for the notation). We often denote $M(X_\bullet/L)$ by $M(X_\bullet)$. For $\mathscr{M}_\bullet$ or $\mathscr{M}$ in $M(X_\bullet/L)$, the fiber over $[i]$ is denoted by $\mathscr{M}_i$. For $\phi\colon [i] \to [j]$, the homomorphism $X(\phi)^*\mathscr{M}_i \to \mathscr{M}_j$ is called the *gluing homomorphism*.

(ii) We denote by $\mathrm{Hol}(X_\bullet/L)$, or simply by $\mathrm{Hol}(X_\bullet)$, the full subcategory of $M(X_\bullet/L)_{\bullet,\mathrm{tot}}$ consisting of $\mathscr{M}_\bullet$ such that $\mathscr{M}_i \in \mathrm{Hol}(X_i/L)$ for any $i \geq 0$.

(iii) We denote by $D^\star_{\mathrm{hol}}(X_\bullet/L)$ or $D^\star_{\mathrm{hol}}(X_\bullet)$ the full subcategory of $D^\star(M(X_\bullet/L))$ whose cohomology objects are in $\mathrm{Hol}(X_\bullet/L)$. We denote $D^\star_{\mathrm{tot}}(M(X_\bullet/L)_\bullet)$ by $D^\star_{\mathrm{tot}}(X_\bullet/L)$ or $D^\star_{\mathrm{tot}}(X_\bullet)$.

2.1.5. Let $\mathscr{M}_\bullet$ and $\mathscr{N}_\bullet$ be in $D^{\mathrm{b}}_{\mathrm{tot}}(X_\bullet/L)$. Then by (A.1.2.1), we have the following spectral sequence:

$$E_1^{p,q} := \mathrm{Ext}^q_{D(X_p)}(\mathscr{M}_p, \mathscr{N}_p) \Rightarrow \mathrm{Ext}^{p+q}_{D(X_\bullet)}(\mathscr{M}_\bullet, \mathscr{N}_\bullet). \tag{2.1.5.1}$$

We have kernels, cokernels, and inductive limits in $M(X_\bullet)$, and they can be calculated termwise; namely, these functors commute with the functor sending $\mathscr{M}_\bullet$ to $\mathscr{M}_i$ for any $i \geq 0$. This is because, for any $\phi\colon [i] \to [j]$, the functors Ker, Coker, $\varinjlim$ commute with $X(\phi)^*$. In particular, $M(X_\bullet)$ is an abelian category. Moreover, projective limits are representable in $M(X_\bullet)$ and they can be calculated termwise as well by using the canonical homomorphism $X(\phi)^* \circ \varprojlim \to \varprojlim \circ X(\phi)^*$.

2.1.6. Let us define three basic functors. Take $i \geq 0$. We define a functor $\rho_i^*\colon M(X_\bullet) \to M(X_i)$ by sending $\mathscr{M}_\bullet \in M(X_\bullet)$ to $\mathscr{M}_i$. Obviously, this is an exact functor. Now, take $\mathscr{N} \in M(X_i)$. We define

$$\rho_{i*}(\mathscr{N}) := \left\{ \prod_{\phi\colon [k]\to[i]} X(\phi)_*(\mathscr{N}) \right\}_k, \qquad \rho_{i!}(\mathscr{N}) := \left\{ \bigoplus_{\phi\colon [i]\to[k]} X(\phi)^*(\mathscr{N}) \right\}_k,$$

and the gluing homomorphisms are defined as follows: for $\psi\colon [k] \to [k']$, the map $X(\psi)^*\rho_{i*}(\mathscr{N})_k \to \rho_{i*}(\mathscr{N})_{k'}$ (resp. $X(\psi)^*\rho_{i!}(\mathscr{N})_k \to \rho_{i!}(\mathscr{N})_{k'}$) is the product (resp. direct sum) of the adjunction (resp. canonical) homomorphisms

$$X(\psi)^*X(\phi)_*(\mathscr{N}) \to X(\phi')_*(\mathscr{N}) \qquad \big(\text{resp. } X(\psi)^*X(\phi)^*(\mathscr{N}) \to X(\phi')^*(\mathscr{N})\big),$$

where $\phi\colon [k] \xrightarrow{\psi} [k'] \xrightarrow{\phi'} [i]$ and 0 if ϕ cannot be factored through ψ (resp. $\phi'\colon [i] \xrightarrow{\phi} [k] \xrightarrow{\psi} [k']$). These data define functors $\rho_{i*}, \rho_{i!}\colon M(X_i) \to M(X_\bullet)$.

Lemma. (i) *We have adjoint pairs* (ρ_i^*, ρ_{i*}) *and* $(\rho_{i!}, \rho_i^*)$.

(ii) *The functors* ρ_i^* *and* $\rho_{i!}$ *are exact. In particular* ρ_i^* *and* ρ_{i*} *preserve injective objects.*

(iii) *The category* $M(X_\bullet)$ *is a Grothendieck category.*

Proof. Since the verification of (i) is standard, we leave the details to the reader. The first claim of (ii) follows from the exactness of $X(\phi)^*$. Let us check (iii). We have an arbitrary inductive limit as observed in 2.1.5, and filtrant inductive limits are exact. We need to show that it has a generator. Let $\mathscr{G}_i$ be a generator of $M(X_i)$. Then $\{\rho_{i!}(\mathscr{G}_i)\}_{i\geq 0}$ is a set of generators. Indeed, let $\mathscr{M} \in M(X_\bullet)$, and assume $\mathrm{Hom}(\rho_{i!}(\mathscr{G}_i), \mathscr{M}) = 0$ for any $i \geq 0$. Then by (i), we have $\mathrm{Hom}(\mathscr{G}_i, \rho_i^*(\mathscr{M})) = 0$, and thus $\mathscr{M}_i = 0$. Thus by definition, $\mathscr{M} = 0$. □

2.1.7. Let $X_\bullet$ be an admissible simplicial space. Let k be a nonnegative integer. Given $\mathscr{M}_\bullet \in M(\mathrm{sk}_k(X_\bullet))$, we get an object in $M(\mathrm{sk}_{k-1}(X_\bullet))$ denoted by $\sigma_k^*(\mathscr{M}_\bullet) \in M(X_\bullet)$ by putting $\bigl(\sigma_k^*(\mathscr{M}_\bullet)\bigr)_i \cong \mathscr{M}_i$ for $i < k$. Let us construct the left adjoint functor of σ_k^*.

Let D_k be the category of homomorphisms $\phi\colon [i] \to [k]$ such that the morphism from ϕ to $\psi\colon [j] \to [k]$ is a morphism $\alpha\colon [i] \to [j]$ such that $\psi \circ \alpha = \phi$. Now, given $\mathscr{M}_\bullet$ in $M(\mathrm{sk}_{k-1}(X_\bullet))$, we can construct an object in $M(\mathrm{sk}_k(X_\bullet))$ as follows: Put

$$\mathscr{M}_k := \varinjlim_{\phi \in D_k} X(\phi)^*(\mathscr{M}_i).$$

Then $\{\mathscr{M}_i\}_{i\leq k}$ with an obvious gluing homomorphism defines the desired object. The functor is denoted by $\sigma_{k!}$. We can check easily that $(\sigma_{k!}, \sigma_k^*)$ is an adjoint pair.

2.1.8. **Remark.** The functors defined in the last two paragraphs have natural interpretation in terms of the language of topos. Let $X_\bullet$ be a (strictly) simplicial topos (cf. [SGA4, $\mathrm{V}^{\mathrm{bis}}$]). Since sheaves $\mathscr{F}$ on $X_\bullet$ can be described as data $\{\mathscr{F}_i, \phi_{ij}\}$ where $\mathscr{F}_i$ is a sheaf on X_i and ϕ_{ij} is a gluing homomorphism, we have a functor sending a sheaf $\mathscr{F}$ on $X_\bullet$ to the sheaf $\mathscr{F}_i$ on X_i. This functor defines a morphism of topos $e_i\colon X_i \to X_\bullet$. The functors $\rho_{i!}$, ρ_i^*, ρ_{i*} are nothing but analogues of the functors $e_{i!}$, e_i^*, e_{i*} (cf. [SGA4, 1.2.8–1.2.12]). The interpretation of $\sigma_{i!}$, σ_i^* is similar.

2.1.9. **Lemma.** *Let us denote by* $\mathrm{Hol}(X_\bullet)_\bullet$ *the cofibered category over* Δ^+ *such that the fiber over* $[i]$ *is* $\mathrm{Hol}(X_i)$. *Let* $D^{\mathrm{b}}_{\mathrm{tot}}(\mathrm{Hol}(X_\bullet)_\bullet)$ *be the derived category defined in* §A.1. *The canonical functor* $D^{\mathrm{b}}_{\mathrm{tot}}(\mathrm{Hol}(X_\bullet)_\bullet) \to D^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet)$ *is an equivalence of categories.*

Proof. We can argue as [KSc, 15.3.1]. It suffices to show that the functor

$$D^{\mathrm{b}}(\mathrm{Hol}(X_\bullet)_\bullet) \to D^{\mathrm{b}}(X_\bullet)$$

is fully faithful. By [KSc, 13.2.8], it suffices to show the following: given a surjection $\mathscr{A} \to \mathscr{M}$ such that $\mathscr{M} \in \mathrm{sec}_+\mathrm{Hol}(X_\bullet)_\bullet$ and $\mathscr{A} \in M(X_\bullet)$, there exists a homomorphism $\mathscr{N} \to \mathscr{A}$ such that $\mathscr{N} \in \mathrm{sec}_+\mathrm{Hol}(X_\bullet)_\bullet$ and the composition $\mathscr{N} \to \mathscr{M}$ is surjective. To check this, it suffices to construct, for each $k \geq 0$, the following $\mathscr{N}_{(k)}$ in $M(\mathrm{sk}_k(X_\bullet))$: 1. $\sigma_k^*(\mathscr{N}_{(k)}) \cong \mathscr{N}_{(k-1)}$; 2. we have a homomorphism $\mathscr{N}_{(k)} \to \sigma_k^*(\mathscr{A})$ such that the composition $\mathscr{N}_{(k)} \to \sigma_k^*(\mathscr{A}) \to \sigma_k^*(\mathscr{M})$ is surjective.

We use the induction on k. For $\mathscr{N}_{(0)}$, take one as in [KSc, 15.3.1]. Assume we have constructed $\mathscr{N}_{(k-1)}$. Take $\mathscr{N}' \to \mathscr{A}_k$ such that $\mathscr{N}' \in \mathrm{Hol}(X_k)$ and the composition with $\mathscr{A}_k \to \mathscr{M}_k$ is surjective, as in [KSc, 15.3.1]. Given $\phi\colon [i] \to [k]$, we have the following diagram:

$$\begin{tikzcd} X(\phi)^*\mathscr{N}_{(k-1),i} \arrow[r] \arrow[d, dotted, "?"'] & X(\phi)^*\mathscr{A}_i \arrow[r] \arrow[d] & X(\phi)^*\mathscr{M}_i \arrow[d] \\ \mathscr{N}' \arrow[r] & \mathscr{A}_k \arrow[r] & \mathscr{M}_k. \end{tikzcd}$$

We need to construct the dotted homomorphism making the diagram commutative, for which we modify $\mathscr{N}'$. We put

$$(\mathscr{N}_{(k)})_i := \begin{cases} (\mathscr{N}_{(k-1)})_i & \text{for } i < k, \\ \mathscr{N}' \oplus (\sigma_{k!}(\mathscr{N}_{(k-1)}))_k & \text{for } i = k. \end{cases}$$

With the obvious gluing homomorphisms, these data define an object $\mathscr{N}_{(k)}$ in $M(\mathrm{sk}_k(X_\bullet))$, which is what we are looking for. $\square$

2.1.10. Let $f_\bullet\colon X_\bullet \to Y_\bullet$ be a *cartesian* morphism of admissible simplicial spaces such that f_i is *finite* for any $i \geq 0$. We call such a morphism a *cartesian finite morphism* for short. For a morphism $\phi\colon [j] \to [k]$, we have the following cartesian diagram:

$$\begin{tikzcd} X_k \arrow[r, "f_k"] \arrow[d, "X(\phi)"'] \arrow[dr, phantom, "\square"] & Y_k \arrow[d, "Y(\phi)"] \\ X_j \arrow[r, "f_j"'] & Y_j, \end{tikzcd}$$

where f_j and f_k are finite. Let $\mathscr{M}_\bullet$ be an object in $M(Y_\bullet)$. For a finite morphism g, we denote $\mathscr{H}^0 g^!$ by g°, which is left exact by 1.2.9. We have a canonical homomorphism

$$X(\phi)^* f_j^\circ(\mathscr{M}_j) \cong f_k^\circ Y(\phi)^*(\mathscr{M}_j) \to f_k^\circ(\mathscr{M}_k)$$

by Lemma 2.1.1. Using this homomorphism, $\{f_k^\circ(\mathscr{M}_k)\}$ defines an object in $M(X_\bullet)$, and it defines a functor $f^\circ\colon M(Y_\bullet) \to M(X_\bullet)$. Since f_k° is left exact, f° is left exact as well. We can take the associated derived functor to get

$$f^! := \mathbb{R}f^\circ\colon D^+(M(Y_\bullet)) \to D^+(M(X_\bullet)).$$

On the other hand, the functor f_{k+} is exact. For $\mathscr{N} \in M(X_\bullet)$, using Lemma 2.1.1, we have a homomorphism

$$Y(\phi)^* f_{j+}(\mathscr{N}_j) \cong f_{k+} X(\phi)^*(\mathscr{N}_j) \to f_{k+}(\mathscr{N}_k),$$

which defines an object $\{f_{k+}(\mathscr{N}_k)\}$ in $M(Y_\bullet)$. The functor is denoted by f_+. Since this functor is exact, we can take the derived functor

$$f_+\colon D(M(X_\bullet)) \to D(M(Y_\bullet)).$$

Lemma. *Let $X_\bullet \xrightarrow{f} Y_\bullet \xrightarrow{g} Z_\bullet$ be cartesian finite morphisms of admissible simplicial spaces.*

(i) *We have a canonical isomorphism $\rho_k^* \circ f^! \cong f_k^! \circ \rho_k^*$. In particular, $f^!$ sends $D^\star_{\mathrm{hol}}(Y_\bullet)$ into $D^\star_{\mathrm{hol}}(X_\bullet)$ for $\star \in \{+, \mathrm{b}\}$.*

(ii) *We have an adjoint pair $(f_+, f^!)$, and f_+ is exact. In particular, f° preserves injective objects.*

(iii) *We have a canonical isomorphism $f^! \circ g^! \cong (g \circ f)^!$.*

Proof. Let us show (i). Since ρ_k^* is exact and preserves injective objects by Lemma 2.1.6, the first isomorphism follows by definition. Let us check the second one. The functor $f^!$ preserves total complexes by Lemma 2.1.1 (i). It remains to show that it preserves holonomicity and boundedness. These are immediate consequences of the isomorphism $\rho_k^* \circ f^! \cong f_k^! \circ \rho_k^*$.

The verification of (ii) is easy. To show (iii), by (ii), it suffices to show that $f^\circ \circ g^\circ \cong (g^\circ \circ f^\circ)$. This follows by definition and the corresponding statement for spaces. □

2.1.11. Now, we use the notation of 2.1.1. Let $f\colon X_\bullet \to S$ be a smooth morphism (cf. 2.1.2) from an admissible simplicial space to a space. In this situation, let us define an adjoint pair of functors $(f^*, \mathbb{R}f_*)$. Let $f_i\colon X_i \to S$ be the induced morphism. The pullback is easy to define: Let $\mathscr{N} \in M(S)$. We put $\mathscr{N}_i := f_i^*(\mathscr{N})$ which is defined in $M(X_i)$. Let $\phi\colon [i] \to [j]$ be a map. Then we define a homomorphism, which is in fact an isomorphism, $X(\phi)^*\mathscr{N}_i \to \mathscr{N}_j$ to be the gluing homomorphism. The object we constructed in $M(X_\bullet)$ is denoted by $f^*(\mathscr{N})$. Thus, we have a functor

$$f^*\colon M(S) \to M(X_\bullet).$$

The functor f^* is exact since each f_i^* is.

Let us define its right adjoint. Take $\mathscr{M}_\bullet$ in $M(X_\bullet)$. For $\phi\colon [i] \to [j]$, we have the homomorphism

$$\alpha_\phi\colon f_{i*}(\mathscr{M}_i) \to f_{j*}(\mathscr{M}_j).$$

We put

$$f_*(\mathscr{M}_\bullet) := \mathrm{Ker}\big(f_{0*}(\mathscr{M}_0) \rightrightarrows f_{1*}(\mathscr{M}_1)\big).$$

Since f_{0*} and f_{1*} are left exact, the functor f_* is left exact as well. Thus we may take the associated derived functor to get

$$\mathbb{R}f_*\colon D^+(X_\bullet) \to D^+(S).$$

These constructions can be generalized to a smooth morphism $f\colon X_{\bullet\bullet} \to S_\bullet$ from a double simplicial space to a simplicial space: Pullback is defined in an obvious manner. Let $\mathscr{M}_{\bullet\bullet}$ in $M(X_{\bullet\bullet})$. Recall that f is a collection of $f_{n\bullet}\colon X_{n\bullet} \to S_n$ satisfying the compatibility conditions. We get $f_{n\bullet*}(\mathscr{M}_{n\bullet})$ in $M(S_n)$. The transition homomorphism is defined by adjunction, and we have $f_{\bullet\bullet*}\colon M(X_{\bullet\bullet}) \to M(S_\bullet)$. This is a left exact functor, and we get the derived functor $\mathbb{R}f_*\colon D^+(X_{\bullet\bullet}) \to D^+(S_\bullet)$.

Lemma. *We have an adjoint pair (f^*, f_*). Thus, the pair $(f^*, \mathbb{R}f_*)$ is also, and the obvious analogue holds for the double simplicial case.*

Proof. The adjointness of $(f^*, \mathbb{R}f_*)$ follows by the first one using Lemma 1.2.1. Let $S_\bullet$ be the constant simplicial space. We have the morphism $f_\bullet\colon X_\bullet \to S_\bullet$, and the adjoint pair (f_i^*, f_{i*}) defines a pair of functors $(f_\bullet^*, f_{\bullet*})$ between $M(X_\bullet)$ and $M(S_\bullet)$. It is straightforward to check that this is an adjoint pair. Thus, it suffices to check the lemma for the morphism $S_\bullet \to S$. This follows from the following general fact: Let $\mathcal{A}$ be an abelian category, and let $\Delta^+\mathcal{A}$ be the category of cosimplicial objects, namely the abelian category of functors $\Delta^+ \to \mathcal{A}$. Let $\rho^*\colon \mathcal{A} \to \Delta^+\mathcal{A}$ be functor assigning the constant object, and let $\rho_*\colon \Delta^+\mathcal{A} \to \mathcal{A}$ be the functor associating $\mathrm{Ker}(M_0 \rightrightarrows M_1)$ to $\{M_i\}$. Then (ρ^*, ρ_*) is an adjoint pair. The verification is straightforward. The double simplicial case is similar. □

2.1.12. The following spectral sequence is one of the keys to show the cohomological descent.

Lemma. *Let $\mathscr{M} \in D^+(X_\bullet)$. Then we have the following spectral sequence:*

$$E_1^{p,q} := \mathbb{R}^q f_{p*}(\mathscr{M}_p) \Rightarrow \mathbb{R}^{p+q} f_*(\mathscr{M}).$$

Proof. The construction is essentially the same as [O3, Corollary 2.7]. Since we use a similar argument again later, we sketch the proof. Let $\mathscr{N} \in M(X_i)$. We have a homomorphism $f_{0*}\bigl(\rho_{i*}(\mathscr{N})_0\bigr) \cong \prod_{[0]\to[i]} f_{i*}(\mathscr{N}) \to f_{i*}(\mathscr{N})$ where the second homomorphism is the projection to the component of the map $\alpha\colon [0] \to [i]$ such that the image is $0 \in [i]$. This induces a homomorphism from the Čech type complex:

$$C_i^\bullet(\mathscr{N}) := \bigl[0 \to f_{0*}\bigl(\rho_0^*\rho_{i*}(\mathscr{N})\bigr) \to f_{1*}\bigl(\rho_1^*\rho_{i*}(\mathscr{N})\bigr) \to f_{2*}\bigl(\rho_2^*\rho_{i*}(\mathscr{N})\bigr) \to \cdots\bigr]$$

to $f_{i*}(\mathscr{N})$. This homomorphism is in fact a homotopy equivalence. Indeed, the cohomologies of the complex $\widetilde{T} := [0 \to \prod_{[0]\to[i]} L \to \prod_{[1]\to[i]} L \to \cdots]$, which is isomorphic to that of the i-simplex, vanish except for degree 0, and since $K(\mathrm{Vec}_L) \cong D(\mathrm{Vec}_L)$, the homomorphism $\widetilde{T} \to L$ of the projection to the α-component is a homotopy equivalence. Since $C_i^\bullet(\mathscr{N}) \cong \widetilde{T} \otimes_L f_{i*}(\mathscr{N})$, we get the claim.

Now, for any $\mathscr{N} \in M(X_\bullet)$, there exists an embedding $\mathscr{N} \hookrightarrow \mathscr{I}$ into an injective object in $M(X_\bullet)$ such that the complex

$$(\star) \qquad 0 \to f_{0*}(\rho_0^*(\mathscr{I})) \to f_{1*}(\rho_1^*(\mathscr{I})) \to f_{2*}(\rho_2^*(\mathscr{I})) \to \cdots$$

is exact away from the degree 0 part. For this, take an embedding $\rho_i^*\mathscr{N} \hookrightarrow \mathscr{I}_{(i)}$ into an injective object in $M(X_i)$, and put $\mathscr{I} := \prod_i \rho_{i*}(\mathscr{I}_{(i)})$. Note that products can be calculated termwise by 2.1.5. Moreover, small products and f_{i*} commute by [KSc, 2.1.10] since f_{i*} admits a left adjoint f_i^*. This implies that $(\star)$ is isomorphic to $\prod_i C_i^\bullet(\mathscr{I}_{(i)})$. Now, each $C_i^\bullet(\mathscr{I}_{(i)})$ is homotopic to $f_{i*}(\mathscr{I}_{(i)})$ by the observation above. Since homotopy equivalence is preserved even after taking product, we get that $(\star)$ is homotopic to $\prod_i f_{i*}\mathscr{I}_{(i)}$; in particular, the complex is exact away from 0. Since ρ_{i*} preserves injective objects by Lemma 2.1.6 and the product of injective objects remains to be injective, $\mathscr{I}$ is a desired object.

Finally, let $\mathscr{M} \to \mathscr{I}^\bullet$ be a resolution of $\mathscr{M}$ consisting of the complex as above. We consider the double complex $\bigl\{f_{p*}\rho_p^*(\mathscr{I}^q)\bigr\}_{p,q\geq 0}$. The acyclicity of $(\star)$ except for degree 0 shows that the total complex is $\mathbb{R}f_*(\mathscr{M})$, and thus the associated spectral sequence is the one we want. □

Recall that by Lemma 1.2.8, or by 2.2.7 in the second read case, $\mathbb{R}f_{i*}$ preserves holonomicity. Thus, we have the following corollary:

Corollary. *The functor $\mathbb{R}f_*$ preserves holonomicity and induces a functor $D^+_{\mathrm{hol}}(X_\bullet) \to D^+_{\mathrm{hol}}(S)$.*

2.1.13. **Proposition.** *Let $X \to S$ be a smooth surjective morphism between spaces, and put $f\colon X_\bullet := \mathrm{cosk}_0(X/S) \to S$. The adjoint pair of functors $(f^*, \mathbb{R}f_*)$ induces an equivalence between $D^\star_{\mathrm{tot}}(X_\bullet)$ and $D^\star(S)$ for $\star \in \{+, \mathrm{b}\}$. Moreover, it induces an equivalence between $D^\star_{\mathrm{hol}}(X_\bullet)$ and $D^\star_{\mathrm{hol}}(S)$.*

Proof. The second claim follows by the first one since f^* and $\mathbb{R}f_*$ preserve the holonomicity by Corollary 2.1.12. Thus, it suffices to show that the canonical homomorphisms $\mathrm{id} \to \mathbb{R}f_*f^*$ and $f^*\mathbb{R}f_* \to \mathrm{id}$ are isomorphisms. For the first one, it suffices to show the equalities after taking $i_s^!$ where s is a closed point of S and $i_s\colon \{s\} \hookrightarrow S$. By taking the fiber product, it induces a morphism of simplicial spaces $i_{X,s}\colon X_{s\bullet} \hookrightarrow X_\bullet$. Using Lemma 2.1.1, we have $\mathbb{R}f_{s*} \circ i^!_{X,s} \cong i^!_s \circ \mathbb{R}f_*$ where $f_s\colon X_{s\bullet} \to s$. Thus, by considering Lemma 1.3.7 (ii), this can be reduced to the situation where we have a section $s\colon S \to X_0$ of f_0. In this case, the verification for the first homomorphism is the same as [Co1, Thm 7.2]. Let us recall the argument briefly. By using Lemma 2.1.12, we have the spectral sequence $E_1^{p,q} = \mathbb{R}^q f_{p*}(\mathscr{M}_p) \Rightarrow \mathbb{R}^{p+q} f_*(\mathscr{M})$. Put $E_1^{-1,q} := \mathbb{R}^q\mathrm{id}_*(\mathscr{M})$, which is $\mathscr{M}$ if $q = 0$ and 0 otherwise. We claim that the complex defined by adjunction $0 \to E_1^{-1,q} \to E_1^{\bullet,q} \to 0$ where $E_1^{-1,q}$ is placed at degree -1, is acyclic. To check this, we construct a concrete homotopy $E_1^{p,q} \to E_1^{p-1,q}$ using the section s. For example, $E_1^{0,q} \to E_1^{-1,q}$ is constructed as follows: We have an isomorphism $s^+ f_0^+ \mathscr{M} \xrightarrow{\sim} \mathscr{M}$, which induces $f_0^+(\mathscr{M}) \to s_+(\mathscr{M})$ by adjunction. By taking f_{0+} and using the isomorphisms $f_{0+}s_+ \cong \mathrm{id}$ and $f_{0+}f_0^+ \cong f_{0*}f_0^*$, we obtain the desired homotopy. See [Co1] for details.

Let us show the second one. It suffices to show that the homomorphism of functors, before taking the derived functors, $f^*f_* \to \mathrm{id}$ is an isomorphism. Indeed, if this is shown, we get that for $\mathscr{M} \in M(X_\bullet)$, we have

$$\mathbb{R}f_*(\mathscr{M}) \xleftarrow{\sim} \mathbb{R}f_*\bigl(f^*f_*(\mathscr{M})\bigr) \xleftarrow{\sim} f_*(\mathscr{M}),$$

where the second quasi-isomorphism follows by the the first part of the proof. Thus we have $\mathbb{R}^i f_*(\mathscr{M}) = 0$ for $i \neq 0$. Finally, let us show $f^*f_*(\mathscr{M}) \to \mathscr{M}$ is an isomorphism. As the proof of the first isomorphism, it suffices to show the claim when S is a point, and in particular there is a section $S \to X$. In this case, it suffices to show that there exists $\mathscr{N} \in M(S)$ such that $f^*(\mathscr{N}) \cong \mathscr{M}$, namely, $\mathscr{M}$ is *effective descent*. Because of the existence of the section, this is automatic (for example, see [Gir, right after 6.15]). □

2.1.14. Let $f\colon X_\bullet \to S$ be a smooth morphism from an admissible simplicial space to a space, and let $g\colon S' \to S$ be a smooth morphism between spaces. Consider the following cartesian diagram:

$$\begin{tikzcd} X'_\bullet \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X_\bullet \arrow[d, "f"] \\ S' \arrow[r, "g"'] & S. \end{tikzcd}$$

For $\mathscr{M} \in M(X_\bullet)$, the system $\bigl\{g_i'^*(\mathscr{M}_i)\bigr\}_i$ defines an object of $M(X'_\bullet)$, denoted by $g'^*(\mathscr{M})$. This functor g'^* is exact and preserves holonomicity. We can take the derived functor $g'^*\colon D^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet) \to D^{\mathrm{b}}_{\mathrm{hol}}(X'_\bullet)$. Similarly, for $\mathscr{N} \in M(X'_\bullet)$, the system $\bigl\{g'_{i*}(\mathscr{N}_i)\bigr\}$ defines an object of $M(X_\bullet)$ by Corollary 1.5.5, or by the following lemma in the second read case, and the assumption that g' is cartesian. This functor is left exact, and we have the right derived functor $\mathbb{R}g'_*$. The couple $(g'^*, \mathbb{R}g'_*)$ is an adjoint pair.

Lemma. *The canonical homomorphism $g^* \circ \mathbb{R}f_* \to \mathbb{R}f'_* \circ g'^*$ is an isomorphism.*

Proof. By using the spectral sequence of Lemma 2.1.12, the verification is reduced to Corollary 1.5.5 or to the present lemma of the first read case in the second read. □

2.1.15. **Proposition.** *Let $\mathfrak{X}$ be a good stack, and let $X_\bullet \to \mathfrak{X}$ be a simplicial space presentation. Then for $\star \in \{\mathrm{b}, +\}$, the categories $D^\star_{\mathrm{tot}}(X_\bullet)$ and $D^\star_{\mathrm{hol}}(X_\bullet)$ do not depend on the choice of the presentation up to canonical equivalence, and the t-structure as well.*

Proof. Let $X_\bullet \to \mathfrak{X}$ and $X'_\bullet \to \mathfrak{X}$ be two presentations. Let $Z_{n,n'} := X_n \times_{\mathfrak{X}} X'_{n'}$. This defines a double simplicial space $Z_{\bullet\bullet}$ with projections $Z_{\bullet\bullet} \to X_\bullet$, and $Z_{\bullet\bullet} \to X'_\bullet$. Thus, it suffices to show the following: given a smooth morphism $f\colon Z_{\bullet\bullet} \to X_\bullet$ such that $(Z_{i\bullet} \to X_i) = \mathrm{cosk}_0(Z_{i0} \to X_i)$, the functors $\mathbb{R}f_*$ and f^* induce an equivalence of categories. First, the functors $\mathbb{R}f_*$ and f^* preserve holonomicity. Indeed, the preservation of holonomicity for f^* is easy, and for $\mathbb{R}f_*$ use (the double simplicial analogue of) Lemma 2.1.6 (ii), Lemma 2.1.14, and Corollary 2.1.12. If f^*, $\mathbb{R}f_*$ yield an equivalence between $D^\star(X_\bullet)$ and $D^\star(Z_{\bullet\bullet})$, since they preserve holonomicity, they induce the equivalence of $D^\star_{\mathrm{hol}}(X_\bullet)$ and $D^\star_{\mathrm{hol}}(Z_{\bullet\bullet})$, and the proposition follows.

Now, it remains to show that for $\mathscr{N} \in M(X_\bullet)$ and $\mathscr{M} \in M(Z_{\bullet\bullet})$, the homomorphisms

$$f^*\mathbb{R}f_*(\mathscr{M}) \to \mathscr{M}, \qquad \mathscr{N} \to \mathbb{R}f_*f^*(\mathscr{N})$$

are isomorphisms. Since it suffices to show the isomorphism for each X_i, this follows from Proposition 2.1.13. □

2.1.16. **Definition.** Let $\mathfrak{X}$ be a good stack. Take a simplicial space presentation $X_\bullet \to \mathfrak{X}$. By the proposition above, for $\star \in \{\mathrm{b}, +\}$, the categories $D^\star_{\mathrm{tot}}(X_\bullet/L_\blacktriangle)$ and $D^\star_{\mathrm{hol}}(X_\bullet/L_\blacktriangle)$ do not depend on the choice of the presentation. We denote these categories by $D^\star(\mathfrak{X}/L_\blacktriangle)$ and $D^\star_{\mathrm{hol}}(\mathfrak{X}/L_\blacktriangle)$, or more precisely $D^\star(\mathfrak{X}/\mathfrak{T}_\blacktriangle)$ and $D^\star_{\mathrm{hol}}(\mathfrak{X}/\mathfrak{T}_\blacktriangle)$. We often omit $(\cdot)_\blacktriangle$ as usual. These categories are endowed with t-structure, and their hearts are denoted by $M(\mathfrak{X}/L)$ and $\mathrm{Hol}(\mathfrak{X}/L)$, respectively. Objects of $\mathrm{Hol}(\mathfrak{X}/L)$ are called *holonomic modules on* $\mathfrak{X}$. As usual, we often even omit "$/L$" from the notation of categories.

Remark. (i) When $\mathfrak{X}$ is a realizable scheme, $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ is equivalent to the one defined in 1.1.1 by Proposition 2.1.13.

(ii) Let $X_\bullet \to \mathfrak{X}$ be a simplicial space presentation. Then ρ_0^* is conservative, namely, $\rho_0^*(\mathscr{M}) = 0$ for $\mathscr{M} \in D^+_{\mathrm{hol}}(\mathfrak{X})$ if and only if $\mathscr{M} = 0$. Indeed, $\rho_0^*(\mathscr{M}) = 0$ implies $\rho_i^*(\mathscr{M}) = 0$, so the only if part holds.

2.1.17. **Definition.** (i) Let $\mathfrak{X}$ be a good stack. Assume further that the associated reduced algebraic stack $\mathfrak{X}_{\mathrm{red}}$ is smooth. Let $X_\bullet \to \mathfrak{X}$ be a simplicial space presentation. The category of *smooth objects* denoted by $\mathrm{Sm}(\mathfrak{X}/L)$ is the full subcategory of $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/L)$ consisting of $\mathscr{M}$ such that, for any i, $\mathrm{for}_L(\rho_i^*\mathscr{M}) \in D^{\mathrm{b}}_{\mathrm{hol}}(X_i/K)$ is in $\mathrm{Sm}(X_i/K)[d_i]$, where d_i denotes the relative dimension function (cf. 0.0.3) of $X \to \mathfrak{X}$, and see 1.1.3 (12) for the notation of $\mathrm{Sm}(X/K)$. It is straightforward to check that the category does not depend on the choice of the presentation.

(ii) Let $\mathfrak{X}$ be a good stack, and let $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X})$. The *support of* $\mathscr{M}$ is the minimum closed subset $Z \subset \mathfrak{X}$ such that the restriction of $\mathscr{M}$ to $\mathfrak{X}\setminus Z$ is 0. The support is denoted by $\mathrm{Supp}(\mathscr{M})$. For $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/L)$, we put $\mathrm{Supp}(\mathscr{M}) := \bigcup_i \mathrm{Supp}(\mathscr{H}^i\mathscr{M})$.

2.1.18. Let $X_\bullet$ be an admissible simplicial space. Since $M(X_\bullet)$ has enough injectives, we have the bifunctor $\mathbb{R}\mathrm{Hom}_{D(X_\bullet)}(-,-)\colon D(X_\bullet)^\circ \times D^+(X_\bullet) \to D(\mathrm{Vec}_L)$ (cf. [Ha, I, §6]). This induces the bifunctor

$$\mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}(-,-)\colon D^+(\mathfrak{X})^\circ \times D^+(\mathfrak{X}) \to D^+(\mathrm{Vec}_L).$$

Indeed, $\mathbb{R}\mathrm{Hom}_{D(X_\bullet)}$ does not depend on the choice of simplicial space presentation $X_\bullet \to \mathfrak{X}$. To check this, let $f\colon Z_{\bullet\bullet} \to X_\bullet$ be as in the proof of Proposition 2.1.15. Then we have a canonical homomorphism $\mathbb{R}\mathrm{Hom}_{D(X_\bullet)}(\mathscr{M}, \mathscr{N}) \to \mathbb{R}\mathrm{Hom}_{D(Z_{\bullet\bullet})}\big(f^*(\mathscr{M}), f^*(\mathscr{N})\big)$ for $\mathscr{M} \in D(X_\bullet)$, $\mathscr{N} \in D^+(X_\bullet)$. It suffices to show that this homomorphism is a quasi-isomorphism when $\mathscr{M}, \mathscr{N} \in D^+_{\mathrm{tot}}(X_\bullet)$. This follows since the pair $(f^*, \mathbb{R}f_*)$ is an equivalence of categories.

> In the following, for simplicity, we use particularly $D^\star_{\mathrm{hol}}(\mathfrak{X}/L)$ even when we can generalize statements or constructions to $D^\star(\mathfrak{X}/L)$ easily.

2.1.19. We conclude this subsection with the following lemma that we use later, whose proof is similar to the proof of Proposition 2.1.15.

Lemma. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good stacks, and let $X_\bullet$ and $Y_\bullet$ be simplicial space presentations. Let $(X \times Y)_{n,n'} := X_n \times Y_{n'}$, which forms a double simplicial space denoted by $(X \times Y)_{\bullet\bullet}$. Then we have a canonical equivalence*

$$D^+_{\mathrm{hol}}(\mathfrak{X} \times \mathfrak{Y}) \xrightarrow{\sim} D^+_{\mathrm{hol}}((X \times Y)_{\bullet\bullet}).$$

2.2. **Cohomological functors.** In this subsection, we define some cohomological functors for algebraic stacks. Even though six functor formalism is expected for algebraic stacks, unfortunately, at this moment, we can obtain full formalism only for admissible stacks (cf. Definition 2.3.1), which is enough for our purposes. In this subsection, we define functors that we can define for general algebraic stacks.

Finite morphism case.

2.2.1. First, we will define the adjoint pair $(f_+, f^!)$ when f is a finite morphism between good stacks. To do this, we only need to translate the functor constructed in 2.1.10 in the language of algebraic stacks.

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism between good stacks. Let us define $f^!$ and f_+. Take a simplicial space presentation $Y_\bullet \to \mathfrak{Y}$. By pulling back, we get a simplicial space presentation $X_\bullet \to \mathfrak{X}$. Let $f_\bullet\colon X_\bullet \to Y_\bullet$ be the finite cartesian morphism. Let $\star \in \{\mathrm{b}, +\}$. We define

$$f_+ \colon D^\star_{\mathrm{hol}}(\mathfrak{X}) \cong D^\star_{\mathrm{hol}}(X_\bullet) \rightleftarrows D^\star_{\mathrm{hol}}(Y_\bullet) \cong D^\star_{\mathrm{hol}}(\mathfrak{Y}) \colon f^!.$$

We need to check the well-definedness, namely independence of the presentation. By the adjointness property, it suffices to show the independence for f_+. As in the proof of Proposition 2.1.15, it suffices to show the following: Consider the cartesian diagram

$$\begin{tikzcd} Z_{\bullet\bullet} \arrow[r, "g_\bullet"] \arrow[d, "p"'] \arrow[dr, phantom, "\square"] & W_{\bullet\bullet} \arrow[d, "q"] \\ X_\bullet \arrow[r, "f_\bullet"'] & Y_\bullet. \end{tikzcd}$$

Then $q^* \circ f_{\bullet+} \cong g_{\bullet+} \circ p^* \colon M(X_\bullet) \to M(W_{\bullet\bullet})$. The verification is straightforward and is left to the reader. We have the pair $(f_+, f^!)$ of adjoint functors between $D^\star_{\mathrm{hol}}(\mathfrak{X})$ and $D^\star_{\mathrm{hol}}(\mathfrak{Y})$.

Now, let $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ be finite morphisms of good stacks. We have canonical isomorphisms

$$c_{(g,f)} \colon f^! \circ g^! \xrightarrow{\sim} (g \circ f)^!, \qquad c^{(g,f)} \colon (g \circ f)_+ \xrightarrow{\sim} g_+ \circ f_+.$$

These isomorphisms are subject to the following two conditions: 1. we have $c_{(f,\mathrm{id})} = c_{(\mathrm{id},f)} = \mathrm{id}$, $c^{(f,\mathrm{id})} = c^{(\mathrm{id},f)} = \mathrm{id}$; 2. given homomorphisms $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{W}$, we have

$$c_{(h,g\circ f)} \circ c_{(g,f)}(h^!) = c_{(h\circ g,f)} \circ f^!(c_{(h,g)}), \qquad h_+ c^{(g,f)} \circ c^{(h,g\circ f)} = c^{(h,g)}(f_+) \circ c^{(h\circ g,f)}.$$

These results can be rephrased by using the language of (co)fibered categories (cf. [SGA1, Exp. VII, end of 7]) as follows. Let $\mathrm{St}^{\mathrm{fin}}(k)$ the be the category of good stacks (we do not consider the 2-morphisms) over k such that the morphisms are *finite morphisms* between good stacks. To a good stack $\mathfrak{X}$, we associate the triangulated category $D^\star_{\mathrm{hol}}(\mathfrak{X})$. For a finite morphism $\mathfrak{X} \to \mathfrak{Y}$, we consider the functor $f^!$, and $c_{(f,g)}$. Then these data form a fibered category $\mathscr{F}^! \to \mathrm{St}^{\mathrm{fin}}(k)$. Considering f_+ and $c^{(f,g)}$, we get a cofibered category $\mathscr{F}_\oplus \to \mathrm{St}^{\mathrm{fin}}(k)$.

Dual functors.

2.2.2. Let $f\colon X \to Y$ be a smooth morphism of relative dimension d between spaces. Then we have a canonical isomorphism $\big(f^* \circ \mathbb{D}_Y\big)(d) \xrightarrow{\sim} \mathbb{D}_X \circ f^*$ by Lemma 2.1.1. Now, let $X_\bullet$ be an admissible simplicial space, and assume we are given a smooth morphism $X_\bullet \to \mathfrak{X}$ to an algebraic stack. We have the dual autofunctor $\mathbb{D}_{X_i}\colon \mathrm{Hol}(X_i)^\circ \xrightarrow{\sim} \mathrm{Hol}(X_i)$. We modify this functor by putting $\widetilde{\mathbb{D}}_i := (d_{X_i/\mathfrak{X}}) \circ \mathbb{D}_{X_i}$, where $d_{X_i/\mathfrak{X}}$ denotes the relative dimension function (cf. 0.0.3).

Now, we use the notation of Lemma 2.1.9 and §A.1. Let $\mathscr{M}_\bullet \in \mathrm{sec}_+(\mathrm{Hol}(X_\bullet)_\bullet)$. For a morphism $\phi\colon [i] \to [j]$, let $\alpha_\phi\colon X(\phi)^*\mathscr{M}_i \to \mathscr{M}_j$ be the gluing homomorphism. Let

$$\beta_\phi\colon \widetilde{\mathbb{D}}_j(\mathscr{M}_j) \to \widetilde{\mathbb{D}}_j\big(X(\phi)^*(\mathscr{M}_i)\big) \xleftarrow{\sim} X(\phi)^*\widetilde{\mathbb{D}}_i(\mathscr{M}_i).$$

The data $\big\{\widetilde{\mathbb{D}}_i(\mathscr{M}_i), \beta_\phi\big\}$ defines an object in $\mathrm{sec}_-(\mathrm{Hol}(X_\bullet)_\bullet)$ and defines a functor

$$\mathbb{D}_{X_\bullet/\mathfrak{X}-}\colon \mathrm{sec}_+(\mathrm{Hol}(X_\bullet)_\bullet)^\circ \to \mathrm{sec}_-(\mathrm{Hol}(X_\bullet)_\bullet).$$

Similarly, we can define the functor $\mathbb{D}_{X_\bullet/\mathfrak{X}+}\colon \mathrm{sec}_-(\mathrm{Hol}(X_\bullet)_\bullet)^\circ \to \mathrm{sec}_+(\mathrm{Hol}(X_\bullet)_\bullet)$, and we have canonical isomorphisms $c_\mp \circ \mathbb{D}_{X_\bullet/\mathfrak{X}\pm} \cong \mathbb{D}_{X_\bullet/\mathfrak{X}\mp} \circ c_\pm$ by definition of $c_\pm$ (cf. [BD, 7.4.2]). These functors are exact since $\widetilde{\mathbb{D}}_i$ is. Then we have

$$\begin{tikzcd}
D_{\mathrm{tot}}(\mathrm{sec}_\pm(\mathrm{Hol}(X_\bullet)_\bullet))^\circ \arrow[r, dash, "\sim"] \arrow[d, "\mathbb{D}_{X_\bullet/\mathfrak{X}\mp}"'] & D_{\mathrm{tot}}(\mathrm{Hol}(X_\bullet)_\bullet)^\circ \arrow[r, "\sim"] & D_{\mathrm{hol}}(X_\bullet)^\circ \arrow[d, dotted, "\mathbb{D}'_{X_\bullet}"] \\
D_{\mathrm{tot}}(\mathrm{sec}_\mp(\mathrm{Hol}(X_\bullet)_\bullet)) \arrow[r, dash, "\sim"'] & D_{\mathrm{tot}}(\mathrm{Hol}(X_\bullet)_\bullet) \arrow[r, "\sim"'] & D_{\mathrm{hol}}(X_\bullet),
\end{tikzcd}$$

where the left horizontal isomorphism follows by §A.1 and the right horizontal isomorphisms follow by Lemma 2.1.9. We define the dotted functor so that the square is commutative. The dotted functor is called the *dual functor* on $D(X_\bullet)$. By construction, the functor is exact. Moreover, we have a canonical isomorphism $\mathbb{D}'_{X_\bullet} \circ \mathbb{D}'_{X_\bullet} \cong \mathrm{id}$.

Let $\mathfrak{X}$ be a good stack. Take a simplicial space presentation $X_\bullet \to \mathfrak{X}$. We can check that $\mathbb{D}_{X_\bullet/\mathfrak{X}}$ does not depend on the choice of presentation. Thus, we get a functor

$$\mathbb{D}'_{\mathfrak{X}}\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})^\circ \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

We have a canonical isomorphism of functors $\mathbb{D}'_{\mathfrak{X}} \circ \mathbb{D}'_{\mathfrak{X}} \cong \mathrm{id}$.

Remark. Later, in 2.3.13, we define another dual functor $\mathbb{D}$. This is because $\mathbb{D}'$ is not suited to show some fundamental properties of dual functors. The reason we introduced $\mathbb{D}'$ at this point is to show the existence of the left adjoint of f_+ for finite morphism f.

2.2.3. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism between good stacks. Then there exists a canonical isomorphism*

$$\mathbb{D}'_{\mathfrak{Y}} \circ f_+ \xrightarrow{\sim} f_+ \circ \mathbb{D}'_{\mathfrak{X}} \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})^{\circ} \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}).$$

Proof. Take a simplicial space presentation $Y_\bullet \to \mathfrak{Y}$, and let $X_\bullet \to \mathfrak{X}$ be the pullback. We denote by $f_k\colon X_k \to Y_k$ the finite morphism induced by f. Since $f_{k+}(\mathscr{M})$ is in $\mathrm{Hol}(Y_k)$ when $\mathscr{M} \in \mathrm{Hol}(X_k)$ and f_{k+} and $X(\phi)^*$ commute canonically, we can define the push-forward functors $f_{\pm *}\colon \mathrm{sec}_{\pm}(\mathrm{Hol}(X_\bullet)_\bullet) \to \mathrm{sec}_{\pm}(\mathrm{Hol}(Y_\bullet)_\bullet)$ by sending $\{\mathscr{M}_i\}$ to $\{f_{i+}\mathscr{M}_i\}$ with an obvious gluing homomorphism. By the definition of the functors $c_{\pm}$, the following diagrams are commutative:

$$\begin{tikzcd} \mathrm{sec}_+(\mathrm{Hol}(X_\bullet)_\bullet) \arrow[r, "f_{+*}"] \arrow[d] & \mathrm{sec}_+(\mathrm{Hol}(Y_\bullet)_\bullet) \arrow[d] \\ M(X_\bullet) \arrow[r, "f_+"'] & M(Y_\bullet), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{sec}_-(\mathrm{Hol}(X_\bullet)_\bullet) \arrow[r, "f_{-*}"] \arrow[d, "c_+"'] & \mathrm{sec}_-(\mathrm{Hol}(Y_\bullet)_\bullet) \arrow[d, "c_+"] \\ C(M(X_\bullet)) \arrow[r, "f_+"'] & C(M(Y_\bullet)). \end{tikzcd}$$

Thus, it is reduced to constructing an isomorphism $\mathbb{D}_{Y_\bullet/\mathfrak{Y}} \circ f_{+*} \cong f_{-*} \circ \mathbb{D}_{X_\bullet/\mathfrak{X}}$. Since all the functors we used are exact, the verification is easy. $\square$

Definition. The lemma shows that f_+ has a left adjoint functor

$$\mathbb{D}'_{\mathfrak{X}} \circ f^! \circ \mathbb{D}'_{\mathfrak{Y}} \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

This right adjoint functor is denoted by f^+. Since $f^!$ is left exact, f^+ is right exact. Summing up, when f is finite, we have two pairs of adjoint functors (f^+, f_+) and $(f_+, f^!)$. By taking the dual to 2.2.1, f^+ yields a fibered category $\mathscr{F}^{\oplus} \to \mathrm{St}^{\mathrm{fin}}(k)$.

2.2.4. **Lemma.** *Let $f\colon \mathfrak{C} \to \mathfrak{C}'$ be a finite morphism in $\mathrm{St}^{\mathrm{fin}}(k)$.*

(i) *Assume f is surjective radicial morphism. Then f_+ and f^+ define an equivalence of categories between $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{C}')$ and $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{C})$, and $\mathrm{Hol}(\mathfrak{C}')$ and $\mathrm{Hol}(\mathfrak{C})$.*

(ii) *Assume f is an étale morphism. Then for any $\mathscr{M} \in \mathrm{Hol}(\mathfrak{C})$, it is a direct factor of $f^+f_+(\mathscr{M})$.*

(iii) *If f is a flat morphism, then for any $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{C}')$, $\mathscr{M}$ is a direct factor of $f_+f^+(\mathscr{M})$.*

Proof. For (i), it suffices to show the claim when $\mathfrak{C}$ and $\mathfrak{C}'$ are schemes. This is nothing but Lemma 1.1.3 in the first read case, and the second read case can be reduced to the first one immediately. For (iii), we can define homomorphisms $f_+f^+(\mathscr{M}) \to \mathscr{M}$ using the trace map of realizable schemes, and the claim follows easily.

For (ii), let $C'_\bullet \to \mathfrak{C}'$ be a simplicial space presentation. Put $\mathscr{M}_i := \rho_i^*(\mathscr{M})$, and $f_i\colon \mathfrak{C} \times_{\mathfrak{C}'} C'_i \to C'_i$. We have the morphisms $\mathscr{M}_i \to f_i^+ f_{i+}(\mathscr{M}_i) \to \mathscr{M}_i$. Here the first morphism is defined by the trace map in the first read case and, in the second read case, we use the homomorphism defined in the first read case for first morphism. The composition is an isomorphism. Indeed, by [AC1, 1.3.11], it is reduced to checking the claim when f_i is $\coprod_{j\in J} \mathrm{Spec}(k') \to \mathrm{Spec}(k')$, where k' is a finite extension of k and J is a finite set. In this case, the verification is easy.

Moreover, these homomorphisms are compatible with gluing homomorphisms, so they define homomorphisms $\alpha\colon \mathscr{M} \to f^+f_+\mathscr{M}$ and $\beta\colon f^+f_+\mathscr{M} \to \mathscr{M}$, and the composition $\beta \circ \alpha$ is an isomorphism, thus the claim follows. $\square$

Exterior tensor product.

2.2.5. Let us define the exterior tensor product. Let $X_\bullet$ and $Y_\bullet$ be admissible simplicial spaces. Given $\mathscr{M}_\bullet$ and $\mathscr{N}_\bullet$ in $M(X_\bullet)$ and $M(Y_\bullet)$, respectively, the collection $\left\{\mathscr{M}_i \boxtimes \mathscr{N}_i\right\}_i$ defines an object in $M(X_\bullet \times Y_\bullet)$. This is denoted by $\mathscr{M} \boxtimes \mathscr{N}$. The functor is exact, and we can take the derived functor to get

$$(-) \boxtimes (-)\colon D(M(X_\bullet)) \times D(M(Y_\bullet)) \to D(M(X_\bullet \times Y_\bullet)).$$

We can check easily that this preserves holonomicity and boundedness.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good stacks, and take simplicial space presentations $X_\bullet \to \mathfrak{X}$ and $Y_\bullet \to \mathfrak{Y}$. Then $X_\bullet \times Y_\bullet$ is a simplicial space presentation of the good stack $\mathfrak{X} \times \mathfrak{Y}$. Since $\boxtimes$ does not depend on the choice of presentation, we get the exterior tensor product for good stacks.

2.2.6. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$, $g\colon \mathfrak{X}' \to \mathfrak{Y}'$ be finite morphisms between good stacks. Then we have canonical isomorphisms $f_+(-)\boxtimes g_+(-) \cong (f\times g)_+\big((-)\boxtimes(-)\big)$, $f^\star(-) \boxtimes g^\star(-) \cong (f \times g)^\star\big((-) \boxtimes (-)\big)$ where $\star \in \left\{+,!\right\}$, and $\mathbb{D}'\big((-) \boxtimes (-)\big) \cong \mathbb{D}'(-) \boxtimes \mathbb{D}'(-)$.*

Proof. For the commutation of external tensor product and dual functors, see [AC1, 1.3.3], and for the pushforward, use Proposition 1.1.7. Since the proofs are straightforward, we leave the details to the reader. $\square$

Smooth morphism case.

2.2.7. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism between good stacks. Take simplicial space presentations $Y_\bullet \to \mathfrak{Y}$ and $X_\bullet \to \mathfrak{X}$. Let $X_{n,m} := X_n \times_{\mathfrak{Y}} Y_m$, which defines a double simplicial space $X_{\bullet\bullet}$ with obvious face morphisms, and morphisms $\tilde{f}\colon X_{\bullet\bullet} \to Y_\bullet$ and $g\colon X_{\bullet\bullet} \to X_\bullet$. By (the double simplicial analogue of) Proposition 2.1.13, g^* and $\mathbb{R}g_*$ induce an equivalence between $D^+(X_{\bullet\bullet})$ and $D^+(X_\bullet) \cong D^+(\mathfrak{X})$. Thus, we have functors

$$\mathbb{R}f_*\colon D^+(\mathfrak{X}) \cong D^+(X_{\bullet\bullet}) \rightleftarrows D^+(Y_\bullet) \cong D^+(\mathfrak{Y})\colon f^*,$$

where the middle functors are induced by $\mathbb{R}\tilde{f}_*$ and $\tilde{f}^*$. These functors preserve holonomicity by Corollary 2.1.12. We need to check that these functors do not depend on the choice of the presentations. By the adjointness property, it suffices to check it for f^*. The verification is easy and is left to the reader. It is also straightforward to check that f^* is an exact functor and satisfies the transitivity, namely given smooth morphisms $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ between good stacks, we have a canonical isomorphism $(g \circ f)^* \cong f^* \circ g^*$.

Lemma. *Assume f is of relative dimension d. We have a canonical isomorphism $\mathbb{D}'_{\mathfrak{X}} \circ f^* \cong (d) \circ f^* \circ \mathbb{D}'_{\mathfrak{Y}}$.*

Proof. The proof is similar to Lemma 2.2.3, using Lemma 2.1.1 (ii). $\square$

2.2.8. Now, assume that f is an *open immersion* of good stacks. Then $X_\bullet := Y_\bullet \times_{\mathfrak{Y}} \mathfrak{X}$ is a simplicial space presentation of $\mathfrak{X}$. The canonical morphism $X_\bullet \to Y_\bullet$ is denoted by h. Then we have the commutative diagram of double simplicial spaces

$$\begin{tikzcd} X_{\bullet\bullet} \arrow[rr, "\tilde{f}"] \arrow[dr, "g"'] & & Y_\bullet \\ & X_\bullet . \arrow[ur, "h"'] & \end{tikzcd}$$

This implies that the composition $D^+(\mathfrak{X}) \cong D^+(X_\bullet) \xrightarrow{\mathbb{R}h_*} D^+(Y_\bullet) \cong D^+(\mathfrak{Y})$ is canonically isomorphic to $\mathbb{R}f_*$, and similarly for f^*. Now we have:

Lemma. *The functor $\mathbb{R}f_*$ sends $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ to $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$.*

Proof. It suffices to show that $\mathbb{R}h_*$ preserves the boundedness. This can be seen similarly to Lemma 2.1.10 (i). □

Changing the notation, we put $j := f$ and $j^+ := j^* \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, $j_+ := \mathbb{R}j_* \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$. We define the functor $j_!$ so that $(j_!, j^+)$ is an adjoint pair. Such a functor exists since by the above lemma and Lemma 2.2.7, and $\mathbb{D}'_{\mathfrak{Y}} \circ j_+ \circ \mathbb{D}'_{\mathfrak{X}}$ is left adjoint to j^+. Thus, we have pairs of adjoint functors (j^+, j_+) and $(j_!, j^+)$. Now, since $j^+ j_+ \cong \mathrm{id}$, we have a canonical homomorphism of functors $j_! \to j_+$. In particular, this induces a functor

$$j_{!+} := \mathrm{Im}\bigl(\mathscr{H}^0 j_! \to \mathscr{H}^0 j_+\bigr) \colon \mathrm{Hol}(\mathfrak{X}) \to \mathrm{Hol}(\mathfrak{Y}),$$

called the *intermediate extension functor*.

