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# A RANDOM SCHRÖDINGER OPERATOR ASSOCIATED WITH THE VERTEX REINFORCED JUMP PROCESS ON INFINITE GRAPHS

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## 1. Introduction

This paper concerns the Vertex Reinforced Jump Process (VRJP) and the Edge Reinforced Random Walk (ERRW) and their relation with a random Schrödinger operator associated with a stationary 1-dependent random potential (i.e., the potential is independent at distance larger than or equal to 2).

The VRJP is a continuous time self-interacting process introduced in [5], investigated on trees in [2,3] and on general graphs in [20,21]. We first recall its definition. Let $\mathcal{G} = (V, E)$ be an undirected graph with finite degree at each vertex. We write $i \sim j$ if $i \in V$, $j \in V$, and $\{i, j\}$ is an edge of the graph. We always assume that the graph is connected and has no trivial loops (i.e., vertex $i$ such that $i \sim i$). Let $(W_{i,j})_{i\sim j}$ be a set of positive conductances, $W_{i,j} > 0$, $W_{i,j} = W_{j,i}$. The VRJP is the continuous time process $(Y_s)_{s\geq 0}$ on $V$, starting at time 0 at some vertex $i_0 \in V$, which, conditionally on the past at time $s$, if $Y_s = i$, jumps to a neighbour $j$ of $i$ at rate

$$W_{i,j}L_j(s),$$

where

$$L_j(s) := 1 + \int_0^s \mathbb{1}_{\{Y_u=j\}}\, du.$$

In [20] Sabot and Tarrès introduced the following time change of the VRJP

$$Z_t = Y_{D^{-1}(t)}, \tag{1.1}$$

where $D(s)$ is the following increasing function

$$D(s) = \sum_{i\in V} (L_i^2(s) - 1).$$

We call this process the VRJP in exchangeable time scale and denote by $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$ its law starting from the vertex $i_0$. When the graph is finite, it is proved in Theorem 2 of [20] that the VRJP in exchangeable time scale $(Z_t)_{t\geq 0}$ is a mixture of Markov

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jump processes. More precisely, there exists a random field $(u_j)_{j\in V}$ such that $Z$ is a mixture of Markov jump processes with jump rates from $i$ to $j$

$$\frac{1}{2}W_{i,j}e^{u_j-u_i}.$$

The law of $(u_j)$ is explicit; cf. [20, Theorem 2] and the forthcoming Theorem B. It appears to be a marginal of a supersymmetric $\sigma$-field which had been investigated previously by Disertori, Spencer, and Zirnbauer (cf. [9], [10], [24]). As a consequence of this representation and of [9], [10], it was proved in [20] that when the graph has bounded degree, there exists a real $\lambda_0 > 0$ such that if $W_{i,j} \leq \lambda_0$ for all $i \sim j$, then the VRJP is positively recurrent; more precisely, $Z$ is a mixture of positive recurrent Markov jump processes. When the graph is the grid $\mathbb{Z}^d$, with $d \geq 3$, there exists $\lambda_1 < +\infty$ such that if $W_{i,j} \geq \lambda_1$ for all $i \sim j$, the VRJP is transient. Hence, it shows a phase transition between recurrence and transience in dimension $d \geq 3$. The question of the representation of the VRJP on infinite graphs as a mixture of Markov jump processes is nontrivial, especially in the transient case. It is possible to prove such a representation by a weak convergence argument, following [16], but it gives little information on the mixing law. In this paper we prove such a representation involving the Green function and a generalized eigenfunction of a random Schrödinger operator.

Let us give a flavour of the main results of the paper in the case of the VRJP on $\mathbb{Z}^d$ with $W_{i,j} = W$ constant. We construct a positive 1-dependent random potential $(\beta_j)_{j\in\mathbb{Z}^d}$ (i.e., two subsets of the $\beta$'s are independent if their indices are at least at distance 2) and with marginal given by inverse of Inverse Gaussian law with parameters $(\frac{1}{dW}, 1)$. This field is a natural extension to infinite graphs of the field defined by Sabot, Tarrès, and Zeng in [22]. We consider the random Schrödinger operator

$$H_\beta = -W\Delta + V,$$

where $\Delta$ is the usual discrete (nonpositive) Laplacian and $V$ is the multiplication operator defined by $V_j = 2\beta_j - 2dW$. Hence, it corresponds to the Anderson model with a random potential which is not i.i.d. but only stationary and 1-dependent. When the VRJP is transient, we prove that there exists a positive generalized eigenfunction $\psi$ of $H_\beta$ with eigenvalue 0, stationary and ergodic. Let $(G(i,j))_{i\in Z^d, j\in\mathbb{Z}^d}$ be defined by

$$G(i,j) = \widehat{G}(i,j) + \frac{1}{2}\gamma^{-1}\psi(i)\psi(j),$$

where $\widehat{G} = (H_\beta)^{-1}$ is the Green function (which happens to be well defined in an appropriate sense) and $\gamma$ is an extra random variable independent of the field $\beta$ with law $\Gamma(\frac{1}{2}, 1)$. We prove the following representation for the VRJP: the VRJP in exchangeable time scale $Z$ starting from the point $i_0$ is a mixture of Markov jump processes with jump rates from $i$ to $j$,

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}. \tag{1.2}$$

When the VRJP is recurrent, the same representation is valid with $\psi = 0$. In fact, the function $\psi$ is almost surely the limit of a martingale, the limit being positive when the VRJP is transient and 0 when the VRJP is recurrent. It is remarkable that when the VRJP is recurrent it can be represented as a mixture with

$\beta$-measurable jump rates, but when the VRJP is transient it involves an extra independent Gamma random variable. This representation extends to infinite graphs the representation given in [22] for finite graphs. A new feature appears in the transient case, where the generalized eigenfunction $\psi$ is involved in the representation. We suspect that recurrence/transience of the VRJP is related to localization/delocalization of the random Schrödinger operator $H_\beta$ at the bottom of the spectrum.

The representation (1.2) has several consequences on the VRJP and the ERRW. The ERRW is a reinforced process introduced by Coppersmith and Diaconis in 1986 (see section 2.5 for a definition). The recurrence of the two-dimensional ERRW is a famous open question raised by Diaconis; see [4, 12, 17, 18] for early references. Important progress has been made recently in the understanding of this process. In particular, in [20], an explicit relation between the ERRW and the VRJP was stated, thus somehow reducing the analysis of the ERRW to that of the VRJP. In [1, 20], it was proved by rather different methods that the ERRW on any graph with bounded degree at strong enough reinforcement is positive recurrent. In [8], it was proved that the ERRW is transient on $\mathbb{Z}^d$, $d \geq 3$, at weak reinforcement.

The representation (1.2) allows us to complete the picture both in dimension 2 and in the transient regime. More precisely, we prove a functional central limit theorem for the ERRW and for the discrete time process associated with the VRJP in dimension $d \geq 3$ at weak reinforcement, using the estimates of [8, 10]. Using the polynomial estimate provided by Merkl and Rolles in [17], we are able to prove recurrence of ERRW on $\mathbb{Z}^2$ for all initial constant weights, hence giving a full answer to the question raised by Diaconis.

## 2. Statements of the results

2.1. **Notation.** We denote by $\mathbb{R}_+$ (resp. $\mathbb{R}_+^*$) the set of nonnegative (resp. positive) reals.

Let $\mathcal{G} = (V, E)$ be an undirected, locally finite, connected graph without trivial loops or multiple edges. For $i, j \in V$, write $i \sim j$ if $i$ is a neighbor of $j$. We write $\mathrm{d}_\mathcal{G}$ for the graph distance in $\mathcal{G}$, and for two subsets $U, U'$ of $V$, we define $\mathrm{d}_\mathcal{G}(U, U') = \inf_{i\in U, j\in U'} \mathrm{d}_\mathcal{G}(i, j)$. We suppose given, for each edge $e = \{i, j\} \in E$, a positive real $W_{i,j} > 0$, which is understood as the conductance of the edge $e$. In this case we call $(\mathcal{G}, (W_e)_{e\in E})$ a graph with conductances.

**Convention.** We adopt the notation $\sum_{i\sim j}$ for the sum on all undirected edges $\{i, j\}$, counting each edge only once.

When $\beta = (\beta_i)_{i\in V} \in \mathbb{R}^V$ is a real vector indexed by the vertices and $U \subset V$, we write $\beta_U$ for the restriction of $\beta$ to $U$, i.e., $\beta_U = (\beta_i)_{i\in U}$. When $A = (A_{i,j})_{i,j\in V} \in \mathbb{R}^{V\times V}$ is a real function on $V \times V$ and $U \subset V$, $U' \subset V$, we write $A_{U,U'}$ for the restriction of $A$ to $U \times U'$, i.e., $A_{U,U'} = (A_{i,j})_{i\in U, j\in U'}$.

It will be convenient to define the continuous time processes that appear in the text on the same canonical space. In the rest of this paper we will denote by $D([0, \infty), V)$ the space of càdlàg functions from $[0, \infty)$ to $V$. The law of the VRJP in an exchangeable time scale, as defined in (1.1) and starting from $i_0$, will be denoted by $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$, which is a probability on $D([0, \infty), V)$. The VRJP will always be defined on the canonical space, and $(Z_t)_{t\in\mathbb{R}_+}$ will denote the canonical process defined by $Z_t(\omega) = \omega(t)$ for $\omega \in D([0, \infty), V)$.

*Remark* 1. We do not allow multiple edges or trivial loops since it does not bring more generality to the VRJP. Indeed, from its definition, it follows that the VRJP on a graph with multiple edges and trivial loops has the same law as the VRJP on the graph where trivial loops are removed and multiple edges are replaced by a single edge by summing the conductances of the multiples edges. Similarly, the law on random potentials that appears in the rest of this paper can always be reduced to graphs without multiple edges or trivial loops. Nevertheless, in section 5 it simplifies notation to allow trivial loops.

2.2. **Representation of the VRJP on infinite graphs.** Define the operator $P = (P_{i,j})_{i,j\in V}$ by

$$P_{i,j} = \begin{cases} W_{i,j}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

We define below a probability distribution on potentials on the graph. A potential on the graph will generically be denoted $\beta = (\beta_i)_{i\in V} \in \mathbb{R}^V$. With the potential $\beta \in \mathbb{R}^V$, we associate the Schrödinger operator on $\mathcal{G}$

$$H_\beta = -P + 2\beta, \tag{2.1}$$

where $\beta$ represents the operator of multiplication by the potential $(\beta_i)$ (or equivalently the diagonal operator with diagonal terms $(\beta_i)_{i\in V}$).

We denote by

$$\mathcal{D}_V^W = \{\beta \in \mathbb{R}^V,\ \ (H_\beta)_{U,U} > 0 \text{ for all finite subsets } U \subset V\}, \tag{2.2}$$

where $(H_\beta)_{U,U} > 0$ means that the restriction of $H_\beta$ to $U \times U$ is positive definite. Obviously, $\mathcal{D}_V^W \subset (\mathbb{R}_+^*)^V$ since when $U = \{i\}$, the restriction of $H_\beta$ is the real $2\beta_i$. We endow $\mathcal{D}_V^W$ with its Borelian $\sigma$-field denoted $\mathcal{B}(\mathcal{D}_V^W)$.

The following statement extends the random potential defined in [22, Theorem 1], to infinite graphs.

**Proposition 1.** *Let $(\mathcal{G}, (W_e)_{e\in E})$ be a graph with conductances as defined in section 2.1. There exists a unique probability distribution $\nu_V^W$ defined on $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W))$, such that for any finite subset $U \subset V$ and any $(\lambda_i)_{i\in U} \in \mathbb{R}_+^U$:*

$$\int e^{-\sum_{i\in U}\lambda_i\beta_i}\nu_V^W(d\beta)$$
$$= e^{-\sum_{i\sim j,\ i,j\in U} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)-\sum_{i\sim j, i\in U, j\notin U} W_{i,j}(\sqrt{1+\lambda_i}-1)}\frac{1}{\prod_{i\in U}\sqrt{1+\lambda_i}}.$$

*In particular, on the probability space $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W), \nu_V^W(d\beta))$, we have the following properties:*

- 1-dependence: *If $U, U' \subset V$ are such that $\mathrm{d}_{\mathcal{G}}(U, U') \geq 2$, then the random variables $\beta \mapsto \beta_U$ and $\beta \mapsto \beta_{U'}$ are independent.*
- Reciprocal inverse Gaussian marginals: *For $i \in V$, the random variable $\beta \mapsto \frac{1}{2\beta_i}$ has an inverse Gaussian distribution with parameter $(\frac{1}{W_i}, 1)$ where $W_i = \sum_{j\sim i} W_{i,j}$.*

*Remark* 2. On finite graphs, the density of $\nu_V^W$ is explicit; cf. [22, Theorem 1], and Theorem A below.

In the rest of this paper, the probability space $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W), \nu_V^W)$ will be considered as the canonical space of random potentials on the graph. We write $\mathbb{E}_{\nu_V^W}$ for the expectation with respect to $\nu_V^W$. We will introduce several random variables on this probability space and adopt the following notation: when $\beta \mapsto X_\beta$ is a measurable function, we will write $X$ for the associated random variable and $X_\beta$ for its realization on the potential $\beta$. In particular, we will write $H$ for the random Schrödinger operator $\beta \mapsto H_\beta$ defined above. By abuse of notation, we sometimes consider $\beta_i$ for $i \in V$ or $\beta_U$ for $U \subset V$ as random variables (more precisely, the random variables are $\beta \mapsto \beta_i$ and $\beta \mapsto \beta_U$).

**Definition 1.** Let $(V_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite connected subsets of $V$ such that

$$\bigcup_{n=0}^{\infty} V_n = V.$$

For $n \in \mathbb{N}$, we define $\mathcal{F}^{(n)} \subset \mathcal{B}(\mathcal{D}_V^W)$ as the sub-$\sigma$-field generated by the random variable $\beta \mapsto \beta_{V_n}$. For $n \in \mathbb{N}$ and $\beta \in \mathcal{D}_V^W$, we define a random operator $(\widehat{G}_\beta^{(n)}(i,j))_{i,j\in V}$ by

$$\widehat{G}_\beta^{(n)}(i,j) = \begin{cases} ((H_\beta)_{V_n,V_n})^{-1}(i,j), & \text{if } i,j \in V_n, \\ 0, & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ and $\beta \in \mathcal{D}_V^W$, we define a random function $(\psi_\beta^{(n)}(i))_{i\in V}$ as the unique solution of the equation

$$\begin{cases} H_\beta(\psi_\beta^{(n)})(i) = 0, & \text{for } i \in V_n, \\ \psi_\beta^{(n)}(i) = 1, & \text{for } i \in V_n^c. \end{cases}$$

By definition, the random variables $\widehat{G}^{(n)} : \beta \mapsto \widehat{G}_\beta^{(n)}$ and $\psi^{(n)} : \beta \mapsto \psi_\beta^{(n)}$ are $\mathcal{F}^{(n)}$-measurable.

The fact that there is a unique solution to the equation defining $\psi_\beta^{(n)}$ is elementary; see the proof in section 4.2.

Our main theorem is the following.

**Theorem 1.**

(i) *For all $i,j \in V$, the sequence of random variables $\widehat{G}^{(n)}(i,j)$ is nondecreasing and converges a.s. to*

$$\widehat{G}(i,j) := \lim_{n\to\infty} \widehat{G}^{(n)}(i,j).$$

*Moreover, $\nu_V^W$-almost surely, $0 < \widehat{G}(i,j) < \infty$, and the limit does not depend on the choice of the sequence of subsets $V_n$.*

(ii) *Under the probability $\nu_V^W$, for all $i \in V$, $\psi^{(n)}(i)$ is a positive $\mathcal{F}^{(n)}$-martingale. It converges a.s. to a random variable $\psi(i)$, such that $\psi(i) \geq 0$ a.s., and the limit does not depend on the choice of the increasing sequence $(V_n)$. Moreover, the quadratic variation of the vectorial martingale $(\psi^{(n)}(i))_{i\in V}$ is given a.s. by*

$$\langle \psi(i), \psi(j)\rangle_n = \widehat{G}^{(n)}(i,j).$$

*In particular, $\psi^{(n)}(i)$ is bounded in $L^2$ if and only if $\mathbb{E}_{\nu_V^W}(\widehat{G}(i,i)) < \infty$.*

(iii) *For any real $\gamma > 0$ and $\beta \in \mathcal{D}_V^W$, we define*

$$G_{\beta,\gamma}(i,j) = \widehat{G}_\beta(i,j) + \frac{1}{2}\gamma^{-1}\psi_\beta(i)\psi_\beta(j).$$

*For $i_0 \in V$ and $x \in V$, denote by $P_x^{\beta,\gamma,i_0}$ the law of the Markov jump process which starts at $x \in V$ and jumps from $i$ to $j$ at rate*

$$\frac{1}{2}W_{i,j}\frac{G_{\beta,\gamma}(i_0,j)}{G_{\beta,\gamma}(i_0,i)}. \tag{2.3}$$

*Then the VRJP in an exchangeable time scale (defined in section 2.1 with conductances $(W_{i,j})$ and starting from $i_0$) is a mixture of these Markov jump processes and has law*

$$\mathbb{P}_{i_0}^{VRJP}(\;\cdot\;) = \int P_{i_0}^{\beta,\gamma,i_0}(\;\cdot\;)\nu_V^W(d\beta)\frac{\mathbb{1}_{\gamma>0}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma. \tag{2.4}$$

(iv) *For $\nu_V^W$-almost all $\beta$, all $\gamma > 0$, and all $i_0 \in V$, we have*
- *the Markov process $P_{i_0}^{\beta,\gamma,i_0}$ is transient if and only if $\psi_\beta(j) > 0$ for all $j \in V$;*
- *the Markov process $P_{i_0}^{\beta,\gamma,i_0}$ is recurrent if and only if $\psi_\beta(j) = 0$ for all $j \in V$.*

**N.B.** Note that $P_x^{\beta,\gamma,i_0}$ is well defined for $\nu_V^W$-almost all $\beta$ and all $\gamma > 0$ by (i) and (ii).

*Notation.* We denote by

$$\nu_V^W(d\beta, d\gamma) := \nu_V^W(d\beta) \otimes \frac{\mathbb{1}_{\gamma>0}}{\sqrt{\pi\gamma}}e^{-\gamma}d\gamma \tag{2.5}$$

the probability distribution which appears in (2.4), under which $\gamma$ is $\Gamma(\frac{1}{2},1)$-distributed and independent of $\beta$. In general, we simply write $G(i,j)$ for $G_{\beta,\gamma}(i,j)$ and consider it as a random variable on the probability space $(\mathcal{D}_V^W \times \mathbb{R}_+^*, \mathcal{B}(\mathcal{D}_V^W) \otimes \mathcal{B}(\mathbb{R}_+^*), \nu_V^W(d\beta, d\gamma))$.

*Remark* 3. When the VRJP is recurrent, then $G = \widehat{G}$, and the representation of the VRJP (2.4) only involves the variable $\beta$ and not $\gamma$.

*Remark* 4. The representation (2.3) extends to infinite graphs the representation provided in [22, Theorem 2], for finite graphs. An interesting new feature appears in the transient regime, where the generalized eigenfunction $\psi$ and the extra gamma random variable enter the expression of $G(i,j)$. As it appears in the proof, the eigenfunction $\psi$ can be interpreted as the mixing field of a VRJP starting from infinity.

Denote by $\tau_{i_0}^+ = \inf\{t \geq 0,\;\; Z_t = i_0,\; \exists s < t \text{ s.t. } Z_s \neq i_0\}$ the first return time to $i_0$ by $(Z_t)_{t\geq 0}$. The point Theorem 1(iv) is in fact a consequence of the following more precise assertion.