2.2.9. **Lemma.** *Let $j \colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ be an open immersion between good stacks, and let $i \colon \mathfrak{Z} \hookrightarrow \mathfrak{X}$ be its complement. Then, we have the following exact triangles:*

$$i_+ i^! \to \mathrm{id} \to j_+ j^+ \xrightarrow{+1}, \qquad j_! j^+ \to \mathrm{id} \to i_+ i^+ \xrightarrow{+1},$$

where the homomorphisms are defined by adjunctions.

Proof. Let us check the left one. Let $X_\bullet \to \mathfrak{X}$ be a simplicial realizable scheme presentation, and let $\mathscr{I}^\bullet$ be a complex of injective objects in $C(X_\bullet)$. By abuse of notation, we denote the open and closed immersions $\mathfrak{U} \times_{\mathfrak{X}} X_k \hookrightarrow X_k$ and $\mathfrak{Z} \times_{\mathfrak{X}} X_k \hookrightarrow X_k$ by j and i, respectively. By construction and Lemma 2.1.10, i_+ and $i^!$ commute with ρ_k^*. By definition, j^+ commutes also with ρ_k^*. Moreover, j^+ commutes with $\rho_{k!}$ as well, which implies that their right adjoint functors j_+ and ρ_k^* commute. Thus, it suffices to show that the sequence

$$0 \to i_+ i^! (\rho_k^*(\mathscr{I}^l)) \to \rho_k^*(\mathscr{I}^j) \to j_+ j^+ (\rho_k^*(\mathscr{I}^l)) \to 0$$

is exact. This follows by Lemma 1.2.11. The right triangle is exact by duality. □

Projection case.

2.2.10. We define the push-forward functor for a projection $\mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Y}$. The method here is close to the definition of $Rf^!$ in [SGA4, XVIII, 3.1]. We start with the following lemma:

Lemma. *Let $X_\bullet$ and $Y_\bullet$ be admissible simplicial spaces, and let $\mathscr{A}$ be an object in $M(X_\bullet)$. Let*

$$p^*_{\mathscr{A}} := \mathscr{A} \boxtimes (-) \colon M(Y_\bullet) \to M((X \times Y)_{\bullet\bullet}),$$

where we use the notation of Lemma 2.1.19. Then there exists a right adjoint denoted by $p_{\mathscr{A}*}$.

Proof. Since the functor $p^*_{\mathscr{A}}$ is exact and commutes with direct sums by definition, it commutes with arbitrary inductive limits by [KSc, 2.2.9]. Since $M(Y_\bullet)$ is a Grothendieck category and $p^*_{\mathscr{A}}$ commutes with inductive limits, the existence follows from the adjoint functor theorem (cf. [KSc, 8.3.27 (iii)]). $\square$

Given a homomorphism $\mathscr{A} \to \mathscr{B}$ in $M(X_\bullet)$, we have a natural homomorphism of functors

$$p^*_{\mathscr{A}} p_{\mathscr{B}*} \to p^*_{\mathscr{B}} p_{\mathscr{B}*} \to \mathrm{id},$$

where the last homomorphism is the adjunction. Taking the adjoint, we get a homomorphism $p_{\mathscr{B}*} \to p_{\mathscr{A}*}$. Obviously, if the homomorphism $\mathscr{A} \to \mathscr{B}$ is 0, the induced morphism of functors is 0 as well. Thus, for a complex $\mathscr{A}^\bullet \in C(M(X_\bullet))$, we have a complex of functors

$$p_{\mathscr{A}^\bullet *} : [\cdots \to p_{\mathscr{A}^{i+1}*} \to p_{\mathscr{A}^i *} \to \cdots],$$

where $p_{\mathscr{A}^i *}$ is placed at the degree $-i$ term.

2.2.11. **Lemma.** *Let $\mathscr{I}$ be an* injective object *in $M((X \times Y)_{\bullet\bullet})$. Then the contravariant functor*

$$p_{-*}(\mathscr{I}) \colon M(X_\bullet)^\circ \to M(Y_\bullet)$$

sending $\mathscr{A}$ to $p_{\mathscr{A}*}(\mathscr{I})$ is exact.

Proof. Let $0 \to \mathscr{A}' \to \mathscr{A} \to \mathscr{A}'' \to 0$ be a short exact sequence. It suffices to show that, for any $\mathscr{N}$ in $M(Y_\bullet)$, the complex

$$0 \to \mathrm{Hom}(\mathscr{N}, p_{\mathscr{A}''*}(\mathscr{I})) \to \mathrm{Hom}(\mathscr{N}, p_{\mathscr{A}*}(\mathscr{I})) \to \mathrm{Hom}(\mathscr{N}, p_{\mathscr{A}'*}(\mathscr{I})) \to 0$$

is exact, which implies in fact that the sequence $0 \to p_{\mathscr{A}''*}(\mathscr{I}) \to p_{\mathscr{A}*}(\mathscr{I}) \to p_{\mathscr{A}'*}(\mathscr{I}) \to 0$ is split exact. This follows by definition. $\square$

2.2.12. **Corollary.** *Let $\mathscr{C}$ be in $C(M(X_\bullet))$, and let $\mathscr{M} \in C((X \times Y)_{\bullet\bullet})$. We have the spectral sequence*

$$E_2^{p,q} := \mathbb{R}^p p_{\mathscr{H}^{-q}(\mathscr{C})*}(\mathscr{M}) \Rightarrow \mathbb{R}^{p+q} p_{\mathscr{C}*}(\mathscr{M}).$$

Proof. The lemma shows that if $\mathscr{I}$ is an injective object in $M((X \times Y)_{\bullet\bullet})$, we have

$$\mathscr{H}^{-i}\big([\cdots \to p_{\mathscr{A}^{i+1}*}(\mathscr{I}) \to p_{\mathscr{A}^i *}(\mathscr{I}) \to p_{\mathscr{A}^{i-1}*}(\mathscr{I}) \to \cdots]\big) \cong p_{\mathscr{H}^i(\mathscr{A}^\bullet)*}(\mathscr{I}).$$

Let $\mathscr{I}^\bullet$ be an injective resolution of $\mathscr{M}$. Then the spectral sequence associated to the double complex $p_{\mathscr{A}^\bullet *}(\mathscr{I}^\bullet)$ is the desired one. $\square$

2.2.13. **Corollary.** *If a homomorphism of complexes $\mathscr{A}^\bullet \to \mathscr{B}^\bullet$ in $C(M(X_\bullet))$ is a quasi-isomorphism, the induced homomorphism of derived functors $\mathbb{R}p_{\mathscr{B}^\bullet *} \to \mathbb{R}p_{\mathscr{A}^\bullet *}$ is a quasi-isomorphism as well.*

Proof. The homomorphism of functors $\mathbb{R}p_{\mathscr{B}^\bullet *} \to \mathbb{R}p_{\mathscr{A}^\bullet *}$ induces the homomorphism of spectral sequences

$$\begin{array}{ccc} {}^{\mathrm{I}}E_2^{p,q} = \mathbb{R}^p p_{\mathscr{H}^{-q}(\mathscr{B})*} & \Longrightarrow & \mathbb{R}^{p+q} p_{\mathscr{B}*} \\ \downarrow & & \downarrow \\ {}^{\mathrm{II}}E_2^{p,q} = \mathbb{R}^p p_{\mathscr{H}^{-q}(\mathscr{A})*} & \Longrightarrow & \mathbb{R}^{p+q} p_{\mathscr{A}*}. \end{array}$$

Since the left vertical homomorphism is an isomorphism, so is the right. $\square$

2.2.14. **Definition.** Let $\star \in \{\emptyset, \mathrm{b}, +, -\}$, and let $\mathscr{C} \in C^\star(M(X_\bullet))$. We can take the derived functor of $p_{\mathscr{C}*}$ to get

$$p_{\mathscr{C}+} := \mathbb{R}p_{\mathscr{C}*} \colon D^+((X\times Y)_{\bullet\bullet}) \to D(Y_\bullet), \qquad p^+_{\mathscr{C}} := p^*_{\mathscr{C}} \colon D^\star(Y_\bullet) \to D^\star((X\times Y)_{\bullet\bullet}).$$

By Corollary 2.2.13, we may even take $\mathscr{C}$ in $D(X_\bullet)$. By definition, the pair $(p^+_{\mathscr{C}}, p_{\mathscr{C}+})$ is an adjoint pair.

Remark. Assume $\mathscr{C} \in M(X_\bullet)$. Then by definition, for $\mathscr{M} \in M((X \times Y)_{\bullet\bullet})$, $\mathscr{H}^i p_{\mathscr{C}+}(\mathscr{M}) = 0$ for $i < 0$, and in particular, it sends D^+ to D^+. Now, let $\mathscr{C} \in D(X_\bullet)$ such that there exists an integer a with $\mathscr{H}^i\mathscr{C} = 0$ for $a < i$. Then by Corollary 2.2.13, for $\mathscr{M} \in M((X \times Y)_{\bullet\bullet})$, we have $\mathscr{H}^i p_{\mathscr{C}+}(\mathscr{M}) = 0$ for $i < -a$. In particular, the functor sends D^+ to D^+ as well.

2.2.15. Let us compute $p_{\mathscr{A}*}$ more concretely when $\mathscr{A}_\bullet \in \mathrm{Hol}(X_\bullet)$. Let $X_\bullet$ be an admissible simplicial space, and let Y be a space. Let $p\colon X_\bullet \times Y \to Y$ be the projection. Let $\mathscr{A}_\bullet \in \mathrm{Hol}(X_\bullet)$. Take $\mathscr{M}_\bullet \in M(X_\bullet \times Y)$. For $\phi\colon [i] \to [j]$, we have the commutative diagram

$$\begin{tikzcd} X_j \times Y \arrow[d, "(X\times Y)(\phi)"'] \arrow[dr, "p_j"] & \\ X_i \times Y \arrow[r, "p_i"'] & Y. \end{tikzcd}$$

Recall 1.2.10. Since $\big((X\times Y)(\phi)^* \circ p^*_{i\mathscr{A}_i}, p_{i\mathscr{A}_i*} \circ (X\times Y)(\phi)_*\big)$ is an adjoint pair and we have the canonical isomorphism $(X \times Y)(\phi)^* \circ p^*_{i\mathscr{A}_i} \cong p^*_{j\mathscr{A}_j}$, we have

$$p_{i\mathscr{A}_i*} \circ (X \times Y)(\phi)_* \cong p_{j\mathscr{A}_j*}.$$

This isomorphism together with the gluing homomorphism of $\mathscr{M}_\bullet$ for ϕ induces a homomorphism $\alpha_\phi\colon p_{i\mathscr{A}_i*}(\mathscr{M}_i) \to p_{j\mathscr{A}_j*}(\mathscr{M}_j)$. With this homomorphism, we define

$$p_{\mathscr{A}\bullet\times}(\mathscr{M}_\bullet) := \mathrm{Ker}\big(p_{0\mathscr{A}_0*}(\mathscr{M}_0) \rightrightarrows p_{1\mathscr{A}_1*}(\mathscr{M}_1)\big).$$

Now, recall the notation of 2.1.19, and let $Y_\bullet$ be an admissible simplicial space, and let $\mathscr{M}_{\bullet\bullet}$ be in $M((X \times Y)_{\bullet\bullet})$. Given a morphism $\psi\colon [k] \to [l]$, consider the following diagram:

(2.2.15.1)
$$\begin{tikzcd} X_\bullet \times Y_l \arrow[r, "p^l"] \arrow[d, "(X\times Y)(\psi)"'] \arrow[dr, phantom, "\square"] & Y_l \arrow[d, "Y(\psi)"] \\ X_\bullet \times Y_k \arrow[r, "p^k"'] & Y_k. \end{tikzcd}$$

Then we have canonical homomorphisms in $M(Y_l)$,

$$Y(\psi)^* p^k_{\mathscr{A}_\bullet\times}(\mathscr{M}_{\bullet k}) \xrightarrow{\sim} p^l_{\mathscr{A}_\bullet\times}\big((X \times Y)(\psi)^*(\mathscr{M}_{\bullet k})\big) \to p^l_{\mathscr{A}_\bullet\times}(\mathscr{M}_{\bullet l}).$$

With this gluing homomorphism, we get a left exact functor

$$p_{\mathscr{A}_\bullet\times} := p^\bullet_{\mathscr{A}_\bullet\times} \colon M((X \times Y)_{\bullet\bullet}) \to M(Y_\bullet).$$

Lemma. *The pair $(p^*_{\mathscr{A}_\bullet}, p_{\mathscr{A}_\bullet\times})$ is an adjoint pair. Thus, we have an isomorphism $p_{\mathscr{A}_\bullet\times} \cong p_{\mathscr{A}_\bullet*}$*

Proof. The proof is essentially the same as that of Lemma 2.1.11, and we do not repeat it here. $\square$

2.2.16. **Lemma.** *Assume $\mathscr{A}_\bullet \in \mathrm{Hol}(X_\bullet)$. Then, there exists the following spectral sequence*
$$E_1^{i,j} = \mathbb{R}^j p_{i\mathscr{A}_i *}(\mathscr{M}_i) \Rightarrow \mathscr{H}^{i+j} p_{\mathscr{A}+}(\mathscr{M}),$$
where $p_i\colon X_i \times Y_\bullet \to Y_\bullet$ is the projection.

Proof. By Lemma 2.2.15, it suffices to construct the spectral sequence for $p_{\mathscr{A}_\bullet \times}$. The construction is the same as that of Lemma 2.1.12, which is analogous to [O3, Corollary 2.7]. □

2.2.17. Now, we use these computations to show finiteness results of the functor defined in Definition 2.2.14:

Proposition. *Let $\mathscr{A}$ be an object of $D^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet)$.*

(i) *The functors $p_{\mathscr{A}+}$ and $p^+_{\mathscr{A}}$ induce functors between $D^+_{\mathrm{hol}}((X \times Y)_{\bullet\bullet})$ and $D^+_{\mathrm{hol}}(Y_\bullet)$.*

(ii) *We have an adjoint pair $(p^+_{\mathscr{A}}, p_{\mathscr{A}+})$ between $D^+_{\mathrm{hol}}(Y_\bullet)$ and $D^+_{\mathrm{hol}}((X \times Y)_{\bullet\bullet})$.*

Proof. Let us show (i). First, let us check that for $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}((X \times Y)_{\bullet\bullet})$ and for integers $i \geq 0$ and n, $\big(\mathscr{H}^n p_{\mathscr{A}+}(\mathscr{M})\big)_i$ is in $\mathrm{Hol}(Y_i)$. By combining the spectral sequences of Corollary 2.2.12 and Lemma 2.2.16, it suffices to check the holonomicity of $\mathbb{R}^q p_{i\mathscr{H}^p(\mathscr{A})*}(\mathscr{M}_i)$ for any integers p, q, i. This follows from 2.1.1 or 2.2.7. It remains to show that $\mathscr{H}^n p_{\mathscr{A}+}(\mathscr{M})$ is a total complex, namely for $\psi\colon [k] \to [l]$, the homomorphism
$$Y(\psi)^* p^k_{\mathscr{A}+}(\mathscr{M}_{\bullet k}) \to p^l_{\mathscr{A}+}(\mathscr{M}_{\bullet l})$$
is a quasi-isomorphism. We may assume $\mathscr{A} \in \mathrm{Hol}(X_\bullet)$. In this case, this follows by smooth base change (cf. Corollary 1.5.5) and Lemma 2.2.16. Using (i), (ii) follows immediately by construction. □

Remark. Unfortunately, the boundedness is not preserved as we can see from the standard example [LM, 18.3.3]. Thus, to get six functor formalism for algebraic stacks, dealing with unbounded derived category is essential as in [LO]. However, we only construct a complete formalism for admissible stacks (cf. Definition 2.3.1), and for a morphism between admissible stacks, the boundedness is preserved, so we do not use unbounded category.

2.2.18. As one can expect, these functors define functors for good stacks. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good stacks, and $\mathscr{A} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. Take simplicial space presentations $X_\bullet \to \mathfrak{X}$ and $Y_\bullet \to \mathfrak{Y}$. Then we have a functor
$$D^+_{\mathrm{hol}}(\mathfrak{X} \times \mathfrak{Y}) \cong D^+_{\mathrm{hol}}((X \times Y)_{\bullet\bullet}) \xrightarrow{p_{\mathscr{A}+}} D^+_{\mathrm{hol}}(Y_\bullet) \cong D^+_{\mathrm{hol}}(\mathfrak{Y}),$$
and the same for $p^+_{\mathscr{A}}\colon D^\star_{\mathrm{hol}}(\mathfrak{Y}) \to D^\star_{\mathrm{hol}}(\mathfrak{X} \times \mathfrak{Y})$. The pair $(p^+_{\mathscr{A}}, p_{\mathscr{A}+})$ is an adjoint pair. Now, we have:

Lemma. *The functors do not depend on the choice of presentation.*

Proof. By the adjointness property, it suffices to show the lemma for $p^+_{\mathscr{A}}$, in which case the verification is easy. □

Remark. When $\mathfrak{X}$ and $\mathfrak{Y}$ are realizable schemes, then $p_{\mathscr{A}+}$ coincides with the functor defined in 1.2.10, which justifies the notation. This follows since both functors are right adjoint to $p^+_{\mathscr{A}}$.

2.2.19. **Proposition.** *Let $p\colon \mathfrak{X} \to \mathrm{Spec}(k)$ be the structural morphism of a good stack. Let $\mathscr{A}$ be an object in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. For any $\mathscr{M}$ in $D^{+}_{\mathrm{hol}}(\mathfrak{X})$, we have a canonical isomorphism* (*recall* 1.4.14 *and* 2.1.18 *for the notation*)

$$\mathbb{R}\Gamma \circ p_{\mathscr{A}+}(\mathscr{M}) \cong \mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}(\mathscr{A}, \mathscr{M}).$$

Proof. Take a simplicial space presentation $X_\bullet \to \mathfrak{X}$. For $\mathscr{A} \in C^{\mathrm{b}}(X_\bullet)$ and $\mathscr{M} \in M(X_\bullet)$, we have

$$\Gamma \circ p_{\mathscr{A}*}(\mathscr{M}) \cong \mathrm{Hom}_{M(\mathrm{Spec}(k))}\bigl(L, p_{\mathscr{A}*}(\mathscr{M})\bigr) \cong \mathrm{Hom}_{M(X_\bullet)}(\mathscr{A}, \mathscr{M}).$$

Now, $p_{\mathscr{A}^i *}$ preserves injective objects since the left adjoint functor $p^*_{\mathscr{A}^i}$ is exact. This shows that $\mathbb{R}\bigl(\Gamma \circ p_{\mathscr{A}*}\bigr) \cong \mathbb{R}\Gamma \circ \mathbb{R}p_{\mathscr{A}*}$. Thus by the definition of $p_{\mathscr{A}+}$, the proposition follows. □

2.2.20. We have defined a pair of adjoint functors $(p^{+}_{\mathscr{A}}, p_{\mathscr{A}+})$, which depends on the choice of the complex $\mathscr{A}$. For the construction of normal pushforward and pullback, we need a *canonical choice* of $\mathscr{A}$, which is nothing but the unit object $L_{\mathfrak{X}}$ when $\mathfrak{X}$ is a smooth realizable scheme. To construct this complex for good stacks, we need the following theorem of [BBD], as in the construction of [LO].

Theorem ([BBD, 3.2.4]). *Let $X_\bullet$ be an admissible simplicial space. Assume given data $\bigl\{\mathscr{C}_i, \alpha_\phi\bigr\}$ where $\mathscr{C}_i \in D^{\mathrm{b}}(X_\bullet)$ and for $\phi\colon [i] \to [j]$, $\alpha_\phi\colon X(\phi)^*\mathscr{C}_i \xrightarrow{\sim} \mathscr{C}_j$ satisfying the cocycle condition. Assume moreover that*

$$\mathbb{R}^i\mathrm{Hom}_{D(X_i)}(\mathscr{C}_i, \mathscr{C}_i) = 0$$

for any $i < 0$. Then there exists a unique $\mathscr{C} \in D^{\mathrm{b}}_{\mathrm{tot}}(X_\bullet)$ such that $\rho_i^*(\mathscr{C}) \cong \mathscr{C}_i$ (cf. 2.1.6) *and the gluing isomorphism is equal to α_ϕ via this isomorphism.*

Proof. For the uniqueness, use the spectral sequence (2.1.5.1). The existence is more difficult. We use a construction of Beilinson and Drinfeld. In [BD, 7.4.10], they define an abelian category $hot_+(M(X_\bullet))$. This category is nothing but $\mathrm{tot}(\mathcal{A}^+)$ in the notation of [BBD, 3.2.7] by taking $\mathcal{A}(n)$ to be $M(X_n)$. In [BD], they construct an equivalence of categories $s_+\colon D\mathrm{sec}_+(M(X_\bullet)) \xrightarrow{\sim} Dhot_+(M(X_\bullet))$ and characterize $D_{\mathrm{tot}}(X_\bullet)$ in terms of $Dhot$. Even though the appearance is slightly different, this is the statement corresponding to [BBD, 3.2.17]. Thus, our task is to construct an object in $K(hot_+(M(X_\bullet)))$. For this, we can copy the argument of [BBD, 3.2.9]. □

2.2.21. **Lemma.** *Let $p\colon X \to \mathrm{Spec}(k)$ be a morphism of spaces. Let $L_X := p^+(L)$. Then we have $\mathbb{R}^i\mathrm{Hom}_{D(X)}(L_X, L_X) = 0$ for $i < 0$.*

Proof. Consider the first read case, namely the case where X is a realizable scheme. Since for_L is conservative, we may assume that $L = K$. We have isomorphisms

$$\mathbb{R}\mathrm{Hom}_{D(X)}(L_X, L_X) \cong \mathbb{R}\Gamma\, p_+\, \mathcal{H}om(L_X, L_X) \cong \mathbb{R}\Gamma\, p_+ p^+(L),$$

where we used Proposition 2.2.19 for the first isomorphism, and the second one follows since p^+ is monoidal and L_X is the unit object. Now the lemma follows by the left c-t-exactness of p_+ (cf. Lemma 1.3.4) and $\mathbb{R}\Gamma$. For the second read case, this can be reduced to the first read case by using (2.1.5.1). □

2.2.22. Let $\mathfrak{X}$ be a good stack, and let $X_\bullet \to \mathfrak{X}$ be a simplicial space presentation. Let us construct the unit complex on $X_\bullet$. The unit complex L_{X_i} in $D^{\mathrm{b}}_{\mathrm{hol}}(X_i/L)$ has already been defined. Let $\phi\colon [i] \to [j]$. Recall the notation in sections 0.0.3 and 0.0.4. We have a canonical isomorphism

$$X(\phi)^*\bigl(L_{X_i}[d_{X_i/\mathfrak{X}}]\bigr) \cong L_{X_j}[d_{X_j/\mathfrak{X}}].$$

By Lemma 2.2.21, the conditions in Theorem 2.2.20 are satisfied. Thus the data $\bigl\{L_{X_i}[d_{X_i/\mathfrak{X}}]\bigr\}_i$ yield an object $L_{X_\bullet/\mathfrak{X}}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet)$.

Lemma. *The object $L_{X_\bullet/\mathfrak{X}}$ does not depend on the choice of simplicial presentation up to canonical isomorphism.*

Proof. The proof is straightforward. □

Definition. (i) We define the *unit complex* $L_{\mathfrak{X}}$ to be the object in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ defined by $L_{X_\bullet/\mathfrak{X}}$ thanks to the lemma above.

(ii) We define the *dualizing complex* $L^{\omega}_{\mathfrak{X}}$ to be $\mathbb{D}'_{\mathfrak{X}}(L_{\mathfrak{X}})$.

Remark. We can construct $L^{\omega}_{\mathfrak{X}}$ similarly to $L_{\mathfrak{X}}$, without using the functor $\mathbb{D}'_{\mathfrak{X}}$.

2.2.23. **Lemma.** (i) *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good stacks. Then we have $L_{\mathfrak{X}\times\mathfrak{Y}} \cong L_{\mathfrak{X}} \boxtimes L_{\mathfrak{Y}}$, and $L^{\omega}_{\mathfrak{X}\times\mathfrak{Y}} \cong L^{\omega}_{\mathfrak{X}} \boxtimes L^{\omega}_{\mathfrak{Y}}$.*

(ii) *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism between good stacks. Then we have an isomorphism $\iota_f\colon f^+(L_{\mathfrak{Y}}) \cong L_{\mathfrak{X}}$ such that given another finite morphism $g\colon \mathfrak{Y} \to \mathfrak{Z}$, the isomorphism is compatible with composition: the composition $(g\circ f)^+(L_{\mathfrak{Z}}) \cong g^+(f^+(L_{\mathfrak{Z}})) \xrightarrow[\sim]{\iota_f} g^+L_{\mathfrak{Y}} \xrightarrow[\sim]{\iota_g} L_{\mathfrak{X}}$ is equal to $\iota_{g\circ f}$.*

Proof. For (i), the first claim follows from the corresponding statement for spaces and the second by Lemma 2.2.6. Verification of (ii) is left to the reader. □

2.2.24. **Definition.** Let $\mathfrak{X}$ and $\mathfrak{Y}$ be good stacks. We put

$$p_+ := p_{L_{\mathfrak{X}}+}\colon D^{+}_{\mathrm{hol}}(\mathfrak{X}\times\mathfrak{Y}) \to D^{+}_{\mathrm{hol}}(\mathfrak{Y}), \qquad p^+ := p^+_{L_{\mathfrak{X}}}\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}\times\mathfrak{Y}).$$

Let $\mathfrak{X}\times\mathfrak{Y}\times\mathfrak{Z} \xrightarrow{f} \mathfrak{Y}\times\mathfrak{Z} \xrightarrow{g} \mathfrak{Z}$ be projections. By using the canonical isomorphism in Lemma 2.2.23, we have a canonical isomorphisms

$$f^+\circ g^+ \cong L_{\mathfrak{X}} \boxtimes \bigl(L_{\mathfrak{Y}} \boxtimes (-)\bigr) \cong L_{\mathfrak{X}\times\mathfrak{Y}} \boxtimes (-) \cong (g\circ f)^+. \tag{2.2.24.1}$$

By taking the adjoint, we also get a canonical isomorphism $(g\circ f)_+ \cong g_+\circ f_+$.

2.2.25. Let $\mathfrak{X}^{(\prime)}$, $\mathfrak{Y}^{(\prime)}$ be good stacks, and take $\mathscr{A}^{(\prime)} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}^{(\prime)})$. Let $p^{(\prime)}\colon \mathfrak{X}^{(\prime)} \times \mathfrak{Y}^{(\prime)} \to \mathfrak{Y}^{(\prime)}$. Let $q\colon (\mathfrak{X}\times\mathfrak{X}')\times(\mathfrak{Y}\times\mathfrak{Y}') \to \mathfrak{Y}\times\mathfrak{Y}'$ be the projection. By definition, we have a canonical isomorphism

$$p^+_{\mathscr{A}}(-) \boxtimes p'^+_{\mathscr{A}'}(-) \cong q^+_{\mathscr{A}\boxtimes\mathscr{A}'}\bigl(- \boxtimes -\bigr). \tag{2.2.25.1}$$

Now, we have

$$q^+_{\mathscr{A}\boxtimes\mathscr{A}'}\bigl(p_{\mathscr{A}+}(-) \boxtimes p'_{\mathscr{A}'+}(-)\bigr) \cong p^+_{\mathscr{A}}p_{\mathscr{A}+}(-) \boxtimes p'^+_{\mathscr{A}'}p'_{\mathscr{A}'+}(-) \to (-)\boxtimes(-), \tag{2.2.25.2}$$

where we used (2.2.25.1) for the first isomorphism.

Lemma. *The homomorphism $p_{\mathscr{A}+}(-)\boxtimes p'_{\mathscr{A}'+}(-) \to q_{\mathscr{A}\boxtimes\mathscr{A}'+}\bigl((-)\boxtimes(-)\bigr)$, defined by taking an adjoint to (2.2.25.2), is an isomorphism.*

Proof. This can easily be reduced to the case where $\mathfrak{X}^{(\prime)}$ and $\mathfrak{Y}^{(\prime)}$ are spaces using the spectral sequences of Corollary 2.2.12 and Lemma 2.2.16. Using the same spectral sequences, we may assume further that $\mathfrak{X}^{(\prime)}$ and $\mathfrak{Y}^{(\prime)}$ are realizable schemes in the second read case. Now, for realizable schemes X and X' and $\mathscr{M}^{(\prime)}$, $\mathscr{N}^{(\prime)}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(X^{(\prime)})$, we have

$$\mathcal{H}om\big(\mathscr{M}\boxtimes\mathscr{M}',\mathscr{N}\boxtimes\mathscr{N}'\big)\cong\mathcal{H}om(\mathscr{M},\mathscr{N})\boxtimes\mathcal{H}om(\mathscr{M}',\mathscr{N}').$$

This follows by the isomorphism $\mathcal{H}om(\mathscr{M},\mathscr{N})\cong\mathbb{D}\big(\mathscr{M}\otimes\mathbb{D}(\mathscr{N})\big)$ (cf. 1.1.4) and the commutativity of $\mathbb{D}$ and $\boxtimes$ (cf. [AC1, 1.3.3 (i)]). Using the Künneth formula for realizable schemes in 1.1.7, the lemma follows. $\square$

Before the second read.

2.2.26. Before moving on to the second read (cf. 2.0.2) to establish the theory for more general algebraic stacks, we need the following proposition, which is an analogue of Beilinson's equivalence [AC2] for Deligne–Mumford stacks.

Proposition. *Let $\mathfrak{X}$ be a Deligne–Mumford stack of finite type. Moreover, assume that $\mathfrak{X}$ is* separated. *Then the canonical functor $D^{\mathrm{b}}(\mathrm{Hol}(\mathfrak{X}))\to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ is an equivalence.*

Proof. First, we note that since the Deligne–Mumford stack is separated and of finite type, the diagonal morphism is finite, and in particular, schematic. Full faithfulness is the only problem. Thus, since the canonical functors $D^{\mathrm{b}}(\mathrm{Hol}(\mathfrak{X}))\to D^{\mathrm{b}}_{\mathrm{hol}}(\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X})))\to D^{+}_{\mathrm{hol}}(\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X})))$ are fully faithful, it suffices to show that the canonical functor $D^{+}(\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X})))\to D^{+}(\mathfrak{X})$ is fully faithful. Let $f\colon X\to Y$ be an affine étale morphism of realizable schemes. Then the pair $(f_!,f^*)$ of functors between $M(X/L)$ and $M(Y/L)$ is an adjoint pair, and $f_!$ is exact by [AC1, 1.3.13]. In particular, f^* sends injective objects to injective objects. Now, let $f\colon X\to\mathfrak{X}$ be a smooth morphism from an affine scheme. The functors in 2.2.7 define a pair of adjoint functors (f^*,f_*) between $\mathrm{Hol}(X/L)$ and $\mathrm{Hol}(\mathfrak{X}/L)$. By passing to the Ind-categories, these functors induce a pair of adjoint functors $(f^{\odot},f_{\odot})$ between $M(X/L)$ and $\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X}/L))$. Then we can define functors

$$\mathbb{R}f_{\odot}\colon D^{+}(X/L)\rightleftarrows D^{+}(\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X}/L)))\colon f^{\odot},$$

as in the scheme case. Consider the following cartesian diagram:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & \mathfrak{X}, \end{tikzcd}$$

where X, Y are affine schemes, and f, g are étale. We can easily check that the canonical homomorphism $g^*\circ f_*\to f'_*\circ g'^*$ is an isomorphism. This extends to an isomorphism $g^{\odot}\circ f_{\odot}\xrightarrow{\sim}f'_*\circ g'^*$. Since $\mathfrak{X}$ is assumed separated, g' is an affine étale morphism, and thus g'^* preserves injective objects. This implies that the canonical homomorphism

$$(\star)\qquad g^{\odot}\circ\mathbb{R}f_{\odot}\to\mathbb{R}f'_*\circ g'^*\colon D^{+}(X/L)\to D^{+}(Y/L)$$

is an isomorphism.

Now, let $f\colon X_\bullet \to \mathfrak{X}$ be a simplicial realizable scheme presentation such that $X_0 \to \mathfrak{X}$ is étale. Then we can define a pair of adjoint functors

$$\mathbb{R}f_\odot\colon D^+(X_\bullet/L) \rightleftarrows D^+(\mathrm{Ind}(\mathrm{Hol}(\mathfrak{X}/L)))\colon f^\odot$$

similarly to 2.1.11. To conclude the proof, we need to show an analogue of Proposition 2.1.13 in this context. For this, take an étale presentation $Y \to \mathfrak{X}$, use the base change ($\star$) above, and this is reduced to the realizable scheme situation we have already treated. □

Remark. The proposition is false for general algebraic stacks. For example, if G is a geometrically connected algebraic group over k, the category $\mathrm{Hol}(BG)$ is equivalent to the category of finite-dimensional L-vector spaces (cf. Lemma 2.4.7). However, extension computed in the category $D^{\mathrm{b}}_{\mathrm{hol}}(BG)$ is not trivial. For example, we can compute as in the classical case (cf. [LM, 18.3.3]) that $H^i(B\mathbb{G}_m, L) \cong H^i(\mathbb{P}^i, L)$ for $i \geq 0$.

2.2.27. For an algebraic stack of finite type over k, we may take a presentation $X \to \mathfrak{X}$ such that X is a separated algebraic space of finite type (or even affine scheme of finite type). Then $\mathrm{cosk}_0(X \to \mathfrak{X})$ consists of *separated* algebraic spaces of finite type, since $\mathfrak{X}$ is assumed to be quasi-separated. In the first read, the starting point of the construction was Theorem 1.1.8. In the second read, Proposition 2.2.26 plays the role of the theorem. Replacing the definitions of the terminologies according to the table in 2.0.2, we can construct the theory for quasi-separated algebraic stacks of finite type over k.

2.2.28. **Remark.** (i) As one can see from the construction of the cohomology theory explained in 2.2.27, quasi-separatedness is important. Otherwise, nonseparated algebraic spaces appear in the simplicial space $\mathrm{cosk}_0(X \to \mathfrak{X})$, and we are not able to apply Proposition 2.2.26. Non-quasi-separated stacks naturally appear in the work of [O1].

(ii) We may treat the non-quasi-compact case without much effort. For a separated scheme X locally of finite type over k, we take an affine covering $\{U_i\}$, and define $M(X/L)$ by gluing $M(U_i/L)$. Note that this category is not equivalent to $\mathrm{Ind}(\mathrm{Hol}(X/L))$ in general. Even though we need to care about finiteness and so on, the constructions in §2.1 and §2.2 can be carried out similarly.

2.2.29. **Question.** Let X be a separated scheme of finite type over k. We may construct a functor $D^{\mathrm{b}}(\mathrm{Con}(X/L)) \to D^{\mathrm{b}}_{\mathrm{hol}}(X/L)$ as in [Bei]. We ask if this is an equivalence of categories.

Remark. (i) If we have a positive answer to this problem, we can define the six functor formalism exactly as in [LO], or more precisely, we may define the pushforward to be the derived functor of ${}^{\mathrm{c}}\mathscr{H}^0 f_+$. If the proof of the question is *motivic*, then suitable six functor formalisms for schemes can be extended to that for algebraic stacks automatically.

(ii) The problem is solved by Nori [N] when $k = \mathbb{C}$ and $\mathrm{Hol}(X/L)$ is replaced by the category of perverse sheaves.

Theory of weights.

2.2.30. Let k be a finite field with $q = p^s$ elements, and we consider the arithmetic situation where the base tuple (cf. 1.4.10) is $\mathfrak{T}_F := (k, R, K, L, s, \sigma = \mathrm{id})$. We fix an isomorphism $\iota\colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$. Let X be a realizable scheme over k. We say that $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$ is *ι-mixed* (resp. *ι-mixed of weight* $\leq w$, ι-mixed of weight $\geq w$) if $\mathrm{for}_L(\mathscr{M}) \in D^{\mathrm{b}}_{\mathrm{hol}}(X/K_F)$ is also. The results [AC1, 4.1.3, 4.2.3] are automatically true also for $D^{\mathrm{b}}_{\mathrm{hol}}(-/L_F)$ since cohomological operators commute with for_L by 1.4.9. The same holds for [AC1, 4.3].

Remark. For $(V, \varphi) \in F\text{-}\mathrm{Vec}_L$, if $f_\varphi(x)$ is the characteristic polynomial of φ and $\{\alpha_i\}$ is the set of eigenvalues, then the set of eigenvalues for $\mathrm{for}_L((V,\varphi)) \in F\text{-}\mathrm{Vec}_K$ is

$$\bigl\{x \in \overline{\mathbb{Q}}_p \mid \sigma(f_\varphi)(x) = 0\bigr\}_{\sigma \in \mathrm{Hom}_K(L, \overline{\mathbb{Q}}_p)} = \{\sigma(\alpha_i)\}_{\sigma \in \mathrm{Hom}_K(L, \overline{\mathbb{Q}}_p)}.$$

2.2.31. Let $\mathfrak{X}$ be an algebraic stack over k. We say that $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X}/L_F)$ is *ι-pure of weight w* (resp. *ι-mixed*, *ι-mixed of weight $\leq w$*, *ι-mixed of weight $\geq w$*) if for any $f\colon X \to \mathfrak{X}$ in $\mathfrak{X}_{\mathrm{sm}}$ (cf. 0.0.2) of relative dimension d, $f^*(\mathscr{M})$ is ι-pure of weight $w + d$ (resp. ι-mixed, ι-mixed of weight $\leq w + d$, ι-mixed of weight $\geq w + d$). By the existence of weight filtration [AC1, 4.3.4], if $\mathscr{M}$ is ι-mixed, then there exists an increasing filtration W such that $\mathrm{gr}^W_i(\mathscr{M})$ is ι-pure of weight i. A complex $\mathscr{C} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/L_F)$ is said to be *ι-mixed complex of weight* $\star \in \{\leq w, \geq w, \emptyset\}$, if $\mathscr{H}^i\mathscr{C}$ is ι-mixed of weight $\star + i$. We say that the complex $\mathscr{C}$ is *ι-pure of weight w* if it is ι-mixed of weight both $\leq w$ and $\geq w$. We can check that ι-mixedness or other relevant notions defined here are compatible with those for the realizable scheme case in 2.2.30.

2.2.32. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of algebraic stacks of finite type over k. We have the following properties:

(1) For any algebraic stack $\mathfrak{X}$, the functor $\mathbb{D}'_{\mathfrak{X}}$ preserves ι-mixed complexes and exchanges ι-mixed complexes of weight $\leq w$ and that of weight $\geq -w$.
(2) Assume f is finite. Then f_+, $f^!$ preserve ι-mixedness. Moreover, f_+ preserves weights, and $f^!$ preserves complexes of weight $\geq w$.
(3) Let $\mathscr{M}$ and $\mathscr{N}$ be ι-mixed complexes in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/L_F)$ and $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}/L_F)$, respectively. Then $\mathscr{M} \boxtimes \mathscr{N}$ is ι-mixed as well. Moreover, if $\mathscr{M}$ and $\mathscr{N}$ are of weight $\geq w$ and $\geq w'$ (resp. $\leq w$ and $\leq w'$), then $\mathscr{M} \boxtimes \mathscr{N}$ is of weight $\geq w + w'$ (resp. $\leq w + w'$).
(4) The complex $L^{\omega}_{\mathfrak{X}}$ (resp. $L_{\mathfrak{X}}$) is ι-mixed of weight ≥ 0 (resp. ≤ 0).
(5) Assume $f =: j$ is an open immersion, and let $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X}/L_F)$ be ι-pure. Then $j_{!+}(\mathscr{M})$ (cf. 2.2.8) is ι-pure with the same weight (cf. [AC1, 4.2.4]).
(6) Assume f is a projection $\mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Y}$, and let $\mathscr{A}$ be an ι-mixed complex of weight $\leq w'$ on $\mathfrak{X}$. Then $f_{\mathscr{A}+}$ sends ι-mixed complexes of weight $\geq w$ to that of weight $\geq w - w'$.

We think only the last property needs a proof. Let $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X} \times \mathfrak{Y})$ be ι-mixed of weight $\geq w$. We may assume $\mathscr{A} \in \mathrm{Hol}(\mathfrak{X})$. Let us use the notation of 2.2.15. We denote by d_i (resp. d'_j) the relative dimension of $X_i \to \mathfrak{X}$ (resp. $Y_j \to \mathfrak{Y}$). By definition, $\mathscr{M}_{i,j}$ is ι-mixed of weight $\geq w + d_i + d'_j$, and $\mathscr{A}_i$ is of weight $\leq w' + d_i$. Recall that

$$\mathbb{R}p_{i\mathscr{A}_i *}(\mathscr{M}_{i,j}) \cong p_{i+}\,\mathcal{H}om(q_i^+\mathscr{A}_i, \mathscr{M}_{i,j}),$$

where $q_i\colon X_i \times Y_j \to Y_j$ is the projection. Using [AC1, 4.1.3], $\mathbb{R}p_{i\mathscr{A}_i*}(\mathscr{M}_{i,j})$ is of weight $\geq (w + d_i + d'_j) - (d_i + w') = w - w' + d'_j$. Now, by the spectral sequence of Lemma 2.2.16, the claim follows.

2.3. **Six functor formalism for admissible stacks.** In this subsection, we construct six functor formalism for admissible stacks, namely algebraic stacks of finite type with finite diagonal morphism.

2.3.1. **Definition.** A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ between algebraic stacks is said to be *admissible* if it is of finite type and the diagonal morphism $\Delta_f\colon \mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is finite. An *admissible stack* (*over* k) is an algebraic stack over k whose structural morphism is admissible.

Remark. (i) An admissible morphism is quasi-compact and separated by definition.

(ii) For an admissible stack $\mathfrak{X}$, there exists a finite covering $\{\mathfrak{U}_i\}$ such that $\mathfrak{U}_i$ possesses a quasi-finite flat morphism $V_i \to \mathfrak{U}_i$ from a scheme. This is possible by [SGA3, V, 7.2].[3] In particular, there is a dense open substack $\mathfrak{U}$ of $\mathfrak{X}$ such that there exists a finite locally free morphism $V \to \mathfrak{U}$ from a scheme.

2.3.2. **Lemma.** *Let $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ be morphisms between algebraic stacks.*

(i) *If f and g are admissible, $g \circ f$ is also.*

(ii) *Let $\mathfrak{Y}' \to \mathfrak{Y}$ be a morphism between algebraic stacks. If f is admissible, then the base change $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ is admissible as well.*

(iii) *If $g \circ f$ is admissible, f is also.*

(iv) *Separated representable morphisms of finite type between algebraic stacks are admissible. In particular, immersions are admissible.*

(v) *Any morphism $X \to \mathfrak{Y}$ from a scheme to an admissible stack is schematic.*

Proof. The proofs for (i) and (ii) are the same as [EGAI, 5.5.1]. Let us show (iii). Consider the factorization of $\Delta_{g\circ f}$ into morphisms

$$\mathfrak{X} \xrightarrow{\Delta_f} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \xrightarrow{p} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{X}.$$

These morphisms are representable and separated by [LM, 7.7]. We need to show that Δ_f is finite. By the definition of an algebraic stack, $\mathfrak{Y} \to \mathfrak{Y} \times \mathfrak{Y}$ is representable and separated. Thus, by [EGAI, 5.5.1 (v)], Δ_g is separated as well. This implies that p is separated since it is the base change of Δ_g. Since the composition $p \circ \Delta_f$ is assumed finite and the morphisms are representable, we conclude that Δ_f is finite by [EGAII, 6.1.5 (v)].

For (iv), assume f is representable and separated. Then [LM, 8.1.2] shows that the diagonal $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a monomorphism, and since f is assumed separated, it is a closed immersion. For the latter assertion, use [EGAI, 5.5.1 (i)].

For (v), factorize the morphism into $X \to X \times \mathfrak{Y} \to \mathfrak{Y}$. The first morphism is finite since $\mathfrak{Y}$ is admissible, so it is schematic. The second one is schematic as well since X is a scheme. $\square$

[3] See also [Co2, 2.1].

2.3.3. The following variant of Chow's lemma for an admissible stack is important for showing fundamental properties of cohomological operations:

Proposition. *Let $\mathfrak{X}$ be an admissible stack. Then there exists a morphism $p\colon X \to \mathfrak{X}$ such that X is a scheme and p is a surjective generically finite proper morphism.*

Proof. We modify slightly the proof of [O2, (1.1)]. From [O2, 2.1–2.4], the argument is the same. In [O2, 2.5], he replaces $\mathfrak{X}$ by the closure of $\mathfrak{U} \hookrightarrow \mathfrak{X} \times \mathbb{P}(V)$. This replacement is not finite, but birational over $\mathfrak{U}$, so this replacement is harmless in our situation. In [O2, 2.6], it suffices to take P' such that $\dim(P') = \dim(\mathfrak{X})$ in addition to the conditions there. If we have a surjective morphism $a\colon P' \to \mathfrak{X}$, then this is generically finite. Indeed, let $q\colon Q \to \mathfrak{X}$ be a smooth presentation, and $P'_Q := P' \times_{\mathfrak{X}} Q$. By generic finiteness and some standard limit argument, there exists an open dense subscheme $V \subset Q$ such that $P'_Q \times_Q V \to V$ is finite. By fpqc descent, a is finite over $q(V) \subset \mathfrak{X}$.

We do not need [O2, 2.7, 2,8]. Take a quasi-finite flat covering $\{V_i \to \mathfrak{X}\}$ as in Remark 2.3.1 (ii). Put $P' = \mathbb{P}^r_P$ where $r := \dim(\mathfrak{X}) - \dim(P)$. By copying the argument of [O2, 2.9] (taking P_1 to be P'), we can shrink V_i and may assume that there exist morphisms $V_i \to P'$ factoring $V_i \to \mathfrak{X} \to P$ and the morphism $\coprod V_i \to P'$ is surjective. Indeed, in [O2, 2.9], he uses only the fact that $V_i \to P$ has equidimensional fibers. In our case, since $V_i \to \mathfrak{X}$ and $\mathfrak{X} \to P$ are flat, the equidimensionality holds. For [O2, 2.10–2.13], we just copy word by word. Since he only takes normalizations and blowups of P', the dimension does not change, and we get the desired morphism. $\square$

Remark. We are not able to take p to be generically étale in general. Indeed, let k be an algebraically closed field of characteristic p, and let G be a connected finite flat group scheme of dimension 0 over k which is not étale (e.g., $\alpha_p := \mathrm{Spec}(k[T])/(T^p)$). Consider the admissible stack $\mathfrak{X} := BG := [\mathrm{Spec}(k)/G]$. Assume there exists a generically finite étale proper surjective morphism $p\colon X \to BG$. Then since $\dim(BG) = 0$, the dimension of X would be 0 as well. Since BG is smooth over $\mathrm{Spec}(k)$ by [Beh, 5.1.2], X is étale over $\mathrm{Spec}(k)$. Thus by taking a connected component, we may assume that $X = \mathrm{Spec}(k)$, since k is assumed algebraically closed. Since any G-torsor on $\mathrm{Spec}(k)$ splits, the category $BG(\mathrm{Spec}(k))$ is a singleton, and p would be nothing but the universal torsor. The morphism p cannot be étale, since if it were, G would be étale.

2.3.4. **Corollary.** *Let $\mathfrak{X}$ be an admissible stack. Then there exists a generically finite proper surjective morphism $X \to \mathfrak{X}$ such that X is a smooth quasi-projective scheme.*

Proof. Use Chow's lemma above, and then use de Jong's alteration theorem. $\square$

2.3.5. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of admissible stacks. When f is a finite morphism (resp. projection), we denote by $f_\oplus$ and $f^\oplus$ the functors f_+ and f^+ defined in 2.2.1 (resp. 2.2.24) for clarification.

Consider the canonical factorization $\mathfrak{X} \xrightarrow{i} \mathfrak{X} \times \mathfrak{Y} \xrightarrow{p} \mathfrak{Y}$. Since $\mathfrak{Y}$ is admissible, i is finite, and the following definitions make sense:

$$f_+ := p_\oplus \circ i_\oplus \colon D^+_{\mathrm{hol}}(\mathfrak{X}) \to D^+_{\mathrm{hol}}(\mathfrak{Y}), \qquad f^+ := i^\oplus \circ p^\oplus \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

We have the adjoint pair (f^+, f_+).

2.3.6. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism between admissible stacks. Then there are canonical isomorphisms $f_{\oplus} \cong f_{+}$ and $f^{\oplus} \cong f^{+}$.*

Proof. Let $\mathfrak{X} \xrightarrow{i} \mathfrak{X} \times \mathfrak{Y} \xrightarrow{p} \mathfrak{Y}$ be the standard factorization. By the adjointness property, it suffices to construct the isomorphism for the pullback. We construct the isomorphism for their duals, namely $f^{!}$ and $i^{!} \circ p^{!}$ where $p^{!} := L^{\omega}_{\mathfrak{X}} \boxtimes (-)$ (cf. Definition 2.2.22). Let $Y_{\bullet} \to \mathfrak{Y}$ be a simplicial realizable scheme presentation, and let $X_{\bullet} \to \mathfrak{X}$ be its pullback. We may assume that $X_0 \to \mathfrak{X}$ and $Y_0 \to \mathfrak{Y}$ are equidimensional. For $\mathscr{M} \in M(Y_{\bullet})$, let us construct an isomorphism $\alpha\colon \mathscr{H}^0 f^{!}(\mathscr{M}) \xrightarrow{\sim} \mathscr{H}^0\bigl(i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{M})\bigr)$. Put $Y = Y_n$ for $n \geq 0$, and $X := \mathfrak{X} \times_{\mathfrak{Y}} Y$. We have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "g"'] & X \times Y \arrow[r, "p'"] \arrow[d, "g'"] & Y \arrow[d, "g''"] \\ \mathfrak{X} \arrow[r, "i"'] & \mathfrak{X} \times \mathfrak{Y} \arrow[r, "p"'] & \mathfrak{Y}. \end{tikzcd}$$

Let $d := d_{g'} - d_{g''}$ where $d_{\star}$ denotes the relative dimension of $\star$. We have canonical isomorphisms

$$\begin{aligned} g^{*} f^{!}(\mathscr{M}) &\cong (p' \circ i')^{!} g''^{*}(\mathscr{M}) \cong i'^{!}(L^{\omega}_{X} \boxtimes g''^{*}\mathscr{M}) \\ &\cong i'^{!} g'^{*}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{M})(d)[d] \cong g^{*} i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{M}). \end{aligned}$$

Apply $\mathscr{H}^0$ to this isomorphism, and since it satisfies the cocycle condition, we have the desired isomorphism α. Moreover, this isomorphism implies that

$$(\star) \qquad \mathscr{H}^n\bigl(i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{I})\bigr) = 0 \quad \text{for } n \neq 0$$

if $\mathscr{I}$ is an injective object in $M(Y_{\bullet})$. Now, for $\mathscr{M} \in D^{+}(X_{\bullet})$, take an injective resolution $\mathscr{M} \to \mathscr{I}^{\bullet}$. We denote by $\mathscr{H}^0\bigl(i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{I}^{\bullet})\bigr)$ the complex whose term in degree n is $\mathscr{H}^0\bigl(i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{I}^{n})\bigr)$. Recall the notation $f^{\circ} := \mathscr{H}^0 f^{!}$ in 2.1.10. We have quasi-isomorphisms

$$f^{!}(\mathscr{M}) \cong f^{\circ}(\mathscr{I}^{\bullet}) \xrightarrow{\sim} \mathscr{H}^0\bigl(i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{I}^{\bullet})\bigr) \xleftarrow{\sim} i^{!}(L^{\omega}_{X_{\bullet}} \boxtimes \mathscr{M}),$$

where the first isomorphism holds since $f^{!} := \mathbb{R}f^{\circ}$, the second one is induced by α, and the third one follows by the vanishing $(\star)$. Thus, the lemma follows. $\square$

2.3.7. **Lemma.** *Let $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ be morphisms of admissible stacks. Then we have canonical isomorphisms of functors*

$$\begin{aligned} &\alpha\colon \mathrm{id}^{+} \xrightarrow{\sim} \mathrm{id}, && \beta\colon \mathrm{id} \xrightarrow{\sim} \mathrm{id}_{+}, \\ &c_{g,f}\colon f^{+} \circ g^{+} \xrightarrow{\sim} (g \circ f)^{+}, && d^{g,f}\colon (g \circ f)_{+} \xrightarrow{\sim} g_{+} \circ f_{+}. \end{aligned}$$

These homomorphisms are subject to the following conditions: 1. *We have identities $c_{f,\mathrm{id}} = \alpha(f^{+})$, $c_{\mathrm{id},f} = f^{+}\alpha$.* 2. *Assume we are given another morphism of admissible stacks $h\colon \mathfrak{Z} \to \mathfrak{W}$. Then we have the equality*

$$c_{h,g\circ f} \circ c_{g,f}(h^{+}) = c_{h\circ g,f} \circ f^{+}(c_{h,g}).$$

We have the similar equalities for β and $d^{g,f}$.

Proof. By the adjointness property, it suffices to show the lemma for the pullback. First, we define α to be the isomorphism of Lemma 2.3.6. Consider the following diagram:

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "c"] \arrow[d, "a"'] \arrow[dr, phantom, "\textcircled{1}"] & & \mathfrak{X}\times\mathfrak{Z} \arrow[dl, "d"] \arrow[dd] \arrow[d, phantom, "\textcircled{2}"] \\
\mathfrak{X}\times\mathfrak{Y} \arrow[r, "b"'] \arrow[d] \arrow[dr, phantom, "\textcircled{3}"] & \mathfrak{X}\times\mathfrak{Y}\times\mathfrak{Z} \arrow[d] \arrow[dr] \arrow[dr, phantom, "\textcircled{4}"] & \\
\mathfrak{Y} \arrow[r] & \mathfrak{Y}\times\mathfrak{Z} \arrow[r] & \mathfrak{Z}.
\end{tikzcd}$$

For a morphism a, we denote by Γ_a the graph morphism. The *transitivity isomorphism* for ①, namely the isomorphism $a^{\oplus}\circ b^{\oplus}\cong c^{\oplus}\circ d^{\oplus}$, is defined by 2.2.1. For ④, use Lemma 2.2.23, and for ③, use Lemma 2.2.6 for finite morphisms $\mathrm{id}\colon \mathfrak{X}\to\mathfrak{X}$ and $\Gamma_g\colon \mathfrak{Y}\to\mathfrak{Y}\times\mathfrak{Z}$. Finally, for the transitivity isomorphism for ②, it suffices to construct $\Gamma_f^{+}(L_{\mathfrak{X}\times\mathfrak{Y}})\cong L_{\mathfrak{X}}$. This follows by Lemma 2.2.23. The verification of the compatibility conditions is straightforward, so we leave it as an exercise. $\square$

Let $\mathrm{St}^{\mathrm{adm}}(k)$ be the full subcategory of the category of algebraic stacks (we do not consider the 2-morphisms) consisting of admissible stacks. For an admissible stack $\mathfrak{X}$, we associate the triangulated category $D^{+}_{\mathrm{hol}}(\mathfrak{X})$ (resp. $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$). By the data of the lemma above, we have a cofibered category $\mathscr{F}_{+}\to\mathrm{St}^{\mathrm{adm}}$ by considering f_{+} (resp. a fibered category $\mathscr{F}^{+}\to\mathrm{St}^{\mathrm{adm}}$ by considering f^{+}). Recall the notation of 2.2.1 and Definition 2.2.3. The isomorphisms of Lemma 2.3.6 yield isomorphisms of fibered and cofibered categories $\mathscr{F}^{\oplus}\cong\mathscr{F}^{+}$ and $\mathscr{F}_{\oplus}\cong\mathscr{F}_{+}$ over the category of an admissible stack with finite morphism $\mathrm{St}^{\mathrm{adm,fin}}$.

2.3.8. **Lemma.** *Let $f\colon \mathfrak{X}\to\mathfrak{Y}$ be a smooth morphism of admissible stacks, and let d be the relative dimension of f. Then $f^{*}\cong f^{+}[d]$, and $\mathbb{R}f_{*}\cong f_{+}[-d]$ (cf. 2.2.7).*

Proof. By adjointness, it suffices to prove $f^{*}\cong f^{+}[d]$. Since the proof is similar to that of Lemma 2.3.6, we only sketch the proof. As the proof of the lemma, we also show the dual claim: $f^{*}(d)\cong f^{!}[-d]$ (cf. Lemma 2.2.7). We may take $X_{\bullet\bullet}$, $Y_{\bullet}$ as in 2.2.7. Let $Y:=Y_i$ and $X:=X_{i,j}$. Consider the following cartesian diagram:

$$\begin{tikzcd}
X\times_{\mathfrak{Y}} Y \arrow[r] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\square"] & X\times Y \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathfrak{X} \arrow[r] & \mathfrak{X}\times\mathfrak{Y} \arrow[r] & \mathfrak{Y}.
\end{tikzcd}$$

For $\mathscr{M}\in M(Y_{\bullet})$, we have $\alpha^{*}f^{*}(\mathscr{M})(d)\cong\alpha^{*}f^{!}[-d](\mathscr{M})$. This follows by using the fact that if g is a smooth morphism of relative dimension d_g between realizable schemes, then $g^{*}(d)\cong g^{!}[-d_g]$ by definition. We finish the proof by the descent argument. $\square$

For a smooth morphism $\rho\colon X\to\mathfrak{X}$ from a realizable scheme to an admissible scheme and $\mathscr{M}\in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, we often denote $\rho^{*}(\mathscr{M})$ by $\mathscr{M}_X$. When f is an open immersion, this lemma justifies the notation of 2.2.8. Recall that in such a case, we have an adjoint pair $(f_{!},f^{+})$ as well as (f^{+},f_{+}).

2.3.9. **Proposition** (Smooth base change). *Consider the following cartesian diagram of admissible stacks:*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r, "g"'] & \mathfrak{Y}, \end{tikzcd}$$

where g is smooth. Then the base change homomorphism of functors $g^+ f_+ \to f'_+ g'^+ \colon D^+_{\mathrm{hol}}(\mathfrak{X}) \to D^+_{\mathrm{hol}}(\mathfrak{Y}')$ is an isomorphism.

Proof. In the verification, it suffices to replace g^+ and g'^+ by g^* and g'^* by Lemma 2.3.8. We use the standard factorization of f into a finite morphism and a projection, and the verification is reduced to these cases separately. In both cases, the verification is straightforward from the definition, so we leave the details to the reader. $\square$

2.3.10. **Proposition.** *Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism between admissible stacks. Then f_+ preserves boundedness, namely it induces a functor $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$.*

Proof. It suffices to show that $f_+(\mathscr{M})$ is bounded for $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X})$. Let us show by the induction on the dimension of the support of $\mathscr{M}$. We may assume that the support of $\mathscr{M}$ is equal to $\mathfrak{X}$. Now, we may shrink $\mathfrak{X}$. Indeed, consider the localization triangle $i_+ i^! \to \mathrm{id} \to j_+ j^+ \xrightarrow{+1}$ (cf. Lemma 2.2.9). We know that $i_+ i^!$ and $j_+ j^+$ preserve boundedness. By the induction hypothesis, we are reduced to showing the proposition for $f_+ j_+ \cong (f \circ j)_+$, and the claim follows. By shrinking $\mathfrak{X}$, we may assume that there exists a finite locally free morphism $X \to \mathfrak{X}$ from an affine scheme X by Remark 2.3.1 (ii). In this situation, since $\mathscr{M}$ is a direct factor of $h_+ h^+(\mathscr{M})$ by Lemma 2.2.4, we may assume that $\mathfrak{X}$ is an affine scheme. Finally, let $g \colon Y \to \mathfrak{Y}$ be a smooth presentation from an affine scheme. Since the functor g^* is conservative by Remark 2.1.16 (ii), it suffices to show the claim for the morphism $\mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$. Since $\mathfrak{X}$ is assumed to be an affine scheme, this is a morphism of realizable schemes, and the boundedness is already known. $\square$

2.3.11. Let $\mathfrak{X}$ be an admissible stack. Then the diagonal morphism $\Delta \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is finite. For $i = 1, 2$, let $p_i \colon \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ be the i-th projection. We define the *internal Hom functor* by

$$\mathcal{H}om(\mathscr{M}, \mathscr{N}) := p_{1\mathscr{M}+}\big(\Delta_+(\mathscr{N})\big) \colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})^{\circ} \times D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

Let $\mathscr{L} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. Since $(p^+_{1\mathscr{M}}, p_{1\mathscr{M}+})$ is an adjoint pair, with Remark 1.2.1, we have

$$\mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}\big(\mathscr{L}, \mathcal{H}om(\mathscr{M}, \mathscr{N})\big) \cong \mathbb{R}\mathrm{Hom}_{D(\mathfrak{X} \times \mathfrak{X})}\big(\mathscr{L} \boxtimes \mathscr{M}, \Delta_+ \mathscr{N}\big).$$

We can check easily that when $\mathfrak{X}$ is a realizable scheme, $\mathcal{H}om$ coincides with that in 1.1.3. Now, let $f \colon \mathfrak{X} \to \mathfrak{Y}$, $g \colon \mathfrak{X}' \to \mathfrak{Y}'$ be morphisms of admissible stacks. Then, we have an isomorphism $(f \times g)^+\big((-) \boxtimes (-)\big) \cong f^+(-) \boxtimes g^+(-)$. This follows by combining Lemma 2.2.6 and (2.2.25.1). Using this, we have

$$\begin{aligned} \mathrm{Hom}\big(\mathscr{L}, f_+ \mathcal{H}om(f^+\mathscr{M}, \mathscr{N})\big) &\cong \mathrm{Hom}\big(f^+(\mathscr{L}) \boxtimes f^+(\mathscr{M}), \Delta_{\mathfrak{X}*}(\mathscr{N})\big) \\ &\cong \mathrm{Hom}\big((f \times f)^+(\mathscr{L} \boxtimes \mathscr{M}), \Delta_{\mathfrak{X}*}(\mathscr{N})\big) \\ &\cong \mathrm{Hom}\big(\mathscr{L}, \mathcal{H}om(\mathscr{M}, f_+\mathscr{N})\big). \end{aligned}$$

Thus, we get an isomorphism

$$f_+ \mathcal{H}om(f^+\mathscr{M}, \mathscr{N}) \cong \mathcal{H}om(\mathscr{M}, f_+\mathscr{N}).$$

2.3.12. **Lemma.** *Let $\mathscr{M}$ and $\mathscr{N}$ be objects of $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. For a presentation $\rho\colon X \to \mathfrak{X}$ from a realizable scheme, there is a canonical isomorphism*

$$\rho^* \mathcal{H}om(\mathscr{M}, \mathscr{N}) \cong \mathcal{H}om\big(\rho^*(\mathscr{M}), \rho^*(\mathscr{N})\big)[d],$$

and d denotes the relative dimension function of ρ.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} & & X \arrow[dll, "\delta"'] \arrow[d, "\Delta'"] \arrow[drr, equal] & & \\ X \times X \arrow[rr, "\rho\times\mathrm{id}"'] & & \mathfrak{X} \times X \arrow[rr, "q"'] & & X \arrow[r, "\rho"'] & \mathfrak{X}, \end{tikzcd}$$

where δ is the diagonal morphism, $\Delta' := (\rho, \mathrm{id})$, and q is the second projection. Put $\mathscr{N}' := \rho^*(\mathscr{N})$. By the definition of the functor $q_{\mathscr{M}+}$, we have

$$(\star) \qquad \rho^* \mathcal{H}om(\mathscr{M}, \mathscr{N}) \cong q_{\mathscr{M}+}(\Delta'_+(\mathscr{N}')).$$

Let $\mathscr{L}$ be an object in $D^{\mathrm{b}}_{\mathrm{hol}}(X)$. For an algebraic stack $\mathfrak{Y}$, we denote $\mathrm{Hom}_{D(\mathfrak{Y})}$ by $\mathrm{Hom}_{\mathfrak{Y}}$. We have

$$\begin{aligned} \mathrm{Hom}_X\big(\mathscr{L}, q_{\mathscr{M}+}(\Delta'_+(\mathscr{N}'))\big) &\cong \mathrm{Hom}_{\mathfrak{X}\times X}\big(\mathscr{M} \boxtimes \mathscr{L}, \Delta'_+(\mathscr{N}')\big) \\ &\cong \mathrm{Hom}_X\big(\delta^+(\rho\times\mathrm{id})^+(\mathscr{M}\boxtimes\mathscr{L}), \mathscr{N}'\big) \cong \mathrm{Hom}_X\big(\rho^+(\mathscr{M})\otimes\mathscr{L}, \mathscr{N}'\big) \\ &\cong \mathrm{Hom}_X\big(\mathscr{L}, \mathcal{H}om(\rho^+(\mathscr{M}), \mathscr{N}')\big), \end{aligned}$$

where the first isomorphism follows by the adjunction $(q^+_{\mathscr{M}}, q_{\mathscr{M}+})$, the second by the adjunction (Δ'^+, Δ'_+), the third by Lemma 2.2.6, and the last by the adjointness property of $\otimes$ and $\mathcal{H}om$ for realizable schemes. Thus, we have a canonical isomorphism $\mathcal{H}om\big(\rho^+(\mathscr{M}), \mathscr{N}'\big) \cong q_{\mathscr{M}+}\big(\Delta'_+(\mathscr{N}')\big)$. Combining with the isomorphism $\rho^+ \cong \rho^*[-d]$ and $(\star)$, the lemma follows. $\square$

2.3.13. Recalling Definition 2.2.22, we define the *dual functor* to be

$$\mathbb{D}_{\mathfrak{X}}(\mathscr{M}) := \mathcal{H}om(\mathscr{M}, L^{\omega}_{\mathfrak{X}})\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})^{\circ} \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

If no confusion may arise, we often omit the subscript $\mathfrak{X}$ from $\mathbb{D}_{\mathfrak{X}}$.

Proposition (Biduality)**.** *There exists a canonical isomorphism of functors*

$$\gamma\colon \mathrm{id} \xrightarrow{\sim} \mathbb{D}_{\mathfrak{X}} \circ \mathbb{D}_{\mathfrak{X}}\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

Proof. We have isomorphisms

(2.3.13.1)

$$\begin{aligned} \mathrm{Hom}\big(\mathbb{D}(\mathscr{M}), \mathbb{D}(\mathscr{M})\big) &\cong \mathrm{Hom}\big(\mathbb{D}(\mathscr{M}) \boxtimes \mathscr{M}, \Delta_+(L^{\omega}_{\mathfrak{X}})\big) \\ &\cong \mathrm{Hom}\big(\mathscr{M} \boxtimes \mathbb{D}(\mathscr{M}), \Delta_+(L^{\omega}_{\mathfrak{X}})\big) \cong \mathrm{Hom}(\mathscr{M}, \mathbb{D}\mathbb{D}(\mathscr{M})), \end{aligned}$$

where the second isomorphism is induced by the morphism $\mathfrak{X} \times \mathfrak{X} \xrightarrow{\sim} \mathfrak{X} \times \mathfrak{X}$ exchanging the first and second factor. The image of the identity homomorphism induces γ in the claim. Let $\rho\colon X \to \mathfrak{X}$ be a presentation from a realizable scheme. By Remark 2.1.16 (ii), it suffices to show that $\rho^*(\gamma)$ is an isomorphism. Since the dual functor is compatible with ρ^* by Lemma 2.3.12, this is reduced to checking the biduality in the realizable scheme case, which is Lemma 1.1.4. $\square$

Remark. (i) We have a canonical isomorphism $\mathscr{H}^i\mathbb{D}_{\mathfrak{X}} \cong \mathscr{H}^i\mathbb{D}'_{\mathfrak{X}}$ for any i. Indeed, for any presentation $\rho\colon X \to \mathfrak{X}$, we can check that $\rho^*\mathbb{D}_{\mathfrak{X}}$ and $\rho^*\mathbb{D}'_{\mathfrak{X}}$ are canonically isomorphic to $\mathbb{D}_X\rho^*$, thus we get the claim by gluing. However, even though there is no doubt that $\mathbb{D}_{\mathfrak{X}}$ and $\mathbb{D}'_{\mathfrak{X}}$ coincide, we do not know how to construct a morphism $\mathbb{D}_{\mathfrak{X}} \to \mathbb{D}'_{\mathfrak{X}}$ in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ compatible with the isomorphisms of i-th cohomologies.

(ii) We have a canonical isomorphism $L^{\omega}_{\mathfrak{X}} \cong \mathbb{D}_{\mathfrak{X}}(L_{\mathfrak{X}})$. This follows since for any $X \in \mathfrak{X}_{\mathrm{sm}}$, we have $(L^{\omega})_X \cong \mathbb{D}(L)_X$, and we have the uniqueness of Theorem 2.2.20.

(iii) When j is an open immersion, we have $j^+\circ\mathbb{D} \cong \mathbb{D}\circ j^+$. Thus, $j_! \cong \mathbb{D}\circ j_+\circ\mathbb{D}$.

(iv) When f is a finite morphism, we prove in Lemma 2.3.17 that we have an isomorphism $f^+ \cong \mathbb{D}\circ f^!\circ\mathbb{D}$.

2.3.14. We define the *tensor product* by

$$(-)\otimes(-) := \Delta^+\big((-)\boxtimes(-)\big)\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})\times D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}) \to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}).$$

Let $\mathscr{M}$, $\mathscr{N}$, $\mathscr{L}$ be objects in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. We have

$$\begin{aligned}\mathbb{R}\mathrm{Hom}\big(\mathscr{M},\mathcal{H}om(\mathscr{N},\mathscr{L})\big) &:= \mathbb{R}\mathrm{Hom}\big(\mathscr{M},p_{1\mathscr{N}+}(\Delta_+\mathscr{L})\big) \cong \mathbb{R}\mathrm{Hom}\big(\mathscr{M}\boxtimes\mathscr{N},\Delta_+\mathscr{L}\big)\\ &\cong \mathbb{R}\mathrm{Hom}\big(\Delta^+(\mathscr{M}\boxtimes\mathscr{N}),\mathscr{L}\big) =: \mathbb{R}\mathrm{Hom}\big(\mathscr{M}\otimes\mathscr{N},\mathscr{L}\big).\end{aligned} \tag{2.3.14.1}$$

The identity homomorphism of $\mathcal{H}om(\mathscr{M},\mathscr{N})$ induces the *evaluation homomorphism*

$$\mathcal{H}om(\mathscr{M},\mathscr{N})\otimes\mathscr{M} \to \mathscr{N}.$$

Now, let $p\colon \mathfrak{X} \to \mathrm{Spec}(k)$ denote the structural morphism. Since $p_2\circ\Delta \cong \mathrm{id}$, we have

$$L_{\mathfrak{X}}\otimes\mathscr{M} = \Delta^+\big(L_{\mathfrak{X}}\boxtimes\mathscr{M}\big) \cong \Delta^+(p_2^+(\mathscr{M})) \cong (p_2\circ\Delta)^+(\mathscr{M}) \cong \mathscr{M},$$

where the isomorphisms hold by the definition and results in 2.2.24. Using these, we have

$$\mathbb{R}\Gamma\circ p_+\,\mathcal{H}om(\mathscr{M},\mathscr{N}) \cong \mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}\big(L_{\mathfrak{X}},\mathcal{H}om(\mathscr{M},\mathscr{N})\big) \cong \mathbb{R}\mathrm{Hom}_{D(\mathfrak{X})}(\mathscr{M},\mathscr{N}),$$

where the first isomorphism holds by Proposition 2.2.19, and the second by what we have just proven.

2.3.15. **Proposition.** *Let $f\colon \mathfrak{X} \to \mathfrak{X}'$, $g\colon \mathfrak{Y} \to \mathfrak{Y}'$ be morphisms of admissible stacks.*

(1) *We have an isomorphism $(f\times g)^+\big((-)\boxtimes(-)\big) \cong f^+(-)\boxtimes g^+(-)$.*
(2) *We have $\mathbb{D}\big((-)\boxtimes(-)\big) \cong \mathbb{D}(-)\boxtimes\mathbb{D}(-)$.*
(3) *We have $f^+\big((-)\otimes(-)\big) \cong f^+(-)\otimes f^+(-)$.*
(4) *We have $\mathcal{H}om(\mathscr{M}\otimes\mathscr{N},\mathscr{L}) \cong \mathcal{H}om\big(\mathscr{M},\mathcal{H}om(\mathscr{N},\mathscr{L})\big)$.*
(5) *We have $\mathcal{H}om\big(\mathbb{D}(\mathscr{M}),\mathbb{D}(\mathscr{N})\big) \cong \mathcal{H}om(\mathscr{N},\mathscr{M})$.*
(6) *We have $\mathcal{H}om(\mathscr{M},\mathscr{N}) \cong \mathbb{D}\big(\mathscr{M}\otimes\mathbb{D}(\mathscr{N})\big)$.*

Proof. The first one is just a reproduction from 2.3.11. The second claim follows by combining Lemma 2.2.6 (the commutativity of f_+ and $\boxtimes$), Lemma 2.2.25, and Lemma 2.2.23 (i). Let us show (3). Let $\mathscr{M}'$, $\mathscr{N}'$ be objects in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}')$. Consider the following commutative diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"] \arrow[d, "f"'] & \mathfrak{X}\times\mathfrak{X} \arrow[d, "f\times f"] \\ \mathfrak{X}' \arrow[r, "\Delta_{\mathfrak{X}'}"'] & \mathfrak{X}'\times\mathfrak{X}'. \end{tikzcd}$$

Using this diagram, we have

$$f^+\big(\mathscr{M}'\otimes\mathscr{N}'\big) \cong f^+\Delta^+_{\mathfrak{X}'}\big(\mathscr{M}'\boxtimes\mathscr{N}'\big) \cong \Delta^+_{\mathfrak{X}}(f\times f)^+\big(\mathscr{M}'\boxtimes\mathscr{N}'\big) \cong f^+(\mathscr{M}')\otimes f^+(\mathscr{N}').$$

Let us show (4). For any $\mathscr{Q}\in D^{\mathrm{b}}(\mathfrak{X})$, we have a canonical isomorphism

$$\mathrm{Hom}\big(\mathscr{Q},\mathcal{H}om(\mathscr{M}\otimes\mathscr{N},\mathscr{L})\big) \cong \mathrm{Hom}\big(\mathscr{Q},\mathcal{H}om(\mathscr{M},\mathcal{H}om(\mathscr{N},\mathscr{L}))\big)$$

by using (2.3.14.1) twice, thus the claim follows. To show (5), the isomorphisms (2.3.13.1) remain to hold even if we replace Hom by $\mathcal{H}om$. For the last claim (6), it suffices to construct a canonical isomorphism $\mathrm{Hom}\big(\mathscr{Q},\mathcal{H}om(\mathscr{M},\mathscr{N})\big) \cong \mathrm{Hom}\big(\mathscr{Q},\mathbb{D}(\mathscr{M}\otimes\mathbb{D}(\mathscr{N}))\big)$. This can be shown by using (4) and (5). $\square$

2.3.16. Let $f\colon \mathfrak{X}\to\mathfrak{Y}$ be a morphism of admissible stacks, and let $\mathscr{M},\mathscr{N}$ be objects in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. The adjunction homomorphism $f^+f_+\to\mathrm{id}$ induces a homomorphism

$$f^+\big(f_+(\mathscr{M})\otimes f_+(\mathscr{N})\big) \cong f^+f_+(\mathscr{M})\otimes f^+f_+(\mathscr{N}) \to \mathscr{M}\otimes\mathscr{N},$$

where the first isomorphism follows by Proposition 2.3.15. This induces the homomorphism

$$f_+(\mathscr{M})\otimes f_+(\mathscr{N}) \to f_+\big(\mathscr{M}\otimes\mathscr{N}\big). \tag{2.3.16.1}$$

Using this, we have a homomorphism

$$f_+\,\mathcal{H}om(\mathscr{M},\mathscr{N})\otimes f_+(\mathscr{M}) \to f_+\big(\mathcal{H}om(\mathscr{M},\mathscr{N})\otimes\mathscr{M}\big) \to f_+(\mathscr{N}),$$

where the second homomorphism is induced by the evaluation homomorphism. Taking the adjunction (2.3.14.1), we get a canonical homomorphism

$$f_+\,\mathcal{H}om\big(\mathscr{M},\mathscr{N}\big) \to \mathcal{H}om(f_+\mathscr{M},f_+\mathscr{N}). \tag{2.3.16.2}$$

Duality results.

2.3.17. First, let us construct the trace map for projective morphisms. Let $f\colon\mathfrak{X}\to\mathfrak{Y}$ be a *projective* morphism between admissible stacks. Let $Y_\bullet\to\mathfrak{Y}$ be a simplicial quasi-projective scheme presentation, and since f is assumed projective, the cartesian product $X_\bullet := \mathfrak{X}\times_{\mathfrak{Y}} Y_\bullet$ is an admissible simplicial quasi-projective scheme as well. Since $f_+L^\omega_{X_\bullet}, L^\omega_{Y_\bullet}$ are in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}_\bullet)$, we have a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{D(Y_p)}(f_{p+}L^\omega_{X_p},L^\omega_{Y_p}) \Rightarrow \mathrm{Ext}^{p+q}_{D(Y_\bullet)}(f_+L^\omega_{X_\bullet},L^\omega_{Y_\bullet})$$

by (2.1.5.1). The usual trace map defines $\mathrm{Tr}_{f_p}\in\mathrm{Hom}_{D(X_p)}(f_{p+}L^\omega_{X_p},L^\omega_{Y_p})$ for each p. By this spectral sequence, the trace map for $p=1$ yields the desired trace map Tr_f. We note that $(\mathrm{Tr}_f)_p$ is nothing but Tr_{f_p} by the compatibility of the trace map.

Now, using this trace map, we define

$$\begin{aligned} f_+\circ\mathbb{D}_{\mathfrak{X}} \cong f_+\,\mathcal{H}om(-,L^\omega_{\mathfrak{X}}) &\xrightarrow{\star} \mathcal{H}om(f_+(-),f_+L^\omega_{\mathfrak{X}}) \\ &\xrightarrow{\mathrm{Tr}_f} \mathcal{H}om(f_+(-),L^\omega_{\mathfrak{Y}}) \cong \mathbb{D}_{\mathfrak{Y}}\circ f_+, \end{aligned}$$

where $\star$ is induced by (2.3.16.2). This homomorphism is in fact an isomorphism. To check this, since the verification is local, this may easily be reduced to the realizable scheme case, and this case follows by Lemma 1.1.5. Summing up, we have the following:

Lemma. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a projective morphism between admissible stacks. Then we have the canonical isomorphism $\mathbb{D}_{\mathfrak{Y}} \circ f_+ \xleftarrow{\sim} f_+ \circ \mathbb{D}_{\mathfrak{X}}$. In particular, putting $f^! := \mathbb{D}_{\mathfrak{X}} \circ f^+ \circ \mathbb{D}_{\mathfrak{Y}}$, the pair $(f_+, f^!)$ is an adjoint pair. Moreover, consider the following cartesian diagram of admissible stacks:*

(2.3.17.1)
$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r, "g"'] & \mathfrak{Y}. \end{tikzcd}$$

Under the assumption that f is projective, the canonical homomorphism $g^+ f_+ \to f'_+ g'^+$ is an isomorphism. If, moreover, g is an open immersion, we have the canonical isomorphism $g_! \circ f'_+ \cong f_+ \circ g'_!$.

2.3.18. **Proposition** (Proper base change). *Consider the cartesian diagram of admissible stacks* (2.3.17.1). *Assume that f is proper* (*where we do* not *assume f to be projective*). *Then the canonical homomorphism $g^+ f_+ \to f'_+ g'^+$ is an isomorphism.*

Proof. It suffices to show that $g^+ f_+(\mathscr{M}) \xrightarrow{\sim} f'_+ g'^+(\mathscr{M})$ for $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X})$. We may assume $\mathfrak{X}$ to be reduced by Lemma 2.2.4 and Lemma 2.3.17. By the smooth base change theorem, Proposition 2.3.9, we may replace $\mathfrak{Y}$ by its smooth presentation. In particular, we may assume $\mathfrak{Y} =: Y$ to be a realizable scheme. Now, we use the induction on the dimension of the support of $\mathscr{M}$. We may assume $\mathrm{Supp}(\mathscr{M}) = \mathfrak{X}$. By the induction hypothesis, it suffices to show the equality for $\mathscr{M} = j_!(\mathscr{N})$, where $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ which is open dense. By Corollary 2.3.4, there is a smooth quasi-projective scheme X that is projective and generically finite over $\mathfrak{X}$, and projective over Y. Since $h\colon X \to \mathfrak{X}$ is projective, we already know the base change by Lemma 2.3.17. We may shrink $\mathfrak{U}$ so that h is finite flat over $\mathfrak{U}$ since $\mathfrak{X}$ is assumed reduced. We denote $h^{-1}(\mathfrak{U}) \to \mathfrak{U}$ by h, abusing the notation. Then we have $h_+ j'_! h^+(\mathscr{N}) \cong j_! h_+ h^+(\mathscr{N})$, where $j'\colon h^{-1}(\mathfrak{U}) \to X$, and this contains $j_!(\mathscr{N})$ as a direct factor by Lemma 2.2.4. Thus, the verification is reduced to the case $\mathscr{M} = j'_!(h^+(\mathscr{N}))$. Indeed, let $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, and let $\mathscr{E}$ be a direct factor of $\mathscr{F}$. For any integer i, we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{H}^i g^+ f_+ \mathscr{E} \arrow[r] \arrow[d] & \mathscr{H}^i g^+ f_+ \mathscr{F} \arrow[r] \arrow[d, "\star"] & \mathscr{H}^i g^+ f_+ \mathscr{E} \arrow[d] \\ \mathscr{H}^i f'_+ g'^+ \mathscr{E} \arrow[r] & \mathscr{H}^i f'_+ g'^+ \mathscr{F} \arrow[r] & \mathscr{H}^i f'_+ g'^+ \mathscr{E}, \end{tikzcd}$$

where the compositions of the horizontal homomorphisms are the identities. If the homomorphism $\star$ is an injection (resp. surjection), the left (resp. right) vertical homomorphism is also, so it suffices to show that $\star$ is an isomorphism.

Thus we may replace $\mathfrak{X}$ by X. In this case, the verification is local with respect to Y, and we may assume it to be an affine scheme. In this situation, X is realizable as well since X is projective over Y, and the proper base change theorem has already been known (cf. 1.1.3 (8)). $\square$

2.3.19. **Definition.** A morphism of admissible stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$ is said to be *compactifiable* if it can be factorized as

$$\mathfrak{X} \xrightarrow{j} \overline{\mathfrak{X}} \xrightarrow{f'} \mathfrak{Y},$$

where $\overline{\mathfrak{X}}$ is admissible, j is an open immersion, and f' is proper. We say that $\mathfrak{X} \to \overline{\mathfrak{X}}$ is a *compactification of f*. An admissible stack $\mathfrak{X}$ is said to be *compactifiable* if the structural morphism is compactifiable. We abbreviate compactifiable admissible stack as *c-admissible stack*.

2.3.20. In this subsection, we fix a subcategory $\mathfrak{S}_{\mathrm{adm}}$ of the category of admissible stacks satisfying the following conditions: 1. open immersions and proper morphisms are morphisms in $\mathfrak{S}_{\mathrm{adm}}$; 2. any morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{S}_{\mathrm{adm}}$ is compactifiable by an object in $\mathfrak{S}_{\mathrm{adm}}$; and 3. for a proper morphism $\mathfrak{X} \to \mathfrak{Y}$ and any morphism $\mathfrak{Y}' \to \mathfrak{Y}$ in $\mathfrak{S}_{\mathrm{adm}}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \to \mathfrak{Y}'$ is in $\mathfrak{S}_{\mathrm{adm}}$. An example of such a category is the following:

Lemma. *The full subcategory of c-admissible stacks satisfies the conditions.*

Proof. We need to show that any morphism between c-admissible stacks is compactifiable. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism between c-admissible stacks, and let $\overline{\mathfrak{X}}$ be a compactification of the structural morphism of $\mathfrak{X}$. Then f is factorized as $\mathfrak{X} \xrightarrow{\Gamma} \overline{\mathfrak{X}} \times \mathfrak{Y} \xrightarrow{p} \mathfrak{Y}$ where Γ is the graph morphism and p is the projection. Since $\overline{\mathfrak{X}}$ is assumed proper, p is proper. Thus, it suffices to show that Γ is compactifiable. Since $\mathfrak{Y}$ is admissible, Γ is a quasi-finite morphism. By [LM, 16.5], any quasi-finite morphism between admissible stacks is compactifiable, and the claim follows. $\square$

Remark. Any *algebraic space* separated of finite type over k is known to be c-admissible by [CLO].

2.3.21. An advantage of considering the category $\mathfrak{S}_{\mathrm{adm}}$ is that it satisfies the conditions of [SGA4, XVII, 3.2.4] if we take (S) to be $\mathfrak{S}_{\mathrm{adm}}$, (S,i) to be the subcategory consisting of open immersions, and (S,p) to be the subcategory consisting of proper morphisms.

Now, for $\mathfrak{X} \in \mathfrak{S}_{\mathrm{adm}}$, we associate the category $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. We shall further endow this with data which satisfy the conditions of [SGA4, XVII, 3.3.1]. For a proper morphism p, we consider the pushforward p_+ and the canonical isomorphism $(q \circ p)_+ \cong q_+ \circ p_+$ for composable morphisms p, q. For an open immersion j, we consider $j_!$ with canonical isomorphisms for compositions. These are the data of [SGA4, (i), (i'), (ii), (ii')]. These functors are subject to the conditions [SGA4, (a), (a'), (b), (b')]. Finally, for [SGA4, (iii)], we use the proper base change 2.3.18 and localization exact triangle 2.2.9. This isomorphism is subject to the conditions [SGA4, (c), (c')]. Thus, we may apply [SGA4, Proposition 3.3.2]. Summing up, we get the following definition.

Definition. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{S}_{\mathrm{adm}}$. Take a compactification $j\colon \mathfrak{X} \hookrightarrow \overline{\mathfrak{X}}$, and let $g\colon \overline{\mathfrak{X}} \to \mathfrak{Y}$ be the proper morphism. Then the functor $g_+ \circ j_!$ does not depend on the choice of the factorization up to canonical equivalence. This functor is denoted by $f_!$. Given composable morphisms f and g in $\mathfrak{S}_{\mathrm{adm}}$, we have a canonical equivalence $(f \circ g)_! \cong f_! \circ g_!$.

2.3.22. **Proposition.** *Consider the cartesian diagram* (2.3.17.1) (*where we do* not *assume f to be projective*). *We assume that the diagram is in $\mathfrak{S}_{\mathrm{adm}}$. Then there exists a canonical isomorphism $g^+ \circ f_! \cong f'_! \circ g'^+$.*

Proof. By definition of $f_!$, it suffices to treat the case where f is proper and an open immersion separately. When f is proper, this is nothing but the proper base change theorem, Proposition 2.3.18. When $f =: j$ is an open immersion, we have the canonical homomorphism $j'_! \circ g'^+ \to g^+ \circ j_!$. By definition, this homomorphism is an isomorphism if we take j'^+. Thus by the localization triangle (cf. Lemma 2.2.9), we get the isomorphism. Finally, we need to show that the resulting isomorphism does not depend on the choice of the factorization. Since the verification is standard, we leave it to the reader. $\square$

2.3.23. Let us construct a trace map, namely a map $f_! L^\omega_{\mathfrak{X}} \to L^\omega_{\mathfrak{Y}}$ for any morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathfrak{S}_{\mathrm{adm}}$. This will be achieved in Theorem 2.3.30. For this, we need to introduce a new t-structure.

Definition. Let X be a realizable scheme over k. For $\star \in \{\geq 0, \leq 0\}$, let ${}^{\mathrm{c}}D^\star$ be the full subcategory of $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_\emptyset)$ consisting of $\mathscr{C}$ such that $\mathrm{for}_L(\mathscr{C}) \in {}^{\mathrm{c}}D^\star_{\mathrm{hol}}(X/K_\emptyset)$, using 1.3.1. The pair $({}^{\mathrm{c}}D^{\leq}, {}^{\mathrm{c}}D^{\geq})$ defines a t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_\emptyset)$ called the *constructible t-structure*. We define ${}^{\mathrm{dc}}D^{\leq} := \mathbb{D}({}^{\mathrm{c}}D^{\geq})$ and ${}^{\mathrm{dc}}D^{\geq} := \mathbb{D}({}^{\mathrm{c}}D^{\leq})$. Then $({}^{\mathrm{dc}}D^{\leq}, {}^{\mathrm{dc}}D^{\geq})$ defines a t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}$. This is called the *dual constructible t-structure*. We also define t-structures $({}^{\mathrm{c}}D^{\leq}, {}^{\mathrm{c}}D^{\geq})$ and $({}^{\mathrm{dc}}D^{\leq}, {}^{\mathrm{dc}}D^{\geq})$ on $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$ such that $\mathscr{C}$ is in one of the full subcategories if and only if $\mathrm{for}_F(\mathscr{C})$ is in the corresponding one of $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_\emptyset)$.

2.3.24. **Definition.** Let $\mathfrak{X}$ be an algebraic stack. Let $X_\bullet \to \mathfrak{X}$ be a simplicial realizable scheme presentation, and let $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(X_\bullet)$. Put $d_i := d_{X_i/\mathfrak{X}}$ (cf. 0.0.3).

(1) A complex $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ is in ${}^{\mathrm{c}}D^\star$ ($\star \in \leq 0, \geq 0$) if and only if $\rho_i^*(\mathscr{M}) \in {}^{\mathrm{c}}D^{\star - d_i}$.

(2) A complex $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ is in ${}^{\mathrm{dc}}D^\star$ ($\star \in \leq 0, \geq 0$) if and only if $\rho_i^*(\mathscr{N}) \in {}^{\mathrm{dc}}D^{\star + d_i}$.

We leave the reader to check that $({}^{\mathrm{c}}D^{\leq 0}, {}^{\mathrm{c}}D^{\geq 0})$ and $({}^{\mathrm{dc}}D^{\leq 0}, {}^{\mathrm{dc}}D^{\geq 0})$ define t-structures, and that they do not depend on the choice of the simplicial schemes. These t-structures are called the *constructible t-structure* and *dual constructible t-structure*, and are abbreviated as *c-t-structure* and *dc-t-structure*, respectively. We denote the cohomology functor for the c-t-structure (resp. dc-t-structure) by ${}^{\mathrm{c}}\mathscr{H}^*$ (resp. ${}^{\mathrm{dc}}\mathscr{H}^*$), and objects in the heart are called *c-modules* (resp. *dc-modules*).

2.3.25. **Lemma.** (i) *We have* ${}^{\mathrm{dc}}\mathscr{H}^i \cong \mathbb{D} \circ {}^{\mathrm{c}}\mathscr{H}^{-i} \circ \mathbb{D}$. *In particular,* $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ *is a c-module if and only if* $\mathbb{D}(\mathscr{M})$ *is a dc-module, and a homomorphism* $f\colon \mathscr{M} \to \mathscr{N}$ *of c-modules is c-injective* (*resp. c-surjective*) *if and only if* $\mathbb{D}(f)$ *is dc-surjective* (*resp. dc-injective*).

(ii) *Let* $f\colon \mathfrak{X} \to \mathfrak{Y}$ *be a smooth morphism. Then the functor* f^+ *is c-t-exact.*

Proof. For (ii), see Lemma 1.3.4. The details are left to the reader. $\square$

2.3.26. **Lemma.** (i) *For an admissible stack* $\mathfrak{X}$, $L^\omega_{\mathfrak{X}}$ *is a dc-module.*

(ii) *Let* $f\colon \mathfrak{X} \to \mathfrak{Y}$ *be a morphism in* $\mathfrak{S}_{\mathrm{adm}}$. *Then* $f_!$ *is right dc-t-exact.*

Proof. To check (i), it suffices to show that $\mathbb{D}(L^\omega_{\mathfrak{X}}) \cong L_{\mathfrak{X}}$ is a c-module. This follows from Lemma 1.3.4 (i). Let us check (ii). First we may assume $\mathfrak{Y} =: Y$ to be a realizable scheme. For a dc-module $\mathscr{M}$ on $\mathfrak{X}$, we need to show that ${}^{\mathrm{dc}}\mathscr{H}^i f_!(\mathscr{M}) = 0$ for $i > 0$. We use induction on the support of $\mathscr{M}$. We may assume $\mathrm{Supp}(\mathscr{M}) = \mathfrak{X}$.

For an open dense substack $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$, it suffices to check that ${}^{\mathrm{dc}}\mathscr{H}^i f_!(j_!j^+\mathscr{M}) = 0$ for $i > 0$. Indeed, consider the localization triangle (cf. 2.2.9),

$$j_!j^+\mathscr{M} \to \mathscr{M} \to i_+i^+\mathscr{M} \xrightarrow{+1},$$

where i is the closed immersion of the complement of $\mathfrak{U}$. Since i_+i^+ is right dc-t-exact by Lemma 1.3.4 and $i_+i^+\mathscr{M}$ is supported on the complement of $\mathfrak{U}$, we know that ${}^{\mathrm{dc}}\mathscr{H}^i f_!(i_+i^+\mathscr{M}) = 0$ for $i > 0$ by the induction hypothesis. Thus, if we know the vanishing for $j_!j^+(\mathscr{M})$, we do for $\mathscr{M}$ also. By shrinking $\mathfrak{X}$, we can take a finite flat morphism $h\colon X \to \mathfrak{X}$ from a realizable scheme. By Lemma 2.2.4, $\mathscr{M}$ is a direct factor of $\mathbb{D}h_+h^+\mathbb{D}(\mathscr{M}) \cong h_!h^!(\mathscr{M})$ (cf. Lemma 2.3.17). Since $h^!$ is dc-t-exact, it remains to prove the right dc-t-exactness of $f \circ h\colon X \to Y$. Since $\mathbb{D} \circ f_! \circ \mathbb{D} \cong f_+$ is left c-t-exact by Lemma 1.3.4, we get the result. □

2.3.27. **Lemma.** *Let $X_\bullet \to \mathfrak{X}$ be an admissible simplicial scheme. Assume we are given $\{\mathscr{M}_i, \alpha_\phi\}$, where $\mathscr{M}_i$ is a dc-module on X_i, and $\alpha_\phi\colon X(\phi)^*(\mathscr{M}_i) \cong \mathscr{M}_j$ for $\phi\colon [i] \to [j]$ satisfying the cocycle condition. Then there exists a unique dc-module $\mathscr{M}$ on $\mathfrak{X}$, the* descent, *such that $\rho_i^*(\mathscr{M}) \cong \mathscr{M}_i$. Moreover, given other data $\{\mathscr{N}_i, \beta_\phi\}$ and its descent $\mathscr{N}$ on $\mathfrak{X}$, homomorphisms $\mathscr{M} \to \mathscr{N}$ correspond bijectively to systems of homomorphisms $\mathscr{M}_i \to \mathscr{N}_i$ compatible in the obvious sense. We also have similar results for c-modules.*

Proof. To check this, it suffices to show that $\mathbb{R}^k\mathrm{Hom}(\mathscr{M}_i, \mathscr{M}_i) = 0$ for $k < 0$ by Theorem 2.2.20. By the definition of dc-t-structure and biduality (cf. Proposition 2.3.13), we may assume that $\mathscr{M}_i$ is a c-module. In this case, since $\mathbb{R}\mathrm{Hom}(-,-)$ is left c-t-exact, the claim follows. □

2.3.28. **Lemma.** *Let $f\colon X \to Y$ be a smooth morphism of realizable schemes. Let $\mathscr{M}'$ be a complex in $D^{\mathrm{b}}_{\mathrm{hol}}(Y)$, and put $\mathscr{M} := f^+(\mathscr{M}')$. Then there exists an open dense subscheme $V \subset Y$ such that* $\mathrm{Supp}(\mathscr{M}') \cap V$ *is dense in* $\mathrm{Supp}(\mathscr{M}')$, *and for any closed immersion from a point $g\colon \{y\} \to V \hookrightarrow Y$, the base change homomorphism $g^+f_+(\mathscr{M}) \to f'_+g'^+(\mathscr{M})$ is an isomorphism, where $f'\colon X' := X \times_Y \{y\} \to \{y\}$ and $g'\colon X' \to X$ are the base changes of f and g.*

Proof. We may assume $\mathscr{M} \in \mathrm{Hol}(X/L)$. By replacing Y by the support of $\mathscr{M}$, we may assume that the support of $\mathscr{M}$ is equal to Y. We may assume Y to be reduced, and by shrinking Y, we may moreover assume that Y is smooth and $\mathscr{M}'$ is smooth on Y. We may take V such that each cohomology of $f_+(\mathscr{M})$ is smooth. Let c be the codimension of $\{y\}$ in Y. In this case, we have $g^+f_+(\mathscr{M}) \cong g^!f_+(\mathscr{M})(c)[2c]$ and $f'_+g'^+(\mathscr{M}) \cong f'_+g'^!(\mathscr{M})(c)[2c]$ by Theorem 1.5.14, and the claim follows by 1.1.3 (8). □

2.3.29. **Lemma.** *Let $X_\bullet \to \mathfrak{X}$ be a simplicial realizable scheme presentation of a c-admissible stack $\mathfrak{X}$. Let $p_i\colon X_i \to \mathfrak{X}$ be the induced morphism. We put $p^0_{i+} := {}^{\mathrm{c}}\mathscr{H}^0 p_{i+}$ (resp. $p^0_{i+} := {}^{\mathrm{dc}}\mathscr{H}^0 p_{i+}$), and similarly for $p^0_{i!}$. For a c-module $\mathscr{M}$ (resp. dc-module $\mathscr{N}$), we denote by $\mathscr{M}_i$ (resp. $\mathscr{N}_i$) the object $\rho_i^*\mathscr{M}$ (resp. $\rho_i^*\mathscr{N}$) on X_i.*

We have the following exact sequences of c-modules (resp. dc-modules):

$$0 \to \mathscr{M} \to p^0_{0+}(\mathscr{M}_0[-d_0]) \to p^0_{1+}(\mathscr{M}_1[-d_1])$$
$$(\textit{resp. } 0 \to \mathscr{N} \to p^0_{0+}(\mathscr{N}_0[-d_0]) \to p^0_{1+}(\mathscr{N}_1[-d_1])),$$
$$p^0_{1!}(\mathscr{M}_1[d_1](d_1)) \to p^0_{0!}(\mathscr{M}_0[d_0](d_0)) \to \mathscr{M} \to 0$$
$$(\textit{resp. } p^0_{1!}(\mathscr{N}_1[d_1](d_1)) \to p^0_{0!}(\mathscr{N}_0[d_0](d_0)) \to \mathscr{N} \to 0).$$

Proof. Let us prove the first sequence for $\mathscr{M}$. The sequence is defined by the adjunction. We only need to show that it is exact. Thus, we may assume $\mathfrak{X} =: X$ to be a scheme. First, let us show that there exists an open subscheme $j\colon U \hookrightarrow X$ which is dense in the support of $\mathscr{M}$ such that $j_+\mathscr{M}|_U$ satisfies the exactness property. We may take U such that p_0 and p_1 possess the base change property by Lemma 2.3.28. Indeed, it suffices to show the exactness after restricting to U since j_+ is left c-t-exact by Lemma 1.3.4. Then, we are reduced to the case where X is a point by the base change. Then c-t-structure coincides with the usual t-structure, and the exactness follows by Proposition 2.1.13.

We show the exactness by the induction on the dimension of the support of $\mathscr{M}$. Take an open dense subscheme U of the support of $\mathscr{M}$ such that the sequence is exact for $j_+\mathscr{M}|_U$ where $j\colon U \hookrightarrow X$. Consider the following diagram of c-modules where we omit shifts and twists:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{C} \arrow[r] \arrow[d] & \mathscr{M} \arrow[r] \arrow[d] & j_+\mathscr{M} \arrow[r] \arrow[d] & \mathscr{C}' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & p^0_{0+}\mathscr{C}_0 \arrow[r] \arrow[d] & p^0_{0+}\mathscr{M}_0 \arrow[r] \arrow[d] & p^0_{0+}j_+\mathscr{M}_0 \arrow[r, dotted] \arrow[d] & p^0_{0+}\mathscr{C}'_0 \arrow[d] & \\
0 \arrow[r] & p^0_{1+}\mathscr{C}_1 \arrow[r] & p^0_{1+}\mathscr{M}_1 \arrow[r] & p^0_{1+}j_+\mathscr{M}_1 \arrow[r, dotted] & p^0_{1+}\mathscr{C}'_1. &
\end{tikzcd}$$

The horizontal sequences are complexes, and the ones with solid arrows are exact. By the induction hypothesis, the vertical sequences are known to be exact except for the one starting from $\mathscr{M}$. Then by diagram chasing, we get that the vertical sequence starting from $\mathscr{M}$ is exact as well, and we get the lemma.

Now, for the exactness of the second sequence for $\mathscr{N}$, we just argue dually. For the second sequence for $\mathscr{M}$, the argument is similar and even simpler: We may assume $\mathfrak{X}$ to be a scheme. We can check the exactness by taking the *stalk*, and reduce to the case where X is a point immediately, in which case we get the exactness by Proposition 2.1.13. We can show dually for the first sequence for $\mathscr{N}$, and we may finish the proof. ☐

2.3.30. **Theorem.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{S}_{\mathrm{adm}}$. Then there exists a unique homomorphism $\mathrm{Tr}^{\mathrm{p}}_f\colon f_!L^{\omega}_{\mathfrak{X}} \to L^{\omega}_{\mathfrak{Y}}$ satisfying the following conditions.*

(I) *Transitivity. Given $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ in $\mathfrak{S}_{\mathrm{adm}}$, the composition of the following homomorphisms is equal to $\mathrm{Tr}^{\mathrm{p}}_{g\circ f}$:*

$$(g\circ f)_!L^{\omega}_{\mathfrak{X}} \cong g_!(f_!L^{\omega}_{\mathfrak{X}}) \xrightarrow{g_!\mathrm{Tr}^{\mathrm{p}}_f} g_!L^{\omega}_{\mathfrak{Y}} \xrightarrow{\mathrm{Tr}^{\mathrm{p}}_g} K^{\omega}_{\mathfrak{Z}}.$$

(II) *When* $\mathfrak{X} =: X$ *and* $\mathfrak{Y} =: Y$ *are realizable schemes and* $L = K$, *then* $\mathrm{Tr}^{\mathrm{p}}_f$ *is the adjunction homomorphism* $f_! K^{\omega}_X \cong f_! f^! K^{\omega}_Y \to K^{\omega}_Y$. *Moreover,* $\mathrm{Tr}^{\mathrm{p}}_f$ *commutes with* for_L.

(III) *The trace map is compatible with smooth pullback on* $\mathfrak{Y}$. *Namely, consider* (2.3.17.1) *in* $\mathfrak{S}_{\mathrm{adm}}$ *such that* g *is smooth of relative dimension* d. *Then the composition*

$$f'_! L^{\omega}_{\mathfrak{X}'} \cong f'_! g'^* L^{\omega}_{\mathfrak{X}}(d)[d] \to g^* f_! L^{\omega}_{\mathfrak{X}}(d)[d] \xrightarrow{g^*\mathrm{Tr}^{\mathrm{p}}_f} g^* L^{\omega}_{\mathfrak{Y}}(d)[d] \cong L^{\omega}_{\mathfrak{Y}'},$$

where the second map is the base change homomorphism, coincides with $\mathrm{Tr}^{\mathrm{p}}_{f'}$.

Proof. We put the dc-t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ and $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$. Since $f_!$ is right dc-t-exact by Lemma 2.3.26, it suffices to construct a morphism of dc-t-modules $f^0_! L^{\omega}_{\mathfrak{X}} \to L^{\omega}_{\mathfrak{Y}}$ where $f^0_! := {}^{\mathrm{dc}}\mathscr{H}^0 f_!$. When $\mathfrak{X}$ and $\mathfrak{Y}$ are realizable schemes, the trace map for $L = K$ extends uniquely to general L by (II). Let us construct this in the case where $\mathfrak{Y} =: Y$ is a realizable scheme. Let $X_0 \to \mathfrak{X}$ be a presentation from a quasi-projective scheme, let $X_1 := X_0 \times_{\mathfrak{X}} X_0$, let $p_0, p_1 \colon X_1 \to X_0$ be the first and second projection, and put $f_i \colon X_i \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$. By the property of the adjunction homomorphism, we have the following commutative diagram:

$$\begin{tikzcd} f^0_{1!} L^{\omega}_{X_1} \arrow[d, shift right, "p_1^*"'] \arrow[d, shift left, "p_0^*"] \arrow[dr, "\mathrm{Tr}^{\mathrm{p}}_{f_1}"] & \\ & L^{\omega}_Y. \\ f^0_{0!} L^{\omega}_{X_0} \arrow[ur, "\mathrm{Tr}^{\mathrm{p}}_{f_0}"'] & \end{tikzcd}$$

Thus by the second exact sequence for $\mathscr{N}$ in Lemma 2.3.29, we have a homomorphism $\mathrm{Tr}^{\mathrm{p}}_f \colon f^0_! L^{\omega}_{\mathfrak{X}} \to L^{\omega}_Y$ as required. By condition (I), this map is uniquely determined. It is straightforward to check that this map does not depend on the choice of the smooth presentation and satisfies (II).

Finally consider the case where $\mathfrak{Y}$ is not a realizable scheme. Take a simplicial realizable scheme presentation $Y_\bullet \to \mathfrak{Y}$. By Lemma 2.3.27, it suffices to construct a homomorphism $(f^0_! L^{\omega}_{\mathfrak{X}})_{Y_i} \to L^{\omega}_{Y_i}$ with compatibility conditions. By condition (III), this map should be the one we have already constructed, and we conclude the proof. $\square$

2.3.31. Let $f \colon \mathfrak{X} \to \mathfrak{Y}$ be a *proper* morphism between admissible stacks. Then we have the homomorphism $f_+ L^{\omega}_{\mathfrak{X}} \xleftarrow{\sim} f_! L^{\omega}_{\mathfrak{X}} \xrightarrow{\mathrm{Tr}^{\mathrm{p}}_f} L^{\omega}_{\mathfrak{Y}}$. This homomorphism induces $f_+ \circ \mathbb{D}_{\mathfrak{X}} \to \mathbb{D}_{\mathfrak{Y}} \circ f_+$ as in 2.3.17. Now, let f be a morphism in $\mathfrak{S}_{\mathrm{adm}}$. Let $\mathfrak{X} \xrightarrow{j} \overline{\mathfrak{X}} \xrightarrow{\overline{f}} \mathfrak{Y}$ be a compactification of f in $\mathfrak{S}_{\mathrm{adm}}$. We have the homomorphism

$$(\star) \qquad f_+ \circ \mathbb{D}_{\mathfrak{X}} \cong \overline{f}_+ \circ j_+ \circ \mathbb{D}_{\mathfrak{X}} \xleftarrow{\sim} \overline{f}_+ \circ \mathbb{D}_{\mathfrak{X}'} \circ j_! \to \mathbb{D}_{\mathfrak{Y}} \circ \overline{f}_+ \circ j_! \cong \mathbb{D}_{\mathfrak{Y}} \circ f_!,$$

where the second isomorphism follows by Remark 2.3.13 (iii). We may check that this homomorphism does not depend on the choice of the factorization up to canonical equivalence.

Theorem (Duality). *For any morphism* f *in* $\mathfrak{S}_{\mathrm{adm}}$, *the homomorphism* $(\star)$ *is, in fact, an isomorphism.*

Proof. First, we may assume $\mathfrak{Y}$ is a scheme by (III) of Theorem 2.3.30. Let us show the isomorphism for $\mathscr{M} \in \mathrm{Hol}(\mathfrak{X})$. We use induction on $\dim \mathrm{Supp}(\mathscr{M})$. Assume the theorem holds for $\dim \mathrm{Supp}(\mathscr{M}) < k$. Let $\mathfrak{Z}$ be the support of $\mathscr{M}$. We may shrink $\mathfrak{X}$ so that $\mathfrak{Z}$ is shrunk by its open dense subscheme. Indeed, let $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ be an open immersion such that $\mathfrak{Z} \cap \mathfrak{U}$ is dense in $\mathfrak{Z}$, and let $i\colon \mathfrak{W} \hookrightarrow \mathfrak{X}$ be its complement. The proposition holds for $i_+ i^+(\mathscr{M})$ by the induction hypothesis. Thus, it suffices to show the theorem only for $\mathscr{M} = j_! j^+(\mathscr{M})$. Since the theorem holds for $f = j$, we may replace $\mathfrak{X}$ by $\mathfrak{U}$.

Shrinking $\mathfrak{X}$ by its open dense substack, we may assume that there exists a finite flat morphism $g\colon X \to \mathfrak{X}$ from a realizable scheme. Since $\mathscr{M}$ is a direct factor of $g_+ g^+ \mathscr{M}$, by arguing as in the proof of Proposition 2.3.18, it suffices to show that the homomorphism $f_+ \circ \mathbb{D}_{\mathfrak{X}}(g_+ g^+ \mathscr{M}) \to \mathbb{D}_{\mathfrak{Y}} \circ f_!(g_+ g^+ \mathscr{M})$ is an isomorphism. By Lemma 2.3.17, it is reduced to the realizable scheme case. $\square$

2.3.32. **Definition.** Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{S}_{\mathrm{adm}}$. We define $f^! := \mathbb{D}_{\mathfrak{X}} \circ f^+ \circ \mathbb{D}_{\mathfrak{Y}}$. The couple $(f_!, f^!)$ is an adjoint pair. Transitivity holds since it holds for f^+.

2.3.33. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathrm{Spec}(k)$ be the structural morphism of a c-admissible stack of dimension d. Then for $\mathscr{M} \in \mathrm{Con}(\mathfrak{X})$, we have $\mathscr{H}^i f_!(\mathscr{M}) = 0$ for $i > 2d$.*

Proof. We may use induction on the dimension of $\mathscr{M}$. By standard *dévissage* using the induction hypothesis, we may shrink $\mathfrak{X}$ by its open dense substack. By shrinking $\mathfrak{X}$ and taking a finite flat morphism from a realizable scheme, we may assume that X is a realizable scheme. Then the proposition is reduced to Lemma 1.3.8. $\square$

2.3.34. **Theorem** (Relative Poincaré duality)**.** *Let $\mathfrak{M}^{\mathrm{st}}_d$ be the set of morphisms $f\colon \mathfrak{X} \to \mathfrak{Y}$ of $\mathfrak{S}_{\mathrm{adm}}$ such that there exists an open substack $\mathfrak{U} \subset \mathfrak{Y}$ such that $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{U} \to \mathfrak{U}$ is flat of relative dimension d, and the dimension of any fiber of $\mathfrak{Y} \setminus \mathfrak{U}$ is $< d$. Then for $f \in \mathfrak{M}^{\mathrm{st}}_d$ there is a unique trace map $\mathrm{Tr}^{\mathrm{sm}}_f\colon f_! f^+(d)[2d] \to \mathrm{id}$ satisfying the following properties.*

(I) *When $\mathfrak{X}$ and $\mathfrak{Y}$ are realizable schemes and $L = K$, then it coincides with the trace map in Theorem* 1.5.1. *Moreover, it commutes with* for_L.

(II) *It commutes with base change in the sense of* (Var 2) *of* 1.5.1 *if we replace the diagram of realizable schemes by that in $\mathfrak{S}_{\mathrm{adm}}$ and $f \in \mathfrak{M}^{\mathrm{st}}_d$.*

(III) *It is transitive with respect to the composition of morphisms in $\mathfrak{M}^{\mathrm{st}}_d$ in the sense of* (Var 3) *of* 1.5.1.

Taking the adjoint, we have a homomorphism $f^+(d)[2d] \to f^!$, which is an isomorphism when f is smooth.

Remark. The superscript of $\mathrm{Tr}^{\mathrm{sm}}$ stands for "smooth". This is because the trace map is used to show that $f^+(d)[2d] \cong f^!$ for *smooth* morphisms. On the other hand, the superscript for the trace map Tr^{p} in Theorem 2.3.30 stands for "proper", since this trace map is related to the isomorphism $f_! \xrightarrow{\sim} f_+$ when f is proper. These two trace maps have a priori no relation. Finally, the superscript of $\mathfrak{M}^{\mathrm{st}}_d$ stands for "stack".

Proof. First, we need to construct the trace map $\mathrm{Tr}^{\mathrm{sm}}_f\colon f_! f^+ L_{\mathfrak{Y}}(d)[2d] \to L_{\mathfrak{Y}}$. Put c-t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$. By Lemma 2.3.33, it suffices to construct a homomorphism ${}^{\mathrm{c}}\mathscr{H}^{2d} f_! f^+ L_{\mathfrak{Y}}(d) \to L_{\mathfrak{Y}}$ of c-modules. When $\mathfrak{X}$ and $\mathfrak{Y} =: Y$ are schemes, this is the trace map of Theorem 1.5.1 when $L = K$, and in general it is defined by extending the scalar. For the careful reader, we remark that, when $\blacktriangle = F$, in Theorem 1.5.1, we used the category $F\text{-}D^{\mathrm{b}}_{\mathrm{hol}}(Y/K)$ to define the trace map. However, the isomorphism defining the Frobenius structure in $D^{\mathrm{b}}_{\mathrm{hol}}(Y/K)$ induces an isomorphism in $\mathrm{Con}(Y/K)$, which defines an object in $\mathrm{Con}(Y/\mathfrak{T}_F)$, so Theorem 1.5.1 is enough to get a trace map in $D^{\mathrm{b}}(Y/\mathfrak{T}_F)$.

For the construction of a trace map in the general case, let $Y_\bullet \to \mathfrak{Y}$ be an admissible simplicial scheme. By Lemma 2.3.27, it suffices to construct the trace map for $\mathfrak{X} \times_{\mathfrak{Y}} Y_i \to Y_i$ for each i compatible with each other. The construction is similar to that of Theorem 2.3.30 using Lemma 2.3.29, so we leave the details to the reader.