**Proposition 2.** *We have, for $\nu_V^W$-almost all $\beta$, for all $\gamma > 0$ and $i_0$, $i \in V$,*

$$P_i^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty) = \begin{cases} \frac{\psi(i_0)^2}{4\gamma\widetilde{\beta}_{i_0}\widehat{G}(i_0,i_0)G(i_0,i_0)}, & \text{if } i = i_0, \\ \frac{\psi(i_0)}{2\gamma}\frac{\widehat{G}(i_0,i_0)\psi(i) - \widehat{G}(i_0,i)\psi(i_0)}{\widehat{G}(i_0,i_0)G(i_0,i)}, & \text{if } i \neq i_0, \end{cases}$$

*where* $\widetilde{\beta}_{i_0} = \sum_{j\sim i_0} \frac{1}{2} W_{i_0,j} \frac{G(i_0,j)}{G(i_0,i_0)}$. *In particular,* $\psi(i_0) = 0$ *if and only if*

$$P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty) = 0.$$

Using Doob's $h$ transform, the law of the process $(Z_t)$ conditioned on the event $\{\tau_0^+ < \infty\}$ or $\{\tau_0^+ = \infty\}$ can be computed, and it takes a rather nice form, both under the law $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$ or under the law $P_{i_0}^{\beta,\gamma,i_0}$ for $\nu_V^W$-almost all $\beta$. We provide these formulae in section 7.

A natural question that emerges from point Theorem 1(iv) is that of a 0-1 law for transience/recurrence. We provide an answer below in the case of vertex transitive graphs with conductances. We say that $(\mathcal{G}, W)$ is vertex transitive if the group of automorphisms of $\mathcal{G}$ that leaves invariant $(W_{i,j})$ is transitive on vertices. In particular, this is the case for the cubic lattice $\mathbb{Z}^d$ with constant conductances $W_{i,j} = W$. Denote by $\mathcal{A}$ the group of automorphisms that leave $W$ invariant.

**Proposition 3.** *If $(\mathcal{G}, W)$ is vertex transitive and $\mathcal{G}$ is infinite, then under the distribution $\nu_V^W(d\beta)$, the random variables $(\beta_i)_{i\in V}$, $(\psi(i))_{i\in V}$, $(\widehat{G}(i,j))_{i,j\in V}$ are stationary and ergodic for the group of transformations $\mathcal{A}$. Moreover, the VRJP is either recurrent or transient, i.e.,*

$$\mathbb{P}_{i_0}^{\mathrm{VRJP}}(\text{ every vertex is visited infinitely often }) = 1$$

*or*

$$\mathbb{P}_{i_0}^{\mathrm{VRJP}}(\text{ every vertex is visited finitely often }) = 1.$$

*In the first case $\psi(i) = 0$ for all $i \in V$, a.s., in the second case $\psi(i) > 0$ for all $i \in V$, a.s.*

**N.B.** The action of $\mathcal{A}$ on $\widehat{G}$ is $(\tau\widehat{G})(i,j) = \widehat{G}(\tau i, \tau j)$ for $\tau \in \mathcal{A}$.

2.3. **Relation with random Schrödinger operators.** Let us now relate Theorem 1 to the properties of the random Schrödinger operator $H : \beta \mapsto H_\beta$ associated with the random potential $(\beta_j)$ under the law $\nu_V^W$, defined in (2.1) and Proposition 1.

**Theorem 2.** *Under $\nu_V^W(d\beta)$ the followoing hold:*

(i) *The spectrum of $H$ is a.s. included in $[0,\infty)$.*
(ii) *The operator $\widehat{G}$ is the inverse of $H$ in the following sense: for all $i,j \in V$ a.s.*

$$\widehat{G}(i,j) = \lim_{\epsilon>0,\epsilon\to 0} (H+\epsilon)^{-1}(i,j).$$

(iii) *We have $(H\psi)(i) = 0$ a.s. for all $i \in V$.*
(iv) *In the case of the grid $\mathbb{Z}^d$ and when $W_{i,j} = W$ is constant, $(\widehat{G}(i,j))$ and $(\psi(i))$ are stationary ergodic for the spacial shift. Moreover, in the transient case, $\psi$ is a.s. a positive generalized eigenfunction with eigenvalue $0$ in the sense that $H\psi = 0$ and $\psi$ has at most polynomial growth. More precisely, for all $p > d$ and $C > 0$, a.s. there exists a random integer $K > 0$ such that*

$$|\psi(i)| \le C\|i\|_\infty^p \quad \forall i \in \mathbb{Z}^d \text{ such that } \|i\|_\infty \ge K.$$

2.4. **Functional central limit theorem.** We denote by $(\widetilde{Z}_n)_{n\in\mathbb{N}}$ the discrete time process that describes the successive jumps of $(Z_t)_{t\in\mathbb{R}_+}$. From Theorem 1(iii), under $\mathbb{P}_{i_0}^{\mathrm{VRJP}}$, $\widetilde{Z}_n$ is a mixture of Markov chains starting from $i_0$ and with conductances

$$W_{i,j}G(i_0,i)G(i_0,j) \tag{2.6}$$

under the probability distribution $\nu_V^W(d\beta, d\gamma)$.

We prove below a functional central limit theorem for the discrete time VRJP on $\mathbb{Z}^d$, $d\geq 3$, at weak reinforcement (i.e., for $W$ large enough).

**Theorem 3.** *Consider the cubic graph $\mathbb{Z}^d$, $d\geq 3$, with constant conductances $W_{i,j}=W$. Denote*

$$\widetilde{Z}_t^{(n)} = \frac{\widetilde{Z}_{[nt]}}{\sqrt{n}}.$$

*There exists $\lambda_2>0$ such that if $W>\lambda_2$, the discrete time VRJP satisfies a functional central limit theorem (i.e., under $\mathbb{P}_0^{VRJP}$) for any real $0<T<\infty$ and $(\widetilde{Z}_t^{(n)})_{t\in[0,T]}$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$ with nondegenerate isotropic diffusion matrix $\sigma^2\,\mathrm{Id}$, for some $0<\sigma^2<\infty$.*

2.5. **Consequences for the ERRW.** The ERRW is a famous discrete time process introduced in 1986 by Coppersmith and Diaconis [4, 12].

Endow the edges of the graph $\mathcal{G}=(V,E)$ with some positive weights $(a_e)_{e\in E}$. Let $(X_n)_{n\in\mathbb{N}}$ be a random process that takes values in $V$, and let $\mathcal{F}_n=\sigma(X_0,\dots,X_n)$ be the filtration of its past. For any $e\in E$, $n\in\mathbb{N}$, let

$$N_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{\{X_{k-1},X_k\}=e\}} \tag{2.7}$$

be the number of crossings of the (undirected) edge $e$ up to time $n$ plus the initial weight $a_e$.

Then $(X_n)_{n\in\mathbb{N}}$ is called the ERRW with starting point $i_0\in V$ and weights $(a_e)_{e\in E}$, if $X_0=i_0$ and, for all $n\in\mathbb{N}$,

$$\mathbb{P}(X_{n+1}=j \mid \mathcal{F}_n) = \mathbb{1}_{\{j\sim X_n\}} \frac{N_n(\{X_n,j\})}{\sum_{k\sim X_n} N_n(\{X_n,k\})}. \tag{2.8}$$

We denote by $\mathbb{P}_{i_0}^{\mathrm{ERRW}}$ the law of the ERRW starting from the initial vertex $i_0$. We will assume that the ERRW is defined on the canonical space $V^{\mathbb{N}}$; i.e., that $(X_n)_{n\in\mathbb{N}}$ is the canonical process on $V^{\mathbb{N}}$.

Important progress has been made in the last ten years in the understanding of this process; cf., e.g., [1, 8, 17, 20]. In particular, in was proved in 2012 by Sabot and Tarrès in [20] and Angel, Crawford, and Kozma in [1], on any graph with bounded degree at strong reinforcement (i.e., for $a_e<\widetilde{\lambda}_0$ for some fixed $\widetilde{\lambda}_0>0$), that the ERRW is a mixture of positive recurrent Markov chains. It was proved by Disertori, Sabot, and Tarrès in [8] that on $\mathbb{Z}^d$, $d\geq 3$, the ERRW is transient at weak reinforcement; i.e., for $a_e>\widetilde{\lambda}_1$ for some fixed $\widetilde{\lambda}_1<\infty$.

From Theorem 1 of [20], we know that the ERRW has the law of a VRJP in independent conductances. More precisely, consider $(W_e)_{e\in E}$ as independent random variables with gamma distribution with parameters $(a_e,1)$. Consider the VRJP in conductances $(W_e)_{e\in E}$ and its underlying discrete time process $(\widetilde{Z}_n)$.

Then the annealed law of $(\widetilde{Z}_n)$ (after expectation with respect to $W$) is that of the ERRW $(X_n)$ with initial weights $(a_e)$. Hence, we can apply Theorem 1 at fixed $W$ and then integrate on $W$. We thus consider the joint law $\widetilde{\nu}_V^a(dW, d\beta, d\gamma)$ on $(\mathbb{R}_+^*)^E \times (\mathbb{R}_+^*)^V \times \mathbb{R}_+^*$ obtained from $\nu_V^W(d\beta, d\gamma)$ after randomization with respect to $W$. More formally, let $\widetilde{\nu}_V^a(dW)$ be the probability distribution on $(\mathbb{R}_+^*)^E$ such that under $\widetilde{\nu}_V^a(dW)$ if the random variables $W \mapsto W_e$ are independent with gamma distribution with parameters $(a_e, 1)$, then $\widetilde{\nu}_V^a(dW, d\beta, d\gamma)$ is the probability distribution on $(\mathbb{R}_+^*)^E \times (\mathbb{R}_+^*)^V \times \mathbb{R}_+^*$ such that for any bounded measurable test function $F$,

$$\int F(W, \beta, \gamma)\widetilde{\nu}_V^a(dW, d\beta, d\gamma) = \int \left(\int F(W, \beta, \gamma)\nu_V^W(d\beta, d\gamma)\right)\widetilde{\nu}_V^a(dW).$$

In the rest of this paper, $\widetilde{\nu}_V^a(dW, d\beta)$, $\widetilde{\nu}_V^a(d\beta)$ will denote the corresponding marginal distributions, and $\widetilde{\nu}_V^a(dW)$ is the $W$ marginal. (By definition, $\widetilde{\nu}_V^a(dW, d\beta)$ is supported on the set of $(W, \beta)$ such that $\beta \in \mathcal{D}_V^W$.) From Theorem 1 we see that the ERRW starting from $i_0$ is a mixture of reversible Markov chains with conductances

$$x_{i,j} = W_{i,j}G(i_0, i)G(i_0, j), \tag{2.9}$$

where $G$ is defined in Theorem 1 and $(W, \beta, \gamma)$ are distributed according to $\widetilde{\nu}_V^a(dW, d\beta, d\gamma)$. More formally, if $\widetilde{P}_{i_0}^x$ denotes the law of the Markov chain starting at $i_0$ and with conductances $(x_{i,j})_{i\sim j}$, then

$$\mathbb{P}_{i_0}^{\mathrm{ERRW}}(\cdot) = \int \widetilde{P}_{i_0}^x(\cdot)\widetilde{\nu}_V^a(dW, d\beta, d\gamma).$$

An important point is that we keep the 1-dependence of the field $\beta$, after taking expectation with respect to $W$.

**Proposition 4.** *Under $\widetilde{\nu}_V^a(d\beta)$, $(\beta_j)_{j\in V}$ is 1-dependent: if $U, U' \subset V$ are such that $\mathrm{d}_{\mathcal{G}}(U, U') \geq 2$, then $(\beta_i)_{i\in U}$ and $(\beta_j)_{j\in U'}$ are independent.*

*Proof.* Indeed, from Proposition 1, the Laplace transform of $(\beta_i)_{i\in U}$ under $\nu_V^W(d\beta)$ only involves the conductances $W_{i,j}$ for $i$ or $j$ in $U$. This implies that, if $\mathrm{d}_{\mathcal{G}}(U, U') \geq 2$, the joint Laplace transform of $(\beta_i)_{i\in U}$ and $(\beta_i)_{i\in U'}$ is still the product of Laplace transforms even after taking expectation with respect to the random variables $(W_e)$, i.e., under $\widetilde{\nu}_V^a(\,\mathrm{d}\beta)$. ☐

This yields a counterpart of Proposition 3 for the ERRW.

**Proposition 5.** *Assume $(\mathcal{G}, (a_{i,j}))$ is vertex transitive with automorphism group $\mathcal{A}$, and $\mathcal{G}$ infinite. Then under the distribution $\widetilde{\nu}_V^a(dW, d\beta)$, the random variables $(W_e)_{e\in E}$, $(\beta_i)_{i\in V}$, $(\psi(i))_{i\in V}$, $(\widehat{G}(i,j))_{i,j\in V}$ are stationary and ergodic for the group of transformations $\mathcal{A}$. Moreover, the ERRW is either recurrent or transient, i.e.,*

$$\mathbb{P}_{i_0}^{ERRW}(\text{ every vertex is visited infinitely often }) = 1$$

*or*

$$\mathbb{P}_{i_0}^{ERRW}(\text{ every vertex is visited finitely often }) = 1.$$

*In the first case $\psi(i) = 0$ for all $i \in V$ a.s. in the second case $\psi(i) > 0$ for all $i \in V$ a.s.*

**N.B.** The action of $\mathcal{A}$ on $\widehat{G}$ and $W$ is $(\tau\widehat{G})(i,j) = \widehat{G}(\tau i, \tau j)$, $\tau W_{i,j} = W_{\tau i,\tau j}$ for $\tau \in \mathcal{A}$.

*Remark* 5. In [16] it was proved on infinite graphs that the ERRW is a mixture of Markov chains, obtained as a weak limit of the mixing law of the ERRW on finite approximating graphs. The difference in the representation we give in (2.9) is that the random variables $\psi$, $\widehat{G}$ are obtained as almost sure limits and hence are measurable functions of the random variables $\beta$. This yields stationarity and ergodicity, which are the key ingredients in the 0-1 law, and in the forthcoming Theorems 4 and 5.

*Remark* 6. It seems that this 0-1 law is new, both for the VRJP and the ERRW. In [16] it was proved that if the ERRW comes back with probability 1 to its starting point, then it visits infinitely often all points a.s., which is a weaker result. This was proved using the representation of the ERRW as mixture of Markov chains of [16]. (A short proof of this last result can also be given, cf. [23].)

We now give a counterpart of Theorem 3 for the ERRW. It is a consequence of Theorem 1 and of the delocalization result proved by Disertori, Sabot, and Tarrès in [8].

**Theorem 4.** *Consider the cubic graph $\mathbb{Z}^d$, $d \geq 3$, with constant weights $a_{i,j} = a$. Denote*

$$X_t^{(n)} = \frac{X_{[nt]}}{\sqrt{n}}.$$

*There exists $\widetilde{\lambda}_2 > 0$ such that if $a > \widetilde{\lambda}_2$, the ERRW satisfies a functional central limit theorem (i.e., under $\mathbb{P}_0^{ERRW}$ for any real $0 < T < \infty$) and $(X_t^{(n)})_{t\in[0,T]}$ converges in law (for the Skorokhod topology) to a d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$ with nondegenerate isotropic diffusion matrix $\sigma^2 \,\mathrm{Id}$, for some $0 < \sigma^2 < \infty$.*

Finally, we can deduce recurrence of the ERRW in dimension 2 from Theorem 1, Proposition 5, and the estimates obtained by Merkl and Rolles in [15, 17].[1]

**Theorem 5.** *The ERRW $(X_n)_{n\geq 0}$ on $\mathbb{Z}^2$ with constant weights $a_{i,j} = a$ is a.s. recurrent, i.e.,*

$$\mathbb{P}_0^{ERRW}(\text{ every vertex is visited infinitely often }) = 1.$$

In [15, 17], by a Mermin–Wagner-type argument, Merkl and Rolles proved a polynomial decrease of the form

$$\mathbb{E}\left(\left(\frac{x_\ell}{x_0}\right)^{\frac{1}{4}}\right) \leq c(a)|\ell|^{-\xi(a)}, \tag{2.10}$$

for some constants $c(a) > 0$, $\xi(a) > 0$, depending only on $a$, and where $x_\ell$ is the conductance at the site $\ell$ for the mixing measure of the ERRW, uniformly for a sequence of finite approximating graphs. When $0 < \xi < 1$, it does not give by itself enough information to prove recurrence. It was used in the case of a diluted two-dimensional graph to prove positive recurrence at strong reinforcement. The extra information given by the representation (2.9) and the stationarity of $\psi$ implies that the polynomial estimate (2.10) is incompatible with $\psi(i) > 0$ and hence is incompatible with transience. Detailed arguments are provided in section 8.

[1]We are grateful to Franz Merkl and Silke Rolles for a useful discussion on that subject.

*Remark* 7. We expect similarly that the two-dimensional VRJP with constant conductances $W_{i,j} = W > 0$ is recurrent. This would be implied by an estimate of the type (2.10) for the mixing field of the VRJP, which is still not available. More precisely, we can see from the proof of Theorem 5 in section 8 that recurrence of the two-dimensional VRJP would be implied by Theorem 1, Proposition 3, and an estimate of the type

$$\mathbb{E}\left(e^{\eta(u_\ell - u_0)}\right) \le \epsilon(\|\ell\|_\infty),$$

for $\eta > 0$ and $\epsilon(n)$ a positive function such that $\lim_{n\to\infty} \epsilon(n) = 0$, where $(u_j)$ is the mixing field of the VRJP starting from 0 (cf. Theorem B) on finite boxes with wired boundary condition as in section 4.2. We learned from Kozma and Peled that they have a proof of such an estimate.

2.6. **Open questions.** The most important question certainly concerns the relation between the properties of the VRJP and the spectral properties of the random Schrödinger operator $H_\beta$. For example on $\mathbb{Z}^d$ with constant weights $W_{i,j} = W$, is recurrence/transience of the VRJP related to the localized/delocalized regimes of $H_\beta$? A more precise question would be, Does the transient regime of the VRJP coincide with the existence of extended states at least at the bottom of the spectrum of $H_\beta$? It might at first seem inconsistent to expect extended states at the bottom of the spectrum since the Anderson model with i.i.d. potential is expected to be localized at the edges of the spectrum (a fact which is proved in several cases). But this localization is a consequence of Lifshitz tails, and there are good reasons to expect that Lifshitz tails fail for the potential $\beta$, which is not i.i.d. but 1-dependent. Indeed, the bottom of the spectrum of $H_\beta$ is 0, and it does not coincide with the minimum of the support of the distribution of $2\beta$ translated by the spectrum of $-P$, as it is the case for i.i.d. potential. In fact, on a finite set, the minimum of the spectrum is reached on the set $\det(2\beta - P) = 0$ which is a set of codimension 1, hence it is "big".