The trace map defines a morphism $f^+(d)[2d] \to f^!$. Let us show that this is an isomorphism when f is smooth. By the base change property, we may assume $\mathfrak{Y}$ to be a scheme. Moreover, it suffices to show the identity after pulling back to schemes which are smooth over $\mathfrak{X}$. Then we are reduced to the scheme case that we have already treated in Theorem 1.5.13. □

2.3.35. Finally, we have the projection formula, whose proof is similar to the proper base change theorem, and is left to the reader:

Proposition. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathfrak{S}_{\mathrm{adm}}$. Then for $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ and $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$, we have a canonical isomorphism:*

$$f_!\mathscr{M} \otimes \mathscr{N} \cong f_!\bigl(\mathscr{M} \otimes f^+\mathscr{N}\bigr).$$

The Künneth formula.

2.3.36. **Proposition.** *Consider morphisms of admissible stacks $f\colon \mathfrak{X} \to \mathfrak{X}'$ and $g\colon \mathfrak{Y} \to \mathfrak{Y}'$. Let $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$ and $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y})$. Then there exists a canonical isomorphism*

$$f_+(\mathscr{M}) \boxtimes g_+(\mathscr{N}) \xrightarrow{\sim} (f \times g)_+\bigl(\mathscr{M} \boxtimes \mathscr{N}\bigr).$$

Moreover, if f and g are in $\mathfrak{S}_{\mathrm{adm}}$, we get an isomorphism

$$f_!(\mathscr{M}) \boxtimes g_!(\mathscr{N}) \xrightarrow{\sim} (f \times g)_!\bigl(\mathscr{M} \boxtimes \mathscr{N}\bigr).$$

Proof. Let us construct the first homomorphism. We have the following homomorphism

$$(f \times g)^+\bigl(f_+(\mathscr{M}) \boxtimes g_+(\mathscr{N})\bigr) \cong f^+f_+(\mathscr{M}) \boxtimes g^+g_+(\mathscr{N}) \to \mathscr{M} \boxtimes \mathscr{N},$$

where the first isomorphism follows by Proposition 2.3.15 and the second homomorphism is by adjunction. By taking the adjunction, we get the homomorphism we are looking for. To check that this homomorphism is an isomorphism, it suffices to treat the finite morphism case and the projection case separately. The finite morphism case follows by Lemma 2.2.6, and the projection case follows by Lemma 2.2.25 and Lemma 2.2.23 (i). By Theorem 2.3.31, we get that $f_! \cong \mathbb{D}_{\mathfrak{X}'} \circ f_+ \circ \mathbb{D}_{\mathfrak{X}}$. Thus, the second isomorphism holds by the first one and the commutativity of $\boxtimes$ and $\mathbb{D}$ by Proposition 2.3.15. □

2.3.37. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{X}' \to \mathfrak{Y}$ be morphisms between admissible stacks. Consider the following cartesian diagram:

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}' \arrow[r, "i"] \arrow[d, "h"'] \arrow[dr, phantom, "\square"] & \mathfrak{X} \times \mathfrak{X}' \arrow[d, "f \times g"] \\ \mathfrak{Y} \arrow[r, "\Delta_{\mathfrak{Y}}"'] & \mathfrak{Y} \times \mathfrak{Y}. \end{tikzcd}$$

For $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}')$, we put

$$\mathscr{M} \boxtimes_{\mathfrak{Y}} \mathscr{N} := i^{+}\bigl(\mathscr{M} \boxtimes \mathscr{N}\bigr).$$

When f and g are the identities, $(-) \boxtimes_{\mathfrak{Y}} (-)$ is nothing but $(-) \otimes (-)$.

Corollary. *Assume f and g are $\mathfrak{S}_{\mathrm{adm}}$. Then, we have a canonical isomorphism*

$$f_{!}(\mathscr{M}) \otimes g_{!}(\mathscr{N}) \cong h_{!}\bigl(\mathscr{M} \boxtimes_{\mathfrak{Y}} \mathscr{N}\bigr).$$

Proof. Use Proposition 2.3.22. □

Theory of weights revisited.

2.3.38. The theory of six functors for c-admissible stacks fits perfectly with the theory of weights. Consider the situation in 2.2.30. The following is a direct consequence of 2.2.32:

Theorem. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism between admissible stacks.*

(i) *Then the functors f_+, f^+, $\mathbb{D}$, $\otimes$ preserve ι-mixed complexes. Moreover, f_+ (resp. f^+) preserves complexes of weight $\geq w$ (resp. $\leq w$), $\mathbb{D}$ exchanges complexes of weight $\leq w$ and $\geq -w$, and $\otimes$ sends complexes of weight $(\leq w, \leq w')$ to $\leq w+w'$.*

(ii) *Assume $f =: j$ is an immersion, and $\mathscr{M}$ is ι-pure of weight w in* $\mathrm{Hol}(\mathfrak{X}/L_F)$. *Then $j_{!+}(\mathscr{M})$ is ι-pure of weight w.*

In particular, by using the duality 2.3.31, if f is proper, f_+ sends a pure complex of weight w to a pure complex of weight w.

2.4. **Miscellaneous results on cohomology theory.** Before proceeding to the next section, we pause a little and collect some miscellaneous results which are used in the proof of the Langlands correspondence. So far, we have established the theory for admissible stacks. However, in the proof of the Langlands correspondence, we sometimes need to deal with nonadmissible stacks. For this, we employ ad hoc constructions of cohomological operations and prove some basic properties. The next theme of this subsection is to show smoothness results using, again, ad hoc construction of the nearby cycle functor. Finally, we collect some properties of Tannakian fundamental groups of isocrystals.

Cohomology theory for algebraic stacks.

2.4.1. We denote by $\mathrm{St}^{\mathrm{lft}}(k)$ the category of algebraic stacks locally of finite type over k. Let $\mathfrak{X}$ be in $\mathrm{St}^{\mathrm{lft}}(k)$. To $X \in \mathfrak{X}_{\mathrm{sm}}$ (cf. 0.0.2), we associate the category $\mathrm{Con}(X/L)$. For $f\colon X \to Y$ in $\mathfrak{X}_{\mathrm{sm}}$, we have the pull-back functor $f^{+}\colon \mathrm{Con}(Y/L) \to \mathrm{Con}(X/L)$. This functor is exact by Lemma 2.3.25 (ii). With the isomorphism $(f \circ g)^{+} \cong g^{+} \circ f^{+}$, these data form a fibered category $\mathrm{Con}_{\mathfrak{X}/L} \to \mathfrak{X}_{\mathrm{sm}}$. Now, we have:

Lemma. *When $\mathfrak{X}$ is* quasi-compact*, the category of c-modules* $\mathrm{Con}(\mathfrak{X}/L)$ *(cf.* 2.3.24*) we have defined so far is equivalent to the category of cartesian sections of the fibered category* $\mathrm{Con}_{\mathfrak{X}/L}$ *over* $\mathfrak{X}_{\mathrm{sm}}$.

Proof. We denote by $\Gamma\mathrm{Con}_{\mathfrak{X}/L}$ the category of cartesian sections of the fibered category $\mathrm{Con}_{\mathfrak{X}/L}$. We construct a functor $F\colon \mathrm{Con}(\mathfrak{X}/L) \to \Gamma\mathrm{Con}_{\mathfrak{X}/L}$. Let $\mathscr{M} \in \mathrm{Con}(\mathfrak{X}/L)$, and let $X \in \mathfrak{X}_{\mathrm{sm}}$. Let $\rho\colon X \to \mathfrak{X}$ be the smooth morphism. Then since X and $\mathfrak{X}$ are both good stacks, in the sense of the second read case, the functor ρ^{+} (which is isomorphic to $\rho^{*}[-d_{X/\mathfrak{X}}]$ by Lemma 2.3.8) is defined, and we put $F(\mathscr{M})_X := \rho^+(\mathscr{M}) \in \mathrm{Con}(X/L)$. Defining the gluing isomorphism by the transitivity of pullbacks, we have $F(\mathscr{M}) \in \Gamma\mathrm{Con}_{\mathfrak{X}/L}$.

Let $\mathcal{X}_\bullet \to \mathfrak{X}$ be a simplicial algebraic space presentation. By associating the category $\mathrm{Con}(\mathcal{X}_i/L)$ to $\mathcal{X}_i$ and considering the pullback, we have the cofibered category $\mathrm{Con}(\mathcal{X}_\bullet/L)_\bullet$ over Δ^+, similar to Definition 2.1.4. By the construction of F, there exists the canonical functor $\mathrm{Con}(\mathfrak{X}/L) \to (\mathrm{Con}(\mathcal{X}_\bullet/L)_\bullet)_{\mathrm{tot}}$. This induces an equivalence of categories: we can check the full faithfulness by using (2.1.5.1), and it is essentially surjective by Theorem 2.2.20.

Let us construct the functor $G\colon \Gamma\mathrm{Con}_{\mathfrak{X}/L} \to \mathrm{Con}(\mathfrak{X}/L)$. First, let $\mathfrak{X} =: \mathcal{X}$ be an algebraic space. Take $X \in \mathcal{X}_{\mathrm{sm}}$, and let $X_\bullet := \mathrm{cosk}_0(X \to \mathcal{X})$. Then since $X_n \in \mathcal{X}_{\mathrm{sm}}$, we have the *restriction* functor $G\colon \Gamma\mathrm{Con}_{\mathcal{X}/L} \to (\mathrm{Con}(X_\bullet/L)_\bullet)_{\mathrm{tot}} \cong \mathrm{Con}(\mathcal{X}/L)$. This does not depend on auxiliary choices, and it is straightforward to check that this is quasi-inverse to F, thus the lemma is shown when $\mathfrak{X}$ is an algebraic space. Now, for a smooth morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks, we have the faithful functor $\mathfrak{X}_{\mathrm{sm}} \to \mathfrak{Y}_{\mathrm{sm}}$ sending $X \to \mathfrak{X}$ to the composition $X \to \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$. This induces a functor $f^+\colon \Gamma\mathrm{Con}_{\mathfrak{Y}/L} \to \Gamma\mathrm{Con}_{\mathfrak{X}/L}$. Let $\rho\colon \mathcal{X} \to \mathfrak{X}$ be a smooth morphism. Then we have the functor $\Gamma\mathrm{Con}_{\mathfrak{X}/L} \xrightarrow{\rho^+} \Gamma\mathrm{Con}_{\mathcal{X}/L} \cong \mathrm{Con}(\mathcal{X}/L)$. Take a simplicial presentation $\mathcal{X}_\bullet$ of $\mathfrak{X}$, then $\mathcal{X}_n$ is an algebraic space so we argue as in the case where $\mathfrak{X}$ is an algebraic space to construct the quasi-inverse G of F. □

This lemma enables us to define $\mathrm{Con}(\mathfrak{X}/L)$, for $\mathfrak{X}$ in $\mathrm{St}^{\mathrm{lft}}(k)$ not necessarily quasi-compact, to be the category of cartesian sections of $\mathrm{Con}_{\mathfrak{X}/L}$. We often denote $\mathrm{Con}(\mathfrak{X}/L)$ by $\mathrm{Con}(\mathfrak{X})$ for simplicity. For a smooth morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in $\mathrm{St}^{\mathrm{lft}}(k)$, we have a faithful functor $\mathfrak{X}_{\mathrm{sm}} \to \mathfrak{Y}_{\mathrm{sm}}$, which induces the pull-back functor $f^+\colon \mathrm{Con}(\mathfrak{Y}/L) \to \mathrm{Con}(\mathfrak{X}/L)$. Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are of finite type, and let d be the relative dimension of f. Then f^+ is canonically equivalent to $f^*[-d]$ using the functor in 2.2.7. For $\mathscr{M} \in \mathrm{Con}(\mathfrak{Y}/L)$, we sometimes denote $f^+\mathscr{M} \in \mathrm{Con}(\mathfrak{X}/L)$ by $\mathscr{M}_{\mathfrak{X}}$.

2.4.2. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a *representable morphism of finite type* in $\mathrm{St}^{\mathrm{lft}}(k)$. Let us construct f^i_+ and $f^i_!$. To construct these, let $Y \in \mathfrak{Y}_{\mathrm{sm}}$. This defines the morphisms $f_Y\colon \mathfrak{X}_Y := \mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$, and $\rho\colon \mathfrak{X}_Y \to \mathfrak{X}$. Note that $\mathfrak{X}_Y$ is an algebraic space by assumption, thus f_Y is compactifiable by Remark 2.3.20. Now, for $\mathscr{M} \in \mathrm{Con}(\mathfrak{X})$, put $\bigl(f^i_+\mathscr{M}\bigr)_Y := {}^{\mathrm{c}}\mathscr{H}^i f_{Y+}\rho^*(\mathscr{M})$, and similarly for $\bigl(f^i_!\mathscr{M}\bigr)_Y$. For a smooth morphism $\phi\colon Y' \to Y$ in $\mathfrak{Y}_{\mathrm{sm}}$, the base change theorem (Proposition 2.3.9) and Proposition 2.3.22 give us the isomorphisms

$$\phi^+\bigl(f^i_+\mathscr{M}\bigr)_Y \cong \bigl(f^i_+\mathscr{M}\bigr)_{Y'}, \qquad \phi^+\bigl(f^i_!\mathscr{M}\bigr)_Y \cong \bigl(f^i_!\mathscr{M}\bigr)_{Y'}.$$

We can check the transitivity of these isomorphisms easily, and we define objects $\bigl\{(f^i_+\mathscr{M})_Y\bigr\}_{Y\in\mathfrak{Y}_{\mathrm{sm}}}$ and $\bigl\{(f^i_!\mathscr{M})_Y\bigr\}_{Y\in\mathfrak{Y}_{\mathrm{sm}}}$ in $\mathrm{Con}(\mathfrak{Y})$. We denote them by $f^i_+\mathscr{M}$ and $f^i_!\mathscr{M}$ respectively. When f is an immersion, $f^0_!$ is c-t-exact and $f^i_! = 0$ for $i \neq 0$ by Lemma 1.3.4. Moreover, $f^0_!$ coincides with that induced by 2.2.8; in this case, we sometimes denote $f^0_!$ by $f_!$.

2.4.3. Let $\mathrm{St}^{\mathrm{lft,sm}}(k)$ be the subcategory of $\mathrm{St}^{\mathrm{lft}}(k)$ such that the objects are the same, and for morphisms, we only consider smooth morphisms. By associating $\mathrm{Con}(\mathfrak{X})$ to $\mathfrak{X} \in \mathrm{St}^{\mathrm{lft,sm}}(k)$, and considering the pullback f^* for smooth morphisms, we have the fibered category $\mathrm{Con} \to \mathrm{St}^{\mathrm{lft,sm}}(k)$. Then we have the following descent result:

Lemma. *Smooth surjective representable morphisms are universal effective descent morphisms in the fibered category* $\mathrm{Con} \to \mathrm{St}^{\mathrm{lft,sm}}(k)$*.*

Proof. Let $\mathrm{St}^{\mathrm{lft,sm,rep}}(k)$ be the subcategory of $\mathrm{St}^{\mathrm{lft,sm}}(k)$ such that the objects are the same, and we only consider representable ones for morphisms between algebraic stacks. We may consider Con as a fibered category over $\mathrm{St}^{\mathrm{lft,sm,rep}}(k)$. We already proved in the proof of Lemma 2.4.1 that a smooth surjective morphism *from an algebraic space* is an effective descent morphism, thus it is a universal effective descent morphism in $\mathrm{St}^{\mathrm{lft,sm,rep}}(k)$. Let $\mathfrak{Y} \to \mathfrak{X}$ be a smooth surjective representable morphism. Take a smooth representable morphism $\mathcal{X} \to \mathfrak{X}$ from an algebraic space. Then we know that $\mathcal{X} \to \mathfrak{X}$ and $\mathfrak{Y} \times_{\mathfrak{X}} \mathcal{X} \to \mathcal{X}$ are universal effective descent morphisms. Universal effective descent morphisms form a topology by [Gir, 6.23], $\mathfrak{Y} \to \mathfrak{X}$ is universal effective descent by *caractère local* (cf. [SGA4, II, 1.1]) of Grothendieck topology. □

2.4.4. We may extend the pull-back functor to an arbitrary morphism between algebraic stacks. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism in $\mathrm{St}^{\mathrm{lft}}(k)$, and take $Y \in \mathfrak{Y}_{\mathrm{sm}}$. Put $f'\colon \mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$. Now, let $X' \in \mathfrak{X}'_{\mathrm{sm}}$. Then we have the morphism $f'_{X'}\colon X' \to Y$. We put

$$f'^{+}(\mathscr{M}_Y)_{X'} := f'^{+}_{X'}(\mathscr{M}_Y).$$

We can check easily that the collection of these modules satisfies the compatibility condition, and it defines a cartesian section of the fibered category $\mathrm{Con}_{\mathfrak{X}'}$, which we define to be the module $f'^{+}(\mathscr{M}_Y)$ in $\mathrm{Con}(\mathfrak{X}')$. These modules yield a descent data with respect to the representable smooth morphism $\mathfrak{X}' \to \mathfrak{X}$. By using Lemma 2.4.3, we get $f^{+}(\mathscr{M}) \in \mathrm{Con}(\mathfrak{X})$. The pullback is exact by Lemma 2.3.25 (ii), and satisfies the transitivity property: for morphisms $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}$ in $\mathrm{St}^{\mathrm{lft}}$, we have a canonical equivalence $(g \circ f)^{+} \cong f^{+} \circ g^{+}$.

Finally, let $\mathfrak{C}^{(\prime)} \to \mathfrak{D}$ be a morphism in $\mathrm{St}^{\mathrm{lft}}(k)$ between smooth stacks. Let $\mathscr{M}^{(\prime)}$ be in $\mathrm{Sm}(\mathfrak{C}^{(\prime)})$. Let $\Delta\colon \mathfrak{C} \times_{\mathfrak{D}} \mathfrak{C}' \to \mathfrak{C} \times \mathfrak{C}'$ be the canonical morphism. We define $\mathscr{M} \boxtimes_{\mathfrak{D}} \mathscr{M}' := \Delta^{+}(\mathscr{M} \boxtimes \mathscr{M}')$ for $\mathscr{M}^{(\prime)} \in \mathrm{Con}(\mathfrak{C}^{(\prime)})$.

2.4.5. **Lemma.** (i) *If $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a representable morphism in $\mathfrak{S}_{\mathrm{adm}}$, then ${}^{\mathrm{c}}\mathscr{H}^i f_! \cong f^i_!$.*

(ii) *Consider the following cartesian diagram in* $\mathrm{St}^{\mathrm{lft}}(k)$*:*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{C}' \arrow[r, "g"'] & \mathfrak{C}. \end{tikzcd}$$

Assume f is representable. Then there exists a canonical isomorphism $g^{+} \circ f^i_! \cong f'^i_! \circ g'^{+}$.

Proof. The first claim follows from the definition, and the verification for (ii) is easy from the base change, Proposition 2.3.22. □

2.4.6. A morphism in $\mathrm{St}^{\mathrm{lft}}(k)$ is said to be *gerb-like* if, locally with respect to fppf-topology, the morphism can be written as the canonical morphism $B(G/\mathcal{X})(=: BG) \to \mathcal{X}$ (cf. [LM, 9.6]) for some flat group space G of finite presentation over $\mathcal{X}$. Recall that such morphisms are smooth by [Beh, 5.1.3, 5.1.5].

Lemma. *Let $\mathcal{X}$ be a algebraic space of finite type, and let G be a flat algebraic group space finite radicial surjective over $\mathcal{X}$. Let $\rho\colon BG \to \mathcal{X}$ be the canonical morphism. Note that ρ is proper and is in $\mathfrak{S}_{\mathrm{adm}}$. Then ρ^+ and $\rho_! \cong \mathscr{H}^0\rho_!$ induce the equivalence of categories between $\mathrm{Con}(\mathcal{X})$ and $\mathrm{Con}(BG)$.*

Proof. Note that ρ is a proper morphism between admissible stacks since G is finite, thus $\rho \in \mathfrak{S}_{\mathrm{adm}}$. We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "u"'] \arrow[rr, bend left, "\mathrm{id}"] & BG \arrow[r, "\rho"'] & \mathcal{X}, \end{tikzcd}$$

where u is the universal G-torsor, which is a universal homeomorphism by assumption. Thus we can use Lemma 2.2.4 (i) to conclude. $\square$

Corollary. *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a gerb-like morphism in $\mathrm{St}^{\mathrm{lft}}(k)$ whose structural group is flat finite and radicial. Then f^+ induces an equivalence of categories $\mathrm{Con}(\mathfrak{X}) \cong \mathrm{Con}(\mathfrak{Y})$. Moreover, when $f \in \mathfrak{S}_{\mathrm{adm}}$, ${}^{\mathrm{c}}\mathscr{H}^i f_! = 0$ for $i \neq 0$, and ${}^{\mathrm{c}}\mathscr{H}^0 f_!$ can be taken as a quasi-inverse to f^+.*

Proof. Since the structural group is flat, there exists a smooth surjective morphism from an algebraic space $P\colon \mathcal{Y}_0 \to \mathfrak{Y}$ such that $f_0\colon \mathfrak{X}_0 := \mathfrak{X} \times_{\mathfrak{Y}} \mathcal{Y}_0 \to \mathcal{Y}_0$ is a neutral gerb by Lemma A.2.1. Let $\mathcal{Y}_1 := \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0$, $\mathcal{Y}_2 := \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0 \times_{\mathfrak{Y}} \mathcal{Y}_0$, and let $f_i\colon \mathfrak{X}_i := \mathcal{Y}_i \times_{\mathfrak{Y}} \mathfrak{X} \to \mathcal{Y}_i$ be the projection. We have the following diagram:

$$\begin{tikzcd} \mathrm{Con}(\mathfrak{Y}) \arrow[r, "P^+"] \arrow[d, "f^+"'] & \mathrm{Con}(\mathcal{Y}_0) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_0^+"] & \mathrm{Con}(\mathcal{Y}_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_1^+"] & \mathrm{Con}(\mathcal{Y}_2) \arrow[d, "f_2^+"] \\ \mathrm{Con}(\mathfrak{X}) \arrow[r, "P^+"] & \mathrm{Con}(\mathfrak{X}_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Con}(\mathfrak{X}_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{Con}(\mathfrak{X}_2). \end{tikzcd}$$

By the assumption on the structural group, f_0^+, f_1^+, f_2^+ are equivalence of categories by Lemma 2.4.6. Since P is a presentation, we may use Lemma 2.4.3 to conclude. $\square$

2.4.7. **Lemma.** *Let $\mathcal{X}$ be an algebraic space of finite type over k such that $\mathcal{X}_{\mathrm{red}}$ is smooth, and let G be a smooth fiberwise connected algebraic group over $\mathcal{X}$. Then the pullback by the structural morphism induces $\mathrm{Sm}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Sm}(BG)$.*

Proof. We may replace $\mathcal{X}$ by $\mathcal{X}_{\mathrm{red}}$ since the derived categories do not change, and we may assume that $\mathcal{X}$ is smooth. The canonical morphism $\mathcal{X} \to BG$ is a smooth presentation and $\mathcal{X} \times_{BG} \mathcal{X} \cong G$ such that the i-th projection $p_i\colon \mathcal{X} \times_{BG} \mathcal{X} \to BG$ is the structural morphism $p\colon G \to \mathcal{X}$ by [LM, 4.6.1]. By Lemma 2.4.3, taking an object of $\mathrm{Sm}(BG)$ is equivalent to taking $\mathcal{E} \in \mathrm{Sm}(\mathcal{X})$ endowed with an isomorphism $\alpha\colon p^+\mathcal{E} \xrightarrow{\sim} p^+\mathcal{E}$ that satisfies the cocycle condition. Let

$$\mathcal{K} := \mathrm{Ker}\bigl(\alpha - \mathrm{id}\colon p^+\mathcal{E} \to p^+\mathcal{E}\bigr).$$

Since $p^+\mathcal{E} \in \mathrm{Hol}(G)$ is smooth, $\mathcal{K}$ is smooth as well. Let $e\colon \mathcal{X} \to G$ be the unit morphism. Since $p^+\mathcal{E}$ is smooth, we get the exact sequence

$$0 \to e^+(\mathcal{K}) \to e^+p^+\mathcal{E} \xrightarrow{e^+(\alpha-\mathrm{id})} e^+p^+\mathcal{E}.$$

By the cocycle condition, $e^+(\alpha - \mathrm{id})$ is 0, and thus the rank of $\mathcal{K}$ is equal to that of $p^+\mathcal{E}$ since G is connected. Thus α is the identity, and we get the lemma. $\square$

Remark. The assumption that $\mathcal{X}_{\mathrm{red}}$ is smooth is made only for the simplicity. In fact, with a little more argument on *dévissage*, the lemma remains true even if we replace Sm by Con, but we do not need this much.

2.4.8. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a diagonally connected gerb-like morphism* (*cf.* [Beh, 5.1.3]). *Shrinking $\mathfrak{Y}$ by its open dense substack if necessary, the functor $f^+\colon \mathrm{Sm}(\mathfrak{Y}) \to \mathrm{Sm}(\mathfrak{X})$ induces an equivalence.*

Proof. Take a presentation $P\colon Y \to \mathfrak{Y}$ from a scheme. There exists an open dense subscheme $U \subset Y$ such that U_{red} is smooth. By replacing $\mathfrak{Y}$ by $P(U) \subset \mathfrak{Y}$, we may assume that Y_{red} is smooth. Now, by using smooth descent, we may assume that $\mathfrak{Y} =: Y$ is a scheme and $\mathfrak{X} = BG$ with a connected flat algebraic group space G over Y. Since the category is stable under universal homeomorphism, we may replace Y by Y_{red} and assume that BG and Y are smooth.

When G is smooth, the lemma follows from Lemma 2.4.7. In the general case, we use the argument of [Beh, 5.1.17]. Take the relative Frobenius $G \twoheadrightarrow G' \hookrightarrow G^{(p)}$. Then $\mathrm{Ker}(G \to G')$ is of height ≤ 1, so this is flat finite radicial by definition (cf. [SGA3, $\mathrm{VII_A}$, 4.1.3]). By Corollary 2.4.6, $\mathrm{Sm}(BG') \xrightarrow{\sim} \mathrm{Sm}(BG)$, so we may replace G by G'. Repeating this, we come down to the case where G is smooth over a dense open subscheme of Y by [SGA3, $\mathrm{VII_A}$, 8.3]. Thus, by shrinking Y, we are reduced to the case where G is smooth. $\square$

A smoothness criterion.

2.4.9. In the proof of the Langlands correspondence, we need smoothness of certain holonomic modules. For this, we need to use the functors of Beilinson. Let k' be a finite extension of k, and put $\mathbb{A}^1_{k'}(= \mathbb{A}^1) := \mathrm{Spec}(k'[x])$. Let $f\colon \mathfrak{X} \to \mathbb{A}^1_{k'}$ be a morphism from a c-admissible stack. We put $i_f\colon \mathfrak{Z}_f := f^{-1}(0) \hookrightarrow \mathfrak{X}$ and $j_f\colon \mathfrak{U}_f := \mathfrak{X} \setminus \mathfrak{Z}_f \hookrightarrow \mathfrak{X}$. Then for any $\mathscr{M} \in \mathrm{Hol}(\mathfrak{U}_f)$ and integers $a \leq b$, holonomic modules $\Pi^{a,b}_{!+}(\mathscr{M}), \Phi^{\mathrm{un}}_f(\mathscr{M}) \in \mathrm{Hol}(\mathfrak{Z}_f)$ are defined in [AC2, §2] using a technique of Beilinson. To clarify f, we denote it by $\Pi^{a,b}_f(\mathscr{M})$. Put $\Pi^{0,0}_f := \Psi^{\mathrm{un}}_f$, the *unipotent nearby cycle functor.*

Explicitly, we can compute using the notation of [AC2, 2.5] that

(2.4.9.1)
$$\Psi^{\mathrm{un}}_f(\mathscr{M}) \cong \varinjlim_s \mathrm{Ker}\big(j_{f!}(\mathscr{M}^{-s,0}) \to j_{f+}(\mathscr{M}^{-s,0})\big) \cong \varinjlim_s \mathscr{H}^{-1} i^+_f j_{f+}(\mathscr{M}^{-s,0}).$$

2.4.10. Let us recall the local theory very briefly (see [AM, 2.1] for a more detailed review and references of the theory). Let $\mathbf{l}$ be a complete discrete valuation field over k. Then the Robba ring (with coefficients in $K \otimes_{W(k)} W(\mathrm{res}(\mathbf{l}))$) denoted by $\mathcal{R}_{\mathbf{l}}$ is defined. When $\mathbf{l} = k(\!(x)\!)$, then

$$\mathcal{R}_{\mathbf{l}} = \left\{ f = \sum_{n\in\mathbb{Z}} a_n x^n \in K[\![x, x^{-1}]\!] \;\middle|\; \begin{array}{l} \text{there exists } 0 \leq \varepsilon < 1 \text{ such that } f \\ \text{converges on } \varepsilon < |x| < 1 \end{array} \right\},$$

where $K[\![x, x^{-1}]\!]$ is the K*-vector space* of formal series. The Robba ring is endowed with derivation. A differential module over $\mathbf{l}$ is a finite free $\mathcal{R}_{\mathbf{l}}$-module endowed with connection. We have the notion of a *solvable differential module* which is an analogue of an overconvergent isocrystal for differential modules, whose category is denoted by $\mathrm{Sol}(\mathcal{R}_{\mathbf{l}})$. We have the s-th Frobenius endomorphism of $\mathcal{R}_{\mathbf{l}}$, and the pullback defines a functor $F^*\colon \mathrm{Sol}(\mathcal{R}_{\mathbf{l}}) \to \mathrm{Sol}(\mathcal{R}_{\mathbf{l}})$, which is known to be an equivalence of categories. Thus, we may apply the construction of §1.4. The category F-$\mathrm{Sol}(\mathcal{R}_{\mathbf{l}})_L$ is denoted by $\mathrm{Hol}(\mathbf{l}/\mathfrak{T}_F)$. As in §1.1, we denote by $\mathrm{Hol}(\mathbf{l}/\mathfrak{T}_\emptyset)$ the thick full subcategory of the category of differential modules over $\mathbf{l}$ generated by differential modules which can be endowed with s'-th Frobenius structure for some positive integer s' divisible by s (but we do not consider Frobenius structure). Since we only use $\mathrm{Hol}(\mathbf{l}/\mathfrak{T}_\emptyset)$ in the following, we denote this simply by $\mathrm{Hol}(\mathbf{l})$. We call the objects of $\mathrm{Hol}(\mathbf{l})$ *holonomic modules on* $\mathbf{l}$. For a separable finite extension $\mathbf{l}'/\mathbf{l}$, we are able to define the *push-forward functor* $\mathrm{Hol}(\mathbf{l}') \to \mathrm{Hol}(\mathbf{l})$ (cf. [AM, at the end of 2.1.4]).

2.4.11. For a Galois extension $\mathbf{l}/k'(\!(x)\!)$, we define $\Psi_{\mathbf{l},f}(\mathscr{M})$ as follows: Let us denote by $\mathcal{L}_{\mathbf{l}}$ the holonomic module on $k'(\!(x)\!)$ defined by taking the pushforward of the trivial module on $\mathbf{l}$ along the extension $\mathbf{l}/k'(\!(x)\!)$. Let $\mathscr{L}_{\mathbf{l}}$ be the canonical extension of $\mathcal{L}_{\mathbf{l}}$ on $\mathbb{G}_{\mathrm{m},k'}$ in the sense of Crew and Matsuda (cf. [AM, 2.1.9]). Then put $\Psi_{\mathbf{l},f}(\mathscr{M}) := \Psi^{\mathrm{un}}_f(\mathscr{M} \otimes f^+\mathscr{L}_{\mathbf{l}})$. We remark that $\mathscr{M} \otimes f^+\mathscr{L}_{\mathbf{l}}$, which is defined a priori in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, is in $\mathrm{Hol}(\mathfrak{X})$. Indeed, it suffices to check this when $\mathfrak{X} =: X$ is a realizable scheme. In this case, we may take a closed immersion $i\colon X \hookrightarrow P$ to a smooth scheme. By shrinking X, we may assume that there exists $g\colon P \to \mathbb{A}^1$ such that $g \circ i = f$. Now, by the projection formula $i_+(\mathscr{M} \otimes f^+\mathscr{L}_{\mathbf{l}}) \cong i_+(\mathscr{M}) \otimes g^+\mathscr{L}_{\mathbf{l}}$, and the latter is in $\mathrm{Hol}(P)$. Since we have the action of $\mathrm{Gal}(\mathbf{l}/k'(\!(x)\!))$ on $\mathscr{L}_{\mathbf{l}}$, it induces the *Galois action* on $\Psi_{\mathbf{l},f}$.

Lemma. *Let $f\colon C \to \mathbb{A}^1_{k'}$ be an étale morphism from a curve such that $f^{-1}(0) = \{s\}$ and $k(s) = k'$. Let $\mathscr{M} \in \mathrm{Hol}(C)$. Assume that $\mathscr{M}$ is smooth outside of s. If $\Phi^{\mathrm{un}}_f(\mathscr{M}) = 0$ and the actions of $\mathrm{Gal}(\mathbf{l}/k'(\!(x)\!))$ and the monodromy operator on $\Psi_{\mathbf{l},f}(\mathscr{M})$ are trivial for any $\mathbf{l}$, then $\mathscr{M}$ is smooth.*

Proof. Since $\Phi^{\mathrm{un}}_f(\mathscr{M}) = 0$, we get that $i^+_f(\mathscr{M})[-1] \cong \Psi^{\mathrm{un}}_f(\mathscr{M})$. Since the rank of $i^!_f(\mathscr{M})$ and $i^+_f(\mathscr{M})$ are the same, it suffices to show that the rank of $\Psi^{\mathrm{un}}_f(\mathscr{M})$ is equal to that of $\mathscr{M}$ by [AM, 4.1.4]. By (2.4.9.1) and [AC1, 1.5.9 (iii)], $\Psi_{\mathbf{l},f}$ depends only on the differential module on the Robba ring around s defined by restricting $\mathscr{M}$, and we can compute $\Psi_{\mathbf{l},f}(\mathscr{M})$ by using the local monodromy theorem. Since the argument is standard, we leave the details to the reader. □

2.4.12. **Lemma.** *Let $f\colon \mathfrak{X} \xrightarrow{h} \mathfrak{Y} \xrightarrow{g} \mathbb{A}^1$ be morphisms between c-admissible stacks. Assume that h is proper. Then we have*

$$\Pi^{a,b}_g(\mathscr{H}^i h_+\mathscr{M}) \cong \mathscr{H}^i h_+\Pi^{a,b}_f(\mathscr{M}).$$

The same isomorphism holds if we replace $\Pi^{a,b}_\star$ by $\Psi_{\mathbf{l},\star}$ or $\Phi^{\mathrm{un}}_\star$.

Proof. Since h is proper, we have $h_+j_{f\star} \cong j_{g\star}h_+$ for $\star \in \{!,+\}$. Thus, by the projection formula, we have $\varinjlim\!\!\!\!\varprojlim\, j_{g\star}(\mathscr{H}^i h_+\mathscr{M})^{\bullet,\bullet} \cong \varinjlim\!\!\!\!\varprojlim\, \mathscr{H}^i h_+ j_{f\star}(\mathscr{M}^{\bullet,\bullet})$ where $\star \in \{!,+\}$. Thus, by construction, we get the commutativity for $\Pi^{a,b}$. Now, let us define $K_f(\mathscr{M}) := \mathrm{Ker}\bigl(\Xi_f(\mathscr{M}) \oplus \mathscr{M} \twoheadrightarrow j_{f+}(\mathscr{M})\bigr)$, where $\Xi_f := \Pi^{0,1}_f$. Since Ξ_f and j_{f+} commute with $\mathscr{H}^i h_+$, we get that $K_g(\mathscr{H}^i h_+(\mathscr{M})) \cong \mathscr{H}^i h_+K_f(\mathscr{M})$. Similarly,

$I_f(\mathscr{M}) := \mathrm{Im}\big(j_{f!}(\mathscr{M}) \to \Xi_f(\mathscr{M}) \oplus \mathscr{M}\big)$ ($\cong j_{f!}(\mathscr{M})$) commutes with $\mathscr{H}^i h_+$ as well, and the lemma for $\Phi^{\mathrm{un}} := K_f/I_f$ follows by definition. $\square$

2.4.13. **Lemma** ([La2, A.9 (i)]). *Let $p_{\mathfrak{X}} \colon \mathfrak{X} \to S$ be a proper morphism from a c-admissible stack to a smooth scheme, and let $\mathrm{Res} \colon \mathfrak{X} \to \mathfrak{C}$ be a morphism to an algebraic stack locally of finite type over k. Assume that $(p_{\mathfrak{X}}, \mathrm{Res}) \colon \mathfrak{X} \to S \times \mathfrak{C}$ is smooth. Then for any $\mathscr{M} \in \mathrm{Con}(\mathfrak{C})$, the complex $p_{\mathfrak{X}+}\mathrm{Res}^+(\mathscr{M})$ is smooth.*

Proof. We put $\mathscr{H}^i_{\mathfrak{X}} := \mathscr{H}^i p_{\mathfrak{X}+}\mathrm{Res}^+(\mathscr{M})$. Assume we are given a smooth morphism $f \colon S \to \mathbb{A}^1$. Put $g := f \circ p_{\mathfrak{X}}$. Since f and $(p_{\mathfrak{X}}, \mathrm{Res})$ are smooth, we get that $\Phi^{\mathrm{un}}_g(\mathrm{Res}^+(\mathscr{M})) = 0$ and the Galois and monodromy action on $\Psi_{\mathbf{1},g}(\mathrm{Res}^+(\mathscr{M}))$ are trivial. Now, since $p_{\mathfrak{X}}$ is assumed proper, $\Phi^{\mathrm{un}}_f(\mathscr{H}^i_{\mathfrak{X}}) = 0$ and the Galois and monodromy action on $\Psi_{\mathbf{1},f}(\mathscr{H}^i_{\mathfrak{X}})$ are trivial by Lemma 2.4.12. This, in particular, implies that $j_{f!+}(\mathscr{H}^i_{\mathfrak{X}}) \cong \mathscr{H}^i_{\mathfrak{X}}$. Moreover, when S is a curve, the lemma holds. Indeed, take an open subscheme $U \subset S$ such that $\mathscr{H}^i_{\mathfrak{X}}$ is smooth on U. We may replace S by $S \otimes_k k'$, and we may assume that $S \setminus U$ is k'-rational. Since the verification is local, we may assume that we are given an étale morphism $f \colon S \to \mathbb{A}^1_{k'}$ such that $f^{-1}(0) = S \setminus U$ consists of one point, and we then apply Lemma 2.4.11.

Let us treat the general case. By *dévissage*, we may assume that $\mathscr{M}$ has a Frobenius structure. Let $c \colon C \hookrightarrow S$ be an immersion from a smooth curve C. By base change and purity, we have $c^!(\mathscr{H}^i_{\mathfrak{X}}) \cong \mathscr{H}^i_{\mathfrak{X} \times_S C}[-r]$ where r is the codimension of C in S. By the curve case we have already treated, this is smooth. Let $U \subset S$ be an open dense subscheme on which $p_{\mathfrak{X}+}\mathrm{Res}^+(\mathscr{M})$ is smooth. By Shiho's cut by curve theorem [S, Thm 0.1][4] and [Ke4, 5.2.1], the smooth module $\mathscr{H}^i_{\mathfrak{X}}|_U$ on U can be extended to a smooth module on S. Since we showed that $j_{f!+}(\mathscr{H}^i_{\mathfrak{X}}) \cong \mathscr{H}^i_{\mathfrak{X}}$ for any f, we get the lemma. $\square$

Isocrystals and their Tannakian fundamental group.

2.4.14. Let us introduce $\overline{\mathbb{Q}}_p$-coefficient cohomology theory. From now on, by saying "a base tuple", we also allow L to be an algebraic extension of K which may *not* be finite, contrary to the definition in 1.4.10. In the definition of an arithmetic tuple, $\sigma \colon L \to L$ should moreover satisfy the following:

> the automorphism σ is an extension of a lifting of s-th Frobenius automorphism on k to K, and there exists a sequence of finite extensions M_n of K in L such that $\sigma(M_n) \subset M_n$ and $\bigcup_n M_n = L$.

We use the 2-inductive limit method of Deligne [De1, 1.1.3] to construct the L-theory. For an algebraic stack $\mathfrak{X}$ (resp. scheme X) of finite type over k, we define

$$D^{\mathrm{b}}(\mathfrak{X}/L_\blacktriangle) := 2\text{-}\varinjlim_{M \supset K} D^{\mathrm{b}}(\mathfrak{X}/M_\blacktriangle), \qquad \mathrm{Sm}(\mathfrak{X}/L_\blacktriangle) := 2\text{-}\varinjlim_{M \supset K} \mathrm{Sm}(\mathfrak{X}/M_\blacktriangle),$$
$$\mathrm{Isoc}^\dagger(X/L_\blacktriangle) := 2\text{-}\varinjlim_{M \supset K} \mathrm{Isoc}^\dagger(X/M_\blacktriangle),$$

where $M = M_n$ in the case of $\blacktriangle = F$. By taking limits, the results we get in this paper can be generalized automatically to these categories, since the cohomological

[4] In [S], there is an assumption that k is uncountable. However, Shiho pointed out to the author that this assumption is not needed if $\mathcal{E}$ in [S, Thm 0.1] is endowed with Frobenius structure. Indeed, let us use the notation of the proof of the theorem in [S, §2.3]. By using [S, Thm 2.5] instead of [S, Thm 2.10], the slope of $E_{\mathcal{E},L}$ is 0. Since we have a Frobenius structure, the exponents are in $\mathbb{Q}$, and we can use [S, Prop 1.20] to conclude.

operators we have defined so far commute with ι_L by 1.4.5. Further details are left to the reader. Let f be the structural morphism of $\mathfrak{X}$. We denote $f^+(L)$ by L_X as usual.

2.4.15. We have often used the category $\mathrm{Sm}(X/L)$, but the category of overconvergent isocrystals $\mathrm{Isoc}^\dagger(X/L)$ is more standard in the literature. Let us clarify the relation between these categories. Consider the situation of 2.0.1, or 2.4.14 if L/K is not finite. Let X be a smooth scheme separated of finite type of dimension d over k. Recall the functor sp_+ in 1.1.3 (12). By extending the scalar and gluing, we have the following functor

$$\widetilde{\mathrm{sp}}_+ := \mathrm{sp}_+(-d)[-d] \colon \mathrm{Isoc}^\dagger(X/L) \xrightarrow{\sim} \mathrm{Sm}(X/L) \subset D^{\mathrm{b}}_{\mathrm{hol}}(X/L).$$

For a morphism $f \colon X \to Y$ between smooth schemes separated of finite type, let $d := \dim(X) - \dim(Y)$. Then, we have a canonical equivalence $\mathrm{sp}_+ \circ f^* \cong (f^![-d]) \circ \mathrm{sp}_+$ compatible with the composition of morphisms between smooth schemes by [Ca5, 6.1.9]. Thus, by Theorem 1.5.13 and Theorem 1.5.14, we have a canonical equivalence $\widetilde{\mathrm{sp}}_+ \circ f^* \cong f^+ \circ \widetilde{\mathrm{sp}}_+$. Via this equivalence, we identify f^* and f^+. We also know that $\widetilde{\mathrm{sp}}_+$ commutes with tensor products by [Ca6, 3.3.5]. By taking the adjoint, the commutation of $\mathcal{H}om$ follows as well. Finally, let $f \colon X \to \mathrm{Spec}(k)$ be the structural morphism of a *smooth realizable* scheme, and let $M \in \mathrm{Isoc}^\dagger(X/L_\emptyset)$ (resp. $M \in \mathrm{Isoc}^\dagger(X/L_F)$). When X is a scheme which has a compactification $\overline{X}$ such that $\overline{X}$ possesses a smooth lifting over R and that $\overline{X} \setminus X$ is a divisor, then by [A1, 5.9] we have canonical isomorphisms

$$H^*_{\mathrm{rig}}(X, M) \cong \mathscr{H}^* f_+\bigl(\widetilde{\mathrm{sp}}_+(M)\bigr), \qquad H^*_{\mathrm{rig,c}}(X, M) \cong \mathscr{H}^* f_!\bigl(\widetilde{\mathrm{sp}}_+(M)\bigr)$$

as objects in Vec_L (resp. $F\text{-}\mathrm{Vec}_L$). Here, H_{rig} and $H_{\mathrm{rig,c}}$ denotes the rigid cohomology extended to L-coefficients in the obvious manner.

2.4.16. **Question.** Unify the rigid cohomology theory into the framework of arithmetic $\mathscr{D}$-modules. Namely, let X be a separated scheme. Define the category of smooth objects $\mathrm{Sm}(X/L)$ in $\mathrm{Con}(X/L)$, and establish an equivalence of categories $\mathrm{Isoc}^\dagger(X/L) \to \mathrm{Sm}(X/L)$. This equivalence should coincide with the one in 2.4.15 when X is smooth. Finally, compare the rigid cohomology and the pushforward in the sense of $\mathscr{D}$-modules in the style of 2.4.15.

2.4.17. In this paragraph, we fix an algebraic closure $\overline{K}$ of K. We denote by $\overline{k}$ the residue field of $\overline{K}$, which is an algebraic closure of k. Now, let X be a smooth scheme of finite type over k, and assume it to be *geometrically connected*. Take a geometric point $\overline{x} \in X(\overline{k})$. Let x be the closed point of X defined by $\overline{x}$, and denote by $i_x \colon x \hookrightarrow X$ the closed immersion. We denote by K_x be the unramified extension of K induced by the finite extension $k(x)$ of k. Then we have the fiber functor

$$\omega_x \colon \mathrm{Isoc}^\dagger(X/K) \xrightarrow{i_x^+} \mathrm{Isoc}^\dagger(k(x)/K) \cong \mathrm{Vec}_{K_x}.$$

Let L be a finite extension of K_x. Since $\mathrm{End}(K_X) \cong K$, by [DM, 3.10.1], ω_x induces the fiber functor

$$\omega_{x/L} \colon \mathrm{Isoc}^\dagger(X/L) \to \mathrm{Vec}_L$$

by sending $\mathcal{E}$ to $i_x^+(\mathcal{E}) \otimes_{i_x^+ L_X} L$. This fiber functor $\omega_{x/L}$ is compatible with extension of scalar, and we may take the 2-inductive limit to define $\omega_{x/L}$ for any algebraic extension L of K_x. Now, the geometric point $\overline{x}$ determines the embedding $K_x \hookrightarrow \overline{K}$

with which we may regard $\overline{K}$ as an extension of K_x, thus, the fiber functor $\omega_{x/\overline{K}}$ makes sense. This fiber functor is denoted by $\omega_{\overline{x}}$.

Let $\pi_1^{\mathrm{isoc}}(X,\overline{x})$ be the *isocrystal fundamental group* $\mathrm{Aut}^{\otimes}(\omega_{\overline{x}})$, which is an affine group scheme over $\overline{K}$. For an algebraic group G over $\overline{K}$, denote by $\mathrm{Rep}_{\overline{K}}(G)$ the category of a finite-dimensional representation of G. By [DM, 3.11] and taking the 2-inductive limit, we have the following equivalence of tensor categories,

$$\mathrm{Isoc}^{\dagger}(X/\overline{K}) \xrightarrow{\sim} \mathrm{Rep}_{\overline{K}}(\pi_1^{\mathrm{isoc}}(X,\overline{x})).$$

Remark. If $X \to \mathrm{Spec}(k)$ is not geometrically connected, then $\mathrm{Isoc}^{\dagger}(X/\overline{K})$ is *not* a Tannakian category over $\overline{K}$. Indeed, $\overline{K}_X$ is an unit object of the tensor category $\mathrm{Isoc}^{\dagger}(X/\overline{K})$ but we have $\mathrm{End}(\overline{K}_X) \cong \overline{K}^{\times c}$ where c is the number of connected components of $X \otimes_k \overline{k}$. Compare also with Lemma 1.4.11 (i).

2.4.18. From now until the end of this subsection, we consider the case where k is a finite field with $q = p^s$ elements. We fix an arithmetic base tuple $\mathfrak{T}_F := (k, R := W(k), K := \mathrm{Frac}(R), \overline{\mathbb{Q}}_p, s, \mathrm{id})$, where $\overline{\mathbb{Q}}_p$ is an algebraic closure of K. Let $\mathfrak{T}_{\emptyset}$ be the associated geometric base tuple. As in the last paragraph, $\overline{k}$ denotes the residue field of $\overline{\mathbb{Q}}_p$, which naturally contains k. To make the notation compatible with [La2], we denote the relative s-th Frobenius endomorphism on X by Frob_X instead of F_X. Let X be a geometrically connected smooth scheme of finite type over k. Take a geometric point $\overline{x} \in X(\overline{k})$, and $i_x \colon x \hookrightarrow X$ denotes the induced closed immersion. Let $\mathcal{E} \in \mathrm{Isoc}^{\dagger}(X/\overline{\mathbb{Q}}_{p,\emptyset})$. Since $K_x \cong W(k(x)) \otimes_{W(k)} K$, the s-th Frobenius automorphism on $W(k(x))$ induces the automorphism $\mathrm{Frob}_x^* \colon K_x \to K_x$. The fiber $i_x^+\mathcal{E}$ can be seen as an $i_x^+\overline{\mathbb{Q}}_{p,X}$-module, where the latter ring is isomorphic to $K_x \otimes_K \overline{\mathbb{Q}}_p$ since X is geometrically connected. Thus, we have isomorphisms

$$\omega_{\overline{x}}(\mathrm{Frob}_X^*\mathcal{E}) \cong \bigl(K_x \otimes_{\mathrm{Frob}_x^* \nwarrow K_x} i_x^+\mathcal{E}\bigr) \otimes_{K_x \otimes_K \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p \underset{\alpha}{\xleftarrow{\sim}} i_x^+\mathcal{E} \otimes_{K_x \otimes_K \overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p \cong \omega_{\overline{x}}(\mathcal{E}),$$

where the homomorphism α sends $e \otimes a$ to $1 \otimes e \otimes a$. Thus, we get the following 2-commutative diagram:

(2.4.18.1)
$$\begin{tikzcd} \mathrm{Isoc}^{\dagger}(X/\overline{\mathbb{Q}}_{p,\emptyset}) \arrow[d, "\mathrm{Frob}_X^*"'] \arrow[dr, "\omega_{\overline{x}}"] & \\ \mathrm{Isoc}^{\dagger}(X/\overline{\mathbb{Q}}_{p,\emptyset}) \arrow[r, "\omega_{\overline{x}}"'] & \mathrm{Vec}_{\overline{\mathbb{Q}}_p}. \end{tikzcd}$$

This diagram induces a homomorphism $\mathrm{Frob}_X^* \colon \pi_1^{\mathrm{isoc}}(X,\overline{x}) \to \pi_1^{\mathrm{isoc}}(X,\overline{x})$. This homomorphism is in fact an isomorphism, since Frob_X^* gives an equivalence of categories by Remark 1.1.3. We define $\rho \colon \mathbb{Z} \to \mathrm{Aut}(\pi_1^{\mathrm{isoc}}(X,\overline{x}))$ to be the homomorphism sending 1 to Frob_X^*. Using this homomorphism, we put $W^{\mathrm{isoc}}(X,\overline{x}) := \pi_1^{\mathrm{isoc}}(X,\overline{x}) \rtimes \mathbb{Z}$, and call it the *isocrystal Weil group of* X. By construction, we have the equivalence of tensor categories,

(2.4.18.2)
$$\mathrm{Isoc}^{\dagger}(X/\overline{\mathbb{Q}}_{p,F}) \xrightarrow{\sim} \mathrm{Rep}_{\overline{\mathbb{Q}}_p}(W^{\mathrm{isoc}}(X,\overline{x})),$$

induced by $\omega_{\overline{x}}$.

In general, let $X \to \mathrm{Spec}(k)$ be a smooth connected scheme of finite type, which may not be geometrically connected, and take a closed point x. The structural morphism factors as $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$, where k' is a finite field extension of k of degree d and X is geometrically connected over k'. Consider the base tuple $\mathfrak{T}'_F := (k', R' := W(k') \otimes_{W(k)} R, K', \overline{\mathbb{Q}}_p, ds, \mathrm{id})$. Then we define $W^{\mathrm{isoc}}(X,\overline{x})$ to be the isocrystal Weil group of X over $\mathfrak{T}'_F$. Despite the base tuple being changed, the equivalence (2.4.18.2) remains true by Corollary 1.4.11. We note that, by definition, $W^{\mathrm{isoc}}(X,\overline{x})$ does not depend on the choice of the base field k.

Assume that X is geometrically connected, and let k' be a Galois extension of k. Take a geometric point $\overline{x}'$ of $X\otimes_k k'$, and let $\overline{x}$ be the projection to X. Then we have the following exact sequence:

$$1\to W^{\mathrm{isoc}}(X\otimes_k k',\overline{x}')\to W^{\mathrm{isoc}}(X,\overline{x})\to \mathrm{Gal}(k'/k)\to 1.$$

Remark. Let X be a geometrically connected smooth scheme of finite type over k. Assume moreover that we have a k-rational point $i_x\colon \mathrm{Spec}(k)\to X$ for simplicity. Since $\mathrm{Isoc}^\dagger(X/\overline{\mathbb{Q}}_{p,F})$ is a neutral Tannakian category over $\overline{\mathbb{Q}}_p$, by using the fiber functor $\omega:=i_x^+$, we could have used $\mathrm{Aut}^\otimes(\omega)$ as the fundamental group. However, this algebraic group is complicated to handle, and we used the simpler substitute W^{isoc} following Crew [Cr].

2.4.19. Let X', X'' be smooth schemes of finite type and geometrically connected over k. Put $X:=X'\times X''$ which is geometrically connected over k as well. Let $U\subset X$ be an open subscheme such that $(\mathrm{Frob}_{X'}\times\mathrm{id}_{X''})(U)\subset U$ where $\mathrm{Frob}_{X'}\times\mathrm{id}_{X''}\colon X'\times X''\to X'\times X''$. Take a geometric point $\overline{x}\in U(\overline{k})$. Arguing as in 2.4.18, the pullback $(\mathrm{Frob}_{X'}\times\mathrm{id})^+$ induces an outer automorphism of $W^{\mathrm{isoc}}(U,\overline{x})$, and it yields a homomorphism $\mathbb{Z}\to\mathrm{Out}(W^{\mathrm{isoc}}(U,\overline{x}))$ sending 1 to $(\mathrm{Frob}_{X'}\times\mathrm{id})^+$. We put $\mathbb{Z}W^{\mathrm{isoc}}(U,\overline{x}):=W^{\mathrm{isoc}}(U,\overline{x})\rtimes\mathbb{Z}$. Representations of $\mathbb{Z}W^{\mathrm{isoc}}(U,\overline{x})$ correspond to pairs $(\mathcal{E},\alpha)$ where $\mathcal{E}\in\mathrm{Isoc}^\dagger(U/\overline{\mathbb{Q}}_{p,F})$ and $\alpha\colon(\mathrm{Frob}_{X'}\times\mathrm{id})^+(\mathcal{E})\cong\mathcal{E}$.

Lemma ([La2, VI.13]). *Take geometric points $\overline{x}'$ and $\overline{x}''$ of X' and X'', and put $\overline{x}:=(\overline{x}',\overline{x}'')$. Then the canonical homomorphism $\mathbb{Z}W^{\mathrm{isoc}}(X,\overline{x})\to W^{\mathrm{isoc}}(X',\overline{x}')\times W^{\mathrm{isoc}}(X'',\overline{x}'')$ is surjective (or more precisely, faithfully flat).*

Proof. Let x' and x'' be the closed points of X' and X'' induced by $\overline{x}'$ and $\overline{x}''$. Let k' be a Galois extension of k, and put $G:=\mathrm{Gal}(k'/k)$. Consider the following diagram, where we omit the basepoints of the Weil groups and W^{isoc} is abbreviated as W:

$$\begin{tikzcd}
1 \arrow[r] & W(X'\otimes k')\times W(X''\otimes k') \arrow[r]\arrow[d] & W(X')\times W(X'') \arrow[r]\arrow[d] & G\times G \arrow[r]\arrow[d, equal] & 1\\
1 \arrow[r] & \mathbb{Z}W(X\otimes k') \arrow[r] & \mathbb{Z}W(X) \arrow[r] & G\times G \arrow[r] & 1.
\end{tikzcd}$$

Thus, we may replace k by k' and may assume that x' and x'' are rational points of X' and X''. These rational points define morphisms $s'\colon X'\to X$, $s''\colon X''\to X$. Let us show that the canonical homomorphism $\alpha\colon\pi_1^{\mathrm{isoc}}(X,\overline{x})\to\pi_1^{\mathrm{isoc}}(X',\overline{x}')\times\pi_1^{\mathrm{isoc}}(X'',\overline{x}'')$ is surjective. To check this, it suffices to show that for any $\overline{K}$-algebra A, the homomorphism of groups $\alpha(A)\colon\pi_1^{\mathrm{isoc}}(X,\overline{x})(A)\to\pi_1^{\mathrm{isoc}}(X',\overline{x}')(A)\times\pi_1^{\mathrm{isoc}}(X'',\overline{x}'')(A)$ is surjective. Now, the morphism s' induces the homomorphism $s'_*\colon\pi_1^{\mathrm{isoc}}(X',\overline{x}')\to\pi_1^{\mathrm{isoc}}(X,\overline{x})$, and the image of $(\alpha\circ s'_*)(A)$ is $\pi_1^{\mathrm{isoc}}(X',\overline{x}')(A)\times\{1\}$. Using s'', the image of $\alpha(A)$ contains $\{1\}\times\pi_1^{\mathrm{isoc}}(X'',\overline{x}'')(A)$ as well, and the surjectivity of $\alpha(A)$ follows as required. Finally, by definition of the Weil groups, the lemma follows. $\square$

2.4.20. **Lemma.** *Let X be a smooth connected scheme, and let $U\subset X$ be an open subscheme. Take a geometric point $x\in U(\overline{k})$. Then the homomorphism $W^{\mathrm{isoc}}(U,\overline{x})\to W^{\mathrm{isoc}}(X,\overline{x})$ is surjective.*

Proof. It suffices to show that the homomorphism $\pi_1^{\mathrm{isoc}}(U,\overline{x}) \to \pi_1^{\mathrm{isoc}}(X,\overline{x})$ induced by the open immersion is surjective. Let $j\colon U \hookrightarrow X$ be the open immersion. By [DM, 2.21], this is equivalent to showing that j^+ is fully faithful and any subobject of $j^+\mathcal{E}$ for an overconvergent isocrystal $\mathcal{E}$ on X is in the image of j^+. The full faithfulness follows by purity (cf. Theorem 1.5.14). It remains to show that if $\mathcal{E}$ is an irreducible overconvergent isocrystal on X, then $\mathcal{E}|_U$ is irreducible. This follows by [AC1, 1.4.6]. □

3. Cycle classes, correspondences, and ℓ-independence

The aim of this section is to prove an ℓ-independence result. This is a key tool for computing the trace of the action of Hecke algebra on the cohomology of certain moduli spaces. In this section, we fix $\blacktriangle \in \{\emptyset, F\}$ and a base tuple as usual. The algebraic extension L/K can be infinite as in 2.4.14. For simplicity, smooth admissible stacks over k are assumed equidimensional.

3.1. **Generalized cycles and correspondences.**

3.1.1. Let $p\colon \mathfrak{X} \to \mathrm{Spec}(k)$ be the structural morphism of a c-admissible stack $\mathfrak{X}$ (cf. Definition 2.3.19). If no confusion may arise, we denote $L_{\mathfrak{X},\blacktriangle}$ by L. For $\mathscr{M} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, we put

$$\begin{aligned} H^i(\mathfrak{X},\mathscr{M}) &:= \mathrm{Hom}_{D(\mathrm{Spec}(k)/L_\blacktriangle)}\bigl(L, p_+(\mathscr{M})[i]\bigr),\\ H^i_{\mathrm{c}}(\mathfrak{X},\mathscr{M}) &:= \mathrm{Hom}_{D(\mathrm{Spec}(k)/L_\blacktriangle)}\bigl(L, p_!(\mathscr{M})[i]\bigr). \end{aligned}$$

Note that when $\blacktriangle = \emptyset$, we have $H^*(\mathfrak{X},\mathscr{M}) \cong \mathscr{H}^*p_+(\mathscr{M})$ and $H^*_{\mathrm{c}}(\mathfrak{X},\mathscr{M}) \cong \mathscr{H}^*p_!(\mathscr{M})$ as vector spaces over L. For a morphism $i\colon \mathfrak{Z} \to \mathfrak{X}$, we define the *local cohomology* to be

$$H^i_{\mathfrak{Z}}(\mathfrak{X},\mathscr{M}) := \mathrm{Hom}_{D(\mathrm{Spec}(k)/L_\blacktriangle)}\bigl(L, p_+i_+i^!(\mathscr{M})[i]\bigr).$$

Furthermore, we put $H^*_\heartsuit(\mathfrak{X}) := H^*_\heartsuit(\mathfrak{X}, L_{\mathfrak{X}})$ where $\heartsuit \in \{\emptyset, \mathrm{c}, \mathfrak{Z}\}$.

Consider the following commutative diagram of c-admissible stacks:

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "f'"] \arrow[d, "i'"'] & \mathfrak{W} \arrow[d, "i"] \\ \mathfrak{X} \arrow[r, "f"'] & \mathfrak{Y}. \end{tikzcd}$$

If this diagram is cartesian, then we have the base change isomorphism $i^!f_+ \cong f'_+i'^!$ by (the dual of) Proposition 2.3.22, which induces the homomorphism $H^*_{\mathfrak{W}}(\mathfrak{Y}) \to H^*_{\mathfrak{Z}}(\mathfrak{X})$. By abuse of notation, we also denote this homomorphism by f^*. If the diagram is merely commutative, f is the identity, and f' is proper, the adjunction $f'_+f'^! \to \mathrm{id}$ induces the push-forward homomorphism $H^*_{\mathfrak{Z}}(\mathfrak{X}) \to H^*_{\mathfrak{W}}(\mathfrak{X})$.

Finally, given a *proper* morphism of c-admissible stacks $f\colon \mathfrak{X} \to \mathfrak{Y}$, we have a homomorphism $f^*\colon H^*_{\mathrm{c}}(\mathfrak{Y}) \to H^*_{\mathrm{c}}(\mathfrak{X})$ induced by the adjunction homomorphism.

3.1.2. First, let $\mathfrak{S}$ be a c-admissible stack. Let $\mathscr{M}$, $\mathscr{N}$ be objects in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{S}/L)$. Denote by $\Delta\colon \mathfrak{S} \to \mathfrak{S} \times \mathfrak{S}$ the diagonal morphism, and denote by p the structural morphism of $\mathfrak{S}$. By identifying $\mathrm{Spec}(k) \times \mathfrak{S}$ and $\mathfrak{S}$, we have a canonical homomorphism

$$p_+(\mathscr{M}) \boxtimes \mathscr{N} \cong (p \times \mathrm{id})_+\bigl(\mathscr{M} \boxtimes \mathscr{N}\bigr) \xrightarrow{\mathrm{adj}} (p \times \mathrm{id})_+\Delta_+\Delta^+\bigl(\mathscr{M} \boxtimes \mathscr{N}\bigr) \cong \mathscr{M} \otimes \mathscr{N},$$

where the first isomorphism is induced by Proposition 2.3.36, adj is the adjunction homomorphism, and the last isomorphism follows since $(p \times \mathrm{id}) \circ \Delta = \mathrm{id}$.

Now, let $f\colon \mathfrak{X} \to \mathfrak{S}$ be a morphism of c-admissible stacks. We take a factorization $\mathfrak{X} \xrightarrow{j} \overline{\mathfrak{X}} \xrightarrow{\overline{f}} \mathfrak{S}$ in $\mathfrak{S}_{\mathrm{adm}}$ such that j is an open immersion and $\overline{f}$ is proper. We have

$$f_+(\mathscr{M}) \otimes f_!(\mathscr{N}) \cong \overline{f}_+ j_+(\mathscr{M}) \otimes \overline{f}_+ j_!(\mathscr{N}) \to \overline{f}_+\bigl(j_+(\mathscr{M}) \otimes j_!(\mathscr{N})\bigr)$$
$$\xleftarrow{\sim} \overline{f}_+ j_!(\mathscr{M} \otimes \mathscr{N}) \cong f_!(\mathscr{M} \otimes \mathscr{N}),$$

where the second homomorphism is (2.3.16.1). This homomorphism does not depend on the choice of compactification.

Finally, let $i\colon \mathfrak{Z} \to \mathfrak{X}$ be a morphism of c-admissible stacks. Then we have the homomorphism $i_!\bigl(i^!\mathscr{M} \otimes i^+\mathscr{N}\bigr) \cong i_! i^! \mathscr{M} \otimes \mathscr{N} \to \mathscr{M} \otimes \mathscr{N}$. Taking the adjoint, we get a homomorphism

$$i^!\mathscr{M} \otimes i^+\mathscr{N} \to i^!(\mathscr{M} \otimes \mathscr{N}).$$

3.1.3. **Definition.** (i) Let $\mathfrak{X}$ be a c-admissible stack of dimension d. A *generalized cycle of codimension* c is a proper morphism $g\colon \Gamma \to \mathfrak{X}$ between c-admissible stacks such that $\dim(\mathfrak{X}) - \dim(\Gamma) = c$.

(ii) Let S be a scheme of finite type over k, let φ be a *proper* endomorphism of S, and let $f^{(\prime)}\colon \mathfrak{X}^{(\prime)} \to S$ be a c-admissible S-stack. Let

$$c_\Gamma\colon \Gamma \to \mathfrak{X} \times_{\varphi,S} \mathfrak{X}'$$

be a morphism between c-admissible stacks, where the fiber product is taken for $\varphi \circ f\colon \mathfrak{X} \to S$ and $f'\colon \mathfrak{X}' \to S$. For $i = 1, 2$, put $p_i := \pi_i \circ c_\Gamma$ where π_i denotes the i-th projection. The morphism c_Γ is said to be a *correspondence over* φ if Γ is equidimensional of dimension $\dim(\mathfrak{X})$ and p_2 is proper, or Γ is the empty stack. We sometimes denote the correspondence by $\Gamma\colon \mathfrak{X} \rightsquigarrow \mathfrak{X}'$. Note that c_Γ is a generalized cycle of codimension $\dim(\mathfrak{X}')$ of $\mathfrak{X} \times_{\varphi,S} \mathfrak{X}'$. From now on in this subsection, we fix S and φ as above, and we use them freely without referring to this paragraph.

3.1.4. In this section, we denote $\mathrm{Tr}^{\mathrm{sm}}_f$ in Theorem 2.3.34 simply by Tr_f. Let $\alpha\colon \mathfrak{X} \to \mathrm{Spec}(k)$ be the structural morphism of a c-admissible stack $\mathfrak{X}$. We often denote Tr_α by $\mathrm{Tr}_{\mathfrak{X}}$. Now, let $f\colon \mathfrak{Y} \to \mathfrak{X}$ be a morphism of c-admissible stacks, and *assume that* $\mathfrak{X}$ *is smooth.* We construct $\mathrm{fTr}_f\colon f_! f^+ L_{\mathfrak{X}}(d)[2d] \to L_{\mathfrak{X}}$ where $d := \dim(\mathfrak{Y}) - \dim(\mathfrak{X})$, which is called the *fake trace map of* f, as follows. Let $p\colon \mathfrak{Y} \to \mathrm{Spec}(k)$ be the structural morphism. We have the isomorphisms

$$\mathrm{Hom}(f_! f^+ L_{\mathfrak{X}}(d)[2d], L_{\mathfrak{X}}) \cong \mathrm{Hom}(f^+ L_{\mathfrak{X}}(d)[2d], f^! L_{\mathfrak{X}})$$
$$\cong \mathrm{Hom}(p^+ L(d_Y)[2d_Y], p^! L) \cong \mathrm{Hom}(p_! p^+ L(d_Y)[2d_Y], L),$$

where $d_Y := \dim(\mathfrak{Y})$, and we used the Poincaré duality (Theorem 2.3.34) for the second isomorphism. The trace map $\mathrm{Tr}_{\mathfrak{Y}}$ is an element on the right-hand side of the isomorphisms. We define fTr_f to be the homomorphism defined by sending this $\mathrm{Tr}_{\mathfrak{Y}}$ to the left-hand side of the isomorphisms. This homomorphism induces a homomorphism

$$f_*\colon H^{*+2d}_{\mathrm{c}}(\mathfrak{Y})(d) \to H^*_{\mathrm{c}}(\mathfrak{X}). \tag{3.1.4.1}$$

(i) Let $i\colon \mathfrak{Z} \to \mathfrak{X}$ be a generalized cycle of codimension c on a smooth c-admissible stack $\mathfrak{X}$. Then by taking the adjoint, fTr_i induces a homomorphism

$$c_{\mathfrak{Z}}\colon i^+ L_X \to i^! L_X(c)[2c].$$

(ii) Let us construct a similar homomorphism when we are given a correspondence. We use the notation of Definition 3.1.3 (ii). We assume further that $\mathfrak{X}$ is smooth. When Γ is nonempty, we have $\mathrm{fTr}_{p_1}\colon p_{1!}p_1^+L_{\mathfrak{X}} \to L_{\mathfrak{X}}$, where we used the assumption that $\dim(\Gamma)=\dim(\mathfrak{X})$. Thus, we have

$$\iota_\Gamma\colon p_2^+L_{\mathfrak{X}'} \cong p_1^+L_{\mathfrak{X}} \to p_1^! L_{\mathfrak{X}},$$

where we used the adjoint of fTr_{p_1} for the second homomorphism. When Γ is empty, we simply put $\iota_\Gamma := 0$.

3.1.5. Let us characterize the fake trace map in the style of [SGA4$\frac{1}{2}$, Cycle]. Let us consider the following diagram of c-admissible stacks:

$$\begin{tikzcd} \mathfrak{Y} \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathfrak{X} \arrow[dl, "g"] \\ & \mathfrak{S}, & \end{tikzcd}$$

where $\mathfrak{X}$ is smooth, and $\mathfrak{X}$ and $\mathfrak{Y}$ are of dimension N and d, respectively. Put $c := N-d$. Combining homomorphisms in 3.1.2, we have

$$p_{\mathfrak{Y}+}i^!L_{\mathfrak{X}} \boxtimes f_!i^+L_{\mathfrak{X}} \to f_+i^!L_{\mathfrak{X}} \otimes f_!i^+L_{\mathfrak{X}} \to f_!\big(i^!L_{\mathfrak{X}} \otimes i^+L_{\mathfrak{X}}\big) \to f_!i^!L_{\mathfrak{X}} \xrightarrow{\mathrm{adj}_i} g_!L_{\mathfrak{X}},$$

where $p_{\mathfrak{Y}}$ is the structural morphism of $\mathfrak{Y}$, and $\mathrm{adj}_i\colon i_!i^! \to \mathrm{id}$ is the adjunction morphism. Since $\mathscr{H}^k p_{\mathfrak{Y}+}i^!L_{\mathfrak{X}} = 0$ for $k < 2c$ (cf. Lemma 2.3.33), this yields a coupling called the *cup product*

$$\cup\colon H^{2c}_{\mathfrak{Y}}(\mathfrak{X})(c) \otimes f_!L_{\mathfrak{Y}}(d)[2d] \to g_!L_{\mathfrak{X}}(N)[2N].$$

Now, by taking the adjoint, we may regard fTr_i as an element in $H^{2c}_{\mathfrak{Y}}(\mathfrak{X})(c)$. We put $\mathfrak{S} = \mathrm{Spec}(k)$, and we have the following characterization of the fake trace map:

Lemma. *The class* $\mathrm{fTr}_i \in H^{2c}_{\mathfrak{Y}}(\mathfrak{X})(c)$ *is the unique element such that for any* $u \in H^{2d}_{\mathrm{c}}(\mathfrak{Y})(d)$,

$$\mathrm{Tr}_f(u) = \mathrm{Tr}_g\big(\mathrm{fTr}_i \cup u\big).$$

Proof. As in [SGA4$\frac{1}{2}$, Cycle, 2.3], we have the following commutative diagram:

$$\begin{tikzcd} H^{2c}_{\mathfrak{Y}}(\mathfrak{X}) \arrow[r] \arrow[dr, "②"'] & \mathrm{Hom}\big(f_!L_{\mathfrak{Y}}(d)[2d], f_!i^!L_{\mathfrak{X}}(N)[2N]\big) \arrow[r, "①"] \arrow[d, "\mathrm{adj}_i"] & \mathrm{Hom}\big(f_!L_{\mathfrak{Y}}(d)[2d], L_{\mathfrak{S}}\big) \arrow[d, equal] \\ & \mathrm{Hom}\big(g_!i_!L_{\mathfrak{Y}}(d)[2d], g_!L_{\mathfrak{X}}(N)[2N]\big) \arrow[r, "\mathrm{Tr}_g"'] & \mathrm{Hom}\big(g_!i_!L_{\mathfrak{Y}}(d)[2d], L_{\mathfrak{S}}\big). \end{tikzcd}$$

Here, ① is the homomorphism induced by the adjunction using the assumption that $\mathfrak{X}$ is smooth. The homomorphism $\mathrm{adj}_i\colon i_!i^! \to \mathrm{id}$ is the adjunction. The homomorphism ② is induced by $\cup$ defined above. By the definition of fake trace map, the upper horizontal homomorphism maps fTr_i to Tr_f. Moreover, the composition of the upper horizontal maps is an isomorphism. Thus we can conclude the proof. □

Remark. The assumption that $\mathfrak{S} = \mathrm{Spec}(k)$ is used only for the existence of the fake trace over $\mathfrak{S}$. It is not hard to generalize the definition of the fake trace to relative situations as [SGA4$\frac{1}{2}$], and we may prove the lemma in this generality, although we are not sure if it is meaningful for our purposes.

Corollary. *Assume that $f\colon \mathfrak{X} \to \mathfrak{Y}$ is a flat morphism of c-admissible stacks such that $\mathfrak{Y}$ is smooth. Then $\mathrm{Tr}_f = \mathrm{fTr}_f$.*

Proof. This follows readily from the characterization lemma of fTr_f above. □

3.1.6. (i) Consider the situation as in 3.1.4 (i). We have an isomorphism

$$\mathrm{Hom}(i^+ L_{\mathfrak{X}}, i^! L_{\mathfrak{X}}(c)[2c]) \cong \mathrm{Hom}(L_{\mathfrak{X}}, i_+ i^! L_{\mathfrak{X}}(c)[2c]) =: H^{2c}_{\mathfrak{Z}}(\mathfrak{X})(c).$$

The image of $c_{\mathfrak{Z}}$ is denoted by $\mathrm{cl}_{\mathfrak{X}}(\mathfrak{Z})$, and called the *cycle class of* $\mathfrak{Z}$. Since the homomorphism $\mathfrak{Z} \to \mathfrak{X}$ is proper, we have the homomorphism $H^*_{\mathfrak{Z}}(\mathfrak{X}) \to H^*(\mathfrak{X})$. The image of $\mathrm{cl}_{\mathfrak{X}}(\mathfrak{Z})$ in $H^{2c}(\mathfrak{X})(c)$ is also called the cycle class. Note that if the morphism $\mathfrak{Z} \to g(\mathfrak{Z})$ is not generically finite, then $H^{2c}_{g(\mathfrak{Z})}(\mathfrak{X}) = 0$, and in particular the cycle class in $H^{2c}(\mathfrak{X})(c)$ is 0.

(ii) Consider the situation as in 3.1.4 (ii). Recall that φ is assumed proper. We have the action of correspondence on the cohomology, which is the composition of the homomorphisms

$$\Gamma^*\colon f'_! L_{\mathfrak{X}'} \to f'_! p_{2+} p_2^+ L_{\mathfrak{X}'} \xleftarrow{\sim} (\varphi \circ f \circ p_1)_! \, p_2^+ L_{\mathfrak{X}'} \xrightarrow{\iota_\Gamma} \varphi_+ f_! p_{1!} p_1^! L_{\mathfrak{X}} \to \varphi_+ f_! L_{\mathfrak{X}}.$$

When S is a point, this is nothing but the following composition using (3.1.4.1):

$$H^*_{\mathrm{c}}(\mathfrak{X}') \xrightarrow{p_2^*} H^*_{\mathrm{c}}(\Gamma) \xrightarrow{p_{1*}} H^*_{\mathrm{c}}(\mathfrak{X}).$$

3.1.7. **Lemma.** *Consider the following cartesian diagram of c-admissible stacks on the left:*

$$\begin{tikzcd} \mathfrak{Y}' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathfrak{Y} \arrow[d, "f"] \\ \mathfrak{X}' \arrow[r, "g"'] & \mathfrak{X}, \end{tikzcd} \qquad \begin{tikzcd} g'^+ f^+ L_{\mathfrak{X}}(d)[2d] \arrow[r, "g'^+ \mathrm{fTr}_f"] \arrow[d, "\sim"'] & g'^+ f^! L_{\mathfrak{X}} \arrow[d] \\ f'^+ L_{\mathfrak{X}'}(d)[2d] \arrow[r, "\mathrm{fTr}_{f'}"'] & f'^! L_{\mathfrak{X}'}, \end{tikzcd}$$

where $\mathfrak{X}$ and $\mathfrak{X}'$ are smooth. Assume moreover that there exists an open substack $\mathfrak{V} \subset \mathfrak{Y}$ such that the morphism $\mathfrak{V} \to \mathfrak{X}$ is flat of relative dimension d, and $g'^{-1}(\mathfrak{V}) \subset \mathfrak{Y}'$ is dense. Then the above diagram on the right is commutative. In particular, if g is proper, we have an equality

$$g^* f_* = f'_* g'^* \colon H^*_{\mathrm{c}}(\mathfrak{Y}) \to H^{*-2d}_{\mathrm{c}}(\mathfrak{X}')(-d).$$

Proof. Since the commutativity of the diagram on the right can be interpreted as coincidence of two elements in $H^{2a}_{\mathrm{c}}(\mathfrak{Y}')(a)^\vee$, where a denotes the dimension of $\mathfrak{Y}'$, we may shrink $\mathfrak{Y}'$ by its open dense substack by Lemma 2.3.33. Thus we may replace $\mathfrak{Y}$ by $\mathfrak{V}$, and may assume that f is flat of relative dimension d. Now the lemma follows by Corollary 3.1.5 and the base change property of the trace map (cf. Theorem 2.3.34 (II)). □

Corollary (Projection formula)**.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of c-admissible stacks, and $\mathfrak{Y}$ is smooth. Assume that there exists an open dense substack $\mathfrak{U} \subset \mathfrak{X}$ such that the morphism $\mathfrak{U} \to \mathfrak{Y}$ is flat of relative dimension d. Then for $\alpha \in H^i_{\mathrm{c}}(\mathfrak{X})$ and $\beta \in H^j_{\mathrm{c}}(\mathfrak{Y})$, we have the following equality in $H^{i+j-2d}_{\mathrm{c}}(\mathfrak{Y})(-d)$:*

$$f_*\big(\alpha \cup f^* \beta\big) = f_* \alpha \cup \beta.$$

Proof. Consider the following commutative diagram of proper morphisms:

$$\begin{tikzcd}
\mathfrak{X} \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[r, "f"] \arrow[d, "\Delta'"'] \arrow[dr, phantom, "\square"] & \mathfrak{Y} \arrow[d, "\Delta"] \\
& \mathfrak{X} \times \mathfrak{X} \arrow[r] & \mathfrak{X} \times \mathfrak{Y} \arrow[r, "f \times \mathrm{id}"'] & \mathfrak{Y} \times \mathfrak{Y}.
\end{tikzcd}$$

By the hypothesis on f, we can apply the lemma above to the right cartesian diagram (take f, g in the lemma to be $\mathrm{id} \times f$ and Δ, respectively). We have

$$f_*\alpha \cup \beta = \Delta^*\big((f \times \mathrm{id})_*(\alpha \boxtimes \beta)\big) = f_*\Delta'^*(\alpha \boxtimes \beta),$$

where we used the lemma for the second equality. The diagram above and the transitivity of the pullback show that $\Delta'^*(\alpha \boxtimes \beta) = \alpha \cup f^*\beta$, and the corollary follows. $\square$

3.1.8. **Lemma.** *Let S be a scheme of finite type over k, let $\mathfrak{X}^{(\prime)}$ be a c-admissible stack, and let $f^{(\prime)} \colon \mathfrak{X}^{(\prime)} \to S$ be a morphism. Assume that $\varphi = \mathrm{id}$. Let $\Gamma \colon \mathfrak{X} \rightsquigarrow \mathfrak{X}'$ over S. Assume there is an open substack $\Gamma' \subset \Gamma$ such that the first projection $\Gamma' \subset \Gamma \xrightarrow{p_1} \mathfrak{X}$ is flat. For a closed point $i_s \colon s \hookrightarrow S$, we denote by $\mathfrak{X}_s$, $\mathfrak{X}'_s$, Γ_s, Γ'_s the fibers over s. If $\Gamma'_s \subset \Gamma_s$ is* dense, *Γ_s is a correspondence $\mathfrak{X}_s \rightsquigarrow \mathfrak{X}'_s$, and the following diagram is commutative:*

$$\begin{tikzcd}
i_s^+ f'_! L_{\mathfrak{X}'} \arrow[r, "\Gamma^*"] \arrow[d] & i_s^+ f_! L_{\mathfrak{X}} \arrow[d] \\
f'_{s!} L_{\mathfrak{X}'_s} \arrow[r, "\Gamma_s^*"'] & f_{s!} L_{\mathfrak{X}_s}.
\end{tikzcd}$$

Here the vertical homomorphisms are the base change maps.

Proof. Since Γ' is flat over S, Γ_s is a correspondence for any $s \in S$. Now, to show the commutativity, it suffices to show the commutativity of the following diagrams:

$$\begin{tikzcd}
i_s^+ p_1^+ L_{\mathfrak{X}} \arrow[r] \arrow[d, "i_s^+(\iota_\Gamma)"'] & p_{1s}^+ L_{\mathfrak{X}_s} \arrow[d, "\iota_{\Gamma_s}"] \\
i_s^+ p_1^! L_{\mathfrak{X}} \arrow[r] & p_{1s}^! L_{\mathfrak{X}_s},
\end{tikzcd}
\qquad
\begin{tikzcd}
i_s^+ L_{\mathfrak{X}'} \arrow[r, "\sim"] \arrow[d] & L_{\mathfrak{X}'_s} \arrow[d] \\
i_s^+ p_{2+} p_2^+ L_{\mathfrak{X}'} \arrow[r, "\sim"'] & p_{2s+} p_{2s}^+ L_{\mathfrak{X}'_s}.
\end{tikzcd}$$

The commutativity of the left diagram follows by Lemma 3.1.7. The commutativity of the right one follows by the fact that (g^+, g_+) is an adjoint pair when g is a morphism of admissible stacks. $\square$

Remark. In general, there exists an open dense subscheme $U \subset S$ such that the condition of the lemma holds for any $s \in U$.

3.1.9. **Lemma.** *Let $\rho \colon \mathfrak{X} \to \mathfrak{X}'$ be a proper morphism over φ such that $\mathfrak{X}$ is smooth. Let Γ_ρ denote the graph of ρ, and regard it as a correspondence $\mathfrak{X} \rightsquigarrow \mathfrak{X}'$. Then $\rho^* = \Gamma_\rho^*$.*

Proof. We have $\mathfrak{X} \xleftarrow[\sim]{p_1} \Gamma_\rho \xrightarrow{p_2} \mathfrak{X}'$. Via the identification $p_{1!}K_{\Gamma_\rho} \xrightarrow{\sim} K_{\mathfrak{X}}$, the fake trace $\mathrm{fTr}_{p_1} \colon p_{1!}p_1^+ K_{\mathfrak{X}} \to K_{\mathfrak{X}}$ is the identity. Thus the lemma follows. $\square$

3.1.10. Let $\mathfrak{X}$, $\mathfrak{Y}$ be c-admissible stacks of dimension d. Let $g\colon \mathfrak{Y} \to \mathrm{Spec}(k)$, $h\colon \mathfrak{X} \to \mathfrak{Y}$, and $f := g \circ h$. Assume that h is proper. Then, we have the canonical homomorphism $(h^*)^\vee\colon (H^{2d}_{\mathrm{c}}(\mathfrak{X})(d))^\vee \to (H^{2d}_{\mathrm{c}}(\mathfrak{Y})(d))^\vee$, where $(-)^\vee$ denotes $\mathrm{Hom}(-,L)$. Note that $\mathrm{Tr}_{\mathfrak{X}} \in H^{2d}_{\mathrm{c}}(\mathfrak{X})(d)^\vee$.

The morphism h is said to be *generically locally free* if there exists a dense open substack $\mathfrak{V} \subset \mathfrak{Y}$ such that the induced morphism $h^{-1}(\mathfrak{V}) \to \mathfrak{V}$ is finite and locally free. Furthermore, h is said to be *generically of constant degree* if all the degrees of $h|_{h^{-1}(\mathfrak{V})}$ over irreducible components of $\mathfrak{V}$ are the same.

Lemma. *Assume that h is a generically locally free morphism of constant degree. Then $(h^*)^\vee$ sends the trace map Tr_f to $\deg(h) \cdot \mathrm{Tr}_g$.*

Proof. We have the following diagram:

$$\begin{tikzcd} H^{2d}_{\mathrm{c}}(\mathfrak{X})(d)^\vee \arrow[r, "(h^*)^\vee"] \arrow[d] & H^{2d}_{\mathrm{c}}(\mathfrak{Y})(d)^\vee \arrow[d, "\sim"] \\ H^{2d}_{\mathrm{c}}(h^{-1}(\mathfrak{V}))(d)^\vee \arrow[r] & H^{2d}_{\mathrm{c}}(\mathfrak{V})(d)^\vee, \end{tikzcd}$$

where the right vertical homomorphism is an isomorphism by Lemma 2.3.33. Thus, we may replace $\mathfrak{Y}$ by $\mathfrak{V}$, and in particular, we may assume that h is locally free. We have the following commutative diagram:

$$\begin{tikzcd} g_! h_! h^+ g^+(L)(d)[2d] \arrow[r, "\mathrm{Tr}_h"] & g_! g^+(L)(d)[2d] \arrow[r, "\mathrm{Tr}_g"] & L. \\ g_! g^+ L(d)[2d] \arrow[u, "\mathrm{adj}_h"] \arrow[ur, "\deg(h)\cdot"'] & & \end{tikzcd}$$

The composition of the first row is Tr_f. By definition, $(h^*)^\vee(\mathrm{Tr}_f) = \mathrm{adj}_h \circ \mathrm{Tr}_f$. Thus the lemma follows. $\square$

Corollary. *Let $\mathfrak{X}$ be a smooth c-admissible stack.*

(i) *Let $\mathfrak{Z}$, $\mathfrak{Z}'$ be generalized cycles of codimension c of $\mathfrak{X}$. Assume we are given a generically locally free morphism of constant degree $\rho\colon \mathfrak{Z}' \to \mathfrak{Z}$ over $\mathfrak{X}$. Then by the push-forward homomorphism $\rho_*\colon H^{2c}_{\mathfrak{Z}'}(\mathfrak{X})(c) \to H^{2c}_{\mathfrak{Z}}(\mathfrak{X})(c)$, $\mathrm{cl}_{\mathfrak{X}}(\mathfrak{Z}')$ is sent to $\deg(\rho) \cdot \mathrm{cl}_{\mathfrak{X}}(\mathfrak{Z})$.*

(ii) *Let $\Gamma, \Gamma'\colon \mathfrak{X} \rightsquigarrow \mathfrak{X}'$ be correspondences. Assume that there exists a morphism $\rho\colon \Gamma \to \Gamma'$ such that $c_{\Gamma'} \circ \rho = c_\Gamma$, and ρ is generically locally free of constant degree. Then $\Gamma'^* \cong \deg(\rho)^{-1} \cdot \Gamma^*$.*

Proof. They are straightforward from the lemma. $\square$

3.1.11. **Definition.** We denote by $\mathrm{Corr}_\varphi(\mathfrak{X}, \mathfrak{X}')$ the $\mathbb{Q}$-vector space freely generated by the set $\{\Gamma\colon \mathfrak{X} \rightsquigarrow \mathfrak{X}' \mid \text{correspondence over } \varphi\}$. We denote $\mathrm{Corr}_{\mathrm{id}}(\mathfrak{X}, \mathfrak{X}')$ by $\mathrm{Corr}_S(\mathfrak{X}, \mathfrak{X}')$, and we denote $\mathrm{Corr}_\star(\mathfrak{X}, \mathfrak{X})$ by $\mathrm{Corr}_\star(\mathfrak{X})$ ($\star = \varphi, S$) for short. We have a homomorphism

$$\mathrm{Corr}_\varphi(\mathfrak{X}, \mathfrak{X}') \to \mathrm{Hom}_S\big(\varphi^+ f'_! L_{\mathfrak{X}'}, f_! L_{\mathfrak{X}}\big)$$

by sending Γ to Γ^*. Let I be the $\mathbb{Q}$-vector subspace of $\mathrm{Corr}_\varphi(\mathfrak{X}, \mathfrak{X}')$ generated by $(\Gamma' - \deg(\rho)^{-1} \cdot \Gamma)$, where Γ, Γ' are correspondences and ρ is a generically locally free morphism of constant degree $\Gamma \to \Gamma'$. When $\mathfrak{X}$ is smooth, by Corollary 3.1.10 (ii), the homomorphism above factors through $\mathrm{Corr}_\varphi(\mathfrak{X}, \mathfrak{X}')/I$.

Let $\mathrm{Corr}^{\star}_{\varphi}(\mathfrak{X},\mathfrak{X}')$ for $\star=\mathrm{et}$ (resp. fin) be the $\mathbb{Q}$-vector subspace of $\mathrm{Corr}_{\varphi}(\mathfrak{X},\mathfrak{X}')$ generated by integral correspondences Γ (i.e., Γ is integral) such that the first projection $\Gamma\to\mathfrak{X}$ is étale (resp. finite). There exists the composition map

$$\circ\colon \mathrm{Corr}^{\mathrm{et}}_{\psi}(\mathfrak{X}',\mathfrak{X}'')\times\mathrm{Corr}^{\mathrm{et}}_{\varphi}(\mathfrak{X},\mathfrak{X}')\to\mathrm{Corr}^{\mathrm{et}}_{\psi\circ\varphi}(\mathfrak{X},\mathfrak{X}'')$$

defined by sending (Γ',Γ) to $\Gamma'\circ\Gamma:=\Gamma\times_{\mathfrak{X}'}\Gamma'$.

Lemma. *Let $\Gamma\colon\mathfrak{X}\rightsquigarrow\mathfrak{X}'$, $\Gamma'\colon\mathfrak{X}'\rightsquigarrow\mathfrak{X}''$ be correspondences over φ and ψ. Assume further that the second projection of Γ or the first projection of Γ' is smooth. When $\mathfrak{X}$ and $\mathfrak{X}'$ are smooth, we have $(\Gamma'\circ\Gamma)^*=\Gamma^*\circ\Gamma'^*$.*

Proof. The verification is standard. See, for example, [La2, A.7]. □

3.1.12. Given a correspondence, the results in this subsection hold in exactly the same manner for ℓ-adic étale cohomology. However, in [La2, §A], he uses slightly different definition of the actions of correspondences on the cohomology, and we need to compare these.

For a smooth admissible stack $\mathfrak{X}$, we denote by $\mathfrak{X}^{\mathrm{gr}}$ the associated coarse moduli algebraic space of Keel and Mori (cf. [La2, A.2]). Now, let $\Gamma\colon\mathfrak{X}\rightsquigarrow\mathfrak{X}'$ be an integral correspondence over φ such that the first projection $p\colon\Gamma\to\mathfrak{X}$ is *generically finite* and dominant. Then the morphism $\rho\colon\Gamma\to\widetilde{\Gamma}^{\mathrm{gr}}:=\Gamma^{\mathrm{gr}}\times_{p\searrow\mathfrak{X}^{\mathrm{gr}}}\mathfrak{X}$ is generically finite locally free. Indeed, let $P\to\mathfrak{X}$ be a presentation. Then $|\Gamma\times_{\mathfrak{X}}P|\to|\widetilde{\Gamma}^{\mathrm{gr}}\times_{\mathfrak{X}}P|=|\Gamma^{\mathrm{gr}}|\times_{|\mathfrak{X}^{\mathrm{gr}}|}|P|$ is surjective. The last equality holds since the fiber product is taken in the category of algebraic spaces. Thus the morphism $\Gamma\times_{\mathfrak{X}}P\to\widetilde{\Gamma}^{\mathrm{gr}}\times_{\mathfrak{X}}P$ is surjective, and $\Gamma\to\widetilde{\Gamma}^{\mathrm{gr}}$ is surjective. This implies that $\widetilde{\Gamma}^{\mathrm{gr}}$ is irreducible. Now, since $\pi_*\mathscr{O}_{\Gamma}\cong\mathscr{O}_{\Gamma^{\mathrm{gr}}}$ (cf. [Co2, 1.1]), Γ^{gr} is reduced. By [Beh, 5.1.12, 13, 14], the morphism $\mathfrak{X}\to\mathfrak{X}^{\mathrm{gr}}$ is a gerb generically over $\mathfrak{X}^{\mathrm{gr}}$, thus the morphism is generically smooth by [Beh, 5.1.5]. This implies that $\widetilde{\Gamma}^{\mathrm{gr}}\to\Gamma^{\mathrm{gr}}$ is smooth generically, and thus $\widetilde{\Gamma}^{\mathrm{gr}}$ is reduced generically over $\mathfrak{X}$. Since $\widetilde{\Gamma}^{\mathrm{gr}}$ is irreducible and dominant over $\mathfrak{X}$, we conclude that $\widetilde{\Gamma}^{\mathrm{gr}}$ is integral. Thus, ρ is generically flat. Since p is assumed generically finite, ρ is generically finite as well, and thus generically finite locally free.

The generic degree of ρ is denoted by d_{Γ}. We put $\mathrm{norm}(\Gamma):=(d_{\Gamma})^{-1}\cdot\Gamma$. We can check easily that for composable correspondences Γ, Γ', we have $d_{\Gamma'\circ\Gamma}=d_{\Gamma}\cdot d_{\Gamma'}$. Then we have a ring homomorphism

$$\mathrm{norm}\colon\mathrm{Corr}^{\mathrm{fin,et}}_{\varphi}(\mathfrak{X})\to\mathrm{Corr}^{\mathrm{fin,et}}_{\varphi}(\mathfrak{X});\ \Gamma\mapsto(d_{\Gamma})^{-1}\cdot\Gamma.$$

Now, let $\mathfrak{Y}$ be a c-admissible stack, and let $q_{\mathfrak{Y}}\colon\mathfrak{Y}\to\mathfrak{Y}^{\mathrm{gr}}$ be the canonical morphism, which is known to be proper (cf. [Co2, 1.1]). By [La2, A.3], the adjunction homomorphism $\mathbb{Q}_\ell\to\mathbb{R}q_{\mathfrak{Y}*}q^*_{\mathfrak{Y}}(\mathbb{Q}_\ell)$ is an isomorphism. For $\Gamma\in\mathrm{Corr}_{\varphi}(\mathfrak{X},\mathfrak{X}')$, we define

$$\Gamma^{*\mathrm{Laf}}\colon\mathcal{H}^*_{\mathrm{c}}(\mathfrak{X}')\xleftarrow{\sim}\mathcal{H}^*_{\mathrm{c}}(\mathfrak{X}'^{\mathrm{gr}})\xrightarrow{\Gamma^{\mathrm{gr}*}}\mathcal{H}^*_{\mathrm{c}}(\mathfrak{X}^{\mathrm{gr}})\xrightarrow{\sim}\mathcal{H}^*_{\mathrm{c}}(\mathfrak{X}),$$

where, for a c-admissible stack $f\colon\mathfrak{Y}\to S$, $\mathcal{H}^*_{\mathrm{c}}(\mathfrak{Y})$ denotes $\mathscr{H}^*f_!\mathbb{Q}_\ell$. This is nothing but the action of correspondence defined in [La2, A.6].

Lemma. *Let Γ be an element of $\mathrm{Corr}_{\varphi}(\mathfrak{X},\mathfrak{X}')$, which is integral, and the first projection $\Gamma\to\mathfrak{X}$ is generically finite. Then we have $\Gamma^{*\mathrm{Laf}}=(\mathrm{norm}(\Gamma))^*$.*

Proof. Consider the following commutative diagram on the left:

$$
\begin{tikzcd}
& \Gamma \arrow[r] \arrow[d] & \mathfrak{X}' \arrow[d] \\
\mathfrak{X} \arrow[d] & \widetilde{\Gamma}^{\mathrm{gr}} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{p}'"] \arrow[d, "q"] \arrow[dl, phantom, "\square"] & \mathfrak{X}'^{\mathrm{gr}} \arrow[d, equal] \\
\mathfrak{X}^{\mathrm{gr}} & \Gamma^{\mathrm{gr}} \arrow[l, "p^{\mathrm{gr}}"] \arrow[r, "p'^{\mathrm{gr}}"'] & \mathfrak{X}'^{\mathrm{gr}},
\end{tikzcd}
\qquad
\begin{tikzcd}
q^*p'^{\mathrm{gr}*}\mathbb{Q}_\ell \arrow[r, "\iota_{\Gamma^{\mathrm{gr}}}"] \arrow[d, "\sim"'] & q^*p^{\mathrm{gr}!}\mathbb{Q}_\ell \arrow[d] \\
\widetilde{p}'^*\mathbb{Q}_\ell \arrow[r, "\iota_{\widetilde{\Gamma}^{\mathrm{gr}}}"'] & \widetilde{p}^!\mathbb{Q}_\ell.
\end{tikzcd}
$$

Then the right diagram is commutative by using Lemma 3.1.7. Thus, the lemma follows by Lemma 3.1.10. Note that even though $\mathfrak{X}^{\mathrm{gr}}$ is not smooth, it is cohomologically smooth by [La2, A.5], and the arguments in 3.1.10 work without any changes. $\square$

3.2. **Independence of ℓ.** We show an ℓ-independence result of the trace of the action of correspondence on the cohomology of a c-admissible stack. In the scheme and ℓ-adic case, the ℓ-independence result is the one proven in [KS].

3.2.1. The main result of this subsection is as follows:

Theorem. *Let $\mathfrak{X}$ be a smooth c-admissible stack over a finite field k, and let $\Gamma \in \mathrm{Corr}_S(\mathfrak{X})$. Then we have*

$$\mathrm{Tr}\big(\Gamma^* : H^*_{\mathrm{c}}(\mathfrak{X} \otimes \overline{k}, \mathbb{Q}_\ell)\big) = \mathrm{Tr}\big(\Gamma^* : H^*_{\mathrm{c}}(\mathfrak{X}, L_{\mathfrak{X},\emptyset})\big).$$

Remark. We assumed k to be finite since some delicate arguments might be needed in the p-adic situation to reduce to the finite field case as in the proof of [KS, 2.3.6.1]. However, there is no reason to doubt that the theorem holds without the assumption on the base field.

The idea of the proof is essentially the same as [KS, 2.3.6], and the proof takes the whole subsection. In the application, we use the following form: Let k be a finite field with $q = p^s$ elements, and let $R = W(k)$ and $\sigma = \mathrm{id}$. Let X be a separated scheme of finite type over k, and let $i_x \colon x \hookrightarrow X$ be a closed point of X. We take an algebraic extension L of $K_x := \mathrm{Frac}(W(k(x)))$, and take $\sigma_L := \mathrm{id}$ for an extension of σ. Put $s' := [k(x) : k] \cdot s$, and let $\mathfrak{T}_{x,F} := (k(x), W(k(x)), K_x, L, s', \mathrm{id})$. We have the following functor

$$\iota^n_x \colon D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F) \xrightarrow{\mathscr{H}^n i^+_x} \mathrm{Hol}(k(x)/L_F) \cong \mathrm{Hol}(k(x)/\mathfrak{T}_{x,F}),$$

where the last equivalence follows by Corollary 1.4.11. The last category is equivalent to the category of finite-dimensional L-vector spaces with automorphism denoted by F_x. Let $\mathscr{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$, and assume we are given an automorphism α. This induces an automorphism on $\iota^n_x(\mathscr{E})$ denoted by α_x. Note that α_x and F_x commute. For $n \in \mathbb{Z}$, we put

$$\mathrm{Tr}\big(\alpha \times F^n_x : \mathscr{E}\big) := \sum_i (-1)^i \cdot \mathrm{Tr}\big(\alpha_x \circ F^n_x : \iota^i_x(\mathscr{E})\big).$$

Corollary. *Let S be a smooth connected scheme of finite type over k, and let $f \colon \mathfrak{X} \to S$ be a smooth morphism between c-admissible stacks. Assume we are given $\Gamma \in \mathrm{Corr}_S(\mathfrak{X})$. For a closed point $x \in S$, let $\mathfrak{X}_x$ and Γ_x denote the fibers over*

x. Then there exists an open dense subscheme $U \subset S$ such that for any closed point $x \in U$ such that $L \supset K_x$ and any integer n, we have

$$\mathrm{Tr}\big(\Gamma^* \times F_x^n : f_! \mathbb{Q}_\ell\big) = \mathrm{Tr}\big(\Gamma^* \times F_x^n : f_! L_{\mathfrak{X}}\big).$$

Proof. Let f_x denote the fiber of f on x. By Lemma 3.1.8 and its remark, we can take $U \subset S$ such that the action of Γ on $f_!\mathbb{Q}_\ell$ and $f_! L_{\mathfrak{X}}$ at the fiber $x \in U$ are equal to the action of Γ_x on $f_{x!}\mathbb{Q}_\ell$ and $f_{x!}L_{\mathfrak{X}_x}$. Then we use our theorem to get the corollary. □

3.2.2. We do not assume k to be finite here, and we fix $\blacktriangle \in \{\emptyset, F\}$.

Lemma ([SGA4$\frac{1}{2}$, Cycle, 2.3.8 (ii)])**.** *Consider the following cartesian diagram of c-admissible stacks over k :*

$$\begin{tikzcd}
\Gamma' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathfrak{X}' \arrow[r, "h'"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & S' \arrow[d] \\
\Gamma \arrow[r, "g"'] & \mathfrak{X} \arrow[r, "h"'] & S,
\end{tikzcd}$$

where $\mathfrak{X}^{(\prime)}$ is smooth (over k), $S^{(\prime)}$ is a scheme, g is a generalized cycle of codimension c, and h and $h \circ g$ are flat and equidimensional. Then g' is a generalized cycle of codimension c as well, and we have $f^*\mathrm{cl}(\Gamma) = \mathrm{cl}(\Gamma') \in H^{2c}_{\Gamma'}(\mathfrak{X}')(c)$.

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(g^+ L_{\mathfrak{X}}, g^! L_{\mathfrak{X}}) \arrow[r, dash, "\sim"] \arrow[d, dotted] & H^{2c}_{\Gamma}(\mathfrak{X})(c) \arrow[d, "f^*"] \\
\mathrm{Hom}(g'^+ L_{\mathfrak{X}}, g'^! L_{\mathfrak{X}}) \arrow[r, dash, "\sim"'] & H^{2c}_{\Gamma'}(\mathfrak{X}')(c),
\end{tikzcd}$$

where the dotted arrow can be described as follows: Let $\phi \in \mathrm{Hom}(g^+ L_{\mathfrak{X}}, g^! L_{\mathfrak{X}})$. Then the image of ϕ is the unique dotted homomorphism in the following diagram on the left, which makes the diagram commutative:

$$\begin{tikzcd}
g'^+ L_{\mathfrak{X}'} \arrow[r, dotted] & g'^! L_{\mathfrak{X}'}(c)[2c] \\
f'^+ g^+ L_{\mathfrak{X}} \arrow[u, "\sim"] \arrow[r, "\sim", "f'^+(\phi)"'] & f'^+ g^! L_{\mathfrak{X}}(c)[2c], \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
g'^+ L_{\mathfrak{X}'} \arrow[r, "\mathrm{Tr}_{\Gamma'}", "\sim"'] & g'^! L_{\mathfrak{X}'}(c)[2c] \\
f'^+ g^+ L_{\mathfrak{X}} \arrow[u, "\sim"] \arrow[r, "\sim", "f'^+\mathrm{Tr}_\Gamma"'] & f'^+ g^! L_{\mathfrak{X}}(c)[2c]. \arrow[u]
\end{tikzcd}$$

Here the right vertical homomorphism is the base change homomorphism. Thus, by taking $\phi = \mathrm{Tr}_\Gamma$, the problem is reduced to showing the commutativity of the right diagram above. Now, Tr_Γ and $\mathrm{Tr}_{\Gamma'}$ can be regarded as $\mathrm{Tr}_{h\circ g}$ and $\mathrm{Tr}_{h'\circ g'}$ by the transitivity of trace. By the base change property of trace, we get the lemma. □

3.2.3. **Lemma.** *Let X be a smooth scheme of dimension d over k, and let $Z \to X$ be a generalized cycle. Then the cycle class map induces a homomorphism $\mathrm{cl}_X \colon \mathrm{CH}_i(Z) \to H^{2d-2i}_Z(X)(d-i)$.*

Proof. Let W be a closed integral subscheme of dimension $i+1$ of Z and $W \to \mathbb{P}^1$ be a dominant morphism (hence flat by [EGAIV, 2.8.2]). Let W_i be the fiber of $i \in \mathbb{P}^1$. By [Fu, Ch.I, Prop 1.6], it suffices to show that $\mathrm{cl}_X(W_0) = \mathrm{cl}_X(W_1)$. Let

W° be the pullback of $\mathbb{A}^1 \subset \mathbb{P}^1$ by the morphism $W \to \mathbb{P}^1$. Note that $W^\circ \to Z \times \mathbb{A}^1$ is a closed immersion. Consider the following commutative diagram:

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W^\circ \arrow[d] \arrow[dr, phantom, "\square"] & W_1 \arrow[l] \arrow[d] \\ Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z \times \mathbb{A}^1 \arrow[d] \arrow[dr, phantom, "\square"] & Z \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 & \{1\}. \arrow[l] \end{tikzcd}$$

This induces the commutative diagram

$$\begin{tikzcd} H^{2j}_{W_0}(X) \arrow[d] & H^{2j}_{W^\circ}(X \times \mathbb{A}^1) \arrow[l, "0^*"'] \arrow[r, "1^*"] \arrow[d] & H^{2j}_{W_1}(X) \arrow[d] \\ H^{2j}_Z(X) & H^{2j}_{Z\times\mathbb{A}^1}(X \times \mathbb{A}^1) \arrow[l, "\sim", "0^*"'] \arrow[r, "\sim"', "1^*"] & H^{2j}_Z(X), \end{tikzcd}$$

where $j := d - i$, and we omit Tate twists. The bottom arrows are isomorphisms since $H^i(\mathbb{A}^1) = 0$ for $i \neq 0$ and $H^0(\mathbb{A}^1) \cong L$. By Lemma 3.2.2 and the flatness of $X^\circ \to \mathbb{A}^1$, $\mathrm{cl}_{X\times\mathbb{A}^1}(W^\circ)$ on the top middle is sent to $\mathrm{cl}_X(W_0)$ and $\mathrm{cl}_X(W_1)$ via 0^* and 1^*, so the lemma follows. $\square$

Remark. When Z is a cycle in X, we believe that the method of [Gil] can be applied to construct the cycle class map. However, the author does not know how we define the Zariski sheaves $\underline{\Gamma}^*(i)$ in [Gil, 1.1].

3.2.4. **Lemma** ([KS, 2.1.1]). *Let $\mathfrak{X}$ be a c-admissible stack,*[5] *let $\mathfrak{U}$ be an open substack of $\mathfrak{X}$ which is smooth, and let Γ be a generalized cycle of codimension d on $\mathfrak{U}$. Consider the following commutative diagram:*

$$\begin{tikzcd} \Gamma \arrow[r, "i"'] \arrow[rr, bend left, "i'"] & \mathfrak{U} \arrow[r, hook, "j"'] & \mathfrak{X}, \end{tikzcd}$$

where j is the open immersion, and i is the generalized cycle. Assume i' is proper. Recall the homomorphism (3.1.4.1). *Using this homomorphism, the composition*

$$c\colon H^*_{\mathrm{c}}(\mathfrak{X}, j_+L_{\mathfrak{U}}) \xrightarrow{i'^*} H^*_{\mathrm{c}}(\Gamma) \xrightarrow{i_*} H^{*+2d}_{\mathrm{c}}(\mathfrak{U})(d)$$

sends u to $u \cup r(\mathrm{cl}_{\mathfrak{U}}(\Gamma))$ (cf. 3.1.2 for $\cup$). Here, r is the homomorphism $H^{2d}_\Gamma(\mathfrak{U})(d) \cong H^{2d}(\mathfrak{X}, j_!i_+i^!L_{\mathfrak{U}})(d) \to H^{2d}(\mathfrak{X}, j_!L_{\mathfrak{U}}(d))$, where the first isomorphism follows since i' is assumed proper as well.

Proof. We can copy the proof of [KS]. $\square$

[5]In the corresponding statement of [KS], they assume X to be smooth. We think that this is a typo and, in fact, is too strong for their and our purposes. Indeed, in the proof of [KS, 2.3.2], they apply [KS, 2.2.1] in the situation where X is not necessarily smooth.

3.2.5. Let $f\colon X \to \mathfrak{Y}$ be a representable l.c.i. morphism from a scheme[6] of finite type purely of dimension n to a c-admissible stack purely of dimension m over k. For example, a representable morphism between smooth c-admissible stacks is l.c.i. We note that f is schematic since $\mathfrak{Y}$ is admissible. Let $\Gamma \to \mathfrak{Y}$ be a generalized cycle of codimension d which is a scheme, and put $\Gamma' := \Gamma \times_{\mathfrak{Y}} X$, which is a generalized cycle of X and is a scheme as well since f is schematic. Let us briefly recall the construction of $f^!([\Gamma]) \in \mathrm{CH}_{n-d}(\Gamma')$ by Kresch [Kr].

Let $f'\colon \mathfrak{X}' \to \mathfrak{Y}'$ be a representable separated morphism between algebraic stacks. In [Kr, 5.1], Kresch constructs[7] $\rho\colon M^{\circ}_{\mathfrak{X}'}\mathfrak{Y}' \to \mathbb{A}^1$, whose fiber over 0 is called the *normal cone* denoted by $C_{\mathfrak{X}'}\mathfrak{Y}' \to \mathfrak{X}'$, and the general fiber is just $\mathfrak{Y}'$. When $f'\colon X' \hookrightarrow Y'$ is a closed immersion of schemes, $M^{\circ}_{X'}Y'$ is nothing but the one introduced in [Fu, Ch.5]. We remark that by construction, there is a canonical morphism $\mathfrak{X}' \times \mathbb{A}^1 \to M^{\circ}_{\mathfrak{X}'}\mathfrak{Y}'$ defined by the strict transform, and ρ is flat by using [Fu, B.6.7]. When f' is l.c.i., $C_{\mathfrak{X}'}\mathfrak{Y}'$ is known to be a vector bundle over $\mathfrak{X}$, in which case we denote it by $N_{f'}$.

Now, assume that X and $\mathfrak{Y}$ are smooth. In this situation, $M^{\circ}_X\mathfrak{Y}$ is smooth. We put $\Gamma' := \Gamma \times_{\mathfrak{Y}} X$. By definition, we have the diagram below on the left:

$$(\star)\quad \begin{tikzcd} M^{\circ}_{\Gamma'}\Gamma \arrow[r, hook, "\alpha"'] & M^{\circ}_X\mathfrak{Y} \times_{\mathfrak{Y}} \Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & M^{\circ}_X\mathfrak{Y} \arrow[d] \\ & \Gamma \arrow[r] & \mathfrak{Y}, \end{tikzcd} \qquad \begin{tikzcd} \Gamma' \times \mathbb{A}^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times \mathbb{A}^1 \arrow[d] \\ M^{\circ}_{\Gamma'}\Gamma \arrow[r] & M^{\circ}_X\mathfrak{Y}. \end{tikzcd}$$

This diagram induces the cartesian diagram on the right.

Let us define $f^!([\Gamma]) \in \mathrm{CH}_{n-d}(\Gamma')$. For the details, see [Fu, 3.1, 5.1]. By taking the pullback by the morphism $X \to \mathfrak{Y}$ of the diagram above on the left, we have the following diagram of *schemes*.

$$(3.2.5.1)\qquad \begin{tikzcd} C_{\Gamma'}\Gamma \arrow[r, hook] & N' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & N_f \arrow[d] \\ & \Gamma' \arrow[r] & X \arrow[u, bend right, "g"'] \end{tikzcd}$$

The closed immersion on the upper left is induced by α in $(\star)$. By definition, $f^!(\Gamma)$ is the image of $[C_{\Gamma'}\Gamma]$ by the homomorphism

$$\mathrm{CH}_{m-d}(C_{\Gamma'}\Gamma) \to \mathrm{CH}_{m-d}(N') \xrightarrow[\sim]{g^!} \mathrm{CH}_{n-d}(\Gamma'),$$

where the last isomorphism[8] follows by [Fu, Theorem 3.3].

3.2.6. **Lemma** ([KS, 2.1.2]). *We preserve the notation. Let $f^*\colon H^{2d}_{\Gamma}(\mathfrak{Y}) \to H^{2d}_{\Gamma'}(X)$ be the pullback. Then the class $\mathrm{cl}(f^!([\Gamma])) \in H^{2d}_{\Gamma'}(X)$ is equal to $f^*\mathrm{cl}(\Gamma)$.*

---

[6] This assumption is made for simplicity. In fact, with suitable changes, similar results can be obtained when X is a c-admissible stack.