Another natural question concerns the uniform integrability of the martingale $\psi^{(n)}(i)$. Let us ask a more precise question: Is it true (at least for $\mathbb{Z}^d$ with constant weights) that transience of the VRJP implies that the martingale $\psi^{(n)}(i)$ is bounded in $L^2$? It is quite natural to expect such a property from relation (5.2) since $\widehat{G}^{(n)}(i,i)$ appears to be the quadratic variation of $\psi^{(n)}(i)$. This would have several consequences. First, it would imply that in dimension $d \ge 3$, the VRJP satisfies a functional central limit theorem as soon as the VRJP is transient, by the same argument as that of the proof of Theorem 3. It would also imply directly that the VRJP is recurrent as soon as the reversible Markov chain in conductances $(W_{i,j})$ is recurrent, if the group of automorphisms of $(\mathcal{G}, W)$ is transitive. Indeed, assume that the property is true and the VRJP is transient. By Theorem 1, the discrete time process $(\widetilde{Z}_n)$ would be represented as a mixture of reversible Markov chains with conductances $W_{i,j}G(0,i)G(0,j)$. From Proposition 2 applied to $i_0 = 0$, we have that

$$\frac{\widehat{G}(0,i)}{\widehat{G}(0,0)} \le \frac{\psi(i)}{\psi(0)}.$$

Hence, $(\widetilde{Z}_n)$ is equivalently a mixture of Markov chains with conductances

$$\frac{\psi(0)^2}{G(0,0)^2} W_{i,j} G(0,i)G(0,j) \le W_{i,j}\psi(i)\psi(j).$$

But $(\psi(i))$ is stationary ergodic, if $\psi_0$ is squared integrable, we would have

$$\mathbb{E}_{\nu_V^W}(W_{i,j}\psi(i)\psi(j)) \le CW_{i,j}$$

for some constant $C > 0$. Usual arguments imply that the Markov chain in conductance $W_{i,j}\psi(i)\psi(j)$ is recurrent if the Markov chain in conductances $(W_{i,j})$ is recurrent (cf., e.g., [14, Exercise 2.75]). We arrive at a contradiction.

2.7. **Organization of the paper.** In section 3 we gather several results in the case of finite graphs, in particular we recall the main results of [22]. In section 4 we define the important notion of restriction with wired boundary condition and the compatibility property. Section 5 is the key step in the paper where the martingale property is proved. In section 6 we prove Theorem 1, Propositions 2 and 3 and Theorem 2. In section 7 we provide extra computations of $h$-transforms. In section 8 we prove recurrence of ERRW in dimension 2 for all initial constant weights. In section 9 we prove functional central limit theorems for the VRJP and the ERRW, Theorems 3 and 4.

## 3. The random potential $\beta$ on finite graphs

In this section we assume that $\mathcal{G} = (V, E)$ is a finite graph and gather several results in this case. Recall that every undirected edge $e = \{i, j\}$ is labeled with a positive conductance $W_e = W_{i,j}$. In the case of a finite graph, the Schrödinger operator $H_\beta$ defined in (2.1) can be represented by the $V \times V$-matrix given by

$$H_\beta(i,j) = \begin{cases} 2\beta_i, & i = j, \\ -W_{i,j}, & i \neq j,\ i \sim j, \\ 0, & \text{otherwise,} \end{cases}$$

and the set $\mathcal{D}_V^W$ defined in (2.2) is the set of potentials $\beta$ such that $H_\beta$ is positive definite.

3.1. **The probability distribution $\nu_V^W$ on finite graphs, and its relation to the VRJP.** We recall Theorem 1 from [22], which defines the probability distribution $\nu_V^W(d\beta)$ by its density on any finite graph.

**Theorem A** ([22, Theorem 1, Definition 1, Proposition 1]). *Let $(\mathcal{G}, (W_e)_{e\in E})$ be a finite graph with conductances. The measure below is a probability on $\mathcal{D}_V^W$:*

$$\nu_V^W(d\beta) := \mathbb{1}_{H_\beta>0} \left(\frac{2}{\pi}\right)^{|V|/2} \exp\Big(-\sum_{i\in V}\beta_i + \sum_{e\in E} W_e\Big) \frac{d\beta_V}{\sqrt{\det H_\beta}} \tag{3.1}$$

*with $d\beta_V = \prod_{i\in V} d\beta_i$, and where $H_\beta > 0$ means that $H_\beta$ is positive definite.*

*The Laplace transform of the probability distribution $\nu_V^W(d\beta)$ is given, for all $(\lambda_i) \in \mathbb{R}_+^V$, by*

(3.2)

$$\int e^{-\langle\lambda,\beta\rangle}\nu_V^W(d\beta) = \exp\left(-\sum_{i\sim j} W_{i,j}\big(\sqrt{(\lambda_i+1)(\lambda_j+1)} - 1\big)\right) \prod_{i\in V} \frac{1}{\sqrt{\lambda_i+1}}.$$

*Moreover, under $\nu_V^W(d\beta)$ we have the following properties:*

- 1-dependence: *If $U, U' \subset V$ are such that $\mathrm{d}_{\mathcal{G}}(U, U') \ge 2$, then the random variables $\beta \mapsto \beta_U$ and $\beta \mapsto \beta_{U'}$ are independent.*

- Reciprocal inverse Gaussian marginals: *For $i \in V$, the random variable $\beta \mapsto \frac{1}{2\beta_i}$ has an inverse Gaussian distribution with parameter $(\frac{1}{W_i}, 1)$ where $W_i = \sum_{j\sim i} W_{i,j}$.*

If we apply formula (3.2) to $(\lambda_i) \in \mathbb{R}_+^V$ such that $\lambda_{V\setminus U} = 0$ for a subset $U \subset V$, we find the expression of Proposition 1. Hence, it implies Proposition 1 in the case of a finite graph.

The field $\beta$ is closely related to the VRJP, as shown in the next two theorems. In [20] it is shown that the VRJP in an exchangeable time scale defined in section 2.1 is a mixture of Markov jump processes; more precisely,

**Theorem B** ([20, Theorem 2])**.** *Assume $V$ finite. The following measure is a probability distribution on the set $\{(u_i)_{i\in V} \in \mathbb{R}^V,\ u_{i_0} = 0\}$,*

(3.3)
$$\mathcal{Q}_{i_0}^W(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \times \exp\left(-\sum_{i\in V} u_i - \sum_{i\sim j} W_{i,j}(\cosh(u_i-u_j)-1)\right)\sqrt{D(W,u)}du_{V\setminus\{i_0\}},$$

*where $du_{V\setminus\{i_0\}} = \prod_{i\in V\setminus\{i_0\}} du_i$ and $D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{i,j}e^{u_i+u_j}$, where the sum is over $\mathcal{T}$, the set of spanning trees of the graph $\mathcal{G}$.*

*For $(u_i)_{i\in V} \in \mathbb{R}^V$, we denote by $P_{i_0}^{(u)}$ the law of the Markov jump process starting at vertex $i_0$ and with jump rates from $i$ to $j$ given by*

$$\frac{1}{2}W_{i,j}e^{u_j-u_i}.$$

*The law of the VRJP in an exchangeable time scale starting at $i_0$ is a mixture of Markov jump processes, with mixing law given by*

$$\mathbb{P}_{i_0}^{VRJP}(\cdot) = \int P_{i_0}^{(u)}(\cdot)\mathcal{Q}_{i_0}^W(du).$$

*Remark* 8. By the matrix-tree theorem, $D(W,u)$ is any diagonal minor of the $|V|\times|V|$ matrix $(m_{i,j})$ with coefficients

$$m_{i,j} = \begin{cases} 0, & \text{if } i \not\sim j,\ i\neq j,\\ -W_{i,j}e^{u_i+u_j}, & \text{if } i\sim j,\ i\neq j,\\ \sum_{k\in V, k\sim i} W_{i,k}e^{u_i+u_k}, & \text{if } i=j.\end{cases}$$

*Remark* 9. The probability measure $\mathcal{Q}_{i_0}^W(du)$ appeared previously to [20] in a rather different context in the work of Disertori, Spencer, and Zirnbauer [10]. In particular, the fact that $\mathcal{Q}_{i_0}^W(du)$ is a probability measure was proved there as a consequence of a Berezin identity applied to a supersymmetric extension of that measure.

On finite graphs, the random environment $(u_i)$ of the previous theorem can be represented thanks to the Green function of the random potential $(\beta_i, i\in V)$ distributed according to $\nu_V^W(d\beta)$. Let us first recall Proposition 1 of [22].

**Proposition A** ([22, Proposition 1]). *Assume $V$ finite. For $\beta \in \mathcal{D}_V^W$, we denote by*

$$G_\beta := (H_\beta)^{-1}$$

*the Green function of the Schrödinger operator $H_\beta$. For $\beta \in \mathcal{D}_V^W$, $i, j \in V$, we define $u_\beta(i,j)$ by*

$$e^{u_\beta(i,j)} = \frac{G_\beta(i,j)}{G_\beta(i,i)}. \tag{3.4}$$

*For $i_0 \in V$, $(u_\beta(i_0,j))_{j\in V}$ is the unique solution of the equation*

$$\begin{cases} u_\beta(i_0,i_0) = 0, \\ \sum_{j\sim i} W_{i,j} e^{u_\beta(i_0,j)-u_\beta(i_0,i)} = \beta_i, \quad \text{if } i \neq i_0. \end{cases} \tag{3.5}$$

*In particular, the function $\beta \mapsto (u_\beta(i_0,j))_{j\in V}$ is $(\beta_j)_{j\in V\setminus\{i_0\}}$-measurable. Moreover, for all $\beta \in \mathcal{D}_V^W$,*

$$\beta_{i_0} = \frac{1}{2G_\beta(i_0,i_0)} + \frac{1}{2}\sum_{j\sim i_0} W_{i,j} e^{u_\beta(i_0,j)-u_\beta(i_0,i_0)}. \tag{3.6}$$

As usual, we simply denote by $G(i,j)$ and $u(i,j)$ the associated random variables on the probability space $(\mathcal{D}_V^W, \mathcal{B}(\mathcal{D}_V^W), \nu_V^W)$. Let us now recall [22, Theorem 3].

**Theorem C** ([22, Theorem 3]). *Assume $V$ finite. For all $i_0 \in V$, under the probability $\nu_V^W(d\beta)$,*

(i) *the random field $(u(i_0,j))_{j\in V}$ has the distribution $\mathcal{Q}_{i_0}^W$ of Theorem* B*;*
(ii) $\frac{1}{2G(i_0,i_0)}$ *has a gamma distribution with parameters* $(1/2, 1)$*;*
(iii) *$G(i_0,i_0)$ is independent of $(\beta_j)_{j\neq i_0}$, hence it is independent of the field $(u(i_0,j))_{j\in V}$.*

*Remark* 10. Here we only consider the VRJP with initial local time 1, in fact, the above correspondence between $\beta$ and VRJP still holds for the process starting with any positive local times $(\phi_i, i \in V)$, in such a case, there is a corresponding density $\nu_V^{W,\phi^2}$, which is defined in [22, Definition 1 and Theorem 3]. We choose here to normalize the initial local time to 1 since it is equivalent to the general case by a change of time and $W$; see [22, Appendix B].

Combining Theorem B and Theorem C gives a representation of the VRJP in an exchangeable time scale starting from different points in terms of the probability on random potentials $\nu_V^W$. We state this representation below.

**Corollary 1.** *Assume $V$ finite. For $\beta \in \mathcal{D}_V^W$, define $P_x^{\beta,i_0}$ as the law of the Markov jump process starting from $x$ and with jump rates from $i$ to $j$ given by*

$$\frac{1}{2} W_{i,j} \frac{G_\beta(i_0,j)}{G_\beta(i_0,i)}.$$

*Then the VRJP in exchangeable time scale is a mixture of the Markov jump processes*

$$\mathbb{P}_{i_0}^{VRJP}(\,\cdot\,) = \int P_{i_0}^{\beta,i_0}(\,\cdot\,)\nu_V^W(d\beta). \tag{3.7}$$

3.2. **Representation as a sum on paths.** We call *path* in $\mathcal{G}$ from $i$ to $j$ a finite sequence $\sigma = (\sigma_0, \dots, \sigma_m)$ in $V$ such that $\sigma_0 = i$, $\sigma_m = j$, and $\sigma_k \sim \sigma_{k+1}$, for $k = 0, \dots, m-1$. The length of $\sigma$ is defined by $|\sigma| = m$. We denote by $\mathcal{P}^V_{i,j}$ the collection of paths in $V$ from $i$ to $j$, and by $\bar{\mathcal{P}}^V_{i,j}$ the collection of paths $\sigma = (\sigma_0 = i, \dots, \sigma_m = j)$ in $V$ from $i$ to $j$ such that $\sigma_k \neq j, k = 0, \dots, m-1$. For a path $\sigma$ and for $\beta \in \mathcal{D}^W_V$, we set

$$(3.8) \qquad W_\sigma = \prod_{k=0}^{m-1} W_{\sigma_k, \sigma_{k+1}}, \qquad (2\beta)_\sigma = \prod_{k=0}^{m} (2\beta_{\sigma_k}), \qquad (2\beta)^-_\sigma = \prod_{k=0}^{m-1} (2\beta_{\sigma_k}).$$

For the trivial path $\sigma = (\sigma_0)$, we define $W_\sigma = 1$, $(2\beta)_\sigma = 2\beta_{\sigma_0}$, $(2\beta_\sigma)^- = 1$. (Note that these definitions make sense also in the case of infinite graphs.)

The following representation of the Green function $G(\cdot, \cdot)$ as a sum on paths will be convenient.

**Proposition 6.** *Assume that $V$ is finite. For all $\beta \in \mathcal{D}^W_V$, we have, with the notations of Theorem* A,

$$(3.9) \qquad G_\beta(i,j) = \sum_{\sigma \in \mathcal{P}^V_{i,j}} \frac{W_\sigma}{(2\beta)_\sigma}, \qquad \exp(u_\beta(i,j)) = \sum_{\sigma \in \bar{\mathcal{P}}^V_{j,i}} \frac{W_\sigma}{(2\beta)^-_\sigma}.$$

*Proof.* Write $D_\beta$ for the diagonal $V \times V$ matrix with $(\beta_i)_{i \in V}$ as diagonal coefficients, then $H_\beta = (\mathrm{Id} - P D_\beta^{-1}) D_\beta$. Since $H_\beta > 0$, by the Perron–Frobenius theorem, we have that $\rho(P D_\beta^{-1}) < 1$, where $\rho(P D_\beta^{-1})$ is the spectral radius of $P D_\beta^{-1}$. Hence, we can write the following convergent expansion,

$$G_\beta = H_\beta^{-1} = D_\beta^{-1} \sum_{k=0}^{\infty} (P D_\beta^{-1})^k,$$

which exactly corresponds to (3.9).

For the expansion of $\exp(u_\beta(i,j))$, note first that $\sum_{\sigma \in \bar{\mathcal{P}}^V_{j,i}} \frac{W_\sigma}{\beta^-_\sigma} \le \beta_i G_\beta(j,i) < \infty$. A path in $\mathcal{P}^V_{j,i}$ can be cut at its first visit to $i$, turning it into the concatenation of a path in $\bar{\mathcal{P}}^V_{j,i}$ and a path in $\mathcal{P}^V_{i,i}$, and this operation is bijective. It implies that

$$(3.10) \qquad \begin{aligned} \left( \sum_{\sigma \in \bar{\mathcal{P}}^V_{j,i}} \frac{W_\sigma}{(2\beta)^-_\sigma} \right) G_\beta(i,i) &= \left( \sum_{\sigma \in \bar{\mathcal{P}}^V_{j,i}} \frac{W_\sigma}{(2\beta)^-_\sigma} \right) \left( \sum_{\sigma \in \mathcal{P}^V_{i,i}} \frac{W_\sigma}{(2\beta)_\sigma} \right) \\ &= \sum_{\sigma \in \mathcal{P}^V_{j,i}} \frac{W_\sigma}{(2\beta)_\sigma} = G_\beta(i,j), \end{aligned}$$

hence the result. □

3.3. **A priori estimates on $\mathcal{Q}^W_{i_0}(du)$.** The following proposition is borrowed from [10, Lemma 3]. For convenience, we give a shorter proof of that estimate based on spanning trees instead of fermionic variables, following the proof of the corresponding result for the ERRW; cf. [8, Lemma 7].

**Proposition 7.** *Let $(\mathcal{G} = (V,E), W)$ be a finite graph with conductances. Fix a vertex $i_0$. Let $\eta > 0$ and let $e_1 = \{\underline{e_1}, \overline{e_1}\}, \dots, e_K = \{\underline{e_K}, \overline{e_K}\}$ be $K$ distinct* *undirected edges such that $W_{e_k} \geq 2\eta$ for all $k = 1, \ldots, K$. Then*

$$\int \exp\left(\eta \sum_{k=1}^{K} \cosh\left(u_{\overline{e_k}} - u_{\underline{e_k}}\right)\right) \mathcal{Q}_{i_0}^{W}(du) \leq e^{\eta K} 2^{K/2},$$

*where $\mathcal{Q}_{i_0}^{W}(du)$ is the probability distribution defined in Theorem* B.

*Proof.* Recall that $\mathcal{Q}_{i_0}^{W}(du)$ is defined by

$$\mathcal{Q}_{i_0}^{W}(du) = \frac{1}{\sqrt{2\pi}^{|V|-1}} \exp(-\sum_i u_i - \sum_{i\sim j} W_{i,j}(\cosh(u_i - u_j) - 1))\sqrt{D(W,u)}du_{V\setminus\{i_0\}},$$

with $du_{V\setminus\{i_0\}} = \prod_{i\neq i_0} du_i$ and $D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in T} W_{i,j}e^{u_i+u_j}$, where the sum is on spanning trees.

Let $\widetilde{W} = W - \eta\sum_{k=1}^{K} \mathbb{1}_{e_k}$; i.e., $\widetilde{W}$ is equal to $W - \eta$ on the edges $e_1, \ldots, e_K$ and is unchanged on the other edges. By assumption, we have $\widetilde{W}_{i,j} > 0$ on the edges, and for all spanning trees $T$, since edges appear at most once,

$$\prod_{\{i,j\}\in T} W_{i,j}e^{u_i+u_j} \leq \left(\prod_{k=1}^{K} \frac{W_{e_k}}{W_{e_k} - \eta}\right) \prod_{\{i,j\}\in T} \widetilde{W}_{i,j}e^{u_i+u_j} \leq 2^K \prod_{\{i,j\}\in T} \widetilde{W}_{i,j}e^{u_i+u_j},$$

which implies $D(W,u) \leq 2^K D(\widetilde{W}, u)$. From the expression of $\mathcal{Q}_{i_0}^{W}(du)$, we deduce that

$$\exp\left(\eta \sum_{k=1}^{K} \cosh\left(u_{\overline{e_k}} - u_{\underline{e_k}}\right)\right) \mathcal{Q}_{i_0}^{W}(du) \leq e^{\eta K} 2^{K/2} \mathcal{Q}_{i_0}^{\widetilde{W}}(du).$$

It implies that

$$\int \exp\left(\eta \sum_{k=1}^{K} \cosh\left(u_{\overline{e_k}} - u_{\underline{e_k}}\right)\right) \mathcal{Q}_{i_0}^{W}(du) \leq e^{\eta K} 2^{K/2} \int \mathcal{Q}_{i_0}^{\widetilde{W}}(du) = e^{\eta K} 2^{K/2}.$$

□

## 4. The wired boundary condition and Kolmogorov extension to infinite graphs

4.1. **Restriction with wired boundary condition.** Our objective is to extend the relations between the VRJP and the $\beta$ field to the case of infinite graphs. To this end, we need an appropriate boundary condition, which turns out to be the wired boundary condition.

**Definition 2.** Let $\mathcal{G} = (V, E)$ be a connected graph with finite degree at each site, and let $V_1$ be a strict finite subset of $V$. We define the restriction of $\mathcal{G}$ to $V_1$ with wired boundary condition as the graph $\mathcal{G}_1 = (\widetilde{V}_1 = V_1 \cup \{\delta\}, E_1)$, where $\delta$ is an extra point and

$$E_1 = \{\{i,j\} \in E, \text{ s.t. } i \in V_1, j \in V_1, i \sim j\} \cup \{\{i, \delta\}, i \in V_1 \text{ s.t. } \exists j \notin V_1, i \sim j\}.$$

If $(W_{i,j})_{\{i,j\}\in E}$ is a set of positive conductances, we define $(W_{i,j}^{(1)})_{\{i,j\}\in E_1}$ as the set of restricted conductances by

$$\begin{cases} W_{i,j}^{(1)} = W_{i,j}, & \text{if } i,j \in V_1,\ \{i,j\} \in E_1, \\ W_{i,\delta}^{(1)} = \sum_{j\notin V_1, j\sim i} W_{i,j}, & \text{if } \{i,\delta\} \in E_1, \\ 0, & \text{otherwise.} \end{cases}$$

*Remark* 11. Intuitively, this restriction corresponds to identifying all points in $V \setminus V_1$ to a single point $\delta$ and to deleting the edges connecting points of $V \setminus V_1$. The new weights are obtained by summing the weights of the edges identified by this procedure.