[7] In fact, he constructs over $\mathbb{P}^1$ instead of $\mathbb{A}^1$. However, for convenience, we restrict his construction over $\mathbb{A}^1$ and, by abusing the notation, we still denote it by $M^{\circ}_{\mathfrak{X}'}\mathfrak{Y}'$.

[8] This isomorphism holds even when X is an admissible stack. Indeed, by [Kr, 3.5.7], an admissible stack admits "a stratification by global quotients". Then by [Kr, 4.3.2], we have the required homotopy invariance property.

Proof. The verification is essentially the same as [KS]. We have the following commutative diagram:

$$\begin{tikzcd} H^{2d}_{\Gamma}(\mathfrak{Y})(d) \arrow[d, "f^*"'] & H^{2d}_{Z}(M^\circ_X\mathfrak{Y})(d) \arrow[l, "1^*"'] \arrow[r, "0^*"] \arrow[d] & H^{2d}_{C'}(N_f)(d) \arrow[d, "g^*"] \\ H^{2d}_{\Gamma'}(X)(d) & H^{2d}_{\Gamma'\times\mathbb{A}^1}(X\times\mathbb{A}^1)(d) \arrow[l, "1^*"', "\sim"] \arrow[r, "0^*", "\sim"'] & H^{2d}_{\Gamma'}(X)(d), \end{tikzcd}$$

where $Z := M^\circ_{\Gamma'}\Gamma$, $C' := C_{\Gamma'}\Gamma$, and the middle vertical homomorphism is induced by the morphism $X \times \mathbb{A}^1 \to M^\circ_X\mathfrak{Y}$ defined by the strict transform. At the upper row, the image of the cycle class $\mathrm{cl}(Z)$ is sent to $\mathrm{cl}(\Gamma)$ and $\mathrm{cl}(C')$ by 1^* and 0^*, respectively, by Lemma 3.2.2 and the flatness of $Z \to \mathbb{A}^1$.

Recall the diagram of schemes (3.2.5.1). It is reduced to showing that $\mathrm{cl}(g^![C']) = g^*\mathrm{cl}(C')$. Using Lemma 3.2.3, this amounts to proving the commutativity of the following diagram on the left:

$$\begin{tikzcd} \mathrm{CH}_{m-d}(N') \arrow[r] \arrow[d, "g^!"'] & H^{2d}_{N'}(N_f)(d) \arrow[d, "g^*"] \\ \mathrm{CH}_{n-d}(\Gamma') \arrow[r] & H^{2d}_{\Gamma'}(X)(d), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{CH}_{n-d}(\Gamma') \arrow[r] \arrow[d, "p^*"'] & H^{2d}_{\Gamma'}(X)(d) \arrow[d, "p^*"] \\ \mathrm{CH}_{m-d}(N') \arrow[r] & H^{2d}_{N'}(N_f)(d). \end{tikzcd}$$

Note that the stacks appearing in these diagrams are, in fact, schemes. Consider the diagram on the right above, where p denotes the projection $N_f \to X$. Since $g^* \circ p^*$ is the identity on $H^{2d}_{\Gamma'}(X)(d)$, and $g^!$ is an isomorphism whose inverse is p^* on the Chow groups, the verification of the commutativity of the left diagram is reduced to that of the right one. There exists an open dense subscheme $U \subset X$ such that $U \cap \Gamma' \subset \Gamma'$ is dense, and $N_f \times_X U \to U$ is a trivial bundle so that we can write $N_f \times_X U \cong U \times \mathbb{A}^n$. Since $H^{2d}_{\Gamma'}(X)(d) \cong H^{2d}_{\Gamma'\cap U}(U)(d)$, we may replace X by U and Γ' by $\Gamma' \cap U$. Thus, the claim follows by Lemma 3.2.2. ☐

3.2.7. Now, we only consider the case where $\blacktriangle = \emptyset$ (but k is still not necessarily finite). Let us recall the construction of [KS, after Lemma 2.3.1]. Let $\mathfrak{U}$ be a smooth c-admissible stack of dimension d, and let Γ be a correspondence on $\mathfrak{U}$ (over $\mathrm{Spec}(k)$). Let $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ be a compactification. Since the second projection $p_2\colon \Gamma \to \mathfrak{U}$ is assumed proper, the morphism $(j \circ p_1, p_2)\colon \Gamma \to \mathfrak{X} \times \mathfrak{U}$ is proper, and we have isomorphisms

$$H^{2d}_{\Gamma}\big(\mathfrak{X}\times\mathfrak{U}, (j\times\mathrm{id})_!L(d)\big) \xrightarrow{\sim} H^{2d}_{\Gamma}\big(\mathfrak{X}\times\mathfrak{U}, L(d)\big) \xrightarrow{\sim} H^{2d}_{\Gamma}\big(\mathfrak{U}\times\mathfrak{U}, L(d)\big).$$

Thus, the cycle class $\mathrm{cl}(\Gamma)$ defined in $H^{2d}_{\Gamma}\big(\mathfrak{U}\times\mathfrak{U}, L(d)\big)$ induces an element in

$$H^{2d}_{!,*}(\mathfrak{U}\times\mathfrak{U})(d) := H^{2d}\big(\mathfrak{X}\times\mathfrak{U}, (j\times\mathrm{id})_!L(d)\big).$$

Since $\blacktriangle = \emptyset$, this cohomology is isomorphic to $\prod_i \mathrm{End}\big(H^i_{\mathrm{c}}(\mathfrak{U})\big)$ as in [KS] using the Künneth formula (cf. Corollary 2.3.37) and the Poincaré duality (cf. Theorem 2.3.31).

Lemma ([KS, 2.3.2])**.** *The action Γ^* can be identified with the class $\mathrm{cl}(\Gamma)$ via this isomorphism.*

Proof. Using Lemma 3.2.4, the proof of [KS] works exactly in the same manner. ☐

3.2.8. *Proof of Theorem* 3.2.1. By Corollary 2.3.4, we can take a proper generically finite surjective morphism $f\colon X \to \mathfrak{X}$ such that X is a smooth scheme. Using the same corollary and Corollary 3.1.10, we may assume Γ to be a scheme. Let $H^*_{\mathrm{c}}(\mathfrak{X})$ be $H^*_{\mathrm{c}}(\mathfrak{X}\otimes\overline{k},\mathbb{Q}_\ell)$ or $H^*_{\mathrm{c}}(\mathfrak{X},L_{\mathfrak{X}})$. Using Lemma 3.2.7 and Corollary 3.1.7, by arguing as [KS, 2.3.3], we have

$$(\star)\qquad \mathrm{Tr}\big(\Gamma^* : H^*_{\mathrm{c}}(\mathfrak{X})\big) = \deg(f)^{-1}\cdot\mathrm{Tr}\big((f\times f)^*\mathrm{cl}_{\mathfrak{X}\times\mathfrak{X}}(\Gamma) : H^*_{\mathrm{c}}(X)\big),$$

where we regard classes in $H^{2d}_{!,*}(\mathfrak{X}\times\mathfrak{X})(d)$ ($d := \dim(\mathfrak{X})$) as endomorphisms of $H^*_{\mathrm{c}}(\mathfrak{X})$ using Lemma 3.2.7. Consider the following commutative diagram:

$$\begin{tikzcd}
\Gamma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X\times X \arrow[r, "\pi_2"] \arrow[d, "f\times f"] & X \arrow[d, "f"] \\
\Gamma \arrow[r] & \mathfrak{X}\times\mathfrak{X} \arrow[r, "\pi_2"'] & \mathfrak{X}.
\end{tikzcd}$$

Since the composition of the horizontal morphisms below is proper by assumption, the composition morphism from Γ' to $\mathfrak{X}$ is proper. Thus the composition of the horizontal morphisms of the first row is proper. By Lemma 3.2.6, $(f\times f)^*\mathrm{cl}_{\mathfrak{X}\times\mathfrak{X}}(\Gamma)$ is equal to $\mathrm{cl}_{X\times X}\big((f\times f)^!(\Gamma)\big)$. Let $\widetilde{\Gamma}'$ be the correspondence defined by $(f\times f)^!(\Gamma)$. By applying Lemma 3.2.7 once again, the trace of the right-hand side of $(\star)$ is equal to $\mathrm{Tr}\big(\widetilde{\Gamma}'\colon H^*_{\mathrm{c}}(X)\big)$. This implies that it suffices to show the theorem in the case where $\mathfrak{X} =: X$ and Γ are schemes. By Corollary 3.1.10, we may replace Γ by its image in $X\times X$, and assume that $\Gamma\subset X\times X$. In this case, we just repeat the argument of [KS, Prop 2.3.6]. Further details are left to the reader. $\square$

4. Langlands correspondence

In this final section, we establish the Langlands correspondence, and in particular, we prove the existence of *petits camarades cristallins* for curves. We wrote this section to be as independent as possible.

4.1. **Preliminaries.** First, let us very briefly recall basic notions of the p-adic cohomology theory and notation of this paper for the convenience of the reader. We do not have new input, and those who have read the previous sections may skip this.

4.1.1. Let k be a perfect field, let R be a complete discrete valuation ring whose residue field is k, and let K be the field of fractions of R. We assume further that the s-th Frobenius automorphism of k can be lifted to an automorphism $\sigma\colon R \xrightarrow{\sim} R$. The induced automorphism between K is also denoted by σ. With this setup, let X be a scheme of finite type over k. Berthelot defined the category of *overconvergent isocrystals* (resp. *overconvergent F-isocrystals*) denoted by $\mathrm{Isoc}^\dagger(X/K)$ (resp. F-$\mathrm{Isoc}^\dagger(X/K)$). We do not try to recall the definition here, but a standard reference is [Ber1], and we may find other references in [Ke3] and [Ke6]. This category is a p-adic analogue of the category of smooth $\mathbb{Q}_\ell$-sheaves over $X\otimes_k\overline{k}$ (resp. over X). In this paper, we denote F-$\mathrm{Isoc}^\dagger(X/K)$ by $\mathrm{Isoc}^\dagger(X/K_F)$. The description of these categories when $X = \mathrm{Spec}(k)$ is simple but important: $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k)/K)$ is canonically equivalent to the category of finite-dimensional K-vector spaces, and $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k)/K_F)$ is canonically equivalent to that of finite-dimensional K-vector spaces V equipped with an isomorphism $K\otimes_{\sigma,K}V \xrightarrow{\sim} V$.

In particular, the trivial vector space K with trivial isomorphism determines an object of $\mathrm{Isoc}^\dagger(\mathrm{Spec}(k)/K_F)$, which is denoted by K, abusing the notation.

Let $f\colon X \to Y$ be a morphism between schemes of finite type over k. Then the pull-back functor $f^+\colon \mathrm{Isoc}^\dagger(Y/K) \to \mathrm{Isoc}^\dagger(X/K)$ (resp. $f^+\colon \mathrm{Isoc}^\dagger(Y/K_F) \to \mathrm{Isoc}^\dagger(X/K_F)$) is defined in [Ber1, 2.3.2 (iv)].[9] If p is the structural morphism of X, we put $K_X := p^+K$, which is also denoted by K. The category is equipped with tensor product $\otimes$, with which $\mathrm{Isoc}^\dagger(X/K)$ (resp. $\mathrm{Isoc}^\dagger(X/K_F)$) forms a tensor category (cf. [Ber1, 2.3.3 (iii)]). The unit object of the tensor category is K_X. It also possesses an internal hom functor $\mathcal{H}om$. Finally, we have the notion of ranks (cf. [Ber1, 2.3.3 (ii)]). This implies that any object in $\mathrm{Isoc}^\dagger(X/K)$ (resp. $\mathrm{Isoc}^\dagger(X/K_F)$) is of finite length.

Remark. The category $\mathrm{Isoc}^\dagger(X/K)$ used in §1–§3 is smaller than the one recalled here, but the category with Frobenius $\mathrm{Isoc}^\dagger(X/K_F)$ is the same; see the paragraph right after 1.1.3 (11). In the following, we only use $\mathrm{Isoc}^\dagger(X/K_F)$, so this does not cause any problems.

4.1.2. In Langlands correspondence, we consider the case where k is a finite field with $q = p^s$ elements, $R := W(k)$, and $\sigma\colon R \xrightarrow{\sim} R$ is the identity. Then we may consider the category $\mathrm{Isoc}^\dagger(X/K_F)$ which is K-abelian. Note that this would be $K^{\sigma=1}$-abelian if σ were not the identity. Now, we need to extend the scalar from K to $\overline{\mathbb{Q}}_p$. This was done in 1.4.10 and 2.4.14, and let us recall the idea briefly:[10] For a finite extension L/K, we define $\mathrm{Isoc}^\dagger(X/L_F)$ to be the category of pairs $(\mathcal{E},\rho)$ where $\mathcal{E} \in \mathrm{Isoc}^\dagger(X/K_F)$, and we let $\rho\colon L \to \mathrm{End}(\mathcal{E})$ be a homomorphism of K-algebras, and the morphisms are defined in the obvious way. To define $\mathrm{Isoc}^\dagger(X/\overline{\mathbb{Q}}_{p,F})$, we take the 2-inductive limit of $\mathrm{Isoc}^\dagger(X/L_F)$ over all finite extensions L of K. We have the scalar extension functor $\otimes\overline{\mathbb{Q}}_p\colon \mathrm{Isoc}^\dagger(X/K_F) \to \mathrm{Isoc}^\dagger(X/\overline{\mathbb{Q}}_{p,F})$. We remark that even though $\mathcal{E} \in \mathrm{Isoc}^\dagger(X/K_F)$ is irreducible, $\mathcal{E} \otimes \overline{\mathbb{Q}}_p$ may not be irreducible in general, and this is why we needed to extend the scalar. For a morphism $f\colon X \to Y$, the pull-back functor can formally be extended to $f^+\colon \mathrm{Isoc}^\dagger(Y/\overline{\mathbb{Q}}_{p,F}) \to \mathrm{Isoc}^\dagger(X/\overline{\mathbb{Q}}_{p,F})$, and similarly for $\otimes$, $\mathcal{H}om$. The data $\mathfrak{T} := (k, R, K, L, s, \sigma = \mathrm{id})$ (where σ is an extension to L of a lifting of s-th Frobenius automorphism on k to K) we used to define $\mathrm{Isoc}^\dagger(X/L_F)$ is called the base tuple. To clarify the base, we also use the notation $\mathrm{Isoc}^\dagger(X/\mathfrak{T})$ for $\mathrm{Isoc}^\dagger(X/L_F)$. See 1.4.10 and 2.4.14 for details.

4.1.3. Before going to the next section, let us briefly recall what we have done so far. Let $\mathfrak{X}$ be a scheme, or more generally, an algebraic stack of finite type over k. We constructed a triangulated category $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ over $\overline{\mathbb{Q}}_p$ with a natural t-structure $\mathscr{H}$ in Definition 2.1.16. When $\mathfrak{X}$ is a smooth separated scheme over k, $\mathrm{Isoc}^\dagger(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ is fully faithfully embedded into $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ by 2.4.15. There is another t-structure on $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$: the constructible t-structure ${}^{\mathrm{c}}\mathscr{H}$ defined in Definition 2.3.24. Philosophically, $\mathscr{H}$ corresponds to the perverse t-structure in the ℓ-adic setting, and ${}^{\mathrm{c}}\mathscr{H}$ corresponds to the standard (constructible) t-structure. In this section, we mostly use ${}^{\mathrm{c}}\mathscr{H}$ since its heart, denoted by $\mathrm{Con}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$, contains $\mathrm{Isoc}^\dagger(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ when $\mathfrak{X}$ is a smooth separated scheme. Objects of Con are called *constructible objects*. Over the category of *compactifiable admissible stacks*

[9] The pull-back functor is denoted by f^* in [Ber1].

[10] Actually the definition presented here is the one in Remark 1.4.9.

(*c-admissible stacks*), we have six functor formalism: $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ is endowed with tensor and dual functors. Given a morphism $f\colon \mathfrak{X}\to\mathfrak{Y}$, we have

$$f_+, f_!\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})\to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}/\overline{\mathbb{Q}}_{p,F}),\quad f^+, f^!\colon D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{Y}/\overline{\mathbb{Q}}_{p,F})\to D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F}),$$

and they satisfy standard properties. Here, f_+ and f^+ are analogues of f_* and f^* in the ℓ-adic theory, and we adopted these notation in order to follow the tradition of the theory of $\mathscr{D}$-modules. We do not need this much in the statement of Langlands correspondence, but these techniques are required in the proof.

4.2. **Langlands correspondence.** The aim of this subsection is to state the main theorem of this paper, namely the Langlands correspondence, and give an overview of the strategy of the proof.

4.2.1. First of all, let us fix the basis (cf. 1.4.10, 2.4.14). We assume k to be a finite field with $q=p^s$ elements. We fix an algebraic closure $\overline{\mathbb{Q}}_p$ of K, and we denote by $\overline{k}$ the residue field of $\overline{\mathbb{Q}}_p$ which is algebraically closed as well. We define an arithmetic base tuple $\mathfrak{T}_k := (k, R := W(k), K := \mathrm{Frac}(R), \overline{\mathbb{Q}}_p, s, \sigma := \mathrm{id})$. Likewise, for a finite extension k' of k in $\overline{k}$, we put $\mathfrak{T}_{k'} := (k', W(k'), \mathrm{Frac}(R'), \overline{\mathbb{Q}}_p, [k':k]\cdot s, \mathrm{id})$. With these data, we may consider the $\overline{\mathbb{Q}}_p$-coefficient cohomology theory (cf. 2.4.14), which we mainly use. For any finite extension k' of k, the category $\mathrm{Isoc}^\dagger(k'/\mathfrak{T}_{k'})$ is equivalent to the category of finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces V endowed with the isomorphism $\mathrm{id}^*(V)\xrightarrow{\sim} V$ (cf. Definition 1.4.10). By identifying $\mathrm{id}^*(V)$ with V, we view the isomorphism as an automorphism of V.

Let X be a smooth scheme over k, and let $i_x\colon x\hookrightarrow X$ be a closed point of X. Then $\mathfrak{T}_{k(x)} = (k(x), W(k(x)), K_x, \overline{\mathbb{Q}}_p, s', \mathrm{id})$ where $s' := [k(x):k]\cdot s$. Let us define the *linearized Frobenius automorphism at x*. Choose a geometric point $\overline{x}\in X(\overline{k})$ lying above x. This defines an embedding $K_x\hookrightarrow\overline{\mathbb{Q}}_p$, and we have the following functor

$$\iota_{\overline{x}}\colon \mathrm{Isoc}^\dagger(X/\mathfrak{T}_k)\xrightarrow{i_x^+}\mathrm{Isoc}^\dagger(k(x)/\mathfrak{T}_k)\cong\mathrm{Isoc}^\dagger(k(x)/\mathfrak{T}_{k(x)}),$$

where the equivalence follows by Corollary 1.4.11 using the embedding. For $\mathcal{E}\in\mathrm{Isoc}^\dagger(X/\mathfrak{T}_k)$, the equipped automorphism on $\iota_{\overline{x}}(\mathcal{E})$ is called the *linearized geometric Frobenius automorphism at x of $\mathcal{E}$*. The inverse of the linearized geometric Frobenius automorphism is denoted by Frob_x and is simply called the *Frobenius automorphism at x*. The multiset of eigenvalues of Frob_x acting on $\iota_{\overline{x}}(\mathcal{E})$ depends only on the choice of x and not on $\overline{x}$. By abuse of language, we call this multiset the *set of Frobenius eigenvalues at x*.

Remark. We defined Frob_x so that the notation is compatible with that of Lafforgue. In ℓ-adic theory, the corresponding automorphism is sometimes called "arithmetic Frobenius".

4.2.2. **Theorem** (Langlands correspondence for isocrystals)**.** *We fix an isomorphism $\overline{\mathbb{Q}}_p\cong\mathbb{C}$. Let X be a geometrically connected proper smooth curve over k. Denote by F the function field of X, and let $\mathbb{A}_F$ be the ring of adèles. For an integer $r\geq 1$, consider the following two sets.*

$\mathcal{I}_r$: *The set of isomorphism classes of irreducible isocrystals of rank r in $2\text{-}\varinjlim \mathrm{Isoc}^\dagger(U/\mathfrak{T}_k)$, where the limit runs over open subschemes $U \subset X$, such that the determinant is of finite order.*

$\mathcal{A}_r$: *The set of isomorphism classes of cuspidal automorphic representations π of $\mathrm{GL}_r(\mathbb{A}_F)$ such that the order of the central character of π is finite.*

(1) *There exist maps*

$$\mathcal{E}_\bullet \colon \mathcal{A}_r \rightleftarrows \mathcal{I}_r \colon \pi_\bullet$$

with which $\mathcal{A}_r$ and $\mathcal{I}_r$ correspond in the sense of Langlands: for $\pi \in \mathcal{A}_r$ (resp. $\mathcal{E} \in \mathcal{I}_r$), the sets of unramified places of π (resp. $\mathcal{E}$) and $\mathcal{E}_\pi$ (resp. $\pi_\mathcal{E}$) coincide, which we denote by U, and for any $x \in |U|$, the set of Frobenius eigenvalues of $\mathcal{E}_\pi$ (resp. $\mathcal{E}$) at x and that of Hecke eigenvalues of π (resp. $\pi_\mathcal{E}$) at x coincide.

(2) *Assume that $\pi^{(\prime)} \in \mathcal{A}_{r^{(\prime)}}$ and $\mathcal{E}^{(\prime)} \in \mathcal{I}_{r^{(\prime)}}$ correspond in the sense of Langlands. Then the local L-functions and local ε-factors of pairs (π, π') and $(\mathcal{E}, \mathcal{E}')$ coincide for any point $x \in |X|$ (cf.* [A2] *and* [La2, VI.9 (ii)]*).*

Remark. The correspondence is unique if it exists.

4.2.3. For a proof of the theorem, we follow the program of Drinfeld and Lafforgue. We briefly recall the outline of the proof to introduce some notation we use in the next subsection. The idea is explained clearly and in detail in the introduction of [La2], so we encourage the reader who is not familiar with Lafforgue's proof to read through it before entering our proof.

In [A2], with the help of the product formula proven in [AM], we have the following theorem, which is nothing but the p-adic version of *principe de récurrence* by Deligne:

Theorem ([A2, §5])**.** *Let n be a positive integer, and assume Theorem* 4.2.2 *is known for $r, r' \leq n$. Then we have the map $\mathcal{I}_{n+1} \to \mathcal{A}_{n+1}$ in the sense of Langlands such that the corresponding cuspidal representation is unramified at the places where the isocrystal is. Moreover, if we have a map $\mathcal{A}_{n+1} \to \mathcal{I}_{n+1}$ in the sense of Langlands such that the corresponding isocrystal is unramified at the places where the cuspidal representation is, then Theorem* 4.2.2 *holds for $r', r \leq n+1$. In other words,* (2) *of the theorem holds automatically once we prove* (1)*.*

4.2.4. Thanks to the theorem above, our task is only to construct a map $\mathcal{A}_r \to \mathcal{I}_r$ such that the corresponding isocrystal is unramified at the places where the cuspidal representation is. A rough idea is to realize this as the relative cohomologies of moduli spaces of shtukas *á la* Drinfeld. Even though the moduli spaces we use here are the same as that of Lafforgue, we take p-adic cohomologies that we have developed in the preceding sections instead of ℓ-adic cohomologies to carry this out.

In this paper, we use the following various types of moduli spaces of shtukas. We remind the reader that the notation is the same as that of Lafforgue. Let r be a positive integer. Let N be a *level* (i.e., a closed subscheme $N = \mathrm{Spec}(\mathcal{O}_N) \hookrightarrow X$ which is not equal to X), let $p \colon [0, r] \to \mathbb{R}$ be a convex polygon, and let $a \in \mathbb{A}_F^\times$ of degree 1. Given these data, we have c-admissible stacks (cf. Definition 2.3.19) over the surface $(X-N)\times(X-N)$ as follows:

	Moduli space	smooth?	proper?	correspondence	Reference
①	$\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}$	○	×	$\triangle^1$	[La2, right after Prop I.3]
②	$\overline{\mathrm{Cht}_N^{r,\overline{p}\leq p}}/a^{\mathbb{Z}}$	×	○	–	[La2, Def III.8]
③	$\overline{\mathrm{Cht}_N^{r,\overline{p}\leq p}}'/a^{\mathbb{Z}}$	○	×	$\triangle^2$	[La2, Cor III.14, Thm V,14]

[1] We have the action of Hecke algebra only after taking the inductive limit over the convex polygon p.
[2] We have the action of Hecke algebra for each element, but the action may not be compatible with the product structure.

In this table, the second column (resp. the third column) refers to the smoothness (resp. properness) over $(X-N)\times(X-N)$ of the corresponding moduli spaces in the first column, and definitions and proofs of the properties listed here can be found in the References. These stacks are c-admissible by [La2, V.1]; more precisely, in the first line of its proof, it is said that these stacks are quasi-projective over $\overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}}$, and the last stack is serene (cf. [La2, Appendix A]) which is proper over $X\times X$. The components of these spaces are indexed by integers $1\leq d\leq r$ called the *degree*. The component corresponding to d is denoted by $\mathrm{Cht}_N^{r,d,\overline{p}\leq p}$, $\overline{\mathrm{Cht}_N^{r,d,\overline{p}\leq p}}$, $\overline{\mathrm{Cht}_N^{r,d,\overline{p}\leq p}}'$.

4.2.5. Let $f\colon \mathfrak{X}\to(X-N)\times(X-N)$ be one of the three moduli spaces of shtukas. Then $\mathcal{H}^*_{\mathrm{c}}:=f_!\overline{\mathbb{Q}}_{p,\mathfrak{X}}$ contains the isocrystals which correspond to cuspidal representations in the set $\{\pi\}^r_N$ (cf. 4.3.7). However, it also contains a lot of "junk" which has already appeared in the Langlands correspondence of lower ranks, and we need to throw these away. The junk is called the *r-negligible part*, and the part we need for the correspondence is called the *essential part*. We first need to show that the essential part is concentrated at a certain degree of $\mathcal{H}^*_{\mathrm{c}}$. For this, we need to use the purity of intersection cohomology, and we need the compact space ②. Still, the essential part is a mixture of isocrystals corresponding to $\{\pi\}^r_N$, and we need to extract the particular isocrystal which corresponds to a given cuspidal representation $\pi\in\{\pi\}^r_N$. For this, we need to define an action of the Hecke algebra $\mathcal{H}^r_N$. We have ring homomorphism from $\mathcal{H}^r_N$ to the ring of correspondences on the moduli space ① if we pass to the limit of p. Since we are passing to the limit to define the action, the resulting stack is not of finite type anymore. For the calculation of the trace of the action of correspondences, we use ③. We note that even though we have the correspondences associated to elements of the Hecke algebra on ③, this map might not be a homomorphism of rings. Finally, we use the ℓ-independence result to calculate the trace, and we extract exactly the information we need.

4.3. **Proof of the theorem.**

4.3.1. In this subsection, the base tuple $\mathfrak{T}_k$ is fixed as in 4.2.1, and we also fix an isomorphism $\iota\colon\overline{\mathbb{Q}}_p\cong\mathbb{C}$ as in Theorem 4.2.2. Let Y be a smooth scheme of finite type and which is geometrically connected over k. We usually omit "$/\mathfrak{T}_k$" (e.g.,

$\mathrm{Isoc}^\dagger(Y)$ instead of $\mathrm{Isoc}^\dagger(Y/\mathfrak{T}_k)$). We denote the category $\mathrm{Isoc}^\dagger(Y)$ of overconvergent $\overline{\mathbb{Q}}_p$-isocrystals with Frobenius structure by $\mathcal{I}(Y)$ to shorten the notation. We identify $\mathcal{I}(Y) = \mathrm{Isoc}^\dagger(Y)$ and $\mathrm{Sm}(Y) \subset D^{\mathrm{b}}_{\mathrm{hol}}(Y)$ via $\widetilde{\mathrm{sp}}_+$, as defined in 2.4.15. Because of this identification, we often say $\mathcal{E} \in \mathrm{Con}(Y)$ is *smooth* if it comes from $\mathcal{I}(Y)$. Let $p\colon \mathfrak{X} \to \mathrm{Spec}(k)$ be the structural morphism of a c-admissible stack. For $\mathcal{E} \in \mathrm{Isoc}^\dagger(\mathfrak{X})$ or more generally an object of $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, we put $H^*(\mathfrak{X}, \mathcal{E}) := \mathscr{H}^* p_+(\mathcal{E})$ and $H^*_{\mathrm{c}}(\mathfrak{X}, \mathcal{E}) := \mathscr{H}^* p_!(\mathcal{E})$ and regard these as $\overline{\mathbb{Q}}_p$-vector spaces with automorphism. We abbreviate ι-pure (resp. ι-mixed, ι-weight, etc.) simply by pure (resp. mixed, weight, etc.).

When we say ℓ-adic sheaf, it refers to the ℓ-adic Weil sheaf (cf. [De1, 1.1.10]). The category of smooth Weil sheaves is denoted by $\mathcal{W}_\ell(Y)$. For a scheme X over k, we denote by $\mathrm{Frob}_X\colon X \to X$ the absolute Frobenius endomorphism; $f \in \mathcal{O}_X$ is sent to f^q. For an abelian category $\mathcal{A}$, we denote by $\mathrm{Gr}(\mathcal{A})$ the Grothendieck group of $\mathcal{A}$, and $\mathbb{Q}\mathrm{Gr}(\mathcal{A}) := \mathrm{Gr}(\mathcal{A}) \otimes \mathbb{Q}$. For an object $X \in \mathcal{A}$ of finite length, we denote by X^{ss} the semisimplification of X, namely the direct sum of constituents of X.

4.3.2. Let X be a smooth scheme of finite type over k, and let $\mathcal{E}$ be in $\mathcal{I}(X)$. Take a closed point $x \in |X|$. We take a geometric point $\overline{x} \in X(\overline{k})$ which lies above x, and recall the functor $\iota_{\overline{x}}$ defined in 4.2.1. The *local L-function at x* is defined to be

$$L_x(\mathcal{E}, Z) := \det\bigl(1 - Z^{\deg(x)}\mathrm{Frob}_x^{-1}; \iota_{\overline{x}}(\mathcal{E})\bigr)^{-1}$$

in $\overline{\mathbb{Q}}_p[\![Z]\!]$, which does not depend on the choice of $\overline{x}$. Using the fixed isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$, we usually consider this series as a series in $\mathbb{C}[\![Z]\!]$. The *global L-function* is defined as

$$L_X(\mathcal{E}, Z) := \prod_{x \in |X|} L_x(\mathcal{E}, Z).$$

Analogous to Grothendieck's formula, this L-function has the following cohomological interpretation:

$$L_X(\mathcal{E}, Z) = \prod_{\nu=0}^{2\dim(X)} \det\bigl(1 - Z \cdot \mathrm{Frob}_X; H^\nu_{\mathrm{c}}(X, \mathcal{E})\bigr)^{(-1)^{\nu+1}}, \tag{4.3.2.1}$$

which is an identity of formal power series (cf. Corollary A.3.3). We refer to §A.3 for further detail.

4.3.3. We use the theory of weights. We refer to 2.2.30–2.2.32, and 2.3.38 for more detail. Let $t \in \mathbb{C}$. Using ι, we may consider q^t as an element in $\overline{\mathbb{Q}}_p$. We have the automorphism of $\overline{\mathbb{Q}}_p$, considered as a $\overline{\mathbb{Q}}_p$-vector space, sending 1 to q^{-t}, which defines an object in $\mathcal{I}(\mathrm{Spec}(k))$ denoted by $\overline{\mathbb{Q}}_p(t)$. This is of weight $-2\,\mathrm{Re}(t)$. When t is an integer, the notation is compatible with Tate twists (cf. Definition 1.4.13). The following standard consequences of the theory of weights are important tools in the proof of Langlands correspondence:

Proposition ([La2, VI.3]). *Let Y be a smooth geometrically connected scheme of finite type over k. Let $\mathcal{E} \in \mathcal{I}(Y)$ be mixed of weight $\leq n$, and let $\mathcal{E}'$ be an irreducible object in $\mathcal{I}(Y)$ pure of weight m.*

(i) *The rational function $L_Y(\mathcal{E} \otimes \mathcal{E}'^\vee, Z)$ does not have zeros in the region $|Z| < q^{\frac{m-n}{2} - \dim(Y) + \frac{1}{2}}$.*

(ii) *Let* $t \in \mathbb{C}$ *such that* $\mathrm{Re}(t) = (m-n)/2$*, and assume that the multiplicity of* $\mathcal{E}'(t)$ *in the semisimplification of* $\mathcal{E}$ *is* $\mu \geq 0$*. Then the L-function appearing in* (i) *has a pole of order* μ *at* $Z = q^{t-\dim(Y)}$*. Moreover, it does not have any pole in* $|Z| < q^{\frac{m-n}{2}-\dim(Y)}$*.*

Proof. Use Theorem 2.3.38 and (4.3.2.1) to show (i). We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{I}(Y)}(\mathcal{E},\mathcal{E}'(t)) &\cong \bigl(H^0\bigl(Y,\mathcal{E}^\vee\otimes\mathcal{E}'(t)\bigr)\bigr)^{\mathrm{Frob}_Y}\\ &\cong \bigl(H_{\mathrm{c}}^{2\dim(Y)}\bigl(Y,\mathcal{E}\otimes(\mathcal{E}'(t))^\vee\bigr)(\dim(Y))\bigr)^{\mathrm{Frob}_Y},\end{aligned}$$

where $(-)^{\mathrm{Frob}_Y}$ denotes the fixed part by the action of Frob_Y, and the first isomorphism is by 2.3.14, the second by Theorem 2.3.34. If $\mathcal{E}$ is semisimple, then the action of Frob_Y on $H^0\bigl(Y,\mathcal{E}^\vee\otimes\mathcal{E}'(t)\bigr)$ is semisimple, thus the fixed part of Frob_Y is equal to the generalized eigenspace of Frob_Y with eigenvalue 1, and we get (ii). □

We usually use this proposition in the following form:

Corollary. *Let* Y *be as in the proposition. Let* $\mathcal{E}$ *be in* $\mathrm{Gr}(\mathcal{I}(Y))$*, and let* $\mathcal{E}'$ *be an irreducible object in* $\mathcal{I}(Y)$*. Assume that any component of* $\mathcal{E}$ *is mixed of weight* $\leq n$*. For* $t \in \mathbb{C}$ *such that* $\mathrm{Re}(t) = (m-n)/2$*, the multiplicity of* $\mathcal{E}'(-t)$ *in* $\mathcal{E}$ *is exactly the order of pole of* $L_Y(\mathcal{E}\otimes\mathcal{E}'^\vee, Z)$ *at* $Z = q^{t-\dim(Y)}$*.*

4.3.4. **Definition.** Let Y be a smooth scheme of finite type and geometrically connected over k, and let $U \subset Y\times Y$ such that $(\mathrm{Frob}_Y\times\mathrm{id}_Y)(U)\subset U$. We denote by $\mathbb{Z}\mathcal{I}(U)$ the category of overconvergent $\overline{\mathbb{Q}}_p$-isocrystals on U with Frobenius structure equipped with an isomorphism $(\mathrm{Frob}_Y\times\mathrm{id}_Y)^+\mathcal{E}\xrightarrow{\sim}\mathcal{E}$. Let F be the function field of Y. We put

$$\mathcal{I}(F) := 2\text{-}\varinjlim_{U\subset Y}\mathcal{I}(U), \qquad \mathbb{Z}\mathcal{I}(F^2) := 2\text{-}\varinjlim_{U\subset Y\times Y}\mathbb{Z}\mathcal{I}(U).$$

Remark. Take a geometric point $\overline{y}\in Y(\overline{k})$. Using the notation of 2.4.18, $\mathcal{I}(Y)$ is equivalent to $\mathrm{Rep}_{\overline{\mathbb{Q}}_p}(W^{\mathrm{isoc}}(Y,\overline{y}))$, the category of finite-dimensional representations of the algebraic group $W^{\mathrm{isoc}}(Y,\overline{y})$ over $\overline{\mathbb{Q}}_p$. Similarly, we have $\mathbb{Z}\mathcal{I}(U)\cong\mathrm{Rep}_{\overline{\mathbb{Q}}_p}(\mathbb{Z}W^{\mathrm{isoc}}(U,\overline{y}))$ by 2.4.19. We often ignore the basepoints of W^{isoc} and $\mathbb{Z}W^{\mathrm{isoc}}$.

4.3.5. We preserve the notation, and let $q', q''\colon Y\times Y\to Y$ be the first and second projections, respectively.

Definition ([La2, VI.14])**.** Let $r\geq 1$ be an integer. An object $\mathcal{E}\in\mathcal{I}(F^2)$ (resp. element of $\mathbb{Q}\mathrm{Gr}\mathcal{I}(F^2)$) is said to be *r-negligible* if any of its subquotient (resp. any of its component) is a direct factor of an object of the form $q'^+\mathcal{E}'\otimes q''^+\mathcal{E}''$ where $\mathcal{E}'$ and $\mathcal{E}''$ are objects of rank $< r$ in $\mathcal{I}(F)$. It is said to be *essential* if all the subquotients are not r-negligible. A semisimple r-negligible object of $\mathcal{I}(F^2)$ (resp. an r-negligible element of $\mathbb{Q}\mathrm{Gr}\mathcal{I}(F^2)$) is said to be *complete* if it is a direct sum (resp. sum) of objects of the form $q'^+\mathcal{E}'\otimes q''^+\mathcal{E}''$.

4.3.6. **Lemma.** (i) *Let* $\mathcal{E}'$*,* $\mathcal{E}''$ *be irreducible objects in* $\mathcal{I}(F)$*. Then* $q'^+\mathcal{E}'\otimes q''^+\mathcal{E}''$ *is irreducible as an object in* $\mathbb{Z}\mathcal{I}(F^2)$*.*

(ii) *A semisimple* r*-negligible object, or an* r*-negligible element of* $\mathbb{Q}\mathrm{Gr}\mathcal{I}(F^2)$*,* $\mathcal{E}$ *is complete if it is invariant under the action of* $(\mathrm{Frob}_Y\times\mathrm{id}_Y)^+$*, namely if there* *exists an isomorphism* $(\mathrm{Frob}_Y \times \mathrm{id}_Y)^+(\mathcal{E}) \cong \mathcal{E}$. *In particular, for a semisimple* r*-negligible object* $\mathcal{E}$, $\bigoplus_{n=1}^{r^2!}(\mathrm{Frob}_Y^n \times \mathrm{id}_Y)^+(\mathcal{E})$ *is complete.*

Proof. Let us prove (i). We may assume $\mathcal{E}'$ and $\mathcal{E}''$ are defined on $U \subset Y$. By Lemma 2.4.20, it suffices to show that $q'^{+}\mathcal{E}' \otimes q''^{+}\mathcal{E}''$ is irreducible as an object in $\mathbb{Z}\mathcal{I}(U \times U)$. This follows by Lemma 2.4.19. Let us check (ii). There exists $U \subset Y$ such that $\mathcal{E}$ is an isocrystal on $U \times U$. Since $\mathcal{E}$ is assumed negligible, there exists a complete r-negligible object $\widehat{\mathcal{E}} \in \mathbb{Z}\mathcal{I}(U \times U)$ such that $\mathcal{E} \subset \widehat{\mathcal{E}}$ in $\mathcal{I}(U \times U)$ by definition. Since $\mathcal{E}$ is invariant under the pullback by $\mathrm{Frob}_Y \times \mathrm{id}$, we may assume that $\mathcal{E}$ is invariant under the equipped isomorphism $\alpha\colon (\mathrm{Frob}_Y \times \mathrm{id}_Y)^+(\widehat{\mathcal{E}}) \cong \widehat{\mathcal{E}}$, by changing the inclusion if necessary. Let $\rho_{\widehat{\mathcal{E}}}$ be the corresponding representation of $\mathbb{Z}W^{\mathrm{isoc}}(U \times U)$, and $\rho_{\mathcal{E}}$ be the subrepresentation of $\rho_{\widehat{\mathcal{E}}}$ corresponding to $\mathcal{E}$ defined since $\mathcal{E}$ is invariant under the isomorphism α. Let

$$K := \mathrm{Ker}\big(\mathbb{Z}W^{\mathrm{isoc}}(U \times U) \to W^{\mathrm{isoc}}(U) \times W^{\mathrm{isoc}}(U)\big).$$

Then since $\widehat{\mathcal{E}}$ is assumed complete, $\rho_{\widehat{\mathcal{E}}}(K) = \mathrm{id}$. Thus, $\rho_{\mathcal{E}}(K) = \mathrm{id}$, which implies that $\rho_{\mathcal{E}}$ is the pullback of a representation of $W^{\mathrm{isoc}}(U) \times W^{\mathrm{isoc}}(U)$, and the first claim follows. To check the last claim, we note that, for any irreducible objects $\mathcal{E}'$, $\mathcal{E}''$ in $\mathcal{I}(F)$ of rank r' and r'', respectively, $q'^{+}\mathcal{E}' \otimes q''^{+}\mathcal{E}''$ is semisimple in $\mathcal{I}(F^2)$ by (i), and the number of constituents N of $q'^{+}\mathcal{E}' \otimes q''^{+}\mathcal{E}''$ is $\leq r'r''$. This implies that for any constituent $\mathcal{F}$ of $q'^{+}\mathcal{E}' \otimes q''^{+}\mathcal{E}''$ and any integer $k > 0$,

$$\bigoplus_{n=1}^{kN}(\mathrm{Frob}_Y^n \times \mathrm{id}_Y)^+(\mathcal{F})$$

is invariant by the action of $(\mathrm{Frob}_Y \times \mathrm{id}_Y)^+$. Since $N \leq r'r'' \leq r^2$, N divides $r^2!$, and the claim follows by the first part of (ii). $\square$

Remark. In (ii) of the corollary, we may take $\bigoplus_{n=1}^{r!}$ as in [La2], but for our purposes, $\bigoplus_{n=1}^{r^2!}$ is enough.

4.3.7. From now on, we use the notation of §4.2 freely. In the following, we fix $a \in \mathbb{A}_F^\times$ of degree 1, and a level $N = \mathrm{Spec}(\mathcal{O}_N) \hookrightarrow X$. Let p be a large enough convex function. For $\pi \in \mathcal{A}^r(F)$, we denote by χ_π the central character of π. We define a set by

$$\{\pi\}_N^r := \big\{\pi \in \mathcal{A}^r(F) \mid \chi_\pi(a) = 1 \text{ and } \pi \cdot \mathbf{1}_N \neq 0\big\},$$

where $\mathbf{1}_N$ is the quotient of the characteristic function of $K_N := \mathrm{Ker}\big(\mathrm{GL}_r(\mathbb{A}_F) \to \mathrm{GL}_r(\mathcal{O}_N)\big)$ by its volume. It suffices to construct isocrystals corresponding to the cuspidal representations belonging to $\{\pi\}_N^r$. We put $S_N := (X - N) \times (X - N)$.

Let $q', q''\colon X \times X \to X$ be the first and second projection, respectively. For a morphism of c-admissible stacks $f\colon \mathfrak{X} \to S_N$, we denote the relative cohomology ${}^{\mathrm{c}}\mathscr{H}^\nu f_!\overline{\mathbb{Q}}_{p,\mathfrak{X},F}$ by $\mathcal{H}_{\mathrm{c}}^\nu(\mathfrak{X})$, where $\overline{\mathbb{Q}}_{p,\mathfrak{X},F}$ denotes the unit object in $\mathrm{Con}(\mathfrak{X})$. This is an object in $\mathrm{Con}(S_N)$. Assume f is proper. There exists an open dense substack $j\colon \mathfrak{U} \hookrightarrow \mathfrak{X}$ such that $\overline{\mathbb{Q}}_{p,\mathfrak{U},F}$ is pure of weight 0. Then we denote by $\mathcal{IH}^\nu(\mathfrak{X}) := {}^{\mathrm{c}}\mathscr{H}^\nu f_+ j_{!+}\overline{\mathbb{Q}}_{p,\mathfrak{U},F}$, which is pure of weight ν.

We also use ℓ-adic cohomologies. We denote by $\mathcal{H}_{\mathrm{c}}^\nu(\mathfrak{X}, \overline{\mathbb{Q}}_\ell) := \mathscr{H}^\nu f_!\overline{\mathbb{Q}}_\ell$ and $\mathcal{IH}^\nu(\mathfrak{X}, \overline{\mathbb{Q}}_\ell) := \mathscr{H}^\nu f_+ j_{!+}\overline{\mathbb{Q}}_\ell$, where $\mathscr{H}^\nu$ denotes the standard (constructible) t-structure. In most cases in this paper, these ℓ-sheaves are smooth, namely an object of $\mathcal{W}_\ell(S_N)$ (cf. [La2, p.165]).

4.3.8. *We start the proof of the theorem from here.* We prove a slightly stronger statement than the theorem. For an integer $n \geq 1$, we call the following two statements $(\mathrm{S})_n$:

(1) Theorem 4.2.2 is true for $r, r' \leq n$. As a consequence, we have $\mathcal{I}_{n+1} \to \mathcal{A}_{n+1}$ as well by Theorem 4.2.3.

(2) For $r' \leq n$, the constructible object $\mathcal{H}^{\nu}_{\mathrm{c}}(\mathrm{Cht}^{r',\overline{p}\leq p}_N/a^{\mathbb{Z}})$ is smooth and $(n+1)$-negligible for any ν, level $N\colon \mathrm{Spec}(\mathcal{O}_N) \hookrightarrow X$, $a \in \mathbb{A}^{\times}_F$ such that $\deg(a) = 1$, and convex polygon p large enough with respect to X and N.

We know that $(\mathrm{S})_1$ holds. Indeed, (1) is nothing but the class field theory together with the theorem of Tsuzuki [T, Thm 7.1.1]; see [A2, 6.5] for more details. Let us check (2). We know that $f\colon \mathrm{Cht}^{1,\overline{p}\leq p}_N = \mathrm{Cht}^1_N/a^{\mathbb{Z}} \to S_N$ is an abelian covering and $f_!\overline{\mathbb{Q}}_\ell = \bigoplus q'^*\chi' \otimes q''^*\chi''$, where χ' and χ'' are smooth sheaf on $X - N$ of rank 1 by [La2, remark of VI.15]. Using (1), let χ'_p and χ''_p be the corresponding isocrystals of rank 1 in $\mathcal{I}(X - N)$. By Proposition 2.3.22 (or Theorem A.5.4 if one prefers), $f_!\overline{\mathbb{Q}}_\ell$ and $f_!\overline{\mathbb{Q}}_p$ have the same Frobenius eigenvalues. Thus, by Čebotarev density A.4.1, we get $f_!\overline{\mathbb{Q}}_p \cong \bigoplus q'^+\chi'_p \otimes q''^+\chi'_p$.

In the following, we fix an integer $r > 1$, and assume that $(\mathrm{S})_{r-1}$ holds. Our goal is to show $(\mathrm{S})_r$ under this assumption, which is attained at the very end of this subsection.

4.3.9. **Definition.** Let k' be the extension of k of degree $d_0 \geq 1$. For a smooth scheme U over k', we put $\mathcal{I}_{d_0}(U) := \mathrm{Isoc}^{\dagger}(U/\mathfrak{T}_{k'})$ (cf. 4.2.1 for the notation of base tuple). We denote by F_{d_0} and $F^2_{d_0}$ the function fields of $X \otimes_k k'$ and $(X \times X) \otimes_k k'$, and we define $\mathcal{I}_{d_0}(F_{d_0})$, $\mathcal{I}_{d_0}(F^2_{d_0})$ accordingly. An irreducible object in $\mathcal{I}(F)$ is said to be *r-negligible* if it is of rank $< r$. An irreducible object $\mathcal{E}$ in $\mathcal{I}_{d_0}(F_{d_0})$ (resp. $\mathcal{I}_{d_0}(F^2_{d_0})$) is said to be *$r$-negligible* if there exists an irreducible r-negligible object $\mathcal{E}'$ in $\mathcal{I}(F)$ (resp. $\mathcal{I}(F^2)$) such that the pullback $\mathcal{E}' \otimes k'$ contains $\mathcal{E}$. Sums of r-negligible objects are said to be r-negligible as well.

4.3.10. **Lemma** ([La2, VI.16]). *Let $d_0 \geq 1$ be an integer. An irreducible object $\mathcal{E}$ in $\mathcal{I}(F^2)$ (resp. $\mathcal{I}(F)$) is r-negligible if $\mathcal{E} \otimes k'$ contains an r-negligible object in $\mathcal{I}_{d_0}(F^2_{d_0})$ (resp. $\mathcal{I}_{d_0}(F_{d_0}0)$).*

Proof. Let $\varphi \in \mathrm{Gal}(k'/k)$ be a generator, and let $\varphi^*\colon U \otimes k' \to U \otimes k'$ be the automorphism *over* k induced by φ for some smooth scheme U over k. Giving an object in $\mathcal{I}_{d_0}(U \otimes k')$ is equivalent to giving $\mathcal{F} \in \mathcal{I}(U \otimes k')$ with isomorphism $\varphi^*(\mathcal{F}) \cong \mathcal{F}$. This observation implies that if $\mathcal{E}, \mathcal{E}' \in \mathcal{I}(F^2)$ (resp. $\mathcal{I}(F)$) are irreducible objects such that $\mathcal{E} \otimes k' \cong \mathcal{E}' \otimes k'$, then there exists a character χ of $\mathrm{Gal}(k'/k) \cong \mathbb{Z}/d_0\mathbb{Z}$ (which can be seen as a rank 1 object of $\mathcal{I}(\mathrm{Spec}(k))$) such that $\mathcal{E} \otimes \chi \cong \mathcal{E}'$, thus the lemma follows. □

4.3.11. We need to show the following technical proposition. In the statement and the proof, the algebraic stack $\mathfrak{C}^{r,N}$ and its variants[11] are used. We remark that these stacks are used only in this proposition and its corollary in 4.3.12. In [La2, III 3a)], Lafforgue defined a morphism $\mathrm{Res}\colon \overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}} \times_{X \times X} S_N \to \mathfrak{C}^{r,N}$ between algebraic stacks locally of finite type over k. We remind the reader that all three stacks in the table of 4.2.4 are defined over the source of Res. The open substack $\mathfrak{C}^{r,N}_{\emptyset}$ of $\mathfrak{C}^{r,N}$

[11] In [La2], Lafforgue uses script fonts (e.g., $\mathcal{C}^{r,N}$).

is defined in [La2] as well, and the complement $\mathfrak{C}^{r,N} - \mathfrak{C}^{r,N}_{\emptyset}$ is called the *boundary.* See Step 3 of the proof of the following proposition for some review of these stacks.

Proposition ([La2, VI.17])**.** *Let p be a convex polygon large enough with respect to X, N, and an integer $1 \leq d \leq r$. Let $\mathfrak{C}$ be an algebraic stack representable and quasi-projective* (*cf.* [LM, 14.3.4]) *over $\mathfrak{C}^{r,N}$, and consider the following cartesian diagram:*

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "\mathrm{Res}"] \arrow[d, "g'"'] & & \mathfrak{C} \arrow[d, "g"] \\ \overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}} \times_{X\times X} S_N \arrow[rr, "\mathrm{Res}"'] \arrow[urr, phantom, "\square"] & & \mathfrak{C}^{r,N}. \end{tikzcd}$$

We denote by $p_{\mathfrak{X}} \colon \mathfrak{X} \to S_N$ the projection.

(i) *Let $\mathscr{M} \in \mathrm{Con}(\mathfrak{C})$. Then ${}^{\mathrm{c}}\mathscr{H}^{\nu} p_{\mathfrak{X}!}\bigl(\mathrm{Res}^{+}\mathscr{M}\bigr)$ is smooth on S_N for any ν.*

(ii) *Assume, moreover, that $\mathscr{M}$ is supported on the boundary of $\mathfrak{C}$, namely $g^{-1}(\mathfrak{C}^{r,N} - \mathfrak{C}^{r,N}_{\emptyset})$. Then ${}^{\mathrm{c}}\mathscr{H}^{\nu} p_{\mathfrak{X}!}\bigl(\mathrm{Res}^{+}\mathscr{M}\bigr)$ is r-negligible as an object in $\mathcal{I}(F^2)$ for any ν.*

Proof. Before beginning the proof, let us remark that $\mathfrak{X}$ is c-admissible since it is quasi-projective over the proper admissible stack $\overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}}$, thus we are able to use full six functor formalism. However, $\mathfrak{C}$ is locally of finite type but not even admissible. Currently, we only have partial formalism in such a situation, so we need to be slightly careful. Nevertheless, most of the cohomological functors are available on the level of the category $\mathrm{Con}(-)$. See the first part of §2.4 for the details. The proof is divided into several steps.

Step 1 (Proof of (i) and the first reduction of (ii))**.** The morphism $(p_{\mathfrak{X}}, \mathrm{Res}) \colon \mathfrak{X} \to S_N \times \mathfrak{C}$ is known to be smooth by [La2, Prop III.7 (ii)], which implies that Res is smooth as well since S_N is smooth. Using Lemma 2.4.5, we have

$${}^{\mathrm{c}}\mathscr{H}^{\nu} g'_{!}\mathrm{Res}^{+}(\mathscr{M}) \cong \mathrm{Res}^{+} g^{\nu}_{!}(\mathscr{M}).$$

Since $p_{\mathfrak{X}!} \cong \mathrm{pr}_{2!} \circ g'_{!}$, for $\mathscr{N} \in D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X})$, ${}^{\mathrm{c}}\mathscr{H}^{\nu} p_{\mathfrak{X}!}(\mathscr{N})$ can be expressed as extensions of subquotients of $\bigoplus_{i+j=\nu} {}^{\mathrm{c}}\mathscr{H}^{i}\mathrm{pr}_{2!}\, {}^{\mathrm{c}}\mathscr{H}^{j} g'_{!}(\mathscr{N})$. As a consequence, it suffices to show (i) and (ii) for $\mathfrak{C} = \mathfrak{C}^{r,N}$. In this case, (i) is an easy consequence of Lemma 2.4.13: the algebraic stack $\overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}} \times_{X\times X} S_N$ is admissible by [La2, Prop V.I], proper over S_N by [La2, Prop III.7 (i)], and we already recalled that $(p_{\mathfrak{X}}, \mathrm{Res})$ is smooth, so the lemma is applicable. We concentrate on proving (ii) from now on. In the following, we initialize the notation $\mathfrak{C}$ and use it for other stacks.

Step 2 (Induction hypothesis)**.** Let $c_0 := \sup\bigl\{\dim(I_x) \mid x \in |\mathrm{Res}(\mathfrak{X})|\bigr\}$, where I_x is the inertia algebraic group space of $\mathfrak{C}^{r,N}$ at x. By the quasi-compactness of $\mathfrak{X}$, we have $c_0 > -\infty$, and the dimension of any locally closed substack of $\mathfrak{C}^{r,N}$ in the image of Res can be bounded below by $-c_0$. We use induction on $k := c_0 + \dim(\mathrm{Supp}(\mathscr{M}))\,(\geq 0)$ (cf. 2.1.17 for the definition of support). Assume that the proposition holds for $k = k_0 \geq -1$. We will show the proposition for modules whose support is of dimension $k = k_0 + 1$. Now, we frequently use the following reduction, which follows by using the localization exact sequence Lemma 2.2.9 and the induction hypothesis:

> (♣) Let $j\colon \mathfrak{U} \subset \mathfrak{S} := \mathrm{Supp}(\mathscr{M})$ be an open substack such that $\dim(\mathfrak{S} \setminus \mathfrak{U}) < \dim(\mathfrak{S}) = k_0 + 1$. Let $\mathscr{M}' \in \mathrm{Con}(\mathfrak{S})$ such that $j^+\mathscr{M} \cong j^+\mathscr{M}'$. Then proving the proposition for $\mathscr{M}$ and $\mathscr{M}'$ are equivalent.

Step 3 (Recall of geometry of moduli spaces)**.** Now, we put aside cohomology theory for a while and summarize the complicated geometry of moduli spaces in [La2] very briefly. Let $\mathfrak{A}^{r,1} := \mathbb{A}^{r-1}$ be the toric variety, and let $\mathfrak{A}^{r,1}_{\emptyset}$ be the torus $\mathbb{G}_m^{r-1}$. Let $\underline{r} = (r_1, \dots, r_k)$ be a partition $r_1 + \cdots + r_k = r$. We define $\mathfrak{A}^{r,1}_{\underline{r}}$ to be the locally closed subscheme of $\mathfrak{A}^{r,1}$ consisting of points whose coordinates indexed by $r_1 + \cdots + r_e$ for $1 \le e < k$ are zero and invertible for other coordinates. These are orbits of $\mathfrak{A}^{r,1}_{\emptyset}$; see [La2, III 1a)] for more information on these schemes. Now, in [La2, III 2a), 3a)], the sequence of algebraic stacks locally of finite type $\mathfrak{C}^{r,N} \subset \widetilde{\mathfrak{C}}^{r,N} \subset \overline{\mathfrak{C}}^{r,N}$ and the morphism $\overline{\mathfrak{C}}^{r,N} \to \mathfrak{A}^{r,1}/\mathfrak{A}^{r,1}_{\emptyset}$ are defined. For a partition $\underline{r}$, the pullbacks of $\mathfrak{A}^{r,1}_{\underline{r}}/\mathfrak{A}^{r,1}_{\emptyset}$ to the three stacks are denoted by $\mathfrak{C}^{r,N}_{\underline{r}} \subset \widetilde{\mathfrak{C}}^{r,N}_{\underline{r}} \subset \overline{\mathfrak{C}}^{r,N}_{\underline{r}}$, and similarly for $\mathfrak{A}^{r,1}_{\emptyset}/\mathfrak{A}^{r,1}_{\emptyset}$. These are stratifications of the boundary.

Let $\mathbb{A}^N$ and $\mathbb{G}_m^N$ be schemes induced from $\mathbb{A}^1$ and $\mathbb{G}_m$ by the Weil restriction from $\mathcal{O}_N$ to k. We also use a variant ${}^r\overline{\mathfrak{C}}^N_{\emptyset}$, and both ${}^r\overline{\mathfrak{C}}^N_{\emptyset}$ and $\overline{\mathfrak{C}}^{r,N}_{\emptyset}$ have natural morphisms to $(\mathbb{A}^N/\mathbb{G}_m^N)^2$ (cf. [La2, II 1a)]). We have the finite surjective radicial morphism $\mathrm{sw}\colon {}^r\overline{\mathfrak{C}}^N_{\emptyset} \to \overline{\mathfrak{C}}^{r,N}_{\emptyset}$ over the endomorphism $\mathrm{id} \times \mathrm{Frob}$ of $(\mathbb{A}^N/\mathbb{G}_m^N)^2$ sending $(\mathcal{F} \leftarrow \mathcal{F}' \to {}^\tau\mathcal{F})$ to $(\mathcal{F}' \to {}^\tau\mathcal{F} \leftarrow {}^\tau\mathcal{F}')$ using the notation of [La2].

We have the morphism $\overline{\mathrm{Cht}^{r,d,\overline{p}\le p}} \to \overline{\mathfrak{C}}^{r,N}$, which factorizes through $\widetilde{\mathfrak{C}}^{r,N}$ (cf. [La2, III 3a)]). The morphism Res in the statement of the proposition is induced by this. The pullback of $\mathfrak{A}^{r,1}_{\underline{r}}$ is denoted by $\mathrm{Cht}^{r,d,\overline{p}\le p}_{\underline{r}}$ following the notation of [La2, III 1c)].

Step 4 (Geometric construction of Lafforgue)**.** Consider the following commutative diagram of solid arrows (which appears in [La2, right before III.6]):

$$\begin{tikzcd}
\mathrm{Cht}^{r,d,\overline{p}\le p}_{\underline{r}} \arrow[r, "\beta"] \arrow[d] & \widetilde{\mathfrak{C}}^{r,N}_{\underline{r}} \arrow[d, "\gamma"'] & \mathfrak{D} \arrow[l, hook', dotted] \arrow[dl, dotted, "\gamma_{\mathfrak{D}}"] \\
\mathrm{Cht}^{\underline{r},d,\overline{p}\le p} \arrow[r] & \overline{\mathfrak{C}}^{\underline{r},N} &
\end{tikzcd}$$

$$(\overline{\mathfrak{C}}^{\underline{r},N} := \overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} {}^{r_2}\overline{\mathfrak{C}}^N_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \cdots \times_{\mathbb{A}^N/\mathbb{G}_m^N} {}^{r_k}\overline{\mathfrak{C}}^N_{\emptyset}),$$

where γ is defined in [La2, III 2b)], $\mathrm{Cht}^{\underline{r},d,\overline{p}\le p}$ is defined in [La2, Prop III.3]. Lafforgue proved the following two claims in the proof of [La2, VI.17]:

(i) Assume that we are given a locally closed substack $\mathfrak{D}$ of $\mathrm{Im}(\beta)$ in $\widetilde{\mathfrak{C}}^{r,N}_{\underline{r}}$. Then there exists an open dense substack $\mathfrak{D}'$ of $\mathfrak{D}$ such that $\gamma_{\mathfrak{D}'}$ can be written as the composition of the morphisms of the following three types: 1. a gerb-like morphism whose structural group is flat fiberwise geometrically connected; 2. a finite flat radicial morphism; 3. a finite étale morphism.

(ii) Consider the compositions

$$(\star)\qquad \mathfrak{C}^{r,N}_{\underline{r}} \xrightarrow{\gamma} \overline{\mathfrak{C}}^{\underline{r},N} \xrightarrow{\mathrm{pr}_1} \overline{\mathfrak{C}}^{r_1,N}_{\emptyset}, \qquad \mathfrak{C}^{r,N}_{\underline{r}} \xrightarrow{\gamma} \overline{\mathfrak{C}}^{\underline{r},N} \xrightarrow{\mathrm{pr}_k} {}^{r_k}\overline{\mathfrak{C}}^{N}_{\emptyset} \xrightarrow{\mathrm{sw}} \overline{\mathfrak{C}}^{r_k,N}_{\emptyset}.$$

The images of these morphisms consist of finitely many points.

A proof of (i) is written at the end of [La2, p.171], and that for (ii) is in the second paragraph of [La2, p.172]. Let us add a little explanation on the proof of (i). By dimension counting, Lafforgue proves that the preimage of a point of $\mathrm{Im}(\gamma\circ\beta)$ by γ consists of finitely many points. To conclude, use Lemma A.2.4.

Step 5 (Cohomological reduction)**.** Now, let us come back to the cohomologies. Let $j\colon \mathfrak{C} \hookrightarrow \mathfrak{S} := \mathrm{Supp}(\mathscr{M})\,(\subset \mathfrak{C}^{r,N})$ be a dense open immersion. By (♣) of Step 2, we may replace $\mathscr{M}$ by $j_!j^+(\mathscr{M})$ (cf. see 2.4.2 for the definition of functors $j_!$ and j^+ in this situation). By shrinking $\mathfrak{C}$ using (♣), we may assume $\mathfrak{C}$ is of dimension k_0+1 in the stratum $\mathfrak{C}^{r,N}_{\underline{r}}$ for some nontrivial partition $\underline{r}$ such that $\mathscr{M}' := j^+\mathscr{M}$ is smooth. Using (i) above and shrinking $\mathfrak{C}$ always using (♣), we may assume $\gamma' := \gamma_{\mathfrak{C}}$ is the composition of morphisms of the three types. We may change $\mathscr{M}'$ by a smooth object which contains $\mathscr{M}'$ as a direct factor, so by using Lemma 2.4.8, Corollary 2.4.6, and Lemma 2.2.4 (ii), we may assume that there exists a smooth object $\mathscr{N}'$ on a locally closed substack of $\overline{\mathfrak{C}}^{\underline{r},N}$ such that $\mathscr{M}' = \gamma'^+(\mathscr{N}')$. By (ii), shrinking $\mathfrak{C}$ if needed, we may assume that the images of $\mathfrak{C}$ by the two morphisms of $(\star)$ consist of unique locally closed points $\widetilde{\mathfrak{B}}_\star \cong \Big[\mathrm{Spec}(\mathbb{F}_{q_0})/\mathrm{Aut}(\widetilde{\mathfrak{B}}_\star)\Big] \in \overline{\mathfrak{C}}^{r_\star,N}_{\emptyset}$, where $\star = 1$ or k, and $\mathbb{F}_{q_0}$ is the field extension of $\mathbb{F}_q$ of degree d_0. We define the Galois coverings $\widetilde{\mathfrak{B}}'_\star$ of $\widetilde{\mathfrak{B}}_\star$ defined by the *discrete part* $\mathrm{Aut}(\widetilde{\mathfrak{B}}_\star)/(\mathrm{Aut}(\widetilde{\mathfrak{B}}_\star))^\circ$ of $\mathrm{Aut}(\widetilde{\mathfrak{B}}_\star)$. We denote by $\alpha\colon \mathfrak{C}' \to \mathfrak{C}$ the Galois covering induced by $\widetilde{\mathfrak{B}}'_1 \times \widetilde{\mathfrak{B}}'_k \to \widetilde{\mathfrak{B}}_1 \times \widetilde{\mathfrak{B}}_k$. We may replace $\mathscr{M}'$ by $\alpha_+\alpha^+\mathscr{M}'$.

Now, consider the finite surjective radicial morphism

$(\star\star)$

$$\mathrm{id}\times\mathrm{sw}\times\cdots\times\mathrm{sw}\colon \overline{\mathfrak{C}}^{\underline{r},N} \to \overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \overline{\mathfrak{C}}^{r_2,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \cdots \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \overline{\mathfrak{C}}^{r_k,N}_{\emptyset}.$$

We may and do identify the holonomic modules on these stacks by Lemma 2.2.4. By considering Lemma 2.4.8, we may assume that $\mathscr{N}' \cong \mathscr{M}_1 \boxtimes_{\mathbb{A}^N/\mathbb{G}_m^N} \mathscr{M}'' \boxtimes_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \mathscr{M}_k$ (cf. 2.4.4 for the notation) where

- $\mathscr{M}_\star$ $(\star = 1, k)$ is the pushforward of the trivial smooth object on $\widetilde{\mathfrak{B}}'_\star$ by the finite étale covering $\widetilde{\mathfrak{B}}'_i \to \widetilde{\mathfrak{B}}_i$.
- $\mathscr{M}''$ is a smooth object on a locally closed substack of $\overline{\mathfrak{C}}^{r_2,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \cdots \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \overline{\mathfrak{C}}^{r_{k-1},N}_{\emptyset}$.

Step 6. We denote by pr_i the i-th projection, and put

$$X^2 := X\times X, \qquad \mathrm{Cht}^\star := \mathrm{Cht}^{r_\star,d_\star,\overline{p}\le p_\star},$$
$$(\mathrm{Cht})' := \mathrm{Cht}^{r_2,d_2,\overline{p}\le p_2} \times_{X,\mathrm{Frob}} \cdots \times_{X,\mathrm{Frob}} \mathrm{Cht}^{r_{k-1},d_{k-1},\overline{p}\le p_{k-1}},$$
$$(\overline{\mathfrak{C}}^{\underline{r}',N}_{\emptyset})' := \overline{\mathfrak{C}}^{r_2,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \cdots \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \overline{\mathfrak{C}}^{r_{k-1},N}_{\emptyset}.$$

Consider the following commutative diagram.

$$
\begin{tikzcd}
\overline{\mathrm{Cht}^{r,d,\overline{p}\leq p}} \arrow[d, "\mathrm{Res}"'] \arrow[dr, phantom, "\square"] & \mathrm{Cht}_{\underline{r}}^{r,d,\overline{p}\leq p} \arrow[l, hook'] \arrow[r, "g"] \arrow[d, "\beta"'] & \mathrm{Cht}^1 \times_X (\mathrm{Cht})' \times_{X,\mathrm{Frob}} \mathrm{Cht}^k \arrow[d, "\beta'"'] \arrow[dd, bend left, "q=q_1\times q''\times q_k"] \\
\overline{\mathfrak{C}}^{r,N} & \widetilde{\mathfrak{C}}^{r,N}_{\underline{r}} \arrow[l] \arrow[r, "g'"] & \overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} (\overline{\mathfrak{C}}^{\underline{r}',N}_{\emptyset})' \times_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \overline{\mathfrak{C}}^{r_k,N}_{\emptyset} \\
 & X^2\times X^2 & X^2 \times_X X^2 \times_{X,\mathrm{Frob}} X^2 \arrow[l, "r=\mathrm{pr}_1\times\mathrm{pr}_3"', "\sim"]
\end{tikzcd}
$$

Moreover, g' is the composition of γ and $(\star\star)$, and the morphism g is the composition of a gerb-like morphism whose structural group is flat finite radicial and a representable universal homeomorphism defined in [La2, Cor III.4]. Thus, we get $g_!g^+ \cong \mathrm{id}$ by Lemma 2.2.4 (i) and Corollary 2.4.6. Put $p' := r\circ q\circ g$, and we can compute

$$
\begin{aligned}
(\star\star\star)\quad {}^{\mathrm{c}}\mathscr{H}^\nu p'_!\beta^+\mathscr{M} &\cong {}^{\mathrm{c}}\mathscr{H}^\nu p'_! g^+\beta'^+(\mathscr{N}') \\
&\cong {}^{\mathrm{c}}\mathscr{H}^\nu r_! q_!\,\beta'^+\big(\mathscr{M}_1 \boxtimes_{\mathbb{A}^N/\mathbb{G}_m^N} \mathscr{M}'' \boxtimes_{\mathbb{A}^N/\mathbb{G}_m^N,\mathrm{Frob}} \mathscr{M}_k\big) \\
&\cong {}^{\mathrm{c}}\mathscr{H}^\nu r_! q_!\big(\mathrm{Res}^+(\mathscr{M}_1) \boxtimes_X \mathrm{Res}^+(\mathscr{M}'') \boxtimes_{X,\mathrm{Frob}} \mathrm{Res}^+\mathscr{M}_k\big) \\
&\cong {}^{\mathrm{c}}\mathscr{H}^\nu r_!\big(q_{1!}\mathrm{Res}^+(\mathscr{M}_1) \boxtimes_X q''_!\mathrm{Res}^+(\mathscr{M}'') \boxtimes_{X,\mathrm{Frob}} q_{k!}\mathrm{Res}^+\mathscr{M}_k\big) \\
&\qquad\qquad (\because \text{by Künneth 2.3.36}) \\
&\cong {}^{\mathrm{c}}\mathscr{H}^\nu\big((q_{1!}\mathrm{Res}^+(\mathscr{M}_1) \boxtimes q_{k!}\mathrm{Res}^+\mathscr{M}_k) \otimes r'^+(q''_!\mathrm{Res}^+(\mathscr{M}''))\big),
\end{aligned}
$$

where we identify $\mathscr{N}'$ and its zero extension, and similarly for other objects to simplify the notation. Moreover, r' is the composition

$$
(X\times X)\times(X\times X) \xrightarrow{\mathrm{pr}_2\times\mathrm{pr}_3} X\times X \underset{\sim}{\xleftarrow{\mathrm{pr}_1\times\mathrm{pr}_4}} X\times_X (X\times X)\times_{X,\mathrm{Frob}} X \xrightarrow{\mathrm{pr}_2\times\mathrm{pr}_3} (X\times X).
$$

Step 7. Let $\star = 1$ or k. Let $f_\star\colon \mathrm{Cht}^{r_\star,d_\star,\overline{p}\leq p_\star} \times_{\overline{\mathfrak{C}}^{r_\star,N}_{\emptyset}} \widetilde{\mathfrak{B}}'_\star \to X\times X$ be the canonical morphism. By definition [La2, II 1a)], the image of this morphism is contained in $X\times(X-N)\cup(X-N)\times X$. We say that $\widetilde{\mathfrak{B}}_1$ *hits a zero* (resp. $\widetilde{\mathfrak{B}}_k$ *hits a pole*) if the image of f_1 (resp. f_k) is contained in $(X-N)\times\{0\}$ for some $0\in N$ (resp. $\{\infty\}\times(X-N)$ for some $\infty\in N$), and it *does not hit a zero* (resp. *does not hit a pole*) otherwise. This is reduced to the following two claims:

- If $\widetilde{\mathfrak{B}}_1$ hits a zero (resp. does not hit a zero), then the relative cohomology $\mathscr{H}^\nu(\mathrm{pr}_1\circ f_1)_!\overline{\mathbb{Q}}_p$ (resp. $\mathscr{H}^\nu f_{1!}\overline{\mathbb{Q}}_p$) over the generic point of X (resp. $X\times X$) is r-negligible in $\mathcal{I}_{d_0}(F_{d_0})$ (resp. $\mathcal{I}_{d_0}(F^2_{d_0})$).
- If $\widetilde{\mathfrak{B}}_k$ hits a pole (resp. does not hit a pole), then the relative cohomology $\mathscr{H}^\nu(\mathrm{pr}_2\circ f_k)_!\overline{\mathbb{Q}}_p$ (resp. $\mathscr{H}^\nu f_{k!}\overline{\mathbb{Q}}_p$) over the generic point of X (resp. $X\times X$) is r-negligible in $\mathcal{I}_{d_0}(F_{d_0})$ (resp. $\mathcal{I}_{d_0}(F^2_{d_0})$).

Indeed, if these hold, any constituent of $q_{\star!}\mathrm{Res}^+(\mathscr{M}_\star)$ is a direct factor of an object of the form $\mathscr{E}_\star\boxtimes\mathscr{F}_\star$ such that $\mathscr{E}_\star$ and $\mathscr{F}_\star$ are irreducible of rank $< r$ or supported on a point. Since $p_{\mathfrak{X}!}\mathrm{Res}^+\mathscr{M} \cong (\mathrm{pr}_1\times\mathrm{pr}_4)_!p'_!\beta^+\mathscr{M}$, using $(\star\star\star)$ and the Künneth formula again, we see that any constituent of $\mathscr{H}^\nu p_{\mathfrak{X}!}\mathrm{Res}^+\mathscr{M}$ is a direct factor of

$$(\mathscr{E}_1 \boxtimes \mathscr{F}_k) \otimes H^\nu\bigl(X \times X, r'^+ q''_! \mathrm{Res}^+(\mathscr{M}'') \otimes (\mathscr{F}_1 \boxtimes \mathscr{E}_k)\bigr).$$

Thus, by considering Lemma 4.3.10, the proposition follows.

Step 8 (Use of induction hypothesis)**.** Let us show the claims in Step 7. We only consider the $\star = 1$ case. When $\widetilde{\mathfrak{B}}_1$ does not hit a zero, $\mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1} \times_{\overline{\mathfrak{C}}^{r_1,N}_{\emptyset}} \widetilde{\mathfrak{B}}'_1$ and $\mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1}_N \otimes_{\mathbb{F}_q} \mathbb{F}_{q_0}$ are isomorphic as written in [La2, p.173]. Since $r_1 < r$, by the induction hypothesis 4.3.8 (2), we get the desired r-negligibility. Let us treat the case where $\widetilde{\mathfrak{B}}_1$ hits a zero $\{0\} \in N$ defined over $\mathbb{F}_{q_0}$. We put $\mathfrak{X}_1^{d_1} := \overline{\mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1}} \times_{X\times X} \bigl((X-N)\times\{0\}\bigr)$. Let $(p_{\mathfrak{X}_1^{d_1}}, \mathrm{Res}_1)\colon \mathfrak{X}_1^{d_1} \to (X-N) \times \bigl(\overline{\mathfrak{C}}^{r_1,N} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}\bigr)$ be the canonical morphism which is smooth by [La2, III.5]. Since $\overline{\mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1}} \times_{\overline{\mathfrak{C}}^{r_1,N}} \overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \cong \mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1}$ by [La2, III.2], we have the following cartesian diagram:

$$\begin{tikzcd}
\mathrm{Cht}^{r_1,d_1,\overline{p}\leq p_1} \times_{\overline{\mathfrak{C}}^{r_1,N}_{\emptyset}} \widetilde{\mathfrak{B}}'_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathfrak{B}}'_1 \arrow[d] \\
\mathfrak{X}_1^{d_1} \arrow[r, "\mathrm{Res}_1"] & \overline{\mathfrak{C}}^{r_1,N} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}.
\end{tikzcd}$$

Step 9. This is now reduced to the following claim:

Claim ([La2, VI.18])**.** For any constructible object $\mathscr{M}_1$ on $\overline{\mathfrak{C}}^{r_1,N} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}$, the relative cohomology $\mathscr{H}^\nu p_{\mathfrak{X}_1^{d_1}} \mathrm{Res}_1^+ \mathscr{M}_1$ is a smooth object and is r-negligible as an object in $\mathcal{I}_{d_0}(F_{d_0})$ for any ν.

Proof. Since $(p_{\mathfrak{X}_1^{d_1}}, \mathrm{Res}_1)$ is smooth, the relative cohomology is smooth by Lemma 2.4.13. We show the r-negligibility by the induction on r_1 (called the *rank*) and the dimension of the support of $\mathscr{M}_1$. Assume that the result is known for rank $< r_1$ and for the dimension of the support being $< c$. It suffices to show the claim for $\mathscr{M}_1$ which is the zero extension of a smooth object defined on a locally closed substack $\mathfrak{C}$ in $\overline{\mathfrak{C}}^{r_1,N} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}$ of dimension c. By the induction hypothesis, we may assume that $\mathfrak{C}$ is in the stratification associated to a partition $\underline{r}_1$ of r_1. When $\underline{r}_1$ is nontrivial, we can reduce this to the claim for smaller ranks by arguments similar to Steps 1–8. Thus, this case is true by the induction hypothesis, and we only need to treat the case where $\underline{r}_1$ is trivial. In this case, we may assume that $\mathfrak{C}$ is a locally closed point $\widetilde{\mathfrak{B}}_1$ of $\overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}$. Denote by $\widetilde{\mathfrak{B}}'_1$ the finite étale covering of $\widetilde{\mathfrak{B}}_1$ constructed as before, and let $\widetilde{\mathfrak{B}}'_1 \xrightarrow{\alpha} \widetilde{\mathfrak{B}}_1 \xrightarrow{j} \overline{\mathfrak{C}}^{r_1,N}_{\emptyset} \times_{\mathbb{A}^N/\mathbb{G}_m^N} \{0\}$. Recalling the intermediate extension from 2.2.8, put $\mathscr{M}'_1 := j_{!+}\alpha_+(\overline{\mathbb{Q}}_p)$, which is pure of weight 0 by the result of the same paragraph. It suffices to show the claim for $\mathscr{M}'_1$ by the induction hypothesis. Since Res_1 is smooth and $p_{\mathfrak{X}_1^{d_1}}$ is proper, ${}^{\mathrm{c}}\mathscr{H}^\nu p_{\mathfrak{X}_1^{d_1}!} \mathrm{Res}_1^+(\mathscr{M}'_1)$ is pure of weight ν by Theorem 2.3.38. Thus, it suffices to show that the alternating sum

$$(\star) \qquad \sum (-1)^\nu \left[{}^{\mathrm{c}}\mathscr{H}^\nu p_{\mathfrak{X}_1^{d_1}!} \mathrm{Res}_1^+(\mathscr{M}'_1)\right]^{\mathrm{ss}}$$

as an object in $\mathrm{Gr}\mathcal{I}_{d_0}(F_{d_0})$ is r-negligible. Let $\mathcal{F}'_1 := j_{!+}\alpha_+(\overline{\mathbb{Q}}_\ell)$, the ℓ-adic counterpart. Since $\mathfrak{X}_1$ is smooth over $(X-N)\otimes_{\mathbb{F}_q}\{0\}$ and admissible by [La2, II.4], we can use Theorem 3.2.1 to see that the Frobenius eigenvalues of each point of $(X-N)\otimes 0$ coincide with those of

$$\sum(-1)^\nu\left[\mathscr{H}^\nu p_{\mathfrak{X}_1^{d_1}!}\mathrm{Res}_1^*(\mathcal{F}'_1)\right]^{\mathrm{ss}}.$$

This sum is known to be r-negligible exactly by the corresponding claim for the ℓ-adic situation, namely [La2, VI.18]. Thus, there exists a finite collection $\{\sigma'_\iota\}$ in $\mathcal{W}_\ell((X-N)\otimes\{0\})$ such that σ'_ι is a constituent of the pullback of an irreducible object $\sigma_\iota\in\mathcal{W}_\ell(X-N)$ of rank $<r$ and integers a_ι such that the sum is equal to $\sum a_\iota\sigma'_\iota$. By the induction hypothesis 4.3.8 (1), there exists $\mathcal{E}'_\iota\in\mathcal{I}((X-N)\otimes\{0\})$ which corresponds to σ'_ι in the sense of Langlands. By Čebotarev density A.4.1, the sum $(\star)$ is equal to $\sum a_\iota\mathcal{E}'_\iota$. Let $\mathcal{E}_\iota\in\mathcal{I}(X-N)$, which corresponds to σ_ι in the sense of Langlands, whose existence is assured by using the induction hypothesis once again. Since $\mathcal{E}'_\iota$ is a constituent of the pullback of $\mathcal{E}_\iota$, we conclude that $(\star)$ is r-negligible as required. □■

4.3.12. Recall the notation of 4.3.7. The proposition is used in the following form:

Corollary. *For any ν, consider the canonical homomorphisms*

$$\mathcal{H}^\nu_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})\to\mathcal{H}^\nu_{\mathrm{c}}(\overline{\mathrm{Cht}_N^{r,\overline{p}\le p}}'/a^{\mathbb{Z}}),\quad \mathcal{H}^\nu_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})\to\mathcal{IH}^\nu(\overline{\mathrm{Cht}_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}}).$$

These four objects are smooth on S_N, and the kernels and cokernels of these homomorphisms are r-negligible.

Proof. To show the claim for the first homomorphism, take $\mathfrak{C}:=\mathfrak{C}'^r_N$ (cf. [La2, III 3b)]) in Proposition 4.3.11, then $\mathfrak{X}$ is $\overline{\mathrm{Cht}_N^{r,d,\overline{p}\le p}}'$ by [La2, Cor III.14]. Taking $\mathscr{M}$ to be the trivial object of the boundary, we get the claim for the first homomorphism by the proposition and the localization sequence in 2.2.9. To check the second one, we take $\mathfrak{C}$ to be $\mathfrak{C}^r_N$, then $\mathfrak{X}$ is $\overline{\mathrm{Cht}_N^{r,d,\overline{p}\le p}}$ by [La2, Def III.8]. Take $\mathscr{M}$ to be the intersection complex of $\mathfrak{C}^r_N$ restricted to the boundary, and we get the claim. □

4.3.13. Let us extract the essential part from the relative cohomology object $\mathcal{H}^*_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})$. Note that this object is smooth on S_N by Corollary 4.3.12. We fix a prime number $\ell\ne p$ until the end of this subsection. Let $\mathcal{H}^*_{N,\mathrm{ess},\ell}$ be the object in $\mathcal{W}_\ell(S_N)$ denoted by $H^*_{N,\mathrm{ess}}$ in [La2, VI.19].

Lemma. *There exists a unique element $\mathcal{H}^*_{N,\mathrm{ess}}$ in $\mathbb{Q}\mathrm{Gr}\mathcal{I}(S_N)$ such that for any closed point $x\in S_N$,*

$$\mathrm{Tr}_{\mathcal{H}^*_{N,\mathrm{ess}}}(\mathrm{Frob}_x^s)=\mathrm{Tr}_{\mathcal{H}^*_{N,\mathrm{ess},\ell}}(\mathrm{Frob}_x^s). \tag{4.3.13.1}$$

Now, put $\mathcal{H}^*_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})^{\mathrm{ss}}:=\sum(-1)^\nu\,\mathcal{H}^\nu_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})^{\mathrm{ss}}$ in $\mathrm{Gr}\mathcal{I}(S_N)$. Then, the formal difference

$$\mathcal{H}^*_{N,\mathrm{ess}}-\frac{1}{r^2!}\sum_{n=1}^{r^2!}\big(\mathrm{Frob}_X^n\times\mathrm{id}_X\big)^+\,\mathcal{H}^*_{\mathrm{c}}(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}})^{\mathrm{ss}}, \tag{$\star$}$$

considered as an element of $\mathbb{Q}\mathrm{Gr}\mathcal{I}(F^2)$, is complete r-negligible.

Proof. Since $\mathcal{H}^*_{N,\mathrm{ess},\ell}$ is pure by [La2, VI.20 (i)], $\mathcal{H}^*_{N,\mathrm{ess}}$ is pure as well if it exists. Thus, the uniqueness readily follows from the Čebotarev density in A.4.1. Let us show the existence. Since $\mathcal{H}^*_{N,\mathrm{ess},\ell}$ is stable under the pullback by $\mathrm{Frob}_X \times \mathrm{id}_X$ (cf. [La2, right after VI.22]) and using [La2, VI.19],

$$(r!)^{-1}\sum_{n=1}^{r!}\bigl(\mathrm{Frob}_X^n \times \mathrm{id}_X\bigr)^* \mathcal{H}^*_{\mathrm{c}}\bigl(\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}, \mathbb{Q}_\ell\bigr)^{\mathrm{ss}}$$

is stable under the pullback as well. This implies that $(r!)^{-1}\sum_{n=1}^{r!}(\cdots) = (r^2!)^{-1}\sum_{n=1}^{r^2!}(\cdots)$. Thus, there exist complete r-negligible ℓ-adic sheaves σ_ι and rational constants c_ι such that

$$\mathcal{H}^*_{N,\mathrm{ess},\ell} - \frac{1}{r^2!}\sum_{n=1}^{r^2!}\bigl(\mathrm{Frob}_X^n \times \mathrm{id}_X\bigr)^* \mathcal{H}^*_{\mathrm{c}}\bigl(\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}, \mathbb{Q}_\ell\bigr)^{\mathrm{ss}} = -\sum_\iota c_\iota \cdot \sigma_\iota$$

by [La2, VI.19 (i)]. Since σ_ι is complete r-negligible for any ι, there exist ℓ-adic sheaves σ' and σ'' on $X - N$ of rank $< r$ such that $\sigma_\iota \cong q'^*\sigma' \otimes q''^*\sigma''$ by definition. By the induction hypothesis in 4.3.8 (1), there exist $\mathcal{E}'$ and $\mathcal{E}''$ in $\mathcal{I}(X-N)$ which correspond to σ' and σ'' in the sense of Langlands. The object $\mathcal{E}_\iota := q'^+\mathcal{E}' \otimes q''^+\mathcal{E}''$ in $\mathcal{I}(S_N)$ corresponds to σ_ι in the sense of Langlands. Note that this $\mathcal{E}_\iota$ is complete r-negligible by construction. Put

$$\mathcal{H}^*_{N,\mathrm{ess}} := \frac{1}{r^2!}\sum_{n=1}^{r^2!}\bigl(\mathrm{Frob}_X^n \times \mathrm{id}_X\bigr)^+ \mathcal{H}^*_{\mathrm{c}}\bigl(\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}\bigr)^{\mathrm{ss}} - \sum_\iota c_\iota \cdot \mathcal{E}_\iota.$$

Now, the difference $(\star)$ is $\sum_\iota c_\iota \cdot \mathcal{E}_\iota$, thus it is complete r-negligible. Since the c-admissible stack $\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}$ is smooth over S_N (cf. 4.2.4), we may use the base change 2.3.22 and Theorem 3.2.1 to show that it satisfies (4.3.13.1), and the element meets our need. ☐

4.3.14. **Proposition** ([La2, VI.20]). (i) *None of the irreducible components of $\mathcal{H}^*_{N,\mathrm{ess}}$ are r-negligible. All the components have positive multiplicity, and pure of weight $2r-2$.*

(ii) *The object $\mathcal{H}^\nu_{\mathrm{c}}\bigl(\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}\bigr)^{\mathrm{ss}}$ is r-negligible for $\nu \neq 2r-2$. Moreover, the following difference is r-negligible:*

$$\mathcal{H}^*_{N,\mathrm{ess}} - \frac{1}{r^2!}\sum_{n=1}^{r^2!}\bigl(\mathrm{Frob}_X^n \times \mathrm{id}_X\bigr)^+ \mathcal{H}^{2r-2}_{\mathrm{c}}\bigl(\mathrm{Cht}_N^{r,\overline{p}\leq p}/a^{\mathbb{Z}}\bigr)^{\mathrm{ss}}.$$

Proof. We have two proofs, both of which use Proposition 4.3.11 substantially. The first one is just to copy the proof of [La2]. The details are left to the reader. The second one is to make use of the Lafforgue's ℓ-adic result. First, we prove (i). For an object A in $\mathbb{Q}\mathrm{Gr}\mathcal{W}_\ell(S_N)$ or $\mathbb{Q}\mathrm{Gr}\mathcal{I}(S_N)$, let $\{A\}$ be the set of constituents of A, and let $\{A\}_{\mathrm{neg}}$ be the subset consisting of r-negligible objects. Writing $A = \sum_{B\in\{A\}} c_B B$ with $c_B \in \mathbb{Q}$, we put $A_{\mathrm{neg}} := \sum_{B\in\{A\}_{\mathrm{neg}}} c_B B$. For objects A and B in $\mathbb{Q}\mathrm{Gr}\mathcal{W}_\ell(S_N)$ or $\mathbb{Q}\mathrm{Gr}\mathcal{I}(S_N)$, we write $A \overset{\mathrm{Tr}}{=} B$ if $\mathrm{Tr}_A(\mathrm{Frob}_x^s) = \mathrm{Tr}_B(\mathrm{Frob}_x^s)$ for any integer s, $x \in |S_N|$. Denoting $\overline{\mathrm{Cht}} := \overline{\mathrm{Cht}_N^{r,\overline{p}\le p}}/a^{\mathbb{Z}}$, we put

$$\mathcal{IH}^\nu := \frac{1}{r^2!}\sum_{n=1}^{r^2!}(\mathrm{Frob}_X^n \times \mathrm{id}_X)^+\mathcal{IH}^\nu(\overline{\mathrm{Cht}})^{\mathrm{ss}},$$

$$\mathcal{IH}^\nu_\ell := \frac{1}{r^2!}\sum_{n=1}^{r^2!}(\mathrm{Frob}_X^n \times \mathrm{id}_X)^*\mathcal{IH}^\nu(\overline{\mathrm{Cht}},\mathbb{Q}_\ell)^{\mathrm{ss}},$$

and $\mathcal{IH}^*_{(\ell)} := \sum_{\nu=1}^{2(2r-2)}(-1)^\nu\mathcal{IH}^\nu_{(\ell)}$. Summarizing [La2, VI.15, 20], we already know the following:

> The difference $\mathcal{IH}^{2r-2}_\ell - \mathcal{H}^*_{N,\mathrm{ess},\ell}$ as well as $\mathcal{IH}^\nu_\ell$ for $\nu \neq 2r-2$ are r-negligible, and any irreducible component of $\mathcal{H}^*_{N,\mathrm{ess},\ell}$ is not r-negligible.

This implies that

$$\mathcal{IH}^{2r-2}_\ell - \mathcal{H}^*_{N,\mathrm{ess},\ell} = (\mathcal{IH}^{2r-2}_\ell)_{\mathrm{neg}}$$

in $\mathbb{Q}\mathrm{Gr}\mathcal{W}_\ell(S_N)$. Now, let us show the following three equalities for any ν:

$$\mathcal{IH}^\nu_\ell \overset{\mathrm{Tr}}{=} \mathcal{IH}^\nu, \qquad (\mathcal{IH}^{2r-2}_\ell)_{\mathrm{neg}} \overset{\mathrm{Tr}}{=} (\mathcal{IH}^{2r-2})_{\mathrm{neg}},$$
$$\mathcal{IH}^{2r-2}_\ell - \mathcal{H}^*_{N,\mathrm{ess},\ell} \overset{\mathrm{Tr}}{=} \mathcal{IH}^{2r-2} - \mathcal{H}^*_{N,\mathrm{ess}}.$$

Indeed, by ℓ-independence Theorem A.5.4, we know that $\mathcal{IH}^*_\ell \overset{\mathrm{Tr}}{=} \mathcal{IH}^*$. By purity (cf. Theorem 2.3.38 (ii)), $\mathcal{H}^\nu_\ell$ and $\mathcal{H}^\nu$ are pure of weight ν, thus we get the first equality. The third one follows by (4.3.13.1) and the first equality, and it remains to show the second one. Let $\sigma \in \mathbb{Q}\mathrm{Gr}\mathcal{W}_\ell(S_N)$ and $\mathcal{E} \in \mathbb{Q}\mathrm{Gr}\mathcal{I}(S_N)$ such that both are invariant under pullback by $\mathrm{Frob}_X \times \mathrm{id}_X$, $\sigma \overset{\mathrm{Tr}}{=} \mathcal{E}$, and both are positive and pure. The second equality follows if we can show $\sigma_{\mathrm{neg}} \overset{\mathrm{Tr}}{=} \mathcal{E}_{\mathrm{neg}}$. Let us check this. Multiplying σ and $\mathcal{E}$ by some integer, we may assume that they are semisimple objects in $\mathcal{W}_\ell(S_N)$ and $\mathcal{I}(S_N)$. Let σ' be an irreducible r-negligible constituent of σ. Since σ is invariant under the pullback by $\mathrm{Frob}_X \times \mathrm{id}$, σ contains a complete r-negligible sheaf $\widetilde{\sigma}'$ which is the external tensor product of irreducible sheaves on $X-N$ and contains σ' as a constituent. By the induction hypothesis 4.3.8 (1), there exists an r-negligible object $\widetilde{\mathcal{E}}'$ in $\mathcal{I}(S_N)$ such that $\widetilde{\sigma}' \overset{\mathrm{Tr}}{=} \widetilde{\mathcal{E}}'$. By Corollary 4.3.3, at least one constituent of $\widetilde{\mathcal{E}}'$ is contained in $\mathcal{E}$, and since $\mathcal{E}$ is stable under pullback by $\mathrm{Frob}_X \times \mathrm{id}$, $\mathcal{E}''$ is contained in $\mathcal{E}$. Similarly, if we are given $\widetilde{\mathcal{E}}'$ which is complete r-negligible and can be written as the external tensor product of irreducible objects on $X-N$, there exists $\widetilde{\sigma}'$ in σ such that $\widetilde{\mathcal{E}}' \overset{\mathrm{Tr}}{=} \widetilde{\sigma}'$. Thus the claim follows.

Combining all of these three equalities, we get

$$\mathcal{IH}^{2r-2} - \mathcal{H}^*_{N,\mathrm{ess}} \overset{\mathrm{Tr}}{=} (\mathcal{IH}^{2r-2})_{\mathrm{neg}},$$

thus $\mathcal{IH}^{2r-2} - \mathcal{H}^*_{N,\mathrm{ess}} = (\mathcal{IH}^{2r-2})_{\mathrm{neg}}$ in $\mathbb{Q}\mathrm{Gr}\mathcal{I}(S_N)$ by Čebotarev density A.4.1.

Now, let us prove (ii). Since $\mathcal{IH}^\nu_\ell$ is r-negligible and $\mathcal{IH}^\nu_\ell \overset{\mathrm{Tr}}{=} \mathcal{IH}^\nu$ as we have already seen, $\mathcal{IH}^\nu$ is r-negligible as well. Since we know that the difference of $\mathcal{IH}^\nu$ and $\mathcal{H}^\nu_{\mathrm{c}}\big(\mathrm{Cht}_N^{r,\overline{p}\le p}/a^{\mathbb{Z}}\big)^{\mathrm{ss}}$ is r-negligible for any ν by Corollary 4.3.12, (ii) follows from (i). $\square$

4.3.15. **Corollary** ([La2, VI.21]). *Let $p \leq q$ be large enough convex polygons. The kernel and cokernel of the induced homomorphisms*

$$\alpha\colon \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}) \to \mathcal{H}^{2r-2}_{\mathrm{c}}(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}),$$
$$\beta\colon \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}) \to \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}}),$$

by the inclusions $\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \hookrightarrow \overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}$ *and* $\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \hookrightarrow \mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}}$ *are r-negligible.*

Proof. Similarly to [La2], we may prove as follows: The claim for α is just a reproduction of Corollary 4.3.12. Let Γ be the correspondence from $\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}$ to $\overline{\mathrm{Cht}^{r,\overline{p}\leq q}_N}'/a^{\mathbb{Z}}$ defined in [La2, V.14 (i)]. This yields the following commutative diagram by Lemma 3.1.11:

$$\begin{tikzcd}
\mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}}) \arrow[r] & \mathcal{H}^{2r-2}_{\mathrm{c}}(\overline{\mathrm{Cht}^{r,\overline{p}\leq q}_N}'/a^{\mathbb{Z}}) \arrow[d, "\Gamma^*"] \\
\mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}) \arrow[u, "\beta"] \arrow[r, "\alpha"'] & \mathcal{H}^{2r-2}_{\mathrm{c}}(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}),
\end{tikzcd}$$

where Γ^* is the action of Γ as in Definition 3.1.11, and the other three arrows are defined by the inclusions. Thus, we have $\mathrm{Ker}(\beta) \subset \mathrm{Ker}(\alpha)$, and $\mathrm{Ker}(\beta)$ is r-negligible since we already know that $\mathrm{Ker}(\alpha)$ is. This implies that, to show the r-negligibility of $\mathrm{Coker}(\beta)$, it suffices to show that the essential parts, namely the set of constituents which are not r-negligible, of the source and the target of β are the same. This follows by the previous proposition, which concludes the proof. $\square$

4.3.16. We digress a little and recall some general nonsense. Let F be an algebraically closed field, and let $\mathcal{A}$ be an abelian category over F such that any object in $\mathcal{A}$ has finite length. We assume, moreover, that for any X, X' in $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, X')$ is a finite-dimensional F-vector space. We note that for an irreducible object X, $\mathrm{End}(X) \cong F$. For $X \in \mathcal{A}$, we have the functor

$$\mathrm{Hom}_{\mathcal{A}}(X, -)\colon \mathcal{A} \to \mathrm{Vec}^{\mathrm{fin}}_F,$$

where $\mathrm{Vec}^{\mathrm{fin}}_F$ is the category of finite-dimensional F-vector spaces. We can check that it has a left adjoint, denoted by $(-) \boxtimes X$. For an integer $n > 0$, we have $F^{\oplus n} \boxtimes X \cong X^{\oplus n}$.

Let G be a group, and we denote by G-$\mathcal{A}$ the category of objects of $\mathcal{A}$ equipped with G-action, namely couples (X, ρ) where $X \in \mathcal{A}$ and a group homomorphism $\rho\colon G \to \mathrm{Aut}(X)$. Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional F-representation of G. Then $V \boxtimes X$ is equipped with action of G determined by ρ, and thus defines an object of G-$\mathcal{A}$. We sometimes denote this by $\rho \boxtimes X$. The following lemma is well known (cf. [EG, 4.15.8]):

Lemma. (i) *For any irreducible F-representation ρ of G and irreducible object X of $\mathcal{A}$, $\rho \boxtimes X$ is an irreducible object of G-$\mathcal{A}$.*

(ii) *Conversely, any irreducible object of G-$\mathcal{A}$ can be written in such a form.*

In practice, we take $\mathcal{A} = \mathcal{I}(S_N)$. Of course, in this case $F = \overline{\mathbb{Q}}_p$. As we have already recalled, any object of $\mathcal{I}(S_N)$ has finite length, and $\mathrm{Hom}_{\mathcal{A}}$ is finite dimensional by the existence of six functor formalism. Thus, the results of this paragraph are applicable to this category.

4.3.17. Next thing we need to do is define a Hecke action on $\mathcal{H}^*_{N,\mathrm{ess}}$. We put $(\mathrm{Cht}^r_N/a^{\mathbb{Z}})_\eta = \varinjlim \mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times_{S_N} U$, where the inductive limit runs over p and open dense subschemes $U \subset X \times X$. Lafforgue constructed a homomorphism of algebras (cf. 3.1.11)

$$\varrho\colon \mathcal{H}^r_N/a^{\mathbb{Z}} \to \mathrm{Corr}^{\mathrm{fin,et}}_\eta\big((\mathrm{Cht}^r_N/a^{\mathbb{Z}})_\eta\big)$$

sending f to Γ_f, which is a finite étale correspondence on a certain open dense subscheme of $X \times X$ (cf. [La2, I 1c), V 2a)] or [La1, I.4, Theorem 5]). To be compatible with [La2], we consider $\varrho' := \mathrm{norm} \circ \varrho$ for the action on the relative cohomology (cf. 3.1.12). Moreover, we have partial Frobenius endomorphisms Frob_∞ and Frob_0 on $\mathrm{Cht}^r_N/a^{\mathbb{Z}}$ over $\mathrm{Frob}_X \times \mathrm{id}_X$ and $\mathrm{id}_X \times \mathrm{Frob}_X$, respectively, such that $\mathrm{Frob}_0 \circ \mathrm{Frob}_\infty = \mathrm{Frob}_\infty \circ \mathrm{Frob}_0 = \mathrm{Frob}$ (cf. [La2, I 1b)]). The action of correspondence on $\mathcal{H}^*_{N,\mathrm{ess}}$ commutes with Frob_∞ and Frob_0 (cf. [La2, I 1c)]). For any convex polygon p, $f \in \mathcal{H}^r_N/a^{\mathbb{Z}}$, $s, u \in \mathbb{N}$, there exists a convex polygon $q \geq p$ such that the correspondence $f \times \mathrm{Frob}^s_\infty \times \mathrm{Frob}^u_0$ sends $\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}$ to $\mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}}$ over some open subscheme $U \subset S_N$. Thus, via ϱ', we have a homomorphism

$$\begin{aligned}(f \times \mathrm{Frob}^s_\infty \times \mathrm{Frob}^u_0)^*\colon (\mathrm{Frob}^s_\infty \times \mathrm{Frob}^u_0)^+ \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}} \times_{S_N} U)&\\ \to \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times_{S_N} U)&\end{aligned}$$

compatible with compositions by Lemma 3.1.11. Note that since $\mathrm{Frob}^* = (\mathrm{Frob}_\infty \times \mathrm{Frob}_0)^*$ is an isomorphism, $(\mathrm{Frob}_\infty \times \mathrm{id})^*$ and $(\mathrm{id} \times \mathrm{Frob}_0)^*$ are isomorphisms.

Let us define a filtration as in [La2, VI 3a)]. Let $\mathcal{E} \in \mathcal{I}(S_N)$. Then we may define a canonical filtration $F^\bullet$ as follows: Put $F^0\mathcal{E} = 0$. Assume $F^{2i}\mathcal{E}$ has already been constructed. We define $F^{2i+1}\mathcal{E}$ to be the largest submodule such that $F^{2i+1}\mathcal{E}/F^{2i}\mathcal{E}$ is *r-negligible*. Then, we put $F^{2i+2}\mathcal{E}$ to be the largest submodule such that $F^{2i+2}\mathcal{E}/F^{2i+1}\mathcal{E}$ is *essential*. This filtration is functorial, namely, given a morphism $\mathcal{E} \to \mathcal{E}'$ in $\mathcal{I}(F^2)$, it induces a morphism $F^i\mathcal{E} \to F^i\mathcal{E}'$. We put the essential part of $\mathcal{E}$,

$$\mathcal{E}_{\mathrm{even}} := \bigoplus_{i\geq 0} F^{2i+2}/F^{2i+1}(\mathcal{E}).$$

For a convex polygon p, we put $\mathcal{H}^{\leq p} := \mathcal{H}^{2r-2}_{\mathrm{c}}(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}})$, which is an object in $\mathcal{I}(S_N)$. For a large enough convex polygon p and $p' \geq p$, Corollary 4.3.15 tells us that the homomorphism induced by the canonical homomorphism $\mathcal{H}^{\leq p} \to \mathcal{H}^{\leq p'}$

$$\mathcal{H}^{\leq p}_{\mathrm{even}} \to \mathcal{H}^{\leq p'}_{\mathrm{even}}$$

is an isomorphism in $\mathcal{I}(S_N)$. Because of the functoriality of the filtration, the action of the correspondence $(f \times \mathrm{Frob}^s_\infty \times \mathrm{Frob}^u_0)^*$ induces an action of $\mathcal{H}^r_N/a^{\mathbb{Z}}$ as well as the invariance by the pullback $(\mathrm{Frob}_X \times \mathrm{id}_X)^+$ on $\mathcal{H}^{\leq p}_{\mathrm{even}}$ as an object of $\mathcal{I}(F^2)$. Summing up, $\mathcal{H}^{\leq p}_{\mathrm{even}}$, for large enough p, can be seen as an object of $\mathcal{H}^r_N/a^{\mathbb{Z}}$-$\mathbb{Z}\mathcal{I}(F^2)$ (not only of $\mathcal{I}(F^2)$!).

Now, the invariance by $(\mathrm{Frob}_X \times \mathrm{id}_X)^+$ shows that

$$\frac{1}{r^2!}\sum_{n=1}^{r^2!} (\mathrm{Frob}^n_X \times \mathrm{id}_X)^+(\mathcal{H}^{\leq p}_{\mathrm{even}}) = \mathcal{H}^{\leq p}_{\mathrm{even}}$$

in $\mathrm{Gr}\mathcal{I}(F^2)$. Thus by Proposition 4.3.14 (ii), it turns out that $\mathcal{H}^{\leq p}_{\mathrm{even}}$ considered in $\mathbb{Q}\mathrm{Gr}\mathcal{I}(F^2)$ coincides with $\mathcal{H}^*_{N,\mathrm{ess}}$. This implies that the coefficients of $\mathcal{H}^*_{N,\mathrm{ess}}$, which are a priori rational numbers by construction, are positive integers. Now, recall that for a smooth scheme Y over k and an open dense subscheme $U \subset Y$, the restriction functor $\mathcal{I}(Y) \to \mathcal{I}(U)$ is fully faithful (cf. [Ke4, Thm 5.2.1]). This implies that, since $\mathcal{H}^{\leq p}_{\mathrm{even}} \in \mathcal{I}(S_N)$, the action of $\mathcal{H}^r_N/a^{\mathbb{Z}}$ and the invariance by $(\mathrm{Frob}_X \times \mathrm{id}_X)^+$ of $\mathcal{H}^{\leq p}_{\mathrm{even}}$ as an object of $\mathcal{I}(F^2)$ can be extended uniquely to an action of $\mathcal{H}^r_N/a^{\mathbb{Z}}$ and an invariance by $(\mathrm{Frob}_X \times \mathrm{id}_X)^+$ as an object of $\mathcal{I}(S_N)$. Thus, $\mathcal{H}^{\leq p}_{\mathrm{even}}$ can be seen as an element of $\mathcal{H}^r_N/a^{\mathbb{Z}}$-$\mathbb{Z}\mathcal{I}(S_N)$. We denote by $\mathcal{H}_{N,\mathrm{ess}}$ the semisimplification of $\mathcal{H}^{\leq p}_{\mathrm{even}}$ (for p large enough, needless to say) as an object of $\mathcal{H}^r_N/a^{\mathbb{Z}}$-$\mathbb{Z}\mathcal{I}(S_N)$, which is equal to $\mathcal{H}^*_{N,\mathrm{ess}}$ considered as elements of $\mathrm{Gr}\mathcal{I}(S_N)$.

4.3.18. We need to calculate the trace of the Hecke action on $\mathcal{H}_{N,\mathrm{ess}}$. Let $f \in \mathcal{H}^r_N/a^{\mathbb{Z}}$. Recall that Γ_f is a correspondence

$$\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times_{S_N} U \rightsquigarrow \mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}} \times_{S_N} U$$

for some open dense subscheme $U \subset S_N$ and $q \geq p$. We denote by Γ'_f the pullback of Γ_f by the open immersion

$$\begin{aligned}&\big(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times \mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}\big) \times_{(S_N\times S_N)} (U\times U)\\ &\qquad\hookrightarrow \big(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times \mathrm{Cht}^{r,\overline{p}\leq q}_N/a^{\mathbb{Z}}\big) \times_{(S_N\times S_N)} (U\times U).\end{aligned}$$

Now, we take the normalization of the morphism

$$\Gamma'_f \to \mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \times \mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}} \hookrightarrow \overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}/a^{\mathbb{Z}} \times \overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}/a^{\mathbb{Z}}.$$

This normalization certainly defines a correspondence on $\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}/a^{\mathbb{Z}}$. A marvelous thing is that, in fact, the correspondence stabilizes $\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}$ as Lafforgue shows in [La2, V.14], and it defines a correspondence on it. Using the filtration of 4.3.17, the action Γ^*_f of the correspondence Γ_f on $\mathcal{H}^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big)$ induces an action on $\big(\mathcal{H}^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big)\big)_{\mathrm{even}}$.

Let us introduce a notation. Let Y be a smooth scheme over k, $\mathcal{E} \in \mathcal{I}(Y)$, and let α be an endomorphism of $\mathcal{E}$. For a closed point x of Y, take a geometric point $\overline{x}$ above x. Recalling the notation of 4.2.1, α induces an endomorphism of $\iota_{\overline{x}}(\mathcal{E})$ which commutes with Frob_x. We denote $\mathrm{Tr}\big(\alpha \circ \mathrm{Frob}^n_x : \iota_{\overline{x}}(\mathcal{E})\big)$ by $\mathrm{Tr}(\alpha \times \mathrm{Frob}^n_x : \mathcal{E})$ or $\mathrm{Tr}_{\mathcal{E}}(\alpha \times \mathrm{Frob}^n_x)$, which does not depend on the choice of $\overline{x}$. Using this notation, we put

$$\mathrm{Tr}^{\leq p}_{\mathcal{H}^*_{N,\mathrm{ess}}}(f \times \mathrm{Frob}^n_x) := \mathrm{Tr}\Big(\Gamma^*_f \times \mathrm{Frob}^n_x : \big(\mathcal{H}^{2r-2}_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big)\big)_{\mathrm{even}}\Big).$$

We need to compare the trace of the action of correspondences on $\mathcal{H}_{N,\mathrm{ess}}$ defined using Cht in the previous paragraph and $\mathrm{Tr}^{\leq p}_{\mathcal{H}^*_{N,\mathrm{ess}}}$.

4.3.19. **Lemma** ([La2, VI.23, 24]). (i) *Let $\mathcal{E}$ be a mixed object in $\mathcal{I}(S_N)$, and let f be an endomorphism on $\mathcal{E}$. Assume that $\mathcal{E}$ has a filtration $F^\bullet\mathcal{E}$ compatible with f. Take $I \subset \mathbb{Z}$, and put $\mathcal{F} := \bigoplus_{i\in I}\mathrm{gr}^i(\mathcal{E})$. Let $\{\mathcal{F}\}$ denote the set of irreducible objects in $\mathcal{I}(S_N)$ appearing in $\mathcal{F}$. Then there exists a unique set of complex numbers $\{c_{\mathcal{E}'}\}$ such that, for any $x \in |S_N|$ and $n \in \mathbb{Z}$, we have*

$$(\star)\qquad \mathrm{Tr}_{\mathcal{F}}(f \times \mathrm{Frob}^n_x) = \sum_{\mathcal{E}'\in\{\mathcal{F}\}} c_{\mathcal{E}'} \cdot \mathrm{Tr}_{\mathcal{E}'}(\mathrm{Frob}^n_x).$$

(ii) *Let $f \in \mathcal{H}^r_N/a^{\mathbb{Z}}$, and take a large enough convex polygon p. We have*

$$\mathrm{Tr}_{\mathcal{H}_{N,\mathrm{ess}}}(f \times \mathrm{Frob}^n_x) = \mathrm{Tr}^{\leq p}_{\mathcal{H}^*_{N,\mathrm{ess}}}(f \times \mathrm{Frob}^n_x),$$

where the action on the left-hand side is the one defined in 4.3.17.

Proof. Let us show (i). Since $\mathcal{E}$ is assumed to be mixed, any object in $\{\mathcal{F}\}$ is pure by [AC1, 4.2.3, 4.3.4], thus the uniqueness follows by the Čebotarev density A.4.1. We can see $\mathcal{F}$ as an object of $\mathbb{Z}$-$\mathcal{I}(S_N)$ where the action of $\mathbb{Z}$ is defined by f. Let $\mathcal{G}$ be an irreducible object in $\mathbb{Z}$-$\mathcal{I}(S_N)$. Then by Lemma 4.3.16, it can be written as $V \boxtimes \mathcal{G}'$ where V is an irreducible representation of $\mathbb{Z}$ and $\mathcal{G}'$ is an irreducible object in $\mathcal{I}(S_N)$. Since $\mathbb{Z}$ is abelian, $\dim(V) = 1$ and f act as multiplication by c. This implies that $\mathrm{Tr}_{\mathcal{G}}(f \times \mathrm{Frob}^n_x) = c \cdot \mathrm{Tr}_{\mathcal{G}'}(\mathrm{Frob}^n_x)$, and $(\star)$ follows.

For a proof of (ii), we copy the argument of [La2]. Let us sketch the proof. The commutative diagram in the proof of Corollary 4.3.15 yields the following commutative diagram for $p'' \geq p'$ sufficiently large (cf. [La2, p.185]):

$$\begin{array}{ccccc}
\mathcal{H}^{2r-2}_{\mathrm{c}}\big(\mathrm{Cht}^{r,\overline{p}\leq p''}_N/a^{\mathbb{Z}}\big) & & \longrightarrow & & \mathcal{H}^{2r-2}_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p''}_N}'/a^{\mathbb{Z}}\big) \\
{\scriptstyle f}\uparrow & & & & \downarrow \\
\mathcal{H}^{2r-2}_{\mathrm{c}}\big(\mathrm{Cht}^{r,\overline{p}\leq p}_N/a^{\mathbb{Z}}\big) & \longrightarrow & \mathcal{H}^{2r-2}_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big) & \xrightarrow[f]{} & \mathcal{H}^{2r-2}_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big),
\end{array}$$

where the homomorphisms marked as f are induced by the correspondence associated to f, and the others are canonical ones. Thus the claim follows by Corollary 4.3.15. $\square$

4.3.20. For an unramified irreducible admissible representation π of $\mathrm{GL}_r(F_x)$ at some place x of F and $t \in \mathbb{Z}$, we put

$$z^t_{\bullet}(\pi) := z_1(\pi)^t + \cdots + z_r(\pi)^t,$$

where $z_i(\pi)$ denotes the Hecke eigenvalue of π. Take a closed point x in $X \times X$. We denote by ∞_x (resp. 0_x) the image of x by the first (resp. second) projection. We note that $\deg(x) = \mathrm{lcm}\big(\deg(\infty_x), \deg(0_x)\big)$.