The following lemma is fundamental and is the justification for the choice of this notion of restriction.

**Lemma 1.** *Let $(\mathcal{G} = (V, E), W)$ be a finite graph with conductances and let $\nu_V^W$ be the associated distribution on random potentials defined in Theorem* A. *Let $V_1$ be a strict subset of $V$, and let $(\mathcal{G}_1 = (\widetilde{V}_1, E_1), W^{(1)})$ be the restriction of $(\mathcal{G}, W)$ to $V_1$ with wired boundary condition. Let $\nu_{\widetilde{V}_1}^{W^{(1)}}$ be the distribution of random potential associated with $(\mathcal{G}_1, W^{(1)})$. We denote by $\left(\nu_V^W\right)_{|V_1}$ and $\left(\nu_{\widetilde{V}_1}^{W^{(1)}}\right)_{|V_1}$ the marginal distributions on $V_1$ of $\nu_V^W$ and $\nu_{\widetilde{V}_1}^{W^{(1)}}$, respectively. Then*

$$\left(\nu_V^W\right)_{|V_1} = \left(\nu_{\widetilde{V}_1}^{W^{(1)}}\right)_{|V_1}.$$

*Remark* 12. Note that there is no such compatibility relation with the more usual notion of restriction of graph. The wired boundary condition is fundamental and in fact will be responsible for the extra gamma random variable that appears in the representation of the VRJP on the infinite graph.

*Proof.* Taking $(\lambda_i)_{i\in V} \in \mathbb{R}_+^V$ such that $\lambda_{V\setminus V_1} = 0$ in Theorem A, we get that

$$\begin{aligned}
&\int e^{-\sum_{i\in V_1}\lambda_i\beta_i}\nu_V^W(d\beta)\\
&\quad = \exp\Bigg(-\sum_{i\sim j, i,j\in V_1} W_{i,j}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)\\
&\qquad\qquad -\sum_{i\sim j, i\in V_1, j\notin V_1}(W_{i,j}(\sqrt{1+\lambda_i}-1))\Bigg)\prod_{i\in V_1}\frac{1}{\sqrt{1+\lambda_i}}.
\end{aligned}$$

Applying Theorem A to the graph $\mathcal{G}_1$ with $(\lambda_i)_{i\in\widetilde{V}_1} \in \mathbb{R}_+^{\widetilde{V}_1}$ such that $\lambda_\delta = 0$, we get

$$\begin{aligned}
&\int e^{-\sum_{i\in V_1}\lambda_i\beta_i}\nu_{\widetilde{V}_1}^{W^{(1)}}(d\beta)\\
&\quad = \exp\Bigg(-\sum_{i\sim j, i,j\in V_1} W_{i,j}^{(1)}(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1)\\
&\qquad\qquad -\sum_{i\in V_1, i\overset{\mathcal{G}_1}{\sim}\delta}(W_{i,\delta}^{(1)}(\sqrt{1+\lambda_i}-1))\Bigg)\prod_{i\in V_1}\frac{1}{\sqrt{1+\lambda_i}}.
\end{aligned}$$

By the definition of $W_{i,j}^{(1)}$ these Laplace transforms are equal, hence the marginal distributions are equal. □

4.2. **Kolmogorov extension: Proof of Proposition 1 and Definition 1.** Let $\mathcal{G} = (V, E)$ be a connected infinite graph with finite degree at each site with conductances $(W_{i,j})$. Recall that $(V_n)_{n\geq 1}$ is an increasing sequence of finite strict subsets of $V$ that exhausts $V$; i.e., $\bigcup_n V_n = V$.

Let $\mathcal{G}_n = (\widetilde{V}_n = V_n \cup \{\delta_n\}, E_n)$ be the restriction of $\mathcal{G}$ to $V_n$ with wired boundary condition, and let $(W^{(n)})$ be the restricted conductances. By construction if $n < m$, then $(\mathcal{G}_n, W^{(n)})$ is the restriction of $(\mathcal{G}_m, W^{(m)})$ to $V_n$ with wired boundary condition. Lemma 1 implies that the sequence of marginal distributions $\left(\nu_{\widetilde{V}_n}^{W^{(n)}}\right)_{|V_n}$ is a compatible sequence of probabilities. By the Kolmogorov extension theorem, it implies that there exists a probability measure $\nu_V^W$ such that

$$\left(\nu_V^W\right)_{|V_n} = \left(\nu_{\widetilde{V}_n}^{W^{(n)}}\right)_{|V_n},$$

for all integers $n$. By Theorem A, $\nu_V^W(\,\mathrm{d}\beta)$ is supported by the set of potentials $\beta$ such that $(H_\beta)_{V_n,V_n}$ is positive definite for all integers $n$, hence by $\mathcal{D}_V^W$. It also implies the other properties of $\nu_V^W(d\beta)$.

The solution of the equation defining $\psi_\beta^{(n)}$ in Definition 1 exists and is unique since it is equivalent to $(\psi_\beta^{(n)})_{V_n^c} = 1$ and

$$(H_\beta)_{V_n,V_n}(\psi_\beta^{(n)})_{V_n}(i) = \sum_{j\sim i, j\in V_n^c} W_{i,j}, \quad \text{for } i \in V_n. \tag{4.1}$$

Since $(H_\beta)_{V_n,V_n}$ is positive definite for $\beta \in \mathcal{D}_V^W$, it defines $\psi_\beta^{(n)}$ uniquely.

4.3. **Coupling lemma: Definition of $G^{(n)}$ and relations with $\widehat{G}^{(n)}$, $\psi^{(n)}$, and $\gamma$.** Consider the probability $\nu_V^W(d\beta, d\gamma)$ defined in (2.5). It will be convenient to couple the measure $\nu_V^W(d\beta, d\gamma)$ and the measure $\nu_{\widetilde{V}_n}^{W^{(n)}}(d\beta)$ in the following way.

**Lemma 2.** *For $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$, we define $\beta^{(n)} \in \mathbb{R}^{\widetilde{V}_n}$ by*

$$\beta_{V_n}^{(n)} = \beta_{V_n}, \ \beta_{\delta_n}^{(n)} = \sum_{j\in V_n, j\sim\delta_n} \frac{1}{2} W_{j,\delta_n}^{(n)} \psi_\beta^{(n)}(j) + \gamma. \tag{4.2}$$

*Then, $\beta^{(n)} \in \mathcal{D}_{\widetilde{V}_n}^{W^{(n)}}$ and under $\nu_V^W(d\beta, d\gamma)$, $\beta^{(n)}$ is distributed according to $\nu_{\widetilde{V}_n}^{W^{(n)}}$.*

*Let $H_{\beta^{(n)}}^{(n)}$ be the Schrödinger operator associated with $\mathcal{G}_n$, $W^{(n)}$, and potential $\beta^{(n)}$. Let $G_{\beta^{(n)}}^{(n)} = (H_{\beta^{(n)}}^{(n)})^{-1}$ be its Green function. Then,*

$$G_{\beta^{(n)}}^{(n)}(\delta_n, \delta_n) = \frac{1}{2\gamma},$$

*and, for all $i \in V_n$,*

$$\psi_\beta^{(n)}(i) = \frac{G_{\beta^{(n)}}^{(n)}(\delta_n, i)}{G_{\beta^{(n)}}^{(n)}(\delta_n, \delta_n)} = e^{u_{\beta^{(n)}}^{(n)}(\delta_n, i)},$$

*where $u_{\beta^{(n)}}^{(n)}$ is the field defined in Proposition* A *for the graph $\mathcal{G}_n$ and with the potential $\beta^{(n)}$.*

*As usual, we often omit the subscript $\beta$ and write $H^{(n)}$, $G^{(n)}$, $u^{(n)}$, and consider them as random variables on $\mathcal{D}_V^W \times \mathbb{R}_+^*$ under $\nu_V^W(d\beta, d\gamma)$.*

*Proof.* Let $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$. Denote in this proof by $(u(j))_{j\in\widetilde{V}_n}$ the vector defined by

$$u(j) = \begin{cases} 0, & \text{if } j = \delta_n, \\ \log \psi_\beta^{(n)}(j), & \text{if } j \in V_n. \end{cases}$$

Then, by definition of $\psi_\beta^{(n)}$ and $\beta^{(n)}$, we have $(H_{\beta^{(n)}}^{(n)}(e^u))_{V_n} = (H_\beta(\psi_\beta^{(n)}))_{V_n} = 0$ and

$$H_{\beta^{(n)}}^{(n)}(e^u)(\delta_n) = 2\beta_{\delta_n}^{(n)} - \sum_{j\in V_n, j\sim\delta_n} W_{\delta_n,j}^{(n)}\psi_\beta^{(n)}(j) = 2\gamma.$$

Since $(e^{u(j)})$ is a vector with positive coefficients, by general results on symmetric $M$-matrices; see [19, Theorem 2.7, p. 141], it implies that $H_{\beta^{(n)}}^{(n)} > 0$. Moreover, it implies that $\frac{1}{2\gamma}e^{u(\cdot)} = G_{\beta^{(n)}}^{(n)}(\delta_n,\cdot)$, hence that $G_{\beta^{(n)}}^{(n)}(\delta_n,\delta_n) = \frac{1}{2\gamma}$ and $e^{u(\cdot)} = \frac{G_{\beta^{(n)}}^{(n)}(\delta_n,\cdot)}{G_{\beta^{(n)}}^{(n)}(\delta_n,\delta_n)} = e^{u_{\beta^{(n)}}^{(n)}(\delta_n,\cdot)}$.

Finally, by Theorem C, the law of $(\beta_{V_n}^{(n)}, G_{\beta^{(n)}}^{(n)}(\delta_n,\delta_n))$ is the same under $\nu_V^W(d\beta,d\gamma)$ and $\nu_{\widetilde{V}_n}^{W^{(n)}}(d\beta^{(n)})$, and since $\beta^{(n)} \mapsto (\beta_{V_n}^{(n)}, G_{\beta^{(n)}}^{(n)}(\delta_n,\delta_n))$ is a bijection by Proposition A, it implies that under $\nu_V^W(d\beta,d\gamma)$, $\beta^{(n)}$ has law $\nu_{\widetilde{V}_n}^{W^{(n)}}$. □

**Proposition 8.** *With the definition of Proposition 2, for all $i,j\in V_n$, all $\beta\in\mathcal{D}_V^W$, and all $\gamma>0$,*

$$G_{\beta^{(n)}}^{(n)}(i,j) = \widehat{G}_\beta^{(n)}(i,j) + \frac{1}{2\gamma}\psi_\beta^{(n)}(i)\psi_\beta^{(n)}(j).$$

*Proof.* For simplicity, we omit the subscripts $\beta$, $\beta^{(n)}$ in $\widehat{G}_\beta^{(n)}$, $\psi_\beta^{(n)}$, $G_{\beta^{(n)}}^{(n)}$ in the expression below. By Proposition 6 and Lemma 2, using $(\beta^{(n)})_{V_n} = \beta_{V_n}$, we find that

$$G^{(n)}(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^{\widetilde{V}_n}} \frac{W_\sigma^{(n)}}{(2\beta^{(n)})_\sigma},\quad \widehat{G}^{(n)}(i,j) = \sum_{\sigma\in\mathcal{P}_{i,j}^{V_n}} \frac{W_\sigma}{(2\beta)_\sigma} \tag{4.3}$$

and

$$\psi^{(n)}(i) = \frac{G^{(n)}(\delta_n,i)}{G^{(n)}(\delta_n,\delta_n)} = \sum_{\sigma\in\overline{\mathcal{P}}_{i,\delta_n}^{\widetilde{V}_n}} \frac{W_\sigma^{(n)}}{(2\beta)_\sigma^-}.$$

Therefore, if we denote $\mathcal{P}_{i,\delta_n,j}^{\widetilde{V}_n}$ the collection of paths on $\widetilde{V}_n$ starting from $i$, visiting $\delta_n$ at least once, and ending at $j$, that is,

$$\mathcal{P}_{i,\delta_n,j}^{\widetilde{V}_n} = \{\sigma = (\sigma_0,\dots,\sigma_m)\in\mathcal{P}_{i,j}^{\widetilde{V}_n},\ \text{such that } \exists 0\le k\le m, \sigma_k = \delta_n,\},$$

then, since $(W^{(n)})_{V_n,V_n} = W_{V_n,V_n}$ and $(\beta^{(n)})_{V_n} = \beta_{V_n}$,

$$\begin{aligned}
&G^{(n)}(i,j) - \widehat{G}^{(n)}(i,j)\\
&\quad= \sum_{\sigma\in\mathcal{P}_{i,\delta_n,j}^{\widetilde{V}_n}} \frac{W_\sigma^{(n)}}{(2\beta^{(n)})_\sigma}\\
&\quad= \Big(\sum_{\sigma\in\bar{\mathcal{P}}_{i,\delta_n}^{\widetilde{V}_n}} \frac{W_\sigma^{(n)}}{(2\beta^{(n)})_\sigma^-}\Big)\cdot\Big(\sum_{\sigma\in\mathcal{P}_{\delta_n,j}^{\widetilde{V}_n}} \frac{W_\sigma^{(n)}}{(2\beta^{(n)})_\sigma}\Big)\\
&\quad= \psi^{(n)}(i)G^{(n)}(\delta_n,j) = \psi^{(n)}(i)\psi^{(n)}(j)G^{(n)}(\delta_n,\delta_n) = \psi^{(n)}(i)\psi^{(n)}(j)\frac{1}{2\gamma},
\end{aligned}$$

where we used Lemma 2 in the last equality. □

## 5. The martingale property

Recall that $\mathcal{F}^{(n)} = \sigma(\beta_i, i \in V_n)$ is the sub-$\sigma$-field generated by the random variables $\beta \mapsto \beta_i$, $i \in V_n$. The following proposition is the key property for the main theorem.

**Proposition 9.** *With the notations of Definition* 1, *for all $n \in \mathbb{N}$, $\psi^{(n)}$ has finite moments. Moreover, we have, $\nu_V^W$-a.s.,*

$$\mathbb{E}_{\nu_V^W}\left(\psi^{(n+1)}(i)|\mathcal{F}^{(n)}\right) = \psi^{(n)}(i) \quad \forall i \in V, \tag{5.1}$$

*and for all $i, j \in V$,*

$$\begin{aligned}&\mathbb{E}_{\nu_V^W}\left(\psi^{(n+1)}(i)\psi^{(n+1)}(j) - \psi^{(n)}(i)\psi^{(n)}(j)|\mathcal{F}^{(n)}\right)\\ &\quad= \mathbb{E}_{\nu_V^W}\left(\widehat{G}^{(n+1)}(i,j) - \widehat{G}^{(n)}(i,j)|\mathcal{F}^{(n)}\right).\end{aligned} \tag{5.2}$$

*Remark* 13. In Theorem B, by the change of variables $\widetilde{u}(\cdot) = u(\cdot) - \frac{\sum_{i\in V} u(i)}{|V|}$, the new variables $(\widetilde{u}(i))_{i\in V}$ are in the space $\{\sum_{i\in V}\widetilde{u}(i) = 0\}$ and the density becomes

$$\widetilde{\mathcal{Q}}_{i_0}^W(d\widetilde{u}) = \frac{1}{\sqrt{2}^{|V|-1}} e^{\widetilde{u}(i_0)} e^{-\sum_{i\sim j} W_{i,j}(\cosh(\widetilde{u}(i)-\widetilde{u}(j))-1)} \sqrt{D(W,\widetilde{u})}\, d\widetilde{u}_{V\setminus\{i_0\}}.$$

We see from this expression that $e^{\widetilde{u}(i)-\widetilde{u}(i_0)} \cdot \widetilde{\mathcal{Q}}_{i_0}^W = \widetilde{\mathcal{Q}}_i^W$, and hence that

$$\int e^{\widetilde{u}(i)-\widetilde{u}(i_0)} \widetilde{\mathcal{Q}}_{i_0}^W(d\widetilde{u}) = 1.$$

Applied to $V = \widetilde{V}_n$, $i_0 = \delta_n$, we get $\mathbb{E}_{\nu_V^W}(\psi^{(n)}(i)) = 1$, which is a particular case of (5.1).

The original proof of that property was rather technical (see the second arXiv version of the present paper). Some time after the first version of this paper was posted on arXiv, a simpler proof of the martingale property (5.1) was given in [7]. Moreover, using some supersymmetric arguments, the following more general property was proved.

**Lemma 3** ([7])**.** *Let $\lambda \in (\mathbb{R}_+)^V$ be a nonnegative function on $V$ with bounded support. Then*

$$\mathbb{E}\left(e^{-\langle\lambda,\psi^{(n+1)}\rangle - \frac{1}{2}\langle\lambda,\widehat{G}^{(n+1)}\lambda\rangle}|\mathcal{F}^{(n)}\right) = e^{-\langle\lambda,\psi^{(n)}\rangle - \frac{1}{2}\langle\lambda,\widehat{G}^{(n)}\lambda\rangle}.$$

We provide here a different proof of this assertion based on elementary computations on the measures $\nu_V^W$ on finite sets. It also provides a simpler proof of the original assertion Proposition 9 by differentiating in $\lambda$.

### 5.1. Marginal and conditional laws of $\nu_V^W$.

In this subsection we suppose that $\mathcal{G} = (V, E)$ is finite. We state some identities on marginal and conditional laws of the distribution $\nu_V^W$, which will be instrumental in the proof of the martingale property in the next subsection.

Let us first remark that the law $\nu_V^W$ defined in Theorem A can be extended to the case where $P = (W_{i,j})_{i,j\in V}$ has nonzero, diagonal coefficients. Indeed, if some diagonal coefficients of $P$ are positive, then changing from variables $(\beta_i)$ to variables $(\beta_i - \frac{1}{2}W_{i,i})$, we get the law $\nu_V^{\widetilde{W}}$ where $(\widetilde{W_{i,j}})$ is obtained from $(W_{i,j})$ by replacing all diagonal entries by 0. While it is not very natural from the point of view of the VRJP to allow nonzero diagonal coefficients, it is convenient in this section to allow this possibility since it simplifies the statements about conditional law.

Recall that for any function $\zeta : V \mapsto \mathbb{R}$ and any subset $U \subset V$, we write $\zeta_U$ for the restriction of $\zeta$ to the subset $U$. Similarly, if $A$ is a $V \times V$ matrix and $U \subset V$, $U' \subset V$, we write $A_{U,U'}$ for its restriction to the block $U \times U'$. We also write $d\beta_U = \prod_{i\in U} d\beta_i$ to denote integration on variables $\beta_U$.

In the next lemma we give an extension of the family of probability distributions $\nu_V^W$. This extension was proposed by Letac, in the unpublished note [13] discussing the integral defined in [22]. We give a proof of this lemma using Theorem A.