Lemma ([La2, VI.25]). *Let $f \in \mathcal{H}^r_N/a^{\mathbb{Z}}$. Then there exists an open dense subscheme $U_f \subset S_N$ such that for any $x \in U_f$, we have*

$$\mathrm{Tr}_{\mathcal{H}_{N,\mathrm{ess}}}\big(f \times \mathrm{Frob}^{-s/\deg(x)}_x\big) = q^{(r-1)s} \sum_{\pi \in \{\pi\}^r_N} \mathrm{Tr}_\pi(f) \cdot z^{-s'}_{\bullet}(\pi_{\infty_x}) \cdot z^{u'}_{\bullet}(\pi_{0_x}),$$

where $s = \deg(\infty_x)s' = \deg(0_x)u' \in \mathbb{Z} \cdot \deg(x)$.

Proof. By Corollary 3.2.1, there exists an open dense subscheme $U'_f \subset S_N$ such that

$$(\star) \quad \mathrm{Tr}\Big(f \times \mathrm{Frob}^{-s/\deg(x)}_x : \mathcal{H}^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}\big)\Big) = \mathrm{Tr}\Big(f \times \mathrm{Frob}^{-s/\deg(x)}_x : H^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell\big)\Big)$$

for any $x \in U'_f$. Let $\mathcal{H}_{N,\mathrm{ess},\ell}$ be $H_{N,\mathrm{ess}}$ defined in [La2, after VI.22]. Note that since $\mathcal{H}^*_{\mathrm{c}}(\overline{\mathrm{Cht}^{r,\overline{p}\leq p}_N}'/a^{\mathbb{Z}})$ is mixed by Theorem 2.3.38, Lemma 4.3.19 (i) is applicable. Let

$\{\mathcal{E}\}$ (resp. $\{\sigma\}$) be the set of r-negligible objects in $\mathcal{I}(S_N)$ (resp. in $\mathcal{W}_\ell(S_N)$). We have

$$\begin{aligned}
\mathrm{Tr}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess}}\big) &= \mathrm{Tr}^{\leq p}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}^*_{N,\mathrm{ess}}\big) \\
(\star\star) \qquad &= \mathrm{Tr}\Big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}_N^{r,\overline{p}\leq p}}'/a^{\mathbb{Z}}\big)\Big) \\
&\qquad + q^{(r-1)s} \sum_{\mathcal{E}\in\{\mathcal{E}\}} c_{\mathcal{E}} \, \mathrm{Tr}_{\mathcal{E}}(\mathrm{Frob}_x^{-s/\deg(x)}) \\
&= \mathrm{Tr}\Big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}^*_{\mathrm{c}}\big(\overline{\mathrm{Cht}_N^{r,\overline{p}\leq p}}'/a^{\mathbb{Z}}, \overline{\mathbb{Q}}_\ell\big)\Big) \\
&\qquad + q^{(r-1)s} \sum_{\mathcal{E}\in\{\mathcal{E}\}} c_{\mathcal{E}} \, \mathrm{Tr}_{\mathcal{E}}(\mathrm{Frob}_x^{-s/\deg(x)}) \\
&= \mathrm{Tr}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess},\ell}\big) \\
&\qquad + q^{(r-1)s} \sum_{A\in\{\mathcal{E}\}\cup\{\sigma\}} c_A \, \mathrm{Tr}_A(\mathrm{Frob}_x^{-s/\deg(x)}),
\end{aligned}$$

where the first and second equality hold by Lemma 4.3.19 (ii) and (i), respectively, the third by $(\star)$, and the last by repeating the corresponding argument in the ℓ-adic situation. Now, for $\mathcal{G} \in \mathcal{I}(S_N)$, an endomorphism f of $\mathcal{G}$, and an integer n, we put

$$\begin{aligned}
\overline{\mathrm{Tr}}(f \times \mathrm{Frob}_x^n : \mathcal{G}) :&= \frac{1}{r^2!} \sum_{k=1}^{r^2!} \mathrm{Tr}\big(f \times \mathrm{Frob}_x^n : (\mathrm{Frob}_X^k \times \mathrm{id}_X)^+(\mathcal{G})\big) \\
&= \frac{1}{r^2!} \sum_{k=1}^{r^2!} \mathrm{Tr}_{\mathcal{G}}\big(f \times \mathrm{Frob}^n_{(\mathrm{Frob}_X^k \times \mathrm{id}_X)(x)}\big),
\end{aligned}$$

and similarly for objects in $\mathcal{W}_\ell(S_N)$. Note that when $\mathcal{G} \in \{\mathcal{E}\}$, there exist $\mathcal{G}'_\iota$ and $\mathcal{G}''_\iota$ of rank $< r$ and pure of weight 0 in $\mathcal{I}(X - N)$, and constants c_ι, λ_ι such that

$$\overline{\mathrm{Tr}}(\mathrm{Frob}_x^n : \mathcal{G}) = \sum_\iota c_\iota \lambda_\iota^n \, \mathrm{Tr}_{q'+\mathcal{G}'_\iota \otimes q'' + \mathcal{G}''_\iota}(\mathrm{Frob}_x^n).$$

Put $U_f := \bigcap_{n=0}^{r^2!} (\mathrm{Frob}_X^n \times \mathrm{id}_X)^{-1}(U'_f)$. Since $\mathcal{H}_{N,\mathrm{ess}}$ and $\mathcal{H}_{N,\mathrm{ess},\ell}$ are invariant under the pullback by $\mathrm{Frob}_X \times \mathrm{id}_X$, the computation $(\star\star)$ implies that for $x \in U_f$,

$$\begin{aligned}
\mathrm{Tr}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess}}\big) &= \overline{\mathrm{Tr}}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess}}\big) \\
&= \overline{\mathrm{Tr}}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess},\ell}\big) \\
&\qquad + q^{(r-1)s} \sum_{A\in\{\mathcal{E}\}\cup\{\sigma\}} c_A \, \overline{\mathrm{Tr}}_A(\mathrm{Frob}_x^{-s/\deg(x)}) \\
&= \mathrm{Tr}\big(f \times \mathrm{Frob}_x^{-s/\deg(x)} : \mathcal{H}_{N,\mathrm{ess},\ell}\big) \\
&\qquad + q^{(r-1)s} \sum_\iota c_\iota \lambda_\iota^s \, \mathrm{Tr}_{q'+\mathcal{E}'_\iota \otimes q'' + \mathcal{E}''_\iota}(\mathrm{Frob}_x^{-s/\deg(x)}),
\end{aligned}$$

where $\mathcal{E}'_\iota$ and $\mathcal{E}''_\iota$ are irreducible isocrystals on $X - N$ of rank $< r$ and pure of weight 0, and we used Langlands correspondence for rank $< r$ for the last equality. By

[La2, VI.25], if we further shrink U_f, we finally obtain

(♡)
$$q^{-(r-1)s}\,\mathrm{Tr}\big(f\times\mathrm{Frob}_x^n:\mathcal{H}_{N,\mathrm{ess}}\big)-\sum_{\pi\in\{\pi\}_N^r}\mathrm{Tr}_\pi(f)\cdot z_\bullet^{-s'}(\pi_{\infty_x})\cdot z_\bullet^{u'}(\pi_{0_x})$$
$$=\sum_\iota c_\iota\lambda_\iota^s\,\mathrm{Tr}_{q'^+\mathcal{E}'_\iota\otimes q''^+\mathcal{E}''_\iota}(\mathrm{Frob}_x^{-s/\deg(x)}).$$

Since $\mathcal{H}_{N,\mathrm{ess}}$ is pure of weight $2r-2$ and we know that $|z_i(\pi_\infty)|$ and $|z_j(\pi_0)|$ are 1, we have $|\lambda_\iota|=1$. We need to show that the right-hand side of (♡) is 0. The argument is essentially the same as [La2], but for the reader, we recall it. Assume otherwise. Then there would exist irreducible isocrystals $\mathcal{E}'$, $\mathcal{E}''$ of rank $<r$ and pure of weight 0 such that the series

$$\sum_\iota c_\iota\frac{d}{dZ}\log L_{U_f}\big((q'^+\mathcal{E}'_\iota\otimes q''^+\mathcal{E}''_\iota)\otimes(q'^+\mathcal{E}'^\vee\otimes q''^+\mathcal{E}''^\vee),\lambda_\iota Z\big)$$

has a pole on $|Z|=q^{-2}$ by Corollary 4.3.3 since $|\lambda_\iota|=1$. On the other hand, the series
$$\frac{d}{dZ}\log L_{U_f}\big(\mathcal{H}\otimes(q'^+\mathcal{E}'^\vee\otimes q''^+\mathcal{E}''^\vee),q^{1-r}Z\big)$$
does not have poles at $|Z|=q^{-2}$ since $\mathcal{H}$ is essential. Now, let π' and π'' be the automorphic cuspidal representations corresponding to $\mathcal{E}'$ and $\mathcal{E}''$. For a locally closed subscheme Y of $(X-N)\times(X-N)$, let us denote by $\mathfrak{C}_{s,Y}$ the subset of $(x,s',u')\in|Y|\times\mathbb{N}\times\mathbb{N}$ such that $s\cdot\deg(\infty_x)^{-1}=s'\in\mathbb{N}$ and $s\cdot\deg(0_x)^{-1}=u'\in\mathbb{N}$, and put

$$\mathrm{Ser}_Y(Z):=\sum_{s\geq1}Z^{s-1}\sum_{(x,s',u')\in\mathfrak{C}_{s,Y}}\Big[\big(\deg(\infty_x)\cdot z_\bullet^{-s'}(\pi_{\infty_x})\cdot z_\bullet^{s'}(\pi'_{\infty_x})\big)$$
$$\times\big(\deg(0_x)\cdot z_\bullet^{u'}(\pi_{0_x})\cdot z_\bullet^{u'}(\pi''_{0_x})\big)\Big].$$

The series
$$\frac{d}{dZ}\log L_{X-N}(\pi\times\pi'^\vee,Z),\qquad\frac{d}{dZ}\log L_{X-N}(\pi^\vee\times\pi''^\vee,Z)$$
do not have poles on $|Z|\leq q^{-1}$ by [La2, B.10]. Thus, the product series Ser_{S_N} does not have poles at $|Z|\leq q^{-2}$. We claim that the series Ser_{U_f} does not have poles at $|Z|\leq q^{-2}$ either. Indeed, putting $W:=S_N\setminus U_f$, we have $\mathrm{Ser}_W=\mathrm{Ser}_{S_N}-\mathrm{Ser}_{U_f}$. Since $|z_i(\pi)|=|z_j(\pi')|=|z_k(\pi'')|=1$, we have
$$|\mathrm{Ser}_W(Z)|\leq\sum_s|Z|^{s-1}\sum_{(x,s',u')\in\mathfrak{C}_W}\deg(\infty_x)\cdot\deg(0_x).$$

Since W is of dimension 1, the latter series converges on $|Z|<q^{-1}$, and thus Ser_W converges absolutely on the same area, which implies the claim. Combining these, if we put
$$\sum_s Z^{s-1}\sum_{\mathfrak{C}_{s,U_f}}\deg(\infty_x)\cdot\deg(0_x)\cdot z^{s'}(\pi'_{\infty'_x})\cdot z^{u'}(\pi''_{0_x})$$

at the head of the both sides of (♡), the left-hand side does not have poles at $|Z|=q^{-2}$, whereas the right-hand side does at $|Z|=q^{-2}$, which is a contradiction. □

4.3.21. **Lemma** ([La2, VI.26]). *As an object of $\mathcal{H}^r_N/a^{\mathbb{Z}}$-$\mathbb{Z}\mathcal{I}(S_N)$, we can write $\mathcal{H}_{N,\mathrm{ess}}$ as*

$$\bigoplus_{\pi\in\{\pi\}^r_N} \pi\boxtimes\mathcal{H}_\pi(1-r),$$

and there exists an open dense subscheme $U_\pi\subset S_N$ for any $\pi\in\{\pi\}^r_N$ such that the following holds: the object $\mathcal{H}_\pi$ is pure of weight 0, and for any closed point $x\in U_\pi$ and $s=\deg(\infty_x)s'=\deg(0_x)u'$, we have

$$\mathrm{Tr}_{\mathcal{H}_\pi}\bigl(\mathrm{Frob}_x^{-s/\deg(x)}\bigr)=z_\bullet^{-s'}(\pi_{\infty_x})\cdot z_\bullet^{u'}(\pi_{0_x}).$$

Proof. This can be proven similarly to the ℓ-adic case. By Lemma 4.3.16, there exists a finite set of irreducible representations $\{\pi'\}$ of $\mathcal{H}^r_N/a^{\mathbb{Z}}$ and semisimple objects $\mathcal{H}_{\pi'}$ in $\mathbb{Z}\mathcal{I}(S_N)$ for each $\pi'\in\{\pi'\}$ such that $\mathcal{H}_{N,\mathrm{ess}}=\bigoplus_{\pi'\in\{\pi'\}^r_N}\pi'\boxtimes\mathcal{H}_{\pi'}(1-r)$. For $\pi\in\{\pi\}^r_N\cup\{\pi'\}$, we can choose $f_\pi\in\mathcal{H}^r_N/a^{\mathbb{Z}}$ such that $\mathrm{Tr}_\pi(f_\pi)=1$ and $\mathrm{Tr}_{\pi'}(f_\pi)=0$ for any $\pi\neq\pi'\in\{\pi\}^r_N\cup\{\pi'\}$ (cf. proof of [La2, VI.26]), and apply Lemma 4.3.20 by taking $f=f_\pi$. Then the lemma holds with $U_\pi:=U_{f_\pi}$. $\square$

4.3.22. **Theorem** ([La2, VI.27]). *For any $\pi\in\{\pi\}^r_N$, we have*

$$\mathcal{H}_\pi=q'^{+}\mathcal{E}_\pi\otimes q''^{+}\mathcal{E}_\pi^\vee$$

as objects in $\mathcal{I}(S_N)$, where $\mathcal{E}_\pi$ is an isocrystal of rank r on $X-N$ pure of weight 0 corresponding to π in the sense of Langlands.

Proof. Take a closed point $x\in U_\pi$, which lies over $(\infty,0)\in|X-N|\times|X-N|$. Let $X^0:=X\times 0\hookrightarrow X\times X$ be the closed immersion, and let $(X-N)^0\subset X^0$ be the pullback of S_N by the closed immersion. Let $\mathcal{E}^0$ be the semisimplification in $\mathcal{I}((X-N)^0)$ of the pullback of $\mathcal{H}_\pi$, which is pure of weight 0. Let $\mathcal{H}^0_\pi$ be the pullback on $S_N\otimes k(0)$. Let χ_i be the character (i.e., rank 1 isocrystal on the point 0) corresponding to the Hecke eigenvalue $z_i(\pi_0)$. Then the two semisimple objects in $\mathcal{I}(S_N\otimes k(0))$

$$q'^{+}\mathcal{E}^0\otimes q''^{+}\mathcal{E}^{0\vee},\qquad \mathcal{H}^0_\pi\otimes\bigoplus_{i=1}^r\chi_i\otimes\bigoplus_{i=1}^r\chi_i^\vee$$

have the same Frobenius eigenvalues at each closed point of $U_\pi\otimes k(0)$ by the explicit description of the Frobenius trace in Lemma 4.3.21. Thus by the Čebotarev density A.4.1, these two objects coincide.

Thus, there exist $\mathcal{E}'$ and $\mathcal{E}''$ on $X-N\subset X$ which are pure of weight 0 such that $q'^{+}\mathcal{E}'\otimes q''^{+}\mathcal{E}''$ and $\mathcal{H}_\pi$ have at least one constituent in common. We may assume that $\mathcal{E}'$ and $\mathcal{E}''$ are irreducible, and since $\mathcal{H}_\pi$ is stable under the action of $(\mathrm{Frob}_X\times\mathrm{id})^+$, we may assume that $q'^{+}\mathcal{E}'\otimes q''^{+}\mathcal{E}''$ is a subobject of $\mathcal{H}_\pi$.

Let us show that $\mathcal{E}'$ is of rank $\geq r$. The argument is similar to the last part of the proof of Lemma 4.3.20. If $\mathcal{E}'$ were of rank $<r$, there would exist an automorphic cuspidal representation π' corresponding to $\mathcal{E}'$. Consider the series

$$\frac{d}{dZ}\log L_{X-N}(\pi\times\pi'^\vee,Z),\qquad \frac{d}{dZ}\log L_{X-N}(\pi^\vee\times\mathcal{E}''^\vee,Z).$$

The first one converges absolutely on $|Z|<q^{-1+\epsilon}$ for some $\epsilon>0$ by [La2, Thm B.10]. Since $|z_i(\pi)|=1$ and $\mathcal{E}''$ is of weight 0, the second one converges absolutely on $|Z| < q^{-1}$. Thus the product series converges absolutely on $|Z| < q^{-2+\epsilon}$. On the other hand, consider the series

$$(\star) \qquad \frac{d}{dZ} \log L_{U_\pi}\bigl(\mathcal{H}_\pi \otimes (q'^{+}\mathcal{E}'^{\vee} \otimes q''^{+}\mathcal{E}''^{\vee})\bigr).$$

Let $C := S_N \setminus U_\pi$. The difference with the product series is nothing but

$$(\star\star) \qquad \sum_{s\geq 1} Z^{s-1} \sum_{(x,s',u')\in\mathfrak{C}_s} \Bigl[\bigl(\deg(\infty_x)\cdot z_\bullet^{-s'}(\pi_{\infty_x}) \cdot z_\bullet^{s'}(\pi'_{\infty_x})\bigr) \times \bigl(\deg(0_x) \cdot z_\bullet^{u'}(\pi_{0_x}) \cdot z_\bullet^{u'}(\mathcal{E}''_{0_x})\bigr)\Bigr],$$

where $\mathfrak{C}_s \subset |C|\times\mathbb{N}\times\mathbb{N}$ such that $s\cdot\deg(\infty_x)^{-1} = s' \in \mathbb{N}$ and $s\cdot\deg(0_x)^{-1} = u' \in \mathbb{N}$. Since C is of dimension 1 and the complex absolute values of $z_i(\pi_x)$, $z_i(\pi'_x)$, $z_i(\mathcal{E}'')$ are 1, we get that the series $(\star\star)$ converges in $|Z| < q^{-1}$. Thus, considering the radius of convergence of the product series, $(\star)$ should converge on $|Z| < q^{-2+\epsilon}$. However, since by Corollary 4.3.3, it should have a pole at $|Z| = q^{-2}$, which is a contradiction.

This shows that the rank of $\mathcal{E}'$ is $\geq r$. By symmetry, the rank of $\mathcal{E}''$ is $\geq r$ as well. Since the rank of $\mathcal{H}_\pi$ is r^2, we get that the rank of $\mathcal{E}'$ and $\mathcal{E}''$ are r, and $\mathcal{H}_\pi \cong q'^{+}\mathcal{E}' \otimes q''^{+}\mathcal{E}''$. By the induction hypothesis 4.3.8 (1), there exist cuspidal automorphic representations π', π'' of $\mathrm{GL}_r(\mathbb{A})$ corresponding to $\mathcal{E}'$ and $\mathcal{E}''$. Now, since $\dim(X-N) = 1$ and $|z_i(\pi)| = |z_j(\pi')| = |z_k(\pi'')| = 1$, the rational functions

$$\frac{d}{dZ}\log L_{X-N}(\pi\times\pi'^{\vee}, Z), \qquad \frac{d}{dZ}\log L_{X-N}(\pi^{\vee}\times\pi''^{\vee}, Z)$$

do not have poles at $|Z| < q^{-1}$. This implies that they have a pole at $Z = q^{-1-s}$ and q^{-1+s}, respectively, for some $s \in \mathbb{C}$ such that $\mathrm{Re}(s) = 0$. Otherwise, the series

$$\frac{d}{dZ}\log L_{U_\pi}\bigl(\mathcal{H}_\pi \otimes (q'^{+}\mathcal{E}'^{\vee} \otimes q''^{+}\mathcal{E}''^{\vee})\bigr) = \frac{d}{dZ}\log L_{U_\pi}\bigl(\mathcal{H}_\pi \otimes \mathcal{H}_\pi^{\vee}\bigr)$$

would not have a pole at q^{-2}. This shows that $\mathcal{E}' \cong \mathcal{E}''^{\vee}$, and $\mathcal{E}_\pi = \mathcal{E}'(s)$ as requested, and $\mathcal{H}_\pi \cong q'^{+}\mathcal{E}_\pi \otimes q''^{+}\mathcal{E}_\pi^{\vee}$. □

4.3.23. *Conclusion of the proof.* By Theorems 4.3.22 and 4.2.3, 4.3.8 (1) holds for $n = r$. Let us check 4.3.8 (2). Theorem 4.3.22 combined with Proposition 4.3.14 (ii) shows that the average $(r^2!)^{-1}\sum_{n=1}^{r^2!}(\mathrm{Frob}_X^n \times \mathrm{id}_X)^{+}\mathcal{H}_{\mathrm{c}}^{2r-2}(\cdots)^{\mathrm{ss}}$ is $(r+1)$-negligible. Since the averaging involves only positive coefficients, $\mathcal{H}_{\mathrm{c}}^{2r-2}(\cdots)$ is $(r+1)$-negligible. Combining with Proposition 4.3.14 (i), 4.3.8 (2) holds for $n = r$, and $(\mathrm{S})_r$ is shown. Thus, we conclude the proof of Theorem 4.2.2 by induction. □

4.4. **A few applications.** To conclude this paper, let us collect some applications of the Langlands correspondence.

4.4.1. **Theorem** ([De1, 1.2.10 (vi)]). *Let X be a smooth* curve *over a finite field k of characteristic p. Let ℓ be a prime number different from p. Then for any irreducible smooth $\overline{\mathbb{Q}}_\ell$-sheaf whose determinant is of finite order, there exists a* petit camarade cristallin.

Proof. Use Lafforgue's result and Theorem 4.2.2. □

4.4.2. In p-adic cohomology theory, Bertini-type results do not seem to be known, as pointed out in [Ke2]. We consider the situation and notation in 2.4.14 *exclusively in this paragraph.* In particular, k does not need to be finite in the following conjecture.

Conjecture (Bertini-type conjecture)**.** *Let X be a smooth scheme over k, and let $\mathcal{E}$ be an irreducible L-isocrystal. Then there exists a dense Zariski open subset U of X such that the following holds: for any $x \in |U|$, there exists an immersion (not necessarily closed) from a smooth curve $i\colon C \hookrightarrow U$ passing through x such that $i^+\mathcal{E}$ remains irreducible.*

4.4.3. **Theorem.** *Assume we are in the situation of* 4.2.1*. Let X be a scheme of finite type over k. Then any complex in $D^{\mathrm{b}}_{\mathrm{hol}}(X/\overline{\mathbb{Q}}_{p,F})$ is mixed if X is of dimension* 1*. If Conjecture* 4.4.2 *is true, we do not need to assume X to be of dimension* 1*.*

Proof. See [A2, 6.3]. □

4.4.4. **Corollary.** *The Čebotarev density theorem for smooth curves holds for overconvergent F-isocrystals. If Conjecture* 4.4.2 *is true, it holds for any smooth variety.*

Proof. Apply Proposition A.4.1 and Theorem 4.4.3. □

4.4.5. **Theorem.** *Let X be a smooth scheme of finite type over a finite field k. Let $\mathcal{E}$ be an irreducible $\overline{\mathbb{Q}}_p$-isocrystal with Frobenius structure on X such that the determinant is of finite order.* Assume that Conjecture 4.4.2 holds.

(i) *There exists a number field $E/\mathbb{Q}$ such that, for any $x \in |X|$, all the coefficients of the Frobenius eigenpolynomial of $\mathcal{E}$ at x are in E.*

(ii) *For any prime $\ell \neq p$, there exists an $\overline{\mathbb{Q}}_\ell$-adic smooth sheaf $\mathscr{F}$ corresponding to $\mathcal{E}$ such that the sets of Frobenius eigenvalues coincide for any closed point of X.*

Proof. Let us show (i). We denote by $\mathcal{I}_r(X)$ the set of $\overline{\mathbb{Q}}_p$-isocrystals of rank r on X up to isomorphism and semisimplification. We use the notation of [EK, §2]. We prove this by induction on the dimension of X. Let $\overline{X}$ be a normal compactification of X, and let $\overline{X}\backslash X$ be a Cartier divisor. Then there exists a map $\mathcal{I}_r(X) \to \mathcal{V}_r(X)$ using the Langlands correspondence. This is injective by the Čebotarev density. We show the following: Let $\mathcal{E} \in \mathcal{I}_r(X)$. Then there exists an dense open subscheme $U \subset X$ and Cartier divisor D of $\overline{X}$ contained in $\overline{X} \setminus U$ such that $\mathcal{E}_U \in \mathcal{V}_r(U, D)$ where $\mathcal{E}_U$ is the restriction of $\mathcal{E}$ on U. Once this is shown, Conjecture 4.4.2 implies that $\mathcal{E}_U$ is irreducible, and we get (i) of the theorem for $\mathcal{E}_U$ by [EK, 8.2], and by the induction hypothesis, we conclude.

When $\overline{X}$ is proper smooth, $\overline{X} \setminus X$ is a simple normal crossing divisor, and $\mathcal{E}$ is log-extendable to $\overline{X}$, then we may take $D = 0$. Indeed, for a smooth curve, let $\mathcal{E}$ be an $\overline{\mathbb{Q}}_p$-isocrystal whose determinant is finite, and let $\mathscr{F}$ be its ℓ-adic companion. Then the ramification of $\mathcal{E}$ and $\mathscr{F}$ at the boundary are the same by Theorem 4.2.2 (2). In general, take a semistable reduction $\overline{Y} \to \overline{X}$ of $\mathcal{E}$ (cf. [Ke5]). We take $U \subset X$ so that $p\colon V := U \times_{\overline{X}} \overline{Y} \to U$ is finite étale. There exists a Cartier divisor D such that the ramification of $p_*\overline{\mathbb{Q}}_\ell$ is in $\mathcal{V}_d(U, D)$ where $d = \deg(V/U)$. Then we can check easily that $\mathcal{E} \in \mathcal{V}_r(X, D)$.

For (ii), copy the proof of Drinfeld [Dr, §2.3]. □

4.4.6. **Remark.** Using the main results of this paper, K. S. Kedlaya recently proved Theorem 4.4.3 as well as Theorem 4.4.5 without assuming Conjecture 4.4.2

in [Ke6]. His argument reduces to the curve case using an ingenious induction. By proving Conjecture 4.4.2 when k is finite, Abe and H. Esnault gave another proof of this Kedlaya's result in [AE]. The existence of crystalline companions on smooth schemes of finite type over k is still an open problem.

Appendix A

A.1. **Beilinson–Drinfeld gluing of the derived category.** In this subsection, we briefly recall the construction and some results of Beilinson and Drinfeld [BD, 7.4], which are used in the main text.

A.1.1. Let $\mathcal{M} \to \Delta^+$ be a cofibered category such that its fiber $\mathcal{M}_i$ over $[i] \in \Delta^+$ is an abelian category, and for $\phi\colon [i] \to [j]$, the pushforward ϕ_* is exact. We denote by $\mathcal{M}_{\mathrm{tot}}$ the abelian category of cartesian sections; the category of collections $\{M_n, \alpha_\phi\}$ such that $M_n \in \mathcal{M}_n$ and for $\phi\colon [i] \to [j]$, $\alpha_\phi\colon \phi_* M_i \xrightarrow{\sim} M_j$ satisfying the cocycle condition. Now, we want to construct a suitable triangulated category associated to $\mathcal{M}$ whose heart is $\mathcal{M}_{\mathrm{tot}}$. For this, we consider the category $\mathrm{sec}_+(\mathcal{M})$. The objects consist of collections $\{M_n, \alpha_\phi\}$ where $M_n \in \mathcal{M}_n$, and for $\phi\colon [i] \to [j]$, $\alpha_\phi\colon \phi_* M_i \to M_j$, satisfying the condition $\alpha_{\phi\circ\psi} = \alpha_\phi \circ \phi_*(\alpha_\psi)$ for composable morphisms ϕ and ψ in Δ^+, and $\alpha_{\mathrm{id}} = \mathrm{id}$. We put $\mathrm{sec}_- := (\mathrm{sec}_+(\mathcal{M}^\circ))^\circ$. A profound observation of Beilinson and Drinfeld is that there are functors

$$c_+\colon C(\mathrm{sec}_-(\mathcal{M})) \to C(\mathrm{sec}_+(\mathcal{M})), \qquad c_-\colon C(\mathrm{sec}_+(\mathcal{M})) \to C(\mathrm{sec}_-(\mathcal{M}))$$

such that (c_+, c_-) is an adjoint pair, and the adjunction homomorphisms $c_+c_- \to \mathrm{id}$ and $\mathrm{id} \to c_-c_+$ are quasi-isomorphisms (cf. [BD, 7.4.4]). With these functors, we are able to identify $D(\mathrm{sec}_+(\mathcal{M}))$ and $D(\mathrm{sec}_-(\mathcal{M}))$. Now, let $C_{\mathrm{tot}\pm} \subset C(\mathrm{sec}_\pm(\mathcal{M}))$ be the full subcategory consisting of complexes M such that $\mathscr{H}^i(M) \in \mathcal{M}_{\mathrm{tot}}$. We denote by $K_{\mathrm{tot}\pm}(\mathcal{M})$ and $D_{\mathrm{tot}\pm}(\mathcal{M})$ the corresponding homotopy and derived categories. By means of $c_\pm$, we are able to identify $D_{\mathrm{tot}+}(\mathcal{M})$ and $D_{\mathrm{tot}-}(\mathcal{M})$, and we denote them by $D_{\mathrm{tot}}(\mathcal{M})$. The functors $\mathscr{H}^i\colon D_{\mathrm{tot}\pm} \to \mathcal{M}_{\mathrm{tot}}$ induce $D_{\mathrm{tot}}(\mathcal{M}) \to \mathcal{M}_{\mathrm{tot}}$.

A.1.2. Now, an important aspect of the theory is the existence of a spectral sequence connecting $\{\mathcal{M}_n\}$ and $\mathcal{M}$. For $N \in D^-(\mathrm{sec}_-(\mathcal{M}))$ and $M \in D^+(\mathrm{sec}_+(\mathcal{M}))$, we have the following spectral sequence by [BD, 7.4.8]:

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathcal{M}_p}(N_p, M_p) \Rightarrow \mathrm{Hom}_{D(\mathcal{M})}\bigl(c_+(N), M[p+q]\bigr) \cong \mathrm{Hom}_{D(\mathcal{M})}\bigl(N, c_-(M)[p+q]\bigr). \tag{A.1.2.1}$$

Remark. Since the proof of [BD] is rather sketchy, it might be hard to follow their argument in some cases. Let us add a short explanation. When $\mathrm{sec}_+(\mathcal{M})$ has enough injectives, then we can take the right derived functors of the functor $\mathrm{Hom}(N, -)\colon K^+\mathrm{sec}_+(\mathcal{M}) \to DF$, where DF denotes the derived category of filtered modules, in a usual way, and we get the spectral sequence as written in [BD]. However, there might be a situation in which the functor does not admit a right derived functor. Even in this case, we can define the derived functor $\mathbb{R}\mathrm{Hom}(N, -)\colon D^+(\mathrm{sec}_+(\mathcal{M})) \to \mathrm{Ind}(DF)$ as in [SGA4, XVII, 1.2]. We have a functor

$$\widetilde{\mathscr{H}^i}\colon \mathrm{Ind}(DF) \xrightarrow{\mathscr{H}^i} \mathrm{Ind}(F\mathrm{Ab}) \xrightarrow{\varinjlim} F\mathrm{Ab}.$$

Using this, we have $\widetilde{\mathscr{H}^i}\mathrm{gr}^n_F\mathbb{R}\mathrm{Hom}(N, M) \cong \mathrm{Hom}_{D(\mathcal{M}_n)}(N_n, M_n[i-n])$. This follows from the fact that we have $\mathrm{gr}^n_F\mathrm{Hom}(N, M) \cong \mathrm{Hom}_{K(\mathcal{M}_n)}(N_n, M_n)[-n]$ by construction of the functor $\mathrm{Hom}(N,-)$, and the functor $M \mapsto M_n$ has an exact right adjoint as in 2.1.6 which implies the existence of g and h in [SGA4]. Since the category of spectral sequences of abelian groups admits inductive limits, we get the desired result.

A.2. **Some properties of algebraic stacks.** The results of this subsection are used implicitly in Lafforgue's proof of Langlands correspondence. Even though we believe that the results are well known to experts, since we are not able to find references, we decided to write down the details.

A.2.1. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a gerb-like morphism* [Beh, 5.1.3] *such that the structural group is flat. Then there exists a presentation $Y \to \mathfrak{Y}$ such that $\mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$ is a neutral gerb.*

Proof. First, we note that f is smooth surjective. Indeed, since the verification is fppf-local, we may assume that f is neutral, and thus $\mathfrak{Y} =: Y$ is a scheme and $\mathfrak{X} = B(G/Y)$. Since G is assumed flat, f is smooth by [Beh, 5.1.2].

Let $P\colon Y \to \mathfrak{X}$ be a presentation, and consider the smooth morphism $Q := f \circ P\colon Y \to \mathfrak{Y}$, which is a presentation of $\mathfrak{Y}$ since f is smooth surjective. We have the morphism $(P, \mathrm{id})\colon Y \to \mathfrak{X} \times_{\mathfrak{Y}} Y$. This defines a section of the second projection $\mathfrak{X} \times_{\mathfrak{Y}} Y \to Y$. $\square$

A.2.2. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of algebraic stacks over an integral scheme S. For the generic point $\eta \in S$, if f_η is separated, then there exists an open subscheme $U \subset S$ such that $f_U\colon \mathfrak{X} \times_S U \to \mathfrak{Y} \times_S U$ is separated.*

Proof. Let $\mathcal{Y} \to \mathfrak{Y}$ be a presentation, and let $\mathcal{X} \to \mathfrak{X}$ be the induced presentation. Let $f'\colon \mathcal{X} \to \mathcal{Y}$ be the induced morphism. The morphism f being separated is equivalent to f' being separated, and thus by [EGAIV, 8.10.5], the lemma follows. $\square$

A.2.3. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a representable morphism of locally noetherian algebraic stacks. Then there exists an open dense substack $\mathfrak{U}$ of $\mathfrak{X}$ such that $f|_{\mathfrak{U}}$ is separated.*

Proof. By [EGAI, 5.5.1 (vi)], we may assume that $\mathfrak{X}$ and $\mathfrak{Y}$ are reduced. By shrinking $\mathfrak{X}$ and $\mathfrak{Y}$, we may assume that $\mathfrak{X} \to \mathcal{X}$ and $\mathfrak{Y} \to \mathcal{Y}$ are gerbs over algebraic spaces by [LM, 11.5]. By shrinking $\mathfrak{X}$ and $\mathfrak{Y}$, we may assume that $\mathcal{X}$ is a separated scheme and $\mathcal{X} \to \mathcal{Y}$ is separated. Consider the following diagram:

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\alpha"] \arrow[dr, bend right] & \mathfrak{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ & \mathfrak{Y} \arrow[r] & \mathcal{Y}. \end{tikzcd}$$

Note that α is representable by [LM, 3.12 (c)]. Let η be a generic point of $\mathcal{X}$. By Lemma A.2.2, it suffices to show that α_η is separated. So the statement is reduced to the following special case of the lemma:

Claim. Let $\mathfrak{X}$ and $\mathfrak{X}'$ be gerbs over $\mathrm{Spec}(K)$, and let $\alpha\colon \mathfrak{X} \to \mathfrak{X}'$ be a representable morphism. Then this is separated.

Proof. Since there exists a scheme $X \to \mathrm{Spec}(K)$ such that $\mathfrak{X}$ and $\mathfrak{X}'$ are neutral over X, by taking a closed point of X, $\mathfrak{X}$ and $\mathfrak{X}'$ are neutral over a finite extension of K. Since the claim is stable under finite extension, we may assume that $\mathfrak{X}$ and $\mathfrak{X}'$ are neutral gerbs over $\mathrm{Spec}(K)$. Thus, by taking the automorphism groups G and G' of $\mathfrak{X}(K)$ and $\mathfrak{X}'(K)$, we have a homomorphism of K-group spaces $\rho\colon G \to G'$ over $\mathcal{X}$, which induces α. We have the cartesian diagram

$$\begin{tikzcd} G \arrow[r, "\rho"] \arrow[d] \arrow[dr, phantom, "\square"] & G' \arrow[d] \\ BG \arrow[r, "\Delta"'] & BG \times_{BG'} BG. \end{tikzcd}$$

Since the diagonal morphism Δ is quasi-compact by [LM, 7.7], ρ is quasi-compact. By [SGA3, $\mathrm{VI_A}$, Cor 6.7], ρ decomposes as $G \twoheadrightarrow G/N \hookrightarrow G'$. This induces a morphism $BG \to B(G/N) \to BG'$. Since α is assumed representable, the morphism $BG \to B(G/N)$ is representable as well. This can only happen when N is trivial. Thus, ρ is a closed immersion by the same corollary of [SGA3]. This shows that Δ is a closed immersion, and thus α is separated. □■

A.2.4. **Lemma.** *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism locally of finite type between* reduced *algebraic stacks such that*

$$\dim\bigl(\mathfrak{Aut}_{\mathfrak{Y}}\,\mathfrak{X}\bigr) = \dim(\mathfrak{Y}) - \dim(\mathfrak{X}).$$

Then there exists a dense open substack $\mathfrak{V} \subset \mathfrak{Y}$ such that $f|_{f^{-1}(\mathfrak{V})}$ can be factorized as $\mathfrak{X} \xrightarrow{p} \mathfrak{X}' \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y}$, where p is a gerb-like morphism with the structure group space $\mathfrak{Aut}_{\mathfrak{Y}}\,\mathfrak{X}$, g is a representable universal homeomorphism, and h is a representable finite étale morphism.

Proof. Locally on $\mathfrak{X}$, f factors as $\mathfrak{X} \xrightarrow{p} \mathfrak{X}' \xrightarrow{\alpha} \mathfrak{Y}$ such that p is gerb-like and α is a representable morphism [Beh, 5.1.13, 5.1.14]. By the assumption on the dimension, α is a representable quasi-finite morphism. By shrinking $\mathfrak{X}$, and by using Lemma A.2.2, we may assume that α is separated. By using Zariski's main theorem (cf. [LM, 16.5]), by shrinking if $\mathfrak{Y}$ necessary, we may assume that α is a finite morphism.

Let $f\colon X \to Y$ be a finite morphism between integral schemes such that X is normal. The finite extension $K(X)/K(Y)$ of fields can be factorized canonically as $K(X)/M/K(Y)$ such that $K(X)/M$ is purely inseparable and $M/K(Y)$ is separable. Let Z be the normalization of Y in $\mathrm{Spec}(M)$. The morphism f factors as the composition of finite morphisms $X \to Z \to Y$. This construction is compatible with smooth base change $Y' \to Y$ by [LM, 16.2]. Thus, given a finite morphism $\mathfrak{X} \to \mathfrak{Y}$ of reduced algebraic stacks, by shrinking $\mathfrak{Y}$ if necessary, we have a factorization $\mathfrak{X} \to \mathfrak{Z} \to \mathfrak{Y}$ such that the first morphism is generically purely inseparable and the second is generically finite étale by [LM, 14.2.4].

Apply this factorization to α, and we get a factorization $\mathfrak{X}' \xrightarrow{g} \mathfrak{Z} \xrightarrow{h} \mathfrak{Y}$ satisfying the condition above. Take a presentation $Z \to \mathfrak{Z}$, and let $X' \to \mathfrak{X}'$ be the pullback. Then by construction, $\widetilde{g}\colon X' \to Z$ is generically purely inseparable. By [EGAIV, 1.8.7], by shrinking Z, we may assume that $\widetilde{g}$ is radicial and surjective for any fiber of Z. Thus $\widetilde{g}$ is radicial and surjective as well, and moreover since $\widetilde{g}$ is finite, it is a universal homeomorphism by [EGAIV, 2.4.5]. By replacing $\mathfrak{Z}$ by the image of Z and $\mathfrak{X}'$ by the pullback of newly constructed $\mathfrak{Z}$, g can be made universally homeomorphic. By removing the ramification locus of h from $\mathfrak{Y}$, we may assume that h is finite étale. □

A.3. **Lefschetz fixed point theorem.** In this subsection, we prove a Lefschetz fixed point theorem for arithmetic $\mathscr{D}$-modules. In the case of realizable schemes, the theorem has already been proven in [AC1]. This can be generalized to the case of a separated scheme of finite type in an obvious manner, but we present it here for the convenience of the reader and for the future reference.

A.3.1. We consider a similar situation as in 4.2.1: Let k be a finite field with $q = p^s$ elements, and fix an algebraic closure $\overline{\mathbb{Q}}_p$ of K. We denote by $\overline{k}$ the residue field of $\overline{\mathbb{Q}}_p$ which is algebraically closed as well. We put $\mathfrak{T}_k := (k, R := W(k), K := \mathrm{Frac}(R), L, s, \sigma := \mathrm{id})$ to be an arithmetic base tuple (cf. 1.4.10, 2.4.14). Let X be a scheme over k, and let $i_x \colon x \hookrightarrow X$ be a closed point of X. Choose a geometric point $\overline{x} \in X(\overline{k})$ lying above x. This defines an embedding $K_x := \mathrm{Frac}(W(k(x))) \hookrightarrow \overline{\mathbb{Q}}_p$. Let L_x be the field generated by K_x and L in $\overline{\mathbb{Q}}_p$. Put $\mathfrak{T}_{x,k} := (k, R, K, L_x, s, \mathrm{id})$, $\mathfrak{T}_{k(x)} := (k(x), R_x, K_x, L_x, [k(x) : k] \cdot s, \mathrm{id})$, and we have the functor

$$\iota_{\overline{x}} \colon D^{\mathrm{b}}_{\mathrm{hol}}(X/\mathfrak{T}_k) \xrightarrow{i_x^+} D^{\mathrm{b}}_{\mathrm{hol}}(k(x)/\mathfrak{T}_k) \xrightarrow{\otimes_L L_x} D^{\mathrm{b}}_{\mathrm{hol}}(k(x)/\mathfrak{T}_{x,k}) \cong D^{\mathrm{b}}_{\mathrm{hol}}(k(x)/\mathfrak{T}_{k(x)}),$$

where the equivalence follows by Corollary 1.4.11 using the embedding. For $\mathscr{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/\mathfrak{T}_k)$, the automorphism on $\iota_{\overline{x}}(\mathscr{E})$ is denoted by F_x. We note that the eigenpolynomial of F_x, which is a priori in $L_x[t]$, only depends on x not on $\overline{x}$ and belongs in fact to $L[t]$. Indeed, the independence of $\overline{x}$ follows from the construction of the equivalence in Corollary 1.4.11. Let us check that the eigenpolynomial, which we denote by $\chi(\iota_{\overline{x}}\mathscr{E}, t)$ for the moment, is in $L[t]$. We may replace L_x by the Galois closure, and we may assume that L_x is a Galois extension. Take an automorphism γ of L_x over L. This induces an endofunctor γ^* on $D^{\mathrm{b}}_{\mathrm{hol}}(k(x)/\mathfrak{T}_{k(x)})$, and $\chi(\gamma^*\iota\mathscr{E}, t) = \gamma(\chi(\iota\mathscr{E}, t))$. However, we have a canonical isomorphism $\gamma^*\iota\mathscr{E} \cong \iota\mathscr{E}$ since we are taking $\otimes_L L_x$ in the definition of $\iota_{\overline{x}}$, which implies that $\gamma(\chi(\iota\mathscr{E}, t)) = \chi(\iota\mathscr{E}, t)$.

Let X be a separated scheme of finite type over k, and let $\mathscr{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/\mathfrak{T}_k)$. We put $\mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E}) := f_!(\mathscr{E})$ where f is the structural morphism of X, and $H^{\nu}_{\mathrm{c}}(X, \mathscr{E}) := \mathscr{H}^{\nu}\mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E})$ as usual. The object $\mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E})$ is equipped with an automorphism, and we denote this automorphism by Frob_X. For an extension L' of K and $\mathscr{F} \in D^{\mathrm{b}}_{\mathrm{fin}}(\mathrm{Vec}_{L'})$ equipped with automorphism φ, we put $\mathrm{Tr}_{L'}(\varphi; \mathscr{F}) := \sum_{\nu \in \mathbb{Z}} (-1)^{\nu} \mathrm{Tr}_{L'}(\varphi; \mathscr{H}^{\nu}(\mathscr{F}))$, and similarly for the determinant except that we take an alternating product instead of sum. By using the same argument as [AC1, 4.3.9], we have the following theorem:

A.3.2. **Theorem** (Essetially due to [EL])**.** *Let X be a separated scheme of finite type over k, and let $\mathscr{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$. Let k_n be the extension of k of degree n. Then we have the following identity in $\overline{\mathbb{Q}}_p$:*

$$\mathrm{Tr}_L\bigl(\mathrm{Frob}^n_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E})\bigr) = \sum_{x \in X(k_n)} \mathrm{Tr}_{L_x}(F^n_x; \iota_{\overline{x}}(\mathscr{E})).$$

Proof. First of all, let us show the theorem in the case $L = K$. We argue by induction on the dimension of X. Since both sides of the formula are multiplicative with respect to exact triangles, we may assume that X is affine by using the localization exact sequence 2.2.9. By the Noether normalization lemma, we may find a finite dominant morphism $f \colon X \to \mathbb{A}^d$. Thus, we may assume that $X = \mathbb{A}^d$. By the induction hypothesis, we may shrink X and assume that X is affine and $\mathscr{E}$ is smooth on $X \subset \mathbb{A}^d$. In this case, we know that $H^{\nu}_{\mathrm{c}}(X, \mathscr{E})$ is isomorphic to the rigid cohomology by 2.4.15, thus the formula is a result of Etesse and Le Stum [EL].

Now, we show the general case. We may assume that the extension L/K is finite. By *dévissage*, we may assume that $X(k_n) = \emptyset$, and we only need to show that the left-hand side of the equality vanishes. Replacing L by its finite extension, we may assume that an n-th root of $\alpha := \mathrm{Tr}_L(\mathrm{Frob}^n_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E}))$ is contained in L. We denote by β an n-th root. Our goal is to show $\alpha = 0$. We assume the contrary, so $\beta \neq 0$. Let $\mathscr{E} = (\mathscr{F}, \Phi_{\mathscr{F}})$, where $\mathscr{F}$ is the underlying object in $\mathrm{Hol}(X/L_{\emptyset})$ and $\Phi_{\mathscr{F}}$ is the Frobenius structure. We define $\mathscr{E}^{(1/\beta)}$ to be an object in $\mathrm{Hol}(X/L_F)$ such that the underlying object is the same as $\mathscr{F}$, and the Frobenius structure is the composition

$$\mathscr{F} \xrightarrow[\sim]{\cdot(1/\beta)} \mathscr{F} \xrightarrow[\sim]{\Phi_{\mathscr{F}}} F^*\mathscr{F}.$$

Then $\mathrm{Tr}_L(\mathrm{Frob}^n_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E}^{(1/\beta)})) = (1/\beta)^n \times \mathrm{Tr}_L(\mathrm{Frob}^n_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E})) = 1$. For any L-vector space V with automorphism ϕ, we have $\mathrm{Tr}_{L/K}\big(\mathrm{Tr}_L(\phi; V)\big) = \mathrm{Tr}_K(\phi; V)$, where $\mathrm{Tr}_{L/K}$ denotes the field trace. Combining these, we have

$$\begin{aligned} 0 = \mathrm{Tr}_K(\mathrm{Frob}_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{F}^{(1/\beta)})) &= \mathrm{Tr}_{L/K}\big(\mathrm{Tr}_L(\mathrm{Frob}_X; \mathbb{R}\Gamma_{\mathrm{c}}(X, \mathscr{E}^{(1/\beta)}))\big) \\ &= \mathrm{Tr}_{L/K}(1) = [L : K], \end{aligned}$$

where the first equality holds by the case $L = K$. This is a contradiction and implies that $\alpha = 0$. □

A.3.3. Now, let $\mathscr{E}$ be an object in $D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$. We define series in $L[\![Z]\!]$

$$L_x(\mathscr{E}, Z) := \det\big(1 - Z^{\deg(x)} F_x; \iota_{\overline{x}}(\mathscr{E})\big)^{-1}, \qquad L_X(\mathscr{E}, Z) := \prod_{x \in |X|} L_x(\mathscr{E}, Z).$$

Since the first one does not depend on the choice of $\overline{x}$, these are well-defined. The first one (resp. second one) is called the *local L-function* (resp. *global L-function*). By a standard argument (cf. for example, [SGA4$\frac{1}{2}$, Rapport, §3]), we have a cohomological interpretation of an L-function as a consequence of the Lefschetz fixed point theorem.

Corollary. *Let X be a separated scheme of finite type over k, and let $\mathscr{E} \in D^{\mathrm{b}}_{\mathrm{hol}}(X/L_F)$. Then we have an identity of formal power series:*

$$L_X(\mathscr{E}, Z) = \prod_{\nu \in \mathbb{Z}} \det\big(1 - Z \cdot \mathrm{Frob}_X; H^{\nu}_{\mathrm{c}}(X, \mathscr{E})\big)^{(-1)^{\nu+1}}.$$

A.4. **Čebotarev density (after N. Tsuzuki).** The Čebotarev density theorem for curves and mixed isocrystals is proven in [A2]. We need the Čebotarev density for surfaces and mixed isocrystals, which we show in this appendix. We could have included it in the main text, but since the author learned the proof from N. Tsuzuki before writing this paper, it has been kept separate. We consider the situation of §A.3.

A.4.1. **Proposition.** *Let X be a smooth variety over a finite field k. Let $\mathcal{E}$ and $\mathcal{E}'$ be ι-mixed overconvergent F-isocrystals in $\mathrm{Isoc}^{\dagger}(X/L_F)$ such that the sets of Frobenius eigenvalues are the same for any closed point of X. Then $\mathcal{E}^{\mathrm{ss}} = \mathcal{E}'^{\mathrm{ss}}$ where the semisimplification is taken in $\mathrm{Isoc}^{\dagger}(X/L_F)$.*

Proof. Since we have weight filtration on $\mathcal{E}$ and $\mathcal{E}'$ by [AC1, 4.3.4], we may assume that $\mathcal{E}$ and $\mathcal{E}'$ are ι-pure. Let $\mathcal{F}$ be an irreducible overconvergent F-isocrystal. Since Frobenius eigenvalues of $\mathcal{E}$ and $\mathcal{E}'$ are the same, we have

$$L(\mathcal{E} \otimes \mathcal{F}^{\vee}, Z) = L(\mathcal{E}' \otimes \mathcal{F}^{\vee}, Z).$$

If μ is the multiplicity of $\mathcal{F}$ in $\mathcal{E}^{\mathrm{ss}}$, Proposition 4.3.3 implies that $L(\mathcal{E}\otimes\mathcal{F},Z)$ has a pole of order μ at $Z=q^{-\dim(X)}$. The equality tells us that $L(\mathcal{E}'\otimes\mathcal{F}^{\vee},Z)$ has a pole of the same order at the same point, and using the proposition again, we get that $\mathcal{E}'^{\mathrm{ss}}$ contains $\mathcal{F}$ with multiplicity μ. Note that Proposition 4.3.3 is stated for $\overline{\mathbb{Q}}_p$-coefficients, but the proof goes through also for L-coefficients with obvious changes. □

A.5. **Gabber–Fujiwara ℓ-independence.** We generalize the Gabber–Fujiwara ℓ-independence results (cf. [F]) to admissible stacks. For a category $\mathcal{C}$, we denote by $[\mathcal{C}]$ the set of isomorphism classes of $\mathcal{C}$.

A.5.1. **Theorem** (Trace formula)**.** *Let $\mathfrak{X}$ be a c-admissible stack over finite field $\mathbb{F}_q$. Let $\mathscr{M}$ be a complex in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$. For $x\in[\mathfrak{X}(\mathbb{F}_q)]$, we denote by $i_x\colon\mathrm{Spec}(\mathbb{F}_q)\to\mathfrak{X}$ the corresponding morphism. Then, for any $n>0$, we have*

$$\sum_{i\in\mathbb{Z}}(-1)^i\cdot\mathrm{Tr}\big(F^n:H^i_{\mathrm{c}}(\mathfrak{X},\mathscr{M})\big)=\sum_{x\in[\mathfrak{X}(\mathbb{F}_q)]}\frac{1}{\#\mathrm{Aut}(x)}\cdot\mathrm{Tr}\big(F^n:i_x^+(\mathscr{M})\big).$$

Remark. Note that both sides of the equality are finite sums since we are dealing with c-admissible stacks, contrary to the case of more general algebraic stacks. This prevents us from struggling with the convergence issues as in [Beh], which makes it much easier to formulate and prove.

Proof. Since we can prove this similarly to [La2, A.14] or [Beh, 6.4.10], we only sketch the proof here. Let us denote the right-hand side of the equality by $L(\mathfrak{X},\mathscr{M})$. For a morphism $f\colon\mathfrak{X}\to\mathfrak{Y}$ of c-admissible stacks, it suffices to show the equality $L(\mathfrak{X},\mathscr{M})=L(\mathfrak{Y},f_!(\mathscr{M}))$ since the theorem is the particular case where $\mathfrak{Y}=\mathrm{Spec}(\mathbb{F}_q)$. When f is a morphism between schemes, then this equality is already known by Theorem A.3.2. Now, by the localization triangle, the verification is local with respect to $\mathfrak{X}$. By using [LM, 11.5] and some standard *dévissage* argument, it suffices to treat the case where f is gerb-like. By definition of $L(-,-)$ combining with [Beh, 6.4.2], it is reduced to showing the theorem in the case $\mathfrak{X}=BG$ with a finite flat group scheme G over $\mathrm{Spec}(\mathbb{F}_q)$, and $\mathscr{M}$ is $\overline{\mathbb{Q}}_p$. Since the morphism $BG_{\mathrm{red}}\to BG$ is a representable universal homeomorphism, we have $H^i(BG,\overline{\mathbb{Q}}_p)\xrightarrow{\sim}H^i(BG_{\mathrm{red}},\overline{\mathbb{Q}}_p)$, and we may assume G to be smooth. By considering the universal torsor $\mathrm{Spec}(\mathbb{F}_q)\to BG$, which is finite since G is, we get $H^i(BG,\overline{\mathbb{Q}}_p)=0$ for $i\neq 0$. The calculation of H^0 is left to the reader. □

A.5.2. **Definition.** Let $\mathfrak{X}$ be an algebraic stack over $\mathbb{F}_q$, and let $\mathscr{E}$ (resp. $\mathcal{F}$) be an object in $D^{\mathrm{b}}_{\mathrm{hol}}(\mathfrak{X}/\overline{\mathbb{Q}}_{p,F})$ (resp. $D^{\mathrm{b}}_{\mathrm{c}}(X/\overline{\mathbb{Q}}_\ell)$). For $x\in[\mathfrak{X}(\mathbb{F}_{q^d})]$, we denote by $i_x\colon\mathrm{Spec}(\mathbb{F}_{q^d})\to\mathfrak{X}$ and $\rho\colon\mathrm{Spec}(\mathbb{F}_{q^d})\to\mathrm{Spec}(\mathbb{F}_q)$ the canonical morphisms. We say that $\mathscr{E}$ and $\mathcal{F}$ are *compatible* if for any point $x\in[\mathfrak{X}(\mathbb{F}_{q^d})]$, the Frobenius trace of $\rho_+\circ i_x^+(\mathscr{E})$ and $\rho_*\circ i_x^*(\mathcal{F})$ are equal.

A.5.3. **Lemma.** *The couple $(\mathscr{E},\mathcal{F})$ are compatible if and only if for any $X\in\mathfrak{X}_{\mathrm{sm}}$, the pullbacks $\mathscr{E}_X$ and $\mathcal{F}_X$ are compatible.*

Proof. Use [LM, 6.3]. □

A.5.4. **Theorem.** (i) *Let $f\colon\mathfrak{X}\to\mathfrak{Y}$ be a morphism between c-admissible stacks. Then f_+, $f_!$, f^+, $f^!$, $\mathbb{D}$, and $\otimes$ preserve compatible systems.*

(ii) *When $j\colon\mathfrak{U}\hookrightarrow\mathfrak{X}$ is an immersion of c-admissible stacks, $j_{!+}$ preserves compatible systems.*

Proof. By Lemma A.5.3, the theorem for f^+, $\mathbb{D}$, $j_{!+}$ follows from [AC1, 4.3.11]. We only need to show the theorem for $f_!$. For this, use the trace formula A.5.1. □

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