**Lemma 4** (Letac, [13]). *Let $V$ be finite, and let $P = (W_{i,j})_{i,j\in V}$ be a symmetric matrix with nonnegative coefficients. Let $(\eta_i)_{i\in V} \in \mathbb{R}_+^V$ be a vector with nonnegative coefficients. Then the following measure on $\mathcal{D}_V^W$*

$$(5.3)\qquad \begin{aligned}\nu_V^{W,\eta}(d\beta) &:= e^{-\frac{1}{2}\left\langle \eta,(H_\beta)^{-1}\eta\right\rangle} e^{\langle \eta,1\rangle}\nu_V^W(d\beta)\\ &= \mathbb{1}_{H_\beta>0}\left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac{1}{2}\langle 1,H_\beta 1\rangle-\frac{1}{2}\left\langle \eta,(H_\beta)^{-1}\eta\right\rangle}\frac{1}{\sqrt{\det H_\beta}}e^{\langle \eta,1\rangle}d\beta_V\end{aligned}$$

*is a probability distribution, where $1$ in the scalar products $\langle 1, H_\beta 1\rangle$ and $\langle \eta, 1\rangle$ is to be understood as the vector $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$. Its Laplace transform is, for any $\lambda \in \mathbb{R}_+^V$,*

$$(5.4)\qquad \int e^{-\langle\lambda,\beta\rangle}\nu_V^{W,\eta}(d\beta)=e^{-\left\langle \eta,\sqrt{\lambda+1}-1\right\rangle-\sum_{i\sim j}W_{i,j}\left(\sqrt{(1+\lambda_i)(1+\lambda_j)}-1\right)}\prod_{i\in V}\frac{1}{\sqrt{1+\lambda_i}},$$

*where $\sqrt{\lambda+1}-1$ should be considered as the vector $(\sqrt{\lambda_i+1}-1)_{i\in V}$.*

It appears in the following lemma that this extension describes all marginal laws of $\nu_V^W$ and that the larger family $\nu_V^{W,\eta}$ is stable by taking marginals and conditional distributions.

**Lemma 5.** *Assume that $V$ is finite, and let $U \subset V$ be a subset. Under $\nu_V^{W,\eta}(d\beta)$,*

(i) *$\beta_U$ is distributed according to $\nu_U^{W_{U,U},\widehat{\eta}}$, where*

$$(5.5)\qquad \widehat{\eta} = \eta_U + P_{U,U^c}(1_{U^c}),$$

(ii) *conditionally on $\beta_U$, $\beta_{U^c}$ is distributed according to $\nu_{U^c}^{\check{W},\check{\eta}}$, where $\check{P} = (\check{W}_{i,j})_{i,j\in U^c}$ and $\check{\eta} \in (\mathbb{R}_+)^{U^c}$ are the matrix and vector defined by*

$$\check{P} = P_{U^c,U^c} + P_{U^c,U}\left((H_\beta)_{U,U}\right)^{-1}P_{U,U^c},\quad \check{\eta} = \eta_{U^c} + P_{U^c,U}\left((H_\beta)_{U,U}\right)^{-1}(\eta_U).$$

*Remark* 14. Note that $\check{P}$ has nonzero diagonal coefficients.

**N.B.** As we can observe, all the quantities with $\check{\cdot}$ are relative to vectors or matrices on $U^c$, while the quantities with $\widehat{\cdot}$ are relative to vectors or matrices on $U$.

**Lemma 6.** *Let $\mathcal{G}=(V,E)$ be a finite connected graph endowed with conductances $P=(W_{i,j})_{i,j\in V}$. Let $(\eta_i)_{i\in V}\in\mathbb{R}_+^V$ be a vector with nonnegative coefficients. Let $U\subset V$. For $\beta\in\mathcal{D}_V^W$, define $\psi_\beta=G_\beta\eta$, where $G_\beta=H_\beta^{-1}$, and define $\widehat{\eta}=\eta_U+P_{U,U^c}1_{U^c}$, $\widehat{G}_\beta^U=((H_\beta)_{U,U})^{-1}$ and $\widehat{\psi}_\beta=\widehat{G}_\beta^U(\widehat{\eta})$. For any $\lambda\in\mathbb{R}_+^V$, we have, $\nu_V^{W,\eta}$ a.s.,*

$$\mathbb{E}_{\nu_V^{W,\eta}}(e^{-\langle\lambda,\psi\rangle-\frac{1}{2}\langle\lambda,G\lambda\rangle}\,|\mathcal{F}_U)=e^{-\left\langle\lambda_U,\widehat{\psi}\right\rangle-\langle\lambda_{U^c},1_{U^c}\rangle-\frac{1}{2}\left\langle\lambda_U,\widehat{G}^U\lambda_U\right\rangle},\tag{5.6}$$

*where $\mathcal{F}_U=\sigma(\beta_i,i\in U)$.*

*Proof of Lemmas* 4 *and* 5. Lemma 4 and the assertions (i) and (ii) of Lemma 5 are consequences of the same decomposition of the measure $\nu_V^{W,\eta}$. It is partially inspired by computations in [13]. We write $H_\beta$ as a block matrix,

$$H_\beta=\begin{pmatrix}H_{U,U} & -P_{U,U^c}\\ -P_{U^c,U} & H_{U^c,U^c}\end{pmatrix}\text{ and define }\widehat{G}^U=(H_{U,U})^{-1}.$$

Now, define the Schur's complement

$$\check{H}^{U^c}=H_{U^c,U^c}-P_{U^c,U}\widehat{G}^UP_{U,U^c}\tag{5.7}$$

and

$$\check{G}^{U^c}=\left(\check{H}^{U^c}\right)^{-1}.$$

We have

$$H_\beta=\begin{pmatrix}I_U & 0\\ -P_{U^c,U}\widehat{G}^U & I_{U^c}\end{pmatrix}\begin{pmatrix}H_{U,U} & 0\\ 0 & \check{H}^{U^c}\end{pmatrix}\begin{pmatrix}I_U & -\widehat{G}^UP_{U,U^c}\\ 0 & I_{U^c}\end{pmatrix}.\tag{5.8}$$

We remark that with notations of Lemma 5(ii), we have

$$\check{H}^{U^c}=2\beta_{U^c}-\check{P}.$$

By (5.8), we have

$$\begin{aligned}\langle 1,H_\beta 1\rangle\\ &=\left\langle 1_{U^c},\check{H}^{U^c}1_{U^c}\right\rangle+\langle 1_U,H_{U,U}1_U\rangle\\ &\quad+\left\langle 1_{U^c},P_{U^c,U}\widehat{G}^UP_{U,U^c}1_{U^c}\right\rangle-2\,\langle 1_U,P_{U,U^c}1_{U^c}\rangle\,.\end{aligned}\tag{5.9}$$

On the other hand, by (5.8) again, we have

$$G_\beta=H_\beta^{-1}=\begin{pmatrix}I_U & \widehat{G}^UP_{U,U^c}\\ 0 & I_{U^c}\end{pmatrix}\begin{pmatrix}\widehat{G}^U & 0\\ 0 & \check{G}^{U^c}\end{pmatrix}\begin{pmatrix}I_U & 0\\ P_{U^c,U}\widehat{G}^U & I_{U^c}\end{pmatrix},\tag{5.10}$$

therefore, since

$$\begin{pmatrix}I_U & 0\\ P_{U^c,U}\widehat{G}^U & I_{U^c}\end{pmatrix}\begin{pmatrix}\eta_U\\ \eta_{U^c}\end{pmatrix}=\begin{pmatrix}\eta_U\\ \check{\eta}\end{pmatrix},$$

we get

$$\langle\eta,G_\beta\eta\rangle=\left\langle\eta_U,\widehat{G}^U\eta_U\right\rangle+\left\langle\check{\eta},\check{G}^{U^c}\check{\eta}\right\rangle.\tag{5.11}$$

Combining (5.9) and (5.11), we have

$$\langle 1, H_\beta 1\rangle + \langle \eta, G_\beta \eta\rangle - 2\langle \eta, 1\rangle = \left\langle 1_{U^c}, \check{H}^{U^c} 1_{U^c}\right\rangle + \left\langle \check{\eta}, \check{G}^{U^c}\check{\eta}\right\rangle - 2\langle \check{\eta}, 1_{U^c}\rangle + \langle 1_U, H_{U,U} 1_U\rangle + \left\langle \widehat{\eta}, \widehat{G}^U \widehat{\eta}\right\rangle - 2\langle \widehat{\eta}, 1_U\rangle . \tag{5.12}$$

By (5.8) we also have

$$\det H_\beta = \det H_{U,U} \det \check{H}^{U^c}, \quad \mathbb{1}_{H_\beta>0} = \mathbb{1}_{H_{U,U}>0}\mathbb{1}_{\check{H}^{U^c}>0}. \tag{5.13}$$

Combining (5.12) and (5.13), we have

$$\begin{aligned} &\left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac12\langle 1,H_\beta 1\rangle - \frac12\langle \eta, G_\beta\eta\rangle + \langle \eta, 1\rangle} \frac{\mathbb{1}_{H_\beta>0}}{\sqrt{\det H_\beta}} \\ &= \left(\frac{2}{\pi}\right)^{|U|/2} e^{-\frac12\langle 1_U,H_{U,U} 1_U\rangle - \frac12\left\langle \widehat{\eta}, \widehat{G}^U\widehat{\eta}\right\rangle + \langle \widehat{\eta}, 1_U\rangle} \frac{\mathbb{1}_{H_{U,U}>0}}{\det H_{U,U}} \\ &\cdot \left(\frac{2}{\pi}\right)^{|U^c|/2} e^{-\frac12\left\langle 1_{U^c},\check{H}^{U^c} 1_{U^c}\right\rangle - \frac12\left\langle \check{\eta}, \check{G}^{U^c}\check{\eta}\right\rangle + \langle \check{\eta}, 1_{U^c}\rangle} \frac{\mathbb{1}_{\check{H}^{U^c}>0}}{\sqrt{\det \check{H}^{U^c}}}. \end{aligned} \tag{5.14}$$

We remark that the left-hand side is the density of $\nu_V^{W,\eta}(d\beta)$, that the first term of the right-hand side corresponds to the density of $\nu_U^{W_{U,U},\widehat{\eta}}(d\beta_U)$, and that, $\beta_U$ being fixed, the second term of the right-hand side is the density of $\nu_{U^c}^{\check{W},\check{\eta}}(d\beta_{U^c})$. (Indeed, as remarked above, $\check{H}^{U^c} = 2\beta_{U^c} - \check{P}$ and $\check{P}$, $\check{\eta}$ are $\beta_U$-measurable.) □

*Proof of Lemma* 4. Take $\eta = 0$. Then $\check{\eta} = 0$. Integrating on $d\beta_{U^c}$ on both sides of (5.14), with $\beta_U$ fixed, gives

$$\begin{aligned} &\int \left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac12\langle 1,H_\beta 1\rangle} \frac{\mathbb{1}_{H_\beta>0}}{\sqrt{\det H_\beta}} d\beta_{U^c} \\ &= \left(\frac{2}{\pi}\right)^{|U|/2} e^{-\frac12\langle 1_U,H_{U,U} 1_U\rangle - \frac12\left\langle \widehat{\eta}, \widehat{G}^U\widehat{\eta}\right\rangle + \langle 1_U, \widehat{\eta}\rangle} \frac{\mathbb{1}_{H_{U,U}>0}}{\det H_{U,U}} \end{aligned}$$

since $\int \nu_{U^c}^{\check{W}}(d\beta_{U^c}) = 1$ by Theorem A. Integrating on $d\beta_U$, it gives $\int \nu_U^{W_{U,U},\widehat{\eta}}(d\beta_U) = 1$, since $\nu_V^W$ is a probability. Hence, $\nu_U^{W_{U,U},\widehat{\eta}}$ is a probability. This implies Lemma 4 since this restriction procedure allows us to obtain all possible parameters of the family of measures $\nu_V^{W,\eta}$. Indeed, for $V$, $W$, $\eta$, consider $\widetilde{V} = V \cup \{\delta\}$ the set obtained by adding an extra point, and define $(\widetilde{W}_{i,j})_{i,j\in\widetilde{V}}$ by $\widetilde{W}_{V,V} = W$ and $W_{i,\delta} = \eta_i$ for $i \in V$. Then if we apply the previous identity to $\widetilde{V}$ and $U := V \subset \widetilde{V}$, we get $\widehat{\eta} = \eta$, and $\nu_V^{W,\eta}$ is a probability by the previous argument. □

*Proof of Lemma* 5. Integrating on $d\beta_{U^c}$ on both sides of (5.14), with $\beta_U$ fixed, gives

$$\begin{aligned} &\int \left(\frac{2}{\pi}\right)^{|V|/2} e^{-\frac12\langle 1,H_\beta 1\rangle - \frac12\langle \eta, G_\beta\eta\rangle + \langle \eta, 1\rangle} \frac{\mathbb{1}_{H_\beta>0}}{\sqrt{\det H_\beta}} (d\beta_{U^c}) \\ &= \left(\frac{2}{\pi}\right)^{|U|/2} e^{-\frac12\langle 1_U,H_{U,U} 1_U\rangle - \frac12\left\langle \widehat{\eta}, \widehat{G}^U\widehat{\eta}\right\rangle + \langle 1_U, \widehat{\eta}\rangle} \frac{\mathbb{1}_{H_{U,U}>0}}{\det H_{U,U}} \end{aligned}$$

since $\int \nu_{U^c}^{\check{W},\check{\eta}}(d\beta_{U^c}) = 1$ by Lemma 4. Hence, the marginal distribution of $\beta_U$ is $\nu_U^{W_{U,U},\widehat{\eta}}$, proving (i). Finally, (ii) is a consequence of the conditional probability density formula. Indeed, if we denote temporarily by $f(\beta)$ the density of $\nu_V^W(d\beta)$, by $f_U(\beta_U)$ its marginal density on $U$, and by $f_{U^c}^{\beta_U}(\beta_{U^c})$ the conditional density of $\beta_{U^c}$ conditioned on $\beta_U$, we have by (5.14) and (i),

$$f_{U^c}^{\beta_U}(\beta_{U^c}) = \frac{f(\beta)}{f_U(\beta_U)} = \left(\frac{2}{\pi}\right)^{|U^c|/2} e^{-\frac{1}{2}\left\langle 1_{U^c}, \check{H}^{U^c} 1_{U^c}\right\rangle - \frac{1}{2}\left\langle \check{\eta}, \check{G}^{U^c}\check{\eta}\right\rangle + \langle \check{\eta}, 1_{U^c}\rangle} \frac{\mathbb{1}_{\check{H}^{U^c}>0}}{\sqrt{\det \check{H}^{U^c}}}.$$

Since $\check{H}^{U^c} = 2\beta_{U^c} - \check{P}$, $\check{G}^{U^c} = (\check{H}^{U^c})^{-1}$ and $\check{P}$, $\check{\eta}$ are $\beta_U$-measurable, it implies that the right-hand side is the density of $\nu_{U^c}^{\check{W},\check{\eta}}(d\beta_{U^c})$. □

*Proof of Lemma* 6. We take the same notations as in the proof of Lemma 5. By Lemma 5, under $\nu_V^W(d\beta)$, the law of $\beta_{U^c}$, conditionally on $\beta_U$, is $\nu_{U^c}^{\check{W},\check{\eta}}$. Now, we set

$$\check{\psi} = \check{G}^{U^c}\check{\eta}.$$

By (5.10) we have

$$\begin{aligned}
\langle \lambda, \psi\rangle + \frac{1}{2}\langle \lambda, G\lambda\rangle &= \langle \lambda, G\eta\rangle + \frac{1}{2}\langle \lambda, G\lambda\rangle \\
&= \left(\lambda_U, \lambda_U \widehat{G}^U P_{U,U^c} + \lambda_{U^c}\right)\begin{pmatrix} \widehat{G}^U & 0 \\ 0 & \check{G}^{U^c}\end{pmatrix}\begin{pmatrix} \eta_U \\ P_{U^c,U}\widehat{G}^U \eta_U + \eta_{U^c}\end{pmatrix} \\
&\quad + \frac{1}{2}\left(\lambda_U, \lambda_U \widehat{G}^U P_{U,U^c} + \lambda_{U^c}\right)\begin{pmatrix} \widehat{G}^U & 0 \\ 0 & \check{G}^{U^c}\end{pmatrix}\begin{pmatrix} \lambda_U \\ P_{U^c,U}\widehat{G}^U \lambda_U + \lambda_{U^c}\end{pmatrix}.
\end{aligned}$$

If we define $\check{\lambda} = \lambda_{U^c} + P_{U^c,U}\widehat{G}^U\lambda_U \in \mathbb{R}_+^{U^c}$, we have

$$\begin{aligned}
&\langle \lambda, \psi\rangle + \frac{1}{2}\langle \lambda, G\lambda\rangle \\
&\quad = \left\langle \check{\lambda}, \check{\psi}\right\rangle + \frac{1}{2}\left\langle \check{\lambda}, \check{G}^{U^c}\check{\lambda}\right\rangle + \left\langle \lambda_U, \widehat{G}^U \eta_U\right\rangle + \frac{1}{2}\left\langle \lambda_U, \widehat{G}^U\lambda_U\right\rangle \\
&\quad = \left\langle \check{\lambda}, \check{\psi}\right\rangle + \frac{1}{2}\left\langle \check{\lambda}, \check{G}^{U^c}\check{\lambda}\right\rangle + \left\langle \lambda_U, \widehat{\psi}\right\rangle + \frac{1}{2}\left\langle \lambda_U, \widehat{G}^U\lambda_U\right\rangle - \left\langle 1_{U^c}, \check{\lambda} - \lambda_{U^c}\right\rangle.
\end{aligned}$$

Now, we remark that

$$\left\langle \check{\lambda}, \check{\psi}\right\rangle + \frac{1}{2}\left\langle \check{\lambda}, \check{G}^{U^c}\check{\lambda}\right\rangle + \frac{1}{2}\left\langle \check{\eta}, \check{G}^{U^c}\check{\eta}\right\rangle = \frac{1}{2}\left\langle \check{\lambda}+\check{\eta}, \check{G}^{U^c}(\check{\lambda}+\check{\eta})\right\rangle.$$

We get

$$\begin{aligned}
&\mathbb{E}_{\nu_V^{W,\eta}}\left(e^{-\langle \lambda,\psi\rangle - \frac{1}{2}\langle \lambda, G\lambda\rangle} | \mathcal{F}_U\right) \\
&\quad = e^{-\left\langle \lambda_U, \widehat{\psi}\right\rangle - \langle \lambda_{U^c}, 1_{U^c}\rangle - \frac{1}{2}\left\langle \lambda_U, \widehat{G}^U\lambda_U\right\rangle}\mathbb{E}_{\nu_{U^c}^{\check{W},\check{\eta}}}\left(e^{-\left\langle \check{\lambda},\check{\psi}\right\rangle - \frac{1}{2}\left\langle \check{\lambda}, \check{G}^{U^c}\check{\lambda}\right\rangle + \left\langle 1_{U^c}, \check{\lambda}\right\rangle}\right) \\
&\quad = e^{-\left\langle \lambda_U, \widehat{\psi}\right\rangle - \langle \lambda_{U^c}, 1_{U^c}\rangle - \frac{1}{2}\left\langle \lambda_U, \widehat{G}^U\lambda_U\right\rangle}\mathbb{E}_{\nu_{U^c}^{\check{W},\check{\lambda}+\check{\eta}}}(1)
\end{aligned}$$

which concludes the proof of the lemma, using that $\nu_{U^c}^{\check{W},\check{\lambda}+\check{\eta}}$ is a probability □

5.2. **Proof of Lemma 3.** We remark that since $\psi^{(n)}$ is defined for all $n$ by

$$\begin{cases} (H_\beta \psi^{(n)})_{V_n} = 0, \\ \psi^{(n)}_{V_n^c} = 1, \end{cases}$$

we have $\psi^{(n)}_{V_n} = ((H_\beta)_{V_n,V_n})^{-1}(\eta^{(n)})$, where $\eta^{(n)} = P_{V_n,V_n^c}(1_{V_n^c})$. Moreover, by Lemma 5(i), under $\nu_V^W(d\beta)$, we know that $\beta_{V_n}$ has law $\nu_{V_n}^{W,\eta^{(n)}}$. Using Lemma 6 applied to $V = V_{n+1}$ and $U = V_n$, we have that $\widehat{G}^{(n+1)}_{V_{n+1},V_{n+1}}$ corresponds to $G_\beta$ in Lemma 6 and $\widehat{G}^{(n)}_{V_n,V_n}$ to $\widehat{G}^U$, $\eta^{(n+1)}$ to $\eta$, and $\eta^{(n)}$ to $\widehat{\eta}$. Hence, we get that a.s.

$$\begin{aligned}
&\mathbb{E}_{\nu_V^W}\left(e^{-\left\langle \lambda_{V_{n+1}},\psi^{(n+1)}_{V_{n+1}}\right\rangle - \frac12\left\langle \lambda_{V_{n+1}},\widehat{G}^{(n+1)}\lambda_{V_{n+1}}\right\rangle}\,\Big|\,\mathcal{F}^{(n)}\right)\\
&\quad= e^{-\left\langle \lambda_{V_n},\psi^{(n)}_{V_n}\right\rangle - \left\langle \lambda_{V_{n+1}\setminus V_n},1_{V_{n+1}\setminus V_n}\right\rangle - \frac12\left\langle \lambda_{V_n},\widehat{G}^{(n)}\lambda_{V_n}\right\rangle}\\
&\quad= e^{-\left\langle \lambda_{V_{n+1}},\psi^{(n)}_{V_{n+1}}\right\rangle - \frac12\left\langle \lambda_{V_n},\widehat{G}^{(n)}\lambda_{V_n}\right\rangle}
\end{aligned}$$

since $\psi^{(n)}_{V_{n+1}\setminus V_n} = 1$. This concludes the proof since $\psi^{(n)}$ and $\psi^{(n+1)}$ both equal 1 on $V^c_{n+1}$.

## 6. Passing to the limit: Proof of Theorem 1, Proposition 2, and Proposition 3

### 6.1. Proof of Theorem 1(i) and (ii).

*Proof of Theorem* 1(i) *and* (ii). By the path representation (4.3) we know that $\widehat{G}^{(n)}_\beta(i,j)$ is nondecreasing for all $i,j \in V$ since the set of paths $\mathcal{P}^{V_n}_{i,j}$ is nondecreasing and that $\widehat{G}^{(n)}_\beta(i,j) \le G^{(n)}(i,j)$ since $\mathcal{P}^{V_n}_{i,j} \subset \mathcal{P}^{\widetilde{V}_n}_{i,j}$. Hence, it convergences a.s. to a random variable $\widehat{G}(i,j)$. Since $\widehat{G}^{(n)}(i,j) > 0$ as soon as $i,j \in V_n$ (indeed, $V_n$ is connected), we have $\widehat{G}(i,j) > 0$ a.s. It remains to prove that $\widehat{G}(i,j) < \infty$. As $\widehat{G}^{(n)}(i,i)$ converges a.s. to $\widehat{G}(i,i)$ and is nondecreasing, for any $h \ge 0$,

$$\begin{aligned}
\nu_V^W(\widehat{G}(i,i) \le h) &= \nu_V^W(\lim_{n\to\infty} \widehat{G}^{(n)}(i,i) \le h)\\
&= \lim_{n\to\infty} \nu_V^W(\widehat{G}^{(n)}(i,i) \le h)\\
&\ge \lim_{n\to\infty} \nu_V^W(G^{(n)}(i,i) \le h)\\
&= \mathbb{P}\left(\frac{1}{2\Gamma} \le h\right),
\end{aligned}$$

where $\Gamma$ is a gamma random variable with parameters $(\frac12, 1)$. In the last equality we used that $\frac{1}{2G^{(n)}(i,i)}$ has gamma law with parameters $(\frac12, 1)$ by Theorem C. Therefore, $\widehat{G}(i,i) < \infty$ a.s. For the of-diagonal term, since $(H_\beta)_{V_n,V_n}$ is positive definite, we have by the Cauchy–Schwarz inequality that

$$\widehat{G}^{(n)}(i,j) = \left\langle \delta_i, \widehat{G}^{(n)}\delta_j\right\rangle \le \sqrt{\left\langle \delta_i, \widehat{G}^{(n)}\delta_i\right\rangle\left\langle \delta_j, \widehat{G}^{(n)}\delta_j\right\rangle} = \sqrt{\widehat{G}^{(n)}(i,i)\widehat{G}^{(n)}(j,j)};$$

therefore, $\widehat{G}(i,j) \le \sqrt{\widehat{G}(i,i)\widehat{G}(j,j)}$ and $\widehat{G}(i,j)$ is a.s. finite.

From Proposition 9, we know that $\psi^{(n)}(i)$ is a positive martingale for all $i \in V$. As a positive martingale, $\psi^{(n)}(i)$ converges a.s. to some nonnegative integrable random variable $\psi(i)$.

It remains to show that the limits $\psi$ and $\widehat{G}$ do not depend on the choice of the exhausting sequence $(V_n)$. Assuming that $(\Omega_n)$ is another increasing exhausting sequence, we can similarly construct the martingale $\phi^{(n)}(i)$ associated with $\Omega_n$.

As $(\Omega_n)$ and $(V_n)$ are exhausting, we can construct a subsequence $n_k$ such that the alternating sequence $V_{n_1}, \Omega_{n_2}, V_{n_3}, \dots$ is increasing and thus the alternating sequence $\psi^{(n_1)}(i), \phi^{(n_2)}(i), \psi^{(n_3)}(i), \dots$ is a martingale for all $i \in V$. This martingale converges a.s. and this identifies the limits of $\psi^{(n)}(i)$ and $\phi^{(n)}(i)$. The argument is the same for $\widehat{G}$ since the sequence of Green functions associated with the alternating sequence of subsets is nondecreasing and converges a.s. □

6.2. **Representation of the VRJP as a mixture on the infinite graphs: Proof of Theorem 1(iii).** With the coupling of section 4.3, by Proposition 8 we have for $\beta \in \mathcal{D}_V^W$ and $\gamma > 0$,

$$G^{(n)}(i,j) = \widehat{G}^{(n)}(i,j) + \frac{1}{2\gamma}\psi^{(n)}(i)\psi^{(n)}(j).$$

By Theorem 1(i) and (ii), we have that $\nu_V^W(d\beta, d\gamma)$-a.s.

$$\lim_{n\to\infty} G^{(n)}(i,j) = G(i,j), \tag{6.1}$$

where $G(i,j)$ is defined in Theorem 1(iii).

The next corollary of Proposition 7 gives the necessary uniform integrability on jump rates to extend the representation of the VRJP for finite graphs to infinite graphs.

**Corollary 2.** *For any $i, j \in V$, there exists $n_0 \in \mathbb{N}$, such that the family of random variables $\left\{\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}\right\}_{n\ge n_0}$ is uniformly integrable under $\nu_V^W(d\beta, d\gamma)$.*

*Proof.* Choose $n_0$ such that $i, j \in V_{n_0}$, and $i$ and $j$ are connected by a path in $V_{n_0}$. Denote by $K$ the distance between $i$ and $j$ for the graph distance in $V_{n_0}$, and let $(\sigma_0 = i, \sigma_1, \dots, \sigma_K = j)$ be a directed path from $i$ to $j$ in $V_{n_0}$. Note that it is also a directed path in any $V_n$ for $n \ge n_0$ since $V_n$ is increasing. Let

$$\eta := \frac{1}{2}\min_{k=0,\dots,K-1}(W_{\sigma_k,\sigma_{k+1}}) > 0.$$

Let $c(\eta) > 0$ be a positive constant depending only on $\eta$ such that $e^{2x} \le c(\eta)e^{\eta\cosh(x)}$ for all reals $x$ (which exists since $2|x| \le \eta\cosh(x)$ for $x$ large enough and since $\cosh(x) \ge 1$ for all $x$). We can write with notation (3.4), for $n \ge n_0$,

$$\begin{aligned}\left(\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}\right)^2 = e^{2(u^{(n)}(i_0,j)-u^{(n)}(i_0,i))} &= \prod_{k=0}^{K-1} e^{2(u^{(n)}(i_0,\sigma_{k+1})-u^{(n)}(i_0,\sigma_k))}\\ &\le c(\eta)^K \prod_{k=0}^{K-1} e^{\eta\cosh(u^{(n)}(i_0,\sigma_{k+1})-u^{(n)}(i_0,\sigma_k))}.\end{aligned}$$

By Theorem C and Lemma 2, under $\nu_V^W(d\beta, d\gamma)$, $u^{(n)}(i_0,\cdot)$ has law $\mathcal{Q}_{i_0}^W$. Proposition 7 then implies that for $n \ge n_0$,

$$\mathbb{E}_{\nu_V^W}\left(\left(\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}\right)^2\right) \le e^{\eta K}2^{K/2}c(\eta)^K.$$

The family is uniformly bounded in $L^2$ and in particular is uniformly integrable. □

Consider now a connected finite subset $\Lambda \subset V$ containing $i_0$, and set

$$\partial^+\Lambda = \{j \in \Lambda^c,\ \exists i \in \Lambda \text{ such that } i \sim j\ \}.$$

Consider also a real $t_0 > 0$. Let $T$ be the stopping time

$$T = t_0 \wedge \inf\{t \geq 0,\ Z_t \notin \Lambda\}.$$

By construction, the law of the VRJP in an exchangeable time scale on $\mathcal{G}$ up to time $T$ equals the law of the VRJP in an exchangeable time scale on $\mathcal{G}_n$ up to time $T$ for all $n$ such that $\Lambda \cup \partial^+\Lambda \subset V_n$. For convenience, in this proof we write $\mathbb{P}_{i_0}^{\mathrm{VRJP},\mathcal{G}}$ for its law on $\mathcal{G}$ and $\mathbb{P}_{i_0}^{\mathrm{VRJP},\mathcal{G}_n}$ for its law on $\mathcal{G}_n$. Hence, our previous discussion formally means that $\mathbb{P}_{i_0}^{\mathrm{VRJP},\mathcal{G}}((Z_t)_{t\leq T} \in \cdot) = \mathbb{P}_{i_0}^{\mathrm{VRJP},\mathcal{G}_n}((Z_t)_{t\leq T} \in \cdot)$ for $n$ large enough. We denote by

$$\ell_i(T) = \int_0^T \mathbb{1}_{Z_u = i}\, du$$

the local time of $Z$ up to time $T$. Using Corollary 1 and the coupling in Lemma 2, the VRJP in an exchangeable time scale on $\mathcal{G}_n$, starting at $i_0$, is a mixture of Markov jump processes with jump rates from $i$ to $j$

$$\frac{1}{2} W_{i,j}^{(n)} \frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)} \tag{6.2}$$

under the law $\nu_V^W(d\beta, d\gamma)$. We denote by

$$\widetilde{\beta}_i^{(n)} = \sum_{j\sim i} \frac{1}{2} W_{i,j}^{(n)} \frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}$$

the holding time at site $i$ (note that $\widetilde{\beta}_i^{(n)} = \beta_i^{(n)}$ for $i \neq i_0$). We denote by $P_{i_0}^{\mathrm{MJP}}$ the law of the Markov jump process with jump rates $\frac{1}{2}W_{i,j}$ starting at $i_0$. The Radon–Nykodim derivative of the law of $(Z_t)_{t\leq T}$ under the law of the Markov jump process with jump rates (6.2) with respect to its law under $P_{i_0}^{\mathrm{MJP}}$ is

$$e^{-\sum_{i\in\Lambda} \ell_i(T)(\widetilde{\beta}_i^{(n)} - \frac{1}{2}W_i)} \frac{G^{(n)}(i_0, Z_T)}{G^{(n)}(i_0,i_0)}, \tag{6.3}$$

where as usual $W_i = \sum_{j\sim i} W_{i,j}$. We postpone the proof of this formula to the end this subsection.

Formula (6.3) implies that for all positive bounded test functions $F$, for $n$ large enough,

(6.4)

$$\begin{aligned}
&\mathbb{E}_{i_0}^{\mathrm{VRJP},\mathcal{G}}(F((Z_t)_{t\leq T})) = \mathbb{E}_{i_0}^{\mathrm{VRJP},\mathcal{G}_n}(F((Z_t)_{t\leq T})) \\
&\quad = \int \sum_{j\in\Lambda\cup\partial^+\Lambda} \\
&\qquad \times E_{i_0}^{\mathrm{MJP}}\left(\mathbb{1}_{Z_T=j} F((Z_t)_{t\leq T}) e^{-\sum_{i\in\Lambda} \ell_i(T)(\widetilde{\beta}_i^{(n)} - \frac{1}{2}W_i)} \frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)}\right) \nu_V^W(d\beta, d\gamma).
\end{aligned}$$

From (6.1), we have a.s.

$$\lim_{n\to\infty} \widetilde{\beta}_i^{(n)} = \widetilde{\beta}_i := \sum_{j\sim i} \frac{1}{2} W_{i,j} \frac{G(i_0,j)}{G(i_0,i)}.$$

We remark that in (6.4), the term $e^{-\sum_{i\in\Lambda}\ell_i(T)(\widetilde{\beta}_i^{(n)}-\frac{1}{2}W_i)}$ is bounded since $\Lambda\cup\partial_+\Lambda$ is finite and $T\le t_0$. Using the uniform integrability of $\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i_0)}$, Corollary 2, we get, letting $n$ tend to $\infty$, that

$$
\begin{aligned}
&\mathbb{E}_{i_0}^{\mathrm{VRJP},\mathcal{G}}(F((Z_t)_{t\le T}))\\
&\quad=\int\sum_{j\in\Lambda\cup\partial^+\Lambda}\\
&\quad\quad\times E_{i_0}^{\mathrm{MJP}}\left(\mathbb{1}_{Z_T=j}F((Z_t)_{t\le T})e^{-\sum_{i\in\Lambda}\frac{1}{2}\ell_i(T)(\widetilde{\beta}_i-\frac{1}{2}W_i)}\frac{G(i_0,j)}{G(i_0,i_0)}\right)\nu_V^W(d\beta,d\gamma)\\
&\quad=\int E_{i_0}^{\beta,\gamma,i_0}(F((Z_t)_{t\le T}))\,\nu_V^W(d\beta,d\gamma),
\end{aligned}
$$

where $E_{i_0}^{\beta,\gamma,i_0}$ is the expectation associated with the probability $P_{i_0}^{\beta,\gamma,i_0}$ defined in Theorem 1. Since $\Lambda$ and $t_0$ can be chosen arbitrarily, the previous identity characterizes the law of $(Z(t))_{t\ge0}$. This concludes the proof of Theorem 1(iii). □

*Proof of formula* (6.3). Consider the Markov jump process on the graph $\mathcal{G}_n$ with jump rates $\frac{1}{2}W^{(n)}$, and denote by $P_{i_0}^{MJP,(n)}$ its law starting from $i_0$. At each vertex $i$, it waits an exponential random time with parameter $\frac{1}{2}W_i^{(n)}$ and then jumps to $j\sim i$ with probability proportional to $W_{i,j}^{(n)}$. On time interval $[0,t]$ the probability that it follows the discrete path $(\sigma_0=i_0,\sigma_1,\dots,\sigma_n)$ and jumps at times $0<s_1<\cdots<s_n<t$ has distribution

$$
\begin{aligned}
&\left(\prod_{k=0}^{n-1}\frac{W_{\sigma_k,\sigma_{k+1}}^{(n)}}{W_{\sigma_k}^{(n)}}\right)\left(e^{-\frac{1}{2}W_{\sigma_n}^{(n)}(t-s_n)}\prod_{k=0}^{n-1}W_{\sigma_k}^{(n)}e^{-\sum_{k=0}^{n-1}\frac{1}{2}W_{\sigma_k}^{(n)}(s_{k+1}-s_k)}\right)ds_1\cdots ds_n\\
&\quad=\left(\prod_{k=0}^{n-1}W_{\sigma_k,\sigma_{k+1}}^{(n)}\right)e^{-\frac{1}{2}\sum_{i\in\tilde{V}_n}W_i^{(n)}\ell_i((\sigma_k),(s_k))}ds_1\cdots ds_n,
\end{aligned}
$$

where $\ell_i((\sigma_k),(s_k))$ is the total time spent at position $i$ by the trajectory with discrete path $(\sigma_k)$ and jump times $(s_k)$. The same formula is true for the Markov jump process with jump rates (6.2) with $\frac{1}{2}W_{i,j}^{(n)}$ replaced by $\frac{1}{2}W_{i,j}^{(n)}\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}$. By cancellation of the ratios $\frac{G^{(n)}(i_0,j)}{G^{(n)}(i_0,i)}$ along the trajectory, it gives that on time interval $[0,t]$, the Radon–Nikodym derivative of the law of the Markov jump process with jump rates (6.2) starting at $i_0$ with respect to $P_{i_0}^{MJP,(n)}$ is

$$
M_t:=e^{-\sum_{i\in\tilde{V}_n}\ell_i(t)(\widetilde{\beta}_i^{(n)}-\frac{1}{2}W_i^{(n)})}\frac{G^{(n)}(i_0,Z_t)}{G^{(n)}(i_0,i_0)}.
$$

Moreover, $M_t$ is a martingale and is bounded on finite time intervals. Since $T$ is a bounded stopping time $T\le t_0$, we have (6.3) for the stopping time $T$ since $W_{V_n,V_n}^{(n)}=W_{V_n,V_n}$. □

### 6.3. **Proof of Proposition 2 and Theorem 1(iv).**

*Proof of Proposition* 2. Recall the notation of section 3.2 and identities (3.9). As $n \mapsto \widehat{G}^{(n)}(i,j)$ is increasing, we have

$$\widehat{G}(i,j) = \sum_{\sigma\in\mathcal{P}^V_{i,j}} \frac{W_\sigma}{(2\beta)_\sigma}. \tag{6.5}$$

By arguments similar to (3.10), we have

$$\frac{\widehat{G}(i_0,i)}{\widehat{G}(i_0,i_0)} = \sum_{\sigma\in\bar{\mathcal{P}}^V_{i,i_0}} \frac{W_\sigma}{(2\beta)^-_\sigma}.$$

We recall that $(\widetilde{Z}_n)_{n\in\mathbb{N}}$ denotes the discrete time process which describes successive jumps of the process $(Z_t)_{t\in\mathbb{R}_+}$. Clearly, $\{\tau^+_{i_0} < \infty\} = \{\exists n \geq 1, \text{ s.t. } \widetilde{Z}_n = i_0\}$. Therefore, if we denote $\{(\widetilde{Z}_n) \sim \sigma\} = \{\widetilde{Z}_0 = \sigma_0, \dots, \widetilde{Z}_m = \sigma_m\}$ with $m = |\sigma|$, then for $i \neq i_0$

$$\begin{aligned} h(i) := P^{\beta,\gamma,i_0}_i(\tau^+_{i_0} < \infty) &= \sum_{\sigma\in\bar{\mathcal{P}}^V_{i,i_0}} P^{\beta,\gamma,i_0}_i((\widetilde{Z}_n)\sim\sigma) \\ &= \sum_{\sigma\in\bar{\mathcal{P}}^V_{i,i_0}} \frac{W_\sigma}{(2\beta)^-_\sigma}\frac{G(i_0,i_0)}{G(i_0,i)} = \frac{\widehat{G}(i_0,i)}{\widehat{G}(i_0,i_0)}\cdot\frac{G(i_0,i_0)}{G(i_0,i)}. \end{aligned} \tag{6.6}$$

It follows from $G(i,j) = \widehat{G}(i,j) + \frac{1}{2\gamma}\psi(i)\psi(j)$ that, for $i \neq i_0$,

$$P^{\beta,\gamma,i_0}_i(\tau^+_{i_0} = \infty) = 1 - h(i) = \frac{\psi(i_0)}{2\gamma}\frac{\widehat{G}(i_0,i_0)\psi(i) - \widehat{G}(i_0,i)\psi(i_0)}{\widehat{G}(i_0,i_0)G(i_0,i)}.$$

Therefore,

$$\begin{aligned} P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} = \infty) &= \sum_{j\sim i_0} \frac{W_{i_0,j}G(i_0,j)}{2\widetilde{\beta}_{i_0}G(i_0,i_0)} P^{\beta,\gamma,i_0}_j(\tau^+_{i_0} = \infty) \\ &= \sum_{j\sim i_0} \frac{\psi(i_0)W_{i_0,j}}{4\gamma\widetilde{\beta}_{i_0}}\frac{\widehat{G}(i_0,i_0)\psi(j) - \widehat{G}(i_0,j)\psi(i_0)}{\widehat{G}(i_0,i_0)G(i_0,i_0)}. \end{aligned} \tag{6.7}$$

By definition, for $n$ large enough, we have $H\widehat{G}^{(n)}(i_0,\cdot) = \mathbb{1}_{i_0}(\cdot)$. Taking the limit $n\to\infty$, we have $H\widehat{G}(i_0,\cdot) = \mathbb{1}_{i_0}(\cdot)$. By Theorem 2(iii) (proved in section 6.5), we have $H\psi(\cdot) = 0$, therefore,

$$\sum_{j\sim i_0} W_{i_0,j}[\psi(j)\widehat{G}(i_0,i_0) - \psi(i_0)\widehat{G}(i_0,j)] = \psi(i_0),$$

hence $P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} = \infty) = \frac{\psi(i_0)^2}{4\gamma\widetilde{\beta}_{i_0}\widehat{G}(i_0,i_0)G(i_0,i_0)}$. □

*Proof of Theorem* 1(iv). From Proposition 2 we see that $P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} = \infty) > 0$ if and only if $\psi(i_0) > 0$. Since the Markov jump process $P^{\beta,\gamma,i_0}_{i_0}$ is irreducible ($\mathcal{G}$ is connected), (iv) follows. □

### 6.4. **Ergodicity and the 0-1 law: Proof of Propositions 3 and 5.**

*Proof of Proposition* 3. From the expression of the Laplace transform of $\beta$ (cf. Proposition 1), we see that under $\nu_V^W(d\beta)$, $(\beta_i)_{i\in V}$ is stationary for the action of $\mathcal{A}$.

By 1-dependence (cf. Proposition 1) it is also ergodic. Indeed, assume that $(\tau_n)\in\mathcal{A}^{\mathbb{N}}$ is a sequence of automorphims such that $d_{\mathcal{G}}(i_0,\tau_n(i_0))\to\infty$ for some vertex $i_0$. We prove that $(\tau_n)$ is mixing in the sense that for all $A,B\in\sigma(\beta_i,i\in V)$,

$$\lim_{n\to\infty}\nu_V^W(\tau_n^{-1}(B)\cap A)=\nu_V^W(A)\nu_V^W(B). \tag{6.8}$$

Assume that $V_1\subset V$ is finite and that $A,B\in\sigma(\beta_j,\ j\in V_1)$. By 1-dependence, $\tau_n^{-1}(B)$ is independent of $A$ for $n$ large enough. This implies that (6.8) is true for all $A,B$ in the algebra $\mathcal{P}=\bigcup_{V_1\text{ finite}}\sigma(\beta_j,\ j\in V_1)$. Now, any measurable set in $\sigma\{\beta_j,\ j\in V\}$ can be approximated by an element in $\mathcal{P}$; i.e., for $\epsilon>0$ and $A,B$ in $\sigma\{\beta_j,\ j\in V\}$, we can find $A_0,B_0\in\mathcal{P}$ such that $\nu_V^W(A\Delta A_0)<\epsilon$ and $\nu_V^W(B\Delta B_0)<\epsilon$ where $\Delta$ is the symmetric difference (see, e.g., [11, Theorem D, Section 13, p. 56]). Hence, also $\nu_V^W(\tau_n^{-1}(B)\Delta\tau_n^{-1}(B_0))<\epsilon$ since $\tau_n$ is measure preserving. This easily implies (6.8) for all $A$ and $B$. Finally, if $A$ is $\tau$-invariant, (6.8) implies $(\nu_V^W(A))^2=\nu_V^W(A)$, hence $\nu_V^W(A)$ equals 0 or 1.

Let us prove stationarity and ergodicity of the random variables $(\widehat{G}(i,j))_{i,j\in V}$. We also denote by $\tau$ the transformation on $\mathbb{R}^{V\times V}$ given by, for $(M(i,j))\in\mathbb{R}^{V\times V}$, $\tau M(i,j)=M(\tau i,\tau j)$. Since the limit $\widehat{G}_\beta(i,j)$ does not depend on the choice of the sequence $V_n$, a.s., we have that $\tau(\widehat{G}_\beta)=\widehat{G}_{\tau(\beta)}$, by choosing the approximating sequences $V_n$ and $\tau V_n$. It implies that $(\widehat{G}(i,j))$ is stationary. Moreover, if $A\in\mathcal{B}(\mathbb{R}^{V\times V})$ is $\tau$-invariant, then the set $\{\beta,\ \widehat{G}_\beta\in A\}$ is a.s. $\tau$-invariant. Hence, $A$ has measure 0 or 1 under the law of $\widehat{G}$. The proof is similar for the random variables $(\psi(i))_{i\in V}$.

The event $\{\psi(i)=0,\ \forall i\in V\}$ is clearly invariant by $\mathcal{A}$, hence has probability 0 or 1. Together with Theorem 1(iv) this concludes the proof of the proposition. ☐

*Proof of Proposition* 5. The proof is similar to the proof of Proposition 3. ☐

### 6.5. **Proof of Theorem 2: Relation with spectral properties of the random Schrödinger operator.**

*Proof of Theorem* 2(i). By Proposition 1, since $\nu_V^W$ is supported on $\mathcal{D}_V^W$, we have a.s. that $H_{V_n\times V_n}>0$, and passing to the limit, we get $H\geq 0$. Hence, $\sigma(H)\subset[0,+\infty)$. ☐

*Proof of Theorem* 2(ii). As $-\varepsilon$ is strictly outside the spectrum of $H$ almost surely and the equation $(H+\varepsilon)\widehat{G}^\varepsilon=\mathrm{Id}$ has a unique finite solution, we can verify that $\sum_{\sigma\in\mathcal{P}_{i,j}^V}\frac{W_\sigma}{(2\beta+\varepsilon)_\sigma}$ is a solution to this equation. Now by (6.5) we have

$$(H+\varepsilon)^{-1}(i,j)=\sum_{\sigma\in\mathcal{P}_{i,j}^V}\frac{W_\sigma}{(2\beta+\varepsilon)_\sigma}\leq\sum_{\sigma\in\mathcal{P}_{i,j}^V}\frac{W_\sigma}{(2\beta)_\sigma}=\widehat{G}(i,j)<\infty.$$

Therefore, as $\sum_{\sigma\in\mathcal{P}_{i,j}}\frac{W_\sigma}{(2\beta+\varepsilon)_\sigma}$ is increasing as $\varepsilon\to 0$, it converges a.s. to $\widehat{G}(i,j)$. ☐

*Proof of Theorem* 2(iii). We have, for all $i \in V_n$, that $\beta \in \mathcal{D}_V^W$, $\psi_\beta^{(n)}(i) = \sum_{j\sim i} \frac{W_{i,j}}{2\beta_i}\psi_\beta^{(n)}(j)$. As $\psi^{(n)}(i)$ converges a.s. to $\psi(i)$, the above equality holds in the limit; i.e., for all $i \in V$, a.s.

$$\psi(i) = \sum_{j\sim i} \frac{W_{i,j}}{2\beta_i}\psi(j).$$

This exactly means $(H\psi)(i) = 0$. □

*Proof of Theorem* 2(iv). By Fatou's lemma the limit $\psi(i)$ satisfies $\mathbb{E}_{\nu_V^W}(\psi(i)) \leq 1$. By the Markov inequality

$$\nu_V^W\left(\psi(i) \geq C\|i\|_\infty^p\right) \leq \frac{1}{C\|i\|_\infty^p}.$$

Let $\partial B(0,n)$ be the sphere of radius $n$ for $\|\cdot\|_\infty$, i.e., $\partial B(0,n) = \{i \in \mathbb{Z}^d,\ \|i\|_\infty = n\}$. When $p > d$,

$$\begin{aligned}\sum_{i\in\mathbb{Z}^d,\ i\neq 0} \nu_V^W(\psi(i) \geq C\|i\|_\infty^p) &= \sum_{n\geq 1}\sum_{i\in\partial B(0,n)} \nu_V^W(\psi(i) \geq C\|i\|_\infty^p) \\ &\leq \sum_{n\geq 1}\sum_{i\in\partial B(0,n)} \frac{1}{C\|i\|_\infty^p} \\ &\leq C'\sum_n \frac{n^{d-1}}{n^p} < \infty\end{aligned}$$

for some constant $C' > 0$. By the Borel–Cantelli lemma, a.s. only a finite number of $i$ satisfies $\psi(i) \geq C\|i\|_\infty^p$. □

## 7. $h$-TRANSFORMS

**Corollary 3.** *Recall that* $\tau_{i_0}^+ = \inf\{t \geq 0,\ \ Z_t = i_0,\ \exists s < t \text{ s.t. } Z_s \neq i_0\}$ *is the first return time to* $i_0$ *of the process* $(Z_t)_{t\geq 0}$.

(i) *For almost all* $\beta$ *and* $i_0 \in V$, *denote by* $\widehat{P}_{i_0}^{\beta,i_0}$ *the law of the Markov jump process with jump rate from* $i$ *to* $j$

$$\begin{cases} \frac{1}{2}W_{i,j}\frac{\widehat{G}(i_0,j)}{\widehat{G}(i_0,i)}, & i \neq i_0, \\ \widetilde{\beta}_{i_0}\frac{W_{i_0,j}\widehat{G}(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\widehat{G}(i_0,k)}, & i = i_0,\ j \sim i_0, \end{cases}$$

*where as before* $\widetilde{\beta}_{i_0} = \sum_{j\sim i_0} \frac{1}{2}W_{i_0,j}\frac{G(i_0,j)}{G(i_0,i_0)}$. *Then, for* $\nu_V^W$*-almost all* $\beta$, *for* $\gamma > 0$,

$$P_{i_0}^{\beta,\gamma,i_0}\left((Z_t)_{t\leq\tau_{i_0}^+} \in \cdot \mid \tau_{i_0}^+ < \infty\right) = \widehat{P}_{i_0}^{\beta,i_0}\left((Z_t)_{t\leq\tau_{i_0}^+} \in \cdot\right).$$

(ii) *The VRJP in an exchangeable time scale, conditionally on* $\{\tau_{i_0}^+ < \infty\}$ *and up to its first return time to* $i_0$, *is given by the following mixture:*

$$\mathbb{P}_{i_0}^{VRJP}((Z_t)_{t\leq\tau_{i_0}^+} \in \cdot\ |\tau_{i_0}^+ < \infty) = \int \widehat{P}_{i_0}^{\beta,i_0}((Z_t)_{t\leq\tau_{i_0}^+} \in \cdot)\frac{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ < \infty)}{\mathbb{P}_{i_0}^{VRJP}(\tau_{i_0}^+ < \infty)}\nu_V^W(d\beta, d\gamma).$$

(iii) *Let $i_0 \in V$. Almost surely on the event $\{\psi(i) > 0,\ \forall i \in V\}$, $\check{G}(i_0, j) := \widehat{G}(i_0, i_0)\psi(j) - \widehat{G}(i_0, j)\psi(i_0)$ is positive for all $j \neq i_0$, and we define $\check{P}^{\beta,\gamma,i_0}_{i_0}$ as the law of the Markov jump process starting at $i_0$ and with jump rate from $i$ to $j$,*

$$\begin{cases} \frac{1}{2}W_{i,j}\frac{\check{G}(i_0,j)}{\check{G}(i_0,i)}, & i \neq i_0,\ j \neq i_0, \\ \widetilde{\beta}_{i_0}\frac{W_{i_0,j}\check{G}(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\check{G}(i_0,k)}, & i = i_0,\ j \sim i_0, \\ 0, & i \sim i_0\ j = i_0. \end{cases}$$

*Then, $\nu_V^W$-almost surely on this event, for $\gamma > 0$,*

$$P^{\beta,\gamma,i_0}_{i_0}\left((Z_t)_{t\geq 0} \in \cdot \mid \tau^+_{i_0} = \infty\right) = \check{P}^{\beta,\gamma,i_0}_{i_0}\left((Z_t)_{t\geq 0} \in \cdot\right).$$

(iv) *The VRJP in an exchangeable time scale, conditionally on the event $\{\tau^+_{i_0} = \infty\}$, is a mixture of Markov jump processes with mixing law*

$$\mathbb{P}^{VRJP}_{i_0}(\ \cdot\ |\tau^+_{i_0} = \infty) = \int \check{P}^{\beta,\gamma,i_0}_{i_0}(\cdot)\frac{P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} = \infty)}{\mathbb{P}^{VRJP}_{i_0}(\tau^+_{i_0} = \infty)}\nu_V^W(d\beta, d\gamma).$$

*Remark* 15. Note that in case (i), the conditional jump rates do not depend on $\gamma$.

*Proof of Corollary* 3. (i) Recall from (6.6) that for $i \neq i_0$

$$h(i) = P^{\beta,\gamma,i_0}_{i}(\tau^+_{i_0} < \infty) = \frac{\widehat{G}(i_0, i)G(i_0, i_0)}{\widehat{G}(i_0, i_0)G(i_0, i)}.$$

For $i \neq i_0$, we have

$$P^{\beta,\gamma,i_0}_{i_0}(X_{t+dt} = j\ |X_t = i,\ t \leq \tau^+_{i_0} < \infty) \sim \frac{h(j)}{h(i)}P^{\beta,\gamma,i_0}_{i_0}(X_{t+dt} = j\ |X_t = i).$$

Hence, the jump rate of $P^{\beta,\gamma,i_0}_{i_0}(\ \cdot\ |\tau^+_{i_0} < \infty)$, up to time $\tau^+_{i_0}$, from $i$ to $j$ is

$$\frac{1}{2}W_{i,j}\frac{G(i_0, j)}{G(i_0, i)}\frac{h(j)}{h(i)} = \frac{1}{2}W_{i,j}\frac{\widehat{G}(i_0, j)}{\widehat{G}(i_0, i)}.$$

The jump rate of $P^{\beta,\gamma,i_0}_{i_0}(\ \cdot\ |\tau^+_{i_0} < \infty)$, up to time $\tau^+_{i_0}$, from $i_0$ to $j$ is given by

$$\frac{1}{2}W_{i_0,j}\frac{G(i_0, j)}{G(i_0, i_0)}\frac{h(j)}{P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} < \infty)} = \widetilde{\beta}_{i_0}\frac{W_{i_0,j}\widehat{G}(i_0, j)}{\sum_{k\sim i_0} W_{i_0,k}\widehat{G}(i_0, k)},$$

where $\widetilde{\beta}_{i_0} = \sum_{l\sim i_0}\frac{1}{2}W_{i_0,l}\frac{G(i_0,l)}{G(i_0,i_0)}$.
(ii) From (i), we have

$$\begin{aligned} &\mathbb{P}^{\mathrm{VRJP}}_{i_0}((Z_t)_{t\leq\tau^+_{i_0}} \in \cdot\ |\tau^+_{i_0} < \infty) \\ &\quad= \int P^{\beta,\gamma,i_0}_{i_0}((Z_t)_{t\leq\tau^+_{i_0}} \in \cdot\ |\tau^+_{i_0} < \infty)\frac{P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} < \infty)}{\mathbb{P}^{\mathrm{VRJP}}_{i_0}(\tau^+_{i_0} < \infty)}\nu_V^W(d\beta, d\gamma) \\ &\quad= \int \widehat{P}^{\beta,i_0}_{i_0}((Z_t)_{t\leq\tau^+_{i_0}} \in \cdot)\frac{P^{\beta,\gamma,i_0}_{i_0}(\tau^+_{i_0} < \infty)}{\mathbb{P}^{\mathrm{VRJP}}_{i_0}(\tau^+_{i_0} < \infty)}\nu_V^W(d\beta, d\gamma). \end{aligned}$$

(iii) The fact that $\check{G}(i_0, j)$ is positive for $j \neq i_0$ when $\psi > 0$ is a consequence of Proposition 2 and Theorem 1(iv). Similarly to (i), for $i \neq i_0$, we have

$$P_{i_0}^{\beta,\gamma,i_0}(X_{t+dt} = j \;|X_t = i,\; \tau_{i_0}^+ = \infty) \sim \frac{1-h(j)}{1-h(i)} P^{\beta,\gamma,i_0}(X_{t+dt} = j \;|X_t = i).$$

Hence, the jump rate of $P_{i_0}^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ = \infty)$, from $i \neq i_0$ to $j$ is

$$\frac{1}{2}W_{i,j}\frac{G(i_0,j)}{G(i_0,i)}\frac{1-h(j)}{1-h(i)} = \frac{1}{2}W_{i,j}\frac{\widehat{G}(i_0,i_0)\psi(j) - \widehat{G}(i_0,j)\psi(i_0)}{\widehat{G}(i_0,i_0)\psi(i) - \widehat{G}(i_0,i)\psi(i_0)} = \frac{1}{2}W_{i,j}\frac{\check{G}(i_0,j)}{\check{G}(i_0,i)}.$$

The jump rate of $P_{i_0}^{\beta,\gamma,i_0}(\,\cdot\,|\tau_{i_0}^+ = \infty)$, from $i_0$ to $j$ is given by

$$\frac{1}{2}W_{i_0,j}\frac{G(i_0,j)}{G(i_0,i_0)}\frac{1-h(j)}{P_{i_0}^{\beta,\gamma,i_0}(\tau_{i_0}^+ = \infty)} = \widetilde{\beta}_{i_0}\frac{W_{i_0,j}\check{G}(i_0,j)}{\sum_{k\sim i_0} W_{i_0,k}\check{G}(i_0,k)},$$

where $\widetilde{\beta}_{i_0} = \sum_{l\sim i_0}\frac{1}{2}W_{i_0,l}\frac{G(i_0,l)}{G(i_0,i_0)}$.
(iv) This follows easily from (iii) in the same way as in (ii).

□

## 8. Proof of recurrence of two-dimensional ERRW: Theorem 5

Consider the square grid $\mathcal{G} = (\mathbb{Z}^2, E)$ with constant edge weight $a_e = a > 0$. From (2.9) in section 2.5, we know that the ERRW on $\mathbb{Z}^2$ is a mixture of reversible Markov chains with conductances

$$x_{i,j} = W_{i,j}G(0,i)G(0,j), \tag{8.1}$$

where $(W, \beta, \gamma)$ are distributed according to $\widetilde{\nu}_V^a(dW, d\beta, d\gamma)$. We will use [15] to prove the following lemma.

**Lemma 7.** *There exists $c(a) > 0$ and $\xi(a) > 0$, depending only on $a$, such that for $\ell \in \mathbb{Z}^2$,*

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left(\left(\frac{x_\ell}{x_0}\right)^{\frac{1}{4}}\right) \le c(a)\|\ell\|_\infty^{-\xi(a)}, \tag{8.2}$$

*where $x_i = \sum_{j\sim i} x_{i,j}$ and $(x_{i,j})$ is defined in* (8.1).

*Proof.* This estimate follows from [15, Theorem 2.8] (it can also be deduced from [17, Lemma 2.5]) which gives a similar estimate on finite boxes. In of [15, Theorem 2.8], the estimate is stated for a periodic torus, but it is clear in the proof that the only necessary ingredient is that the finite graph with conductances is invariant by the reflection exchanging 0 and $\ell$. For this reason we choose the approximating sequence $V_n = B(\frac{\ell}{2}, n) \cap \mathbb{Z}^2$, where $B(\frac{\ell}{2}, n)$ is the ball with center $\ell/2$ and radius $n$. Consider as in section 4.2 the graph $\mathcal{G}_n = (\widetilde{V}_n = V_n \cup \{\delta_n\}, E_n)$ and the associated weights $(a_e^{(n)})_{e\in E_n}$ obtained by restriction of $(\mathcal{G}, (a_e)_{e\in E})$ to $V_n$ with wired boundary condition. Clearly, central symmetry with respect to $\frac{\ell}{2}$ (mapping $\delta_n$ to itself) leaves $(\mathcal{G}_n, a^{(n)})$ invariant and exchanges 0 and $\ell$.

With the coupling defined in section 4.3, we define for $i \sim j$, $i, j$ in $\widetilde{V}_n$,

$$x_{i,j}^{(n)} = W_{i,j}^{(n)} G^{(n)}(0,i) G^{(n)}(0,j),$$

where $W^{(n)}$ is obtained by restriction with wired boundary condition from $W$. By additivity of Gamma random variables, under $\widetilde{\nu}_V^a$, $(W_e^{(n)})_{e\in E_n}$ are independent

Gamma random variables with parameters $(a_e^{(n)})_{e\in E_n}$. Hence, the ERRW on $\mathcal{G}_n$ with initial weights $a^{(n)}$ and starting from 0, is a mixture of reversible Markov chains with conductances $(x_e^{(n)})_{e\in E_n}$.

From Theorem 1, with the coupling defined in section 4.3, we have that for all $i,j\in\mathbb{Z}^2$, $i\sim j$, a.s.

$$\lim_{n\to\infty} x_{i,j}^{(n)} = x_{i,j}. \tag{8.3}$$

The proof of [15, Theorem 2.8], can be readily adapted to prove the following estimate.

**Lemma 8.** *There exists $c(a)>0$ and $\xi(a)>0$ only depending on $a$ such that for $\ell\in\mathbb{Z}^2$ and $n$ large enough,*

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left(\left(\frac{x_\ell^{(n)}}{x_0^{(n)}}\right)^{\frac14}\right) \le c(a)\|\ell\|_\infty^{-\xi(a)},$$

*where, with the usual convention, $x_\ell^{(n)} = \sum_{j\sim\ell} x_{\ell,j}^{(n)}$.*

Then Lemma 7 follows from Lemma 8, (8.3), and Fatou's lemma. □

We now deduce recurrence of the ERRW from the estimate (8.2) and from Theorem 1 and Proposition 5. We have, for $\ell\neq 0$,

$$x_\ell = \sum_{j\sim\ell} W_{\ell,j}G(0,\ell)G(0,j) = 2\beta_\ell G(0,\ell)^2 \ge \frac{\beta_\ell}{2\gamma^2}\psi(0)^2\psi(\ell)^2.$$

Similarly,

$$x_0 = \sum_{j\sim 0} W_{0,j}G(0,0)G(0,j) = G(0,0)(2\beta_0 G(0,0)-1).$$

Hence,

$$\frac{x_\ell}{x_0} \ge \frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)}\beta_\ell\psi(\ell)^2. \tag{8.4}$$

Assume the ERRW is transient. By Proposition 5 it implies that, a.s., $\psi(i)>0$ for all $i$. Choose first $\eta>0$ such that

$$\widetilde{\nu}_V^a\left(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} \le \eta\right) \le \frac12.$$

For all $\epsilon>0$, we have by (8.2)

$$\widetilde{\nu}_V^a\left(\frac{x_\ell}{x_0}\ge\epsilon\right) \le \frac{1}{\epsilon^{\frac14}} c(a)\|\ell\|_\infty^{-\xi(a)}. \tag{8.5}$$

On the other hand, we have by (8.4)

(8.6)

$$\begin{aligned}\widetilde{\nu}_V^a\left(\frac{x_\ell}{x_0}\ge\epsilon\right) &\ge \widetilde{\nu}_V^a\left(\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} > \eta,\ \ \beta_\ell\psi(\ell)^2 > \frac{\epsilon}{\eta}\right)\\ &= 1-\widetilde{\nu}_V^a\left(\left\{\frac{\psi(0)^2}{2\gamma^2 G(0,0)(2\beta_0 G(0,0)-1)} \le \eta\right\}\cup\left\{\beta_\ell\psi(\ell)^2\le\frac{\epsilon}{\eta}\right\}\right)\\ &\ge \frac12 - \widetilde{\nu}_V^a\left(\beta_\ell\psi(\ell)^2\le\frac{\epsilon}{\eta}\right).\end{aligned}$$

By Proposition 5, $\beta_\ell\psi(\ell)^2$ is stationary with respect to translations. Together with (8.6) and (8.5), it implies that

$$\widetilde{\nu}_V^a\left(\beta_0\psi(0)^2 \le \frac{\epsilon}{\eta}\right) = \widetilde{\nu}_V^a\left(\beta_\ell\psi(\ell)^2 \le \frac{\epsilon}{\eta}\right) \ge \frac{1}{2} - \frac{1}{\epsilon^{\frac{1}{4}}}c(a)\|\ell\|_\infty^{-\xi(a)}.$$

By sending $\ell$ to infinity, we get $\widetilde{\nu}_V^a\left(\beta_0\psi(0)^2 \le \frac{\epsilon}{\eta}\right) \ge \frac{1}{2}$. Letting $\epsilon \to 0$, this is incompatible with $\psi(0) > 0$ a.s., hence with transience of ERRW.

## 9. Proof of Functional central limit theorems for the VRJP and the ERRW: Theorems 3 and 4

*Proof of Theorems* 3 *and* 4. Let us start by the VRJP on $\mathbb{Z}^d$, $d \ge 3$, with constant weights $W_{i,j} = W$. Assume that the VRJP is transient.

Recall that $(X_n)_{n\in\mathbb{N}}$ is the canonical discrete process on $(\mathbb{Z}^d)^{\mathbb{N}}$. For $\nu_V^W$-almost all $\beta$, let us define $\widetilde{P}_x^\psi$ to be the law of the reversible Markov chain, starting at $x$, with conductances $W_{i,j}\psi(i)\psi(j)$, i.e., with transition probabilities

$$\widetilde{P}_x^\psi(X_{n+1} = j | X_n = i) = \frac{W_{i,j}\psi(j)}{\sum_{l\sim i} W_{i,l}\psi(l)}.$$

Denote by $\widetilde{P}_x^{\beta,\gamma,0}$ the law of the underlying discrete time process associated with the Markov jump process $P_x^{\beta,\gamma,0}$, so that for $i \sim j$

$$\widetilde{P}_x^{\beta,\gamma,0}(X_{n+1} = j | X_n = i) = \frac{W_{i,j}G(0,j)}{\sum_{l\sim i} W_{i,l}G(0,l)}.$$

As $\psi$ is a generalized eigenfunction of $H_\beta$, for any $i \in V$,

$$\beta_i = \sum_{j\sim i} \frac{1}{2} W_{i,j}\frac{\psi(j)}{\psi(i)}.$$

It then follows by Proposition 6 that, for $i \ne 0$,

$$h^\psi(i) := \widetilde{P}_i^\psi(\tau_0^+ < \infty) = \sum_{\sigma\in\bar{\mathcal{P}}_{i,0}^V} \widetilde{P}_i^\psi(X_n \sim \sigma) = \sum_{\sigma\in\bar{\mathcal{P}}_{i,0}^V} \frac{W_\sigma}{(2\beta)_\sigma}\frac{\psi(0)}{\psi(i)} = \frac{\widehat{G}(0,i)}{\widehat{G}(0,0)}\frac{\psi(0)}{\psi(i)}.$$

(Recall that $\bar{\mathcal{P}}_{i,0}^V$ is defined in section 3.2.)

Consider the Markov chain $\widetilde{P}_0^\psi(\,\cdot\,|\tau_0^+ = \infty)$ (Doob's $(1-h^\psi)$-transform). By a similar computation as in the proof of Proposition 3, we have that the transition probability of $\widetilde{P}_0^\psi(\,\cdot\,|\tau_0^+ = \infty)$ from $i$ to $j$ is

$$\frac{W_{i,j}\psi(j)(1-h^\psi(j))}{\sum_{l\sim i} W_{i,l}\psi(l)(1-h^\psi(l))} = \frac{W_{i,j}\check{G}(0,j)}{\sum_{l\sim i} W_{i,l}\check{G}(0,l)}$$

for $j \ne 0$, and 0 when $j = 0$. Therefore, we see that the transition probabilities of $\widetilde{P}_0^\psi(\,\cdot\,|\tau_0^+ = \infty)$ are the same as those of $\widetilde{P}_0^{\beta,\gamma,0}(\,\cdot\,|\,\tau_0^+ = \infty)$; cf. Proposition 3(iii). Moreover, if we denote

$$\xi_0 = \sup\{n;\ X_n = 0\},$$

then, by the strong Markov property

$$\widetilde{P}_0^{\psi}(X_n \in \cdot | \tau_0^+ = \infty) = \widetilde{P}_0^{\psi}((X \circ \theta_{\xi_0})_n \in \cdot),$$

$$\widetilde{P}_0^{\beta,\gamma,0}(X_n \in \cdot | \tau_0^+ = \infty) = \widetilde{P}_0^{\beta,\gamma,0}((X \circ \theta_{\xi_0})_n \in \cdot),$$

where $\theta_n$ is the shift in time by $n$. It follows that $(X \circ \theta_{\xi_0})_n$ has the same law under $\widetilde{P}_0^{\psi}$ and under $\widetilde{P}_0^{\beta,\gamma,0}$.

We remark also, from Proposition 3, that $W_{i,j}\psi(i)\psi(j)$ are stationary and ergodic conductances under $\nu_V^W(d\beta)$. We can thus apply [6, Theorems 4.5 and 4.6]. In order to have a functional central limit theorem, we need to show that (cf. [6, Theorem 4.5])

$$\mathbb{E}_{\nu_V^W}(W_{i,j}\psi(i)\psi(j)) < \infty. \tag{9.1}$$

In order to show that it has nondegenerate asymptotic covariance we need to show that (cf. [6, Theorem 4.6 and identity (4.20)])

$$\mathbb{E}_{\nu_V^W}\left(\frac{1}{W_{i,j}\psi(i)\psi(j)}\right) < \infty. \tag{9.2}$$

By invariance of the law of the conductances by symmetries of $\mathbb{Z}^d$, we know that the limit diffusion matrix is of the form $\sigma^2\mathrm{Id}$.

The same reasoning works in the case of the ERRW with constant weights $a_{i,j} = a$: in this case $(W_{i,j})$ are i.i.d. but, as shown in Proposition 5, $W_{i,j}\psi(i)\psi(j)$ is also stationary and ergodic under $\widetilde{\nu}_V^a(dW, d\beta)$.

Estimates (9.1) and (9.2) are provided by [10] in the VRJP case, and by [8] in the ERRW case. This is summarized in the following lemma.

**Lemma 9.**

(i) VRJP case. *Consider the VRJP on $\mathbb{Z}^d$, for $d \geq 3$, with constant weights $W_{ij} = W$. There exists $0 < \lambda_2 < \infty$ such that for $W > \lambda_2$, the VRJP is transient and such that* (9.1) *and* (9.2) *are true under $\nu_V^W(d\beta)$.*

(ii) ERRW case. *Consider the ERRW on $\mathbb{Z}^d$, for $d \geq 3$, with constant weights $a_{ij} = a$. There exists $0 < \widetilde{\lambda}_2 < \infty$ such that for $a > \widetilde{\lambda}_2$, the ERRW is transient and* (9.1) *and* (9.2) *are true under $\widetilde{\nu}_V^a(dW, d\beta)$.*

The proof of that lemma is given below. We first apply it to prove the functional central limit theorem. Consider the VRJP case. Assume that the condition of the lemma is satisfied. Define

$$X_t^{(n)} = \frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}.$$

From [6] we know that there exists $0 < \sigma^2 < \infty$ such that for all bounded Lipschitz functions $F$ for the Skorokhod topology, for all $\epsilon > 0$, for all $0 < T < \infty$,

$$\lim_{n\to\infty} Q^*\left(\left|\widetilde{E}_0^{\psi}(F(X_{0\le t\le T}^{(n)})) - \mathbb{E}(F(B_{0\le t\le T}))\right| \geq \epsilon\right) = 0, \tag{9.3}$$

where $B_t$ is a $d$-dimensional Brownian motion with covariance $\sigma^2\mathrm{Id}$, and where $Q^*$ is the invariant measure for the processes viewed from the particle

$$Q^*(d\beta) = \frac{\sum_{j\sim 0} W_{0,j}\psi(0)\psi(j)}{\mathbb{E}_{\nu_V^W}(\sum_{j\sim 0} W_{0,j}\psi(0)\psi(j))} \cdot \nu_V^W(d\beta).$$

It is clear, since $Q^*$ and $\nu_V^W$ are equivalent probability distributions, that (9.3) is also true when $Q^*$ is replaced by $\nu_V^W$. This implies an annealed functional central limit theorem for the process $(X_n)$ under the annealed law $\mathbb{E}_{\nu_V^W}\left(\widetilde{P}_0^{\psi}(\cdot)\right)$:

$$\lim_{n\to\infty}\left|\mathbb{E}_{\nu_V^W}\left(\widetilde{E}_0^{\psi}(F(X^{(n)}_{0\le t\le T})\right) - \mathbb{E}\left(F(B_{0\le t\le T})\right)\right| = 0. \tag{9.4}$$

Let $\Upsilon_t^{(n)} := \frac{1}{\sqrt{n}}(X\circ\theta_{\xi_0})_{[nt]}$. Denote $d^\circ$ the Skorohod metric on $D([0,\infty),\mathbb{R}^d)$, and denote the space of càdlàg functions $f:[0,\infty)\to\mathbb{R}^d$. As

$$|X_t^{(n)} - \Upsilon_t^{(n)}| = \frac{1}{\sqrt{n}}|X_{[nt]} - X_{[nt+\xi_0]}| \le \frac{|\xi_0|}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0,$$

we have

$$d^\circ(X^{(n)},\Upsilon^{(n)}) \to 0. \tag{9.5}$$

Recall that $F$ is a bounded Lipschitz function for the Skorohod topology; therefore,

$$|F(X_t^{(n)}) - F(\Upsilon_t^{(n)})| \to 0,$$

and (9.4) is valid for $X^{(n)}$ replaced by $\Upsilon^{(n)}$. But $\Upsilon^{(n)}$ has the same law under $\widetilde{P}_0^{\psi}$ and $\widetilde{P}_0^{\beta,\gamma,0}$. This implies the functional central limit theorem (9.4), for the law $\mathbb{E}_{\nu_V^W}\left(\widetilde{P}_0^{\beta,\gamma,0}(\cdot)\right)$ in place of $\mathbb{E}_{\nu_V^W}\left(\widetilde{P}_0^{\psi}(\cdot)\right)$ starting from 0. By Theorem 1, the law $\mathbb{E}_{\nu_V^W}\left(\widetilde{P}_0^{\beta,\gamma,0}(\cdot)\right)$ is that of the discrete time process $(\widetilde{Z}_n)$ under $\mathbb{P}_0^{\mathrm{VRJP}}$.

The proof is exactly the same for the ERRW, one just needs to replace the law $\nu_V^W(d\beta)$ by the law $\widetilde{\nu}_V^a(dW,d\beta)$. □

*Proof of Lemma* 9. Let us start with the ERRW case (ii). Consider the sequence of subsets of $\mathbb{Z}^d$, $V_n = [-n,n]^d$. Recall that

$$\psi^{(n)}(j) = e^{u^{(n)}(\delta_n,j)},$$

when $j\in V_n$. Consider the point $y_n = (-n,0,\ldots,0)$, so that $y_n$ is at the boundary of the set, $y_n\sim\delta_n$. By of [8, Lemma 7] (which is the ERRW counterpart of Proposition 7, section 3.3), we have for $a>16$,

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left(\cosh(u(\delta_n,y_n))^8\right) \le 2. \tag{9.6}$$

(Indeed, the proof does not depend on the graph structure, nor on the choice of the rooting.)

From [8, Theorem 4] there exists $0<\widetilde{\lambda}_2<\infty$ such that if $a>\widetilde{\lambda}_2$, then for all $i,j$ in $V_n$,

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left(\left(\cosh(u^{(n)}(\delta_n,i) - u^{(n)}(\delta_n,j))\right)^8\right) \le 2. \tag{9.7}$$

We remark that in [8], the rooting of the field is at 0 and the graph is the restriction of the graph $\mathbb{Z}^d$ to $V_n$. But an attentive reading of the proof shows that the result is also valid for the graph $\mathcal{G}_n = (V_n\cup\{\delta_n\},E_n)$ and rooting $\delta_n$ as well. Indeed, the estimate is based on the protected Ward estimates, [8, Lemma 4], which remain valid for diamonds inside the set $V_n$ and on the estimate on effective conductances, [8, Proposition 3], which is in fact an estimate inside a "diamond". We remark that the estimate (9.7) is also valid when $i$ or $j$ is at the boundary of the set $V_n$ (in fact the proof is written in the case where the diamond $R_{i,j}$ is inside the set $V_n$, which is the case when $j = y_n$ and $i \in \mathbb{Z}^d$ fixed for $n$ large enough). Specified to $j = y_n$ and $i \in \mathbb{Z}^d$ fixed, it gives for $n$ large enough

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left(\left(\cosh(u^{(n)}(\delta_n, i) - u^{(n)}(\delta_n, y_n))\right)^8\right) \le 2. \tag{9.8}$$

By the Cauchy–Schwartz inequality and by (9.6) and (9.8), we get that

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left((\psi^{(n)}(i))^{\pm 4}\right) \le \mathbb{E}_{\widetilde{\nu}_V^a}\left(e^{\pm 8u^{(n)}(\delta_n,y_n)}\right)^{\frac{1}{2}} \mathbb{E}_{\widetilde{\nu}_V^a}\left(e^{\pm 8(u^{(n)}(\delta_n,i)-u^{(n)}(\delta_n,y_n))}\right)^{\frac{1}{2}} \le C_\pm$$

for some constant $C_\pm > 0$ independent of $n$. From this we deduce by Fatou's lemma for all $i, j$ in $\mathbb{Z}^d$, $i \sim j$,

$$\mathbb{E}_{\widetilde{\nu}_V^a}\left((W_{i,j}\psi(i)\psi(j))^{\pm 1}\right) \le \mathbb{E}_{\widetilde{\nu}_V^a}\left((W_{i,j})^{\pm 2}\right)^{\frac{1}{2}} \mathbb{E}_{\widetilde{\nu}_V^a}\left((\psi(0))^{\pm 4}\right)^{\frac{1}{2}} < \infty$$

for $a$ large enough.

The proof is very similar in the VRJP case, and it uses [10, Theorem 1]. As previously, the estimate is valid in the case we are interested in, that is for the graph $\mathcal{G}_n$, rooted at $\delta_n$, and for $x \in \mathbb{Z}^d$, $y = y_n$ for $n$ large enough. $\square$


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